

# LEAFWISE QUASIGEODESIC FOLIATIONS IN DIMENSION THREE AND THE FUNNEL PROPERTY

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**ABSTRACT.** We construct one dimensional foliations which are subfoliations of two dimensional foliations in 3-manifolds. The subfoliation is by quasigeodesics in each two dimensional leaf, but it is not funnel: not all quasigeodesics share a common ideal point in most leaves.

## 1. INTRODUCTION

The goal of this article is to analyze whether certain geometric conditions imply that a one dimensional foliation in a 3-manifold is the foliation by flow lines of a topological Anosov flow. We do this analysis for one dimensional foliations whose leaves lie inside leaves of two dimensional foliations and whose leaves are quasigeodesics in these two dimensional foliations. In other words the goal of this article is to analyse whether some strictly geometric behavior implies strong dynamical systems behavior in this setting. This has important connections with partial hyperbolicity in dimension 3.

A foliation  $\mathcal{G}$  subfoliates a foliation  $\mathcal{F}$  if each leaf of  $\mathcal{F}$  has a foliation made up of leaves of  $\mathcal{G}$ . We call  $\mathcal{G}$  the subfoliation and  $\mathcal{F}$  the super foliation. This situation is very common, for example if  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are two foliations which are transverse to each other everywhere, then their intersection forms a subfoliation of each of them. This article aims to study geometric properties of leaves of subfoliations inside the leaves of the super foliation.

One very common and extremely important example is the following: let  $\Phi$  be an Anosov flow and let  $\mathcal{F}^{ws}, \mathcal{F}^{wu}$  be the weak stable and weak unstable foliations of  $\Phi$  [Ano63, KH95]. Then  $\mathcal{F}^{ws}, \mathcal{F}^{wu}$  are transverse to each other – the intersection is the foliation by flow lines of  $\Phi$  which is a subfoliation of each of them. This example has connections with geometry or large scale geometry: the leaves of  $\mathcal{F}^{ws}, \mathcal{F}^{wu}$  are Gromov hyperbolic. In rough terms this means that they are negatively curved. The subfoliation by flow lines in, say  $\mathcal{F}^{ws}$ , satisfies an additional strong geometric property: in each leaf of  $\mathcal{F}^{ws}$  the flow lines are quasigeodesics. This means that when lifted to the universal cover of the leaves, the flow lines are uniformly efficient up to a bounded multiplicative distortion in measuring length in the weak stable leaves. In other words the flow lines are quasi-isometrically embedded in these weak stable leaves. The quasigeodesic property has many important consequences, for example the flow lines are within bounded distance from length minimizing geodesics when lifted to the universal cover of their respective weak stable leaves ([Thu82, Thu97, Gro87]). Hence the flow lines have well defined distinct ideal points in the Gromov boundary of the weak stable leaves in both directions. These properties and others

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are very strong and useful in many contexts. Obviously this also works for the flow subfoliation of the weak unstable foliation.

A (one dimensional) subfoliation made of quasigeodesics in the leaves of a super foliation by Gromov hyperbolic leaves is called a *leafwise quasigeodesic foliation*.

The Anosov case has an additional geometric property: in (say) a weak stable leaf all flow lines are forward asymptotic, this is a defining property of the weak stable foliation. In particular all flow lines in a given weak stable leaf have the same forward ideal point in the ideal boundary of the weak stable leaf (when lifted to the universal cover).

When all leaves of a leafwise quasigeodesic subfoliation in a leaf of the super foliation have a common ideal point we call that leaf a *funnel leaf*. If all leaves of the super foliation are funnel leaves then the leafwise quasigeodesic foliation is said to have the *funnel property*.

The motivation for this article is the following question: is the funnel property an additional property or is it a consequence of the leafwise quasigeodesic property? The importance of this is the following: in dimension 3 we have a much stronger connection between some of these properties as follows. Suppose that  $\mathcal{G}$  is a leafwise quasigeodesic foliation (which is a one dimensional subfoliation of a two dimensional foliation) which has the funnel property. The ambient manifold is 3-dimensional. Suppose that the foliation  $\mathcal{G}$  is orientable, or in other words it is the foliation of a non singular flow. Then one can prove (we refer to [BFP20] for definitions of the terms used here and for detailed proofs) that the flow in question is expansive. This implies that the flow is a topological Anosov flow ([IM90, Theorem 15.], [Pat93, Lemma 7]). If the flow is transitive (the union of periodic orbits is dense) then the topological Anosov flow is in addition orbitally equivalent to a (smooth) Anosov flow ([Sha20]). This means that if the leafwise quasigeodesic property implies the funnel property, then this purely geometric condition implies a very strong dynamical systems property: the foliation is the flow foliation of an Anosov flow, up to topological equivalence..

In this article we prove that the funnel property is not a consequence of leafwise quasigeodesic behavior:

**Theorem 1.1.** *There are examples of leafwise quasigeodesic foliations in dimension 3 which do not have the funnel property.*

We now briefly explain one class of examples: start with the Franks-Williams example of a non transitive Anosov flow  $\Phi$ . This is obtained as follows: start with a suspension Anosov flow and do a DA (derived from Anosov) blow up of a periodic orbit transforming it into (say) a repelling orbit  $\alpha$ . Remove a tubular neighborhood of  $\alpha$  so that the resulting semiflow is incoming in the complement of the removed tubular neighborhood of  $\alpha$ . Glue this manifold with boundary with a copy of it which has a reversed flow. One fundamental result is that the ensuing flow  $\Phi$  in the final manifold  $\mathcal{M}$  is Anosov [FW80, BBY17]. This holds for certain isotopy classes of gluings and certain gluing maps satisfying transversality conditions. These were the first examples of non transitive Anosov flows in dimension 3. Our examples use this flow. There is a smooth torus  $T$  in  $\mathcal{M}$  transverse to the flow. There is a single two dimensional attractor and a single two dimensional repeller of the flow  $\Phi$  in  $M$ . Start with a one dimensional foliation  $Z$  in  $T$  which is transverse to the intersections of both the weak stable and the weak unstable foliations of  $\Phi$  with  $T$ . Saturate  $Z$  by the flow producing a collection of two dimensional sets embedded in  $\mathcal{M}$ . The flow saturation of  $T$  is an open subset  $V$  of  $M$ , and the collection of the two dimensional

subsets described is a two dimensional foliation in  $V$ . In addition  $V$  is exactly the complement of the union of the attractor and the repeller of  $\Phi$ . Complete the foliation in  $V$  to a foliation  $\mathcal{F}$  in  $\mathcal{M}$  which is the weak unstable foliation of  $\Phi$  in the attractor of  $\Phi$  and the weak stable foliation in the repeller of  $\Phi$ . The proof that this is in fact a foliation of  $\mathcal{M}$  depends on a careful choice of the one dimensional foliation  $Z$  in  $T$ . There is a subtle point here in that if one chooses an arbitrary foliation  $Z$  in  $T$ , then when lifting to  $\widetilde{\mathcal{M}}$  the lifted sets may not be properly embedded in  $\widetilde{\mathcal{M}}$  and so  $\mathcal{F}$  would not be a foliation in  $\mathcal{M}$ . This is carefully analyzed in section 3 and there we prove that for appropriate choices of  $Z$  the object  $\mathcal{F}$  we construct is a foliation. The super foliation is this two dimensional foliation  $\mathcal{F}$ . The subfoliation  $\mathcal{G}$  of  $\mathcal{F}$  is the foliation by flow lines of  $\Phi$ . Each leaf of  $\mathcal{F}$  is saturated by flow lines. We prove that  $\mathcal{G}$  is a leafwise quasigeodesic subfoliation of  $\mathcal{F}$ , but  $\mathcal{G}$  does not have the funnel property. There is an Anosov flow  $\Phi$  in this example, however notice that the super foliation  $\mathcal{F}$  is neither the weak stable nor the weak unstable foliation of  $\Phi$ , but rather a different foliation. In fact in the same way one can construct an infinite number of inequivalent examples with the same starting flow  $\Phi$ . The foliations are pairwise distinguished because of how they intersect the torus  $T$  in foliations which are not equivalent.

In this article we consider more general examples. We prove that one can construct examples starting with any non transitive Anosov flow  $\Phi$  in dimension 3 so that all the basic sets have dimension 2. As in the case of the Franks-Williams example, we construct super foliations which have Gromov hyperbolic leaves and whose leaves are saturated by flow lines of  $\Phi$ . We show that the subfoliation  $\mathcal{G}$  by flow lines of  $\Phi$  is by quasigeodesics in each leaf of the super foliation  $\mathcal{F}$ . This is the hardest step to prove. This involves a very careful analysis of the geometry in these examples. The proof that  $\mathcal{G}$  is not funnel is simpler than proving it is leafwise quasigeodesic as a subfoliation of  $\mathcal{F}$ .

We finish this introduction mentioning another reason why we analyzed this question: this comes from partially hyperbolic dynamics. Let  $f$  be a partially hyperbolic diffeomorphism in a closed 3-manifold  $M$  (we refer to [BFP20] for definitions and properties of partially hyperbolic diffeomorphisms). Under very general orientability conditions, there is a pair of transverse two dimensional branching foliations (center stable and center unstable foliations) associated with the partially hyperbolic diffeomorphism which intersect in an one dimensional branching foliation, called the center foliation [BI08]. The center foliations subfoliates both the center stable and center unstable foliations. In some situations ([BFP20]) it is shown that the center foliation is a leafwise quasigeodesic subfoliation of both the center stable and center unstable foliations. But in [BFP20] it is proved that in the partially hyperbolic setting the leafwise quasigeodesic property implies that the center foliation has the funnel property (as a subfoliation of both super foliations). The proof of this also uses dynamical systems properties, namely partial hyperbolicity. An open question from the article [BFP20] was to whether the funnel property could be derived strictly from the leafwise quasigeodesic property in (say) the center stable foliation. In this article we prove that this is not the case, by constructing counterexamples for general foliations.

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## 2. PRELIMINARIES

A  $C^1$ -flow  $\Phi_t : \mathcal{M} \rightarrow \mathcal{M}$  on a Riemannian manifold  $\mathcal{M}$  is *Anosov* if the tangent bundle  $T\mathcal{M}$  splits into three  $D\Phi_t$ -invariant sub-bundles  $T\mathcal{M} = E^s \oplus E^0 \oplus E^u$  and there exists two constants  $C, \lambda > 0$  such that

- $E^0$  is generated by the non-zero vector field defined by the flow  $\Phi_t$ ;
- For any  $v \in E^s$  and  $t > 0$ ,

$$\|D\Phi_t(v)\| \leq Ce^{-\lambda t}\|v\|$$

- For any  $w \in E^u$  and  $t > 0$ ,

$$\|D\Phi_t(w)\| \geq Ce^{\lambda t}\|w\|$$

The definition is independent of the choice of the Riemannian metric  $\|\cdot\|$  as the underlying manifold  $\mathcal{M}$  is compact. For a point  $x \in \mathcal{M}$ , the set  $\gamma_x = \{\Phi_t(x) | t \in \mathbb{R}\}$  is called the *flow line* of  $x$ . The collection of all flow lines of a flow defines a one-dimensional foliation on  $\mathcal{M}$ . For an Anosov flow there are several flow invariant foliations associated to the flow and these foliations play a key role in the study of Anosov flows.

**Property 2.1** ([Ano63]). *For an Anosov flow  $\Phi_t$  on  $\mathcal{M}$ , the distributions  $E^u$ ,  $E^s$ ,  $E^0 \oplus E^u$  and  $E^0 \oplus E^s$  are uniquely integrable. The associated foliations are denoted by  $\mathcal{F}^u$ ,  $\mathcal{F}^s$ ,  $\mathcal{F}^{wu}$  and  $\mathcal{F}^{ws}$  respectively and they are called the strong unstable, strong stable, weak unstable and weak stable foliation on  $\mathcal{M}$ .*

**For the remainder of this article we will assume that  $\mathcal{M}$  is a closed three dimensional Riemannian manifold.**

We also assume that  $\mathcal{M}$  is equipped with an Anosov flow  $\Phi_t$  and  $\tilde{\Phi}_t$  is the lift of the flow  $\Phi_t$  in  $\tilde{\mathcal{M}}$ , the universal cover of  $\mathcal{M}$ . The strong unstable, strong stable, weak unstable and weak stable foliation of  $\tilde{\Phi}$  are the lifts of the foliations  $\mathcal{F}^u$ ,  $\mathcal{F}^s$ ,  $\mathcal{F}^{wu}$  and  $\mathcal{F}^{ws}$  in the universal cover  $\tilde{\mathcal{M}}$ , and these foliations in  $\tilde{\mathcal{M}}$  are denoted by  $\tilde{\mathcal{F}}^u$ ,  $\tilde{\mathcal{F}}^s$ ,  $\tilde{\mathcal{F}}^{wu}$  and  $\tilde{\mathcal{F}}^{ws}$  respectively.

A map  $f : (X_1, d_1) \rightarrow (X_2, d_2)$  is called a  $(K, s)$ -*quasi-isometric embedding* if there exists  $K > 1$  and  $s > 0$  such that for all  $x, y \in X_1$

$$\frac{1}{K}d_1(x, y) - s \leq d_2(f(x), f(y)) \leq Kd_1(x, y) + s$$

A  $(K, s)$ -*quasigeodesic* in  $X$  is the image of a  $(K, s)$ -*quasi-isometric embedding*  $\gamma : [a, b] \rightarrow X$  where  $[a, b]$  is a closed interval on  $\mathbb{R}$  with the Euclidean metric. The interval could be infinite (that is  $a = -\infty$ ,  $b = \infty$  or both), in which case the notation would be of a half open or open interval. If we have a map  $\mathbb{R} \rightarrow X$  with rectifiable image we consider the arclength metric in the domain  $\mathbb{R}$ .

**Lemma 2.2.** *Flow lines on the leaves in  $\tilde{\mathcal{F}}^{ws}$  and  $\tilde{\mathcal{F}}^{wu}$  are quasigeodesics with respect to the metric induced from  $\tilde{\mathcal{M}}$  in their respective leaves.*

*Proof.* Reparametrize the flow to have unit speed. The new flow is still Anosov [Ano63] with the same flow lines and the same weak stable and weak unstable foliations. However the strong stable and strong unstable leaves may change.

Any leaf  $L$  of  $\tilde{\mathcal{F}}^{wu}$  is subfoliated by  $\tilde{\mathcal{F}}^u$  and by the flow lines, these two foliations are transversal to each other. We can define a metric  $ds'$  on  $L$  by  $ds' = dw + dy$  where

$dw$  measures length along flow lines and  $dy$  measures length along unstable curves. Suppose  $ds$  is the Riemannian metric induced on  $L^{wu}$  from  $\tilde{\mathcal{M}}$ . The two path metrics induced in  $L$  from  $ds'$  and  $ds$  on are uniformly quasi-isometric to each other [Fen94]. Moreover each flow line in the leaf  $L$  is a length minimizing curve in the  $ds'$  metric, hence flow lines are uniform quasigeodesics with respect to the metric induced by  $ds$ . Similarly it can be shown that flow lines on leaves in  $\tilde{\mathcal{F}}^{ws}$  are quasigeodesic with respect to the induced metric on their respective leaves.  $\square$

**Definition 2.3.** Suppose  $\mathcal{F}$  is a two dimensional foliation on  $\mathcal{M}$  with Gromov hyperbolic leaves when lifted to the universal cover. Suppose that  $\mathcal{G}$  is a one dimensional foliation on  $\mathcal{M}$  which subfoliates  $\mathcal{F}$ . In this situation we say that leaves of  $\mathcal{G}$  are leafwise quasigeodesic in  $\mathcal{F}$  if every leaf of  $\mathcal{G}$  is a quasigeodesic in the respective leaf of  $\mathcal{F}$  containing it when lifted to the universal cover of the leaf. In that case we say that  $\mathcal{G}$  is a leafwise quasigeodesic subfoliation of  $\mathcal{F}$ .

In Lemma 2.2 the flow lines of  $\Phi_t$  are shown to be leafwise quasigeodesics in the leaves of  $\mathcal{F}^{ws}$  and  $\mathcal{F}^{wu}$ .

The leaves in  $\tilde{\mathcal{F}}^{ws}$  and  $\tilde{\mathcal{F}}^{wu}$  are Gromov hyperbolic with respect to the Riemannian metric on the leaves induced from the metric on  $\tilde{\mathcal{M}}$  [Fen94]. Suppose that  $L$  is a leaf either in  $\tilde{\mathcal{F}}^{ws}$  or in  $\tilde{\mathcal{F}}^{wu}$ . As the leaves are Gromov hyperbolic, we can define the ideal boundary of  $L$  which is homeomorphic to the circle and we denote it as  $S^1(L)$ . The compactification  $L \cup S^1(L)$  is homeomorphic to a closed disk. As the flow lines are quasigeodesics in  $L$ , they define two distinct ideal points on  $S^1(L)$ : If  $\gamma$  is a flow line in  $L$  then the forward ray of  $\gamma$  defines an unique ideal point on  $S^1(L)$  as  $\gamma$  is a quasigeodesic, which is called the *forward or positive ideal point* of  $\gamma$ . Similarly we define the *backward or negative ideal point* as the limit of the ray in the backwards direction. The following statement describes the equivalence between the forward and backward flow rays in the leaves of  $\tilde{\mathcal{F}}^{ws}$  and  $\tilde{\mathcal{F}}^{wu}$  and the points on their ideal boundaries.

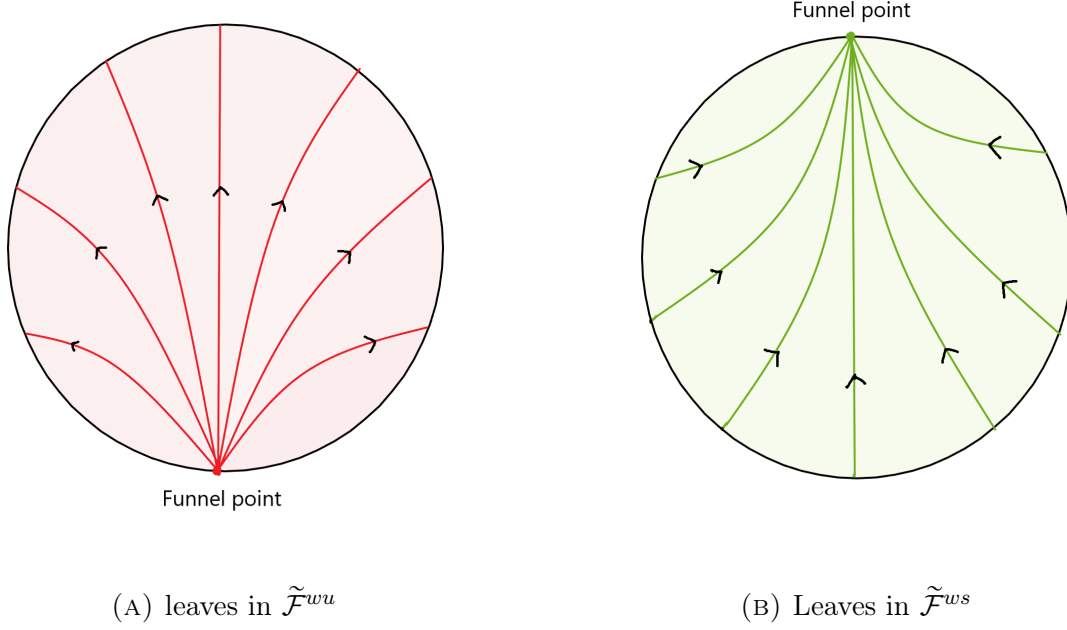
**Property 2.4** ([Fen94]). For a leaf  $L$  either in  $\tilde{\mathcal{F}}^{ws}$  or  $\tilde{\mathcal{F}}^{wu}$ , all the points on  $S^1(L)$  correspond to forward or backward flow rays on  $L$ . If  $L \in \tilde{\mathcal{F}}^{ws}$  then all the flow lines on  $L$  have a common forward ideal point and all the other ideal points are backward ideal points on  $S^1(L)$  of the flow lines. No two different flow lines define a common negative or backward ideal point.

If  $L \in \tilde{\mathcal{F}}^{wu}$  then all the flow lines have a common backward ideal point and all the forward flow lines defines all the other ideal points on  $S^1(L)$ . No two different flow lines define the same positive or forward ideal point.

The property for forward ideal points in  $\tilde{\mathcal{F}}^{ws}$  is immediate as these flow lines are forward asymptotic, a direct consequence of the definitions. The property for backward ideal points in leaves of  $\tilde{\mathcal{F}}^{ws}$  is not as immediate and is proved in [Fen94].

**Definition 2.5.** Suppose that  $\mathcal{G}$  is a leafwise quasigeodesic subfoliation of  $\mathcal{F}$ . If a leaf  $L$  of  $\tilde{\mathcal{F}}$  has all leaves of  $\tilde{\mathcal{G}}$  in it sharing a common ideal point then the projected leaf  $\pi(L)$  of  $\mathcal{F}$  in  $\mathcal{M}$  is called a *funnel leaf*. In this case the common ideal points shared by all the flow lines in  $L$  is called the *funnel point* of  $L$ .

**Corollary 2.6.** By property 2.4, for an Anosov flow  $\Phi_t$  on a three manifold  $\mathcal{M}$ , with the flow foliation a leafwise quasigeodesic subfoliaton of both  $\mathcal{F}^{ws}$  and  $\mathcal{F}^{wu}$  the following happens: all the leaves in weak stable foliation  $\mathcal{F}^{ws}$  and weak unstable foliation  $\mathcal{F}^{wu}$  are funnel leaves as shown in figure 1.

FIGURE 1. Geometry of flow lines on the leaves in  $\tilde{\mathcal{F}}^{wu}$  and  $\tilde{\mathcal{F}}^{ws}$ 

**Basic sets of Anosov flows on three manifolds:** The Anosov flow  $\Phi$  is called *transitive* if there exists a flow line  $\gamma$  dense in  $\mathcal{M}$ , otherwise the flow is *non transitive*. The first example of a non transitive Anosov flow was constructed by John Franks and Bob Williams in their 1980's article [FW80].

A point  $x \in \mathcal{M}$  is called *nonwandering* if for any open neighborhood  $U$  of  $x$  and any  $t_0 > 0$ , there exists  $t > t_0$  such that  $\Phi_t(U) \cap U \neq \emptyset$ , the set of all nonwandering points is denoted by  $\Omega(\Phi)$ . For a non transitive Anosov flow  $\Phi_t$  the nonwandering set  $\Omega(\Phi)$  is not equal to the whole manifold  $\mathcal{M}$  and according to *Spectral Decomposition Theorem* [Sma67],  $\Omega(\Phi)$  is decomposed into finitely many closed, disjoint,  $\Phi_t$ -invariant and transitive *basic sets*  $\{\Lambda_i, i = 1, \dots, n\}$ , so  $\Omega(\Phi) = \bigsqcup_{i=1}^n \Lambda_i$ .

Suppose  $\Lambda$  is a basic set of a non transitive Anosov flow  $\Phi_t$  on a three manifold. Then  $\Lambda$  can be characterised in four different types [Sma67, Bru93],

- $\dim(\Lambda) = 2$  and the basic set  $\Lambda$  is an *attractor*, i.e. there exists an open set  $U$  containing  $\Lambda$  such that  $\bigcap_{t>0} \Phi_t(U) = \Lambda$ .
- $\dim(\Lambda) = 2$  and the basic set  $\Lambda$  is a *repeller*, i.e  $\Lambda$  is an attractor for the reversed flow  $\Psi_t = \Phi_{-t}$ .
- $\dim(\Lambda) = 1$  and  $\Lambda$  is a saddle with local cross section a Cantor set.
- $\dim(\Lambda) = 1$  and  $\lambda$  is a hyperbolic periodic orbit.

**Property 2.7.** *If  $\Lambda$  is an attractor then  $\Lambda$  is saturated by weak unstable leaves. If  $\Lambda$  is a repeller then  $\Lambda$  is saturated by weak stable leaves.*

From now on we assume the following:

**Assumption 2.8.** We assume throughout that the Anosov flow  $\Phi$  on  $\mathcal{M}$  is non transitive and its nonwandering set  $\Omega$  consists of two dimensional basic sets only.

In other words we assume that  $\Phi$  has no one dimensional basic set. As  $\mathcal{M}$  is compact there exists at least one attracting basic set and one repelling basic set. Suppose  $\mathcal{A}$  denotes the union of all attracting basic sets and  $\mathcal{R}$  denotes the union of all repelling basic sets. We will denote the collection of all lifts of  $\mathcal{A}$  in  $\tilde{\mathcal{M}}$  by  $\tilde{\mathcal{A}}$ .  $\tilde{\mathcal{A}}$  is the attracting set for  $\tilde{\Phi}_t$  defined on  $\tilde{\mathcal{M}}$ . The union of all lifts of  $\mathcal{R}$  is denoted by  $\tilde{\mathcal{R}}$  similarly.

**Property 2.9.** [KH95] *Suppose  $\gamma$  is a flow line not contained in  $\mathcal{A}$  or  $\mathcal{R}$ . Then there exists a flow line in  $\mathcal{A}$ , say  $\alpha$ , such that the forward rays of  $\gamma$  and  $\alpha$  are asymptotic in  $\mathcal{M}$ . Similarly there exists a flow line  $\beta$  in  $\mathcal{R}$  such that the backward rays of  $\gamma$  and  $\beta$  are asymptotic in  $\mathcal{M}$ .*

*Proof.* This is classical [KH95], we explain briefly. Given the orbit  $\gamma$  it gets closer and closer to the attractor  $\mathcal{A}$  in future time. Fix  $x$  in  $\gamma$ . Every point in the attractor has a local product structure, see for example Proposition 6.4.21 of [KH95]. Hence for a  $t$  sufficiently big  $\Phi_t(x)$  is  $\epsilon$  near the attractor where  $\epsilon$  is smaller than the size of product boxes of the hyperbolic set  $\mathcal{A}$ . Hence  $\Phi_t(x)$  is  $\epsilon$  near some point  $z$  in  $\mathcal{A}$  and there is  $w$  in  $\mathcal{A}$  near  $z$  so that  $\Phi_t(x)$  is in the stable manifold of  $w$  because of the local product structure in sets of size  $\epsilon$ . This proves the result.  $\square$

The attractor is saturated by leaves of  $\mathcal{F}^{wu}$  and the repeller saturated by leaves of  $\mathcal{F}^{ws}$ . In the property above, one can choose the flow line  $\alpha$  in the attractor  $\mathcal{A}$  to be in the boundary of the attractor. This means the following: let  $x \in \alpha$  and  $L$  the  $\mathcal{F}^{ws}$  leaf through  $x$ . Let  $D$  be a small disk in  $L$  with  $x$  in the interior. The local flow line of  $x$  cuts  $D$  into two components  $D_1, D_2$  (which are also disks). The condition is that one of  $D_1$  or  $D_2$  does not intersect the attractor  $\mathcal{A}$ . Suppose it is  $D_1$ . The “ $D_1$  side” of  $\alpha$  in  $L$  is the side so that  $\gamma$  is getting closer and closer to  $\alpha$ .

### 3. THE FOLIATION $\mathcal{F}$

Throughout the article we will fix a non transitive Anosov flow  $\Phi$  as in the previous section, that is,  $\Phi$  has only two dimensional basic sets.

To prove our results we will consider a two dimensional foliation  $\mathcal{F}$  in  $\mathcal{M}$  such that:

- on the attractor  $\mathcal{A}$ ,  $\mathcal{F}|_{\mathcal{A}} = \mathcal{F}^{wu}|_{\mathcal{A}}$
- on the repeller  $\mathcal{R}$ ,  $\mathcal{F}|_{\mathcal{R}} = \mathcal{F}^{ws}|_{\mathcal{R}}$
- on  $\mathcal{M} \setminus \{\mathcal{A} \cup \mathcal{R}\}$ ,  $\mathcal{F}$  is transversal to both  $\mathcal{F}^{ws}$  and  $\mathcal{F}^{wu}$ .
- every leaf  $L \in \mathcal{F}$  is subfoliated by the flow lines of  $\Phi$ , i.e every leaf  $L$  is  $\Phi_{\mathbb{R}}$ -invariant.

We will denote the lift of  $\mathcal{F}$  in the universal cover  $\tilde{\mathcal{M}}$  by  $\tilde{\mathcal{F}}$ . Leaves of  $\tilde{\mathcal{F}}$  in  $\tilde{\mathcal{A}}$  and  $\tilde{\mathcal{R}}$  look like the leaves in figure 1. Leaves in  $\tilde{\mathcal{M}} \setminus (\tilde{\mathcal{A}} \cup \tilde{\mathcal{R}})$  are described in figure 2. It is not immediate from definition why the leaves not contained in  $\tilde{\mathcal{A}}$  and  $\tilde{\mathcal{R}}$  are as described in figure 2, we will prove this later in this article.

**Theorem 3.1.** *There are foliations  $\mathcal{F}$  with the properties described above.*

*Proof.* We start with an Anosov flow as described above. For simplicity assume that  $M$  is orientable as well. There is a collection of disjoint tori  $\{T_i\}$  transverse to the flow  $\Phi$  which separate the basic sets [Sma67, Bru93]. We choose  $T_i$  to be smooth. The collection of tori is supposed to be minimal with the property that if an orbit is not in  $\mathcal{R}$  or  $\mathcal{A}$  then it intersects one of the  $\{T_i\}$ . Let  $\gamma$  be such an orbit intersecting a specific  $T_i$ , let  $x$  be a point in the intersection. Then the forward orbit of  $x$  is

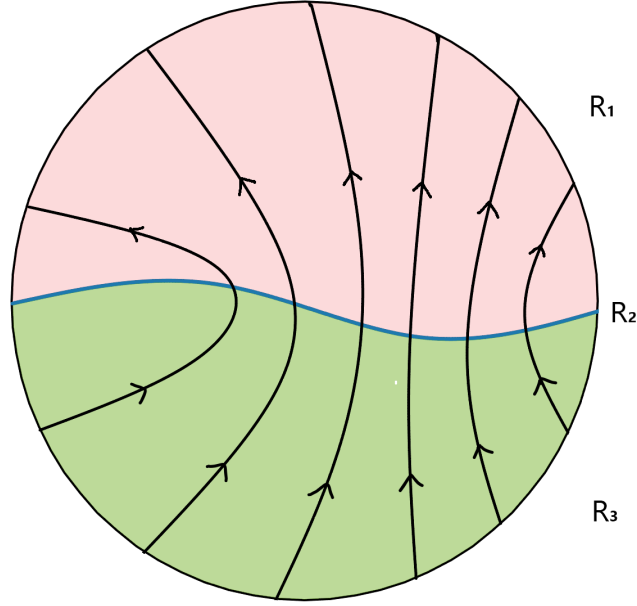


FIGURE 2. An example of a leaf  $L \in \tilde{\mathcal{F}}$  not contained in  $\tilde{\mathcal{A}}$  or  $\tilde{\mathcal{R}}$ . In  $R_1$  forward rays are asymptotic to  $\tilde{\mathcal{A}}$ ; in  $R_3$  backward rays are asymptotic to  $\tilde{\mathcal{R}}$ ;  $R_2$ , the blue line, represents the intersection of  $L$  with some lift  $\tilde{T}_i$  of some torus  $T_i$ .

asymptotic to a component  $A$  of the attractor  $\mathcal{A}$  – this uses the fact that there are no one dimensional components of the non wandering set of  $\Phi$ . The set of such  $x$  so that the forward ray of  $x$  is asymptotic to  $A$  is open in  $T_i$ . This holds for any component  $A$  of the attractor  $\mathcal{A}$ . Since the union over such components of  $\mathcal{A}$  is all of  $T_i$  and  $T_i$  is connected, it follows that all orbits in  $T_i$  are forward asymptotic to a single component  $A$  of  $\mathcal{A}$ .

In a similar way one proves that if  $T_1, T_2$  are tori contained in the complement of the union of the attractor and repeller, and  $T_1, T_2$  intersect a common orbit of  $\Phi$ , then  $T_1, T_2$  intersect exactly the same set of orbits of  $\Phi$ . In other words, if  $B$  is a component of  $M - (\mathcal{A} \cup \mathcal{R})$ , then there is a torus  $T$  contained in  $B$ , transverse to  $\Phi$  so that  $B$  is the flow saturation of  $T$ . Hence we can choose a minimal collection  $\{T_i\}$  of tori transverse to  $\Phi$  and intersecting all orbits in the complement of  $\mathcal{A} \cup \mathcal{R}$ , and any such orbit intersects a unique  $T_i$  and only once.

### Construction of $\mathcal{F}$

Now we construct the foliation  $\mathcal{F}$ . The foliations  $\mathcal{F}^{ws}, \mathcal{F}^{wu}$  are  $C^1$  [KH95], and so are the intersections with each  $T_i$ . On each  $T_i$  choose a one dimensional  $C^1$  foliation  $F_i$  transverse to both

$$\mathcal{F}^{ws} \cap T_i, \quad \text{and} \quad \mathcal{F}^{wu} \cap T_i.$$

Saturate  $F_i$  by the flow to produce a two dimensional foliation in the flow saturation of  $T_i$ . Let  $\mathcal{F}$  be this foliation in the complement of the attractor union the repeller. Figure 2 describes a possible leaf in the lift  $\tilde{\mathcal{F}}$  of  $\mathcal{F}$  to  $\tilde{\mathcal{M}}$ ; where  $R_2$ , the blue line, represents its intersection with some lift of  $T_i$ . The figure depicts the following several

properties that we are going to prove later and that are essential to the results of this article: 1) We will show later that leaves of  $\tilde{\mathcal{F}}$  are Gromov hyperbolic, 2) We will also show that for  $L \in \tilde{\mathcal{F}}$  not in the lift of the attractor or repeller then each flow ray in  $L$  converges to a single point in  $S^1(L)$ , and that distinct flow rays do not forward converge to the same ideal point in  $S^1(L)$ . Similarly for backward flow rays.

A flow line that does not intersect any  $T_i$  has to be either in the attractor ( $\mathcal{A}$ ) or the repeller ( $\mathcal{R}$ ). We define  $\mathcal{F}$  to be  $\mathcal{F}^{wu}$  in the attractor,  $\mathcal{F}^{ws}$  in the repeller, and the saturation of the  $F_i$  everywhere else.

We claim that  $\mathcal{F}$  is a foliation. Clearly it is a foliation in the complement of the union of the attractor and the repeller, because this is an open set and because of the definition of  $\mathcal{F}$ : each component  $\mathcal{C}$  of  $\mathcal{M} \setminus (\mathcal{A} \cup \mathcal{R})$  is equal to  $\Phi_{\mathbb{R}}(T_i)$  for some  $T_i$  and this is homeomorphic to  $T_i \times \mathbb{R}$  with the product topology (the topology in  $T_i$  is induced from  $\mathcal{M}$ ). The foliation  $\mathcal{F}$  in  $\mathcal{C}$  is equivalent to the foliation  $F_i \times \mathbb{R}$  in  $T_i \times \mathbb{R}$ .

There is a subtle point here. Let  $\tilde{\mathcal{F}}$  be the lift of  $\mathcal{F}$  to  $\mathcal{M}$ . If  $\mathcal{F}$  is a foliation, then  $\tilde{\mathcal{F}}$  is a foliation of  $\tilde{\mathcal{M}}$  by properly embedded planes. By construction the leaves of  $\mathcal{F}$  intersecting the attractor are contained in the attractor and similarly for the repeller. Therefore the leaves of  $\mathcal{F}$  in the complement of  $\mathcal{A} \cup \mathcal{R}$  are entirely contained in the complement of  $\mathcal{A} \cup \mathcal{R}$  as well. In particular if  $L$  is a lift of a leaf of  $\mathcal{F}$  in the complement of the attractor and repeller, then it should be properly embedded in  $\tilde{\mathcal{M}}$ . As it turns out this property is not true if one starts with an arbitrary foliation  $F_i$  in  $T_i$ . Let us review the construction: we start with a foliation  $F_i$  in  $T_i$  and saturate it by the flow to produce a foliation in an open set in  $\mathcal{M}$ . Then consider a lift  $L$  of a leaf of this foliation to the universal cover. Is  $L$  always properly embedded in  $\tilde{\mathcal{M}}$ ? In general this is not true. For example start with the Franks-Williams non transitive flow [FW80], consider a smooth torus  $T$  which separates the attractor and repeller and start with say the intersection of the unstable foliation of  $\Phi$  with  $T$ , which we call  $F$ . Then for some of the leaves of  $F$ , it follows that if  $L$  is a lift of the flow saturation to  $\tilde{\mathcal{M}}$ , then  $L$  is not properly embedded in  $\mathcal{M}$ . For example [FW80, Fig. 3, page 164] depicts the foliations induced by the weak stable and unstable foliations in  $T$  for the Franks-Williams flow. Each has two Reeb components. Take  $\alpha$  to be a leaf of the unstable foliation which is not in the interior of a Reeb component, that is a horizontal line in the figure. Lift it to  $\tilde{\alpha}$  in  $\tilde{\mathcal{M}}$ . If  $C$  is the flow saturation of  $\tilde{\alpha}$ , then  $C$  is not properly embedded in  $\tilde{\mathcal{M}}$ : there is an orbit  $\gamma$  of  $\tilde{\Phi}$  which is not in  $C$  but is contained in the closure of  $C$ . This orbit  $\gamma$  is the lift of a periodic orbit contained in the attractor of the Franks-Williams flow.

The reason why our construction of a foliation  $\mathcal{F}$  as above works is because we start with a foliation  $F_i$  which is transverse to both the stable and unstable foliations in  $T_i$ . We state this as a separate result.

**Lemma 3.2.** *Let  $\ell$  be a leaf of  $F_i$  and let  $L$  be a lift to  $\tilde{\mathcal{M}}$  of the flow saturation of  $\ell$  in  $\mathcal{M}$ . Then  $L$  is a properly embedded plane in  $\tilde{\mathcal{M}}$ .*

*Proof.* Let  $E$  be the flow saturation of  $\ell$ . Since  $\ell$  is smooth and the flow is  $C^1$  it follows that  $E$  is  $C^1$ . For any  $x, y$  in  $\ell$  if

$$\Phi_t(x) = \Phi_s(y),$$

then  $x = y$  and  $t = s$ , since the component of  $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$  containing  $\ell$  is homeomorphic to  $T \times \mathbb{R}$  and  $\ell$  is injectively immersed in  $T$ . Hence  $E$  is parametrized as  $\ell \times \mathbb{R}$ , that is every point  $p$  in  $E$  can be represented as  $(x, t)$  where  $x \in \ell$  and  $t \in \mathbb{R}$ .

The Riemannian metric in  $\mathcal{M}$  induces a Riemannian metric in  $E$  and a path metric in  $E$ . We show that with this path metric  $E$  is complete. In particular this implies that the lift  $L$  to  $\widetilde{\mathcal{M}}$  is a properly embedded plane. What we prove is the following:

**Claim 0.** *There is  $a_0 > 0$  so that any point  $p = (x, t)$  in  $E$  is the center of a metric disk of radius  $a_0$  in  $E$ .*

*Proof.* This is obvious for any point  $p$  in  $\ell$  or in other words if  $t = 0$ .

We now prove the claim for  $t > 0$  using the unstable foliation. The analogous proof shows the result for  $t < 0$  using the stable foliation. The foliation  $F_i$  is transverse to both the stable and unstable foliations induced in  $T_i$ , hence uniformly transverse to these foliations. Given any smoothly embedded curve  $\alpha$  in  $M$  let  $l_u(\alpha)$  be its unstable length: we integrate only the component of the tangent vector in the direction of the unstable bundle. For example if  $\alpha$  is contained in a weak stable leaf then  $l_u(\alpha)$  is zero, while if  $\alpha$  is contained in a strong unstable leaf then  $l_u(\alpha)$  is the same as its length under the Riemannian metric of  $\mathcal{M}$ . In particular if  $\alpha$  is a curve not contained in a strong stable or unstable leaf, then original length  $l(\alpha)$  is always strictly greater than the unstable length  $l_u(\alpha)$ .

By the definition of an Anosov flow, there exist constants  $C > 0, \lambda > 1$  such that if we flow forward a segment with  $t$  amount of time, the new unstable length is at least  $C\lambda^t$ -times of the original unstable length. Hence if we let  $a_1 = C$  then for any smooth segment, any flow forward of that segment has unstable length which is at least  $a_1$  times of the original unstable length.

Since  $F_i$  is uniformly transverse to  $\mathcal{F}^{ws} \cap T_i$  by our construction, it follows that any point  $x$  in  $T_i$  is the midpoint of a segment  $\beta$  in its leaf of  $F_i$  of unstable length 2. For any  $t \geq 0$  the unstable length of  $\Phi_t(\beta)$  is at least  $2a_1$ . This constant  $a_1$  is defined globally. In addition if  $v$  is a non zero vector tangent to  $\beta$  then  $v$  makes a definite positive angle with the flow direction. Since flowing forward increases the size of unstable vectors more than the size of tangent vectors (for  $t > t_0 > 0$ ) it follows that there is a global constant  $\theta > 0$  so that  $D\Phi_t v$  also makes an angle  $> \theta$  with the tangent to the flow. Consider the infinitesimal arclengths  $dt, ds, du$  along the flow, stable and unstable bundles. The (non Riemannian) metric

$$|dt| + |ds| + |du|$$

is quasicomparable with the Riemannian metric in  $M$ : there is  $a_2 > 0$  so that the Riemannian length is at least  $a_2$  times the length in this metric. Consider the following set:

$$A = \Phi_{[t-1, t+1]}(\beta)$$

for  $t \geq 0$ . The segment  $\beta$  of  $F_i$  is contained in the leaf  $E$  of  $\mathcal{F}$ . From any point in the boundary of  $A$  to  $\Phi_t(x)$  along  $E$  one has to have at least  $a_1$  unstable length, and flow length of at least 1. It follows that there is a global constant  $a_0$  (depending only on  $a_1$ ) so that  $A$  contains a disk in the Riemannian metric, of radius  $a_0$  and centered at  $\Phi_t(x)$ .

For  $t < 0$  we use the stable foliation and flow backwards instead of forwards.

This finishes the proof of the claim.  $\square$

The claim shows that  $E$  is complete and finishes the proof of the lemma.  $\square$

**Proof of Theorem 3.1** – We consider a foliation  $\mathcal{F}$  as constructed in the beginning of this section. This object  $\mathcal{F}$  is a foliation restricted to  $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$ , and this is an open set. The leaves of  $\mathcal{F}$  in this set lift to properly embedded planes in  $\widetilde{\mathcal{M}}$  by Lemma 3.2. It follows that  $\widetilde{\mathcal{F}}$  describes  $\widetilde{\mathcal{M}}$  as the disjoint union of properly embedded planes which form a foliation in the complement of the lift of the union of the attractor and the repeller.

The only remaining thing to prove is that if a sequence  $x_n$  in  $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$  converges to  $x$  in  $\mathcal{A} \cup \mathcal{R}$  then the leaves of  $\mathcal{F}$  through  $x_n$  converge to the leaf of  $\mathcal{F}$  through  $x$ . Without loss of generality we may assume that  $x$  is in the attractor.

Let  $p_n \in T_i$  so that  $x_n$  are in  $\Phi_{\mathbb{R}}(p_n)$ . There are  $t_n \in \mathbb{R}$  with  $x_n = \Phi_{t_n}(p_n)$ . Since  $x$  is in the attractor then  $t_n$  converges to positive  $\infty$ . The leaf of  $\mathcal{F}$  through  $p_n$  is the  $\Phi$  flow saturation of the leaf of  $F_i$  through  $p_n$ . The tangent to this two dimensional set through  $p_n$  is generated by the Anosov vector field generating  $\Phi$  and a tangent vector  $v$  to  $F_i$  at  $p_n$ . The leaf of  $\mathcal{F}$  is  $\Phi$ -flow invariant. Flowing forward, the flow vector remains invariant. The vector  $v$  is transverse to the weak stable foliation and hence it flows more and more to the weak unstable direction. So flowing forward these leaves become more and more tangent to the  $E^0 \oplus E^u$  bundle, and limit to leaves of  $\mathcal{F}^{wu}$ . Since flowing forward limits to the attractor, this shows that the leaves of  $\mathcal{F}$  through  $x_n$  converge to the leaf of  $\mathcal{F}$  through  $x$ . This shows that  $\mathcal{F}$  defines a foliation. This finishes the proof of Theorem 3.1.  $\square$

We remark that the construction of  $\mathcal{F}$  highlights why our methods do not work when there are one dimensional basis sets. For simplicity suppose that there is a basic set which is a periodic orbit  $\gamma$ . There is a torus  $T$  so that negative saturation limits on  $\gamma$ . If we start with  $F$  in  $T$  transverse to both  $\mathcal{F}^{ws} \cap T$  and  $\mathcal{F}^{wu} \cap T$  then flowing backwards will make it limit to the weak stable leaf of  $\gamma$ . So the weak stable leaf of  $\gamma$  is in the collection  $\mathcal{F}$  so constructed. But there is also a torus  $T'$  so that the forward flow saturation limits on  $\gamma$ . The similar argument shows that the weak unstable foliation of  $\gamma$  also has to be in the collection  $\mathcal{F}$ . Hence the collection  $\mathcal{F}$  has sets which intersect transversely and cannot be a foliation.

The structure of the proof of Theorem 1.1 is as follows: We will prove the following properties for such a foliation  $\mathcal{F}$ :

- (1) The flow lines are *leafwise quasigeodesics* in leaves of  $\mathcal{F}$ ,
- (2) Every leaf of  $\mathcal{F}$  not contained in  $\mathcal{A}$  or  $\mathcal{R}$  is a non funnel leaf as in figure 2.

#### 4. GROMOV HYPERBOLICITY OF THE LEAVES OF $\mathcal{F}$

We will consider a foliation  $\mathcal{F}$  as constructed in the previous section.

In this section we will show that there exists a Riemannian metric  $g$  such that every leaf of the foliation  $\mathcal{F}$  is Gromov hyperbolic. By *Candel's Uniformization Theorem*, this condition is equivalent to the fact that every holonomy invariant non trivial measure  $\mu$  on  $\mathcal{M}$  has Euler characteristic  $\chi_{\mu}(\mathcal{M}, \mathcal{F}) < 0$ , which includes the case when there exists no invariant measure. For more details about Euler characteristic see [Can93] or [CC00]. In our context we will prove that there is no holonomy invariant transverse measure. In fact, under these conditions, Candel proved that there is a metric in  $\mathcal{M}$  inducing a smooth metric in the leaves so that curvature in each leaf of  $\mathcal{F}$  is constant equal to  $-1$ . A precise statements can be found at [Can93],[CC00] or

[Cal07]. We call such a metric a *Candel metric*. This Candel metric is not smooth in the transverse direction.

Here is the precise statement on the equivalence of Gromov hyperbolicity of leaves of a foliation and negative Euler characteristic of a positive invariant measure:

**Proposition 4.1** ([Can93]). *Let  $(M, F)$  be a compact oriented surface lamination with a Riemannian metric  $g$ . Then  $\chi(M, \mu) < 0$  for every positive invariant transverse measure  $\mu$  if and only if there is a metric in  $M$  which induces a metric in each leaf of  $F$  which makes it into a hyperbolic surface. In particular, this holds true if  $M$  has no invariant measure.*

To prove that all the leaves of  $\mathcal{F}$  are hyperbolic, we will show that there does not exist any invariant measure. We will argue by contradiction, we assume that there exist a invariant measure  $\mu$  and we will attain a contradiction.

The *support* of  $\mu$  on  $\mathcal{M}$ , denoted by  $\text{supp}(\mu)$ , is defined as the collection of all points  $x \in \mathcal{M}$  such that if  $\tau$  is a one dimensional manifold transverse to  $\mathcal{F}$  which contains  $x$  in its interior then  $\mu(\tau) > 0$ . The support of a holonomy invariant transverse measure is a closed set and it is saturated by  $\mathcal{F}$ , which means  $\text{supp}(\mu)$  is a union of leaves of  $\mathcal{F}$ . The orientation hypothesis is not essential as it can be achieved by a double cover. The double cover does not change the conformal type of any leaf.

**Lemma 4.2.** *The support of  $\mu$  on  $\mathcal{M}$  contains at least one leaf from the attractor  $\mathcal{A}$  or the repeller  $\mathcal{R}$ .*

*Proof.* Consider a point  $x \in \text{supp}(\mu)$  and suppose  $L_x$  is the leaf in  $\mathcal{F}$  which contains  $x$ , then  $L_x \subset \text{supp}(\mu)$  as  $\text{supp}(\mu)$  is  $\mathcal{F}$ -saturated. If  $x \in \mathcal{A}$ , then  $L_x \subset \mathcal{A}$  and the claim is true. Similarly if  $x$  is in  $\mathcal{R}$  then its leaf is contained in  $\text{supp}(\mu)$ . Finally suppose that  $x \notin (\mathcal{A} \cup \mathcal{R})$ . Then consider the sequence  $\{\Phi_n(x)\}$  as  $n \rightarrow \infty$ . Let  $z$  be an accumulation point of  $\{\Phi_n(x)\}$ . As  $\text{supp}(\mu)$  is closed,  $z$  is in  $\text{supp}(\mu)$  and hence  $L_z \subset \text{supp}(\mu)$ . Since  $z$  is an accumulation point of  $\Phi_n(x)$ , it implies that  $z$  is a non wandering point, hence  $z \in \mathcal{A} \cup \mathcal{R}$  and  $L_z \subset (\mathcal{A} \cup \mathcal{R}) \cap \text{supp}(\mu)$ . In fact since  $n \rightarrow \infty$ , it follows that  $z$  is in the attractor, so  $L_z \subset \mathcal{A}$ .  $\square$

Suppose  $L$  is a leaf in  $\text{supp}(\mu)$  which is contained in  $\mathcal{A}$  (assume in  $\mathcal{A}$  without loss of generality). By [Pla75, Theorem 6.3], we know that if  $\mu$  is a holonomy invariant transverse measure on a compact manifold foliated by a codimension one foliation  $\mathcal{F}$  then any leaf contained in  $\text{supp}(\mu)$  has polynomial growth. Then the leaf  $L_z$  in the attractor  $\mathcal{A}$  as obtained in the previous paragraph has polynomial growth. Recall that the leaves of  $\mathcal{F}$  are either planes or annuli. At the same time,  $L_z$  is contained in the attractor and each leaf in the attractor belongs to the weak unstable foliation of the Anosov flow  $\Phi$ . But weak stable and weak unstable leaves of Anosov flows have exponential growth, a contradiction. This contradiction shows that each leaf of  $\mathcal{F}$  is Gromov hyperbolic.

As each leaf  $L \in \tilde{\mathcal{F}}$  is Gromov hyperbolic with respect to the path metric from the induced Riemannian metric from  $\tilde{\mathcal{M}}$ , we can define the circle at infinity or the ideal boundary  $S^1(L)$  of each leaf  $L$ .

Next we will describe the topology we will use on the spaces

$$S^1(\tilde{\mathcal{M}}) = \bigcup_{L \in \tilde{\mathcal{F}}} S^1(L) \text{ and}$$

$$\widetilde{\mathcal{M}} \cup S^1(\widetilde{\mathcal{M}}) = \bigcup_{L \in \widetilde{\mathcal{F}}} (L \cup S^1(L))$$

For this we will assume first that  $M$  has a Candel metric.

Suppose  $\tau$  is an open segment homeomorphic to  $(0,1)$  and transversal to  $\widetilde{\mathcal{F}}$ . We define the the following sets

$$\mathcal{P}_\tau = \bigcup_{y \in \tau} S^1(L_y) \text{ and } \mathcal{Q}_\tau = \bigcup_{y \in \tau} (L_y \cup S^1(L_y))$$

If  $T^1(\tau)$  denotes the unit tangent bundle of  $\widetilde{\mathcal{F}}$  restricted to  $\tau$ , then  $T^1(\tau)$  is naturally homeomorphic to the standard cylinder. The natural identification between  $T^1(\tau)$  and  $\mathcal{P}_\tau$  induces the topology on  $\mathcal{P}_\tau$  homeomorphic to the standard annulus. In [Fen02] it is proved that this is independent of the transversal  $\tau$  that is chosen intersecting the same sets of leaves of  $\widetilde{\mathcal{F}}$ . This is because the metrics induced in  $S^1(L)$  from the visual metric in any point are Hölder equivalent.

Similarly  $\mathcal{Q}_\tau$  has a natural topology homeomorphic to the standard solid cylinder.

The collection of all  $\mathcal{P}_\tau$ 's over a  $\pi_1(M)$ -invariant discrete collection of transversals defines a topology on  $S^1(\widetilde{\mathcal{M}})$ . Similarly the collection of  $\mathcal{Q}_\tau$ -s over the same collection of transversals defines a topology on  $\widetilde{\mathcal{M}} \cup S^1(\widetilde{\mathcal{M}})$ . Deck transformations act by homeomorphisms on both sets. For more details see [Fen02], [Cal00] or [Cal07].

After the fact it is easy to see that the topologies described are independent of the specific metric in  $M$  chosen and also work for any Riemannian metric in  $M$ .

## 5. PROPERTIES OF FLOW LINES

This section describes the behavior of forward rays of flow lines, in particular their asymptotic behavior towards the the boundary at infinity  $\bigcup_{L \in \widetilde{\mathcal{F}}} S^1(L)$ . In particular

we will prove that the rays are quasigeodesics in their respective leaves of  $\widetilde{\mathcal{F}}$ . Notice that this is definitely much weaker than saying that full flow lines are quasigeodesics in their respective leaves. We will also show that in some leaves, the forward ideal points are pairwise distinct and the negative ideal points are also pairwise distinct. In particular even if the flow foliation is a leafwise quasigeodesic subfoliation of  $\mathcal{F}$  it will not have the funnel property.

We now introduce a family of sets in  $\widetilde{\mathcal{M}}$  which will be extremely useful for us:

### The sets $\mathcal{U}$

Fix a point  $x \in \widetilde{\mathcal{A}} \subset \widetilde{\mathcal{M}}$  and the forward ray from  $x$  which is  $\gamma_x^+ = \widetilde{\Phi}_{[0,\infty)}(x)$  starting at  $x$ , let  $L_x \subset \widetilde{\mathcal{A}}$  be the leaf containing  $\gamma_x^+$ . Recall that in the attractor  $\mathcal{A}$  the foliation  $\mathcal{F}$  is equal to  $\mathcal{F}^{wu}$ , hence transverse to  $\mathcal{F}^{ws}$ . Therefore the foliations  $\widetilde{\mathcal{F}}$  and  $\widetilde{\mathcal{F}}^{ws}$  are transversal to each other near  $\widetilde{\mathcal{A}}$ .

Let  $U$  be a compact rectangle transverse to the flow and with  $x$  in the interior of  $U$ . We assume that  $U$  is contained in foliation boxes of all foliations, that  $U$  is made up of a union of stable segments, every one of which intersects the local strong unstable segment through  $x$ . Consider the set

$$\mathcal{U} = \widetilde{\Phi}_{[0,\infty)}(U)$$

The set  $\mathcal{U}$  is a neighborhood of the forward ray  $\widetilde{\Phi}_{[0,\infty)}(x)$ . We can assume that  $\mathcal{U}$  is homeomorphic to  $[-1,1] \times [-1,1] \times [0,\infty)$  with  $x = (0,0,0)$  and we can define coordinates on  $\mathcal{U}$  such that

- $U$  is identified with  $[-1, 1] \times [-1, 1] \times \{0\}$  and points on  $U$  are represented as  $(r, s, 0)$  for  $r, s \in [-1, 1]$ . In particular,  $x = (0, 0, 0)$ .
- for a point  $y = (r, s, 0)$ ,  $\tilde{\Phi}_t(y)$  has coordinates  $(r, s, t)$ , that is, the ray  $\{(r, s, t) | t \in [0, \infty)\}$  represents the ray  $\tilde{\Phi}_{[0, \infty)}(y)$ .
- for a point  $y' = (r', s, t') \in \mathcal{U}$ ,  $P_{y'}$  denotes the horizontal infinite strip

$$P_{y'} = \{(r, s, t) | r \in [-1, 1], t \in [0, \infty)\}$$

The infinite strip  $P_{y'}$  is contained in the leaf  $L_{y'} \in \tilde{\mathcal{F}}$  which contains  $y'$ .

- for a point  $y' = (r, s', t') \in \mathcal{U}$ ,  $Q_{y'}$  denotes the vertical infinite strip

$$Q_{y'} = \{(r, s, t) | s \in [-1, 1], t \in [0, \infty)\}.$$

The infinite strip  $Q_{y'}$  is contained in the leaf  $E_{y'} \in \tilde{\mathcal{F}}^{ws}$  which contains  $y'$ .

As  $x = (0, 0, 0) \in \tilde{\mathcal{A}}$  the leaf of  $\tilde{\mathcal{F}}$  through  $x$  is actually the weak unstable leaf of  $\tilde{\Phi}$  through  $x$ , hence  $P_x$  is contained in the  $\tilde{\mathcal{F}}^{wu}$  leaf through  $x$ .

The sets  $\mathcal{U}$  will be used throughout this section.

We can define a projection map  $\Pi : \mathcal{U} \rightarrow P_x$  by the formula  $\Pi(y) = S_y \cap P_x$  where  $S_y$  is the one dimensional leaf of the strong stable foliation  $\tilde{\mathcal{F}}^s$  containing  $y$ . This is possible because one can do that in the original rectangle  $U$  as it is a union of strong stable segments, and then  $\mathcal{U}$  is the flow forward saturation of  $U$  and the maps  $\tilde{\Phi}_t$  preserve the strong stable foliation in  $\tilde{\mathcal{M}}$ . These projection maps are well defined and continuous because of the foliation structures on  $\mathcal{U}$ .

For any  $y \in \mathcal{U}$ , the rays  $\tilde{\Phi}_{[0, \infty)}(y)$  and  $\tilde{\Phi}_{[0, \infty)}(\Pi(y))$  are asymptotic as they lie on the same weak stable leaf. We assume that lengths of all the line segments  $\{(r, s, 0) | s \in [-1, 1]\}$  are less than a fixed  $\epsilon > 0$  in  $\tilde{\mathcal{M}}$ .

The line segment  $\lambda = \{(0, s, 0) | s \in [-1, 1]\}$  is transversal to  $\tilde{\mathcal{F}}$ . Consider the open sets  $\mathcal{V} = \bigcup_{x' \in \lambda} S^1(L_{x'})$  and  $\mathcal{W} = \bigcup_{x' \in \lambda} (L_{x'} \cup S^1(L_{x'}))$ .

**Definition 5.1.** Let  $\gamma$  be a flow line of  $\tilde{\Phi}$  contained in a leaf  $L$  of  $\tilde{\mathcal{F}}$ . Given  $x$  in  $\gamma$  if the forward ray converges to a single point of  $S^1(L)$  we let this be  $\eta^+(x)$ . Similarly define  $\eta^-(x)$ . In addition given a point  $a$  in  $\tilde{\mathcal{M}}$  let  $\gamma_a$  be the flow line of  $\tilde{\Phi}$  containing  $a$ .

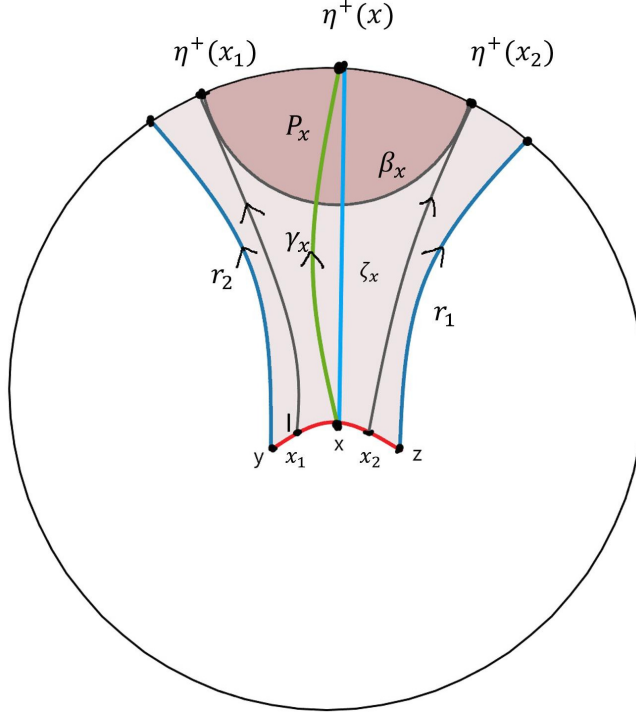
**Lemma 5.2.** For any  $w \in \tilde{\mathcal{M}}$ , the forward and the backward rays of the flow line  $\gamma_w = \tilde{\Phi}_{\mathbb{R}}(w)$  are quasigeodesics on the leaf  $L_w$  in  $\tilde{\mathcal{F}}$  which contains the flow line.

*Proof.* To prove this Lemma we assume a Candel metric in  $\mathcal{M}$  so that leaves of  $\mathcal{F}$  are hyperbolic surfaces. This metric is not Riemannian, but the result is independent of the metric.

By lemma 2.2 every forward or backward ray in a leaf in  $\tilde{\mathcal{F}}^{wu}$  or  $\tilde{\mathcal{F}}^{ws}$  is quasigeodesic in its respective leaf. In particular every flow line is a quasigeodesic in the respective leaf of  $\mathcal{F}$  if contained in the attractor or repeller.

So we may assume that the ray is in a leaf not in the attractor or repeller. Property 2.9 shows that the ray is asymptotic with a ray in the attractor. By taking a subray we may assume that the ray is in the weak stable leaf of a point  $x$  in the attractor and the initial point  $w$  of the ray is very near  $x$ . Hence we may assume that the initial point is contained in a local cross section  $U$  to  $\tilde{\Phi}$  centered at  $x$  as described above. Let  $L_x$  be the leaf of  $\tilde{\mathcal{F}}$  containing  $x$ , and similarly define  $L_w$ .

Recall that  $L_x$  is also the weak unstable leaf of  $\tilde{\Phi}$  containing  $x$ .

FIGURE 3. the region  $A_x$  in  $L_x$  and the half-space  $P_x$ 

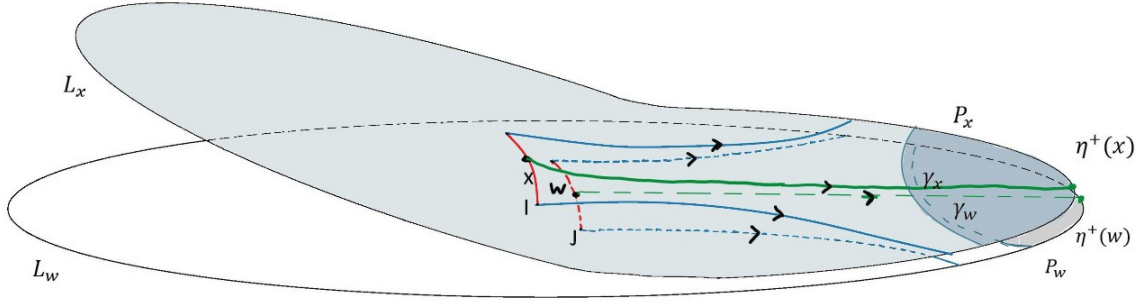
Therefore it is sufficient to show that every forward ray in the set  $\mathcal{U}$  described above is quasigeodesic in its respective leaf of  $\tilde{\mathcal{F}}$ . In the  $\tilde{\mathcal{F}}$  leaf through  $x$  we consider the following curve. Let  $I$  be the compact unstable segment  $U \cap L_x$  which has endpoints  $z, y$ . Let  $r_1, r_2$  be the forward rays of  $\tilde{\mathcal{F}}$  through  $z, y$ . Then  $c := r_1 \cup I \cup r_2$  is a bi-infinite curve as shown in figure 3. The two rays  $r_1, r_2$  are quasigeodesics in  $L_x$  and they converge to distinct ideal points in  $S^1(L)$ .

The curve  $c$  bounds a region  $A$  in  $L_x$  (as in figure 3) which is exactly  $\tilde{\Phi}_{[0, \infty)}(I)$ . This is contained in  $\mathcal{U}$ . This region contains a half plane in  $L_x$ .

Recall that we are considering  $w$  a point in  $U \cap \tilde{\mathcal{F}}^s(x)$ , where  $\tilde{\mathcal{F}}^s(x)$  is the strong stable leaf of  $x$ . Let  $J$  be the intersection of  $L_w \cap U$ , where  $L_w$  is the leaf of  $\tilde{\mathcal{F}}$  through  $w$ . Then  $B := \tilde{\Phi}_{[0, \infty)}(J)$  is contained in  $L_w$  and contained in  $\mathcal{U}$ . In addition since every point in  $J$  is in the strong stable leaf of a point in  $I$ , it follows that every flow ray in  $B$  is asymptotic to a flow ray in  $A$ . In fact as points leave compact sets in  $B$  they become closer and closer to  $A$ .

The induced metrics on the leaves  $\tilde{\mathcal{F}}$  vary continuously and the ray  $r_x = \tilde{\Phi}_{[0, \infty)}(x)$  is quasigeodesic in its leaf and asymptotic to the ray  $r_w = \tilde{\Phi}_{[0, \infty)}(w)$ , it follows that the other ray is also a quasigeodesic in its  $\tilde{\mathcal{F}}$  leaf.

Since this is a very subtle point we provide specific details. In the leaf  $L_x$  choose two points  $x_1, x_2$  in  $I$  with  $x$  in between them so that the geodesic  $\beta_x$  in  $L_x$  with ideal points  $\eta^+(x_1), \eta^+(x_2)$  is contained in the interior of  $A$ . This is possible since the flow lines in  $L_x$  are uniform quasigeodesics and they spread out in the forward direction. We stress that in general it is not possible to choose  $x_1, x_2$  as the endpoints of  $I$  as the flow lines are only quasigeodesics and not geodesics in  $L_x$ . Let  $P_x$  be the half plane of  $L_x$  bounded by  $\beta_x$  and containing a forward ray from  $x$ . We also may assume

FIGURE 4.  $P_x$  in  $L_x$  is asymptotic to  $P_w$  in  $L_w$ 

that every point in  $P_x$  is  $\epsilon_1$  close to  $L_w$  with  $\epsilon_1$  very close to zero. Then  $\beta_x$  is  $\epsilon_1$  close to a curve  $\beta'$  in  $L_w$  which has geodesic curvature very close to zero. To obtain this property of  $\beta'$  with small geodesic curvature in  $L_w$  was one of the reasons to choose a Candel metric with hyperbolic leaves varying continuously. In particular this curve  $\beta'$  is very close in  $L_w$  to an actual geodesic in  $L_w$ , and this geodesic is denoted by  $\beta_w$ . Let  $P_w$  be the half plane of  $L_w$  which is very close to  $P_x$  as shown in figure 4.

Now we prove that the ray  $r_w$  is quasigeodesic in  $L_w$ . The ray  $r_x$  in  $L_x$  satisfies the quasi-isometric property for some  $(K, s)$ . The ray  $r_w$  is asymptotic to  $r_x$  so length along  $r_w$  is extremely well approximated by length along  $r_x$  when moving forward. But distance in  $L_w$  between points in  $r_w$  is also boundedly approximated by the corresponding distance in  $L_x$ . This is because if one gets a length minimizing path in  $L_w$  connecting the endpoints of a segment in  $r_w$ , then this segment is contained in the half plane  $P_w$  as above if the points are far enough in  $L_w$  from  $w$ . This is why we constructed  $P_w$ . Since  $P_w$  is  $\epsilon_1$  close to  $P_x$  there is a corresponding segment in  $P_x$  whose length is multiplicatively increased by at most a factor of  $1 + \epsilon_2$ , where  $\epsilon_2$  is very small. But the approximating segment in a flow line in  $A$  is a quasigeodesic in  $L_x$  with constants  $(K, s)$ , hence the length of the approximating path is bounded below, and so is the length of the length minimizing original path in  $P_w$ .

This proves that  $r_w$  is a quasigeodesic in  $L_w$ .

If we reverse the flow every backward ray becomes forward ray, hence leafwise quasigeodesic.  $\square$

By compactness and continuity there is  $K_0, s_0$  so that given any ray in a flow line, there is a subray of it that is a  $(K_0, s_0)$  quasigeodesic in its leaf of  $\tilde{\mathcal{F}}$ . As the flow rays are quasigeodesics they define unique points on the ideal boundaries.

In the next proposition we consider the sets  $P_y$  contained in  $\mathcal{U}$ .

**Proposition 5.3.** *Suppose  $a, b \in P_y$  but  $\gamma_a \neq \gamma_b$ , then  $\eta^+(a) \neq \eta^+(b)$  in  $S^1(L_y)$ .*

*Proof.* By the previous lemma we already know that all rays are quasigeodesics in their respective leaves. We do the proof by contradiction and assume that  $\eta^+(a) = \eta^+(b)$ . Since the rays  $\tilde{\Phi}_{[0, \infty)}(a), \tilde{\Phi}_{[0, \infty)}(b)$  are quasigeodesics in  $L_y$  and by assumption they have the same ideal point in  $S^1(L_y)$ , the following happens: there is  $d_0 > 0$  and points  $p_i, q_i$  in  $\tilde{\Phi}_{[0, \infty)}(a), \tilde{\Phi}_{[0, \infty)}(b)$  respectively, escaping in the rays and so that  $d_{L_y}(p_i, q_i) < d_0$ . Consider the points  $\Pi(a)$  and  $\Pi(b)$  on  $P_x$ . Since

$$\tilde{\Phi}_{[0, \infty)}(a), \tilde{\Phi}_{[0, \infty)}(\Pi(a))$$

are asymptotic in the weak stable leaf of  $\tilde{\Phi}$  in  $\tilde{\mathcal{M}}$ , there are  $p'_i$  in  $\tilde{\Phi}_{[0,\infty)}(\Pi(a))$  with  $d(p_i, p'_i) \rightarrow 0$ . Here  $d$  is ambient distance in  $\tilde{\mathcal{M}}$ . Similarly there are  $q'_i$  in  $\tilde{\Phi}_{[0,\infty)}(\Pi(b))$  with  $d(q_i, q'_i) \rightarrow 0$ . By the local product structure of the foliation  $\mathcal{F}$  it follows that  $d_{L_x}(p'_i, q'_i) < d_0 + 1$  for  $i$  sufficiently big. Therefore the rays  $\tilde{\Phi}_{[0,\infty)}(\Pi(a))$ ,  $\tilde{\Phi}_{[0,\infty)}(\Pi(b))$  converge to the same ideal point in  $S^1(L_x)$ . Here we are using that  $L_x$  is also a weak unstable leaf of  $\tilde{\Phi}$  and the flow lines are quasigeodesics in the weak unstable leaves of  $\tilde{\Phi}$ . But the flow lines  $\tilde{\Phi}_{\mathbb{R}}(\Pi(a))$ ,  $\tilde{\Phi}_{\mathbb{R}}(\Pi(b))$  are distinct flow lines in  $L_x$ . Again by the description 2.4 of ideal points of flow lines in weak unstable leaves, the forward limit points are distinct, that is,

$$\eta^+(\Pi(a)) \neq \eta^+(\Pi(b)) \quad \text{in } S^1(L_x)$$

This is a contradiction and shows that  $\eta^+(a) \neq \eta^+(b)$  in  $S^1(L_y)$ .  $\square$

**Lemma 5.4.** *In each leaf  $L$  of  $\tilde{\mathcal{F}}$  the leaf space of the flow foliation is Hausdorff and homeomorphic to the reals  $\mathbb{R}$ .*

*Proof.* In the lifts of leaves in the attractor and repeller this is obvious since the foliation by flow lines satisfies this property in weak stable and weak unstable leaves of Anosov flows [Fen94]. Any other leaf  $L$  of  $\tilde{\mathcal{F}}$  is the lift of a leaf of  $\mathcal{F}$  which intersects a torus  $T$  from the collection of tori  $\{T_i\}$  which separates  $\mathcal{A}$  and  $\mathcal{R}$ . Hence  $L$  intersects a lift  $\tilde{T}$  of  $T$  in a curve  $\beta$ . The flow saturation of  $\beta$  is exactly  $L$ , since every flow line in  $M$  is either in the attractor or repeller; or intersects a torus in  $\{T_i\}$ . The curve  $\beta$  is transverse to the weak stable and weak unstable foliations, hence intersects a flow line exactly once. Hence  $\beta$  parametrizes the set of center leaves in  $L$ . This proves the result.  $\square$

For each  $L$  of  $\tilde{\mathcal{F}}$  the map  $\eta^+$  induces a map from the flow foliation leaf space in  $L$  (which is  $\cong \mathbb{R}$ ) to  $S^1(L)$ . Since flow lines are disjoint in their leaves, this map is weakly monotone.

**Corollary 5.5.** *For all  $y \in \mathcal{U}$ ,  $\eta^+(y) \neq \eta^-(y)$  in  $S^1(L_y)$ .*

*Proof.* If  $\eta^+(y) = \eta^-(y)$  then the flow line  $\gamma_y$  bounds a disk  $\mathcal{D}$  on  $L_y \cup S^1(L_y)$  such that the closure of  $\mathcal{D}$  in  $L \cup S^1(L)$  intersects  $S^1(L)$  only in  $\eta^+(y) = \eta^-(y)$ . For any  $z$  in interior of  $\mathcal{D}$ , the flow line  $\gamma_z$  is contained in  $\mathcal{D}$ , hence  $\eta^+(z) = \eta^-(z) = \eta^+(y) = \eta^-(y)$ . This contradicts the proposition 5.3, because if  $z, y \in \mathcal{U}$  and  $\gamma_z \neq \gamma_y$ , then  $\eta^+(z) \neq \eta^+(y)$ .  $\square$

We now extend the map  $\eta^+$  to a map from  $\tilde{\mathcal{M}}$  to  $S^1(\tilde{\mathcal{M}})$ . For each  $x$  in  $\tilde{\mathcal{M}}$  then  $\eta^+(x)$  is in  $S^1(L_x) \subset S^1(\tilde{\mathcal{M}})$ .

**Proposition 5.6.**  *$\eta^+$  and  $\eta^-$  are continuous on  $\tilde{\mathcal{M}}$ .*

*Proof.* In this proof we again use a Candel metric in  $M$ .

Suppose  $x_i \rightarrow x_0$  in  $\tilde{\mathcal{M}}$ . We will show that  $\eta^+(x_i) \rightarrow \eta^+(x_0)$  in  $S^1(\tilde{\mathcal{M}})$ . There are two different cases depending on whether  $x_0 \in \tilde{\mathcal{R}}$  or  $x_0 \notin \tilde{\mathcal{R}}$ .

We first prove the result for  $x_0 \notin \tilde{\mathcal{R}}$ . As  $x_0 \notin \tilde{\mathcal{R}}$  the forward ray starting at  $x_0$  is asymptotic to a forward flow ray in  $\tilde{\mathcal{A}}$ . Therefore it is enough to assume that  $\{x_i\}$  and  $x_0$  belong to a neighborhood  $\mathcal{U}$  as constructed above, since this is true for every ray asymptotic to  $\tilde{\mathcal{A}}$ .

For  $z$  in  $\mathcal{M}$  let  $L_z$  be the leaf of  $\tilde{\mathcal{F}}$  containing  $z$ .

For for  $i \in \mathbb{N} \cup \{0\}$ , let  $\gamma_i^+$  denote the forward flow ray starting from  $x_i$  and let  $\zeta_i$  denote the geodesic ray on  $L_{x_i}$  starting at  $x_i$  and with ideal point  $\eta^+(x_i)$  in  $S^1(L_{x_i})$ . Each  $\zeta_i$  defines the ideal point  $\eta^+(x_i)$  on  $S^1(L_{x_i})$ , therefore it is enough to show that any convergent subsequence of  $(\zeta_i)$  converges to  $\zeta_0$  in the compact open topology. Since  $x_i$  are in a compact subset of  $\tilde{\mathcal{M}}$ , existence of convergent subsequences of  $\{\zeta_i\}$  is assured.

Suppose that a subsequence  $(\zeta_{i(k)})$  converges to  $\zeta'$ . We have to prove that  $\zeta' = \zeta_0$ . We assume that the neighborhood  $\mathcal{U}$  constructed above has a point  $x \in \tilde{\mathcal{A}}$  as in the construction of  $\mathcal{U}$ . Then all flow rays in  $L_x \cap \mathcal{U}$  are  $(K, s)$ -quasigeodesics in  $L_x$  for some fixed  $K, s$ . Since all flow rays in  $\mathcal{U}$  are forward asymptotic to flow rays in  $L_x$  there are  $K', s'$  so that all flow rays in  $\mathcal{U}$  are  $(K', s')$ -quasigeodesics in their respective  $\tilde{\mathcal{F}}$  leaves. It follows that there exists a uniform  $d > 0$  such that

$$\gamma_{i(k)}^+ \subset \mathcal{N}_d(\zeta_{i(k)}) \quad \text{and} \quad \gamma_0^+ \subset \mathcal{N}_d(\zeta_0),$$

where  $\mathcal{N}_d$  denotes the neighborhood of radius  $d$  in the respective leaf of  $\tilde{\mathcal{F}}$ . For any  $d_1 > 0$ , the segment of length  $d_1$  on  $\gamma_{i(k)}^+$  starting at  $x_{i(k)}$  is within  $d$ -distance from  $\zeta_{i(k)}$ . Therefore in the limit the segment of  $\gamma_0^+$  of length  $d_1$  starting from  $x_0$  is contained in  $d$  distant neighborhood from  $\zeta'$  in the respective leaf. This is true for all  $d_1$ , so  $\zeta'$  is at Hausdorff distance  $d$  from  $\gamma_0^+$  on  $L_{x_0}$ . But  $\gamma_0^+$  is also at bounded distance from  $\zeta_0$  on  $L_{x_0}$ , therefore  $\zeta'$  and  $\zeta_0$  are at a finite Hausdorff distance from each other on  $L_{x_0}$ . Hence  $\zeta' = \zeta_0$ , because they have the same starting point. As this is true for all convergent subsequences of  $(\zeta_i)$ , we get our result for  $x_0$  not in  $\tilde{\mathcal{R}}$ .

Before dealing with the remaining case let us note the following:

**Observation 5.7.** *By the construction of  $\mathcal{U}$  starting with  $x$  in  $\tilde{\mathcal{A}}$ , and continuity of  $\eta^+$  near  $\tilde{\mathcal{A}}$  we observe that the set  $\mathcal{U} \cup \{\eta^+(z) | z \in \mathcal{U}\}$  is homeomorphic to  $[0, 1] \times [0, 1] \times [0, 1]$  inside  $\mathcal{W} = \bigcup_{y \in \lambda} (L_y \cup S^1(L_y))$ , which is homeomorphic to a compact solid cylinder  $[0, 1] \times \{\text{the unit disc } \mathbb{D}\}$ .*

*The set  $\mathcal{U} \cup \{\eta^+(z) | z \in \mathcal{U}\}$  above is saturated by forward flow lines and all the ideal points contained in this neighborhood are defined by forward flow rays. This is true for example for any forward ideal point  $p$  on any  $S^1(L)$  where  $L$  is a leaf of  $\tilde{\mathcal{F}}$  in  $\tilde{\mathcal{A}}$ . Here  $\lambda$  is a transversal to  $\tilde{\mathcal{F}}$  intersecting exactly the leaves of  $\tilde{\mathcal{F}}$  which intersect  $\mathcal{U}$ . This in particular implies that for any  $y$  in  $\lambda$  and ideal point  $v$  in  $S^1(L_y)$  which is a forward ideal point of  $\tilde{\Phi}_{\mathbb{R}}(z)$  with  $z$  in  $L_y \cap \mathcal{U}$  and in the interior of  $\mathcal{U}$  then  $\eta^+(z)$  is an interior point of the interval  $I_y$  of  $S^1(L_y)$  associated to all forward ideal points of flow lines in  $L_y \cap \mathcal{U}$ . In particular we stress the important fact that any ideal point in this interval  $I_y$  in  $S^1(L_y)$  is an ideal point of a forward flow ray, but it is not an ideal point of a backwards flow ray.*

*In an analogous way the corresponding property is true for any backward ideal point  $q$  on any  $S^1(E)$  where  $E \subset \tilde{\mathcal{R}}$ , but the difference is that the neighborhood around a backward flow ray defining  $q$  is saturated by backward flowrays and all the ideal points in that neighborhood are defined by backward flowrays and no such ideal point is an ideal point of a forward flow ray.*

To continue the proof of Proposition 5.6 we next assume that  $x_0 \in \tilde{\mathcal{R}}$ . Suppose that a subsequence  $(\eta^+(x_{i(k)}))$  converges to  $q$  where  $q$  is not  $\eta^+(x_0)$ . As  $x_0$  is in  $\tilde{\mathcal{R}}$ , then  $L_{x_0}$  is a leaf of the weak stable foliation  $\tilde{\mathcal{F}}^{ws}$ . Hence by property 2.4 on  $L_{x_0}$  all

the forward flow rays converge to a single ideal point in  $S^1(L_{x_0})$  and all the other ideal points in  $S^1(L_{x_0})$  are ideal points of backward flow rays. As  $q \neq \eta^+(x_0)$ ,  $q$  is defined by a backward ray, that is  $q = \eta^-(z)$  for some  $z$  in  $L_{x_0}$ . By Observation 5.7 starting with  $z$  in  $\tilde{\mathcal{R}}$  (notice that  $z$  is in the repeller, not the attractor), there exists a neighborhood  $\mathcal{V}$  saturated by backward flow rays around  $z$  in  $\bigcup_{y \in \lambda'} (L_y \cup S^1(L_y))$  for some transversal  $\lambda'$ . By Observation 5.7 all limit points are backward ideal points in  $\mathcal{V}$  and no limit point is a forward ideal point. This contradicts the fact that the forward rays  $\gamma_{i(k)}^+$  have ideal points in these intervals of ideal points for  $k$  big enough by construction. This contradiction shows that a subsequence  $(\eta^+(x_{i(k)}))$  converging to  $q \neq \eta^+(x_0)$  is not possible, hence  $q = \eta^+(x_0)$ .

Hence  $\eta^+$  is continuous on  $\tilde{\mathcal{M}}$ . If we consider the flow  $\Psi_t = \Phi_{-t}$  then backward ideal points of  $\Phi_t$  are forward ideal points of  $\Psi_{-t}$  and the continuity of  $\eta^-$  follows. This completes the proof of Proposition 5.6.  $\square$

In the next lemma we combine all the above results and describe a key property that will be used to show that all the flow lines are quasigeodesic on their respective leaves of  $\tilde{\mathcal{F}}$ .

Again we use a Candel metric. As before, given  $x$  in  $\tilde{\mathcal{M}}$ , let  $\gamma_x$  be the  $\tilde{\Phi}$  flow line through it. In addition if  $L_x$  is the leaf of  $\tilde{\mathcal{F}}$  containing  $x$ , let  $g_x$  be the geodesic in  $L_x$  with ideal points  $\eta^+(x), \eta^-(x)$ . Notice that we already proved that  $\eta^+(x), \eta^-(x)$  exist and are distinct from each other. This follows from Corollary 5.5.

**Lemma 5.8.** *There exists  $d > 0$  such that for all  $x \in \tilde{\mathcal{M}}$  we have that*

$$\gamma_x \subset \mathcal{N}_d(g_x),$$

where  $g_x$  is the geodesic on  $L_x$  connecting  $\eta^+(x)$  and  $\eta^-(x)$  and  $\mathcal{N}_d(g_x)$  is the  $d$ -neighborhood of  $g_x$  on  $L_x$ .

*Proof.* Suppose that there does not exist any such  $d$ . Then there exists a sequence  $(x_i)$  in  $\tilde{\mathcal{M}}$  with  $x_i$  in leaves  $L_{x_i}$  of  $\tilde{\mathcal{F}}$  such that  $d_{L_{x_i}}(x_i, g_{x_i}) > i$ . Up to deck transformations there exists a convergent subsequence of  $(x_i)$  which we assume is the original sequence, and we assume  $x_i \rightarrow x$ . By lemma 5.6 we know that

$$\eta^+(x_i) \rightarrow \eta^+(x) \quad \text{and} \quad \eta^-(x_i) \rightarrow \eta^-(x).$$

Since  $x_i$  converges to  $x$  we assume that all  $x_i$  are in leaves of  $\tilde{\mathcal{F}}$  which intersect a fixed transversal  $\lambda$  to  $\tilde{\mathcal{F}}$ .

Since  $\eta^+(x_i)$  converges to  $\eta^+(x)$ ,  $\eta^-(x_i)$  converges to  $\eta^-(x)$  and  $\eta^+(x) \neq \eta^-(x)$  it follows that  $\{g_{x_i}\}$  converges to  $g_x$ . This uses that the topology defined on  $\bigcup_{y \in \lambda} (S^1(L_y))$

is given by the trivialization of the unit tangent bundle to  $\tilde{\mathcal{F}}$  along  $\lambda$ . By convergence we mean convergence in the compact open topology. But this contradicts that  $d_{L_{x_i}}(x_i, g_{x_i})$  converges to infinity, since  $d_{L_x}(x, g_x)$  is finite and the sequence converges to it. This finishes the proof.  $\square$

## 6. FLOW LINES ARE LEAFWISE QUASIGEODESIC

We first prove a weak quasigeodesic property of the flow lines on the leaves of  $\tilde{\mathcal{F}}$  containing them.

**Proposition 6.1.** *For all  $b > 0$  there exists a constant  $c_b > 0$  depending on  $b$  such that if  $\gamma$  is a flow segment connecting  $x$  and  $y$  with  $\text{length}(\gamma) > c_b$  then  $d_{L_x}(x, y) > b$ , where  $L_x$  is the leaf of  $\tilde{\mathcal{F}}$  which contains  $x$ .*

*Proof.* Fix  $b > 0$ . We do the proof by contradiction. Suppose the statement is not true for some  $b > 0$ . Then for all  $i \in \mathbb{N}$  there exists two points  $x_i$  and  $y_i$  in leaves  $L_i$  of  $\tilde{\mathcal{F}}$ , with  $x_i, y_i$  in the same flow line defining a flow line segment  $\gamma_i$  satisfying  $\text{length}(\gamma_i) > i$  but  $d_{L_i}(x_i, y_i) < b$ . Up to deck transformations and a subsequence, we assume that  $(x_i)$  is convergent and  $x_i \rightarrow x_0$ . Since  $d_{L_i}(x_i, y_i) < b$  we can similarly assume that  $(y_i)$  is convergent and let  $y_i \rightarrow y_0$ .

**Claim 1.**  $x_0$  and  $y_0$  are on the same leaf  $L_0$  of  $\tilde{\mathcal{F}}$ .

*Proof.* If we consider a compact ball  $B_{x_0}$  on  $L_0$  containing  $x_0$  and a product neighborhood of  $N(B_{x_0})$  of  $\tilde{\mathcal{F}}$ , then for all large  $i$ ,  $L_i$  intersects  $N(B_{x_0})$  and  $x_i \in L_i \cap N(B_{x_0})$ . If we consider  $B_{x_0}$  sufficiently large, the assumption  $d_{L_i}(x_i, y_i) < b$  for all  $i$  forces that  $y_i$  has to be contained in  $N(B_{x_0})$ . Hence by the product structure on  $N(B_{x_0})$ ,  $y_0$  also has to be on  $L_0$  as  $y_i \rightarrow y_0$ .  $\square$

**Claim 2.**  $x_0$  and  $y_0$  cannot be on the same flow line in  $L_0$ .

*Proof.* If not, then there exists a flow line segment  $\gamma$  connecting  $x_0$  and  $y_0$  and consider a compact neighborhood  $\mathcal{N}$  around  $\gamma$  which has a product structure with respect to the flow lines. This is the crucial fact. As  $x_i \rightarrow x_0$  and  $y_i \rightarrow y_0$ , the flow segments  $\gamma_i$  are contained in  $\mathcal{N}$  for all large  $i$ . By continuity of length of flow lines,  $\text{length}(\gamma_i) \rightarrow \text{length}(\gamma)$ . But that is not possible as  $\text{length}(\gamma_i) \rightarrow \infty$  and  $\gamma$  is compact, a contradiction.  $\square$

**Claim 3.**  $x_0$  and  $y_0$  can not be connected by a curve on  $L_0$  everywhere transversal to the flow lines in  $L_0$ .

*Proof.* Suppose that there exists a line segment  $\sigma$  on  $L_0$  everywhere transversal to the flow lines on  $L_0$  and connecting  $x_0$  and  $y_0$ . By the local product structure of  $\tilde{\mathcal{F}}$  near  $\sigma \in L_0$ , there should be a segment  $\sigma_i$  in  $L_i$  connecting  $x_i$  and  $y_i$  and everywhere transversal to flow lines on  $L_i$ . Up to taking a sub-segment of  $\gamma_i$  if necessary and then a sub-segment of  $\sigma_i$  we may assume that  $\gamma_i$  does not intersect the interior of  $\sigma_i$ . It follows that the union of  $\sigma_i$  and  $\gamma_i$  bounds a disk  $\mathcal{D}_i$  on  $L_i$  as their end points are same. All the flow lines which enter  $\mathcal{D}_i$  transversally intersecting  $\sigma_i$  have to exit  $\mathcal{D}_i$  transversally intersecting  $\gamma_i$ . Poincaré-Hopf theorem says that there exists at least one flow line tangent to  $\sigma_i$ , a contradiction.  $\square$

By Lemma 5.4 the leaf space of the flow foliation in  $L_0$  is homeomorphic to the reals. Hence any two distinct flow lines in  $L_0$  are connected by a transversal.

This contradiction proves Proposition 5.1.  $\square$

Now we are ready to prove our final claim:

**Theorem 6.2.** *The flow lines are uniformly quasigeodesics in their respective leaves of  $\tilde{\mathcal{F}}$ .*

*Proof.* We prove the theorem by contradiction. Again we use a Candel metric in  $\mathcal{F}$ . We assume that the geodesics are not uniform quasigeodesic on their leaves. From

this assumption we will construct sequence of pairs  $\{(x_i, y_i)\}$  such that  $x_i$  and  $y_i$  are connected by a flow segment  $\gamma_i$  where  $\text{length}(\gamma_i) \rightarrow \infty$  but  $d_{L_i}(x_i, y_i)$  is bounded. Here  $L_i$  is the  $\tilde{\mathcal{F}}$  leaf containing both  $x_i, y_i$ . But that will contradict the ‘weak quasigeodesic property’ in proposition 6.1. A very similar result was proved in [FM01], we reconstruct the same arguments in our specific case.

By our assumption that flow lines are not uniform quasigeodesics, we get that for any  $K > 0$ , there exists a flow segment  $\gamma_{[x,y]}$  on a leaf  $L_x$  such that

$$\text{length}(\gamma_{[x,y]})/d_{L_x}(x, y) > 2K \text{ and } \text{length}(\gamma_{[x,y]}) > K$$

Here the  $x, y, L_x$  depend of the  $K$ , we omit the explicit dependence. Consider the geodesic  $g_x = g_y$  on  $L_x$  with ideal points

$$\eta^+(x) = \eta^+(y) \text{ and } \eta^-(x) = \eta^-(y) \text{ on } S^1(L_x).$$

By Lemma 5.8, there exists  $d > 0$  such that  $\gamma \subset \mathcal{N}_d(g_x)$ , where the neighborhood is in  $L_x$ . This  $d$  is global. Let  $\rho : L_x \rightarrow g_x$  be the ‘closest point map’, which means  $\rho(p)$  is the closest point on  $g_x$  from  $p \in L_x$ . This is the orthogonal projection in  $L_x$  to  $g_x$ . It follows that:

$$d_{L_x}(\rho(x), \rho(y)) \leq d_{L_x}(x, y) \leq d_{L_x}(\rho(x), \rho(y)) + 2d \text{ (*)}$$

Let us assume that  $d_{L_x}(x, y) > 1 + 2d$ . Hence  $d_{L_x}(\rho(x), \rho(y)) > 1$  by (\*) and

$$\frac{\text{length}(\gamma_{[x,y]})}{d_{L_x}(\rho(x), \rho(y))} \geq \frac{\text{length}(\gamma_{[x,y]})}{d_{L_x}(x, y)} \geq 2K > K + \frac{K}{d_{L_x}(\rho(x), \rho(y))}$$

Therefore

$$\frac{\text{length}(\gamma_{[x,y]})}{K} > d_{L_x}(\rho(x), \rho(y)) + 1 > [d_{L_x}(\rho(x), \rho(y))]$$

where  $[a]$  denotes the integer  $n$  such that  $n - 1 < a \leq n$ .

Suppose  $n_0 = [d_{L_x}(\rho(x), \rho(y))]$ , then  $\text{length}(\gamma_{[x,y]}) > n_0 K$ . Also

$$n_0 - 1 < [d_{L_x}(\rho(x), \rho(y))] \leq n_0,$$

hence we can construct a sequence  $\{\rho(x) = z_0, z_1, \dots, z_n = \rho(y)\}$  of points in  $g_x$ , such that  $d_{L_x}(z_{i-1}, z_i) = 1$  for all  $i < n_0$  and  $d_{L_x}(z_{n_0-1}, z_{n_0}) \leq 1$ . Next we consider the sequence  $x = x_0, x_1, \dots, x_{n_0}$  where  $x_i$  is the last point on  $\gamma_{[x,y]}$  such that  $\rho(x_i) = z_i$ .

If  $\gamma_i$  denote the flow segment joining  $x_{i-1}$  and  $x_i$ , we have  $\gamma_{[x,y]} = \gamma_1 * \gamma_2 * \dots * \gamma_{n_0}$ . Hence

$$\sum_{n=1}^{n_0} \text{length}(\gamma_i) = \text{length}(\gamma_{[x,y]}) > n_0 K$$

By the pigeonhole principle there exists  $x_{i-1}$  and  $x_i$  such that  $\text{length}(\gamma_{[x_{i-1}, x_i]}) > K$ . But from (\*) we get that for all  $i$ ,

$$d_{L_x}(x_{i-1}, x_i) \leq d_{L_x}(\rho(x_{i-1}, x_i)) + 2d = d_{L_x}(z_{i-1}, z_i) + 2d < 1 + 2d.$$

As the choice of  $K > 0$  was arbitrary, this proves that the ‘weak quasigeodesic property’ in lemma 6.1 is not true for  $b = 1 + 2d$ , a contradiction. We conclude that flow lines are uniformly quasigeodesic on their respective leaves of  $\tilde{\mathcal{F}}$ .

This finishes the proof of the theorem 6.2.  $\square$

**Conclusion:** Section 4 shows that every leaf in  $\mathcal{F}$  is Gromov hyperbolic when lifted to the universal cover. Theorem 6.2 proves that the flow foliation is a leafwise quasigeodesic subfoliation of  $\tilde{\mathcal{F}}$ . Moreover Proposition 5.3 proves that all leaves of  $\mathcal{F}$  which

are not contained in  $\mathcal{A}$  or  $\mathcal{R}$  are *non funnel*, whereas all leaves in  $\mathcal{A}$  or  $\mathcal{R}$  are *funnel* by corollary 2.6. This completes the proof of the Main Theorem 1.1.  $\square$

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