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The Need to Renormalize the Cosmological Constant

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ABSTRACT

We consider the massless, minimally coupled scalar on de Sitter background. Although the 1-loop divergences of the graviton 1PI 2-point function are canceled by the usual Weyl (C^2) and Eddington (R^2) counterterms, there is still a finite, nonzero contribution to the graviton 1-point function. Unless this is canceled by a finite renormalization of the cosmological constant, the 1PI 2-point function will not be conserved, nor will the parameter "H" correspond to the actual Hubble constant. We argue that a similar finite renormalization of the cosmological constant is necessary in pure gravity, and that this must be done when solving the effective field equations for 1-loop corrections to the graviton wave function and to the force of gravity.

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1 Introduction

One of the peculiarities of quantum gravity derives from the fact that the inverse metric and the determinant of the metric are infinite order in the graviton field. In consequence, counterterms generally affect all 1PI (one-particle-irreducible) n-point functions. For example, the R^2 counterterm which is needed to remove dimensionally regulated divergences in the 1PI 2-graviton function [1] also makes a finite, nonzero contribution to the 1-point function on de Sitter background to recover the conformal anomaly [2].

The gravitational divergences induced by a single loop of massless matter fields can all be removed using two quadratic curvature counterterms,

$$\Delta\mathcal{L}_1 \equiv \alpha C_{\rho\sigma\mu\nu} C^{\rho\sigma\mu\nu} \sqrt{-g} \qquad , \qquad \Delta\mathcal{L}_2 \equiv \beta R^2 \sqrt{-g} \; , \tag{1}$$

where $C_{\rho\sigma\mu\nu}$ is the Weyl tensor and R is the Ricci scalar [3]. For a massless, minimally coupled scalar the coefficients α and β can be chosen as,

$$\alpha = \frac{\mu^{D-4}\Gamma(\frac{D}{2})}{2^8 \pi^{\frac{D}{2}}} \frac{2}{(D+1)(D-1)(D-3)^2(D-4)} \; , \tag{2}$$

$$\beta = \frac{\mu^{D-4}\Gamma(\frac{D}{2})}{2^8 \pi^{\frac{D}{2}}} \frac{(D-2)}{(D-1)^2(D-3)(D-4)} \; , \tag{3}$$

where D is the dimension of spacetime and μ is the mass scale of dimensional regularization [4, 5]. On de Sitter background the Weyl (C^2) term makes no change in the 1-point function while the Eddington (R^2) term makes a finite change which, however, leaves a nonzero remainder. The purpose of this note is to show that preserving the Ward identity for the 1PI 2-point function requires that we make an additional, finite renormalization of the cosmological constant which nulls the 1-point function.

In section 2 we derive the connection between the 1-point function and the Ward identity for the primitive contribution to the 1PI 2-point function. Section 3 analyzes the contributions from the two required counterterms (1), and from a cosmological counterterm. Our conlcusions are given in section 4, including a discussion of the implications for pure gravity.

2 Primitive Obstacle to Conservation

The purpose of this section is to derive the connection between a nonzero 1-point function and the failure of the primitive 1PI 2-point function to obey the Ward identity. We begin by defining the graviton field and giving the scalar action and its variations. The obstacle is derived using a functional integral representation for the 1PI 2-point function which is valid for an arbitrary homogeneous and isotropic background. The section closes by evaluating the obstacle for de Sitter background.

2.1 Preliminaries

We define the graviton field $h_{\mu\nu}(x)$ by conformally rescaling with the time-dependent scale factor $a(x^0)$,

$$g_{\mu\nu} \equiv a^2 \widetilde{g}_{\mu\nu} \equiv a^2 \Bigl[\eta_{\mu\nu} + \kappa h_{\mu\nu}\Bigr] \qquad , \qquad \kappa^2 \equiv 16\pi G \quad , \quad H \equiv \frac{\partial_0 a}{a^2} \; . \tag{4}$$

We do not at this stage take the de Sitter limit of constant Hubble parameter H. The massless, minimally coupled scalar action is definied in D spacetime dimensions to facilitate the use of dimensional regularization,

$$S[\phi, h] = -\frac{1}{2} \int \! d^D x \, a^{D-2} \sqrt{-\widetilde{g}} \, \widetilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \; . \tag{5}$$

After taking its variations with respect to the scalar and the graviton we set the graviton field to zero,

$$\frac{\delta S[\phi,0]}{\delta \phi(x)} = \partial_\alpha \Bigl[a^{D-2} \partial^\alpha \phi\Bigr] \quad , \quad \frac{\delta S[\phi,0]}{\delta h_{\mu\nu}(x)} = \frac{\kappa}{2} a^{D-2} \Bigl[\partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi\Bigr] \; . \tag{6}$$

Note that derivative indices are raised and lowered with the Minkowski metric, $\partial^\mu \equiv \eta^{\mu\nu} \partial_\nu$. The second variation with respect to the graviton is,

$$\frac{\delta^2 S[\phi,0]}{\delta h_{\mu\nu}(x) \delta h_{\rho\sigma}(x')} = \frac{\kappa^2}{2} a^{D-2} \Bigl\{ -\frac{1}{4} \eta^{\alpha\beta} \eta^{\rho\sigma} \eta^{\mu\nu} + \frac{1}{2} \eta^{\alpha\beta} \eta^{\mu(\rho} \eta^{\sigma)\nu} + \frac{1}{2} \eta^{\alpha\rho} \eta^{\beta\sigma} \eta^{\mu\nu} \\ + \frac{1}{2} \eta^{\alpha\mu} \eta^{\beta\nu} \eta^{\rho\sigma} - 2 \eta^{\alpha(\rho} \eta^{\sigma)(\mu} \eta^{\nu)\beta} \Bigr\} \partial_\alpha \phi \partial_\beta \phi \delta^D(x-x') \; . \tag{7}$$

2.2 Deriving the Obstacle

It is most convenient to use a functional integral definition for the primitive contribution to the 1PI graviton 2-point function,

$$-i \Bigl[{}^{\mu\nu} \Sigma^{\rho\sigma}_{\rm prim}\Bigr](x;x') = \Bigl[[d\phi] e^{i S[\phi,0]} \Bigl\{ \frac{i \delta S[\phi,0]}{\delta h_{\mu\nu}(x)} \frac{i \delta S[\phi,0]}{\delta h_{\rho\sigma}(x')} + \frac{i \delta^2 S[\phi,0]}{\delta h_{\mu\nu}(x) \delta h_{\rho\sigma}(x')} \Bigr\}. \tag{8}$$

Note that the integration is solely with respect to the scalar, even though the integrand involves variations with respect to the graviton. The Ward identity for a matter loop contribution to the graviton self-energy follows from stress-energy conservation and takes the form,

$$\mathcal{W}^{\mu}{}_{\alpha\beta} \times -i\Big[{}^{\alpha\beta}\Sigma^{\rho\sigma}_{\rm total}\Big](x;x') = 0 \; , \tag{9}$$

where we define the Ward operator as,[1]

$$\mathcal{W}^{\mu}{}_{\alpha\beta} \equiv \delta^{\mu}{}_{(\alpha}\partial_{\beta)} + aH\delta^{\mu}{}_{0}\eta_{\alpha\beta} \; . \tag{10}$$

Note that stress-energy conservation implies that the Ward operator annihilates the graviton variation when the scalar obeys its equation of motion,

$$\mathcal{W}^{\mu}{}_{\alpha\beta} \times \frac{i\delta S[\phi,0]}{\delta h_{\alpha\beta}(x)} = \frac{\kappa}{2}\partial^{\mu}\phi(x) \times \frac{i\delta S[\phi,0]}{\delta\phi(x)} \; . \tag{11}$$

The simplest and most general derivation of the obstacle is to perform a functional partial integration of the Ward operator acting on the first term,

$$\frac{\kappa}{2}\partial^{\mu}\phi(x)\frac{i\delta S[\phi,0]}{\delta\phi(x)}\frac{i\delta S[\phi,0]}{\delta h_{\rho\sigma}(x')}e^{iS[\phi,0]} = \frac{\kappa}{2}\partial^{\mu}\phi(x)\frac{\delta}{\delta\phi(x)}\Big[e^{iS[\phi,0]}\Big]\frac{i\delta S[\phi,0]}{\delta h_{\rho\sigma}(x')}$$
$$\longrightarrow -\frac{\kappa}{2}e^{iS[\phi,0]}\frac{\delta}{\delta\phi(x)}\Big[\partial^{\mu}\phi(x)\frac{i\delta S[\phi,0]}{\delta h_{\rho\sigma}(x')}\Big] \; . \tag{12}$$

There is no contribution from the variation of the $\partial^{\mu}\phi(x)$ term in dimensional regularization but the variation of the second term gives,

$$-\frac{\kappa}{2}\partial^{\mu}\phi(x) \times \frac{i\delta^2 S[\phi,0]}{\delta\phi(x)\delta h_{\rho\sigma}(x')}$$
$$= \frac{i\kappa^2}{2}\partial^{\mu}\phi\Big\{\partial^{(\rho}\Big[a^{D-2}\partial^{\sigma)}\phi\delta^D(x-x')\Big] - \frac{1}{2}\eta^{\rho\sigma}\partial_{\alpha}\Big[a^{D-2}\partial^{\alpha}\phi\delta^D(x-x')\Big]\Big\} . \tag{13}$$

[1]In the absence of mixing between the graviton and matter, each 1-loop matter contribution to the graviton self-energy must separately obey relation (9), without the need to include contributions from other matter fields or gravitons, because each distinct 1-loop contribution is independent of every other one. In particular, there is no mechanism for the 1-loop graviton contribution to depend upon the number or type of matter fields which are coupled to gravity. For example, suppose there are N independent, massless, minimally coupled scalars. Each of these scalars makes the same 1-loop contribution to $-i[{}^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ so, if the graviton loop happened to cancel the obstacle for one choice of N, it would not do so for any other choice.

Acting the Ward operator on the second part of (8) gives,

$$\mathcal{W}^{\mu}{}_{\alpha\beta}\times\frac{i\delta^2 S[\phi,0]}{\delta h_{\alpha\beta}(x)\delta h_{\rho\sigma}(x')}=\frac{i\kappa^2}{2}\Bigg\{-\partial_\alpha\Big[a^{D-2}\partial^\alpha\phi\eta^{\mu(\rho}\partial^{\sigma)}\phi\delta^D(x-x')\Big]$$
$$+\frac{1}{2}\eta^{\mu(\rho}\partial^{\sigma)}\Big[a^{D-2}\partial_\alpha\phi\partial^\alpha\phi\delta^D(x-x')\Big]-\partial^{(\rho}\Big[a^{D-2}\partial^{\sigma)}\phi\partial^\mu\phi\delta^D(x-x')\Big]$$
$$+\frac{1}{2}\eta^{\rho\sigma}\partial_\alpha\Big[a^{D-2}\partial^\alpha\phi\partial^\mu\phi\delta^D(x-x')\Big]+\frac{1}{2}a^{D-4}\partial^\mu\Big[a^2\partial^\rho\phi\partial^\sigma\phi\delta^D(x-x')\Big]$$
$$-\frac{1}{4}a^{D-4}\eta^{\rho\sigma}\partial^\mu\Big[a^2\partial_\alpha\phi\partial^\alpha\phi\delta^D(x-x')\Big]\Bigg\}. \quad (14)$$

After some judicious manipulations, the sum of (13) and (14) becomes,

$$\mathcal{W}^{\mu}{}_{\alpha\beta}\times -i\Big[{}^{\alpha\beta}\Sigma^{\rho\sigma}_{\rm prim}\Big](x;x')=-\kappa\int[d\phi]e^{iS[\phi,0]}$$
$$\times\Bigg\{\eta^{\mu(\rho}\delta^{\sigma)}{}_\alpha\partial_\beta\Big[\frac{i\delta S[\phi,0]}{\delta h_{\alpha\beta}(x)}\delta^D(x-x')\Big]-\frac{1}{2a^2}\frac{i\delta S[\phi,0]}{\delta h_{\rho\sigma}(x)}\partial^\mu\Big[a^2\delta^D(x-x')\Big]\Bigg\}. \quad (15)$$

Note that this expression does not require de Sitter; it is valid for an arbitrary cosmological background.

2.3 Evaluating the Obstacle on de Sitter

The obstacle (15) can be expressed in terms of the scalar propagator $i\Delta(x;x')$,

$$\mathcal{W}^{\mu}{}_{\alpha\beta}\times -i\Big[{}^{\alpha\beta}\Sigma^{\rho\sigma}_{\rm prim}\Big](x;x')$$
$$=\frac{i\kappa^2}{2}\Bigg\{-\eta^{\mu(\rho}\delta^{\sigma)}{}_\alpha\partial_\beta\Big[a^{D-2}\Big(\eta^{\alpha\gamma}\eta^{\beta\delta}-\frac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\delta}\Big)\partial_\gamma\partial'_\delta i\Delta(x;x')\delta^D(x-x')\Big]$$
$$+\frac{a^{D-4}}{2}\Big(\eta^{\rho\gamma}\eta^{\sigma\delta}-\frac{1}{2}\eta^{\rho\sigma}\eta^{\gamma\delta}\Big)\partial_\gamma\partial'{}_\delta i\Delta(x;x')\Big|_{x'=x}\partial^\mu\Big[a^2\delta^D(x-x')\Big]\Bigg\}. \quad (16)$$

This expression is still valid for a general cosmological background.

We now specialize to de Sitter for which the Hubble parameter is constant. The scalar propagator is quite complicated in D dimensions, however, the coincidence limit of its mixed second derivative is simple [6, 7],

$$\partial_\mu\partial'_\nu i\Delta(x;x')\Big|_{x'=x}=-\Big(\frac{D-1}{D}\Big)kH^2a^2\eta_{\mu\nu}\quad,\quad k\equiv\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\;. \quad (17)$$

Substituting (17) in (16) gives the obstacle for de Sitter,

$$\mathcal{W}^{\mu}{}_{\alpha\beta} \times -i \Bigl[{}^{\alpha\beta}\Sigma^{\rho\sigma}_{\rm prim}\Bigr](x;x') \longrightarrow i\kappa^2 H^2 k \frac{(D-2)(D-1)}{4D} \times \Biggl\{ -\eta^{\mu(\rho}\partial^{\sigma)} \Bigl[a^D \delta^D(x-x')\Bigr] + \frac{1}{2} a^{D-2} \eta^{\rho\sigma} \partial^\mu \Bigl[a^2 \delta^D(x-x')\Bigr] \Biggr\} . \quad (18)$$

We have retained dimensional regularization even though the result is finite.

3 Counterterms

The purpose of this section is to consider how counterterms affect the obstacle. We begin by demonstrating that the Weyl counterterm does not affect the obstacle at all, while the Eddington counterterm does not completely eliminate it. The section closes by showing that a finite cosmological counterterm absorbs the obstacle.

3.1 Weyl

The Weyl counterterm has the most complicated tensor structure but it makes the simplest contribution to the graviton self-energy. We define a second order tensor differential operator $\mathcal{C}_{\alpha\beta\gamma\delta}{}^{\mu\nu}$ by expanding the Weyl tensor of the conformally rescaled metric $\widetilde{g}_{\mu\nu}$ in powers of the graviton field,

$$\widetilde{C}_{\alpha\beta\gamma\delta} = \mathcal{C}_{\alpha\beta\gamma\delta}{}^{\mu\nu} \times \kappa h_{\mu\nu} + O(\kappa^2 h^2) \; . \quad (19)$$

The explicit form of this operator can be found in section 3.2 of [8] but we require only the fact that it is both transverse and traceless. The first two variations of the Weyl counterterm are,

$$\frac{i\delta\Delta S_1[0]}{\delta h_{\mu\nu}(x)} = 0 \;\; , \;\; \frac{i\delta^2\Delta S_1[0]}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')} = 2i\alpha\kappa^2 \mathcal{C}^{\alpha\beta\gamma\delta\mu\nu} \Bigl[a^{D-4} \mathcal{C}_{\alpha\beta\gamma\delta}{}^{\rho\sigma} \delta^D(x-x')\Bigr] \; . \quad (20)$$

It follows that the Ward operator annihilates the contribution from the Weyl counterterm,

$$\mathcal{W}^{\mu}{}_{\alpha\beta} \times -i \Bigl[{}^{\alpha\beta}\Sigma^{\rho\sigma}_1\Bigr](x;x') = 0 \; . \quad (21)$$

Note that all of these relations pertain for any cosmological background.

3.2 Eddington

It is best specialize the Eddington counterterm to de Sitter background (with constant H) and to break it up into three terms involving the cosmological constant $\Lambda \equiv (D-1)H^2$,

$$R^2 = \Bigl[R - D\Lambda\Bigr]^2 + 2D\Lambda\Bigl[R - (D-2)\Lambda\Bigr] + D(D-4)\Lambda^2 \; . \qquad (22)$$

This leads to three counterterms,

$$\Delta\mathcal{L}_{2a} = \beta\Bigl[R - D\Lambda\Bigr]^2\sqrt{-g} \; , \qquad (23)$$

$$\Delta\mathcal{L}_{2b} = 2D\beta\Lambda\Bigl[R - (D-2)\Lambda\Bigr]\sqrt{-g} \; , \qquad (24)$$

$$\Delta\mathcal{L}_{2c} = D(D-4)\beta\Lambda^2\sqrt{-g} \; . \qquad (25)$$

Because $\Delta\mathcal{L}_{2c}$ is the same as a cosmological constant, we will postpone its consideration until the next subsection.

To analyze $\Delta\mathcal{L}_{2a}$ we expand $R - D\Lambda$ in powers of the graviton field, like what we did with the conformally rescaled Weyl tensor in expression (19),

$$R - D\Lambda = \frac{1}{a^2}\overline{\mathcal{F}}^{\mu\nu} \times \kappa h_{\mu\nu} + O(\kappa^2 h^2) \; . \qquad (26)$$

The tensor differential operator $\overline{\mathcal{F}}^{\mu\nu}$ is,

$$\overline{\mathcal{F}}^{\mu\nu} \equiv \partial^\mu\partial^\nu - \eta^{\mu\nu}\Bigl[\partial^2 - (D-1)aH\partial_0\Bigr] \\ -2(D-1)aH\delta^{(\mu}{}_0\partial^{\nu)} + D(D-1)a^2H^2\delta^\mu{}_0\delta^\nu{}_0 \; . \qquad (27)$$

Variation of ΔS_{2a} results in this operator being partially integrated to give,

$$\mathcal{F}^{\mu\nu} = \partial^\mu\partial^\nu - \eta^{\mu\nu}\Bigl[\partial^2 + (D-1)aH\partial_0 + (D-1)a^2H^2\Bigr] \\ +2(D-1)aH\delta^{(\mu}{}_0\partial^{\nu)} + (D-2)(D-1)a^2H^2\delta^\mu{}_0\delta^\nu{}_0 \; . \qquad (28)$$

Because $\Delta\mathcal{L}_{2a}$ is quadratic in the graviton, its first variation vanishes at $h_{\mu\nu} = 0$. Its second variation is,

$$\frac{i\delta^2\Delta S_{2a}[0]}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')} = 2i\kappa^2\beta\mathcal{F}^{\mu\nu}\Bigl[a^{D-4}\mathcal{F}^{\rho\sigma}\delta^D(x-x')\Bigr] \; . \qquad (29)$$

The Ward operator annihilates this contribution,

$$\mathcal{W}^{\mu}{}_{\alpha\beta} \times -i\Bigl[{}^{\alpha\beta}\Sigma^{\rho\sigma}_{2a}\Bigr](x;x') = 0 \; . \tag{30}$$

The middle part of the Eddington counterterm $\Delta\mathcal{L}_{2b}$ is proportional to the Einstein-Hilbert Lagrangian so its first variation vanishes at $h_{\mu\nu} = 0$. Its second variation is,

$$\frac{i\delta^2 \Delta S_{2b}[0]}{\delta h_{\mu\nu}(x) \delta h_{\rho\sigma}(x')} = D(D\!-\!1) i\kappa^2 H^2 \beta \Biggl\{ \Bigl[\eta^{\mu(\rho} \eta^{\sigma)\nu} \!-\! \eta^{\mu\nu} \eta^{\rho\sigma}\Bigr] \partial^{\alpha} \Bigl[a^{D-2} \partial_{\alpha} \delta^D(x\!-\!x')\Bigr] \\ + \Bigl[2 \partial'^{(\mu} \eta^{\nu)(\rho} \partial^{\sigma)} \!+\! \eta^{\mu\nu} \partial^{\rho} \partial^{\sigma} \!+\! \partial'^{\mu} \partial'^{\nu} \eta^{\rho\sigma}\Bigr] \Bigl[a^{D-2} \delta^D(x\!-\!x')\Bigr]\Biggr\} . \tag{31}$$

The Ward operator also annihilates this contribution,

$$\mathcal{W}^{\mu}{}_{\alpha\beta} \times -i\Bigl[{}^{\alpha\beta}\Sigma^{\rho\sigma}_{2b}\Bigr](x;x') = 0 \; . \tag{32}$$

3.3 Cosmological

The cosmological counterterm is,

$$\Delta\mathcal{L}_3 \equiv \gamma \sqrt{-g} = \gamma a^D \sqrt{-\widetilde{g}} \; . \tag{33}$$

This term can obviously be combined with $\Delta\mathcal{L}_{2c}$, leading to a counterterm of the same form as (33) but with coefficient,

$$\gamma' = \gamma + D(D\!-\!1)^2 (D\!-\!4) H^4 \beta \; . \tag{34}$$

The first variation is,

$$\frac{i\delta \Delta S_{2c+3}[0]}{\delta h_{\mu\nu}} = \frac{i\gamma' \kappa}{2} a^D \eta^{\mu\nu} \qquad \Longrightarrow \qquad \mathcal{W}^{\mu}{}_{\alpha\beta} \times \frac{i\delta \Delta S_{2c+3}[0]}{\delta h_{\alpha\beta}(x)} = 0 \; . \tag{35}$$

The second variation is,

$$\frac{i\delta^2 \Delta S_{2c+3}[0]}{\delta h_{\mu\nu}(x) \delta h_{\rho\sigma}(x')} = \frac{i\gamma' \kappa^2}{2} a^D \Bigl[-\eta^{\mu(\rho} \eta^{\sigma)\nu} + \frac{1}{2} \eta^{\mu\nu} \eta^{\rho\sigma}\Bigr] \delta^D(x\!-\!x') \; . \tag{36}$$

Acting the Ward operator on this gives,

$$\mathcal{W}^{\mu}{}_{\alpha\beta} \times \frac{i\delta^2 \Delta S_{2c+3}[0]}{\delta h_{\alpha\beta}(x) \delta h_{\rho\sigma}(x')} \\ = \frac{i\gamma' \kappa^2}{2} \Biggl\{ -\eta^{\mu(\rho} \partial^{\sigma)} \Bigl[a^D \delta^D(x\!-\!x')\Bigr] \!+\! \frac{1}{2} a^{D-2} \eta^{\rho\sigma} \partial^{\mu} \Bigl[a^2 \delta^D(x\!-\!x')\Bigr]\Biggr\} . \tag{37}$$

Because the obstruction from a cosmological constant (37) takes the same form as the primitive obstruction (18), they can be made to cancel by choosing,

$$\gamma' = -\frac{(D-2)(D-1)}{2D} k H^2 \; , \tag{38}$$

where the constant k was defined in expression (17). From (38) we see that this corresponds to a renormalization of the cosmological constant which is finite in $D = 4$ dimensions,

$$\gamma = -\frac{\mu^{D-4} H^4}{2^7 \pi^{\frac{D}{2}}} \frac{(D-2)\Gamma(\frac{D}{2}+1)}{D-3} - \frac{H^D}{(4\pi)^{\frac{D}{2}}} \frac{(D-2)\Gamma(D)}{4\Gamma(\frac{D}{2}+1)} \longrightarrow -\frac{H^4}{8\pi^2} \; . \tag{39}$$

4 Conclusions

We considered contributions to the graviton self-energy $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ from a loop of massless, minimally coupled scalars, at first on a general cosmological background and then specialized to de Sitter. Conservation of the scalar stress tensor is usually thought to imply that this quantity should obey a Ward identity (9). However, the primitive contribution to $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ suffers a delta function obstruction (15), whose specialization to de Sitter is (18). This obstruction is proportional to the graviton 1-point function on a general cosmological background, and is finite on de Sitter.

Of course the primitive contribution contains divergences which must be subtracted using counterterms (1) proportional to the squares of the Weyl tensor and the Ricci scalar, with coefficients (2-3). The Weyl (C^2) counterterm has no effect on the obstruction. The Eddington (R^2) counterterm alters the obstruction but does not cancel it. Owing to global scale invariance in $D = 4$ dimensions, the Eddington contribution to the obstruction, and to the graviton 1-point function, is finite on de Sitter. This means that the finite part of the coefficient (3) has no effect on either the obstruction or the 1-point function; only the divergent part matters. Full cancellation of the obstruction requires an additional renormalization of the cosmological constant (33) with finite (on de Sitter) coefficient (39). Making this renormalization also causes the graviton 1-point function to vanish.

Of course we must subtract divergences but finite renormalizations are usually considered to be optional, so we should explain the strong motivation for making this one. First, is the fact that it is needed to remove the obstruction to the Ward identity (9). This is not a sterile, mathematical problem, it compromises our ability to use the renormalized graviton self-energy to solve for quantum corrections to gravitational radiation and to the force of gravity using the linearized effective field equations [9],

$$\mathcal{D}^{\mu\nu\rho\sigma} \times \kappa h_{\mu\nu}(x) - \int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}\Big](x;x')\kappa h_{\rho\sigma}(x') = 8\pi G T^{\mu\nu}(x) \; . \tag{40}$$

Here $T^{\mu\nu}(x)$ is the stress tensor density and $\mathcal{D}^{\mu\nu\rho\sigma}$ is the Lichnerowitz operator. We need conservation because the Ward operator (10) annihilates both the stress tensor density and the Lichnerowitz operator for de Sitter,

$$\mathcal{D}^{\mu\nu\rho\sigma} = \frac{a^2}{2}\Big[(\eta^{\mu(\rho}\eta^{\sigma)\nu} - \eta^{\mu\nu}\eta^{\rho\sigma})\partial^2 + \eta^{\mu\nu}\partial^\rho\partial^\sigma + \eta^{\rho\sigma}\partial^\mu\partial^\nu - 2\partial^{(\mu}\eta^{\nu)(\rho}\partial^{\sigma)}\Big] + Ha^3$$
$$\times\Big[(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu(\rho}\eta^{\sigma)\nu})\partial_0 - 2\eta^{\mu\nu}\delta^{(\rho}{}_0\partial^{\sigma)} + 2\delta^{(\rho}{}_0\eta^{\sigma)(\mu}\partial^{\nu)}\Big] + 3H^2a^4\eta^{\mu\nu}\delta^\rho{}_0\delta^\sigma{}_0 \; . \tag{41}$$

Hence the equation (40) would not be consistent if the Ward operator did not also annihilate $-i[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}](x;x')$.

An additional motivation comes from the connection (15) between the obstacle and the graviton 1-point function. If the 1-point function fails to vanish then back-reaction will change the true expansion rate so that it no longer coincides with the quantity we call the "Hubble parameter" [10]. This does not represent a fine tuning but is instead similar to the on-shell renormalization condition that the quantity "m" in the electron propagator of quantum electrodynamics should represent the actual electron mass.

Relation (15) applies as well to a massive, minimally coupled scalar. However, the greatest relevance of this work is to the study of how loops of inflationary gravitons affect the graviton mode function [11] and the force of gravity [12]. When graviton loops are involved one must act the Ward operator (10) on each of the two points in order to reach zero [13],

$$\mathcal{W}^\mu{}_{\alpha\beta} \times \mathcal{W}'^\rho{}_{\gamma\delta} \times -i\Big[{}^{\alpha\beta}\Sigma^{\gamma\delta}_{\rm ren}\Big](x;x') = 0 \; . \tag{42}$$

An explicit computation shows that this relation is obeyed for $x^\mu \neq x'^\mu$ [14]. However, it seems inevitable that there will be a delta function obstacle, similar to the scalar relation (15), and that removing this obstacle will require a finite renormalization to cancel graviton loop contributions to the 1-point function.

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