Spikes and accretion of unbound, collisionless matter around black holes

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We consider the steady-state density and velocity dispersion profiles of collisionless matter around a Schwarzschild black hole (BH) and its associated rate of accretion onto the BH. We treat matter, which could be stars or dark matter particles, whose orbits are *unbound* to the BH, but still governed by its gravitational field. We consider two opposite spatial geometries for the matter distributions: an infinite, three-dimensional cluster and a two-dimensional razor-thin disk, both with zero net angular momentum. We demonstrate that the results depend critically on the adopted geometry, even in the absence of angular momentum. We adopt a simple monoenergetic, isotropic, phase-space distribution function for the matter as a convenient example to illustrate this dependence. The effect of breaking strict isotropy by incorporating an unreplenished loss cone due to the BH capture of low-angular momentum matter is also considered. Calculations are all analytic and performed in full general relativity, though key results are also evaluated in the Newtonian limit. We consider one application to show that the rate of BH accretion from an ambient cluster can be significantly less than that from a thin disk to which it may collapse, although both rates are considerably smaller than Bondi accretion for a (collisional) fluid with a similar asymptotic particle density and velocity dispersion (i.e., sound speed).

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I. INTRODUCTION

A black hole (BH), and especially a supermassive black hole (SMBH), typically will steepen the density profile of stars and/or dark matter (DM) within the hole's sphere of influence, i.e., within a radius $r_h \approx GM/v_\infty^2$. Here, M is the mass of the hole and v_∞ is the velocity dispersion in the cluster or galaxy core outside r_h. The density profile of this spike depends on the properties of the matter, as well as the formation and evolution history of the BH. Most of the treatments to date focus on matter *bound* to the BH and hence moving on orbits that are confined inside r_h. In some analyses the matter is treated as completely collisionless, as was assumed for stars in typical galaxies in Refs. [1,2], and for cold DM in galaxies and halos in Ref. [3], where in both of these cases the orbiting species were assumed bound to adiabatically growing BHs. In other cases the matter is assumed to be weakly collisional and slowly evolving on long relaxation timescales. Such relaxation may be governed by distant, cumulative, two-body, gravitational encounters (Coulomb scattering) for stars bound to a central, massive BH in a globular cluster or to a SMBH in a dense galaxy core [4–8]. For collisionless DM bound to a SMBH in a galaxy core, relaxation may occur due to gravitational encounters with ambient stars [9–11], or, in the case of self-interacting DM, to particle self-interactions [12–14], and/or annihilations [15,16]. In all of these instances the orbiting matter forms power-law density spikes inside r_h.

Here we consider collisionless matter, either dark matter or stars (both of which we shall refer to as "particles") that move on orbits *unbound* to the BH. These unbound orbits can take particles infinitely far away from the BH, but particles that plunge inside r_h on such orbits also may, in some instances, generate a density spike around the BH. Those particles traveling inward with sufficiently low angular momentum about the BH are eventually captured. By collisionless here we mean that the particles interact solely with the gravitational field of the central BH and that any scattering due to, or collisions with, their neighbors, themselves, or other intruders remain generally unimportant over the age of the system. We take the BH to be a nonspinning Schwarzschild BH with a mass M fixed in time, and we determine the density profile and accretion rate of the unbound ambient matter. Collisionless particles move on geodesic orbits about the BH, and for this analysis we assume their self-gravity is unimportant. We take all of the particles to have the same mass, and to describe them we adopt a simple, monoenergetic, phase-space distribution function of the form $f = f(E)$, where E is the conserved "energy at infinity" per unit mass of a particle. Such a distribution function has been adopted in several previous investigations of unbound particles about a BH [17–20], and, though idealized, it is sufficient to illustrate our main conclusions. Any distribution function that depends solely on conserved integrals of the motion, such as E, automatically satisfies the time-independent, collisionless Boltzmann

(Vlasov) equation. Hence the profiles and accretion rates we determine from our adopted distribution function yield steady-state solutions.

Given our adopted distribution function we compare two opposite spatial geometries for the unbound matter—infinite three-dimensional (3D) clusters and two-dimensional (2D) razor-thin disks. The adopted disks, though extreme, serve to highlight the differences between extended, spherical-like vs thin disklike density distributions for unbound, collisionless particles orbiting a central BH and their associated rates of accretion. We note that collisionless dark matter in the early universe, and even the first generation stars, may, in fact, form thin sheets or "pancakes" [21,22], so our extreme disks may mimic some of their features, should massive BHs reside within them. In both our cluster and disk cases we analyze the case where the orbital velocities are isotropic at every point. We then consider the anisotropic velocity distribution associated with an unreplenished loss cone that can arise from BH capture of particles with sufficiently low angular momentum. In all cases the net angular momentum of the orbiting matter is assumed to be zero, so we are only examining the effect of spatial geometry on the profiles and accretion rates.

Infinite, unbound collisionless clusters around Schwarzschild BHs have been previously investigated with a different approach in Refs. [23,24], who adopted a Maxwell-Jünter phase-space distribution function as an application. A treatment involving thin (equatorial plane) disks around Kerr black holes was recently provided by Ref. [25], again with a Maxwell-Jünter distribution function and a different approach from the one adopted here. We compare the accretion rates found by these studies with the ones calculated here.

The significance of determining the density profiles and accretion rates of stars and DM near BHs is that it provides clues as to the nature, formation history, and evolution of the systems in which they are found. The presence of a sufficiently steep density spike not only conveys the presence of a massive, central BH but, in the case of DM, may lead to an observable excess of gamma rays or other form of radiation, if particle annihilation occurs at a sufficient rate in the innermost regions where the density is highest (see [3,26–31] and references therein). These applications motivate our adopting a simple illustration to point out the importance of unbound collisionless particles and their global spatial geometry in determining the existence of a stellar, or DM, spike around a BH and their consumption rate by the BH.

In Sec. II we consider an infinite, unbound, monoenergetic, 3D cluster of collisionless particles that orbit a Schwarzschild BH. We derive the particle density and velocity profiles and the associated particle accretion rate onto the BH. We consider both an isotropic velocity distribution and one with an unreplenished loss cone. The derivations and quoted quantities are analytic and fully general relativistic, but the final expressions are also quoted for the slow-velocity, weak-field (i.e., Newtonian) limit. In Sec. III we repeat the analysis in Sec. I, but now for particles confined to a 2D razor-thin disk. In Sec. IV we compare the accretion rates found for the clusters and disks. In the Appendix we provide two alternative derivations for the cluster accretion rate, all arriving at the same answer. We adopt geometric units throughout, setting $G = 1 = c$, unless otherwise noted.

II. UNBOUND, MONOENERGETIC CLUSTER

A. Isotropic distribution

The gravitational field of our system is governed by the Schwarzschild BH, and we adopt the following familiar form for the spacetime metric:

$$ds^2 = -(1-2M/r)dt^2 + \frac{dr^2}{1-2M/r} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{1}$$

The monoenergetic phase-space distribution function we shall employ everywhere in space is given by

$$f(E) = K\delta(E - E_\infty), \qquad K, E_\infty \text{ const}, \tag{2}$$

where $E = -\mathbf{p}\cdot\mathbf{e_t}$ is the "energy at infinity" of a particle per unit mass, $\mathbf{p}$ is the particle energy-momentum 4-vector, divided by its rest mass, and $\mathbf{e_t} = \partial/\partial t$ is the time coordinate basis vector. All particles are assumed to have the same rest mass m. The constant E_∞ satisfies $E_\infty > 1$, appropriate for an unbound orbit, while the constant K, given E_∞, is determined by the asymptotic density at infinity [see Eq. (9) below]. This distribution function, specifically chosen to describe a homogeneous density of unbound, collisionless particles moving randomly at the same speed far from the BH in fact applies everywhere as a consequence of Liouville's theorem. Our adopted function, though simple, may also be thought of as a Green's function for other energy distribution functions (e.g., Maxwellians or power laws) and our results below in which E_∞ appears can always be integrated over E_∞ for these alternative energy distributions.

1. Density

We now repeat the derivation of the number density of particles at all radii r we provided previously (see Appendix in Ref. [19]), so that we may refer to and modify some of the equations in subsequent sections where we change the boundary conditions or contrast the results with other cases. The particle energy per unit mass measured by a local, static, orthonormal observer with 4-velocity $\mathbf{u} = \mathbf{e}_{\hat{t}}$ at radius r is given by

$$E_{\rm local} \equiv p^{\hat{t}} = -\mathbf{p}\cdot\mathbf{e}_{\hat{t}} = \frac{E}{(1-2M/r)^{1/2}}, \tag{3}$$

where a caret on a variable denotes an orthonormal component. The number density of particles at r measured by this observer is then

$$n(r) = \int f(E)d^3\hat{p} \tag{4}$$

$$= 4\pi \int f(E)\hat{p}^2 d\hat{p}, \tag{5}$$

where

$$\hat{p} = [(p^{\hat{t}})^2 - 1]^{1/2} \tag{6}$$

is the particle 3-momentum per unit mass. Using Eqs. (3) and (6) we obtain

$$\hat{p}^2 d\hat{p} = \frac{\hat{p}EdE}{1-2M/r} = \frac{(E^2-1+2M/r)^{1/2}}{(1-2M/r)^{3/2}}EdE. \tag{7}$$

Substituting Eqs. (2) and (7) into (5) gives

$$n(r) = 4\pi K \frac{(E_\infty^2 - 1 + 2M/r)^{1/2}E_\infty}{(1-2M/r)^{3/2}}. \tag{8}$$

Evaluating Eq. (8) at $r = \infty$ determines K:

$$K = \frac{n_\infty}{4\pi(E_\infty^2-1)^{1/2}E_\infty}. \tag{9}$$

So in general the density profile is

$$\frac{n(r)}{n_\infty} = \frac{(E_\infty^2 - 1 + 2M/r)^{1/2}}{(E_\infty^2-1)^{1/2}(1-2M/r)^{3/2}}. \tag{10}$$

An important special case applies to nonrelativistic particles at infinity moving with velocity $\hat{v}_\infty \ll 1$, for which

$$E_\infty \approx 1 + \frac{1}{2}\hat{v}_\infty^2, \qquad E_\infty^2 - 1 \approx 2(E_\infty - 1) \approx \hat{v}_\infty^2. \tag{11}$$

In this limit Eq. (9) reduces to

$$K \approx \frac{n_\infty}{4\pi\hat{v}_\infty} \qquad (\hat{v}_\infty \ll 1), \tag{12}$$

and Eq. (8) becomes

$$\frac{n(r)}{n_\infty} \approx \frac{[1+2M/(r\hat{v}_\infty^2)]^{1/2}}{(1-2M/r)^{3/2}} \qquad (\hat{v}_\infty \ll 1). \tag{13}$$

We note that Eq. (13) in the weak-field domain, wherein $M/r \ll 1$, is consistent with earlier Newtonian derivations in Refs. [17,18].

2. Velocity dispersion

The velocity dispersion may be calculated formally from

$$\langle \hat{v}^2(r)\rangle = \frac{\int \hat{v}^2 f(E)d^3\hat{p}}{\int f(E)d^3\hat{p}}, \tag{14}$$

where the locally measured velocity is $\hat{v} = \hat{p}/E_{\rm local}$ or, using Eqs. (3) and (6),

$$\hat{v} = \frac{(E^2-1+2M/r)^{1/2}}{E}. \tag{15}$$

Substituting Eqs. (2) and (15) into (14) and integrating yields

$$\langle \hat{v}^2(r)\rangle = \hat{v}^2(E_\infty) = \frac{(E_\infty^2 - 1 + 2M/r)^{1/2}}{E_\infty}. \tag{16}$$

The first equality in Eq. (16) also may be arrived at trivially by noting that all particles move on geodesics with the same specific energy E_∞, and the second equality also can be obtained by invoking $E_{\rm local} = \gamma = 1/(1-\hat{v}^2)^{\frac{1}{2}}$, replacing $E_{\rm local}$ by E using Eq. (3), and inverting for $\hat{v}$.

3. Accretion rate

Here we generalize our earlier Newtonian derivation of the DM accretion rate onto the BH [see Ref. [18], Eqs. (14.2.13)–(14.2.20)] to one that is fully general relativistic. The phase-space momentum element $d^3\hat{p}$ may be expressed as

$$d^3\hat{p} = 2\pi p^{\hat{\perp}}dp^{\hat{\perp}}dp^{\hat{r}} = \frac{4\pi J dJ dE}{(1-2M/r)^{1/2}|v^{\hat{r}}|r^2}, \tag{17}$$

where $\perp$ denotes directions perpendicular to the radial direction (whereby $J = rp^{\hat{\perp}}$ is the particle angular momentum per unit mass) and where Eqs. (12.4.9), (12.4.13), and (12.4.16) of Ref. [18] were employed to relate $dp^{\hat{r}}$ to dE:

$$dp^{\hat{r}} = \frac{dE}{(1-2M/r)^{1/2}|v^{\hat{r}}|}. \tag{18}$$

An additional factor of 2 arises in Eq. (17) since for a given E, $p^{\hat{r}}$ can be either positive or negative. Let $N^-(r,E,J)$ be the number of particles per interval dr, dE, and dJ with *inward*-directed radial velocity:

$$N^-(r,E,J)drdEdJ = \frac{1}{2} f(E,J) d^3 r d^3 \hat{p} = \frac{8\pi^2 J f(E,J)}{|v^{\hat{r}}|(1-2M/r)^{1/2}} drdEdJ. \quad (19)$$

The resulting total capture rate for particles onto the central BH as measured by the local observer is then

$$\frac{dN_{\rm tot}}{d\tau} = \int_{(1-\frac{2M}{r})^{\frac{1}{2}}}^{\infty} dE \int_0^{J_{\rm min}(E)} dJ |v^{\hat{r}}| N^-(r,E,J) = 8\pi^2 \int_{(1-\frac{2M}{r})^{\frac{1}{2}}}^{\infty} dE \int_0^{J_{\rm min}(E)} dJ \frac{Jf}{(1-\frac{2M}{r})^{1/2}}, \quad (20)$$

where particles moving with specific angular momentum less than a critical value $J_{\rm min}$ will be captured by BH as they approach the pericenter. The region $J < J_{\rm min}$ thus defines a capture loss cone. For nonrelativistic (NR) particles at large distances with $E - 1 \ll 1$, which is the case for typical stars in clusters of galaxies, and cold dark matter, we note that

$$J_{\rm min}(E) = 4M \qquad (\text{NR particles}, \hat{v}_\infty \ll 1). \quad (21)$$

Substituting Eqs. (2) and (9) into Eq. (20) and evaluating the integral yields

$$\frac{dN_{\rm tot}}{d\tau} = \frac{\pi n_\infty J^2_{\rm min}(E_\infty)}{E_\infty (E^2_\infty - 1)^{1/2}} \frac{1}{(1-2M/r)^{1/2}}. \quad (22)$$

The particle capture rate as measured by a static observer at infinity is then given by

$$\frac{dN_{\rm tot}}{dt} = \frac{dN_{\rm tot}}{d\tau}\frac{d\tau}{dt} = \frac{dN_{\rm tot}}{d\tau}\left(1 - \frac{2M}{r}\right)^{1/2} \quad (23)$$

or

$$\frac{dN_{\rm tot}}{dt} = \frac{\pi n_\infty J^2_{\rm min}(E_\infty)}{E_\infty (E^2_\infty - 1)^{1/2}}. \quad (24)$$

It is this observer who measures the steady depletion of particles from the ambient cluster and their acquisition by the BH.

The corresponding *rest-mass* accretion rate measured by this observer, $dM_0/dt = m dN_{\rm tot}/dt$, is then given by

$$\frac{dM_0}{dt} = \frac{\pi \rho_\infty J^2_{\rm min}(E_\infty)}{E_\infty (E^2_\infty - 1)^{1/2}}, \quad (25)$$

where $\rho_\infty \equiv m n_\infty$ is the asymptotic *rest-mass* density. Equation (25) agrees with the result obtained by a different route in Ref. [20] [see their Eq. (76), setting $\gamma_\infty = E_\infty$ and $L_c(E_\infty) = m J_{\rm min}(E_\infty)$]. The rate of accretion of *total mass-energy* onto the black hole is then given by $dM/dt = m E_\infty dN/dt$ or

$$\frac{dM}{dt} = \frac{\pi \rho_\infty J^2_{\rm min}(E_\infty)}{(E^2_\infty - 1)^{1/2}}. \quad (26)$$

Evaluating Eqs. (25) and (26) for NR particles at infinity, using Eqs. (11) and (21), yields

$$\frac{dM}{dt} \approx \frac{dM_0}{dt} \approx \frac{16\pi M^2 \rho_\infty}{\hat{v}_\infty} \quad (\text{NR particles}, \hat{v}_\infty \ll 1), \quad (27)$$

which agrees with the previous Newtonian result we derived in Ref. [18] [see Eq. (14.2.20)]. Since the steady-state accretion rate can be evaluated at large $r \gg M$, as we did in Ref. [18], it is no surprise that the Newtonian derivation for the rate provided there for nonrelativistic particles at infinity yields the exact same result derived in General Relativity (GR) for the limiting case given by Eq. (27). Note that to convert back from geometric to physical units, one simply multiplies the right-hand sides of Eqs. (25)–(27) by G^2/c^2.

It is interesting to note that the particle density as measured by a locally static observer *blows up* when the observer is stationed arbitrarily close to the BH horizon [see Eq. (10) as $r \to 2M$]. (However, recall that static observers cannot exist at or inside the horizon.) Using Eq. (12.4.17) of Ref. [18], for the particle radial velocity, i.e.,

$$|v^{\hat{r}}| = \left[1 - \frac{1}{E^2}\left(1 - \frac{2M}{r}\right)\left(1 + \frac{J^2}{r^2}\right)\right]^{1/2}, \quad (28)$$

it is seen that the particles all have velocities which are measured by this observer to approach the speed of light and move in the radial direction as they approach the horizon. Based on these observations one might naively take the radial matter flux to be $\sim n(r)\hat{v}$, and then estimate the accretion rate by multiplying this product by the invariant area $4\pi r^2$. Doing so yields a rate that blows up near the horizon. However, this estimate is too naive, as the actual steady-state depletion rate of unbound particles from the ambient gas must be obtained by more careful considerations, such as those we incorporated above, leading to the net accretion rates onto the BH as measured by a distant, static observer, Eqs. (24)–(26). The latter remain perfectly *finite*. This contrast is sufficiently striking that we are motivated to provide two alternative derivations in the Appendix for the net accretion rate, leading to the same equations obtained above.

The integration of an isotropic distribution function $f(E)$ over all phase space allows for so-called "white hole" orbits, i.e., outgoing trajectories at the horizon. Eliminating these orbits leads to a density profile of the form $n(r) \sim 1/(1-2M/r)^{1/2}$ as $r \to 2M$. Thus, we still have $n(r)$ blowing up near the horizon, but only with a mild ("redshift") factor instead of the $1/(1-2M/r)^{3/2}$ behavior exhibited by Eq. (10). Using this "modified" density near the horizon to measure the inward flux then yields the correct locally measured accretion rate, Eq. (22). Furthermore, choosing to measure the density by an observer comoving with the net flow should lead to a finite density at the horizon [23,24]. However, as these modifications do not affect our computations of the accretion rates, which are our primary targets, we do not implement them here.

4. Massless and extremely relativistic particles

To evaluate Eq. (26) for massless particles (e.g., photons), we may first set $E_\infty = (1-\hat{v}_\infty^2)^{-1/2}$ and then note that $J_{\min} = b(E_\infty^2 - 1)^{1/2} = \hat{v}_\infty(1-\hat{v}_\infty^2)^{-1/2}$, where b is the critical impact parameter for particle capture. Now $b = 3\sqrt{3}$ is the critical impact parameter for massless particles [see, e.g., Ref. [18], Eqs. (12.4.36) and (12.5.11)]. Next let $n_\infty m E_\infty \equiv \epsilon_\infty^r$ be the energy density of particles far from the BH. Then taking the limit as $\hat{v}_\infty \to 1$ with these substitutions gives

$$\frac{dM}{dt} = 27\pi M^2 \epsilon_\infty^r \qquad (m=0), \tag{29}$$

which agrees with the result quoted in Exercise 14.4 in Ref. [18] that was obtained by an alternative approach.

We also note that Eq. (29) also applies to extremely relativistic (ER) particles with nonzero rest mass m. For such particles the accretion rates may also be written as

$$\frac{dM}{dt} = E_\infty \frac{dM_0}{dt} \approx 27\pi M^2 E_\infty \rho_\infty \qquad (\text{ER particles}, \hat{v}_\infty \gg 1). \tag{30}$$

B. Loss-cone effect

Given that particles having nearly radial, inward velocities result in their being captured by the BH, it is interesting to examine a scenario in which, after a sufficient time has passed and steady state is achieved, there is a depletion of particles in a low-angular momentum capture loss cone about the BH. For perfectly collisionless gas in the complete absence of perturbations, the loss cone cannot be replenished, since particle self-interactions and gravitational scattering by stars or other perturbers are assumed absent. (For scenarios in which self-interactions of dark matter particles around BHs may be important, see, e.g., Refs. [12–14], and where their gravitational scattering off stars may be significant, see, e.g., Refs. [9–11] and references therein.) In reality, the slightest gravitational perturbations, at large distances, due, for example, to small density anisotropies or to intruders, or the weakest self-interactions, will likely be sufficient to replenish the narrow loss cone. Our analysis above of the matter profiles and accretion rate will thus apply in these situations. But immediately below we will consider a perturbation-free, perfectly collisionless cluster that has depleted its loss cone and cannot refill it.

To describe this idealized situation a minimal modification to our adopted distribution function will suffice:

$$\begin{aligned} f(E) &= K\delta(E - E_\infty), \qquad J_{\min}(E) \le J \le J_{\max}(E), \\ &= 0, \qquad 0 \le J \le J_{\min}(E), \end{aligned} \tag{31}$$

where again K and $E_\infty > 1$ are constants.

1. Density

To determine the density profile we again evaluate Eq. (4), employing Eq. (17), which yields

$$n(r) = \int_{(1-\frac{2M}{r})^{\frac{1}{2}}}^{\infty} f(E)dE \int_{J_{\min}(E)}^{J_{\max}(E)} \frac{4\pi J dJ}{r^2 |v^{\hat{r}}| (1-\frac{2M}{r})^{1/2}}, \tag{32}$$

where $|v^{\hat{r}}|$ is given by Eq. (28) and $J_{\max}(E)$ is the maximum specific angular momentum that a particle moving on a geodesic with specific energy E can have,

$$J_{\max}(E) = r\left[\frac{(E^2 - 1 + \frac{2M}{r})}{(1-\frac{2M}{r})}\right]^{1/2} \tag{33}$$

[set $dr/d\tau = 0$ and $\tilde{l} = J_{\max}(E)$ in Eq. (12.4.13) in Ref. [18]]. Substituting Eq. (31) into (32) and integrating yields

$$\frac{n(r)}{n_\infty} = \frac{[E_\infty^2 - (1-2M/r)(1+(J_{\min}(E)/r)^2)]^{1/2}}{(E_\infty^2 - 1)^{1/2}(1-2M/r)^{3/2}}, \tag{34}$$

where the normalization constant K is again given by Eq. (9). Comparing Eqs. (10) and (34) shows that the spike density is somewhat lower everywhere, but mostly around $r \lesssim J_{\min}$, if a loss cone is established and unreplenished.

Equation (34) reduces to Eq. (10) in the absence of a loss cone, i.e., when the captured particle distribution is assumed to be continually replenished so that $J_{\min}(E) = 0$ in Eq. (31). For nonrelativistic particles with $\hat{v}_\infty \ll 1$ and an empty loss cone with $J_{\min} = 4M$ the density becomes

$$\frac{n(r)}{n_\infty} \approx \frac{[1 + 2M/(r\hat{v}_\infty^2)(1 - \frac{8M}{r} + (\frac{4M}{r})^2)]^{1/2}}{(1-2M/r)^{3/2}} \qquad (\text{NR particles}, \hat{v}_\infty \ll 1). \tag{35}$$

For sufficiently small $\hat{v}_\infty \ll 1$ the density exhibits a minimum at $r \approx 4M$ outside the event horizon.

2. Velocity dispersion

As all particles have the same energy E_∞, they have the same velocity and velocity dispersion profile as in the absence of a loss cone, i.e., Eq. (16).

3. Accretion rate

Given a loss cone that is unreplenished with particles once they have been captured, the accretion rate becomes zero in steady state.

III. UNBOUND, MONOENERGETIC, RAZOR-THIN DISK

A. Isotropic distribution

We now derive the surface density profile around the BH for unbound, collisionless particles assumed to reside in a razor-thin disk. Since the distribution function is zero outside of the disk, we will simply work in the plane of the disk and drop the vertical dependence of any quantity. For comparison purposes we adopt the same distribution function adopted for the large 3D cluster, but now confined to the 2D disk plane, i.e., we again use Eq. (2) to describe unbound particles moving isotropically in the plane of the disk.

1. Density

The surface number density $\Sigma^*(r)$ measured by a locally static observer is then given by

$$\Sigma^*(r) = \int f(E) d^2\hat{p} = 2\pi \int f(E) \frac{EdE}{1 - 2M/r}, \quad (36)$$

where we used Eq. (7) to evaluate $d^2\hat{p} = 2\pi\hat{p}d\hat{p}$. Substituting Eq. (2) in (36) and integrating yields

$$\frac{\Sigma^*(r)}{\Sigma^*_\infty} = \frac{\Sigma(r)}{\Sigma_\infty} = \frac{1}{1 - 2M/r}, \quad (37)$$

where $\Sigma(r) = m\Sigma^*(r)$ is the surface rest-mass density, m is the particle rest mass, and $K = \Sigma_\infty/(2\pi m E_\infty)$. Far from the BH the surface density is thus seen to be flat. Once again, eliminating the “white hole” orbits softens the blowup near the horizon, whereby the modified surface density scales as $\Sigma(r) \sim 1/(1 - 2M/r)^{1/2}$ as $r \to 2M$.

2. Velocity dispersion

As all particles again have the same energy E_∞, they have the same velocity and velocity dispersion profile given by Eq. (16).

3. Accretion rate

To derive the accretion rate we adapt Eq. (19) to a plane, yielding

$$N^-(r, E, J) dr dE dJ = \frac{1}{2} f(E, J) d^2r d^2\hat{p}$$
$$= 4\pi \frac{dJdEdr}{(1 - 2M/r)^{1/2}} \frac{f(E, J)}{|v^{\hat{r}}|}, \quad (38)$$

where we used $d^2r = 2\pi r dr$, $J = p^{\hat{\perp}} r$, Eq. (18), and

$$d^2\hat{p} = dp^{\hat{\perp}} dp^{\hat{r}} = 4 \frac{dJ}{r} \frac{dE}{(1 - 2M/r)^{1/2} |v^{\hat{r}}|}. \quad (39)$$

The factor of 4 is inserted since for a given E and J, both $p^{\hat{r}}$ and $p^{\hat{\perp}}$ can be either positive or negative.

The resulting capture rate measured by the local observer is then given by

$$\frac{dN_{\text{tot}}}{d\tau} = \int_{(1-\frac{2M}{r})^{\frac{1}{2}}}^{\infty} dE \int_0^{J_{\min}(E)} dJ |v^{\hat{r}}| N^-(r, E, J)$$
$$= 4\pi \int_{(1-\frac{2M}{r})^{\frac{1}{2}}}^{\infty} dE \int_0^{J_{\min}(E)} dJ \frac{f}{(1 - \frac{2M}{r})^{1/2}}, \quad (40)$$

yielding

$$\frac{dN_{\text{tot}}}{d\tau} = \frac{2\Sigma^*_\infty J_{\min}(E_\infty)}{E_\infty} \frac{1}{(1 - 2M/r)^{1/2}}. \quad (41)$$

The depletion rate of the disk as measured by a distant static observer is then obtained using Eq. (23), which gives

$$\frac{dN_{\text{tot}}}{dt} = \frac{2\Sigma^*_\infty J_{\min}(E_\infty)}{E_\infty}. \quad (42)$$

The corresponding rates of accretion of rest mass and total mass-energy onto the BH are then

$$\frac{dM_0}{dt} = \frac{2\Sigma_\infty J_{\min}(E_\infty)}{E_\infty} \quad (43)$$

and

$$\frac{dM}{dt} = 2\Sigma_\infty J_{\min}(E_\infty), \quad (44)$$

respectively. The rate for NR particles at infinity, using Eq. (21), is then

$$dM/dt \approx 8M\Sigma_\infty \qquad (\text{NR particles}, \hat{v}_\infty \ll 1). \quad (45)$$

We note that Eq. (45) agrees with Eq. (93) in Ref. [25] for Maxwell-Jüttner particle temperatures approaching zero and Schwarzschild BHs [set $\alpha \equiv a/M = 0$ and note $\rho_{s,\infty} \equiv \Sigma_\infty$ in Eq. (93)].

It is interesting to observe that in this slow-velocity limit, the above rate of mass-energy accretion for a 2D razor-thin disk depends only on the surface density of distant particles and not on their velocity dispersion, in contrast to accretion from a large 3D cluster, which depends on both the asymptotic density and the velocity dispersion [see Eq. (27)].

We note that to convert back from geometric to physical units, one simply multiplies the right-hand sides of Eqs. (43)–(45) by G^2/c.

4. Massless and extremely relativistic particles

Defining the asymptotic particle surface energy density to be $\mathcal{E}^r_\infty \equiv \Sigma_\infty E_\infty$ and noting that $J_{\min} \approx 3\sqrt{3} M E_\infty$ when $E_\infty \gg 1$, we can evaluate Eq. (44) for massless particles, yielding

$$\frac{dM}{dt} = 6\sqrt{3} M \mathcal{E}^r_\infty \qquad (m = 0). \tag{46}$$

We again note that Eq. (46) also applies to ER particles with nonzero m. For such particles the accretion rates may also be written as

$$\frac{dM}{dt} = E_\infty \frac{dM_0}{dt} \approx 6\sqrt{3} M \Sigma_\infty E_\infty \qquad (\text{ER particles}, \hat{v}_\infty \gg 1). \tag{47}$$

We point out that Eq. (47) agrees with Eqs. (124) and (125) in Ref. [25] for Maxwell-Jüttner particle temperatures approaching infinity and Schwarzschild BHs [again set $\alpha \equiv a/M = 0$ and note $\rho_{s,\infty} \equiv \Sigma_\infty$ in Eq. (124) and $\epsilon_{s,\infty} = \Sigma_\infty E_\infty$ in Eq. (125)].

B. Loss-cone effect

1. Density

Here we treat the scenario whereby collisionless particles in the loss cone are never replenished once captured, whereby the distribution function may again be represented by Eq. (31). Using Eq. (39), the surface density is then given by

$$\Sigma^*(r) = \int f(E) d^2\hat{p} = \int_{(1-\frac{2M}{r})^{\frac{1}{2}}}^{\infty} f(E) dE \int_{J_{\min}(E)}^{J_{\max}(E)} \frac{4dJ}{r|v^{\hat{r}}|(1-\frac{2M}{r})^{1/2}}. \tag{48}$$

Substituting Eqs. (31) and (33) and integrating yields

$$\frac{\Sigma(r)}{\Sigma_\infty} = \frac{1 - \frac{2}{\pi}\arctan\left(\frac{J_{\min}/r}{\left[\frac{E_\infty^2 - 1 + 2M/r}{1-2M/r} - (J_{\min}/r)^2\right]^{1/2}}\right)}{1 - 2M/r}. \tag{49}$$

For NR particles in Newtonian gravitation, Eq. (49) reduces to

$$\frac{\Sigma(r)}{\Sigma_\infty} \approx 1 - \frac{2}{\pi}\arcsin\left(\frac{4M}{r\hat{v}_\infty}\right) \qquad (\hat{v}_\infty \ll 1, M/r \ll 1). \tag{50}$$

2. Velocity dispersion

As all particles again have the same energy E_∞, they have the same velocity and velocity dispersion profile given by Eq. (16).

3. Accretion rate

Given a loss cone that is unreplenished with particles once they have been captured, the accretion rate becomes zero in steady state.

IV. COMPARISON OF ACCRETION RATES

Comparing the accretion rates of 3D clusters and 2D thin disks with filled loss cones is not entirely straightforward, given their different geometries and defining parameters. The results will depend on the different physical systems that are compared. To give one example, let us consider a large, homogeneous spherical cluster of particles with density ρ_∞, isotropic velocity dispersion v_∞, radius $R_c \gg r_h$, and total mass M_c. Imagine that it undergoes collapse parallel to the z-axis to a thin pancake in the x-y plane, preserving its surface density along cylinders. This scenario might mimic one way that thin sheets of collisionless particles form in the early universe. We will compare the unbound, collisionless particle accretion rates for the large spherical cluster and the pancake for NR particles with $v_\infty \ll 1$.

The surface density in the pancake is

$$\Sigma_\infty(r_\perp) = 2\rho_\infty R_c \left(1 - \frac{r_\perp^2}{R_c^2}\right)^{1/2}, \tag{51}$$

where $r_\perp$ is the radius in the x-y plane measured from the center of the pancake. Taking the surface density at $r_\perp$ in the central core, requiring $r_h \ll r_\perp \ll R_c$, whereby the particles remain largely unperturbed by the central BH, gives

$$\Sigma_\infty \approx 2\rho_\infty R_c. \tag{52}$$

The ratio of BH accretion rates in the spherical cluster vs the thin disk is then approximated by

$$\frac{\dot{M}_c}{\dot{M}_d} \approx \pi \frac{M}{R_c} \frac{1}{v_\infty}, \tag{53}$$

where $\dot{M}_c$ is given by Eq. (27) and $\dot{M}_d$ is given by Eq. (45), substituting Eq. (52) for Σ_∞. We can evaluate v_∞ if we assume that it is some fraction of the virial value, whereby

$$v_\infty^2 \approx \frac{M_c}{R_c}. \tag{54}$$

Equation (53) then yields

$$\frac{\dot{M}_c}{\dot{M}_d} \approx \pi \left(\frac{M}{M_c}\right)^{1/2} \left(\frac{M}{R_c}\right)^{1/2} \ll 1. \tag{55}$$

The strong inequality above for this one extreme example shows the dominance of disk vs cluster accretion, thereby demonstrating that *the rate of unbound, collisionless particles can depend significantly on the geometry of the ambient particle cloud, other parameters being equal*.

It is also interesting to compare the collisionless particle rates to the standard spherical Bondi accretion rate for particles comprising a true fluid. The Bondi rate for an adiabatic gas with $1 \leq \Gamma \leq 5/3$ [32] (see also [18] for a relativistic treatment) is given by

$$\frac{dM}{dt} = \dot{M}_b = 4\pi\lambda \frac{M^2 \rho_\infty}{a_\infty^3}, \tag{56}$$

where a_∞ is the asymptotic sound speed of the fluid and λ is a constant of order unity that depends on Γ. We will equate a_∞ to v_∞ for comparison below, obtaining

$$\frac{\dot{M}_b}{\dot{M}_c} \approx \frac{\lambda}{4} \frac{1}{v_\infty^2} \gg 1 \tag{57}$$

and

$$\frac{\dot{M}_b}{\dot{M}_d} \approx \frac{\pi\lambda}{4} \frac{M}{R_c} \frac{1}{v_\infty^3} \approx \frac{\pi\lambda}{4} \left(\frac{M}{M_c}\right) \left(\frac{R_c}{M_c}\right)^{1/2}, \tag{58}$$

where we used Eq. (54) to obtain the second equality in Eq. (58). Equation (57) shows that Bondi accretion dominates collisionless particle accretion for a large 3D cluster. However, Eq. (58) suggests that the ratio for an extended 2D thin disk depends on the particular system, since the first factor in parentheses on the right-hand side may be much smaller than unity, but the second factor is much bigger than unity. Once again, spatial geometry counts.

A. Applications

There may exist a near "universal" value of the surface density for DM that spans, within a factor of 2, over at least nine (and possibly more) galaxy magnitudes and across several different Hubble types [33,34]:

$$\Sigma_\infty^u \approx 140 M_\odot \text{ pc}^{-2}. \tag{59}$$

The cosmological implications of this observation are not yet resolved, but Eq. (45) suggests that a thin disk with the surface density Σ_∞^u and a central BH would have an accretion rate given by

$$\frac{dM_d^u}{dt} \approx 3.3 \times 10^{-2} M_\odot \text{ yr}^{-1} \left(\frac{M}{10^6 M_\odot}\right) \left(\frac{\Sigma_\infty^u}{140 M_\odot \text{ yr}^{-1}}\right). \tag{60}$$

The surface density Σ_∞^u is also within a factor of 2 of estimates of DM in the Galactic neighborhood, where $\rho_D \sim 0.008 M_\odot$ pc^{-3} and $D \sim 8.5$ kpc [35], yielding $\Sigma_\infty^{\rm Gal} \sim \rho_D D \approx 68 M_\odot$ pc^{-2}. Since ρ_D may scale as r^{-1} should it obey an NFW profile [36], the Galactic DM surface density estimated as $\sim\rho_D r$ would be constant all the way down to the BH sphere of influence at r_h. If it were to reside in a thin disk near the Galactic Center, its accretion rate would be comparable to the universal value given by Eq. (60) for a BH of mass 4.3×10^6 [37,38]. If instead it were to occupy a large spherical cluster outside r_h and move with a velocity dispersion of $\sim$100 km s^{-1}, then Eq. (27) suggests it will accrete at a much smaller rate of $dM^c/dt \approx 2.4 \times 10^{-7} M_\odot$ yr^{-1}.

By comparison the Bondi value for the baryon accretion rate onto Sgr A* at the Galactic Center, which is determined from the gas density and temperature inferred from the diffuse x-ray emission observed by *Chandra* at $\sim$2 arcsec ($\sim$0.1 pc) from the black hole, is $dM^b/dt \sim 2 \times 10^{-5} M_\odot$ yr^{-1}. In fact, the baryon accretion rate is believed to be $\sim 10^{-8} M_\odot$ yr^{-1}, or roughly 3 orders of magnitude below the Bondi value as determined from polarization measurements [39] and models of the near-horizon accretion flow and emitted luminosity [40,41]. This difference may be due to the angular momentum of the stellar winds that may be supplying the gas, or possibly to more exotic effects such as the heating of the gas by DM annihilation in the spike about the BH [42].

V. SUMMARY AND CONCLUSIONS

We have examined the steady-state density and velocity profiles, and the associated accretion rates, of collisionless particles (e.g., stars or DM) moving around a central Schwarzschild black hole in unbound orbits. We considered two distinct spatial geometries for the particle: an infinite 3D cluster and a 2D razor-thin disk, both without net angular momentum. We adopted the same simple monoenergetic, phase-space distribution function for the particles for both cases, arguing that, though idealized, this assignment was sufficient to illustrate the features that might distinguish nonrotating spherical-like and disklike collisionless systems orbiting a black hole. We treated both a totally isotropic velocity profile at each point and one in which an empty loss cone is present due to the capture of low-angular momentum particles that are captured by the BH and not replenished. In all cases the net angular momentum of the systems was assumed to be zero so that the only differences were due to the different spatial geometries and velocity anisotropies adopted.

We found that even in the weak-field region, where $r \gg M$, a mild spike arises in the locally measured particle density $n(r)$ for the 3D cluster but that the surface density Σ^* remains constant with r for the 2D razor-thin disk. We also found that, at least for one simple application, the rate of accretion of the disk was much larger than that of the cluster. However, both rates were much lower than the Bondi accretion rate for a fluid with a comparable particle density and velocity dispersion (i.e., sound speed) far from the BH.

While these differences may not be so striking when more realistic phase-space distribution functions and geometries are considered, the results do suggest that the spatial distribution of particles around a black hole is a feature that affects the resulting steady-state particle profiles and accretion rates significantly. So this is just one other factor that must be accounted for, in addition to knowing what the nature of the particles are (e.g., collisionless or collisional fluid matter, or mildly collisional by virtue of self-interactions and/or annihilations) and the global properties of their distributions (bound or unbound, with or without net angular momentum, subject or not to gravitational intruders, etc.) in assessing their profiles and capture rates about a black hole.

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APPENDIX: CLUSTER ACCRETION RATES: ALTERNATIVE DERIVATIONS

1. Alternative derivation 1

Here we obtain the accretion rate for an unbound, monoenergetic, nonrotating cluster as measured by a locally static observer by calculating the total inward particle flux across a sphere of radius $r\times$ the area of the sphere $\times$ the fraction of these particles that move within the loss cone and are thus captured:

$$\frac{dN_{\rm tot}}{d\tau} = \left(\frac{1}{4} n\hat{v}\right) \times (4\pi r^2) \times \mathcal{P}, \tag{A1}$$

where $\hat{v}$ is the magnitude of the isotropic 3-velocity at r given by Eq. (16) and $\mathcal{P}$ is the fraction of particles captured. For $\mathcal{P}$ we have

$$\mathcal{P} = \frac{\int_0^{J_{\rm min}(E_\infty)} dJJ}{\int_0^{J_{\rm max}(E_\infty)} dJJ} = \frac{J^2_{\rm min}(E_\infty)}{J^2_{\rm max}(E_\infty)}, \tag{A2}$$

while the density n is given by Eq. (10) and $J_{\rm max}(E)$ is given by Eq. (33). Assembling the factors in Eq. (A1) then yields Eq. (22) for the locally measured accretion rate at r, $dN_{\rm tot}/d\tau$, from which, using Eq. (23), the rates measured by a distant observer, Eqs. (24)–(26) for $dN_{\rm tot}/dt$, dM_0/dt, and dM/dt, respectively, follow immediately.

2. Alternative derivation 2

Here we provide yet another derivation of the accretion rate found above. The maximum impact parameter $b_{\rm max}$ for a particle of energy E_∞ falling inward from infinity to be captured by the BH is given by

$$b^2_{\rm max} = \frac{J^2_{\rm min}(E_\infty)}{E^2_\infty - 1} \tag{A3}$$

[see, e.g., Ref. [18], Eq. (12.4.35), with the typo corrected for the missing square on E_∞], whereby the capture cross section is

$$\sigma_{\rm cap} = \pi b^2_{\rm max} = \frac{\pi J^2_{\rm min}(E_\infty)}{E^2_\infty - 1}. \tag{A4}$$

So far from the BH the accretion rate is obtained as the intensity of particles for an isotropic distribution $\times$ the area of a large sphere about the BH $\times$ the solid angle within which a particle is captured:

$$\frac{dN_{\rm tot}}{dt} = \left(\frac{n_\infty \hat{v}_\infty}{4\pi}\right)(4\pi r^2)(\Delta\Omega_{\rm cap}), \tag{A5}$$

where using Eq. (16) at $r \to \infty$, we have

$$\hat{v}_\infty = \frac{(E^2_\infty - 1)^{1/2}}{E_\infty} \tag{A6}$$

and where

$$\Delta\Omega_{\rm cap} = \frac{\sigma_{\rm cap}}{r^2}. \tag{A7}$$

Assembling the factors in Eq. (A5) again yields Eq. (24) for $dN_{\rm tot}/dt$, from which Eq. (22) for $dN_{\rm tot}/d\tau$ and Eqs. (25) and (26) for dM_0/dt and dM/dt, respectively, again follow immediately.

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