Joint Estimation of Continuous and Discrete States in Randomly Switched Linear Systems With Unobservable Subsystems

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***Abstract*—This article investigates the problem of joint continuous and discrete state estimation of randomly switched linear systems in which subsystems may not be observable. Estimation of both continuous state and discrete sequence simultaneously based on the same output observations is a challenging task that is inherently nonlinear and often infinite dimensional. This article presents necessary and sufficient conditions when joint estimation is possible without using a probing input. When such conditions are not satisfied, a suitably designed input must be used to achieve the goal of jointly detecting the subsystem and estimating the internal state. This article employs certain structures of randomly switched linear systems to develop algorithms that use finite-dimensional estimators for continuous states and sampled data for detecting the discrete states. The convergence analysis shows that this framework can achieve convergence. Examples and simulation case studies are presented to illustrate the main results of this article. The findings of this article can be used to form a supporting foundation for robust control.**

***Index Terms*—Convergence, hybrid system, input design, observability, observer design, randomly switched linear system (RSLS), stochastic distinguishability, stochastic joint observability.**

I. Introduction

THIS article studies the problem of joint continuous and discrete state estimation of randomly switched linear systems (RSLSs). RSLSs are an important class of hybrid systems that cover diversified applications in which linear continuous dynamics and discrete events interact to generate unique time-varying systems [1], [2], [3], [4], [5], [6], [7], [8]. RSLSs are common in practical systems, such as platoon reformulation in autonomous vehicles, contingency and faults in smart grids, interruptions and packet loss in networked communication systems, etc.

When the sequence of subsystems is known or can be directly measured, one may treat a switched linear system as a linear time-varying system and design control systems accordingly. In addition, if each subsystem is observable, then following the detection of the subsystem, observers can be designed to estimate the continuous states. The literature on observability, observer design, and other related properties in deterministic hybrid system frameworks is quite extensive including observability and controllability [2], observability notions and testing conditions [9], [10], [11], geometric methods [12], hybrid observability under input probing [13], observability, detectability, attractivity, observer design, and related convergence analysis [14], [15], [16], [17], among many others. A related study on the distinguishability of linear systems and switched linear systems was presented in [18] and [19] with a broad class of input signals and control schemes. A different notion of observability and distinguishability with an unknown switching time of jump linear systems was treated in [20].

In contrast, to capture the random nature of system switching in applications [21], [22], RSLSs were treated as stochastic hybrid systems, regime-switching systems, and hybrid switching diffusions [23], [24], [25, pp. 137–157], [26].

This article treats RSLSs with a different perspective from the aforementioned literature. When the switching sequence is random and unknown and subsystems are unobservable, the corresponding RSLSs are no longer amenable to treatment using existing techniques, and the associated estimation problems become highly challenging. The estimation problem is inherently nonlinear and often infinite dimensional; see the work on Wonham filters [27].

The main premise of this article reflects data collection and estimation scenarios in typical large-scale complex systems that involve many sensors for monitoring internal states and detecting faults. Using modern power systems (MPSs) as examples, cyber-physical contingencies include loss of distribution and transmission lines from natural causes, loss of generators,

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failure of communication systems, trips of line switches, etc. Mathematically, these sudden changes can be represented by jumps in system models from one configuration to another as discrete events. Due to the unpredictable and random nature of contingency and adversary events, they are stochastic and can be modeled as finite-state stochastic processes. MPSs employ phasor measurement units (PMUs), advanced metering infrastructures (AMIs), voltage transformers (VTs), and current transformers (CTs), and many other sensors to monitor system states, detect contingency, and support system operation.

In such systems, any single sensor or a local sensor cluster cannot provide sufficient information for state estimation of the entire MPS, resulting in unobservable subsystems. An information processing center (such as power system control centers) collects data from different sensor clusters and determines the current subsystem and then gradually obtains accurate and evolving continuous state estimation by using the data collaboratively. Within each time interval, only the observable subspace of the active subsystem can be estimated. The same subspace may become unobservable when the system is switched to another subsystem; hence, its estimator must run open-loop using dynamic models without correction and the estimation errors on the substates can grow exponentially.

In our recent paper [28], state estimation of RSLSs with unobservable subsystems was investigated under noise-free observations, known subsystems, and feedback-based linear observers. Properly designed subsystem observers and their suitable organization were shown to achieve strong and exponential convergence.

This article investigates the joint estimation of continuous and discrete states of RSLSs in which subsystems may not be observable. For such systems, both the continuous and discrete states must be determined from input–output data. The problem treated in this article encounters some fundamental new challenges. First, when the active subsystem is unknown, one cannot estimate the continuous state since the corresponding system matrices are unavailable. Second, it is possible that for different subsystems, certain special initial states may produce exactly the same output. The issue is further complicated when subsystems are unobservable, namely some initial states may produce zero outputs. Third, if one can use the input to probe subsystems, the ability to distinguish subsystems can be potentially enhanced. This added capability introduces new issues on input design, subsystem distinguishability, joint observability of both continuous and discrete states, and algorithm development.

This article contains the following original contributions.

i) It presents some necessary and sufficient conditions on joint observability by the output observation without probing inputs.
ii) The probing capability of the input is investigated under zero initial conditions. When the conditions in (i) are not met, the input can be used. It is shown that as long as the subsystem transfer functions are distinct, all nonvanishing input signals with strictly proper rational Laplace transforms can distinguish subsystems in an infinitesimal time.
iii) For practical implementation, data must be sampled. A sampling theorem is presented that provides sampling rate conditions for distinguishing subsystems in a finite time interval.
iv) Joint observability for both continuous and discrete states is further developed for estimating the subsystems and continuous states simultaneously under probing inputs. Mode-modulated and magnitude-modulated input design methods are introduced.
v) A time-division framework and the corresponding algorithms for jointly estimating continuous and discrete states are developed, and their convergence properties are established.

The rest of the article is organized as follows. Section II contains notations, system descriptions, and basic definitions of distinguishability and joint observability for RSLSs. Section III studies joint observability without a probing input. Necessary and sufficient conditions are presented for both observable and unobservable subsystems. The distinguishability of subsystems by using probing inputs under zero initial states is discussed in Section IV. Section V explores input design, estimation algorithms, sampling theorems, and joint observability under a probing input. Also, mode-modulated and magnitude-modulated input design methods are detailed, and a time-division framework and estimation algorithms are developed. The main results of this article are illustrated by some examples in Section VII. Finally, the main findings and their potential extensions are summarized in Section VIII.

II. Preliminaries

Denote by $\mathbb{R}$ the field of real numbers and $\mathbb{C}$ the field of complex numbers. For a column vector $v \in \mathbb{R}^n$, $\|v\|$ is its Euclidean norm. For a matrix $M \in \mathbb{R}^{n\times m}$, M' is its transpose, $\lambda(M)$ an eigenvalue of M, $\sigma(M) = \sqrt{\lambda(M'M)}$ a singular value of M, $\sigma_{\min}(M)$ its minimum singular value, and $\sigma_{\max}(M)$ its largest singular value. The value $\sigma_{\max}(M)$ is also its operator norm induced by the Euclidean norm $\sigma_{\max}(M) = \|M\| = \sup_{\|v\|=1} \|Mv\|$. The kernel or null space of $M \in \mathbb{R}^{n\times m}$ is $\ker(M) = \{x \in \mathbb{R}^m : Mx = 0\}$ and its range is $\text{Range}(M) = \{y = Mx : x \in \mathbb{R}^m\}$.

For a subspace $\mathbb{U} \subseteq \mathbb{R}^n$ of dimension $p \le n$, a matrix $M \in \mathbb{R}^{n\times p}$ is said to be a base matrix of $\mathbb{U}$, written as $M = \text{Base}(\mathbb{U})$, if the column vectors of M are linearly independent and $\text{Range}(M) = \mathbb{U}$.

A function $y(t) \in \mathbb{R}$ in a time interval $[0, T)$ is piecewise continuously differentiable if $[0, T)$ can be divided into a finite number of subintervals $[t_{k-1}, t_k)$, $k = 1, \ldots, \ell$, $t_0 = 0$, $t_\ell = T$ such that $y(t)$ is right continuous in $[t_{k-1}, t_k)$ and continuously differentiable, to any order as needed, in (t_{k-1}, t_k). The space of such functions is denoted by $\mathcal{C}[0, T)$.

Consider a continuous-time single-input-single-output RSLS

$$\begin{cases} \dot{x}(t) = A(\alpha(t))x(t) + B(\alpha(t))u(t) \\ y(t) = C(\alpha(t))x(t) \end{cases} \tag{1}$$

where $u(t) \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$ are the input, state, and output, respectively.

The system matrices depend on the randomly switching process $\alpha(t)$ that takes m possible values in a discrete state space $\mathcal{S} = \{1, \ldots, m\}$. For each given value $i \in \mathcal{S}$, the corresponding

(deterministic) linear time invariant (LTI) system in (1) with constant matrices $(C(i), A(i), B(i))$ is called *the ith subsystem* of the RSLS.

Assumption 2.1: Given $\tau > 0$, (i) the switching process $\alpha(t)$ can switch only at the sampling instants $k\tau$, $k = 0, 1, \ldots,$ that generates a stochastic sequence $\alpha(t) = \alpha_k$, $t \in [k\tau, (k+1)\tau)$. (ii) The sequence $\{\alpha_k\}$ is independent and identically distributed (i.i.d.) such that the probability $\Pr\{\cdot\}$ satisfies $\Pr\{\alpha_k = i\} = p_i > 0, i \in \mathcal{S}$ and $\sum_{i=1}^{m} p_i = 1$. (iii) α_k cannot be directly measured.

Remark 2.1: Although physical systems can change their structures and parameters at any time instant, all management platforms for practical systems have an interval (an internal clock) for data sampling and transfer, and for decision making. The interval depends on sensor, communication, and computer hardware/software systems. For example, for data acquisition accuracy and antialiasing requirements in PMU data, the data rate of the Power Xpert Meter is 1024 samples per cycle in power systems. For contingency management, 160 ms is the IEEE-imposed limit for voltage sag/surge. For power dispatch, a decision interval of 5 min is commonly used in practice. Mathematically, under Assumption 2.1, the randomly switching process can be treated as a discrete-time stochastic sequence, rather than a continuous-time process. Random switching time in continuous-time stochastic processes is beyond the scope of this article.

Under Assumption 2.1, the following stochastic matrix sequences are induced by α_k

$$A_k = A(\alpha_k) = \sum_{i=1}^{m} A(i)\mathbf{1}_{\{\alpha_k = i\}}$$

$$B_k = B(\alpha_k) = \sum_{i=1}^{m} B(i)\mathbf{1}_{\{\alpha_k = i\}}$$

$$C_k = C(\alpha_k) = \sum_{i=1}^{m} C(i)\mathbf{1}_{\{\alpha_k = i\}}$$

where $\mathbf{1}_V$ is the indicator function of the event V: $\mathbf{1}_V = 1$ if V is true; and $\mathbf{1}_V = 0$, otherwise. The sampled values of the signals are denoted by $x_k = x(k\tau)$ and $y_k = y(k\tau)$.

For constant $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{1\times n}$, and for a finite time interval $[0, \tau)$, consider the mapping $\mathcal{G} : \mathbb{R}^n \to \mathcal{C}[0, \tau)$ from the initial state $x(0) = x_0$ to $y(t)$, $y(t) = \mathcal{G}(x_0)(t) = Ce^{At}x_0, \quad t \in [0, \tau)$. The kernel of $\mathcal{G}$ is defined as $\mathrm{Ker}(\mathcal{G}) = \{x_0 \in \mathbb{R}^n : y(t) \equiv 0, t \in [0, \tau)\}$. Let W be the observability matrix of (C, A)

$$W = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

The following result is classical and well known.

Lemma 2.1: For any $\tau > 0$, $\mathrm{Ker}(\mathcal{G}) = \ker(W)$.

The premise of this article is to treat RSLSs whose initial states are unknown and whose switching sequence cannot be directly measured. As a result, both the continuous state x_k and discrete state α_k must be estimated from the known input $u(t)$ and observed output $y(t)$. The available dataset in a time interval $[0, T)$ is given by the noise-free dataset $\mathcal{D}_T = \{y(t), t \in [0, T)\}$ for a given $T > 0$.

Consider the RSLS in (1). For a given time interval $[0, \ell\tau)$, the sample path $\{\alpha_k, k = 0, \ldots, \ell - 1\}$ is unknown.

Definition 2.1:

1) The system is said to be *stochastically distinguishable* if $\{\alpha_k, k = 0, \ldots, \ell - 1\}$ can be uniquely determined from the dataset $\mathcal{D}_{\ell\tau}$.
2) The system is said to be *stochastically and jointly observable* if $\{\alpha_k, k = 0, \ldots, \ell - 1\}$ and $x(0)$ can be uniquely determined from the dataset $\mathcal{D}_{\ell\tau}$.

Remark 2.2: Since the stochastic distinguishability and stochastic joint observability are conditioned on the sample path α_k, they are random quantities. During system implementation, the sample path occurs (but unknown) and $\{\alpha_k, k = 0, \ldots, \ell - 1\}$ is to be estimated from the dataset $\mathcal{D}_{\ell\tau}$. In the subsequent analysis, we will focus on one time segment $[k\tau, (k+1)\tau)$. In particular, without loss of generality, we will use the first segment $[0, \tau)$ as a generic case for our study, focusing on the determination of α_0 and $x(0)$.

In RSLSs with unobservable subsystems, we must identify the subsystem even if we cannot estimate the continuous state accurately. In contrast, there exist several different definitions of distinguishability, such as these in [18], [19], and [20] that require the determination of both continuous states and discrete states simultaneously, which imply observable subsystems.[1] We will show that input assistance and design are critical. For conciseness, "stochastic distinguishability" will be abbreviated to "distinguishability" and "stochastic joint observability" to "joint observability."

Remark 2.3: Similar to the classical observability definition, if an RSLS is distinguishable and/or jointly observable, then theoretically the unique determination of α_0 and $x(0)$ can be achieved over an infinitesimal time interval. On the other hand, classical observer design uses the feedback-based observer structure for its computational simplicity, robustness, and much reduced memory complexity. Our theoretical results on the distinguishability and joint observability are stated for any $\tau > 0$ (so it can be infinitesimally small), but our observer algorithms will be feedback-based. Our subsequent convergence analysis shows that this observer structure is, indeed, suitable for the joint estimation tasks.

III. Joint Observability Without Probing Input

We start by considering the case of zero-input systems. The following example shows that in general, without input assistance, joint observability may be lost.

Example 3.1: Consider two first-order subsystems

$$\begin{cases} \dot{x}_1(t) = ax_1(t) + u(t) \\ y_1(t) = x_1(t). \end{cases}; \quad \begin{cases} \dot{x}_2(t) = ax_2(t) + u(t) \\ y_2(t) = 2x_2(t). \end{cases}$$

[1]The papers [18], [19], [20] consider more comprehensive problems of switching time estimation, smooth inputs, etc. These are beyond the scope of this article.

Although these two subsystems are different, they share the same eigenvalue a. Under the zero-input condition, their respective outputs are $y_1(t) = e^{at}x_1(0),\ y_2(t) = 2e^{at}x_2(0),\ t \in [0,\tau)$. For any initial state $x_1(0) \neq 0$, there exists $x_2(0) = 0.5x_1(0)$ such that $y_1(t) \equiv y_2(t),\quad t \in [0,\tau)$. Consequently, α_0 and $x(0)$ cannot be uniquely determined from the dataset $\mathcal{D}_\tau$.

On the other hand, we will show that the common eigenvalues are the only factor that will result in the loss of joint observability. Consider the zero-input case $u(t) \equiv 0$

$$\begin{cases} \dot{x}(t) = A(\alpha(t))x(t) \\ y(t) = C(\alpha(t))x(t). \end{cases} \tag{2}$$

For the ith subsystem in $\mathcal{S}$, $A(i)$ and $C(i)$ are constant matrices, and its observability matrix is

$$W(i) = \begin{bmatrix} C(i) \\ C(i)A(i) \\ \vdots \\ C(i)(A(i))^{n-1} \end{bmatrix}, \quad i = 1, \ldots, m. \tag{3}$$

The combined matrix for the set $\mathcal{S}$ is

$$W_{\mathcal{S}} = \begin{bmatrix} W(1) \\ W(2) \\ \vdots \\ W(m) \end{bmatrix}. \tag{4}$$

We note that $W(i)$ and $W_{\mathcal{S}}$ are deterministic matrices that contain only information on subsystems. They do not involve actual switching sequences. As a result, they can be evaluated offline.

A. Observable Subsystems

We first consider RSLSs whose subsystems are observable. Hence, $W(i)$, $i = 1, \ldots, m$, are full rank.

1) Joint Observability: Suppose that the RSLS starts from $x(0) = x_0$ and α_0 in the first interval $[0,\tau)$. Both x_0 and α_0 are unknown and must be jointly estimated from the output observation $y(t)$, $t \in [0,\tau)$ for any given $\tau > 0$. The unknown pair is denoted by $\xi_0 = (x_0, \alpha_0)$.

The output $y(t)$ is said to be *nonvanishing in* $[0,\tau)$, if $y(t) \not\equiv 0$, $t \in [0,\tau)$. It is noted that if $y(t) \equiv 0$, $t \in [0,\tau)$, it is obviously not possible to determine α_0, although $x(0) = 0$ can be concluded since all subsystems are observable.

Definition 3.1: The system in (2) is said to be *jointly observable*, if for any $\tau > 0$, ξ_0 can be uniquely determined from any nonvanishing $y(t)$, $t \in [0,\tau)$.

We first establish a basic necessary and sufficient condition for the separation of two subsystems. Consider any two subsystems with matrices $(C(i), A(i))$ and $(C(j), A(j))$, $i \neq j$. Their corresponding observability matrices are $W(i)$ and $W(j)$.

Lemma 3.1 is similar to a result in [20],[2] but with a different statement and a different and self-contained proof.

Lemma 3.1: Suppose that both subsystems are observable, namely $W(i)$ and $W(j)$ are full rank. Then ξ_0 can be uniquely determined from non-vanishing $y(t)$, $t \in [0,\tau)$ for any $\tau > 0$ if and only if the combined matrix

$$W = \begin{bmatrix} W(i) & -W(j) \\ W(i)(A(i))^n & -W(j)(A(j))^n \end{bmatrix} \in \mathbb{R}^{2n} \tag{5}$$

is full rank.

Proof: Without loss of generality, assume that the true ξ_0 is (x_0^i, i).

Consider the system

$$\begin{cases} \dot{z} = Az \\ v = Cz \end{cases} \tag{6}$$

where $A = \begin{bmatrix} A(i) & 0 \\ 0 & A(j) \end{bmatrix}$, $C = [C(i), -C(j)]$. Then, W in (5) is the observability matrix of (6). From the initial condition $z_0 = \begin{bmatrix} x_0^i \\ x_0^j \end{bmatrix}$, we have $v(t) = C(i)e^{A(i)t}x_0^i - C(j)e^{A(j)t}x_0^j$.

The "Only If" Part: Suppose that W is not full rank. Then, there exists initial value $z_0 \in \ker(W)$ with $z_0 \neq 0$ such that

$$v(t) = C(i)e^{A(i)t}x_0^i - C(j)e^{A(j)t}x_0^j \equiv 0, t \in [0,\tau).$$

This implies that if the true $\xi_0 = (x_0^i, i)$, then another $\bar{\xi}_0 = (x_0^j, j)$ will generate exactly the same $y(t), t \in [0,\tau)$. As a result, ξ_0 cannot be uniquely determined from $y(t)$, $t \in [0,\tau)$.

The "If" Part: Suppose that W is full rank. If there are (x_0^i, i) and (x_0^j, j) that generate the same $y(t) \not\equiv 0$, $t \in [0,\tau)$, then

$$v(t) = C(i)e^{A(i)t}x_0^i - C(j)e^{A(j)t}x_0^j \equiv 0, t \in [0,\tau).$$

Since the combined system (6) is observable, this implies $z_0 = 0$, namely, $x_0^i = 0$ and $x_0^j = 0$. Therefore, $y(t) \equiv 0$, $t \in [0,\tau)$, which is a contradiction since y is nonvanishing. As a result, α_0 can be uniquely determined. Once α_0 is determined, x_0^i can also be uniquely determined since $(C(i), A(i))$ is observable. ■

Theorem 3.1: Suppose that m subsystems $(C(i), A(i))$, $i = 1, \ldots, m$ are individually observable. Then, ξ_0 can be uniquely determined from nonvanishing $y(t)$, $t \in [0,\tau)$ if and only if $A(i)$ and $A(j)$ do not have common eigenvalues, for all $i \neq j$.

Proof: Let $(C(i), A(i))$ and $(C(j), A(j))$, $i \neq j$, be any two subsystems.

The "Only If" Part: Suppose that λ is a common real-valued eigenvalue of $A(i)$ and $A(j)$. Then, there exist $v_i \neq 0$ and $v_j \neq 0$ such that

$$A(i)v_i = \lambda v_i, \quad A(j)v_j = \lambda v_j.$$

Since $W(i)$ and $W(j)$ are full rank, $a_i = C(i)v_i \neq 0$ and $a_j = C(j)v_j \neq 0$.[3] Let $z_0 = \begin{bmatrix} a_j v_i \\ a_i v_j \end{bmatrix} \neq 0$. It can be directly verified that

$$Wz_0 = \begin{bmatrix} W(i) & -W(j) \\ W(i)(A(i))^n & -W(j)(A(j))^n \end{bmatrix} \begin{bmatrix} a_j v_i \\ a_i v_j \end{bmatrix}$$

[2]The paper [20] also treated the identification of switching time, which is beyond the scope of this article.

[3]If $a_i = C(i)v_i = 0$, then $C(i)(A(i))^j v_i = \lambda^j a_i = 0$, for all j, which implies that $W(i)$ is not full rank.

Algorithm 1:

1) For $i = 1, \ldots, m$, compute the observability Gramians: $\Gamma_i = \int_0^\tau e^{A'(i)t} C'(i) C(i) e^{A(i)t} dt$, and $Q_i = \int_0^\tau e^{A'(i)t} C'(i) y(t) dt$. Then, the estimate of x_0^i is $\widehat{x}_0^i = \Gamma_i^{-1} Q_i$. It is noted that since $(C(i), A(i))$ is observable, Γ_i is invertible for any $\tau > 0$.
2) Generate m errors:

$$e_i = \sup_{t \in [0,\tau)} \|y(t) - C(i) e^{A(i)t} \widehat{x}_0^i\|, \quad i = 1, \ldots, m.$$

3) Estimate ξ_0:

$$\widehat{\alpha}_0 = i^* = \arg \min_{i=1,\ldots,m} e_i, \qquad \text{Discrete State Estimation}$$

$$\widehat{x}_0 = x_0^{i^*}, \qquad \text{Initial State Estimation}$$

$$= \begin{bmatrix} a_j a_i - a_i a_j \\ (a_j a_i - a_i a_j)\lambda \\ \vdots \\ (a_j a_i - a_i a_j)\lambda^{2n-1} \end{bmatrix}$$

$$= 0.$$

This implies that W is not full rank. By Lemma 3.1, the system is not jointly observable. The proof for common complex-valued eigenvalues is similar and omitted.

The "If" Part: Suppose that $A(i)$ and $A(j)$ do not have common eigenvalues. Now, assume that there are (x_0^i, i) and (x_0^j, j) that generate the same $y(t)$. This implies that

$$v(t) = C(i) e^{A(i)t} x_0^i - C(j) e^{A(j)t} x_0^j \equiv 0, t \in [0, \tau).$$

Since both subsystems are observable, the modes in $C(i)e^{A(i)t}x_0^i$ and $C(j)e^{A(j)t}x_0^j$ are linearly independent, individually.

Furthermore, since $A(i)$ and $A(j)$ do not have common eigenvalues, the modes in $v(t) = C(i)e^{A(i)t}x_0^i - C(j)e^{A(j)t}x_0^j$ are jointly linearly independent. Consequently, the fact "$v(t) \equiv 0,\ t \in [0, T)$" implies that all coefficients are zero. Therefore

$$C(i) e^{A(i)t} x_0^i \equiv 0 \text{ and } C(j) e^{A(j)t} x_0^j \equiv 0, t \in [0, \tau).$$

By hypothesis, both subsystems are observable, which implies that $x_0^i = 0$ and $x_0^j = 0$. As a result, W is full rank. By Lemma 3.1, the system is jointly observable. □

2) Joint Estimation Algorithm: When the conditions of Theorem 3.1 are satisfied, Algorithm 1 can be used to compute ξ_0.

Note that if the true $\alpha_0 = i^*$, then $e_{i^*} = 0$. Under the conditions of Theorem 3.1, all other $e_j > 0$, $j \neq i^*$. Consequently, Algorithm 1 selects the true α_0, which then implies that the continuous state's initial value can be uniquely determined since the subsystem is observable.

We comment that Algorithm 1 is infinite dimensional since it involves the integration of data in the interval $[0, \tau)$.

B. Unobservable Subsystems

The conditions of Theorem 3.1 do not cover RSLSs whose subsystems are unobservable. Typically, one sensor does not have sufficient data for estimating the entire state. Hence, subsystems are typically not observable, violating the conditions of Theorem 3.1.

Assumption 3.1: (i) All subsystems are unobservable, namely, $\text{Rank}(W(i)) = n_i < n$, $i \in \mathcal{S}$. (ii) $W_{\mathcal{S}}$ is full column rank.

Remark 3.1: We assume that all subsystems are unobservable. If some subsystems are actually observable, then condition (ii) of Assumption 3.1 will be trivially satisfied. As a result, we focus on the nontrivial scenario given by condition (i) of Assumption 3.1. Condition (ii) of Assumption 3.1 ensures that their collective observable subspaces cover $\mathbb{R}^n$. This condition of Assumption 3.1 is a necessary condition in our development. To see this, if $W_{\mathcal{S}}$ is not full rank, then the kernel of $W_{\mathcal{S}}$ cannot be observed by any subsystem, see an example in [28].

Since the ith subsystem is not observable, namely $\text{Rank}(W(i)) = n_i < n$, we construct $M_i = \text{Base}(\ker(W(i))) \in \mathbb{R}^{n \times (n-n_i)}$ and select any $N_i \in \mathbb{R}^{n \times n_i}$ such that $T_i = [M_i, N_i]$ is invertible. The inverse of T_i is decomposed into $T_i^{-1} = \begin{bmatrix} K_i \\ F_i \end{bmatrix}$, where $K_i \in \mathbb{R}^{(n-n_i)\times n}$ and $F_i \in \mathbb{R}^{n_i \times n}$.

The state transformation $\widetilde{z}^i = T_i^{-1} x$ can be decomposed into $\widetilde{z}^i = T_i^{-1} x = \begin{bmatrix} K_i x \\ F_i x \end{bmatrix} = \begin{bmatrix} v^i \\ z^i \end{bmatrix}$ where $z^i \in \mathbb{R}^{n_i}$. Correspondingly, this coordinate transformation leads to the transformed matrices $A^i = T_i^{-1} A(i) T_i$, $C^i = C(i) T_i$, and A^i and C^i have the structure $A^i = \begin{bmatrix} A_{11}^i & A_{12}^i \\ 0 & A_{22}^i \end{bmatrix}$, $C^i = [0, C_2^i]$ with $A_{22}^i \in \mathbb{R}^{n_i \times n_i}$ and $C_2^i \in \mathbb{R}^{1 \times n_i}$.

As a result, if we focus only on the dynamics of the observable partial state z^i, we have

$$\begin{cases} \dot{z}^i = A_{22}^i z^i \\ y = C_2^i z^i \end{cases} \tag{7}$$

and (C_2^i, A_{22}^i) is observable.

Define the true observable substates $z = \begin{bmatrix} z^1 \\ \vdots \\ z^m \end{bmatrix} \in \mathbb{R}^{n_s}$ and their estimates, to be designed later, as $\widehat{z} = \begin{bmatrix} \widehat{z}^1 \\ \vdots \\ \widehat{z}^m \end{bmatrix} \in \mathbb{R}^{n_s}$. Let $z_k^i = z^i(k\tau)$.

Assumption 3.2: For all $i \neq j$, A_{22}^i and A_{22}^j do not have common eigenvalues.

Theorem 3.2: Under Assumption 3.2, if $y(t) \not\equiv 0, t \in [k\tau, (k+1)\tau)$, then $\alpha_k = i^*$ can be uniquely determined from

$\mathcal{D}_\tau = \{y(t), t \in [k\tau, (k+1)\tau)\}$. Furthermore, the observable partial state z_k^i can be uniquely determined.

Proof: Since $y(t) \not\equiv 0$, $[\mathrm{t} \in k\tau, (k+1)\tau)$, we have

$$y(t) = C_2^i e^{A_{22}^i(t-k\tau)} z_k^i.$$

Under Assumption 3.2, by limiting all subsystems to their observable partial states, the conditions of Theorem 3.1 hold. As a result, by applying Algorithm 1 to the observable partial states, $\alpha_k = i^*$ can be uniquely determined. Furthermore, since (C_2^i, A_{22}^i) is observable, z_k^i can be uniquely determined. ■

Example 3.2: Consider an RSLS with $\alpha_k \in \{1, 2, 3\}$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} x(t) \\ y(t) = C(\alpha(t))x(t) \end{cases} \tag{8}$$

with distinct a_1, a_2, a_3, and $C(1) = [1, 0, 0]$, $C(2) = [0, 1, 0]$, $C(3) = [0, 0, 1]$.

All three subsystems are unobservable. In this case, transformation matrices are

$$T_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with observable subspaces $(C_2^1, A_{22}^1) = (1, a_1)$, $(C_2^2, A_{22}^2) = (1, a_2)$, $(C_2^3, A_{22}^3) = (1, a_3)$.

Since a_1, a_2, a_3 are distinct, Assumption 3.2 is satisfied. It follows that α_k can be uniquely determined in each time interval under nonvanishing data.

IV. Distinguishability With Probing Input Under Zero Initial State

A. Distinguishability

Assumption 3.2 for subsystems is restrictive since a complex system often contains overlapping subspaces for different subsystems. In interconnected complex systems, switching of systems may be caused by an interruption of a communication channel, a sensor failure, a switching control, among many other scenarios. In such networked systems, switching usually affects only part of the system, leaving most of the system unchanged. These unchanged system components imply that some subsystems often have common components and therefore share common eigenvalues, violating the conditions of Theorem 3.1. In such situations, it is necessary to use probing inputs to assist in determining α_0 and $x(0)$. In this section, we explore the determination of the switching sequence with assistance from a probing input.

The algebra of proper rational functions of s over the field $\mathbb{R}$ is denoted by $\mathcal{R} = \left\{G(s) : G(s) = \frac{b(s)}{a(s)}\right\}$, where $a(s)$ and $b(s)$ are polynomials of s with real coefficients, and the order of $b(s)$ is less than or equal to the order of $a(s)$. The vanishing function $G(s) \equiv 0$ for all s is denoted by $\mathbf{0}$. The subalgebra of $\mathcal{R}$ that consists of nonvanishing and strictly proper (namely, the order of $b(s)$ is strictly less than the order of $a(s)$) rational functions is denoted by $\mathcal{R}_0$. It is easy to verify that $\mathcal{R}_0$ is an algebra over the field $\mathbb{R}$. Also, if $G(s) \in \mathcal{R}$, $G(s) \neq \mathbf{0}$, and $U(s) \in \mathcal{R}_0$, then $Y(s) = G(s)U(s) \in \mathcal{R}_0$.

For a one-sided time function $g(t)$, $t \geq 0$, its Laplace transform is $G(s) = \mathcal{L}\{g\}$, and correspondingly $g = \mathcal{L}^{-1}\{G(s)\}$. We say $g \in \mathcal{R}$ (or $g \in \mathcal{R}_0$) if and only if $G(s) = \mathcal{L}\{g\} \in \mathcal{R}$ (or $G(s) \in \mathcal{R}_0$).

Lemma 4.1: Suppose that $Y(s) \in \mathcal{R}_0$ and $y = \mathcal{L}^{-1}\{Y(s)\}$. For any $\tau > 0$, $y(t) \equiv 0$, $t \in [0, \tau)$ if and only if $Y(s) = \mathbf{0}$.

Proof: The "If" Part: Apparently, if $Y(s) = \mathbf{0}$, then $y(t) = \mathcal{L}^{-1}\{Y(s)\} \equiv 0$, $t \geq 0$.

Proof: The "Only If" Part: Suppose that $Y(s) \neq \mathbf{0}$. For $Y(s) = \frac{b(s)}{a(s)}$, suppose that the order of $a(s)$ is r. Let $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, q$ be the distinct roots of $a(s)$ of multiplicity r_i, $\sum_{i=1}^q r_i = r$. Since $Y(s)$ is rational and strictly proper, $y(t)$ is a linear combination of exponential polynomial functions

$$y(t) = \sum_{i=1}^{q} \sum_{j=1}^{r_i} c_{ij} t^{j-1} e^{\lambda_i t}$$

for some $c_{ij} \in \mathbb{C}$.[4] Since $Y(s) \neq \mathbf{0}$, in any time interval of finite length, $y(t)$ has only a finite number of zeros. It follows that $y(t) \not\equiv 0$, $t \in [0, \tau)$ for any $\tau > 0$. ■

In this article, the "input response" and the "zero-state response" have the same meaning, as do the "zero-input response" and the "initial-state response."

Consider two nonvanishing rational transfer functions $G_1(s), G_2(s) \in \mathcal{R}$. Suppose that the input $u(t)$, $t \geq 0$ has its Laplace transform $U(s) \in \mathcal{R}_0$, and the zero-state responses are denoted by $y_1(t) = G_1 u$ and $y_2(t) = G_2 u$, respectively.

Theorem 4.1: For any $\tau > 0$, $y_1(t) \equiv y_2(t)$, $t \in [0, \tau)$, if and only if $G(s) = G_1(s) - G_2(s) = \mathbf{0}$.

Proof: Obviously, if $G_1(s) - G_2(s) = \mathbf{0}$, then $y_1(t) \equiv y_2(t)$, $t \geq 0$.

Conversely, suppose that $G(s) = G_1(s) - G_2(s) \neq \mathbf{0}$. Then, $G(s) \in \mathcal{R}_0$. Since $U(s) \in \mathcal{R}_0$ and $\mathcal{R}_0$ is an algebra, $Y(s) = Y_1(s) - Y_2(s) = G(s)U(s) \in \mathcal{R}_0$, namely, $Y(s) \neq \mathbf{0}$. By Lemma 4.1, $y(t) \not\equiv 0$, $t \in [0, \tau)$, for any $\tau > 0$. ■

Theorem 4.1 immediately implies the following conclusion on distinguishability by using a rational probing input under zero initial conditions.

Corollary 4.1: Consider a set of m nonvanishing and distinct rational transfer functions $\mathbf{G} = \{G_1(s), \ldots, G_m(s)\}$ with $G_i(s) \in \mathcal{R}$, $i = 1, \ldots, m$, and $G_i(s) - G_j(s) \neq \mathbf{0}$ if $i \neq j$. Let $u(t)$ be a probing input whose Laplace transform $U(s) \in \mathcal{R}_0$. Suppose that $G^* \in \mathbf{G}$ is the true system and $y^* = G^* u$ is the true output under the zero initial condition. Then, for any $\tau > 0$, the observation data $y^*(t)$ in $[0, \tau)$ can uniquely determine G^*.

Remark 4.1: The condition that the probing input belongs to $\mathcal{R}_0$ is essential. Indeed, if $u(t)$ is a delayed input of $u_1(t)$, $t \geq 0$, with starting time $T > 0$ and $U_1(s) \in \mathcal{R}_0$, then its Laplace transform is $e^{-Ts}U_1(s) \notin \mathcal{R}_0$. In the time domain, $u(t) = u_1(t - T)$, $t \geq T$. Under this input, for all $G_i \in \mathbf{G}$, $y_i(t) = G_i u \equiv 0$, $t \in [0, T)$. As a result, the conclusions of Theorem 4.1 are no longer true.

[4] Although $\lambda_i \in \mathbb{C}$ and $c_{ij} \in \mathbb{C}$, $y(t)$ is still real valued.

Under the conditions of Theorem 4.1, a subsystem can be uniquely determined over an infinitesimal time interval. We should emphasize that this conclusion is under the zero initial condition. The combination of nonzero initial conditions and probing input significantly complicates the analysis and will be studied in Section V.

B. Sampling Theorem for Distinguishability

For practical implementation, data need to be sampled. When α_k is known, the feedback-based observer algorithms in [28] are shown to converge strongly and exponentially. In this section, we focus only on accurately determining α_k.

Let $0 < \tau_0 \ll \tau$. We will use data in $[0, \tau_0)$ to estimate α_0, so that the correct observer feedback can be designed and implemented in $[\tau_0, \tau)$. Let $\mathcal{D}_{\tau_0} = \{y(t), t \in [0, \tau_0)\}$. Take N samples of $y(t)$ in $[0, \tau_0)$ with sampling interval $\upsilon = \tau_0/N$, where N needs to be decided. Denote by $\widetilde{\mathcal{D}}_{\tau_0}$ the sampled dataset $\widetilde{\mathcal{D}}_{\tau_0} = \{y(\ell\upsilon), \ell = 0, 1, \ldots, N-1\}$. We will answer the following key questions: Will $\widetilde{\mathcal{D}}_{\tau_0}$ be sufficient for uniquely determining α_0? How many data points are needed? How can we estimate α_0? Can we estimate α_0 recursively?

We first establish a result on the number of zeros of exponential polynomial functions in a finite time interval. Suppose that a system $G(s) \in \mathcal{R}$ is non-vanishing, $U(s) \in \mathcal{R}_0$. Then, $Y(s) = G(s)U(s) \in \mathcal{R}_0$. Let the distinct poles of $G(s)U(s)$ be $\lambda_i \in \mathbb{C}$ with multiplicity m_i, $i = 1, \ldots, q$, whose real parts are $\Re(\lambda_i)$ and imaginary parts are $\Im(\lambda_i) \geq 0$. The order of $Y(s)$ is $r = \sum_{i=1}^{q} m_i$. Define $\omega_{\max} = \max_{i=1,\ldots,q} \Im(\lambda_i)$, and

$$\mu_{\tau_0} = 2(r-1) + \left\lceil \frac{\tau_0}{2\pi}\omega_{\max} \right\rceil \tag{9}$$

where $\lceil a \rceil$ is the smallest integer that is larger than or equal to the value a.

Since $Y(s) \in \mathcal{R}_0$, $y(t) = Gu$ is a linear combination of the modes from $\{\lambda_i, i = 1, \ldots, q\}$. As a result, it belongs to the class of exponential polynomials: for any $t \in [0, \tau_0)$

$$y(t) = \sum_{i=1}^{q} \sum_{j=1}^{m_i} v_{i,j} \frac{t^{j-1}}{(j-1)!} e^{\lambda_i t}. \tag{10}$$

Since the modes are linearly independent and Y is nonvanishing, the coefficients are not all zeros. The following key lemma on the number of zeros of exponential polynomials can be derived from [29].

Lemma 4.2: The number N_{τ_0} of zeros in $[0, \tau_0)$ of a nonvanishing exponential polynomial y defined in (10) is bounded by $N_{\tau_0} \leq \mu_{\tau_0}$.

Assumption 4.1: (i) The subsystem transfer functions $G_i \in \mathcal{R}$, $i = 1, \ldots, n$ are nonvanishing and distinct. (ii) The input $u \in \mathcal{R}_0$.

Under a given input u satisfying Assumption 4.1, the subsystem output $y_i = G_i u \in \mathcal{R}_0$. Let the poles of $G_i(s)U(s)$ be denoted by a_{ij}, $j = 1, \ldots, \zeta_i$ with real parts $\Re(a_{ij})$ and imaginary parts $\Im(a_{ij}) \geq 0$, and the order of $G_i(s)U(s)$ be r_i. Define

$$r_{\max} = \max_{i=1,\ldots,m} r_i$$

Algorithm 2: Estimation of α_0 Under Zero Initial State.

1) Initial Set of Discrete State: $\mathcal{S}(0) = \mathcal{S} = \{1, \ldots, m\}$. For $\ell = 0, \ldots, N-1$, perform the following recursion.
2) Observation Error Calculation: Calculate $|\delta_i(\ell\upsilon)| = |y_i^0(\ell\upsilon) - y^0(\ell\upsilon)|$, $i \in \mathcal{S}(\ell)$. Define $\Delta(\ell) = \{i \in \mathcal{S}(\ell) : |\delta_i(\ell\upsilon)| \neq 0\}$.
3) Discrete State Set Reduction: $\mathcal{S}(\ell+1) = \mathcal{S}(\ell) \ominus \Delta(\ell)$, where $\ominus$ is the set subtraction.
4) Identification of α_0: $\mathcal{S}(N)$ is a singleton, containing the estimate of the true i^*.

$$\omega_{\max} = \max_{i=1,\ldots,m} \max_{j=1,\ldots,\zeta_i} \Im(a_{ij}).$$

Define

$$N_0 = 2(2r_{\max} - 1) + \left\lceil \frac{\tau_0}{2\pi}\omega_{\max} \right\rceil.$$

Theorem 4.2: Under Assumption 4.1, if $N > N_0$, then α_0 can be uniquely determined from $\widetilde{\mathcal{D}}_{\tau_0}$.

Proof: Without loss of generality, suppose that the true $\alpha_0 = 1$. The true input response is $y_1^0(t) = G_1 u$, $t \in [0, \tau_0)$.

For any $i \neq 1$, the input response of the ith subsystem is $y_i^0(t) = G_i u$, $t \in [0, \tau_0)$. The difference is $\delta_i(t) = y_i^0(t) - y_1^0(t) = (G_i - G_1)u, t \in [0, \tau_0)$. Here, $(G_i - G_1)U \in \mathcal{R}_0$ whose order is less than or equal to $2r_{\max}$. By Lemma 4.2, the number of zeros of $\delta_i(t)$ is bounded by N_0. It follows that if the number N of samples in $[0, \tau_0)$ exceeds N_0, $\delta_i(\ell\upsilon) \not\equiv 0, \ell = 0, 1, \ldots, N-1$. This is sufficient to exclude G_i.

Since $i \neq 1$ is arbitrary, this proves that G_1 can be uniquely determined from $\widetilde{\mathcal{D}}_{\tau_0}$. ■

C. Recursive Estimation of α_k

Take $N > N_0$ samples of $y(t)$ in $[0, \tau_0)$ to obtain the sampled dataset $\widetilde{\mathcal{D}}_{\tau_0} = \{y(\ell\upsilon), \ell = 0, 1, \ldots, N-1\}$.

By Theorem 4.2, this algorithm will always terminate with a singleton $\mathcal{S}(N)$ that contains the true discrete state α_0. We emphasize that this conclusion is valid under the condition that the initial state is zero and the sampled dataset is noise free.

V. Joint Observability With Probing Input

When the input is applied to an unknown subsystem with the unknown initial state, the situation becomes substantially more complicated. To illustrate the complications involved in joint observability with the unknown initial state, we use an example to show that even with input assistance, joint observability can still be lost.

Example 5.1: Consider the following two subsystems, both are observable

$$\begin{cases} \dot{x}_1(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x_1(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y_1(t) = [1\ 2] x_1(t) \end{cases}$$

$$\begin{cases} \dot{x}_2(t) = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} x_2(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y_2(t) = [1\ 3] x_2(t). \end{cases}$$

Since the two systems share the common eigenvalue -1, they cannot be distinguished without assistance from a probing input. Their respective transfer functions are $G_1(s) = \frac{1}{s+1} + \frac{2}{s+2} = \frac{3s+4}{(s+1)(s+2)} \in \mathcal{R}$; $G_2(s) = \frac{1}{s+1} + \frac{3}{s+3} = \frac{4s+6}{(s+1)(s+3)} \in \mathcal{R}$.

Suppose that the input is the unit step $U(s) = \frac{1}{s} \in \mathcal{R}_0$. The total respective responses of the two subsystems are

$$y_1(t) = a_1 e^{-t} + a_2 e^{-2t} + 2 - e^{-t} - e^{-2t}, t \in [0, \tau)$$
$$y_2(t) = b_1 e^{-t} + b_2 e^{-2t} + 2 - e^{-t} - e^{-3t}, t \in [0, \tau)$$

where a_1, a_2, b_1, b_2 are determined by the initial states.

Their difference

$$\begin{aligned} \delta(t) &= y_1(t) - y_2(t) \\ &= (a_1 - b_1)e^{-t} + (a_2 - 1)e^{-2t} + (b_2 - 1)e^{-3t} \end{aligned}$$

becomes $\delta(t) \equiv 0, t \in [0, \tau)$ if $a_1 = b_1$, $a_2 = 1$, $b_2 = 1$. This implies that under this input, the dataset $\mathcal{D}_\tau$ cannot uniquely determine α_0 and $x(0)$.

In this section, we will develop methods on input design for joint observability with suitable conditions that resolve the issue demonstrated by Example 5.2. For simplicity, we assume that all subsystems are individually observable in this section. The same conclusions can be obtained by using the methods in Section III for unobservable subsystems.

A. Mode-Modulated Input Design

Consider the set $G = \{G_i, i = 1, \ldots, m\}$ of m distinct subsystems. The set of poles of G_i (or equivalently the eigenvalues of $A(i)$) is Λ_i and $\Lambda = \cup_{i=1}^m \Lambda_i$.

Assumption 5.1: Let $\mathcal{U} \subset \mathcal{R}_0$ be the set of nonvanishing inputs u whose Laplace transforms $U(s)$ satisfy the following conditions. (i) $U(s) = \frac{b(s)}{a(s)}$ is coprime, namely, no common pole-zero pairs (i.e., no pole-zero cancellation). (ii) $U(s)$ contains at least one pole λ of any multiplicity $q \geq 1$ such that $\lambda \notin \Lambda$ and $G_i(\lambda)$, $i = 1, \ldots, m$, are distinct.

Theorem 5.1: For the set of distinct subsystems $G = \{G_i, i = 1, \ldots, m\}$, if the input $u \in \mathcal{U}$, then for any $\tau > 0$, the true subsystem can be uniquely determined from the dataset $\mathcal{D}_\tau = \{y(t) \not\equiv 0, t \in [0, \tau)\}$, regardless of the actual initial state $x(0)$.

Proof: Without loss of generality, assume $G_i \in G$ is the true subsystem.

Since u satisfies Assumption 5.1, $U(s)$ contains a pole $\lambda \notin \Lambda$ that is of multiplicity q.

Case 1: Real-valued λ. In this case $U(s) = \frac{1}{(s-\lambda)^q} U_1(s)$ such that λ is not a pole of $U_1(s)$. Since $U(s)$ is coprime, $U_1(\lambda) \neq 0$. Consider any $G_j, j \neq i$. Under any input $u \in \mathcal{U}$ with real-valued λ, the input responses are

$$\begin{aligned} Y_i^{\text{input}}(s) &= \frac{1}{(s-\lambda)^q} G_i(s) U_1(s) \\ &= \frac{G_i(\lambda) U_1(\lambda)}{(s-\lambda)^q} + \cdots + \frac{a}{s-\lambda} + \widetilde{Y}_i^{\text{input}}(s) \\ Y_j^{\text{input}}(s) &= \frac{1}{(s-\lambda)^q} G_j(s) U_1(s) \\ &= \frac{G_j(\lambda) U_1(\lambda)}{(s-\lambda)^q} + \cdots + \frac{b}{s-\lambda} + \widetilde{Y}_j^{\text{input}}(s) \end{aligned}$$

for some a and b, where $\widetilde{Y}_i^{\text{input}}(s)$ (and $\widetilde{Y}_j^{\text{input}}(s)$) contains all other terms of the partial fraction expansions. Their corresponding modes in time-domain expressions are

$$y_i^{\text{input}}(t) = G_i(\lambda) U_1(\lambda) t^{q-1} e^{\lambda t} + \cdots + a e^{\lambda t} + \tilde{y}_i^{\text{input}}(t),$$
$$y_j^{\text{input}}(t) = G_j(\lambda) U_1(\lambda) t^{q-1} e^{\lambda t} + \cdots + b e^{\lambda t} + \tilde{y}_j^{\text{input}}(t),$$

where $\tilde{y}_i^{\text{input}}(t)$ (and $\tilde{y}_j^{\text{input}}(t)$) contains modes from other poles. Furthermore, the initial-state responses

$$y_i^{\text{state}}(t) = C(i) e^{A(i)t} x_i(0),\ y_j^{\text{state}}(t) = C(j) e^{A(j)t} x_j(0)$$

consist of modes from the eigenvalues of $A(i)$ and $A(j)$ that are in Λ.

It is well known that the mode $t^{q-1} e^{\lambda t}$ is linearly independent of $\{e^{\lambda t}, \ldots, t^{q-2} e^{\lambda t}\}$ and the modes corresponding to different poles are linearly independent [30]. As a result, their total responses

$$y_i(t) = G_i(\lambda) U_1(\lambda) t^{q-1} e^{\lambda t} + \delta_i(t)$$
$$y_j(t) = G_j(\lambda) U_1(\lambda) t^{q-1} e^{\lambda t} + \delta_j(t)$$

contain the same mode $t^{q-1} e^{\lambda t}$, which is linearly independent of all modes in $\delta_i(t)$ and $\delta_j(t)$. Their difference is

$$\begin{aligned} \Delta(t) &= y_i(t) - y_j(t) \\ &= [G_i(\lambda) - G_j(\lambda)] U_1(\lambda) t^{q-1} e^{\lambda t} + \delta_i(t) - \delta_j(t). \end{aligned}$$

Under Assumption 5.1, $U_1(\lambda) \neq 0$ and is finite, $G_i(\lambda) - G_j(\lambda) \neq 0$. Hence, $[G_i(\lambda) - G_j(\lambda)] U_1(\lambda) t^{q-1} e^{\lambda t} \not\equiv 0$ in $t \in [0, \tau)$ for any $\tau > 0$. Furthermore, since $\lambda \notin \Lambda$ and $U_1(s)$ does not have poles at λ, $t^{q-1} e^{\lambda t}$ is linearly independent of the modes in $\delta_i(t) - \delta_j(t)$ that consist of the modes from Λ, $U_1(s)$, and $\{e^{\lambda t}, \ldots, t^{q-2} e^{\lambda t}\}$.

This implies that $\Delta(t) \not\equiv 0$, $t \in [0, \tau)$. Therefore, G_i can be uniquely determined from $\mathcal{D}_\tau$.

Case 2: Complex-valued $\lambda = \alpha + j\omega$. Since the subsystems have real coefficients whose complex poles are in pairs, $\bar{\lambda} = \alpha - j\omega$ is also a pole. By grouping factors corresponding to λ and $\bar{\lambda}$, the input can be expressed as $U(s) = \frac{1}{(s^2 - 2\alpha s + \alpha^2 + \omega^2)^q} U_1(s)$ such that λ and $\bar{\lambda}$ are not poles of $U_1(s)$. The modes in the total responses of G_i and G_j contain the same mode $t^{q-1} e^{\alpha t} \cos(\omega t + \theta)$ but with different coefficients in G_i and G_j. The remaining steps of the proof are nearly identical to Case 1, with $t^{q-1} e^{\lambda t}$ replaced by $t^{q-1} e^{\alpha t} \cos(\omega t + \theta)$ and, hence, are omitted. ■

Example 5.2: Consider the two subsystems in Example 5.2

$$G_1(s) = \frac{1}{s+1} + \frac{2}{s+2}; G_2(s) = \frac{1}{s+1} + \frac{3}{s+3}.$$

$\Lambda_1 = \{-1, -2\}$, $\Lambda_2 = \{-1, -3\}$, $\Lambda = \{-1, -2, -3\}$. Choose $\lambda = -4 \notin \Lambda$ and $U(s) = \frac{1}{s+4}$. Since $G_1(-4) = -\frac{10}{3}$,

$G_2(-4) = -\frac{4}{3}$, Assumption 5.1 is satisfied. As a result, this input can be used to distinguish G_1 and G_2, regardless of what the initial state is.

On the other hand, $\lambda = 0 \notin \Lambda$ but $G_1(0) = 2 = G_2(0)$. It means $U(s) = \frac{1}{s}$ violates Assumption 5.1. Example 5.2 has already demonstrated that $U(s) = \frac{1}{s}$ (the unit step) cannot be used to distinguish G_1 and G_2.

B. Magnitude-Modulated Input Design

Suppose that we have an RSLS with m subsystems, represented by the set of distinct rational transfer functions $\mathbf{G} = \{G_i(s) = C(i)(sI - A(i))^{-1}B(i), i = 1, \ldots, m\}$ with $G_i(s) \in \mathcal{R}$, $i = 1, \ldots, m$, and $G_i(s) - G_j(s) \neq \mathbf{0}$ if $i \neq j$. Let the true subsystem be i^* and the corresponding true system be $G^* \in \mathbf{G}$ with the unknown initial state $x^*(0)$. The input $u(t) = Ru_1(t)$ with $U_1(s) \in \mathcal{R}_0$, and the magnitude $R > 0$ will be selected later.

For a selected $0 < \tau_0 \ll \tau$, denote the dataset $\mathcal{D}_{\tau_0} = \{y(t), t \in [0, \tau_0]\}$.[5]

Assumption 5.2: (i) $\|x^*(0)\| \leq \mu_0$. (ii) $\max_{i=1,\ldots,m} \max_{t\in[0,\tau_0]} \|C(i)e^{A(i)t}\| \leq \mu_1$.

Theorem 5.2: Under Assumption 5.2, there exists R_0, depending on τ_0, such that if $R > R_0$, then the discrete state α_0 of the RSLS can be uniquely determined from the dataset $\mathcal{D}_{\tau_0}$.

Proof: Under the input $u_1(t)$, the corresponding zero-state responses $y_i^0(t) = G_i u_1$, $i = 1, \ldots, m$, are exponential polynomial functions and, hence, are continuous. The true zero-state output is

$$y_*^0(t) = \int_0^t e^{A(i^*)(t-\theta)} u_1(\theta) d\theta, \ t \in [0, \tau_0].$$

By Theorem 5.2, if $i \neq i^*$, then $y_i^0(t) - y_*^0(t) \not\equiv 0, t \in [0, \tau_0]$. By continuity, $\delta_i = \max_{t\in[0,\tau_0]} |y_i^0(t) - y_*^0(t)| > 0$ and $\delta_{\min} = \min_{i\neq i^*} \delta_i > 0$.

The total observed true response from the input $u = Ru_1$ and the initial state $x^*(0)$ is

$$y_*(t) = C(i^*)e^{A(i^*)t}x^*(0) + Ry_*^0(t)$$

and for $i \neq i^*$

$$y_i(t) = C(i)e^{A(i)t}x_i(0) + Ry_i^0(t).$$

It follows that

$$\begin{aligned}
&\max_{t\in[0,\tau_0]} |y_i(t) - y_*(t)| \\
&\geq R \max_{t\in[0,\tau_0]} |y_i^0(t) - y_*^0(t)| \\
&\quad - |C(i^*)e^{A(i^*)t}x^*(0) - C(i)e^{A(i)t}x_i(0)| \\
&= R\delta_i - |C(i^*)e^{A(i^*)t}x^*(0) - C(i)e^{A(i)t}x_i(0)| \\
&\geq R\delta_i - 2\mu_0\mu_1 \\
&\geq R\delta_{min} - 2\mu_0\mu_1.
\end{aligned}$$

[5]The closed set $[0, \tau_0]$ is used here so that it becomes a compact set for the maximum to exist in the subsequent expressions. It does not have any further implications on the results.

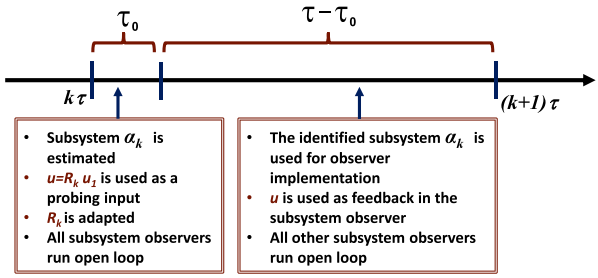

Fig. 1. Time-division framework for estimating the unknown subsystem and unknown continuous state of the identified subsystem.

Define $R_0 = \frac{2\mu_0\mu_1}{\delta_{\min}}$. Then for $i \neq i^*$, if $R > R_0$, we have

$$\min_{i\neq i^*} \max_{t\in[0,\tau_0]} |y_i(t) - y_*(t)| > 0.$$

Consequently, the estimate

$$\widehat{i} = \arg \min_{i=1,\ldots,m} \max_{t\in[0,\tau_0]} |y_i(t) - y_*(t)| = i^*$$

namely, i^* can be uniquely determined. ■

VI. Observer Design and Stochastic Convergence Analysis

A. Time-Division Framework

Our observers will be implemented in a time-division framework. Each time segment $[k\tau, (k+1)\tau)$ is divided into two intervals. The first (much smaller) interval $[k\tau, k\tau + \tau_0]$ is designated for estimating α_k (that is, identifying the active subsystem). Using the magnitude-modulated input design as an example, during this interval, the input signal $u = Ru_1$ with R satisfying Theorem 5.2 is used as a probing input to assist in the determination of α_k. Since the information on the continuous state improves over time, the magnitude R may be broadened to a sequence of time-varying R_k and adapted accordingly. Since the true α_k is unknown in this interval, all subsystem observers run open loop on the basis of system models only.

Once $\alpha_k = i$ is correctly estimated, in the second interval $[k\tau + \tau_0, (k+1)\tau)$, a feedback-based observer is implemented for the ith subsystem to estimate its observable substate z^i. All other subsystem observers still run open loop. This framework is depicted in Fig. 1.

B. Estimation Algorithm for α_k Using Data in $[k\tau, k\tau + \tau_0]$.

Let $0 < \tau_0 \ll \tau$. We will use data in $[k\tau, k\tau + \tau_0]$ to estimate α_k first. Let the observation dataset be given as $\mathcal{D}^k = \{y(t) : t \in [k\tau, k\tau + \tau_0)\}$. Compute the value R_0^k as in Theorem 5.2. Choose $R_k > R_0^k$.

Algorithm 3: Estimation of α_k Under Unknown Initial State

1) Compute the input responses of the subsystems (assuming zero initial condition): $y_i(t) = G_i u, \quad t \in [k\tau, k\tau + \tau_0], \ i = 1, \ldots, m$
2) Calculate the errors: $\varepsilon_i = \sup_{t\in[k\tau, k\tau+\tau_0]} |y_i(t) - y(t)|$.
3) Determine α_k: $\widehat{\alpha}_k = \arg\min_{i=1,\ldots,m} \varepsilon_i$.

In Algorithm 3, we ignore the unknown true zero-input response of the ith subsystem in computing the outputs of subsystems although we know that the actual output $y(t)$ is affected by the unknown initial state. Since the input response is dominant due to the suitably selected magnitude R_k, by Theorem 5.2, this algorithm will always produce the correct α_k.

C. Observer Design

After determining $\alpha_k = i$ correctly, an observer can be designed to estimate $z^i(k\tau + \tau_0)$. Since the observer feedback developed in [28] can be designed and implemented in $[k\tau + \tau_0, (k+1)\tau)$ when $\alpha_k = i$ is identified, in principle, the design considerations in [28] remain valid. Also, since τ_0 can be made very small, its impact on convergence properties in [28] can be tolerated. We refer the reader to [28] for observer design and the corresponding properties.

However, design parameters must be modified to accommodate the time required for estimating α_k, the effect on the unknown state due to probing inputs, and relationships between the input response and initial-state response. These new aspects in observer design and convergence analysis will be covered in this section.

For simplicity and clarity, we will focus on the scenario in [28, Sec. VI], which treats RSLSs with independent subsystem error dynamics. General systems can be treated in a similar way as in [28, Sec. VII]. The errors in estimating z^i and z are denoted by $e_i = z^i - \widehat{z}^i$ and $e = z - \widehat{z}$, respectively. Denote $\mu^i(t) = \|e_i(t)\|$, $\mu^i_k = \|e_i(k\tau)\|$, $\mu(t) = \|e(t)\|$, $\mu_k = \|e(k\tau)\|$.

Assumption 6.1: (i) The RSLS has independent subspace error dynamics, namely $\dot{z}^i$ depends on z^i only, independent of α_k, under zero input. For such systems, the *subsystem state equation* will be $\dot{e}_i = A^i_{22} e_i$ in open loop without input. (ii) B_k is known.

We consider three cases in error analysis.

Case 1: $t \in [k\tau, k\tau + \tau_0]$

In this time interval, all subsystem observers are running open loop. Since a probing input is applied, under Assumption 6.1 the dynamics of z^i are

$$\dot{z}^i = F_i \dot{x} = F_i A_k x + F_i B_k u = A^i_{22} z^i + F_i B_k u.$$

The observer is $\dot{\widehat{z}}_i = A^i_{22}\widehat{z}^i + F_i B_k u$. It follows that the error dynamics are $\dot{e}_i = A^i_{22} e_i$ and $\|e_i(k\tau + \tau_0)\| \le \gamma^i_0$ for some $\gamma^i_0 > 0$. Let $\gamma_0 = \max_{i=1,\dots,m} \gamma^i_0$.

Case 2: $t \in [k\tau + \tau_0, (k+1)\tau)$ and $\alpha_k \neq i$.

In the interval $t \in [k\tau + \tau_0, (k+1)\tau)$, the input $u \equiv 0$. When the ith subsystem is running open loop, we have the error bound $\mu^i_{k+1} \le \gamma^i_1 \|e_i(k\tau + \tau_0)\| \le \gamma^i_1 \gamma^i_0 \mu^i_k, \quad \alpha_k \neq i$ for some constant γ^i_1. Let $\gamma_1 = \max_{i=1,\dots,m} \gamma^i_1$.

Case 3: $t \in [k\tau + \tau_0, (k+1)\tau)$ and $\alpha_k = i$.

Observe that if $\alpha_k = i$, the observer error dynamics for the ith subsystem are $\dot{e}_i = (A^i_{22} - L_i C^i_2) e_i = A^i_c e_i$. By designing the observer gain properly, $A^i_c = A^i_{22} - L_i C^i_2$ can have n_i eigenvalues with real part less than $-a_i$ with $a_i > 0$. Under the given τ, for some $c > 0$, $\|e^{A^i_c \tau}\| \le c e^{-a_i \tau}$, which can be made arbitrarily small by choosing sufficiently large a_i. Consequently, $\mu^i_{k+1} \le \gamma^i_c \|e_i(k\tau + \tau_0)\| \le \gamma^i_c \gamma^i_0 \mu^i_k$, where γ^i_c can be made arbitrarily small. Denote $\gamma_c = \max_{i=1,\dots,m} \gamma^i_c$. The actual value γ_c will be selected later to ensure convergence of the organized observer for the entire system.

Total Errors

Combining the three cases, we have

$$\dot{e}_i = \begin{cases} A^i_{22} e_i, & t \in [k\tau, k\tau + \tau_0] \\ I_{\{\alpha_k = i\}} A^i_c e_i + I_{\{\alpha_k \neq i\}} A^i_{22} e_i, & t \in [k\tau + \tau_0, (k+1)\tau). \end{cases}$$

It follows that the errors are bounded by $\mu^i_{k+1} \le \gamma^i_k \mu^i_k$, with $\gamma^i_k = I_{\{\alpha_k = i\}} \gamma^i_c \gamma^i_o + I_{\{\alpha_k \neq i\}} \gamma^i_1 \gamma^i_o$. Consequently, $\mu^i_k \le (\Pi^k_{j=1} \gamma^i_j) \mu^i_0$.

Under Assumption 2.1, the process $\{\gamma^i_k\}$ is i.i.d. with $P(\gamma^i_k = \gamma^i_c \gamma^i_o) = p_i, P(\gamma^i_k = \gamma^i_1 \gamma^i_o) = 1 - p_i$.

Lemma 6.1: Under Assumption 2.1, for any $\gamma_* < 1$, the pole positions in the observer design can be selected such that

$$\gamma^i = (\gamma^i_c \gamma^i_o)^{p_i} (\gamma^i_1 \gamma^i_o)^{(1-p_i)} \le \gamma_* < 1. \tag{11}$$

Subsystem observers are designed to satisfy (11).

Convergence Analysis

Assumption 6.2: $\alpha(t)$ is independent of $e_i(0)$.

Recall that for a positive-valued stochastic process $\{s_k\}$, it is said to converge to 0 strongly if $\lim_{k\to\infty} s_k = 0$ w.p.1. The convergence rate is said to be exponential if $\lim_{k\to\infty} \frac{1}{k} \ln s_k = -r$ w.p.1. for some $r > 0$.

Define the continuous-time error $\mu^i(t) = \|e_i(t)\|$, which is a scalar stochastic process. Although the rate expressions are different in this article, the proofs for the following theorems are similar to those in [28] and will be omitted.

Theorem 6.1: Under Assumption 6.2 and the observer design in Lemma 6.1, (i) μ^i_k converges strongly and exponentially to 0, as $k \to \infty$. (ii) $\mu^i(t)$ converges strongly and exponentially to 0, as $t \to \infty$.

Define $e(t) = [e_1(t), \dots, e_m(t)]'$. The estimation error on x is $\epsilon(t) = x(t) - \widehat{x}(t)$ with error norm $\mu(t) = \|\epsilon(t)\|$.

Theorem 6.2: Under the same assumptions as Theorem 4.1, the estimation error on $x(t)$ satisfies $\mu(t) \to 0$ exponentially as $t \to \infty$.

VII. Illustrative Examples

Example 7.1: Consider an RSLS with two scalar subsystems

$$\dot{x}_1 = a_1 x_1 + b_1 u$$
$$\dot{x}_2 = a_2 x_2 + b_2 u$$
$$y(t) = \mathbf{1}_{\{\alpha(t)=1\}} x_1 + \mathbf{1}_{\{\alpha(t)=2\}} x_2$$

where $a_1 = 4, a_2 = 3$. Hence, $A_1 = A_2 = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, C_1 = [1, 0]$, $C_2 = [0, 1]$, $W_1 = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$, $W_2 = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$, $W_{\mathcal{S}} = \begin{bmatrix} 1 & 0 \\ 4 & 0 \\ 0 & 1 \\ 0 & 3 \end{bmatrix}$. Since $\mathrm{Rank}(W_1) = 1$ and $\mathrm{Rank}(W_2) = 1$, both subsystems are unobservable. But $\mathrm{Rank}(W) = 2$, satisfying Assumption 3.1.

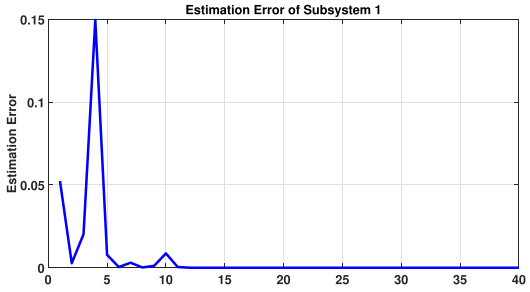

Fig. 2. Error trajectory of the observer for Subsystem 1.

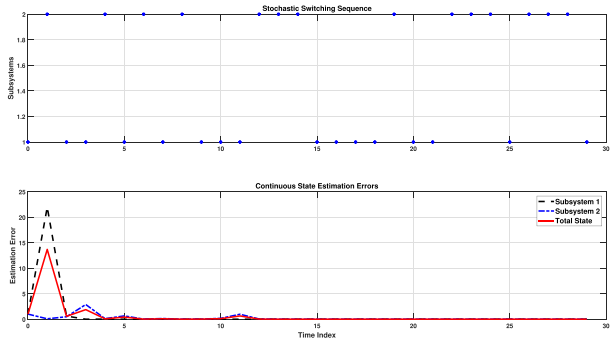

Fig. 3. Estimation error trajectories for the subsystems and for the entire state.

Now consider the observable parts of the two subsystems with $A_{22}^1 = 4$, $A_{22}^2 = 3$. Since these two eigenvalues are distinct, Assumption 3.2 is satisfied, implying that the input is not needed for distinguishing the two subsystems. As a result, $u \equiv 0$. Suppose that the interval $\tau = 0.5$ s. Within each time segment, the first interval of length $\tau_0 = 0.05$ is used to estimate α_k in which all subsystems run open loop. Suppose that the true initial states are $x_1(0) = 10$ and $x_2(0) = 5$. As an example, suppose that the true $\alpha_0 = 1$. Then, in the first time segment [0,0.5), we use the data in [0,0.05] to estimate α_0. The true output is $y(t) = 10e^{4t}$, $t \in [0, 0.05]$. By Algorithm 1, applied to the subinterval [0,0.05] on the observable subsystems, we have

$$\begin{aligned}
\Gamma_1 &= \int_0^{0.05} e^{8t}dt = 0.0615, Q_1 = \int_0^{0.05} 10e^{8t}dt = 0.615\\
\widehat{x}_0^1 &= Q_1/\Gamma_1 = 10\\
\Gamma_2 &= \int_0^{0.05} e^{6t}dt = 0.0583, Q_2 = \int_0^{0.05} 10e^{7t}dt = 0.5987\\
\widehat{x}_0^2 &= Q_2/\Gamma_2 = 10.2670.
\end{aligned}$$

Apparently, we have

$$\begin{aligned}
e_1 &= \sup_{t\in[0,0.05)} \|y(t) - e^{4t}10\| = 0\\
e_2 &= \sup_{t\in[0,0.05)} \|y(t) - 10.2670e^{3t}\|\\
&= \sup_{t\in[0,0.05)} \|10e^{4t} - 10.2670e^{3t}\| > 0.
\end{aligned}$$

As a result, $\widehat{\alpha}_0 = \arg\min\{e_1, e_2\} = 1$, namely the correct α_0 is identified.

This algorithm is applied to all time segments. As a demonstration, we show the error trajectories for estimating x_1 in this simulation. When $\alpha_k = 1$, a feedback-based observer is designed such that the closed-loop system has the eigenvalue $a_c = -7$. These parameters lead to $\gamma_0 = e^{a_1\tau_0} = 1.2214$, $\gamma_1 = e^{a_c(\tau-\tau_0)} = 0.0429$, and $\gamma_2 = e^{a_1(\tau-\tau_0)} = 6.0496$.

The random switching with $P\{\alpha(t) = 1\} = 0.5$ results in the value in (11) as $\gamma = (\gamma_0\gamma_1)^p(\gamma_0\gamma_2)^{1-p} = 0.6219$. Since $\gamma < 1$, this implies that the RSLS observer for Subsystem 1 is convergent. The same conclusion can be reached for Subsystem 2. Fig. 2 demonstrates the convergence of state estimation errors on x_1.

Example 7.2: Consider the following two subsystems, similar to an example in [28] but with the unknown sequence now, $C(1) = [1, 0, 0]$, $A(1) = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, $W(1) = \begin{bmatrix} C(1) \\ C(1)A(1) \\ C(1)(A(1))^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 11 & 8 & 0 \end{bmatrix}$ with $\text{Rank}(W(1)) = 2$, $C(2) = [0, 0, 2]$, $A(2) = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 2 & 0 \end{bmatrix}$, $W(2) = \begin{bmatrix} C(2) \\ C(2)A(2) \\ C(2)(A(2))^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 12 & 4 \end{bmatrix}$ with $\text{Rank}(W(2)) = 2$,

$$W_{\mathcal{S}} = \begin{bmatrix} W(1) \\ W(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 11 & 8 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 12 & 4 \end{bmatrix} \text{ with } \text{Rank}(W_{\mathcal{S}}) = 3.$$

The eigenvalues of A_1 are $\{-1.3166, 5.3166, 8.0000\}$ and the eigenvalues of A_2 are $\{-0.5616, 3.5616, 5.0000\}$. Without common eigenvalues, the two subsystems can be separated by using y without probing input.

For observer design on subsystems, we can obtain

$$M_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, T_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, F_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, F_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $F = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. F is full column rank.

Suppose that we choose $\tau = 0.5$ s as the decision interval. Within this interval, the first part of $\tau_0 = 0.05$ is used for detecting subsystems (estimation of α_k). After the correct α_k is detected, the remaining part of the interval with length $\tau - \tau_0 = 0.45$ is used for estimating the continuous state. The poles for the observer of Subsystem 1 are selected as $-15 \pm 100j$, and the poles for the observer for Subsystem 2 are selected as

$-15, -15.1$. The subsystem states and the total states are related by $\widetilde{z}_1 = F_1 x, \widetilde{z}_2 = F_2 x, x = \Phi \widetilde{z}$ where $\widetilde{z} = [\widetilde{z}_1{}' \, \widetilde{z}_2{}']'$.

For each time interval $[k\tau, (k+1)\tau)$, during $[k\tau, k\tau + \tau_0)$, both observers run open-loop. During $[k\tau + \tau_0, (k+1)\tau)$, the observer for the identified subsystem runs closed loop but the other observer still runs open loop. Starting from the initial condition $x(0) = [1, 1, 1]'$, we ran simulations and recorded the error norms. One sample path is shown in Fig. 3 . The result demonstrates convergence of state estimation after integrating the discrete state estimator with continuous state observers.

VIII. Conclusion

The joint estimation of continuous state and discrete switching sequence for RSLSs with unobservable subsystems involves intriguing technical complications on distinguishability, joint observability, input design, observer design, and convergence analysis. Under noise-free observations, this article has established testing conditions, input design methods, a sampling theorem, observer design algorithms, and convergence analysis. One immediate step along the direction of this article is to include noisy observations, which will be pursued in the near future. Furthermore, the integration of the results of this article with optimization or control in complex systems is an important topic to explore.

References

[1] A. Nerode and W. Kohn, "Models for hybrid systems: Automata, topologies, controllability, observability," in *Hybrid Systems*. Berlin, Germany: Springer, 1992, pp. 317–356.

[2] J. Ezzine and A. H. Haddad, "Controllability and observability of hybrid systems," *Int. J. Control*, vol. 49, no. 6, pp. 2045–2055, 1989.

[3] J. Lunze and F. Lamnabhi-Lagarrigue, *Handbook of Hybrid Systems Control: Theory, Tools, Applications*. Cambridge, U.K.: Cambridge Univ. Press, 2009.

[4] M. Babaali and M. Egerstedt, "Observability of switched linear systems," in *Proc. Int. Workshop Hybrid Syst., Computation Control*, 2004, pp. 48–63.

[5] L. Sellami and K. Abderrahim, "State estimation for switched linear systems," in *Proc. IEEE 12th Int. Multi-Conf. Syst., Signals Devices*, 2015, pp. 1–5.

[6] L. Y. Wang, P. P. Khargonekar, and A. Beydoun, "Robust control of hybrid systems: Performance guided strategies," in *Proc. Int. Hybrid Syst. Workshop*, 1997, pp. 356–390.

[7] M. Fliess, C. Join, and W. Perruquetti, "Real-time estimation for switched linear systems," in *Proc. IEEE 47th Conf. Decis. Control*, 2008, pp. 941–946.

[8] X. Zhao, H. Liu, J. Zhang, and H. Li, "Multiple-mode observer design for a class of switched linear systems," *IEEE Trans. Automat. Sci. Eng.*, vol. 12, no. 1, pp. 272–280, Jan. 2015.

[9] S. C. Johnson, "Observability and observer design for switched linear systems," Ph.D. dissertation, Dept. Elect. Comput. Eng., Purdue Univ., West Lafayette, IN, USA, 2016.

[10] K. Haddadi, N. Gazzam, and A. Benalia, "Observability of switched linear systems based geometrical conditions: An application to dc/dc multilevel converters," in *Proc. Int. Conf. Adv. Elect. Eng.*, 2019, pp. 1–5.

[11] F. Küsters and S. Trenn, "Switch observability for switched linear systems," *Automatica*, vol. 87, pp. 121–127, 2018.

[12] D. Gomez-Gutierrez, A. Ramirez-Trevino, J. Ruiz-Leon, and S. Di Gennaro, "Observability of switched linear systems: A geometric approach," in *Proc. IEEE 49th Conf. Decis. Control*, 2010, pp. 5636–5642.

[13] M. Babaali and G. J. Pappas, "Observability of switched linear systems in continuous time," in *Proc. Int. Workshop Hybrid Syst., Computation Control*, 2005, pp. 103–117.

[14] R. G. Sanfelice, R. Goebel, and A. R. Teel, "Invariance principles for hybrid systems with connections to detectability and asymptotic stability," *IEEE Trans. Autom. Control*, vol. 52, no. 12, pp. 2282–2297, Dec. 2007.

[15] P. Bernard and R. G. Sanfelice, "On notions of detectability and observers for hybrid systems," in *Proc. IEEE 59th Conf. Decis. Control*, 2020, pp. 5767–5772.

[16] H. Ríos, J. Dávila, and A. R. Teel, "State estimation for linear hybrid systems with periodic jumps and unknown inputs," *Int. J. Robust Nonlinear Control*, vol. 30, no. 15, pp. 5966–5988, 2020.

[17] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems*. Princeton, NJ, USA: Princeton Univ. Press, 2012.

[18] H. Lou and R. Yang, "Conditions for distinguishability and observability of switched linear systems," *Nonlinear Anal., Hybrid Syst.*, vol. 5, pp. 427–445, 2011.

[19] H.-W. Lou and R. Yang, "Necessary and sufficient conditions for distinguishability of linear control systems," *Acta Mathematicae Applicatae Sinica, English Ser.*, vol. 30, no. 2, pp. 473–482, 2014.

[20] R. Vidal, A. Chiuso, S. Soatto, and S. Sastry, "Observability of linear hybrid systems," in *Proc. Int. Workshop Hybrid Syst., Computation Control*, 2003, pp. 526–539.

[21] G. Yin, Y. Sun, and L. Y. Wang, "Asymptotic properties of consensus-type algorithms for networked systems with regime-switching topologies," *Automatica*, vol. 47, no. 7, pp. 1366–1378, 2011.

[22] T. Li and J.-F. Zhang, "Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises," *IEEE Trans. Autom. Control*, vol. 55, no. 9, pp. 2043–2057, Sep. 2010.

[23] J. Lygeros and M. Prandini, "Stochastic hybrid systems: A powerful framework for complex, large scale applications," *Eur. J. Control*, vol. 16, no. 6, pp. 583–594, 2010.

[24] C. G. Cassandras and J. Lygeros, *Stochastic Hybrid Systems*. Boca Raton, FL, USA: CRC Press, 2018.

[25] G. Yin and C. Zhu, *Hybrid Switching Diffusions*. New York, NY, USA: Springer, 2010.

[26] G. Yin, L. Y. Wang, and Y. Sun, "Stochastic recursive algorithms for networked systems with delay and random switching: Multiscale formulations and asymptotic properties," *Multiscale Model. Simul.*, vol. 9, no. 3, pp. 1087–1112, 2011.

[27] W. M. Wonham, "Some applications of stochastic differential equations to optimal nonlinear filtering," *SIAM J. Control*, vol. 2, pp. 347–369, 1965.

[28] L. Y. Wang, G. Yin, F. Lin, M. P. Polis, and W. Chen, "Stochastic observability and convergent analog state estimation of randomly switched linear systems with unobservable subsystems," *IEEE Trans. Autom. Control*, to be published, doi: 10.1109/TAC.2022.3148602.

[29] C. A. Berenstein and R. Gay, *Complex Analysis and Special Topics in Harmonic Analysis*. New York, NY, USA: Springer, 1995.

[30] T. Kailath, *Linear Systems*, vol. 156. Englewood Cliffs, NJ, USA: Prentice-Hall, 1980.

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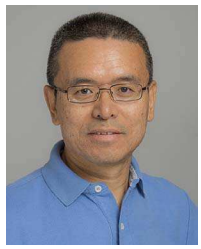

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