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Numerical solutions of optimal stopping problems for a class of hybrid stochastic systems ☆

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ABSTRACT

This paper is devoted to numerically solving a class of optimal stopping problems for stochastic hybrid systems involving both continuous states and discrete events. The motivation for solving this class of problems stems from quickest event detection problems of stochastic hybrid systems in broad application domains. We solve the optimal stopping problems numerically by constructing feasible algorithms using Markov chain approximation techniques. The key tasks we undertake include designing and constructing discrete-time Markov chains that are locally consistent with switching diffusions, proving the convergence of suitably scaled sequences, and obtaining convergence for the cost and value functions. Finally, numerical results are provided to demonstrate the performance of the algorithms.

1. Introduction

Optimal stopping problems are motivated by a wide range of applications in which sudden changes in system structures and parameters must be detected and managed. These sudden changes can be triggered by variables passing certain thresholds and boundaries in financial and engineering systems, cyber attacks on infrastructures, and disruptive changes in biological systems. In modern emerging technologies such as modern power systems, autonomous systems, etc., prompt detection of events is especially important for fault and contingency diagnosis to achieve reliability enhancement and risk mitigation. For example, in modern power systems, when a contingency (such as a line fault, a load jump, or a cyber attack) occurs, it is important to detect its occurrence as quickly and accurately as possible so that the system's stability and operation can be maintained [1–3].

Timely detection or "quickest detection" of abrupt changes is vitally important for risk management of systems. The theory of optimal stopping is often employed to solve such quickest detection problems. The framework of optimal stopping consists of a decision maker observing an underlying stochastic process and who, at each stopping time τ, must make one of two decisions: (i) stop observing the process and collect the "reward" at τ or (ii) continue observing the process. For a review of the theory of sequential detection/optimal stopping for discrete-time processes, the reader is referred to [4,5]. For continuous-time processes, the reader is referred to the seminal books of Peskir and Shiryaev [6,7].

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Quickest detection problems are usually reformulated as optimal stopping problems by introducing an appropriate cost function. This allows one to formulate an optimization problem whose solution is given by the stopping time τ^* at which the infimum of the cost function is realized. Solving an optimal stopping problem may also be viewed as solving an optimal stochastic control problem where the control is the stopping time τ. The solution to an optimal stochastic control problem typically requires deriving a partial differential equation (PDE) and verifying that the solution to the PDE is indeed the value function (the optimum). To solve this PDE, one also needs to identify the auxiliary conditions (boundary conditions). In contrast, an optimal stopping problem is usually worked out by solving a free-boundary problem standing in a one-to-one correspondence with the optimal stopping problem.

Perhaps the most classical quickest detection problem is that in which one observes a trajectory of the Wiener process with a drift changing from 0 to $\mu \neq 0$ at some random time θ. The task then becomes to find the stopping time τ^* that is as "close as possible" to the unknown time θ [6,8]. In lieu of the standard one-dimensional diffusions, the diffusions that we shall consider in the present paper are multi-dimensional hybrid switching diffusions in which continuous states and discrete events coexist. The discrete events, which are formulated as continuous-time Markov chains, are utilized to represent the switching dynamic feature. Indeed, finding both analytic and numerical solutions to optimal stopping problems for hybrid switching diffusions becomes substantially more difficult due to the presence of regime switching.

The present paper is in part motivated by the recent work of Ernst and Mei [9], which considered a multi-dimensional switching diffusion system given by the following dynamics

$$dX(t) = \widetilde{F}(\alpha(t))X(t)dt + \sum_{a=1}^{n} \widetilde{G}_a(\alpha(t))X(t)dB_a(t), \;\; X(0) = x, \;\; \alpha(0) = \imath, \tag{1.1}$$

where $\alpha(\cdot)$ is a continuous-time Markov chain with a finite state space $\mathcal{M}$. For each $a = 1, \dots, n$, B_a is a real-valued standard Brownian motion, and for each $i \in \mathcal{M}$, $\widetilde{F}(i)$ and $\widetilde{G}_a(i)$ are $d \times d$ matrix-valued coefficients. Under suitable conditions, Eq. (1.1) has a unique positive solution and the pair $(X(\cdot), \alpha(\cdot))$ is a strong Markov process; see Yin and Zhu [10, Chapter 2] as well as the recent work of Nguyen et al. [11,12]. We denote by τ a stopping time for the stochastic system and define a cost function

$$J(x, \imath, \tau) := \mathbb{E}_{x,\imath} \int_0^{\tau} \exp\Big(-\int_0^t \lambda(\alpha(s))ds\Big) H(X(t), \alpha(t))dt, \tag{1.2}$$

where $\mathbb{E}_{x,\imath}$ is the expectation with respect to the initial data $(x, \imath)$, $\lambda(\cdot)$ is the so-called discount rate function, and H is the running cost rate function. The value function is then given as

$$V(x, \imath) = \inf_{\tau} J(x, \imath, \tau), \tag{1.3}$$

where the infimum is taken over $\mathcal{T}_{X,\alpha}$, the collection of all possible stopping times of $(X(\cdot), \alpha(\cdot))$ with respect to the natural filtration $\{\mathcal{F}_t : t > 0\}$ augmented with all $\mathbb{P}$-null sets; see [9] for further details. The solution to the optimal stopping problem in (1.3) can be represented by a system of Hamilton–Jacobi–Bellman (HJB) equations, and the stopping boundary is a free boundary to be specified. The presence of the continuous-time Markov chain $\alpha(t)$ allows us to work with a system of PDEs. Section 6 of [9] considered an application to quickest detection of Brownian coordinate drifts with $\mathcal{M} = \{1, 2\}$ and $d = 2$; in this example, the stopping boundary is identified as the solution of a system of integral equations. However, for a more general set $\mathcal{M}$ with more than two discrete states and with dimension $d > 2$, an analytic solution is not attainable. Further, even in the simplified setting of $\mathcal{M} = \{1, 2\}$ and $d = 2$ in [9, Section 6], the system of integral equations cannot be solved in closed form for many practical applications.

The above limitations reinforce the importance of developing robust numerical approximations to optimal stopping problems. The present work proposes an approximation scheme based on a Markov chain approximation method, initiated by Kushner, continued by Kushner and Dupuis [13], and further extended for controlled switching diffusions and games; see Song et al. [14,15]. A preliminary exploration of the numerical solutions to the optimal stopping problem was in [16], whereas the current paper develops the method further together with a complete convergence analysis of the algorithm. The objectives and methods used in [9] and the current paper are different. The main effort in [9] is on the analysis aspect. It is devoted to analyzing the optimal stopping problem, where as the current paper provides a computational method for solving such optimal stopping problem numerically.

To simplify the discussion, in what follows, we shall assume that $\alpha(\cdot)$ is known. The case of hidden Markov chains can be handled by the methods proposed in this paper together with the use of Wonham-type filter algorithms; see Wonham [17].

The remainder of this paper is organized as follows. Section 2 presents a formulation of the optimal stopping problem under consideration. Section 3 is devoted to Markov chain approximation, in which we design an algorithm by constructing a suitable Markov chain. Sections 4 and 5 provide convergence analysis. Section 6 offers numerical examples to demonstrate the performance of the algorithm.

2. Formulation and basic setup

Consider a finite set $\mathcal{M} = \{1, \dots, m_0\}$ as the state space of a continuous-time finite-state Markov chain $\alpha(\cdot)$. Suppose that $F : \mathbb{R}^d \times \mathcal{M} \to \mathbb{R}^d$ and $G : \mathbb{R}^d \times \mathcal{M} \to \mathbb{R}^{d\times d}$ are appropriate nonlinear functions satisfying suitable conditions. We shall consider nonlinear systems which are more general than (1.1) and are of the form

$$dX(t) = F(X(t), \alpha(t))dt + G(X(t), \alpha(t))dB(t), \;\; X(0) = x, \;\; \alpha(0) = \imath, \tag{2.1}$$

where $B(\cdot)$ is a d-dimensional standard Brownian motion, $\alpha(\cdot)$ is a continuous time finite-state Markov chain with state space $\mathcal{M}$, and $\alpha(\cdot)$ is assumed independent of $B(\cdot)$. The generator of the continuous-time Markov chain is denoted by $Q = (q_{ij})$, where $\sum_{j=1}^{m_0} q_{ij} = 0$ for each $i \in \mathcal{M}$ and $q_{ij} \geq 0$ for $i \neq j$.

We shall utilize the same cost function as that of [9]; see (1.2). The corresponding value function is denoted as

$$V(x,\imath) := \inf_{\tau} J(x,\imath,\tau). \tag{2.2}$$

In numerical computations, the process will be confined to a compact set $\overline{\mathcal{O}}$ in order to make it computationally feasible. That is, $\overline{\mathcal{O}}$ is the closure of its interior (open set) $\mathcal{O}$, and the process must stop at the first exit time from $\mathcal{O}$, *i.e.*,

$$\tau_0 = \inf\{t : (X(t),\alpha(t)) \notin \mathcal{O}\times\mathcal{M}\}, \tag{2.3}$$

if it has not stopped earlier. For further discussions on the requirement of τ_0 and the associated conditions for finiteness, we refer the reader to Remark 4.3 in this paper.

The problem is now to find the stopping time $\tau \le \tau_0$ which minimizes the cost function in (1.2). The optimal stopping problem under consideration becomes

$$V(x,\imath) = \inf_{\tau\le\tau_0} J(x,\imath,\tau). \tag{2.4}$$

Our study begins with the following preliminary assumption.

(H0) Suppose that $F(\cdot)$ and $G(\cdot)$ are suitable functions such that (2.1) has a unique (in the sense of distribution) solution for each initial condition. We also assume that the optimal stopping problem has a unique solution in the weak sense.

Note that we are using the weak solution (or solutions of the martingale problems). Sufficient conditions for the strong solution of the system can be provided. However, this is not the focus of the present paper. As we stated in Section 1, our main concern is to design feasible computational procedures for the optimal stopping problem given that the problem has a solution.

For each $i \in \mathcal{M}$ and $y \in \overline{\mathcal{O}}$, define $\mathcal{L}$, the operator for a twice continuously differentiable function $\psi(\cdot,i)$, as

$$\mathcal{L}\psi(y,i) = [\nabla\psi(y,i)]'F(y,i) + \frac{1}{2}\mathrm{tr}[\nabla^2\psi(y,i)G(y,i)G'(y,i)] + Q\psi(y,\cdot)(i), \tag{2.5}$$

where z' denotes the transpose of z,

$$Q\psi(y,\cdot)(i) = \sum_{j=1}^{m_0} q_{ij}\psi(y,j),$$

and $\nabla\psi(y,i)$ and $\nabla^2\psi(y,i)$ denote the gradient and Hessian of $\psi(y,i)$ with respect to y, respectively. The system of Hamilton–Jacobi–Bellman (HJB) equations satisfied by the value function in (2.4) is

$$\min\{\mathcal{L}V(x,\imath) - \lambda(\imath)V(x,\imath) + H(x,\imath), -V(x,\imath)\} = 0. \tag{2.6}$$

The original derivation of the systems of HJBs (2.6) can be found in [18]. Because the motivation was for a mathematical finance problem, maximization was used; see also the related reference [19]. This notation was also used in [9] although the quickest detection is for minimization of the cost function. In this paper, we inherit the notation as in [9]. The key purpose of the present paper is to construct algorithms which approximate the solution to the key optimal stopping problem under consideration in (2.4).

3. Markov chain approximation

3.1. Discrete grids and approximation

Let $h > 0$ be a small step size and S_h be the h-grid of $\mathbb{R}^d$ defined by

$$S_h = \Big\{y : y = \sum_{j=1}^{d} n_j e_j h, j = 1,\dots,d,\ n_j \in \mathbb{Z}\Big\}, \tag{3.1}$$

where e_j is the standard unit vector in the jth coordinate direction. We construct a discrete-time two-component Markov chain $\{(\xi_k^h,\alpha_k^h) : k < \infty\}$ on the discrete state space $S_h\times\mathcal{M}$ with transition probabilities $p^h((y,i),(z,\ell))$ from the state $(y,i)\in S_h\times\mathcal{M}$ to the state $(z,\ell)\in S_h\times\mathcal{M}$. This setting for switching diffusions originates from the work of Song et al. [14,15]. The intuition here is that $\{\xi_k^h\}$ should "approximate" $X(\cdot)$ and $\{\alpha_k^h\}$ should "approximate" $\alpha(\cdot)$. We approximate the stopping time τ by N^h using stepsize h and approximate the cost function $J(x,\imath,\tau)$ by denoting

$$J^h(x,\imath,N^h) := \mathbb{E}_{x,\imath}\sum_{k=0}^{N^h-1}\exp\Big(-\sum_{j=0}^{k}\lambda(\alpha_j^h)\Big)H(\xi_k^h,\alpha_k^h)\Delta t_k^h, \tag{3.2}$$

with Δt_k^h to be specified in (3.4) below. The discounting is constant on the interval $[t_k^h, t_{k+1}^h)$. The corresponding value function of the approximating Markov chain is

$$V^h(x,\imath) = \inf_{N^h} J^h(x,\imath,N^h).$$

The associated dynamic programming equation in (2.6) will then have the following discrete form

$$V^h(x,\imath) = \min\Big\{e^{-\lambda(\imath)\Delta t^h(x,\imath)}\sum_{(y,i)} p^h\big((x,\imath),(y,i)\big)V^h(y,i) + H(x,\imath)\Delta t^h(x,\imath),\ 0\Big\}. \tag{3.3}$$

For ease of flow, we postpone the derivation of (3.3); it will appear immediately after the proof of Lemma 3.2.

In our analysis, in order to establish convergence, we shall utilize continuous-time interpolations. Suppose that we have an interpolation interval $\Delta t^h(\cdot,\cdot) > 0$ on $S_h \times \mathcal{M}$; the construction of this interval will be given in (3.9). Let

$$\Delta t_k^h := \Delta t^h(\xi_k^h, \alpha_k^h). \tag{3.4}$$

The interpolated time is defined as

$$t_k^h = \sum_{j=0}^{k-1} \Delta t_j^h(\xi_j^h, \alpha_j^h).$$

Some more notation is necessary for what follows below. Let us denote $\mathcal{F}_t^h$ as the σ-algebra generated by $\{\xi^h(s), \alpha^h(s), z^h(s) : s \le t\}$ and let τ^h be a $\mathcal{F}_t^h$-stopping time.

To ensure the approximation is in line with the dynamics of (2.1), we need to check that our construction is "locally consistent"; that is, the conditional mean and conditional covariance of the constructed discrete-time Markov chain $\{\xi_k^h, \alpha_k^h\}$ "match" that of the switching diffusion in Eq. (2.1) and, further, that the error tends to 0 as $h \to 0$. Similar to the work of Song et al. [14, Definition 1], we will show that the constructed Markov chain leads to the correct limit. With the constructed Markov chain in hand, we now can define the interpolated processes as

$$\begin{aligned}
&\xi^h(t) = \xi_k^h,\ \ \alpha^h(t) = \alpha_k^h,\ \ z^h(t) = k \ \text{ for } \ t \in [t_k^h, t_{k+1}^h),\ \ \tau^h = t_{N^h}^h,\\
&J^h(x,\imath,\tau^h) = E_{x,\imath} \int_0^{\tau^h} \exp\Big(-\int_0^t \lambda(\alpha^h(s))ds\Big) H\left(\xi^h(t), \alpha^h(t)\right) dt,\\
&V^h(x,\imath) = \inf_{\tau^h} J^h(x,\imath,\tau^h).
\end{aligned}$$

3.2. Construction of the controlled Markov chains

We begin with the construction of the Markov chain $\{\xi_k^h, \alpha_k^h\}$ in order to approximate the switching diffusion in Eq. (2.1). Define the covariance matrix $C(y,i)$ as

$$C(y,i) := G(y,i)G'(y,i) = (c_{jr}(y,i)). \tag{3.5}$$

Note that for the general diffusion coefficient G, the matrix C is not diagonal. We define the transition probabilities by invoking the finite difference method for approximating the first and second derivative of $V(\cdot,i), i \in \mathcal{M}$. This is done as follows

$$\begin{aligned}
V_{x_j}(y,i) \quad &\to [V(y+e_jh,i) - V(y,i)]/h, \ \text{ if } F_j(y,i) \ge 0,\\
V_{x_j}(y,i) \quad &\to [V(y,i) - V(y-e_jh,i)]/h, \ \text{ if } F_j(y,i) < 0,\\
V_{x_jx_j}(y,i) \quad &\to [V(y+e_jh,i) + V(y-e_jh,i) - 2V(y,i)]/h^2,\\
V_{x_jx_r}(y,i) \quad &\to [V(y+e_jh+e_rh,i) + V(y-e_jh-e_rh,i) + 2V(y,i)]/2h^2\\
&\quad -[V(y+e_jh,i) + V(y-e_jh,i) + V(y+e_rh,i) + V(y-e_rh,i)]/2h^2\\
&\qquad\qquad \text{if } c_{jr}(y,i) \ge 0,\\
V_{x_jx_r}(y,i) \quad &\to -[V(y+e_jh-e_rh,i) + V(y-e_jh+e_rh,i) + 2V(y,i)]/2h^2\\
&\quad +[V(y+e_jh,i) + V(y-e_jh,i) + V(y+e_rh,i) + V(y-e_rh,i)]/2h^2\\
&\qquad\qquad \text{if } c_{jr}(y,i) < 0,
\end{aligned} \tag{3.6}$$

where the e_j represent a d-dimensional unit vector that has zeros in all components expect for the jth component being equal to 1. We note that the approximation of mixed second order partial derivatives depends on the sign of $c_{jr}(y,i)$. This guarantees that the coefficients of $V^h(y,i)$ will be nonnegative and sum to unity so that it can be invoked as the transition probabilities of a Markov chain. We also assume, for all $y \in \mathbb{R}^d$ and $i \in \mathcal{M}$, that

$$c_{jj}(y,i) - \sum_{r:r\ne j} |c_{jr}(y,i)| \ge 0. \tag{3.7}$$

The condition in (3.7) requires the matrix to be diagonally dominant, which is a convenient condition for our construction in the present work. A condition of this sort actually depends on the coordinate system we are using and there are several ways to relax it; see Kushner and Dupuis [13, pp. 110–111] for details. Denote by $p^h((y,i),(z,\ell))$ the transition probabilities from a state $(y,i) \in S_h \times \mathcal{M}$ to a state $(z,\ell) \in S_h \times \mathcal{M}$. Substituting these approximation (3.6) into (2.6), we are able to define the transition probabilities as follows

$$\begin{aligned}
&p^h\big((y,i),(y \pm e_jh,i)\big) = \Big[c_{jj}(y,i)/2 - \sum_{r\ne j} |c_{jr}(y,i)|/2 + hF_j^{\pm}(y,i)\Big]/Q^h(y,i),\\
&p^h\big((y,i),(y+e_jh+e_rh,i)\big) = p^h\big((y,i),(y-e_jh-e_rh,i)\big) = c_{jr}^+(y,i)/2Q^h(y,i) \text{ if } r \ne j,\\
&p^h\big((y,i),(y-e_jh+e_rh,i)\big) = p^h\big((y,i),(y+e_jh-e_rh,i)\big) = c_{jr}^-(y,i)/2Q^h(y,i) \text{ if } r \ne j,\\
&p^h\big((y,i),(z,\ell)\big) = (h^2 q_{i\ell})/Q^h(y,i) \ \text{ if } \ell \ne i,\\
&p^h\big((y,i),(z,\ell)\big) = 0 \text{ otherwise,}
\end{aligned} \tag{3.8}$$

where

$$
\begin{aligned}
Q^h(y,i) &= \sum_{j=1}^{d} c_{jj}(y,i) - \sum_{j,r:j\neq r} |c_{jr}(y,i)|/2 + h\sum_{j=1}^{d} |F_j(y,i)| - h^2 q_{ii},\\
\Delta t^h(y,i) &= \frac{h^2}{Q^h(y,i)+h^2\lambda(i)}.
\end{aligned}
\tag{3.9}
$$

In Lemma 3.2 below, we shall show that the above construction of the Markov chain retains the local behavior of system (2.1). That is, the "local mean" and "local covariance" of the Markov chain $\{\xi_k^h, \alpha_k^h\}$ match that of the regime switching diffusion Eq. (2.1). We call this match "local consistency" (see Definition 3.1). Let $\Delta\xi_n^h = \xi_{n+1}^h - \xi_n^h$, and denote by $\mathbb{P}_{y,i,n}^h$, $\mathbb{E}_{y,i,n}^h$, and $\mathbb{C}\mathrm{ov}_{y,i,n}^h$ the conditional probability, conditional expectation, and conditional covariance with respect to the σ-algebra generated by

$$\{\xi_k^h, \alpha_k^h; k \le n, \xi_n^h = y, \alpha_n^h = i\}.$$

We now proceed with the statement of Definition 3.1.

Definition 3.1 (*Local Consistency*)**.** The sequence $\{\xi_k^h, \alpha_k^h\}$ with $\Delta t^h(y,i)$ defined in (3.8) is said to be "locally consistent" with the system in (2.1) if, given $\xi_n^h = y$ and $\alpha_n^h = i$, the following five conditions hold

$$
\begin{aligned}
&\mathbb{E}_{y,i,n}^h \Delta\xi_n^h = F(y,i)\Delta t^h(y,i) + o(\Delta t^h(y,i)),\\
&\mathbb{C}\mathrm{ov}_{y,i,n}^h \Delta\xi_n^h = G(y,i)G'(y,i)\Delta t^h(y,i) + o(\Delta t^h(y,i)),\\
&\mathbb{P}_{y,i,n}^h(\alpha_{n+1}^h = \ell) = q_{i\ell}\Delta t^h(y,i) + o(\Delta t^h(y,i)),\ \ \ell \neq i,\\
&\mathbb{P}_{y,i,n}^h(\alpha_{n+1}^h = i) = 1 + q_{ii}\Delta t^h(y,i) + o(\Delta t^h(y,i)),\\
&\sup_{n,\omega} |\Delta\xi_n^h| \to 0 \ \text{ w.p.1 as } h \to 0.
\end{aligned}
\tag{3.10}
$$

We now proceed with Lemma 3.2.

Lemma 3.2. *The constructed Markov chain $\{\xi_k^h, \alpha_k^h\}$ with transition probabilities defined in (3.8) is locally consistent with Eq. (2.1).*

Proof. The verbatim proof is omitted for brevity. It is straightforward to directly verify each of the conditions in Definition 3.1 by using the transition probabilities defined in (3.8). □

The local consistency of the Markov chain $\{\xi_k^h, \alpha_k^h\}$ leads to Proposition 3.3 below.

Proposition 3.3. *The dynamic programming equation for system (2.1) becomes*

$$V^h(x,\imath) = \min\Big\{ e^{-\lambda(\imath)\Delta t^h(x,\imath)} \sum_{(y,i)} p^h((x,\imath),(y,i))V^h(y,i) + H(x,\imath)\Delta t^h(x,\imath),\ \ 0 \Big\}.$$

Proof. We begin by discretizing the original HJB Eq. (2.6) using (3.6), (3.8) and (3.9). Doing so gives the discretized form

$$
\begin{aligned}
\min\Big\{ \sum_{(y,i)} Q^h(x,\imath)p^h((x,\imath),(y,i))V^h(y,i) + h^2 H(x,\imath) \ \ &-(Q^h(x,\imath)+h^2\lambda(\imath))V^h(x,\imath),\\
&-h^2 V^h(x,\imath) \Big\} = 0.
\end{aligned}
\tag{3.11}
$$

Recall

$$\Delta t^h(x,\imath) = \frac{h^2}{Q^h(x,\imath)+h^2\lambda(\imath)}. \tag{3.12}$$

Dividing by $Q^h(x,\imath) + h^2\lambda(\imath)$ and then adding $V^h(x,\imath)$ to both sides of Eq. (3.11), we have

$$
\begin{aligned}
\min\Big\{ \sum_{(y,i)} (1-\lambda(\imath)\Delta t^h(x,\imath))p^h((x,\imath),(y,i))V^h(y,i) \ \ &+\Delta t^h(x,\imath)H(x,\imath),\\
&(1-\Delta t^h(x,\imath))V^h(x,\imath) \Big\} = V^h(x,\imath).
\end{aligned}
\tag{3.13}
$$

Note that the right-hand side of Eq. (3.12) is order $O(h^2)$. It allows us to regard the term $1-\lambda(\imath)\Delta t^h(x,\imath)$ as an approximation of $e^{-\lambda(\imath)\Delta t^h(x,\imath)}$ with an error of the order $o(h^4)$. Eq. (3.13) becomes

$$
\begin{aligned}
V^h(x,\imath) = \min\Big\{ e^{-\lambda(\imath)\Delta t^h(x,\imath)} \sum_{(y,i)} p^h((x,\imath),(y,i))V^h(y,i) \ \ &+\Delta t^h(x,\imath)H(x,\imath),\\
&(1-\Delta t^h(x,\imath))V^h(x,\imath) \Big\}.
\end{aligned}
$$

Since the value function takes values in $(-\infty, 0]$, we have

$$
\begin{aligned}
&V^h(x,\imath)\\
&=\begin{cases} e^{-\lambda(\imath)\Delta t^h(x,\imath)} \sum_{(y,i)} Q^h(x,\imath)p^h((x,\imath),(y,i))V^h(y,i) + \Delta t^h(x,\imath)H(x,\imath) & \text{if } V^h(x,\imath) < 0,\\ 0 & \text{if } V^h(x,\imath) = 0. \end{cases}
\end{aligned}
$$

We thus conclude that the HJB equation associated with the approximated Markov chain $\{\xi_k^h, \alpha_k^h\}$ has the desired form. This finishes the proof. $\square$

We now preview the key results in Section 4 and in Section 5. We will show that, as $h \to 0$,

(a) the sequence $(\xi^h(\cdot), \alpha^h(\cdot), \tau^h)$ converges weakly to $(X(\cdot), \alpha(\cdot), \tau)$. We shall do so by showing that the limit of $(\xi^h(\cdot), \alpha^h(\cdot))$ is a solution of the martingale problem with operator $\mathcal{L}$ defined in (2.5), which allows us to conclude that $(\xi^h(\cdot), \alpha^h(\cdot))$ converges weakly to $(X(\cdot), \alpha(\cdot))$. This is accomplished in Section 4.2.
(b) $J^h(x, \imath, \tau^h) \to J(x, \imath, \tau)$ and $V^h(x, \imath) \to V(x, \imath)$. This is accomplished in Theorems 5.2 and 5.3.

4. Convergence of Markov chain approximation

This section is devoted to proving the convergence of the Markov chain approximation procedure. To proceed, we impose Assumption 1 below.

Assumption 1. We assume that the following conditions hold.

(H1) For each $i \in \mathcal{M}$, we have that $F(\cdot, i)$, $G(\cdot, i)$, and $H(\cdot, i)$ are continuous functions on $\overline{\mathcal{O}}$.
(H2) For each $i \in \mathcal{M}$ and each $y \in \overline{\mathcal{O}}$, $G(y, i)$ is invertible.
(H3) $\mathbb{E}\tau_0 < \infty$.

Remark 4.1. Recall that in Assumption (H0) from Section 2, we have already assumed that there is a unique solution of (2.1) in the "weak sense". This means that, for each initial condition, the probability law of an admissible $\{\alpha(\cdot), B(\cdot), \tau\}$ determines the probability law of any solution $\{X(\cdot), \alpha(\cdot), B(\cdot), \tau\}$ of (2.1) regardless of the probability space.

Condition (H2) ensures the matrix $G(\cdot, i)$ is non-singular in $\overline{\mathcal{O}}$. In the case where G is not invertible, we can "modify" its inverse. This is a useful trick for the martingale problem representation; for further details, see Kushner and Dupuis [13, p. 288].

Remark 4.2. In Kushner and Dupuis [13], it is assumed that the drift, diffusion, and the running cost function are bounded and continuous. The main rationale is that the computation must be done in a bounded set, and the continuity yields the boundedness. Therefore, it suffices to consider bounded functions. In this paper, we take a similar approach to [13] by assuming that the drift, diffusion, and running cost are continuous on $\overline{\mathcal{O}}$, which implies the boundedness on $\overline{\mathcal{O}}$.

Remark 4.3. Condition (H3) is not strong. In fact, if we assume the drift and diffusion coefficients are Lipschitz continuous in the first variable, that is, for some $\kappa_1 > 0$,

$$|F(y,i) - F(z,i)| + |G(y,i) - G(z,i)| \le \kappa_1 |y - z|,$$

then the following assertions hold

(i) For each $i \in \mathcal{M}$ and $y \in \overline{\mathcal{O}}$, both F and G grow at most linearly; that is,

$$|F(y,i)| \le K(1 + |y|), \ \ |G(y,i)| \le K(1 + |y|).$$

(ii) Eq. (2.1) has a unique solution in the pathwise sense.
(iii) $\mathbb{E}\tau_0 < \infty$ and $\tau_0 < \infty$ a.s.

Because the system is time homogeneous, the Lipschitz condition implies assertion (i). Assertion (ii) is standard and follows from the Lipschitz condition and linear growth in (i). For a proof of assertion (iii), we refer the reader to Zhu et al. [20, Lemma 3.2]. It should be noted that [20] considers a more general system, namely, switching jump diffusions, in which the switching depends on continuous state and another Poisson type jump process. In the present paper, rather than assuming Lipschitz continuity, we assume (for purposes of simplicity) assumption (H0) in Section 2. This enables us to work with a larger class of systems where the Lipschitz condition may not hold.

4.1. Continuous-time interpolations

The purpose of this section is to define the appropriate interpolated cost function. We begin by defining

$$t_n^h = \sum_{k=0}^{n-1} \Delta t^h(\xi_k^h, \alpha_k^h).$$

For $t \in [t_n^h, t_{n+1}^h)$, we define the continuous-time interpolation

$$\xi^h(t) = \xi_n^h, \ \ \alpha^h(t) = \alpha_n^h, \ \ \tau^h(t) = \tau_n^h, \ \ n^h(t) = n, \tag{4.1}$$

where $n^h(t)$ is the "look back" function of time. The local consistency in (3.10) gives

$$\begin{aligned}\xi^h(t) &= x + \sum_{k=0}^{n^h(t)-1} \Delta\xi_k^h(t)\\ &= x + \sum_{k=0}^{n^h(t)-1} \mathbb{E}_k^h \Delta\xi_k^h + \sum_{k=0}^{n^h(t)-1} \left(\Delta\xi_k^h - \mathbb{E}_k^h \Delta\xi_k^h\right)\\ &= x + \sum_{k=0}^{n^h(t)-1} \left[F(\xi_k^h,\alpha_k^h)\Delta t^h(\xi_k^h,\alpha_k^h) + o(\Delta t^h(\xi_k^h,\alpha_k^h))\right] + \sum_{k=0}^{n^h(t)-1} (\Delta\xi_k^h - \mathbb{E}_k^h \Delta\xi_k^h)\\ &= x + \int_0^t F(\xi^h(s),\alpha^h(s))ds - (t-t_n^h)F(\xi_n^h,\alpha_n^h) + \sum_{k=0}^{n^h(t)-1} o(\Delta t^h(\xi_k^h,\alpha_k^h)) + M^h(t),\end{aligned}$$

where $\mathbb{E}_k^h$ denotes the conditional expectation with respect to the σ-algebra generated by $\{\xi_j^h, \alpha_j^h, \tau_j^h; j \le k\}$, and

$$M^h(t) := \sum_{k=0}^{n^h(t)-1} \left(\Delta\xi_k^h - \mathbb{E}_k^h \Delta\xi_k^h\right).$$

Defining

$$W^h(t) = \sum_{k=0}^{n^h(t)-1} G^{-1}(\xi_k^h,\alpha_k^h)\left(\Delta\xi_k^h - \mathbb{E}_k^h \Delta\xi_k^h\right), \tag{4.2}$$

we then have that

$$\xi^h(t) = x + \int_0^t F(\xi^h(s),\alpha^h(s))ds + \int_0^t G(\xi^h(s),\alpha^h(s))dW^h(s) + \epsilon^h(t), \tag{4.3}$$

where the term

$$\epsilon^h(t) := \sum_{k=0}^{n^h(t)-1} \left[o(\Delta t^h(\xi_k^h,\alpha_k^h))\right] - (t-t_n^h)F(\xi_n^h,\alpha_n^h) - (t-t_n^h)(\Delta\xi_n^h - \mathbb{E}_n^h \Delta\xi_n^h),$$

represents negligible error tending to zero as $h \to 0$. Invoking similar logic to that used above, we obtain the following interpolated cost function and its associated value function

$$\begin{aligned} J^h(x,\imath,\tau^h) &= \mathbb{E}_{x,\imath} \sum_{k=0}^{N^h-1} \exp\Big(-\sum_{j=0}^{k} \lambda(\alpha_j^h)\Big) H(\xi_k^h,\alpha_k^h)\Delta t^h(\xi_k^h,\alpha_k^h)\\ &= \mathbb{E}_{x,\imath}\Big[\int_0^{\tau^h} \exp\big(-\int_0^t \lambda(\alpha^h(s))ds\big) H(\xi^h(t),\alpha^h(t))dt\Big],\\ V^h(x,\imath) &= \inf_{\tau^h} J^h(x,\imath,\tau^h).\end{aligned} \tag{4.4}$$

4.2. Weak convergence

In this subsection, we prove the weak convergence of the constructed Markov chain. We first verify the tightness of continuous-time processes $\{\xi^h(\cdot), \alpha^h(\cdot), \tau^h, W^h(\cdot)\}$. We then characterize the limit of $W^h(\cdot)$ as a standard Brownian motion, the limit of $\alpha^h(\cdot)$ as the Markov chain $\alpha(\cdot)$, and the limit of $\xi^h(\cdot)$ as the solution of (2.1).

4.2.1. Tightness

Lemma 4.4. *The continuous-time interpolated process $\{\alpha^h(\cdot)\}$ converges weakly to the Markov chain $\alpha(\cdot)$ with generator Q.*

Proof. The proof of this lemma can be found in Yin et al. [21, Theorem 3.1, pp. 457–458]. We thus briefly provide an outline of the proof. One begins by proving that $\alpha^h(\cdot)$ is tight. Invoking the interpolation of (4.1) and the Markov property of the discrete-time Markov chain, standard calculations yield

$$\mathbb{E}\Big[\big(\alpha^h(t+s) - \alpha^h(s)\big)^2 | \mathcal{F}_s^h\Big] \le \gamma^h(t) \text{ and } \lim_{t\to 0}\limsup_{h\to 0} \mathbb{E}\gamma^h(t) = 0,$$

where $\gamma^h(t) \ge 0$ is an $\mathcal{F}_s^h$-measurable function. The tightness of $\{\alpha^h(\cdot)\}$ is then guaranteed by the tightness criterion in Kushner [22, p. 47]. One then shows that the limit of $\alpha^h(\cdot)$ is the solution of a martingale problem with operator Q, and this completes the proof. □

Lemma 4.5. *Consider the approximating Markov chain $\{\xi_k^h, \alpha_k^h\}$ with transition probabilities defined in (3.8). Then the interpolated process $\{\xi^h(\cdot), \alpha^h(\cdot), \tau^h, W^h(\cdot)\}$ is tight.*

Proof. Step 1: Tightness of $\xi^h(\cdot)$. Recalling the local consistency definition in (3.10), we have

$$\begin{aligned}
\mathbb{E}^h_{x,\iota}|\xi^h(t)-x|^2 &= \mathbb{E}^h_{x,\iota}\Big|\sum_{k=0}^{n^h(t)-1}\mathbb{E}^h_k\Delta\xi^h_k + (\Delta\xi^h_k - \mathbb{E}^h_k\Delta\xi^h_k)\Big|^2\\
&\le 2\mathbb{E}^h_{x,\iota}\Big|\sum_{k=0}^{n^h(t)-1}\mathbb{E}^h_k\Delta\xi^h_k\Big|^2 + 2\mathbb{E}^h_{x,\iota}\Big|\sum_{k=0}^{n^h(t)-1}(\Delta\xi^h_k - \mathbb{E}^h_k\Delta\xi^h_k)\Big|^2\\
&\le 2\mathbb{E}^h_{x,\iota}\Big|\sum_{k=0}^{n^h(t)-1}F(\xi^h_k,\alpha^h_k)\Delta t^h(\xi^h_k,\alpha^h_k) + o(\Delta t^h(\xi^h_k,\alpha^h_k))\Big|^2\\
&\quad + 2\mathbb{E}^h_{x,\iota}\sum_{k=0}^{n^h(t)-1}\Big[C(\xi^h_k,\alpha^h_k)\Delta t^h(\xi^h_k,\alpha^h_k) + o(\Delta t^h(\xi^h_k,\alpha^h_k))\Big]\\
&\le 2K^2t^2 + 2Kt + 4\mathbb{E}^h_{x,\iota}\sum_{k=0}^{n^h(t)-1}o(\Delta t^h(\xi^h_k,\alpha^h_k)),
\end{aligned}$$

where K is a bound for $|F(y,i)| \vee |C(y,i)|$ for $y \in \overline{\mathcal{O}}$ and $i \in \mathcal{M}$. For each $t \in [0,\infty)$ and $\delta > 0$, Chebyshev's inequality implies there exists a sufficiently large $C_{t,\delta} > 0$ such that

$$\sup_h\big\{\mathbb{P}_{x,\iota}\big(|\xi^h(t)| > C_{t,\delta}\big)\big\} \le \delta.$$

According to Kushner and Dupuis [13, Theorem 2.1, p. 251], we also need to show that

$$\lim_{\delta\to 0}\limsup_{h\to 0}\sup_{\eta\in\mathcal{T}^h_T}\mathbb{E}^h_{x,\iota}\Big(1\wedge|\xi^h(\eta+\delta)-\xi^h(\eta)|\Big) = 0, \tag{4.5}$$

where $\mathcal{T}^h_T$ is the set of $\mathcal{F}^h_t$-stopping times less than or equal to T w.p.1. for any $T \in [0,\infty)$. By the strong Markov property of $\{\xi^h_k,\alpha^h_k\}$, for any $\eta \in \mathcal{T}^h_T$, we have

$$\begin{aligned}
\mathbb{E}^h_{x,\iota}\big(1\wedge|\xi^h(\eta+\delta)-\xi^h(\eta)|\big) &\le \big(\mathbb{E}^h_{x,\iota}|\xi^h(\eta+\delta)-\xi^h(\eta)|^2\big)^{1/2}\\
&\le \Big(2K^2\delta^2 + 2K\delta + 4\mathbb{E}^h_{x,\iota}\sum_{k=n^h(\eta)}^{n^h(\eta+\delta)-1}o(\Delta t^h(\xi^h_k,\alpha^h_k))\Big).
\end{aligned}$$

Eq. (4.5) is thus verified and so $\xi^h(\cdot)$ is tight.

Step 2: Tightness of $W^h(\cdot)$. Recall the definition of $W^h(t)$ defined in (4.2). To show that $W^h(\cdot)$ is also tight, we invoke the same approach in dealing with the terms $\xi^h(\cdot)$ and observing the boundedness of $G(y,i)$ for $y \in \overline{\mathcal{O}}$ and $i \in \mathcal{M}$.

Step 3: Tightness of τ^h. Since the stopping time τ^h has the potential to be unbounded, we must consider weak convergence for sequences of random variables with values in $[0,\infty]$. Note that $[0,\infty]$ is the one point of compactification of $[0,\infty)$, *i.e.*, the point of $\{\infty\}$ is added to the set $[0,\infty)$ as the limit of any increasing and unbounded sequence. Due to the compactness of $[0,\infty]$, the sequence of stopping times $\{\tau^h\}$ is tight. Together with Lemma 4.4, the proof is now complete. □

4.2.2. Characterization of the limits

In this subsection, we characterize the limit of the processes $\{\xi^h(\cdot),\alpha^h(\cdot),\tau^h,W^h(\cdot)\}$. Since the sequence of this quadruple is tight, we consider a convergent subsequence still denoted by $\{\xi^h(\cdot),\alpha^h(\cdot),\tau^h,W^h(\cdot)\}$. We represent the limit as $\{X(\cdot),\alpha(\cdot),\bar\tau,W(\cdot)\}$. By virtue of the Skorokhod representation theorem, we have that

$$\{\xi^h(\cdot),\alpha^h(\cdot),\tau^h,W^h(\cdot)\} \to \{X(\cdot),\alpha(\cdot),\bar\tau,W(\cdot)\}, \quad \text{w.p.1.}$$

This leads to Theorem 4.6 below.

Theorem 4.6. *$W(\cdot)$ is an $\mathcal{F}_t$-measurable standard Brownian motion. Letting $\bar\tau$ be $\mathcal{F}_t$-measurable stopping time, we have that*

$$X(t) = x + \int_0^t F(X(s),\alpha(s))ds + \int_0^t G(X(s),\alpha(s))dW(s),$$

where $\mathcal{F}_t = \sigma(X(s),\alpha(s),\bar\tau I_{\bar\tau\le s},W(s); s \le t)$.

Proof. Step 1: Characterization of the limit of $W^h(\cdot)$. We first prove that $W(\cdot)$ is indeed an $\mathcal{F}_t$-Wiener process. Let $\rho(\cdot)$ be a real-valued bounded and continuous function, let p be any positive integer, and let $t_k \le t$ with $k \le p$. We have

$$\mathbb{E}\rho\big(\xi^h(t_k),W^h(t_k),\tau^h I_{\{\tau^h\le t\}}; k\le p\big)\big[W^h(t+s)-W^h(t)\big] = 0.$$

Taking $h \to 0$ yields

$$\mathbb{E}\rho\big(X(t_k),W(t_k),\bar\tau I_{\{\bar\tau\le t_k\}}; k\le p\big)\big[W(t+s)-W(t)\big] = 0.$$

It follows that $W(\cdot)$ is an $\mathcal{F}_t$-martingale. To show $W(\cdot)$ is indeed a standard Brownian motion, we prove that $W(\cdot)$ is a continuous process and its quadratic variation is tI, where I is the $d \times d$ identity matrix. The Lévy characterization then yields the desired result. We note that

$$\begin{aligned}&\mathbb{E}\rho\big(\xi^h(t_k), W^h(t_k), \tau^h I_{\{\tau^h \le t_k\}}; k \le p\big)\Big[W^h(t+s)W^{h,\prime}(t+s) - W^h(t)(W^h(t))'\Big]\\ &= \mathbb{E}\rho(\xi^h(t_k), W^h(t_k), \tau^h I_{\{\tau^h \le t_k\}}; k \le p)\Big[\big(W^h(t+s) - W^h(t)\big)\big(W^h(t+s) - W^h(t)\big)'\Big].\end{aligned}$$

By the direct computation using the definition of $W^h(t)$ in (4.2), and taking $h \to 0$, weak convergence and the Skorokhod representation yield

$$\mathbb{E}\rho\big(X(t_k), W(t_k), \bar{\tau} I_{\{\bar{\tau} \le t_k\}}; k \le p\big)\big[W(t+s)W(t+s)' - W(t)W'(t)\big] = sI.$$

We thus conclude that $W(\cdot)$ is a d-dimensional standard Brownian motion.

Step 2: Characterization the limit of $\xi^h(\cdot)$. From the tightness of $\{\xi^h(\cdot)\}$ in Lemma 4.5, we may assume that $\xi^h(\cdot) \to X(\cdot)$ with probability one using the Skorokhod representation. For each $\delta > 0$, if $t \in [j\delta, (j+1)\delta)$, we define

$$\xi^h_\delta(t) = \xi^h(j\delta),\ \ \alpha^h_\delta(t) = \alpha^h(j\delta),\ \ X_\delta(t) = X(j\delta).$$

It now becomes useful to recall the representation in (4.3),

$$\xi^h(t) = x + \int_0^t F(\xi^h(s), \alpha^h(s))ds + \sum_{k=0}^{n^h(t)-1} G(\xi^h_k, \alpha^h_k)(W^h(t^h_{k+1}) - W^h(t^h_k)) + \varepsilon^h_1(t).$$

By the above representation, the continuity and boundedness of $F(y,i)$ and $G(y,i)$ for $y \in \overline{\mathcal{O}}, i \in \mathcal{M}$, and the tightness of $\{\xi^h(\cdot), \alpha^h(\cdot)\}$, we obtain

$$\begin{aligned}\xi^h_\delta(t) \ &= x + \int_0^t F\big(\xi^h_\delta(s), \alpha^h_\delta(s)\big)ds\\ &\quad + \sum_{j=0}^{[t/\delta]} G\big(\xi^h_\delta(j\delta), \alpha^h_\delta(j\delta)\big)\big[W^h(j\delta+\delta) - W^h(j\delta)\big] + \varepsilon^h_{\delta,t} + O(h^2),\end{aligned}$$

where $[s]$ denotes the integer part of s and $\mathbb{E}|\varepsilon^h_{\delta,t}| \to 0$ as $\delta \to 0$, uniformly in $h > 0$ and t in any bounded interval. Taking $h \to 0$ implies

$$X_\delta(t) = x + \int_0^t F\big(X_\delta(s), \alpha_\delta(s)\big)ds + \sum_{j=0}^{[t/\delta]} G\big(X_\delta(j\delta), \alpha_\delta(j\delta)\big)\big[W(j\delta+\delta) - W(j\delta)\big] + \varepsilon_{\delta,t},$$

where $\mathbb{E}|\varepsilon_{\delta,t}| \to 0$ as $\delta \to 0$. The boundedness of G and standard properties of the Wiener process give

$$X_\delta(t) = x + \int_0^t F(X_\delta(s), \alpha_\delta(s))ds + \int_0^t G(X_\delta(s), \alpha_\delta(s))dW(s) + \bar{\varepsilon}_{\delta,t}.$$

Here, note that $\mathbb{E}|\bar{\varepsilon}_{\delta,t}| \to 0$ as $\delta \to 0$. Taking $\delta \to 0$ yields

$$X(t) = x + \int_0^t F(X(s), \alpha(s))ds + \int_0^t G(X(s), \alpha(s))dW(s),$$

which completes the proof. □

5. Convergence of cost and value functions

In the previous section, we established the weak convergence of the continuous-time interpolated process $\{\xi^h(\cdot), \alpha^h(\cdot), \tau^h\}$ and characterized its limit as a solution to the system in (2.1). In this section, we shall focus on the convergence of the cost and value functions.

For a function $\phi \in D([0,\infty); \mathbb{R}^d)$ (*i.e.*, functions defined on $[0,\infty)$ taking values in $\mathbb{R}^d$ that are right continuous having left limits equipped with the Skorokhod topology), we define

$$\widehat{\tau}(\phi) = \begin{cases} \inf\big\{t : \phi(t) \notin \mathcal{O}\big\}, \\ \infty \ \text{ if } \ \phi(t) \in \mathcal{O} \ \text{ for all } \ t < \infty. \end{cases} \tag{5.1}$$

We are now prepared to state Lemma 5.1 below.

Lemma 5.1. *Assume* (H0) *and* (H1)–(H3). *Then the following assertions hold*

(i) *$\widehat{\tau}(X(\cdot)) < \infty$ and $\widehat{\tau}(X(\cdot))$ is continuous w.p.1.*
(ii) *For sufficiently small $h > 0$, $\{\tau^h\}$ is uniformly integrable. That is, there is an $h_0 > 0$ such that for all $h < h_0$, $\{\tau^h : h < h_0\}$ is uniformly integrable.*

Proof. The proof of the lemma for systems of diffusions was in Kushner and Dupuis [13, pp. 259–261]. For our case, we need to modify it for the switching diffusions. We shall only highlight the main difference below. In a Markov switching diffusion, we have diffusions between the Markovian jump times. It suffices to examine the system until the first jump. We focus on the case where no switching occurs before the first escape time of $\mathcal{O}$. Let σ denote the first jump time of the continuous-time Markov chain $\alpha(t)$. Suppose that before the first jump time σ, $\alpha(t) = i \in \mathcal{M}$. Note that before the first jump time, $X(t) \in \mathcal{O}$ and it is a diffusion

$$\begin{cases} dX(t) = F(X(t), i)dt + G(X(t), i)dB(t), \\ X_0 = x. \end{cases}$$

Denote by $\widetilde{\tau}_{x,i}$ the $\mathcal{F}_t^{X(t)}$-stopping time for the above diffusion

$$\widetilde{\tau}_{x,i} = \inf\{t > 0, X(t) \notin \mathcal{O}\}.$$

By virtue of [13, pp. 263], there exists a $\delta_1 > 0$ so that $\inf_{(x,i)\in\mathcal{O}} \mathbb{P}_{x,i}(\widetilde{\tau}_{x,i} \le T) \ge \delta_1$. Thus,

$$\begin{aligned} \inf_{(x,i)\in\mathcal{O}\times\mathcal{M}} \mathbb{P}_{x,i}(\widehat{\tau} \le T) &\ge \inf_{(x,i)\in\mathcal{O}\times\mathcal{M}} \mathbb{P}_{x,i}(\widehat{\tau} \le T | \sigma > T)\mathbb{P}_{x,i}(\sigma > T) \\ &= \inf_{(x,i)\in\mathcal{O}\times\mathcal{M}} \mathbb{P}_{x,i}(\widetilde{\tau}_{x,i} \le T)e^{q_{ii}T} \\ &\ge \inf_{i\in\mathcal{M}} e^{q_{ii}T}\delta_1 > 0, \end{aligned}$$

where we used the fact that the first jump distribution of σ is an exponential random variable with parameter $-q_{ii}$. The rest of the details are omitted. □

5.1. Convergence of the cost functions

The key result of this section is as follows.

Theorem 5.2. *Suppose that the assumptions of Lemma 5.1 hold. Then the cost function $J^h(x, \imath, \tau^h)$ in (4.4) converges to $J(x, \imath, \tau)$ as $h \to 0$.*

Proof. First note that by assertion (i) in Lemma 5.1, $\widehat{\tau}(X(\cdot)) < \infty$ leads to $\sup_{(x,i)\in\overline{\mathcal{O}}\times\mathcal{M}} \mathbb{E}_{x,\imath}\tau_0 < \infty$, where

$$\tau_0 = \inf\{t : (X(t), \alpha(t)) \notin \mathcal{O} \times \mathcal{M}\}.$$

Because $\{\xi^h(\cdot), \alpha^h(\cdot), \tau^h\}$ is tight, we may extract a convergent subsequence, with its corresponding limit denoted by $(X(\cdot), \alpha(\cdot), \bar{\tau})$. Note that the stopping time τ satisfies $\tau \le \tau_0$, where τ_0 is the first exit time from $\mathcal{O}$. Thus τ still has the possibility to be the exit time τ_0. This implies that the tangency problem may still occur; see Kushner and Dupuis [13, pp. 276–278] for the detailed description of tangency problem. By the Skorokhod representation, with a slight abuse of notation, we may assume that the convergence is with probability one, *i.e.*,

$$(\xi^h(\cdot), \alpha^h(\cdot), \tau^h) \to (X(\cdot), \alpha(\cdot), \bar{\tau}) \text{ w.p.1.}$$

Proving $J^h(x, \imath, \tau^h)$ converges to $J(x, \imath, \tau)$ requires first showing $\tau = \overline{\tau}$ w.p.1. In view of Lemma 5.1, we have

$$\overline{\tau} = \lim_{h\to 0} \tau^h = \lim_{h\to 0} \widehat{\tau}(\xi^h(\cdot)) = \widehat{\tau}(X(\cdot)) = \tau \text{ w.p.1.}$$

In addition, it needs to be demonstrated that the mapping

$$(X(\cdot), \alpha(\cdot), \tau) \mapsto \int_0^\tau \exp\Big(-\int_0^t \lambda(\alpha(s))ds\Big)H(X(t), \alpha(t))dt \tag{5.2}$$

is continuous w.p.1 with respect to the measure induced by $(X(\cdot), \alpha(\cdot), \tau)$. However, this is already implied by the continuity of $H(\cdot, \cdot)$ and Lemma 5.1. We thus have

$$\begin{aligned} &\int_0^{\tau^h} \exp\Big(-\int_0^t \lambda(\alpha^h(s))ds\Big)H(\xi^h(t), \alpha^h(t))dt \\ &\quad\to \int_0^\tau \exp\Big(-\int_0^t \lambda(\alpha(s))ds\Big)H(X(t), \alpha(t))dt \text{ w.p.1.} \end{aligned} \tag{5.3}$$

In order to obtain the desired convergence for the cost functional, one first takes expectations on both sides of (5.3) and then takes limits. This requires the uniform integrability, for some $h_0 > 0$, of the set

$$\Big\{\int_0^{\tau^h} \exp\Big(-\int_0^t \lambda(\alpha^h(s))ds\Big)H(\xi^h(t), \alpha^h(t))dt; 0 < h < h_0\Big\}.$$

Due to the boundedness of H in the set $\overline{\mathcal{O}}$ and the positivity of $\lambda(\cdot)$, we only need uniform integrability of $\{\tau^h; h < h_0\}$. However, this has already been demonstrated in Lemma 5.1. We thus have that $J^h(x, \imath, \tau^h) \to J(x, \imath, \tau)$ as $h \to 0$ as desired. □

5.2. Convergence of the value functions

In what follows, for $\Delta > 0$, we say that $(X(\cdot), \alpha(\cdot), W(\cdot), \tau^{\Delta})$ with initial data $(x, \imath)$ is a Δ-optimal process satisfying (2.1) if

$$J(x, \imath, \tau^{\Delta}) < V(x, \imath) + \Delta.$$

The random variable τ^{Δ} is called a Δ-optimal stopping time.

Since τ appears in the upper limit of integration of the cost function, the ε-optimal stopping times (the "control") will not influence the dynamics of the regime switching diffusions. In other words, if we restrict the stopping times τ to a certain set, we will still have the same constructed Markov chain $\{\xi_k^h, \alpha_k^h\}$ and its continuous time interpolated process $\{\xi^h(\cdot), \alpha^h(\cdot)\}$; see (5.5) below. Therefore, in contrast to numerical approximation in standard stochastic optimal control problems, in which the control variable will influence the underlying dynamics, we do not need to use the chattering lemma (cf. [23,24]).

Theorem 5.3. *Under the conditions of Theorem 5.2, the value function $V^h(x, \imath)$ in (4.4) converges to $V(x, \imath)$ as $h \to 0$.*

Proof. By Theorem 5.2, we have that

$$J^h(x, \imath, \tau^h) \to J(x, \imath, \tau) \geq V(x, \imath).$$

Thus $\liminf_{h\to 0} V^h(x, \imath) \geq V(x, \imath)$. To complete the proof, we now need only prove the reserve inequality

$$\limsup_{h\to 0} V^h(x, \imath) \leq V(x, \imath) = \inf_{\tau \leq \tau_0} J(x, \imath, \tau).$$

The idea is to find an ε-optimal stopping time for the process $(X(\cdot), \alpha(\cdot))$ and adapt it to the Markov chain, and then to use the minimality of $V^h(x, \imath)$ and weak convergence. We now formally proceed. By the continuity and boundedness of H, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that we can restrict the stopping times for (2.1) to take values on $\{n\delta; n\delta < T\}$ and for which the cost (1.2) increases by at most ε. Denote by τ_ε the optimal stopping time with above restriction. That is, τ_ε is taken to be one of the values in the set $\{n\delta; n\delta \leq T\}$. The restriction on stopping times implies

$$J(x, \imath, \tau_\varepsilon) \leq V(x, \imath) + \varepsilon. \tag{5.4}$$

For the optimal stopping problem associated with the cost function (1.2), where the stopping time is restricted to take values on $\{n\delta; n\delta \leq T\}$ as above, we have

$$(\xi^h(\cdot), \alpha^h(\cdot), W^h(\cdot), \tau_\varepsilon^h) \Rightarrow (X, \alpha(\cdot), W(\cdot), \tau_\varepsilon), \tag{5.5}$$

where τ_ε^h is the time approximating stopping time τ_ε. Then Theorem 5.2 and the inequality in (5.4) yield

$$V^h(x, \imath) \leq J^h(x, \imath, \tau_\varepsilon^h) \leq J(x, \imath, \tau_\varepsilon) + \delta_1(\varepsilon) \leq V(x, \imath) + \delta_1(\varepsilon) + \varepsilon.$$

Letting $h \to 0$, $\delta_1(\varepsilon) \to 0$. Taking $\limsup_{h\to 0}$ and letting $\varepsilon \to 0$ now completes the proof. □

6. Numerical examples and remarks

In this section, we illustrate how to use the numerical algorithm developed in this paper with computational examples. We shall apply our algorithm to obtain both the stopping region and the continuation region for a two-dimensional system. We then conclude with some remarks.

6.1. Numerical example

We first provide a motivation of the study from power systems. Quickest detection of faults in power systems has always been a critical issue in power system reliability studies and technology development. In power systems, a common scenario of contingencies is a class of line faults, such as line-to-ground faults, line-to-line faults, over-current, loss of a transformer, line disconnection, etc. Starting from the occurrence time of a contingency, there is a critical maximum time interval in which the fault must be detected and cleared. Beyond this time interval, synchronous generators of the power system lose their synchronism and stability, and must be taken off the grid; see Kothari and Nagrath [2]. Since line faults are represented as switches in hybrid system models for power systems, the prompt and accurate switching time estimation investigated in the present paper has the potential to be an essential and promising tool.

Mathematically, a line fault can be represented by a sudden change in line impedance, from a normal value to certain extreme values (near zero or extremely high values) depending on the types of the fault. For instance, without direct measurement devices on every segment of a long distance transmission line, the command center must rely on available measurement devices such as phaser measurement units on buses to diagnosis such faults. The algorithm of this paper provides an advanced method to detect such line faults based on system dynamics and Markov chain information.

Before continuing with numerical experiments, we outline the value iteration policy in order to approximate the value function in (3.3), as follows.

1. Initialization. For each $(x, \imath) \in S_h \times \mathcal{M}$, we set the initial value $V_0^h(x, \imath)$ to be 1.
2. Value iteration. Given the value of $V_n^h(x, \imath)$, we find the next value $V_{n+1}^h(x, \imath)$ using (3.3) as

$$V_{n+1}^h(x, \imath) = \min\left\{ e^{-\lambda(\imath)\Delta t^h(x,\imath)} \sum_{(y,i)} p^h\big((x, \imath), (y, i)\big) V_n^h(y, i) + H(x, \imath)\Delta t^h(x, \imath),\ 0 \right\}. \tag{6.1}$$

3. Error bounds and stopping criterion. If $|V_{n+1}^h - V_n^h| \le$ tolerance level, then the iteration stops; otherwise, we continue the value iteration procedure.
4. Stopping region and optimal policy. Given the estimated optimal value $\overline{V}^h(x, \imath)$ and the tolerance level ε, the stopping region will be

$$\overline{D}^{h,\varepsilon} = \{(x, \imath) \in S_h \times \mathcal{M} : \overline{V}^h(x, \imath) > -\epsilon\},$$

and the estimated optimal stopping time will be

$$\overline{\tau}_D^{h,\epsilon} = \inf\{t \ge 0 : \xi^h(t) \in \overline{D}^{h,\epsilon}\}.$$

Example 6.1 (*Quickest Detection of a Markovian Drift*)**.** In this example, we assume that one can observe a sample path of $X = (X^1, X^2)$. Namely, there is no drift initially. Then at a random and unobservable time $\theta > 0$ following an exponential distribution $\exp(\lambda)$ with parameter $\lambda > 0$, one of the coordinate processes of X obtains a (known) nonzero drift μ depending on a Markov chain α, where $\alpha = \{\alpha_t; t \ge 0\}$ takes values in a finite state space $\mathcal{M} = \{1, \dots, m_0\}$ and the Q matrix $Q = (q_{ij})$ satisfying $\sum_{j\in\mathcal{M}} q_{ij} = 0$ and $q_{ij} \ge 0$ for all $i \ne j$. In this example, we take $m_0 = 3$. Let the probability of θ taking zero value equals $\pi \in [0, 1)$. Therefore, the quickest detection problem in such case aims to find the time θ as "accurate" as possible.

Formally, $X = (X^1, X^2)$ satisfies the following stochastic differential equations:

$$\begin{cases} dX_t^1 = \mu(\alpha_t)\mathbf{1}_{\{t\ge\theta,\beta=1\}}dt + dB_t^1, & X_0^1 = x_1, \\ dX_t^2 = \mu(\alpha_t)\mathbf{1}_{\{t\ge\theta,\beta=2\}}dt + dB_t^2, & X_2^0 = x_2, \end{cases} \tag{6.2}$$

where α is the Markov chain with initial state $\alpha_0 = \imath$. The B_t^1, B_t^2 are two independent standard Brownian motions. The parameter β is used to represent the coordinate number whose process obtains the Markovian drift. We assume $\beta \overset{d}{=} B(1, p)$, the binomial distribution with parameter $p \in (0, 1)$, and we also assume that $\alpha, B_t^1, B_t^2, \theta, \beta$ are all independent.

We use the term "accurate" to mean that we seek to find a stopping time τ^* that is as "close as possible" to θ. Following [9], this means that we would like to minimize a cost functional $J(\cdot)$ with respect to the stopping time τ defined as follows

$$J(\tau) = \mathbb{P}(\tau < \theta) + c\ \mathbb{E}_{\imath,\pi}\left[\mathbf{1}_{\{\tau>\theta\}}(e^{\gamma(\tau-\theta)} - 1)\right], \quad \gamma > 0, \tag{6.3}$$

where τ are stopping times adapted to the natural filtering $\mathcal{F}_t^{X,\alpha} = \sigma\{X_s, \alpha_s; 0 \le s \le t\}$ augmented with all $\mathbb{P}$-null set. The first term in the right of (6.3) represents the probability of false alarm, while the expectation term is the expected exponentially penalized cost for detection delay. The constant c is a trade-off coefficient. Denote by the value function V as

$$V(\imath, \pi) = \inf_{\tau} J(\tau), \tag{6.4}$$

where the infimum is taken for all (bounded) stopping times τ.

To proceed, we perform the change of measure techniques in Ernst and Mei [9, Section 6.1]. The original detection problem can be reformulated as the following linear switching diffusion

$$\begin{cases} dY_t^1 = [\lambda + (\lambda+\gamma)Y_t^1]dt + \mu(\alpha_t)Y_t^1 d\widehat{B}_t^1, & Y_0^1 = y_1 \\ dY_t^2 = [\lambda + (\lambda+\gamma)Y_t^2]dt + \mu(\alpha_t)Y_t^2 d\widehat{B}_t^2, & Y_0^2 = y_2, \end{cases} \tag{6.5}$$

where $\alpha_0 = \imath$. The corresponding value function in the new measure has the equivalent form

$$\widehat{V}(\imath, y_1, y_2) = \inf_{\tau} \mathbb{E}_{\imath,y_1,y_2} \int_0^{\tau} e^{-\lambda t}\Big(pY_t^1 + (1-p)Y_t^2 - \frac{\lambda}{c\gamma}\Big)dt. \tag{6.6}$$

In our numerical example, we concentrate on solving (6.5) and (6.6), *i.e.*, the dynamics after the measure change. We take $n = 3, \lambda = 0.5, \gamma = 0.4, p = 0.4, \mu(1) = 5, \mu(2) = 1, \mu(3) = -5$ and $c = 1, \pi = 0.25$. Furthermore, we set $\overline{\mathcal{O}} = [0, 5] \times [0, 5]$ and let the step size $h = 0.01$. The Markov chain has three states $\mathcal{M} = \{1, 2, 3\}$ with

$$Q \in \mathbb{R}^{3\times3} = \{q_{ij}, i, j \le 3\} = \begin{pmatrix} -3 & 1 & 2 \\ 4 & -5 & 1 \\ 3 & 1 & -4 \end{pmatrix}.$$

Figs. 1–3 below display the optimal stopping boundaries with Markov chain state $\{1, 2, 3\}$, respectively. For each of these figures, the optimal stopping boundary is depicted by the dark solid line. The blue area is the continuation region and the red area represents the stopping region. Starting from a point in the continuation region, the underlying process evolves until it hits the optimal stopping boundary. The corresponding hitting time is the optimal stopping time that minimizes the cost function.

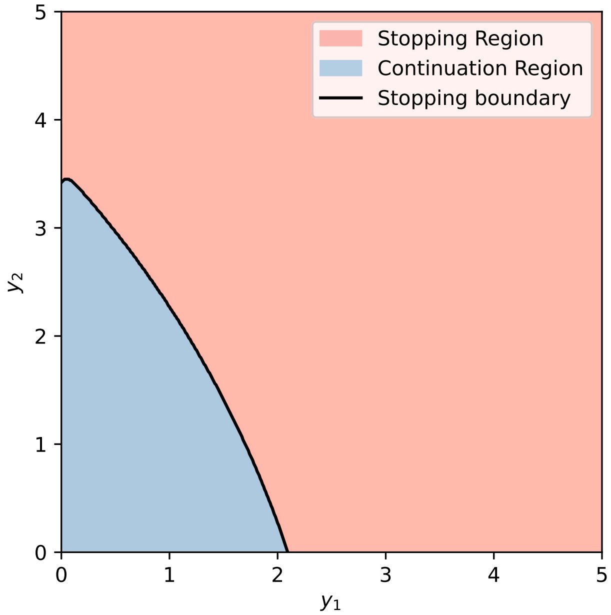

Fig. 1. Optimal stopping boundary, continuation region, and stopping region: $\alpha = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

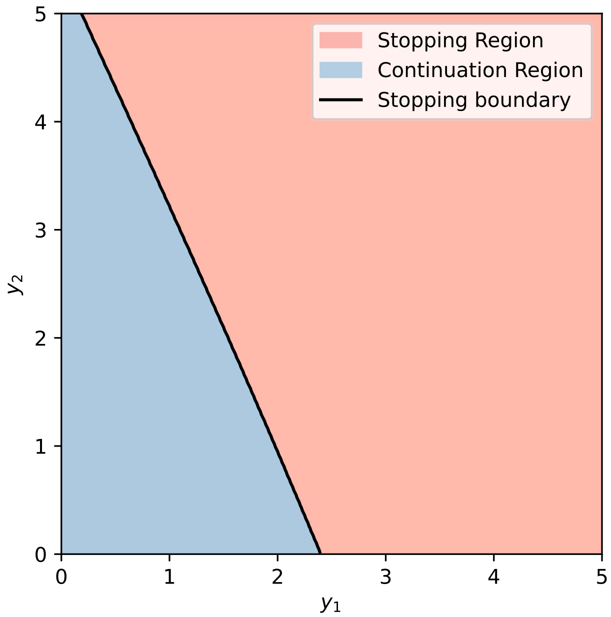

Fig. 2. Optimal stopping boundary, continuation region, and stopping region: $\alpha = 2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

6.2. Final remarks

This work has been devoted to numerically approximating optimal stopping problems for a class of stochastic hybrid systems. We note that for a number of large and complex systems, using time-scale separation and singular perturbations, we can reduce a large-scale and complex system to a much simplified and reduced order system. Although a simple numerical example has been demonstrated, the developed algorithms can certainly be employed for a wide range of applications.

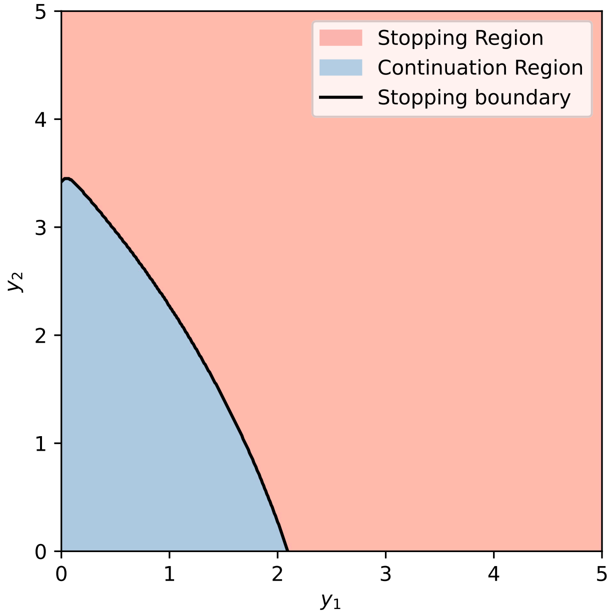

Fig. 3. Optimal stopping boundary, continuation region, and stopping region: $\alpha = 3$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

CRediT authorship contribution statement

Philip A. Ernst: Writing – review & editing. **Xiaohang Ma:** Writing – review & editing. **Masoud H. Nazari:** Writing – review & editing. **Hongjiang Qian:** Writing – review & editing. **Le Yi Wang:** Writing – review & editing. **George Yin:** Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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