Almost-Sure Robust Stabilization of Randomly Switched Linear Systems With Uncontrollable Subsystems

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***Abstract*—This article investigates state feedback design for achieving almost-sure robust stabilization of randomly switched linear systems whose subsystems are uncontrollable and whose models are subject to nonlinear modeling errors due to, for example, linearization. Complex systems are often uncontrollable under a fixed configuration from a single control input. However, when control actions can be used sequentially and collaboratively through different system configurations, stabilization can be potentially achieved. This design problem encounters some fundamental issues that must be resolved, involving mostly suitable coordinated implementations of state decomposition, feedback pole-placement design, usage of stochastic information on the switching process, coupling of controllable and uncontrollable substates, subsystem interactions, and modeling errors. The common state feedback on controllable substates can lead to unstable closed-loop systems, due to substate coupling. A modified control algorithm is introduced that decouples substates and designs feedback gains simultaneously. Further complications arise when subsystem interaction destabilizes the system. Some structural conditions are shown to be essential for achieving almost-sure stabilization. A design procedure that integrates feedback gain selection and switching information is introduced to achieve almost-sure stability for the closed-loop system. Robustness of the design procedure is established. Examples and simulation case studies are presented to illustrate the main algorithms and stabilization properties.**

***Index Terms*—Almost-sure stabilization, hybrid system, randomly switched linear system (RSLS), robust stability, state feedback, uncontrollable system.**

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I. Introduction

RANDOMLY switched linear systems (RSLSs) form an important class of stochastic hybrid systems (SHSs) and have appeared in broad application domains that involve random contingencies, system interactions, communication networks, etc. They are especially common in emerging technologies such as autonomous vehicles, robotics, modern power systems, energy networks, smart buildings, human–machine teams, etc. In terms of model structures, RSLSs include linear continuous dynamics that are modulated by stochastic switching processes [1], [2], [3], [4], [5].

RSLSs and SHSs are common in emerging interconnected complex systems. For example, modern power systems include many diversified subsystems such as renewable generators, battery systems, controllable loads, among many other types. Due to random physical system switches in structures and parameters, caused by transmission line faults, generator failures, sensor malfunctioning, communication packet losses, and many other issues, dynamic power systems become intertwined with random discrete events, leading to SHSs. Consequently, reliability analysis of operation, robust voltage/frequency regulations, and optimal power dispatch in modern power systems must be carried out under an SHS environment. Similar scenarios emerge in autonomous systems in many domains such as autonomous vehicles, autonomous buildings, automated lighting in communities, etc. The continuous dynamics and logic-based supervisory control systems introduce naturally SHSs.

In our recent work, observability and observer design problems for RSLSs were studied, including continuous state estimation for achieving almost-sure convergent estimators [6], joint estimation of both continuous and discrete states for convergence [7], and mean-square (MS) convergence of continuous state estimation [8]. There is an extensive literature of state estimation problems for hybrid systems; see [9], [10], [11], [12], [13], [14], [15], [16], [17], and [18] for hybrid systems in deterministic settings and [19], [20], [21], and [22] for SHSs.

This article investigates robust stabilization problems for RSLSs whose subsystems are uncontrollable. Control of hybrid systems has been studied extensively in both deterministic and stochastic frameworks with applications [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32]. In deterministic systems, models and optimal control problems in hybrid systems were investigated in [33], [34], and [35]. Various notions of controllability in deterministic hybrid systems were introduced with testing conditions in [2], [36], [37], [38], [39], [40], and [41]. Stability of deterministic hybrid systems was established in [27], [39], [42], [43], and [44]. Stabilization of SHSs was covered comprehensively in [21] and [45]. Stability analysis of randomly switching systems was studied in [23] and [24]. Observed-based feedback design was investigated for robust tracking problems in aircraft systems that involve system couplings [25].

To the best of our knowledge, there are no known results on stochastic control of hybrid systems involving all uncontrollable subsystems. Although small-scale physical systems are often controllable, complex systems that involve many interconnected local dynamic systems are often uncontrollable by a single control input. Due to the random nature of system faults, failures, contingencies, and network interruptions, controllable substates are also randomly changing. This article aims to answer the following questions: Is it possible to stabilize an RSLS by a suitable state feedback? How can controllers be designed? What is the impact of coupling between controllable and uncontrollable substates? What is the influence of subsystem interaction? What is the relevance of the stochastic information of the switching process on stabilization?

Randomly switched systems are stochastic systems whose stability can be studied in different modes of convergence. This article studies almost-sure stability. Almost-sure exponential convergence ensures that (almost) every individual realization of the state trajectories converges to zero exponentially fast. In contrast, other convergence notions such as MS convergence, convergence in probability, or convergence in distribution, target convergence properties of the entire ensemble (the whole population), leaving a certain chance, albeit diminishingly small, for individual realizations of the system to fail to converge during implementation.[1]

Due to the stochastic and time-varying nature of RSLSs, the common duality properties for linear time invariant (LTI) systems do not hold for RSLSs. As a result, our previous results on state observers of RSLSs [6], [7], [8] cannot be directly used here, and unique feedback design procedures must be developed to achieve stabilization of the continuous states. This article exhibits several fundamental issues that must be resolved to guarantee the stability in the sample-path sense, including state decomposition, feedback design, usage of the stochastic information, coupling of controllable and uncontrollable substates, and subsystem interactions. When subsystems are uncontrollable and switching sequences are random, the common state feedback on controllable substates are shown to be potentially divergent due to substate coupling. Further complications arise when subsystem interactions are involved. These unique features raise critical and challenging issues and will be resolved in this article.

[1]In stochastic systems, the terms "almost-sure convergence," "convergence with probability one (w.p.1)," and "strong convergence" are used exchangeably. It should not be confused with the term "strong stabilization" in deterministic systems, which means stabilization by using a stable controller. To avoid confusion, this article uses "almost-sure" convergence and stabilization.

This article contains the following original contributions.

1) It introduces a framework of feedback design for RSLSs with uncontrollable subsystems.
2) It reveals several fundamental destabilizing factors in RSLSs that limit the ability of feedback to achieve stabilization.
3) A modified control algorithm is introduced that decouples substates and designs the feedback gains simultaneously.
4) A design procedure is developed for almost-sure stabilization by integrating controllable system dynamics, pole assignment, switching information, and uncontrollable dynamics.
5) Some structural conditions are introduced for achieving almost-sure stabilization.
6) The design method is shown to be robust against nonlinear modeling errors, such as those from linearization, that satisfy some linear growth rate and error bound conditions.

The rest of this article is organized as follows. Section II contains notations, system descriptions, and basic definitions. Almost-sure stabilization problems are investigated in Section III. Instability issues from substate coupling and subsystem interaction are demonstrated with examples. A modified control design that plays the dual roles of substate decoupling and feedback control is introduced. Algorithms are developed and their almost-sure stabilization properties are established. Section IV is focused on robustness analysis. Under (potentially nonlinear) modeling errors that satisfy certain linear growth conditions, the error bounds are derived under which our design algorithms can achieve almost-sure, exponential, and robust stabilization. Examples and simulation case studies are presented in various sections and in Section V to illustrate the main algorithms and their stabilization properties. Finally, Section VI concludes this article.

II. Preliminaries

A. Notation

For a column vector $v \in \mathbb{R}^n$, $\|v\|$ is its Euclidean norm. For a matrix $M \in \mathbb{R}^{n\times m}$, M' is its transpose, $\lambda(M)$ an eigenvalue of M, $\sigma(M) = \sqrt{\lambda(M'M)}$ a singular value of M, $\sigma_{\min}(M)$ its minimum singular value, and $\sigma_{\max}(M)$ its largest singular value. The value $\sigma_{\max}(M)$ is also the operator norm of M induced by the Euclidean norm

$$\sigma_{\max}(M) = \|M\| = \sup_{\|v\|=1} \|Mv\|.$$

The kernel or null space of $M \in \mathbb{R}^{n\times m}$ is $\mathrm{Ker}(M) = \{x \in \mathbb{R}^m : Mx = 0\}$ and its range is $\mathrm{Range}(M) = \{y = Mx : x \in \mathbb{R}^m\}$. For a p-dimensional subspace U of vectors in $\mathbb{R}^n$, a matrix $M \in \mathbb{R}^{n\times p}$ is said to be a base matrix of U, written as $M = \mathrm{Base}(U)$, if the column vectors of M are linearly independent, and $\mathrm{Range}(M) = U$.

B. Systems

Consider the following RSLS:

$$\dot{x}(t) = A(\alpha(t))x(t) + B(\alpha(t))u(t) \tag{1}$$

where $u(t) \in \mathbb{R}^\rho$ is the control input, $x(t) \in \mathbb{R}^n$ is the state, and $\alpha(t)$ is a randomly switching process taking m possible values in a discrete state space $\mathcal{S} = \{1, \ldots, m\}$. The system matrices $A(\cdot)$ and $B(\cdot)$ depend on the process $\alpha(t)$. For each given value $i \in \mathcal{S}$, the corresponding LTI system in (1) with matrices $A(i)$ and $B(i)$ will be called the *ith subsystem of the RSLS*.

The discrete state process $\alpha(t)$ is a piecewise-constant process satisfying the following assumption.

Assumption 2.1: For a given fixed interval τ, with $0 < \underline{\tau} \leq \tau \leq \overline{\tau} < \infty$,

1) the switching process $\alpha(t)$ can switch only at the sampling instants $k\tau$, $k = 1, 2, \ldots$, generating a stochastic sequence $\{\alpha_k = \alpha(k\tau)\}$, which will be termed a *skeleton sequence*;
2) the sequence $\{\alpha_k\}$ is independent and identically distributed (i.i.d.) with

$$P\{\alpha_k = i\} = p_i > 0, i \in \mathcal{S}, \text{ and } \sum_{i=1}^{m} p_i = 1. \tag{2}$$

Remark 2.1: In real-world systems, an interval τ is physically determined by device platforms, system dynamics, software packages, sensor and actuator response speeds, and other physical constraints. For example, all measurement devices with digital data streams have a sampling time interval, which is commonly small. Similarly, communication systems often use an interval of fixed duration in time-division multiplexing protocols for transmitting and receiving independent signals over a shared signal path. Also, control and decision at system levels always impose a decision updating interval for synchronized decision implementation at all nodes of network systems. Since physical contingencies occur infrequently in comparison to data rate, such contingencies do not occur twice in this small interval. These common scenarios form a basic motivation of using a small interval τ as a background data platform to develop the results of this article.

It is noted that the design methods developed in this article employ this information, and as a result the controllers depend on τ also. To ensure convergence rates, it is necessary to bound τ below and above. Mathematically, it means that the frequency $f = 1/\tau$ should not be too high or too low. It is known that without suitable control design, high-frequency switching in hybrid systems may cause instability; see [45].

The results in this article capture certain critical aspects of convergence analysis that can be extended to treat ergodic Markov chains (such as irreducible and aperiodic Markov chains). Such Markov chains have stationary distributions that contain the same information as an i.i.d. process. The control design in this article requires only the information on the stationary distribution. However, Markov chains represent a much richer class of stochastic processes than i.i.d. processes. The comprehensive usage of the information provided by Markov chains will be treated in separate articles.

System (1) may be viewed as a linear parameter varying (LPV) system, with time-varying jumps in system parameters and structures. However, typical deterministic LPV systems assume the knowledge on the entire time-varying system, and hence, a feedback design can potentially use this future information on system dynamics to achieve stabilization. In our formulation, the occurrence of the jumping sequence α_k cannot be predicted since it is i.i.d. For a causal design, one cannot use any future information on the dynamic system.

For a sample-path skeleton sequence α_k, $k = 1, \ldots$, denote the corresponding stochastic matrix sequences by $A_k = A(\alpha_k)$ and $B_k = B(\alpha_k)$. Under Assumption 2.1

$$A_k = A(\alpha_k) = \sum_{i=1}^{m} A(i)\mathbf{1}_{\{\alpha_k = i\}}$$

$$B_k = B(\alpha_k) = \sum_{i=1}^{m} B(i)\mathbf{1}_{\{\alpha_k = i\}}$$

where $\mathbf{1}_G$ is the indicator function of the event G. These matrices are random.

For each $i = 1, \ldots, m$, the controllability matrix for the ith subsystem with $(A(i), B(i))$ is

$$W(i) = [B(i), A(i)B(i), \ldots, (A(i))^{n-1}B(i)] \in \mathbb{R}^{n \times n\rho}. \tag{3}$$

The combined controllability matrix for the set $\mathcal{S}$ is

$$W_{\mathcal{S}} = [W(1), W(2), \ldots, W(m)] \in \mathbb{R}^{n \times mn\rho}. \tag{4}$$

The matrices $W(i)$ and $W_{\mathcal{S}}$ are constant matrices.

Assumption 2.2:

1) All subsystems are uncontrollable, namely, $\mathrm{Rank}(W(i)) = n_i < n, i \in \mathcal{S}$.[2]
2) $W_{\mathcal{S}}$ is full row rank.

Remark 2.2: Assumption 2.2 is important. The first assumption raises a challenging issue since it implies that subsystem coordination in a stochastic setting becomes mandatory. In most application problems, the second assumption is a necessary condition for convergent feedback controllers to exist. This is easy to understand: If a subspace does not belong to the controllable subspaces of any subsystem, then its dynamics cannot be controlled. It follows that if some of the substates on this subspace are governed by unstable dynamics, they can never be convergent, regardless of how the feedback controllers are designed.

III. Almost-Sure Stabilization of RSLSs

A. Controllable Substates

It is well known that if an LTI system is uncontrollable, in general, it is not possible to achieve stabilization by state feedback, unless all uncontrollable subspaces are stable. Since this article treats subsystems that are all uncontrollable, the

[2]We focus on the more general and complicated scenario in which all subsystems are uncontrollable. If some subsystems are controllable, the analysis can be simplified.

stochastic information of α_k must be used to coordinate subsystem designs so that the closed-loop system becomes almost surely stable.

It is well known [46], [47], [48] that if the ith subsystem with matrices $(A(i), B(i))$ is not controllable, namely $\mathrm{Rank}(W(i)) = n_i < n$, there exists a nonsingular matrix $T_i \in \mathbb{R}^{n\times n}$ such that the transformed system has the new state variable

$$\widetilde{x}^i = T_i^{-1}x = \begin{bmatrix}\widetilde{x}_1^i\\ \widetilde{x}_2^i\end{bmatrix}$$

with $\widetilde{x}_1^i \in \mathbb{R}^{n_i}$ that satisfies $\dot{\widetilde{x}}^i = \widetilde{A}^i\widetilde{x}^i + \widetilde{B}^i u$, where $\widetilde{A}^i = T_i^{-1}A(i)T_i = \begin{bmatrix}A_{11}^i & A_{12}^i\\ 0 & A_{22}^i\end{bmatrix}$, $\widetilde{B}^i = T_i^{-1}B(i) = \begin{bmatrix}B_1^i\\ 0\end{bmatrix}$, and (A_{11}^i, B_1^i) is controllable.

The construction of $T_i = [M_i, N_i]$ involves first defining $M_i = \mathrm{Base}(\mathrm{Range}(W(i))) \in \mathbb{R}^{n\times n_i}$, and then, selecting N_i to make T_i invertible. For $\rho = 1$, M_i may be selected as $M_i = [B(i), A(i)B(i), \ldots, (A(i))^{n_i-1}B(i)]$ which will put the transformed A_{11}^i, B_1^i in a controllable canonical form; see [46, p. 131] for details. Denote $T_i^{-1} = \begin{bmatrix}G_i\\ F_i\end{bmatrix}$. It follows that $\widetilde{x}_1^i = G_i x \in \mathbb{R}^{n_i}$. Write

$$M = [M_1, \ldots, M_m], \quad G = \begin{bmatrix}G_1\\ \vdots\\ G_m\end{bmatrix}.$$

Lemma 3.1: Under Assumption 2.2, G is full rank.

Proof: By the construction of M_i, $\mathrm{Range}(M) = \mathrm{Range}(W_S)$. By Assumption 2.2, M is full rank.

Since $T_i^{-1}T_i = I_n$, we have $G_iM_i = I_{n_i}$. Let $M_i = [v_1^i, \ldots, v_{n_i}^i]$. Then, $G_i v_j^i = e_j^i \neq 0$, where e_j^i is the jth column of I_{n_i}.

Since M is full rank, we can select n column vectors $\{v_1, \ldots, v_n\}$ from M that are linearly independent. For each v_j, there exists ℓ such that v_j is a vector of M_ℓ. As a result, $G_\ell v_j \neq 0$. This implies that $Gv_j \neq 0$.

Since this is valid for all $j = 1, \ldots, n$ and $\{v_1, \ldots, v_n\}$ are linearly independent, $\mathrm{Ker}(G) = \{0\}$. Therefore, G is full rank. ■

Consequently, $$\widetilde{x} = \begin{bmatrix}\widetilde{x}_1^1\\ \vdots\\ \widetilde{x}_1^m\end{bmatrix} = Gx, \quad \text{and} \quad x = (G'G)^{-1}G'\widetilde{x}.$$

This confirms that under Assumption 2.2, the convergence of controllable substates from all subsystems implies the convergence of x.

B. Complications in State Feedback Design

When $\alpha_k = i$, $\widetilde{x}_2^i$ cannot be controlled. It is then natural to concentrate only on controlling $\widetilde{x}_1^i$. If a linear state feedback gain $L^i \in \mathbb{R}^{\rho\times n_i}$ is used on $\widetilde{x}_1^i$, then $u = -L^i\widetilde{x}_1^i$ and the closed-loop system is

$$\dot{\widetilde{x}}_1^i = (A_{11}^i - B_1^iL^i)\widetilde{x}_1^i + A_{12}^i\widetilde{x}_2^i$$
$$\dot{\widetilde{x}}_2^i = A_{22}^i\widetilde{x}_2^i.$$

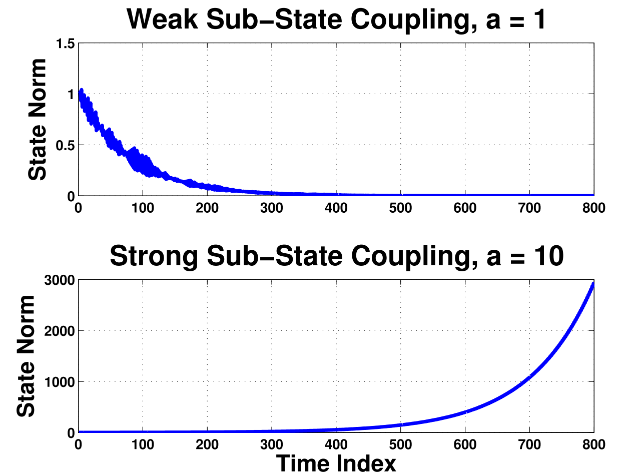

Fig. 1. Impact of substate coupling on the closed-loop system stability.

The coupling between the substates $\widetilde{x}_2^i$ and $\widetilde{x}_1^i$ introduces a critical issue in stabilization of RSLSs. In general, the substate coupling may destabilize the closed-loop system, as shown by the following example.

Example 3.1: Consider an RSLS with two subsystems

$$A(1) = \begin{bmatrix}2 & a\\ 0 & 2\end{bmatrix}, \quad B(1) = \begin{bmatrix}1\\ 0\end{bmatrix}$$
$$A(2) = \begin{bmatrix}2 & 0\\ a & 2\end{bmatrix}, \quad B(2) = \begin{bmatrix}0\\ 1\end{bmatrix}$$

where the value a represents the level of coupling between the controllable and uncontrollable substates. Since

$$W(1) = \begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix} \text{ and } W(2) = \begin{bmatrix}0 & 0\\ 1 & 2\end{bmatrix}$$

have rank 1, both subsystems are uncontrollable. But $W_S = [W(1), W(2)]$ is full rank. It is easy to derive $T_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $T_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, and the controllable substates are $\widetilde{x}_1^1 = x_1$, $\widetilde{x}_2^1 = x_2$, $\widetilde{x}_1^2 = x_2$, and $\widetilde{x}_2^2 = x_1$.

When $\alpha_k = 1$, $u = -L^1\widetilde{x}_1^1$, resulting in

$$\begin{cases}\dot{\widetilde{x}}_1^1 = (2 - L^1)\widetilde{x}_1^1 + a\widetilde{x}_2^1\\ \dot{\widetilde{x}}_2^1 = 2\widetilde{x}_2^1\end{cases}$$

and when $\alpha_k = 2$, $u = -L^2\widetilde{x}_1^2$ with

$$\begin{cases}\dot{\widetilde{x}}_2^2 = 2\widetilde{x}_2^2\\ \dot{\widetilde{x}}_1^2 = a\widetilde{x}_2^2 + (2 - L^2)\widetilde{x}_1^2.\end{cases}$$

Suppose that $L^1 = L^2 = 10$, $\tau = 0.01$, $p_1 = 0.5$, and $p_2 = 0.5$. Then, the closed-loop system is

$$\dot{x}(t) = \left(I_{\{\alpha_k=1\}}\begin{bmatrix}-8 & a\\ 0 & 2\end{bmatrix} + I_{\{\alpha_k=2\}}\begin{bmatrix}2 & 0\\ a & -8\end{bmatrix}\right)x(t)$$

for $t \in [k\tau, (k+1)\tau)$, $k = 0, 1, \ldots$.

Fig. 1 shows sample-path trajectories for two values of a: $a = 1$, representing relatively weak coupling of substates, and

$a = 10$, representing stronger coupling. It demonstrates that under weak coupling, the local state feedback design results in a stable closed-loop systems; but when the substate coupling increases, the same design becomes unstable. This example shows the complications when local state feedback is designed on the controllable substates only. It is noted that switching and unobservable subsystems contribute to potential jumps in discrete state sequences x_k at the sampling points. These are reflected in the jumps in Fig. 3 as perturbation bands in the trajectories during transient periods.

C. Modified Control Design for Substate Decoupling

The main issue shown in Example 3.1 is fundamental and must be resolved. In this subsection, we will introduce a modified design for subsystem control that eliminates substate coupling at sampling points.

When $\alpha_k = i$, the ith subsystem has the dynamics

$$\begin{cases} \dot{\widetilde{x}}_1^i = A_{11}^i \widetilde{x}_1^i + A_{12}^i \widetilde{x}_2^i + B_1^i u \\ \dot{\widetilde{x}}_2^i = A_{22}^i \widetilde{x}_2^i \end{cases}$$

for $t \in [k\tau, (k+1)\tau)$. The coupling term $A_{12}^i \widetilde{x}_2^i$ on the controllable substate $\widetilde{x}_1^i$ will now be included in the control design, beyond the local state feedback $u = -L^i \widetilde{x}_1^i$.

The solution to $\dot{\widetilde{x}}_2^i = A_{22}^i \widetilde{x}_2^i$ is $\widetilde{x}_2^i(t) = e^{A_{22}^i (t-k\tau)} \widetilde{x}_2^i(k\tau)$, $t \in [k\tau, (k+1)\tau)$. Substituting this solution to $\dot{\widetilde{x}}_1^i = A_{11}^i \widetilde{x}_1^i + A_{12}^i \widetilde{x}_2^i + B_1^i u$ gives the solution of $\widetilde{x}_1^i$ as

$$\widetilde{x}_1^i(t) = e^{A_{11}^i (t-k\tau)} \widetilde{x}_1^i(k\tau) + v^i(t) + \int_{k\tau}^{t} e^{A_{11}^i (t-\theta)} B_1^i u(\theta) d\theta$$

where $v^i(t)$ is the response to $\widetilde{x}_2^i(t)$

$$v^i(t) = \int_{k\tau}^{t} e^{A_{11}^i (t-\theta)} A_{12}^i e^{A_{22}^i (\theta - k\tau)} d\theta \, \widetilde{x}_2^i(k\tau).$$

The final value of $v^i(t)$ at $t = (k+1)\tau$ is

$$\begin{aligned} v^i((k+1)\tau) &= \int_{k\tau}^{(k+1)\tau} e^{A_{11}^i ((k+1)\tau-\theta)} A_{12}^i e^{A_{22}^i (\theta-k\tau)} d\theta \, \widetilde{x}_2^i(k\tau) \\ &= \int_0^{\tau} e^{A_{11}^i (\tau-\theta)} A_{12}^i e^{A_{22}^i \theta} d\theta \, \widetilde{x}_2^i(k\tau) \\ &= V^i \widetilde{x}_2^i(k\tau) \end{aligned}$$

where $V^i = \int_0^{\tau} e^{A_{11}^i (\tau-\theta)} A_{12}^i e^{A_{22}^i \theta} d\theta$ can be calculated offline.

We modify the input to contain two components as follows:

$$u(t) = \widetilde{u}^i(t) - L^i \widetilde{x}_1^i(t), \; t \in [k\tau, (k+1)\tau) \tag{5}$$

where $\widetilde{u}^i(t) = -(B_1^i)' e^{(A_{11}^i)'((k+1)\tau - t)} (\Gamma^i)^{-1} v^i((k+1)\tau)$ and Γ^i is the controllability Gramian

$$\begin{aligned} \Gamma^i &= \int_{k\tau}^{(k+1)\tau} e^{A_{11}^i ((k+1)\tau-\theta)} B_1^i (B_1^i)' e^{(A_{11}^i)'((k+1)\tau-\theta)} d\theta \\ &= \int_0^{\tau} e^{A_{11}^i \theta} B_1^i (B_1^i)' e^{(A_{11}^i)'\theta} d\theta. \end{aligned}$$

The controllability Gramian Γ^i is full rank for any $\tau > 0$ since (A_{11}^i, B_1^i) is controllable. Also, Γ^i can be calculated offline.

As a result, at $t = (k+1)\tau$

$$\begin{aligned} \widetilde{x}_1^i((k+1)\tau)) &= e^{A_{11}^i \tau} \widetilde{x}_1^i(k\tau) + v^i((k+1)\tau) \\ &\quad + \int_{k\tau}^{(k+1)\tau} e^{A_{11}^i ((k+1)\tau-\theta)} B_1^i \widetilde{u}^i(\theta) d\theta \\ &\quad - \int_{k\tau}^{(k+1)\tau} e^{A_{11}^i ((k+1)\tau-\theta)} B_1^i L^i \widetilde{x}_1^i(\theta) d\theta. \end{aligned}$$

Since

$$\begin{aligned} &\int_{k\tau}^{(k+1)\tau} e^{A_{11}^i ((k+1)\tau-\theta)} B_1^i \widetilde{u}^i(\theta) d\theta \\ &= -\Gamma^i (\Gamma^i)^{-1} v((k+1)\tau) \\ &= -v^i((k+1)\tau) \end{aligned}$$

we have

$$\begin{aligned} \widetilde{x}_1^i((k+1)\tau)) &= e^{A_{11}^i \tau} \widetilde{x}_1^i(k\tau) \\ &\quad - \int_{k\tau}^{(k+1)\tau} e^{A_{11}^i ((k+1)\tau-\theta)} B_1^i L^i \widetilde{x}_1^i(\theta) d\theta \end{aligned}$$

which is the solution of $\dot{\widetilde{x}}_1^i = A_c^i \widetilde{x}_1^i$, where $A_c^i = A_{11}^i - B_1^i L^i)$, at $t = (k+1)\tau$, namely, $\widetilde{x}_1^i((k+1)\tau)) = e^{A_c^i \tau} \widetilde{x}_1^i(k\tau)$. Since (A_{11}^i, B_1^i) is controllable, the poles of A_c^i can be arbitrarily assigned. This implies that for any $0 < \gamma_c^i < 1$, we can design L^i such that $\|A_c^i\| \leq \gamma_c^i$, and γ_c^i will be selected later.[3]

We comment that although $\widetilde{u}^i(t)$ cannot cancel $v^i(t)$, which represents the impact of the coupling term $A_{12}^i \widetilde{x}_2^i(t)$, for all t, it cancels its impact at the final time $t = (k+1)\tau$. Consequently, it eliminates the coupling at the sampling points.

D. Almost-Sure Stability

On the other hand, when $\alpha_k \neq i$, the dynamics of $\widetilde{x}_1^i$ needs to be localized to resolve the issue of instability from subsystem interactions, since it cannot be controlled. This is stated as the following assumption.

Assumption 3.1: When $\alpha_k = j \neq i$, the dynamics of $\widetilde{x}_1^i$ is local, namely

$$\dot{\widetilde{x}}_1^i(t) = A_j^i \widetilde{x}_1^i(t), \quad j \neq i, t \in [k\tau, (k+1)\tau) \tag{6}$$

for some matrices A_j^i.

Remark 3.1: The matrix A_j^i depends on the physical system, control design, and system decomposition. Since the system in (6) will run open loop without feedback correction, its actual values are only relevant for obtaining its growth-rate bound used in the pole placement design.

By using the modified control design (5) with suitably selected L^i and under Assumption 3.1, the sampled values of the substate $\widetilde{x}_1^i$ at $k\tau$, $k = 0, 1, \ldots$, satisfy the closed-loop dynamics

$$\widetilde{x}_1^i((k+1)\tau) = \Phi_k^i \widetilde{x}_1^i(k\tau) \tag{7}$$

[3]When the dimension of the input $\rho > 1$, for the selected pole positions of A_c^i, the feedback gain L^i is not unique. This nonuniqueness does not affect the convergence results.

where

$$\Phi_k^i = I_{\{\alpha_k=i\}} e^{A_c^i \tau} + \sum_{j\neq i} I_{\{\alpha_k=j\}} e^{A_j^i \tau}. \tag{8}$$

If we concentrate on the controllable substates

$$\widetilde{x}_1 = \begin{bmatrix} \widetilde{x}_1^1 \\ \vdots \\ \widetilde{x}_1^m \end{bmatrix}$$

then Assumption 3.1 implies

$$\widetilde{x}_1((k+1)\tau) = \begin{bmatrix} \Phi_k^1 & & 0 \\ & \ddots & \\ 0 & & \Phi_k^m \end{bmatrix} \widetilde{x}_1(k\tau). \tag{9}$$

In other words, the local dynamics given by (6) and the decoupling control design imply that the controllable subsystems form a diagonal structure. This structure will be extended to a triangularly structured interaction later.

Denote

$$\gamma_o^i = \max_{j\neq i} \max_{\underline{\tau}\le\tau\le\overline{\tau}} \|e^{A_j^i \tau}\| \text{ and } \gamma_c^i = \max_{\underline{\tau}\le\tau\le\overline{\tau}} \|e^{A_c^i \tau}\|. \tag{10}$$

Since the eigenvalues of A_c^i can be arbitrarily placed, for any $0<\gamma^*<1$, there exists a feedback gain L^i such that

$$\gamma^i = (\gamma_c^i)^{p_i} (\gamma_o^i)^{(1-p_i)} \le \gamma^* < 1. \tag{11}$$

We recall that for a positive-valued stochastic process $\{\eta_k\}$, it is said to converge to 0 almost surely and exponentially if $\lim_{k\to\infty} \frac{1}{k} \ln \eta_k = -r$, w.p.1 for some $r>0$.

Theorem 3.1: Under Assumptions 3.1 and (11), $\widetilde{x}_1^i(k\tau)$ converges to 0 almost surely and exponentially.

Proof: From the expression of Φ_k^i in (7), we have

$$\gamma_k = \|\Phi_k^i\| \le I_{\{\alpha_k=i\}} \gamma_c^i + I_{\{\alpha_k\neq i\}} \gamma_o^i.$$

Denote $e_k^i = \|\widetilde{x}_1^i(k\tau)\|$. From $\widetilde{x}_1^i((k+1)\tau) = \Phi_k^i \widetilde{x}_1^i(k\tau)$, we have $e_{k+1}^i \le \gamma_k e_k$ and

$$e_K^i \le \left(\Pi_{k=0}^{K-1} \gamma_k\right) e_0^i.$$

It follows that

$$\begin{aligned} \frac{1}{K} \ln e_K^i &\le \frac{1}{K}\left(\sum_{k=0}^{K-1} \ln \gamma_k + \ln e_0^i\right) \\ &\le \frac{1}{K} \sum_{k=0}^{K-1} \left(I_{\{\alpha_k=i\}} \ln \gamma_c^i + I_{\{\alpha_k\neq i\}} \ln \gamma_o^i\right) + \frac{1}{K} \ln e_0^i. \end{aligned}$$

Since α_k is i.i.d., by the Strong Laws of Large Numbers

$$\begin{aligned} \frac{1}{K} \ln e_K^i &\le \frac{1}{K} \sum_{k=0}^{K-1} \left(I_{\{\alpha_k=i\}} \ln \gamma_c^i + I_{\{\alpha_k\neq i\}} \ln \gamma_o^i\right) + \frac{1}{K} \ln e_0^i \\ &\to E(I_{\{\alpha_1=i\}} \ln \gamma_c^i + I_{\{\alpha_1\neq i\}} \ln \gamma_o^i) \text{ w.p.1 as } K\to\infty \\ &= p_i \ln \gamma_c^i + (1-p_i) \ln \gamma_o^i \\ &= \ln (\gamma_c^i)^{p_i} (\gamma_o^i)^{(1-p_i)} \\ &\le \ln \gamma^* \\ &< 0. \end{aligned}$$

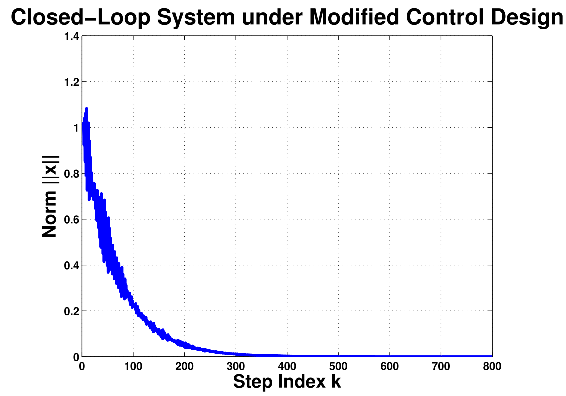

Fig. 2. Closed-loop system trajectories under the modified control design.

As a result, e_K^i converges to 0 almost surely and exponentially. ■

Example 3.2: Consider the system in Example 3.1, with the same $\tau=0.01$, $p_1=0.5$, and $p_2=0.5$. This system satisfies Assumption 3.1.

If we concentrate on $\widetilde{x}_1^1 = x_1$, its dynamics is governed by

$$\begin{aligned} \alpha_k = 1: \quad \dot{x}_1 &= 2x_1 + ax_2 + u \\ \dot{x}_2 &= 2x_2 \\ \alpha_k = 2: \quad \dot{x}_1 &= 2x_1. \end{aligned}$$

To eliminate the interaction term when $\alpha_k=1$, we employ the modified input design. It can be calculated as

$$\Gamma^1 = \int_0^\tau e^{4\theta} d\theta = \frac{1}{4}(e^{4\tau}-1),$$

$$V^1 = a\int_0^\tau e^{2(\tau-\theta)} e^{2\theta} dt = a e^{2\tau}\tau.$$

As a result, $u(t) = -10x_1(t) - a e^{2((k+1)\tau - t)} \frac{4e^{2\tau}\tau}{e^{4\tau}-1} x_2(k\tau), t\in k\tau,(k+1)\tau)$. The closed-loop system for $x_1(k\tau)$ is

$$x_1((k+1)\tau) = \left(I_{\{\alpha_k=1\}} e^{-8\tau} + I_{\{\alpha_k=2\}} e^{2\tau}\right) x_1(k\tau).$$

Similarly

$$x_2((k+1)\tau) = \left(I_{\{\alpha_k=1\}} e^{2\tau} + I_{\{\alpha_k=2\}} e^{-8\tau}\right) x_2(k\tau).$$

The resulting closed-loop system is independent of a, namely the substate coupling has been eliminated.

Fig. 2 shows a sample-path trajectory for the closed-loop system. Now the closed-loop system is stable for any a. The value $a=10$ is used in this simulation.

E. Triangular System Structure and Stability Analysis

Example 3.1 shows that substate coupling can destabilize the closed-loop system. The modified control design in (5) and Assumption 3.1 provide a sufficient condition for achieving stabilization. On the other hand, Assumption 3.1 is sometimes restrictive. This subsection will extend the permitted system structure to a triangular structure defined as follows, which is substantially more general than the diagonal structure imposed by Assumption 3.1. Without loss of generality, the following definition is given in the order $1,2,\ldots,m$.

Assumption 3.2: The dynamics of the controllable substate $\widetilde{x}_1^i$ of the ith subsystem satisfies the following triangular interaction structure: When $\alpha_k = j \neq i$

$$\dot{\widetilde{x}}_1^i(t) = A^i_{j,i}\widetilde{x}^i_i + \sum_{\ell=1}^{i-1} A^i_{j,\ell}\widetilde{x}^\ell_1, \quad i = 1, \ldots, m.$$

This structure means that when $\alpha_k = j \neq i$, $\widetilde{x}^i_1$ will run open-loop and this open-loop dynamics depends on the controllable substates $\widetilde{x}^\ell_1$ of the other subsystems only for $\ell < i$. This implies that for $\alpha_k = j \neq i$

$$\widetilde{x}^i_1((k+1)\tau) = e^{A^i_{j,i}\tau}\widetilde{x}^i_1(k\tau) + \sum_{\ell=1}^{i-1} H^i_{j,\ell}\widetilde{x}^\ell_1(k\tau) \tag{12}$$

for some matrices $H^i_{j,\ell}$. Now, the diagonal structure of (9) is expanded into

$$\widetilde{x}_1((k+1)\tau) = \begin{bmatrix} \Phi_{11} & 0 & \cdots & 0 \\ \Phi_{21} & \Phi_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \Phi_{m1} & \Phi_{m2} & \cdots & \Phi_{mm} \end{bmatrix} \widetilde{x}_1(k\tau) \tag{13}$$

where

$$\Phi_{ii} = I_{\{\alpha_k=i\}}e^{A^i_c\tau} + \sum_{j\neq i} I_{\{\alpha_k=j\}}e^{A^i_j\tau}, \quad i = 1, \ldots, m$$

$$\Phi_{ij} = \sum_{\ell\neq i} I_{\{\alpha_k=\ell\}}H^i_{\ell,j}, j < i$$

which is a triangular structure of subsystem interactions.

Theorem 3.2: Under Assumption 3.2 and the modified control design satisfying (11), there exist feedback gains $L^i, i \in \mathcal{S}$, such that $x(k\tau)$ converges to 0 exponentially as $k \to \infty$ almost surely.

Proof: We prove this theorem by induction on i. To simplify the statements, "converges" in this proof means "converges to 0 almost surely and exponentially."

For $i = 1$,

$$\|\widetilde{x}^1_1((k+1)\tau)\| \leq \left(I_{\{\alpha_k=1\}}\|e^{A^1_c\tau}\| + I_{\{\alpha_k\neq 1\}}\gamma^1_o\right)\|\widetilde{x}^1_1(k\tau)\|.$$

Since this actually satisfies Assumption 3.1, there exists L^1 such that $\|\widetilde{x}^1_1(k\tau)\|$ converges.

Suppose that for $i = 1, \ldots, \ell$, $\widetilde{x}^\ell_1$ converges. Then, for $i = \ell + 1$

$$\begin{aligned} &\|\widetilde{x}^{i+1}_1((k+1)\tau)\| \\ &\leq (I_{\{\alpha_k=i+1\}}\gamma^{i+1}_c + I_{\{\alpha_k\neq i+1\}}\gamma^{i+1}_o)\|\widetilde{x}^{i+1}_1(k\tau)\| \\ &\quad + I_{\{\alpha_k\neq i+1\}}h^{i+1}\max_{\ell=1,\ldots,i-1}\|\widetilde{x}^\ell_1(k\tau)\| \end{aligned} \tag{14}$$

for some $h^{i+1} \geq 0$ and γ^{i+1}_c can be designed to be arbitrarily small. Consequently, following the same arguments as in the proof of Theorem 3.1, L^{i+1} can be designed so that the uncoupled $\widetilde{x}^{i+1}(k\tau)$, namely the solution of (14) when $h^{i+1} = 0$, converges.

Furthermore, in (14), $\|\widetilde{x}^\ell_1(k\tau)\|$, $\ell = 1, \ldots, i-1$, plays the role of exponentially decaying inputs. Consequently, as the response of an exponentially stable system to this input, $\|\widetilde{x}^{i+1}_1(k\tau)\|$ converges. As a result, for all $i = 1, \ldots, m$, $\widetilde{x}^i_1(k\tau)$ converges. This implies that $\widetilde{x}(k\tau)$ converges.

Finally, by Assumption 2.2, $W_{\mathcal{S}}$ is full rank. By Lemma 3.1, G is full column rank. From $x(k\tau) = (G'G)^{-1}G'\widetilde{x}(k\tau)$, we conclude that $x(k\tau)$ converges. ■

IV. Robust Stabilization Under Modeling Errors

In this section, we take into consideration modeling errors and establish the robustness of the control design developed in the previous sections. This problem is motivated by the typical scenario of local linearization: when one starts with a nonlinear system $\dot{x} = f_0(x, u)$ where $f_0(\cdot, \cdot)$ is continuously differentiable, by using a state feedback $u = q(x)$, the closed-loop system $\dot{x} = f_0(x, u) = f_0(x, q(x)) = f(x)$ is nonlinear. Its linearization around an equilibrium point x_0, namely $f(x_0) = 0$, can be represented by a dynamic system on the perturbation variables $\Delta x = x - x_0$ with $\dot{\Delta x} = A\Delta x + \delta(\Delta x)$, where the matrix A is the Jacobian matrix at x_0 and the modeling error term $\delta(\Delta x)$ is nonlinear. For our results to be applicable to this ubiquitous practical situation, we must establish robustness against (small) nonlinear modeling errors.

For simplicity and clarity, we consider the systems that satisfy Assumption 3.1 (namely, systems with diagonal interactions among subsystems).

For the ith subsystem, consider the (potentially nonlinear) modeling errors δ

$$\begin{aligned} &\text{when} \quad \alpha_k = j \neq i: \\ &\qquad \dot{\widetilde{x}}^i_1(t) = A^i_j\widetilde{x}^i_1(t) + \delta^i_j(\widetilde{x}^i_1(t)), \quad t \in [k\tau, (k+1)\tau) \\ &\text{when} \quad \alpha_k = i: \\ &\qquad \dot{\widetilde{x}}^i_1(t) = A^i_c\widetilde{x}^i_1(t) + \delta^i(\widetilde{x}^i_1(t)), \quad t \in [k\tau, (k+1)\tau). \end{aligned}$$

The modeling errors satisfy the following conditions on their growth rates and error bounds.

Assumption 4.1:

1) For some $\epsilon^i_j > 0$, $\|\delta^i_j(\widetilde{x}^i_1)\| \leq \epsilon^i_j\|\widetilde{x}^i_1\|$, $i = 1, \ldots, m, j \neq i$.
2) For some $\epsilon^i > 0$, $\|\delta^i(\widetilde{x}^i_1)\| \leq \epsilon^i\|\widetilde{x}^i_1\|$, $i = 1, \ldots, m$.

Let $\bar{\lambda}^i_j$ be the largest eigenvalue of A^i_j. The following lemma establishes the maximum growth rate of $\widetilde{x}^i_1(k\tau)$ when $\alpha_k = j \neq i$, namely when the ith subsystem runs open loop. It is noted that the robustness, defined by the error bound κ^i_j in Lemma 4.1, depends on A^i_j.

Lemma 4.1: For any given $\lambda > \bar{\lambda}^i_j$, there exist $\kappa^i_j > 0$ such that if $\epsilon^i_j < \kappa^i_j$, then for some $c > 0$,

$$\|\widetilde{x}^i_1((k+1)\tau)\| \leq ce^{\lambda\tau}\|\widetilde{x}^i_1(k\tau)\|.$$

Proof: Consider the nominal system $\dot{\widetilde{x}}^i_1(t) = A^i_j\widetilde{x}^i_1(t)$. Define $y(t) = e^{-\lambda t}\widetilde{x}^i_1(t)$. Then

$$\begin{aligned} \dot{y}(t) &= e^{-\lambda t}\dot{\widetilde{x}}^i_1 - \lambda e^{-\lambda t}\widetilde{x}^i_1(t) \\ &= e^{-\lambda t}A^i_j\widetilde{x}^i_1(t) - \lambda y(t) \\ &= (A^i_j - \lambda I)y(t) \\ &= \widetilde{A}^i_j y \end{aligned}$$

where by hypothesis, all eigenvalues of $\widetilde{A}^i_j = A^i_j - \lambda I$ are in the open left half plane. As a result, the Lyapunov equation $(\widetilde{A}^i_j)^T P^i_j + P^i_j \widetilde{A}^i_j = -I$ has a unique positive definite solution $P^i_j > 0$. Let the largest eigenvalue of P^i_j be η^i_j.

Now, consider the system with modeling errors $\dot{\widetilde{x}}^i_1(t) = A^i_j \widetilde{x}^i_1(t) + \delta^i_j(\widetilde{x}^i_1(t))$. Define the Lyapunov candidate $V = y^T P^i_j y$, which is globally positive definite and radially unbounded. We have

$$\begin{aligned}\dot{V} &= \dot{y}^T P^i_j y + y^T P^i_j \dot{y}\\ &= (\widetilde{A}^i_j y + e^{-\lambda t}\delta^i_j(\widetilde{x}^i_1))^T P^i_j y + y^T P^i_j(\widetilde{A}^i_j y + e^{-\lambda t}\delta^i_j(\widetilde{x}^i_1))\\ &= -y^T y + 2y^T P^i_j e^{-\lambda t}\delta^i_j(\widetilde{x}^i_1).\end{aligned}$$

Furthermore

$$\begin{aligned}|2y^T P^i_j e^{-\lambda t}\delta^i_j(\widetilde{x}^i_1)| &\le 2\|y\|\eta^i_j e^{-\lambda t}\|\delta^i_j(\widetilde{x}^i_1)\|\\ &\le 2\|y\|\eta^i_j e^{-\lambda t}\epsilon^i_j\|\widetilde{x}^i_1\|\\ &= 2\|y\|\eta^i_j e^{-\lambda t}\epsilon^i_j e^{\lambda t}\|y\|\\ &= 2\eta^i_j\epsilon^i_j y^t y.\end{aligned}$$

Define $\kappa^i_j = \frac{1}{2\eta^i_j}$. If $\epsilon^i_j < \kappa^i_j$, then $1 - 2\eta^i_j\epsilon^i_j > 0$ and

$$\dot{V} \le -(1 - 2\eta^i_j\epsilon^i_j)y^T y$$

is globally negative definite. Consequently, the y system is globally asymptotically stable, which implies that $\|y((k+1)\tau)\| \le c\|y(k\tau)\|$ for some constant c. It follows that

$$\begin{aligned}\|\widetilde{x}^i_1((k+1)\tau)\| &= e^{\lambda(k+1)\tau}\|y((k+1)\tau)\|\\ &\le e^{\lambda(k+1)\tau}c\|y(k\tau)\|\\ &= e^{\lambda(k+1)\tau}ce^{-\lambda k\tau}\|x(k\tau)\|\\ &= ce^{\lambda\tau}\|x(k\tau)\|.\end{aligned}$$

■

Since c and λ in Lemma 4.1 depend on A^i_j, we relabel them as c^i_j and λ^i_j, and denote $\gamma^i_j = c^i_j e^{\lambda^i_j \tau}$.

On the other hand, when $\alpha_k = i$, the (stable) eigenvalues of A^i_c can be arbitrarily assigned. Select the eigenvalues of A^i_c as $\{-\lambda_1, \ldots, -\lambda_n\}$ with $\lambda_j > 0$, $j = 1, \ldots, n$, and $\bar{\lambda}^i = \min\{\lambda_1, \ldots, \lambda_n\}$. Since the eigenvalues of A^i_c can be arbitrarily assigned, so is $\bar{\lambda}^i$. The following lemma establishes the least decaying rate of $\widetilde{x}^i_1(k\tau)$ when $\alpha_k = i$. It is noted that the robustness, defined by the error bound κ^i in Lemma 4.2, depends on A^i_{11} and also the feedback design.

Lemma 4.2: For any given $0 < \lambda < \bar{\lambda}^i$, there exists $\kappa^i > 0$ such that if $\epsilon^i < \kappa^i$, then for some $c > 0$

$$\|\widetilde{x}^i_1((k+1)\tau)\| \le ce^{-\lambda\tau}\|\widetilde{x}^i_1(k\tau)\|.$$

Since the proof of Lemma 4.2 is nearly identical to that of Lemma 4.1, it is omitted. Also, since c and λ in Lemma 4.2 depend on A^i_c, we relabel them as c^i_c and λ^i_c, and denote $\gamma^i_c = c^i_c e^{\lambda^i_c \tau}$.

Theorem 4.1: Under Assumption 4.1, there exist $\kappa^i_j > 0$ and $\kappa^i > 0$ such that if $\epsilon^i_j < \kappa^i_j$ and $\epsilon^i_j < \kappa^i_j$, then the control design specified in the previous sections achieves convergence of $\widetilde{x}^i_1(k\tau)$ to 0 almost surely, exponentially, and robustly.

Proof: Under the hypothesis, by Lemmas 4.1 and 4.2, we have

$$\begin{aligned}&\text{when}\quad \alpha_k = j \ne i:\\ &\qquad \|\widetilde{x}^i_1((k+1)\tau) \le \gamma^i_j\|\widetilde{x}^i_1(k\tau)\|\\ &\text{when}\quad \alpha_k = i:\\ &\qquad \|\widetilde{x}^i_1((k+1)\tau) \le \gamma^i_c\|\widetilde{x}^i_1(k\tau)\|.\end{aligned}$$

Since $\gamma^i_c > 0$ can be arbitrarily selected, for any $0 < \gamma^* < 1$, it is always possible to achieve

$$\gamma_i = (\gamma^i_c)^{p_i}(\gamma^i_j)^{1-p_i} \le \gamma^* < 1. \tag{15}$$

Consequently, the critical condition (11) is satisfied robustly for all uncertain systems that satisfy Assumption 4.1. By Theorem 3.1, $\widetilde{x}^i_1(k\tau)$ converges to 0 almost surely, exponentially, and robustly. ■

V. Illustrative Example

Example 5.1: We consider an RSLS with two subsystems of dimension 4. The subsystem matrices are

$$A(1) = \begin{bmatrix}-3.5 & 2 & 4.5 & 0\\ 3 & -0.5 & -7.5 & 1.5\\ -4.5 & 2 & 5.5 & 0\\ -9.5 & 4.5 & 5 & 2.5\end{bmatrix}, \quad B(1) = \begin{bmatrix}1\\2\\1\\5\end{bmatrix}$$

$$A(2) = A(1), \quad B(2) = \begin{bmatrix}4\\11\\0\\13\end{bmatrix}.$$

The transformation matrix can be constructed as

$$T = \begin{bmatrix}1 & 0 & 2 & 0\\ 0 & 1 & 4 & 1\\ 1 & 0 & 0 & 0\\ 3 & 1 & 2 & 3\end{bmatrix}$$

with transformed matrices

$$\widetilde{A}^1 = T^{-1}A(1)T = \begin{bmatrix}1 & 2 & -1 & 2\\ 0 & 1 & 2 & 3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}, \quad \widetilde{B}^1 = \begin{bmatrix}1\\2\\0\\0\end{bmatrix}$$

$$\widetilde{A}^2 = \widetilde{A}^1, \quad \widetilde{B}^2 = \begin{bmatrix}0\\0\\2\\3\end{bmatrix}.$$

We use the modified control design for state feedback and substate decoupling with

$\alpha_k = 1:$ The poles for A^1_c are $\{-10, -10\}$, resulting in

$$L^1 = [30.25, -4.125], \quad A^1_c = \begin{bmatrix}-29.25 & 6.125\\ -60.5 & 9.25\end{bmatrix};$$

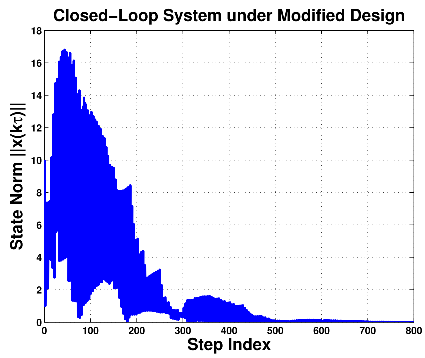

Fig. 3. Closed-loop system trajectories under the modified control design.

$\alpha_k = 2:$ The poles for A_c^2 are $\{-12, -12\}$, resulting in

$$L^2 = [-113.75, 84.5], \quad A_c^2 = \begin{bmatrix} 228.5 & -169 \\ 342.25 & -252.5 \end{bmatrix}.$$

For this system, since the same T is used, we have $\widetilde{x}^1 = \widetilde{x}^2 := \widetilde{x}$. The total closed-loop system for $\widetilde{x}$ is

$$\alpha_k = 1 : \dot{\widetilde{x}} = M_1 \widetilde{x}, \quad M_1 = \begin{bmatrix} -29.25 & 6.125 & 0 & 0 \\ -60.5 & 9.25 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\alpha_k = 2 : \dot{\widetilde{x}} = M_2 \widetilde{x}, \quad M_2 = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 228.5 & -169 \\ 0 & 0 & 342.25 & -252.5 \end{bmatrix}.$$

If we focus on the controllable substates

$$\widetilde{x}_1 = \begin{bmatrix} \widetilde{x}_1^1 \\ \widetilde{x}_1^2 \end{bmatrix}$$

the aforementioned subsystems imply that

$$\dot{\widetilde{x}}_1 = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ 0 & \Phi_{22} \end{bmatrix}$$

where

$$\Phi_{11} = I_{\{\alpha_k = 1\}} \begin{bmatrix} -29.25 & 6.125 \\ -60.5 & 9.25 \end{bmatrix} + I_{\{\alpha_k \neq 1\}} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\Phi_{22} = I_{\{\alpha_k = 2\}} \begin{bmatrix} 228.5 & -169 \\ 342.25 & -252.5 \end{bmatrix} + I_{\{\alpha_k \neq 2\}} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\Phi_{12} = I_{\{\alpha_k \neq 1\}} \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}.$$

This structure satisfies (13) (with an equivalent upper triangular structure). As a result, the closed-loop system is not decoupled, but it is in a triangular form.

Under $\tau = 0.01$, $p_1 = 0.5$, and $p_2 = 0.5$, Fig. 3 shows that the closed-loop system is almost surely stable. Due to switching and unobservable subsystems, the discrete state sequences x_k at the sampling points show jumps in their values. Consequently the trajectory in Fig. 3 shows jumping perturbations during transient periods.

VI. Conclusion

Complex RSLSs are often uncontrollable in a fixed configuration and from one control input. This article has revealed some distinct and fundamental complications in their almost-sure stabilization. Modified control strategies have been introduced and their stabilization capabilities established.

This article leaves many interesting open issues. First, it will be highly useful to apply the algorithms of this article to representative practical RSLSs, especially in emerging technology areas such as renewable power systems, smart grids, autonomous systems, etc. Furthermore, this article is limited to stabilization problems. Optimal control in RSLSs with uncontrollable subsystems is an important and technically challenging future direction.

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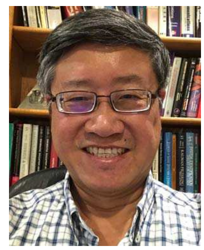

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