

Pearson's goodness-of-fit tests for sparse distributions

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Abstract. Pearson's chi-squared test is widely used to test the goodness of fit between categorical data and a given discrete distribution function. When the number of sets of the categorical data, say k , is a fixed integer, Pearson's chi-squared test statistic converges in distribution to a chi-squared distribution with $k - 1$ degrees of freedom when the sample size n goes to infinity. In real applications, the number k often changes with n and may be even much larger than n . By using the martingale techniques, we prove that Pearson's chi-squared test statistic converges to the normal under quite general conditions. We also propose a new test statistic which is more powerful than chi-squared test statistic based on our simulation study. A real application to lottery data is provided to illustrate our methodology.

Keywords: Goodness-of-fit; discrete distribution; sparse distribution; normal approximation; chi-square approximation

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1 Introduction

Consider an experiment which can result in k possible events, say, $E_1, \dots, E_k$, where $k \geq 2$ is an integer and $E_1, \dots, E_k$ form a partition of the sample space. Repeat the experiment n times independently and let o_i denote the observed frequency of event E_i for $i \in \{1, \dots, k\}$. Then $(o_1, \dots, o_k)$ has a multinomial distribution. To test the null hypothesis that the probabilities for events $E_1, \dots, E_k$ are equal to $p_1, \dots, p_k$, respectively, where $p_1, \dots, p_k$ are k specified positive numbers with $p_1 + \dots + p_k = 1$, define the following chi-squared test statistic

$$\chi_n^2 = \sum_{i=1}^k \frac{(o_i - e_i)^2}{e_i}, \quad (1.1)$$

where $e_i = np_i$ is the expected number of events E_i to occur in the n trials of the experiment for $i \in \{1, \dots, k\}$. The limiting distribution of χ_n^2 is a chi-squared distribution with $k - 1$ degrees of freedom when k is a fixed integer. This is the well-known chi-squared goodness-of-fit test proposed by Pearson [15]. A test with approximate size α rejects the null hypothesis if $\chi_n^2 > \chi_{k-1}^2(\alpha)$, where $\chi_{k-1}^2(\alpha)$ denotes the α -level critical value of a chi-squared distribution with $k - 1$ degrees of freedom for $\alpha \in (0, 1)$.

As a statistical method, Pearson's chi-squared goodness-of-fit test is one of the most popular topics offered in college statistics courses. The above testing problem can be restated in a different form. Let f_0 be a discrete probability mass function defined over $\{x_i, 1 \leq i \leq k\}$ and set $p_i = f_0(x_i)$ for $1 \leq i \leq k$. Assume that a random sample of size n , $X_1, \dots, X_n$, is drawn from the distribution of a discrete random variable X , where X is a discrete random variable having a probability mass function $f(x)$ for $x = x_1, \dots, x_k$. Now we can define $E_i = \{x_i\}$ for $1 \leq i \leq k$ and set $o_i = \sum_{j=1}^n I(X_j \in E_i)$ for $1 \leq i \leq k$, and $e_i = np_i$. Then Pearson's test statistic χ_n^2 defined in (1.1) can be used to test hypothesis $H_0 : f = f_0$; i.e. $f(x_i) = f_0(x_i)$ for $1 \leq i \leq k$. Traditionally, Pearson's chi-squared goodness-of-fit test is suggested to use only if the value of k is relatively small compared with the sample size n . When there are infinite many values for a discrete random variable X or the number of distinct values of X is too large compared with the sample size n , one can first select a proper integer k and then re-group values of X into k categories by putting the values of X with small probabilities (under the null hypothesis) into one category. When f_0 is a probability density function, one can discretize the variable X so that Pearson's chi-squared goodness-of-fit test can be used to test whether the density function of X is equal to f_0 .

When a probability function or density function f_0 is not fully specified, that is, f_0 depends some unknown parameters, say θ , the probabilities $p_1, \dots, p_k$ depend on θ . We can replace θ with some estimators such as the maximum likelihood estimator, then χ_n^2 still converges in distribution to a chi-squared distribution with $k - r - 1$ degrees of freedom where r is the dimension of θ . For more topics and their developments related to Pearson's test statistics, we refer to Voinov et al. [19].

When the sample size n is small or k is relatively large, some expected frequencies e_i may become too

small. A variety of estimates of the discrete probability distribution of Pearson's test statistics have been discussed in the literature, see, e.g. Cochran [2], Yarnold [20], Larntz [11], Lawal [12], Hutchinson [8] and references therein. Baglivo et al. [1] derived formulas for the exact distributions and significance levels of Pearson's goodness of fit test statistics. Cressie and Read [4] provided a comprehensive review for Pearson's goodness-of-fit test and the likelihood ratio test.

In this paper, we are interested in the goodness-of-fit test when both n and k go to infinity, that is, we allow that $k = k_n$ changes with n and k_n can be even much larger than n . We note that asymptotic normality of Pearson's chi-squared test statistics has been obtained by Tumanyan [18] and Holst [7] when $n/k_n \rightarrow a \in (0, \infty)$ and some restrictive conditions are held. A recent work by Rempała and Wesołowski [17] extended this scope by imposing conditions on the following decomposition of Pearson's test statistics:

$$\mathcal{X}_n^2 = S_{n1} + S_{n2}, \quad \text{where } S_{n1} = \sum_{i=1}^{k_n} \frac{(o_i - e_i)^2}{e_i} - \sum_{i=1}^{k_n} \frac{o_i - e_i}{e_i}, \quad S_{n2} = \sum_{i=1}^{k_n} \frac{o_i - e_i}{e_i}. \quad (1.2)$$

By assuming that S_{n2} is negligible, Rempała and Wesołowski [17] showed that $\mathcal{X}_n^2$ is asymptotically normal if $n^2/k_n \rightarrow \infty$ as $n \rightarrow \infty$. The conditions imposed in Rempała and Wesołowski [17] will be discussed further in Section 2. Since the negligibility condition is trivially true for equiprobable cells, that is, $p_1 = \dots = p_{k_n}$, $\mathcal{X}_n^2$ has a normal limit, and furthermore, Rempała and Wesołowski [17] showed in this case that $\mathcal{X}_n^2$, after properly normalized, converges in distribution to a Poisson distribution if $n^2/k_n \rightarrow \lambda \in (0, \infty)$.

Pearson's chi-squared test has been proven to be unbiased if one uses equiprobable cells, see, e.g. Mann and Wald [13] and Cohen and Sackrowitz [3]. Koehler and Larntz [9] provided empirical evidence for the accuracy of the normal approximation when n^2/k_n is reasonably large.

If one does not use equiprobable cells, Haberman [6] noted that Pearson's test can be biased when some expected frequencies become too small. And this is the case if k_n is too large compared with n . To overcome this drawback, Zelterman [21, 22] proposed to use D^2 statistic for the test, i.e. S_{n1} in the decomposition (1.2). Kim et al. [10] compared some asymptotic properties of $\mathcal{X}_n^2$ statistic and D^2 statistic for large sparse multinomial distributions.

In this paper, we investigate the limiting distribution for Pearson's goodness-of-fit test statistic $\mathcal{X}_n^2$ and D^2 statistic. By using the decomposition (1.2) for Pearson's goodness-of-fit test statistic we propose some new test statistics which are more powerful in general.

The rest of the paper is organized as follows. In Section 2, we investigate the limiting distributions of Pearson's goodness-of-fit test statistics and new test statistics. In Section 3, we carry out a simulation study to compare the performance of these test statistics in terms of the size and the power of the tests. In Section 4, we apply Pearson's goodness-of-fit test statistics to test whether the winning numbers from Minnesota Lottery Game Daily 3 were randomly selected with equal probabilities. Then we summarize the paper with some concluding remarks. All proofs are given in the [Supplement](#).

2 Main results

Throughout, we always assume that $k = k_n \rightarrow \infty$ as $n \rightarrow \infty$. We adopt some notations as follows. The symbol $\xrightarrow{d}$ denotes the convergence in distribution, $N(0, 1)$ denotes a standard normal random variable, and $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x e^{-t^2/2} dt$ is the cumulative distribution function of the standard normal. We also define

$$\sigma_{n1}^2 = \frac{2(k_n - 1)(n - 1)}{n}, \quad \sigma_{n2}^2 = \frac{1}{n} \left(\sum_{i=1}^{k_n} \frac{1}{p_i} - k_n^2 \right), \quad \sigma_n^2 = \sigma_{n1}^2 + \sigma_{n2}^2, \quad (2.1)$$

where σ_n^2 , σ_{n1}^2 and σ_{n2}^2 are the variances of $\mathcal{X}_n^2$, S_{n1} and S_{n2} , respectively. In Read and Cressie [16], the first three asymptotic moments have been derived for the so-called power-divergence statistics which include Pearson's $\mathcal{X}_n^2$ statistic as a special case.

We need to impose the following conditions in deriving the limiting distributions for Pearson's test statistic $\mathcal{X}_n^2$ and some new test statistics that we will propose in the paper:

We first investigate the asymptotic properties of S_{n1} (i.e. D^2 statistic) and $\mathcal{X}_n^2$.

$$\frac{1}{n^2 k_n^2} \sum_{i=1}^{k_n} \frac{1}{p_i^2} \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (2.2)$$

$$\min \left(\frac{\sum_{i=1}^{k_n} \frac{1}{p_i^3} - k_n^4}{n^3 \sigma_{n2}^4} I(\sigma_{n2}^2 > 0), \frac{\sigma_{n2}^2}{k_n} \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.3)$$

Theorem 2.1. *If (2.2) holds, then we have*

$$\frac{S_{n1} - (k_n - 1)}{\sigma_{n1}} \xrightarrow{d} N(0, 1) \quad \text{as } n \rightarrow \infty, \quad (2.4)$$

where σ_{n1} is defined in (2.1).

Theorem 2.2. *Under conditions (2.2) and (2.3) we have*

$$\frac{\mathcal{X}_n^2 - (k_n - 1)}{\sigma_n} \xrightarrow{d} N(0, 1) \quad \text{as } n \rightarrow \infty, \quad (2.5)$$

where σ_n is defined in (2.1).

Based on the normal approximation (2.5), a test with approximate size α rejects the null hypothesis if $\mathcal{X}_n^2 > k_n - 1 + \sigma_n z_\alpha$, where z_α denotes the α -level critical value of the standard normal distribution for each $\alpha \in (0, 1)$. Based on (2.4), a test with approximate size α rejects the null hypothesis if $S_{n1} > k_n - 1 + \sigma_{n1} z_\alpha$.

A test of size α is said to be unbiased if the power of the test is at least α under alternative hypotheses. The test based on the statistic $\mathcal{X}_n^2$ is not unbiased under some alternatives as

pointed out by Haberman [6]. More seriously, our simulation study indicates that the test has a nearly zero power under some alternatives, that is, the test loses its power completely in those cases; see Table 2. In order to understand why this happens, we will look at the decomposition (1.2) for the goodness-of-fit test statistic $\mathcal{X}_n^2$. From Lemma A.1 in the [Supplement](#), we have under the null hypothesis $P(E_i) = p_i$ for $1 \leq i \leq k_n$ that

$$\mathbb{E}(S_{n1} - (k_n - 1)) = 0 \text{ and } \mathbb{E}(S_{n2}) = 0,$$

and under the alternative H_1 : $P(E_i) = p'_i$ for $1 \leq i \leq k_n$ that

$$\mathbb{E}(S_{n1} - (k_n - 1)|H_1) = (n - 1) \sum_{i=1}^{k_n} \frac{(p'_i - p_i)^2}{p_i} \text{ and } \mathbb{E}(S_{n2}|H_1) = \sum_{i=1}^{k_n} \frac{p'_i - p_i}{p_i}. \quad (2.6)$$

The variances under H_1 can be calculated for both S_{n1} and S_{n2} . Since the rejection region of the goodness-of-fit test is one-sided, the test gains its power from a shift to right in location of the test statistic $\mathcal{X}_n^2$ under the alternative. In S_{n1} , the effect of a shift is always positive, but the sign of $\sum_{i=1}^{k_n} \frac{p'_i - p_i}{p_i}$, the location shift in S_{n2} , can be negative. If this shift in location to left in S_{n2} is overwhelming, the observed values for $\mathcal{X}_n^2$ can be very small and will result in rejecting the alternative hypotheses.

Since we are considering the situation when both n and k_n are large, from (2.6), $|\mathbb{E}(S_{n2}|H_1)|$ can be very large compared with $\mathbb{E}(S_{n1} - (k_n - 1)|H_1)$ when k_n is much larger than n . This indicates that using $|S_{n2}|$ in the test statistics can be more powerful than S_{n2} . We propose a class of test statistics $S_{n1} + c|S_{n2}|$, where $c \geq 0$ is a constant. Their limiting distributions are given as follows.

Theorem 2.3. *Under conditions (2.2) and (2.3) we have as $n \rightarrow \infty$*

$$\sup_x \left| P\left(\frac{S_{n1} + c|S_{n2}| - (k_n - 1)}{\sigma_{n1}} \leq x\right) - P\left(Z_1 + \frac{c\sigma_{n2}}{\sigma_{n1}}|Z_2| \leq x\right) \right| \rightarrow 0, \quad (2.7)$$

where Z_1 and Z_2 are independent random variables with the standard normal distribution, and $c \geq 0$ is any given constant.

For each s , define $\Psi(x, s)$ as the cumulative distribution function of $Z_1 + s|Z_2|$, i.e.

$$\Psi(x, s) = P(Z_1 + s|Z_2| \leq x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \Phi(x - st) \exp(-t^2/2) dt. \quad (2.8)$$

For each $\alpha \in (0, 1)$, let $\psi_\alpha(s)$ denote an α -level critical value of $\Psi(\cdot, s)$, that is, $1 - \Psi(\psi_\alpha(s), s) = \alpha$. The integral in (2.8) has no close form solution but it can be evaluated numerically by using

function ‘integrate’ in $\mathbf{R}$. Critical values $\psi_\alpha(s)$ can be solved via the Newton-Raphson method. Note that $\Psi(x, 0) = \Phi(x)$ and thus $\psi_\alpha(0) = z_\alpha$ for $\alpha \in (0, 1)$.

Three test statistics, S_{n1} , $\mathcal{X}_n^2$, and $S_{n1} + c|S_{n2}|$ with $c \geq 0$, can be used to test the null hypothesis that $P(E_i) = p_i$ for $1 \leq i \leq k_n$, and their rejection regions at level α , according to equations (2.4), (2.5) and (2.7), are given by

$$\begin{aligned}\mathcal{R}_0 &= \left\{ S_{n1} > k_n - 1 + \sigma_{n1} z_\alpha \right\}, \\ \mathcal{R} &= \left\{ \mathcal{X}_n^2 > k_n - 1 + \sigma_n z_\alpha \right\},\end{aligned}\tag{2.9}$$

and

$$\mathcal{R}_c = \left\{ S_{n1} + c|S_{n2}| > k_n - 1 + \sigma_{n1} \psi_\alpha\left(\frac{c\sigma_{n2}}{\sigma_{n1}}\right) \right\}\tag{2.10}$$

for $c \geq 0$. Note that test $\mathcal{R}_0$ can be considered as a special case of $\mathcal{R}_c$ defined in (2.10) with $c = 0$.

The aforementioned test statistics (or their corresponding rejection regions) are the same when $p_1 = \dots = p_{k_n}$ since $\sigma_{n2}^2 = 0$ and $S_{n2} = 0$ in this case. Note that Theorem 2.1 can be considered as a special case of Theorem 2.3 with $c = 0$, but in Theorem 2.1 we impose only condition (2.2) which is less restrictive than conditions in Theorem 2.3. If we assume $p_1 = \dots = p_{k_n}$, condition (2.2) is equivalent to $\lim_{n \rightarrow \infty} k_n/n^2 = 0$. Immediately we have the following corollary.

Corollary 2.1. *Assume that $\{k_n\}$ is a sequence of positive integers such that $k_n \rightarrow \infty$ and $k_n = o(n^2)$ as $n \rightarrow \infty$. Then under the assumption that $p_1 = \dots = p_{k_n}$, we have*

$$\frac{\mathcal{X}_n^2 - (k_n - 1)}{\sigma_{n1}} \xrightarrow{d} N(0, 1) \text{ as } n \rightarrow \infty.$$

Under the assumption of the equiprobable cells with $p_1 = \dots = p_{k_n}$, we have $S_{n2} = 0 = \sum_{i=1}^{k_n} \frac{o_i - e_i}{e_i}$ for any samples, regardless of how large for any single term $\frac{o_i - e_i}{e_i}$. As a remedy, we can assign a weight for each term such that the weighted sum is not degenerate. Now we introduce a weighted version for S_{n2} as follows

$$\bar{S}_{n2} = \sum_{i=1}^{k_n} \frac{c_i(o_i - e_i)}{e_i},\tag{2.11}$$

where $c_i \geq 0$ for $1 \leq i \leq k_n$ and $\sum_{i=1}^{k_n} c_i = k_n$. Obviously, S_{n2} is a special case of $\bar{S}_{n2}$ with $c_1 = \dots = c_{k_n} = 1$.

We can verify that

$$\bar{\sigma}_{n2}^2 = \mathbb{E}(\bar{S}_{n2}^2) = \frac{1}{n} \left(\sum_{i=1}^{k_n} \frac{c_i^2}{p_i} - k_n^2 \right) \quad \text{and} \quad \mathbb{E}(\bar{S}_{n2}) = 0; \quad (2.12)$$

see (A.31) in the [Supplement](#).

Now we propose a new class of test statistics $S_{n1} + c|\bar{S}_{n2}|$, where $c \geq 0$ is a constant. We need the following condition for the asymptotic normality for those test statistics.

$$\min \left(\frac{\sum_{i=1}^{k_n} \frac{c_i^4}{p_i^3} - k_n^4}{n^3 \bar{\sigma}_{n2}^4} I(\bar{\sigma}_{n2}^2 > 0), \frac{\bar{\sigma}_{n2}^2}{k_n} \right) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.13)$$

Theorem 2.4. *Under conditions (2.2) and (2.13) we have as $n \rightarrow \infty$*

$$\sup_x \left| P \left(\frac{S_{n1} + c|\bar{S}_{n2}| - (k_n - 1)}{\sigma_{n1}} \leq x \right) - P \left(Z_1 + \frac{c\bar{\sigma}_{n2}}{\sigma_{n1}} |Z_2| \leq x \right) \right| \rightarrow 0, \quad (2.14)$$

where Z_1 and Z_2 are independent random variables with the standard normal distribution, and $c \geq 0$ is any given constant.

For given weights $c_1, \dots, c_{k_n}$ and constant $c \geq 0$, a test of size α based on approximation (2.14) for test statistic $S_{n1} + c|\bar{S}_{n2}|$ has the following rejection region

$$\bar{\mathcal{R}}_c = \{S_{n1} + c|\bar{S}_{n2}| > k_n - 1 + \sigma_{n1} \psi_\alpha \left(\frac{c\bar{\sigma}_{n2}}{\sigma_{n1}} \right)\}, \quad (2.15)$$

where $\bar{\sigma}_{n2}$ is defined in (2.12).

Of particular interest, we offer a discussion for the selection on weights $c_1, \dots, c_{k_n}$ so that $\bar{S}_{n2}$ is non-degenerate and condition (2.13) is satisfied for the equiprobable cells. When $p_1 = \dots = p_{k_n} = \frac{1}{k_n}$, we select weights $c_1, \dots, c_{k_n}$ such that they are not identically equal to 1. This ensures $\bar{\sigma}_{n2}^2 > 0$.

A very simple way is to select an integer k_0 such that $k_0 \sim h k_n$ for some $h \in (0, 1)$ and assign a value k_n/k_0 to k_0 of c_i 's and 0 to the remaining $k_n - k_0$ weights. Then for any integer $r \geq 1$, we have

$$\sum_{i=1}^{k_n} \frac{c_i^{r+1}}{p_i^r} - k_n^{r+1} = k_n^r k_0 \left(\frac{k_n}{k_0} \right)^{r+1} - k_n^{r+1} = \left(\left(\frac{k_n}{k_0} \right)^r - 1 \right) k_n^{r+1} \sim (h^{-r} - 1) k_n^{r+1}.$$

Then the first term in the parentheses in (2.13) approximately equals

$$\frac{(h^{-3} - 1) k_n^4}{n((h^{-1} - 1) k_n^2)^2} \sim \frac{h^{-3} - 1}{(h^{-1} - 1)^2} \frac{1}{n} \rightarrow 0$$

as $n \rightarrow \infty$. That is, (2.13) holds.

3 A simulation study

In this section, we compare the performance of the test statistics defined in Section 2 through some simulations.

We first compare the sizes and powers of the test statistics $\mathcal{X}_n^2$, S_{n1} and $S_{n1} + c|S_{n2}|$ under some general null hypotheses. Then we compare the performance of $\mathcal{X}_n^2$ and $S_{n1} + c|\bar{S}_{n2}|$ under equiprobable cells with $p_1 = \dots = p_{k_n}$.

Firstly, we consider the following five tests, including $\mathcal{R}$, $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_3$ and $\mathcal{R}_5$ as defined in (2.9) and (2.10) with selection of $\alpha = 0.05$ and several combinations of n and k_n . For each case, the simulation is repeated 10000 times by using **R** package, and the sizes and powers of these tests are estimated.

We assume k_n is an even integer and define

$$p_1 = \dots = p_{\frac{k_n}{2}} = \frac{r}{k_n}, \quad p_{\frac{k_n}{2}+1} = \dots = p_{k_n} = \frac{2-r}{k_n} \quad (3.1)$$

for $r \in (0, 2)$. For given n and k_n , each of the above probability distributions is uniquely determined by r . We note that $p_1 = \dots = p_{k_n} = \frac{1}{k_n}$ if and only if $r = 1$.

Table 1 contains estimated sizes for the five tests $\mathcal{R}$, $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_3$ and $\mathcal{R}_5$ with $n = 100, 1000$ and some selected values for k_n . For each combination of (n, k_n) , we take three probability distributions from (3.1) with $r = 0.1, 0.2, 0.6$ and 1.0 , respectively. When $r = 1.0$, all the five tests are the same. From Table 1 the estimated sizes for all five tests are very close to the nominal level 0.05 , and thus, we conclude all the five tests perform very well in terms of the accuracy in type I error.

To assess the overall performance of the distributional approximations to the standardized test statistics $(S_{n1} + c|S_{n2}| - (k_n - 1))/\sigma_{n1}$ under the null hypothesis, we compare the empirical distributions of the test statistics based on 10000 samples and their theoretical cumulative distribution functions under the null hypothesis with a distribution from family (3.1). Figure 1 contains plots for both the empirical distributions and the approximate distributions of the test statistics with $c = 0, 1, 3$. The parameter of the distribution under H_0 is set to be $r = 0.2$. We discover from Figure 1 that all three theoretical distributions for the test statistics fit the empirical distributions very well, and their accuracies improve when sample size is getting large. Results for other distributions are similar and are not reported here.

To estimate the power for these tests, for each combination of n and k_n , we choose probability distribution (3.1) as the null hypothesis with $r = 0.2$ (or $r = 0.6$), and use probability

distribution (3.1) with $r = 0.2 \pm 0.1$ (or accordingly, $r = 0.6 \pm 0.1$) as alternative hypotheses from which random samples are generated. Table 2 lists the estimated powers for the five tests. Surprisingly, the power of test $\mathcal{R}$ is nearly zero when the value of r in the true alternative hypothesis is smaller than the value of r specified in the null hypothesis, that is, Pearson's goodness-of-fit test is seriously biased in those cases. As we have explained below equation (1.2), this is mainly due to $\sum_{i=1}^{k_n} \frac{p'_i - p_i}{p_i}$. For example, if $r = 0.2$ for the distribution under the null hypothesis and $r = 0.1$ under the alternative, we have

$$\sum_{i=1}^{k_n} \frac{p'_i - p_i}{p_i} = \left(\frac{-0.1}{0.2} + \frac{0.1}{1.8} \right) \frac{k_n}{2} = -\frac{2k_n}{9}.$$

We can also estimate the standard deviation of $\mathcal{X}_n^2$ under the alternative hypothesis and find out that it is much smaller than order k_n . This explains the incapability of Pearson's goodness-of-fit test in detecting an alternative in this case. We also notice that the performance of $\mathcal{R}$ is quite regular when $r = 0.2$ for the distribution under the null hypothesis and $r = 0.3$ under the alternative. Since $\sum_{i=1}^{k_n} \frac{p'_i - p_i}{p_i} = \frac{2k_n}{9} > 0$ in this case, Pearson's goodness-of-fit test gains its power. In both examples, the power of test $\mathcal{R}_c$ increases with c . In the first example, test $\mathcal{R}_c$ is superior to $\mathcal{R}$ for all $c \geq 0$. In the second example, test $\mathcal{R}_c$ outperforms $\mathcal{R}$ when $c \geq 3$.

It seems plausible that test $\mathcal{R}_c$ with $c > 0$ improves upon $\mathcal{R}$ when $\sum_{i=1}^{k_n} \frac{p'_i - p_i}{p_i}$ is quite different from zero. It is interesting to know how much improvement can be made when $\sum_{i=1}^{k_n} \frac{p'_i - p_i}{p_i}$ is zero or very close to zero. To make an empirical comparison, we introduce a new family of probability distributions. For convenience, we assume that k_n is divisible by 4 and define a family of probability distributions

$$p'_1 = \cdots = p'_{\frac{k_n}{4}} = \frac{1.5r'}{k_n}, p'_{\frac{k_n}{4}+1} = \cdots = p'_{\frac{k_n}{2}} = \frac{0.5r'}{k_n}, p'_{\frac{k_n}{2}+1} = \cdots = p'_{k_n} = \frac{2-r'}{k_n} \quad (3.2)$$

for $r' \in (0, 2)$. For given n and k_n , each probability distribution above is uniquely determined by r' . It is easy to see that for a probability distribution $(p_1, \dots, p_{k_n})$ from (3.1) and a probability distribution $(p'_1, \dots, p'_{k_n})$ from (3.2), $\sum_{i=1}^{k_n} \frac{p'_i - p_i}{p_i} = 0$ if $r = r'$. Table 3 includes estimated powers of the five tests for several combinations of n and k_n with $r = r' = 0.6, 1.4$. The probability distribution under the null hypothesis is from family (3.1) with parameter r and the true probability distribution under the alternative is from (3.2) with parameter $r' = r$. From Table 3, the power of $\mathcal{R}_c$ decreases with c for large c . We note that the constant c represents the weight of $|S_{n2}|$ we take into account in the test, and S_{n1} has always a positive shift in location

under the alternative, and thus it is more likely to detect the alternative if the weight of $|S_{n2}|$ in the test is smaller. In other words, increasing the weight c can decrease the power of $\mathcal{R}_c$ in this case. Therefore, we do not recommend to use a large c in general. In Table 3, we have used least favorable distributions to the use of S_{n2} under alternatives since the expectations of S_{n2} under the alternatives are zero. Overall, the performance of test $\mathcal{R}_0$ is slightly better than test $\mathcal{R}_1$ from Table 3.

Now we compare $\mathcal{X}_n^2$ and $S_{n1} + c|\bar{S}_{n2}|$ for the case of equiprobable cells. Recall that $\mathcal{X}_n = S_{n1}$ in this case. We define $\bar{S}_{n2}$ by using the method discussed at the end of Section 2, that is, we define $k_0 = 0.8k_n$ and set $c_i = \frac{k_n}{k_0} = \frac{1}{0.8} = 1.25$ for $1 \leq i \leq k_0$ and 0 otherwise. This time, we consider only two tests, $\mathcal{R}_0$ and $\bar{\mathcal{R}}_1$, as defined in (2.9) and (2.15) with $c = 1$. For several combinations of n and k_n , the sizes for the two tests are estimated based on 10000 replicates. The powers for the two tests are also estimated when the distributions under the alternatives are from family (3.1) with $r = 0.8, 1.2$ and 1.4 or from family (3.2) with $r' = 0.8, 1.1$ and 1.2 . The estimated sizes and powers are reported in Table 4.

From Table 4, we conclude that the sizes for both $\mathcal{R}_1$ and $\bar{\mathcal{R}}_1$ are close to the nominal level 0.05, and in general, $\bar{\mathcal{R}}_1$ is more powerful than $\mathcal{R}_1$. These empirical results are consistent with Theorem 2.4, and adding the term $\bar{S}_{n2}$ defined in (2.11), a non-trivial linear combination of the terms $(o_i - e_i)/e_i$, can improve the power of the test significantly.

Next, we extend our comparison of S_{n1} and $S_{n1} + c|\bar{S}_{n2}|$ to some none-equiprobable cases. We also use the method discussed at the end of Section 2 to define $\bar{S}_{n2}$ by setting $k_0 = 0.40k_n$ this time. We consider the distributions from family (3.2) and use the same settings as in Table 3. Only for an illustration purpose, we demonstrate the weighted test statistics can improve the power of the test $\mathcal{R}_1$ when the expectation of S_{n2} under H_1 in (2.6) is zero. Both the sizes and powers for tests $\mathcal{R}_0$ and $\mathcal{R}_1$ are reported in Table 5. From the table, we see that both the tests maintain reasonable sizes for all combinations of n and k_n . From Tables 3 and 5, we can conclude that test $\bar{\mathcal{R}}_1$ performs significantly better than $\mathcal{R}_0$ and $\mathcal{R}_1$ in terms of power.

Finally, we compare the performance of these tests under sparsity. To this end, we introduce a class of distributions as follows

$$p_1 = \cdots = p_{0.95k_n} = \frac{r}{8k_n}, \quad p_{0.95k_n+1} = \cdots = p_{k_n} = \frac{160 - 19r}{8k_n} \quad (3.3)$$

for $r \in (0, 8)$, where k_n is a multiple of 20. We see that only 5% of probabilities p_i 's take a larger value $\frac{160-19r}{8k_n}$ in (3.3). In our study, we select $r = 2$ for the distribution under the null

hypothesis and estimate the sizes of the five tests considered in Table 1 and estimate the powers of the tests under the alternatives $r = 1$ and $r = 3$ for some combinations of n and k_n . The estimated sizes and powers are reported in Table 6.

Results in Table 6 are quite similar to Tables 1 and 2 in the following aspects: **a.** the estimated type I errors are close to nominal level 0.05 for all five tests; **b.** Pearson's test $\mathcal{R}$ loses its power totally for some distributions under the alternative while tests $\mathcal{R}_1$, $\mathcal{R}_3$, and $\mathcal{R}_5$ outperform with large powers. Although test $\mathcal{R}_0$ is better than Pearson's test, its overall performance is not quite satisfactory. For example, for some distributions under the alternative, its powers are smaller than the type I errors. We examine the results in Table 6 when $n = 100$ with $r = 1$ under the alternative and find out that the power of the test decreases from 0.0416 to 0.0246 when k_n increases from 100 to 400. The same phenomenon can also be observed when $n = 1000$.

It is worth mentioning that conditions (2.2) and (2.3) that ensure the asymptotic normality of S_{n1} and $\mathcal{X}_n^2$ may be moderately violated if k_n is too large compared with n . This is the case for some combinations of n and k_n and for some distributions used in Table 6. In our study, the sizes (type I errors) of all five tests are reasonably close to the nominal level 0.05; see Tables 1 and 6. In terms of power, test $\mathcal{R}_1$, $\mathcal{R}_3$, and $\mathcal{R}_5$ are also very robust as they gain good powers from Tables 2 and 6.

To conclude this section, we present more discussion on selection of c . Our simulation study indicates that there is no answer for optimal section of c in general. As we have pointed out, c represents the weight of $|S_{n2}|$ in test $\mathcal{R}_c$. When c is large, the test $\mathcal{R}_c$ is almost the same as $\{|S_{n2}|/\sigma_{n2} > z_{\alpha/2}\}$, where $z_{\alpha/2}$ is the $\alpha/2$ -level critical value of the standard normal distribution. On the one hand, the power of test $\mathcal{R}_c$ increases with c in most cases in Tables 2 and 6. By comparing the powers of the test $\mathcal{R}_c$ for various values of c in our simulation study, we find out that the increment in the power for the test $\mathcal{R}_c$ is very limited when c is larger than 3, and the power of $\mathcal{R}_2$ is close to that of $\mathcal{R}_3$ in most cases. On the other hand, Table 3 indicates that one may prefer to employ a test $\mathcal{R}_c$ with a smaller value c in the worst scenarios such as those distributions given in (3.2). We observe from Table 3 that the power of $\mathcal{R}_1$ is very close to that of $\mathcal{R}_0$ in most cases. Our simulation study also shows that the power of $\mathcal{R}_2$ is only slightly smaller than that of $\mathcal{R}_1$ in most cases. Intuitively, the power of test $\mathcal{R}_c$ depends on the probability distributions under both the null and alternative hypotheses as well as the relative convergence rate of n and k_n . A theoretical investigation on how the power function of the test

$\mathcal{R}_c$ depends on these factors can be very helpful but may be very complicated. In practice, the distributions under the alternative are unknown, and the optimal choice of c that works for all distributions does not exist. To balance different situations, one can use tests $\mathcal{R}_1$ or $\mathcal{R}_2$. As a general recommendation, one can calculate tests $\mathcal{R}$, $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_2$ for comparison purpose. One should be cautious about accepting the null hypothesis based on $\mathcal{R}$ or $\mathcal{R}_0$ since the powers of the two tests may be much smaller than their sizes or type I errors. In other words, tests $\mathcal{R}$ and $\mathcal{R}_0$ may reject the alternative hypotheses with a probability close to one when the null hypotheses are not true.

4 A real data application

As an application, we study the winning numbers from the Minnesota Lottery Game *Daily 3*. The game has been played for many years, and a winning number consisting of three digits is drawn daily. The three digits are drawn from digits $0, 1, \dots, 9$. Minnesota Lottery does not reveal how the three digits are selected in its official website. According to the State Lottery Report Card (available at the site <https://www.lotterypost.com/lottery-report-card.asp>), the winning numbers for Minnesota *Daily 3* are drawn by computer programs.

In Game *Daily 3*, there are 1000 possible drawing outcomes. We are interested in whether the drawing mechanism for *Daily 3* is random, this is, whether all 1000 possible outcomes are equally likely. Recent winning numbers for this game can be found at the Minnesota Lottery web site https://www.mnlottery.com/games/lotto_games/daily_3/winning_s/. As an application, we examine some early data from Game *Daily 3*. We have collected a total of 2919 data points from August 14, 1990 to August 13, 1998. The winning numbers were drawn every day except Christmas days in years 1990, 1991 and 1992. This old dataset was obtained from the official Minnesota lottery website but now it is no longer available. One may find it from <https://www.lotterypost.com/results>.

Since our null hypothesis is that $p_1 = \dots = p_{1000} = 1/1000$, all three test statistics, $\mathcal{X}_n^2$, S_{n1} , and $S_{n1} + c|S_{n2}|$, are the same. We first apply all 2919 data for the test. The observed $\mathcal{X}_n^2$ is 978.5677 with mean 999 and standard deviation 44.69, and the standardized statistic is $(978.5677 - 999)/44.69 = -0.4572$, which has a p-value 0.6962. To apply the new test $\overline{\mathcal{R}}_1$, we assign a value 1.25 to these 800 weights c_i to the cells associated to the drawing numbers whose first digits are smaller than 8. Then we have an observed value -0.1774 for $(S_{n1} + |\overline{S}_{n2}| - (k_n - 1))/\sigma_n$, and the corresponding p-value is 0.6927. Therefore, at the 5% significance level, we

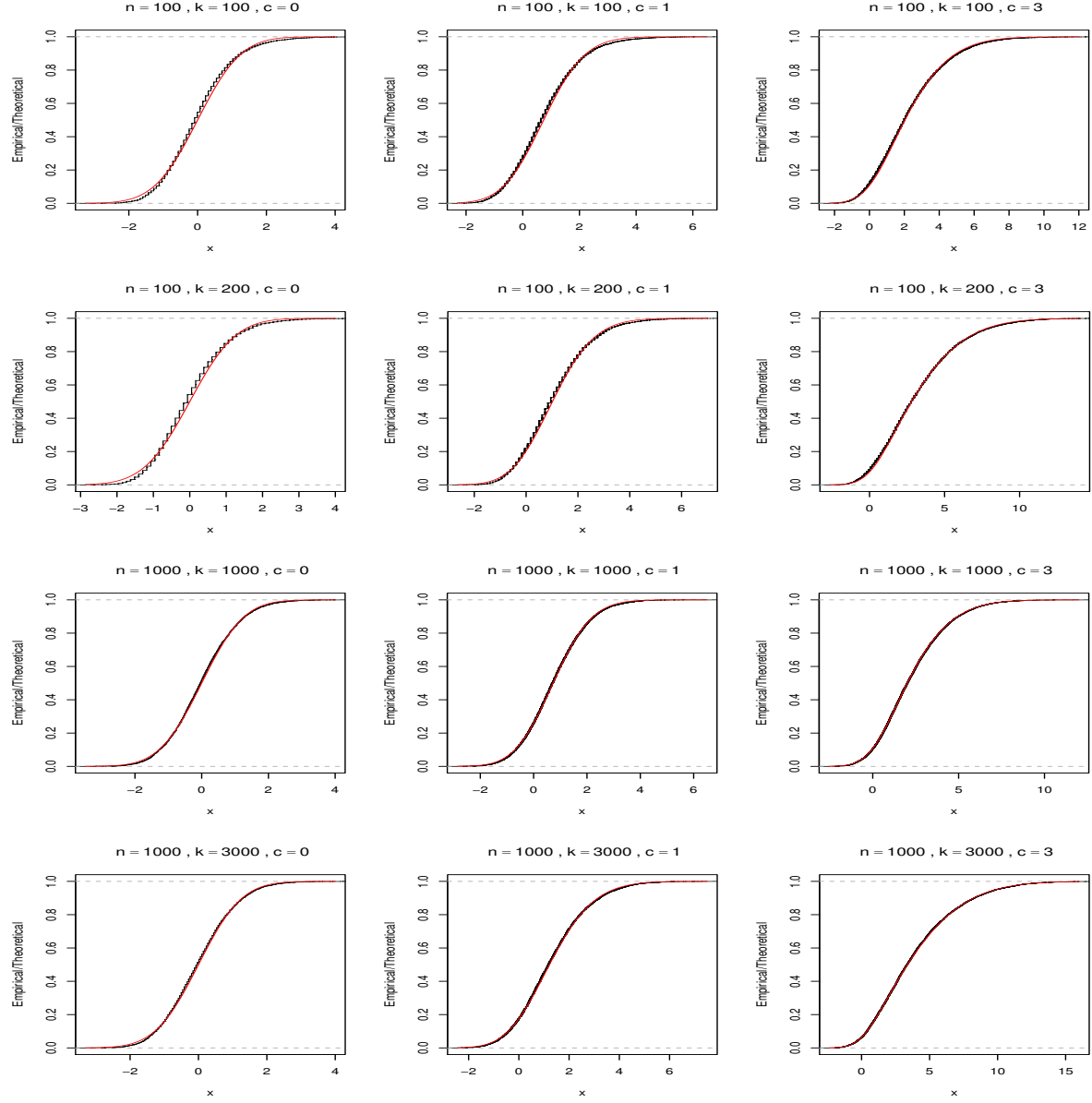


Figure 1: Plots of the empirical distribution for the standardized test statistics $(S_{n1} + c|S_{n2}| - (k_n - 1))/\sigma_{n1}$ and their theoretical limiting cumulative distribution $\Psi(x, c\sigma_{n2}/\sigma_{n1})$ under the null hypothesis from family (3.1) with $r = 0.2$. Ψ is defined in (2.8). We select $c = 0, 1$, and 3 , corresponding to the tests $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_3$. In these plots, the red/smooth lines represent the theoretical cumulative distributions $\Psi(x, c\sigma_{n2}/\sigma_{n1})$.

Table 1: **Estimated Sizes of Tests (Nominal Level $\alpha = 0.05$):** Probability distributions under null hypotheses and the true distributions that are used to generate samples are from family (3.1) with different values for parameter r

Sample Size (n)	k_n	Distribution under H_0	True distribution	Probability of Rejecting H_0 for Tests				
				$\mathcal{R}$	$\mathcal{R}_0$	$\mathcal{R}_1$	$\mathcal{R}_3$	$\mathcal{R}_5$
100	50	$r = 0.1$	$r = 0.1$	0.0679	0.0638	0.0642	0.0536	0.0500
100	50	0.2	0.2	0.0663	0.0614	0.0668	0.0633	0.0599
100	50	0.6	0.6	0.0609	0.0593	0.0628	0.0672	0.0674
100	50	1.0	1.0	0.0576	0.0576	0.0576	0.0576	0.0576
100	100	$r = 0.1$	$r = 0.1$	0.0610	0.0595	0.0569	0.0482	0.0450
100	100	0.2	0.2	0.0667	0.0610	0.0594	0.0567	0.0526
100	100	0.6	0.6	0.0579	0.0562	0.0594	0.0616	0.0642
100	100	1.0	1.0	0.0531	0.0531	0.0531	0.0531	0.0531
100	200	$r = 0.1$	$r = 0.1$	0.0644	0.0679	0.0567	0.0491	0.0413
100	200	0.2	0.2	0.0619	0.0579	0.0622	0.0536	0.0544
100	200	0.6	0.6	0.0580	0.0550	0.0606	0.0579	0.0539
100	200	1.0	1.0	0.0691	0.0691	0.0691	0.0691	0.0691
1000	300	$r = 0.1$	$r = 0.1$	0.0571	0.0543	0.0574	0.0521	0.0519
1000	300	0.2	0.2	0.0552	0.0523	0.0567	0.0562	0.0538
1000	300	0.6	0.6	0.0515	0.0518	0.0527	0.0536	0.0556
1000	300	1.0	1.0	0.0514	0.0514	0.0514	0.0514	0.0514
1000	1000	$r = 0.1$	$r = 0.1$	0.0570	0.0594	0.0583	0.0533	0.0543
1000	1000	0.2	0.2	0.0525	0.0548	0.0526	0.0523	0.0489
1000	1000	0.6	0.6	0.0538	0.0534	0.0526	0.0505	0.0516
1000	1000	1.0	1.0	0.0530	0.0530	0.0530	0.0530	0.0530
1000	3000	$r = 0.1$	$r = 0.1$	0.0542	0.0600	0.0515	0.0501	0.0505
1000	3000	0.2	0.2	0.0549	0.0568	0.0570	0.0511	0.0510
1000	3000	0.6	0.6	0.0544	0.0546	0.0561	0.0553	0.0532
1000	3000	1.0	1.0	0.0565	0.0565	0.0565	0.0565	0.0565
1000	10000	$r = 0.1$	$r = 0.1$	0.0485	0.0648	0.0447	0.0468	0.0472
1000	10000	0.2	0.2	0.0544	0.0614	0.0506	0.0493	0.0494
1000	10000	0.6	0.6	0.0520	0.0590	0.0567	0.0491	0.0486
1000	10000	1.0	1.0	0.0567	0.0567	0.0567	0.0567	0.0567

Table 2: **Estimated Powers of Tests** ($\alpha = 0.05$): Probability distributions under null hypotheses and the true distributions that are used to generate samples are from family (3.1) with different values for parameter r

Sample Size (n)	k_n	Distribution under H_0	True distribution	Probability of Rejecting H_0 for Tests				
				$\mathcal{R}$	$\mathcal{R}_0$	$\mathcal{R}_1$	$\mathcal{R}_3$	$\mathcal{R}_5$
100	50	$r = 0.1$	$r = 0.05$	0.0042	0.0455	0.1044	0.1408	0.1315
100	50	0.2	0.1	0.0041	0.0677	0.1998	0.3646	0.3864
100	50	0.2	0.3	0.3207	0.1377	0.2608	0.3679	0.3884
100	50	0.6	0.5	0.0436	0.0699	0.0909	0.1316	0.1614
100	50	0.6	0.7	0.1247	0.0856	0.1066	0.1463	0.1768
100	200	$r = 0.1$	$r = 0.05$	0.0025	0.0372	0.1120	0.1096	0.0893
100	200	0.2	0.1	0.0010	0.0413	0.2627	0.3577	0.3730
100	200	0.2	0.3	0.3890	0.1156	0.3178	0.3831	0.3935
100	200	0.6	0.5	0.0231	0.0573	0.0924	0.1526	0.1743
100	200	0.6	0.7	0.1399	0.0723	0.1111	0.1678	0.1946
1000	300	$r = 0.1$	$r = 0.05$	0.0000	0.0936	0.7955	0.9723	0.9823
1000	300	0.2	0.1	0.0001	0.2739	0.9465	0.9996	1.0000
1000	300	0.2	0.3	0.8806	0.3361	0.8425	0.9876	0.9951
1000	300	0.6	0.5	0.0369	0.1202	0.2360	0.5269	0.7207
1000	300	0.6	0.7	0.2917	0.1464	0.2591	0.5150	0.6971
1000	1000	$r = 0.1$	$r = 0.05$	0.0000	0.0577	0.9308	0.9825	0.9856
1000	1000	0.2	0.1	0.0000	0.1123	0.9937	0.9999	1.0000
1000	1000	0.2	0.3	0.9678	0.2041	0.9463	0.9953	0.9968
1000	1000	0.6	0.5	0.0072	0.0814	0.2860	0.7165	0.8531
1000	1000	0.6	0.7	0.3585	0.1000	0.2880	0.6896	0.8311
1000	3000	$r = 0.1$	$r = 0.05$	0.0000	0.0423	0.9741	0.9874	0.9880
1000	3000	0.2	0.1	0.0000	0.0657	0.9996	1.0000	1.0000
1000	3000	0.2	0.3	0.9951	0.1545	0.9902	0.9974	0.9979
1000	3000	0.6	0.5	0.0004	0.0632	0.4394	0.8597	0.9174
1000	3000	0.6	0.7	0.5278	0.0826	0.4314	0.8272	0.8948
1000	10000	$r = 0.1$	$r = 0.05$	0.0000	0.0431	0.9838	0.9883	0.9891
1000	10000	0.2	0.1	0.0000	0.0476	1.0000	1.0000	1.0000
1000	10000	0.2	0.3	0.9974	0.1292	0.9952	0.9967	0.9969
1000	10000	0.6	0.5	0.0000	0.0575	0.7122	0.9226	0.9377
1000	10000	0.6	0.7	0.7677	0.0777	0.6766	0.8997	0.9165

Table 3: **Estimated Powers of Tests** ($\alpha = 0.05$): Probability distributions under null hypotheses and the true distributions that are used to generate samples are from family (3.1) with parameter r and family (3.2) with parameter r' , respectively

Sample Size (n)	k_n	Distribution under H_0	Distribution under H_1	Probability of Rejecting H_0 for Tests				
				$\mathcal{R}$	$\mathcal{R}_0$	$\mathcal{R}_1$	$\mathcal{R}_3$	$\mathcal{R}_5$
100	100	$r = 0.6$	$r' = 0.6$	0.1562	0.1511	0.1536	0.1400	0.1173
100	100	1.4	1.4	0.3179	0.3418	0.3386	0.2895	0.2168
100	200	$r = 0.6$	$r' = 0.6$	0.1197	0.1238	0.1215	0.0992	0.0808
100	200	1.4	1.4	0.1940	0.2260	0.2241	0.1647	0.1185
1000	300	$r = 0.6$	$r' = 0.6$	0.8606	0.8739	0.8716	0.8513	0.8083
1000	300	1.4	1.4	0.9999	1.0000	1.0000	1.0000	1.0000
1000	1000	$r = 0.6$	$r' = 0.6$	0.4694	0.4939	0.4833	0.4016	0.2968
1000	1000	1.4	1.4	0.9713	0.9758	0.9721	0.9403	0.8441
1000	3000	$r = 0.6$	$r' = 0.6$	0.2252	0.2575	0.2370	0.1556	0.1104
1000	3000	1.4	1.4	0.6195	0.7019	0.6605	0.4225	0.2482
1000	10000	$r = 0.6$	$r' = 0.6$	0.1156	0.1547	0.1292	0.0794	0.0671
1000	10000	1.4	1.4	0.2212	0.3412	0.2761	0.1275	0.0924

Table 4: **Estimated Sizes and Powers for Tests $\mathcal{R}_0$ and $\overline{\mathcal{R}}_1$ for Equiprobable Cells** ($\alpha = 0.05$). The distributions under alternatives are from family (3.1) with parameter r and from family (3.2) with parameter r' , respectively

Sample Size (n)	k_n	Tests	Sizes	Power under family (3.1)			Power under family (3.2)		
				$r = 0.8$	$r = 1.2$	$r = 1.4$	$r' = 0.8$	$r' = 1.1$	$r' = 1.2$
100	100	$\mathcal{R}_0$	0.0531	0.0964	0.0900	0.3009	0.2013	0.3030	0.4264
100	100	$\overline{\mathcal{R}}_1$	0.0593	0.1306	0.1262	0.4449	0.2544	0.3208	0.4656
100	200	$\mathcal{R}_0$	0.0691	0.0976	0.0973	0.2342	0.1830	0.2428	0.3349
100	200	$\overline{\mathcal{R}}_1$	0.0604	0.1256	0.1140	0.3720	0.2084	0.2234	0.3450
1000	300	$\mathcal{R}_0$	0.0514	0.4804	0.4739	0.9999	0.9978	0.9998	1.0000
1000	300	$\overline{\mathcal{R}}_1$	0.0556	0.6341	0.6264	1.0000	0.9990	0.9998	1.0000
1000	1000	$\mathcal{R}_0$	0.0530	0.2249	0.2238	0.9515	0.9501	0.9998	0.9945
1000	1000	$\overline{\mathcal{R}}_1$	0.0552	0.5072	0.4986	0.9993	0.9604	0.9998	0.9988
1000	3000	$\mathcal{R}_0$	0.0565	0.1384	0.1435	0.6444	0.4710	0.6459	0.8444
1000	3000	$\overline{\mathcal{R}}_1$	0.0561	0.5650	0.5587	0.9984	0.8342	0.7316	0.9675
1000	10000	$\mathcal{R}_0$	0.0567	0.0963	0.0972	0.3176	0.2268	0.3136	0.4599
1000	10000	$\overline{\mathcal{R}}_1$	0.0494	0.7127	0.7153	1.0000	0.8199	0.4869	0.9097

Table 5: **Estimated Sizes and Powers of Tests $\mathcal{R}_0$ and $\overline{\mathcal{R}}_1$** ($\alpha = 0.05$): Probability distributions under null hypotheses are from family (3.1) with parameter r and the distributions under alternatives are from family (3.2) with parameter r'

Sample Size (n)	k_n	Distribution under H_0	Sizes for Tests		Distribution under H_1	Powers for Tests	
			$\mathcal{R}_0$	$\overline{\mathcal{R}}_1$		$\mathcal{R}_0$	$\overline{\mathcal{R}}_1$
100	100	$r = 0.6$	0.0562	0.0612	$r' = 0.6$	0.1511	0.1844
100	100	1.4	0.0612	0.0665	1.4	0.3418	0.4478
100	200	$r = 0.6$	0.0550	0.0515	$r' = 0.6$	0.1238	0.1525
100	200	1.4	0.0564	0.0606	1.4	0.2260	0.3704
1000	300	$r = 0.6$	0.0518	0.0560	$r' = 0.6$	0.8739	0.9227
1000	300	1.4	0.0537	0.0548	1.4	1.0000	1.0000
1000	1000	$r = 0.6$	0.0534	0.0547	$r' = 0.6$	0.4939	0.7565
1000	1000	1.4	0.0573	0.0560	1.4	0.9758	0.9995
1000	3000	$r = 0.6$	0.0546	0.0542	$r' = 0.6$	0.2575	0.6527
1000	3000	1.4	0.0515	0.0547	1.4	0.7019	0.9949
1000	10000	$r = 0.6$	0.0590	0.0497	$r' = 0.6$	0.1547	0.6164
1000	10000	1.4	0.0576	0.0505	1.4	0.3412	0.9947

couldn't reject the null hypothesis and conclude that the 1000 possible winning numbers may be drawn with equal probability.

Next, we test the hypothesis that $p_1 = \dots = p_{1000} = 1/1000$ based on each of eight periods. A period starts from August 14 in one year and ends next August 13. Namely, Period 1 is from August 14, 1990 to August 13, 1991, and Period 2 is from August 14, 1991 to August 13, 1993, so on. For each of the first seven periods, both tests result in large p-values. For Period 8, that is, during Aug 14, 1997 to Aug 13, 1998, the p-value for test $\mathcal{R}$ is 0.00085, and the p-value from test $\overline{\mathcal{R}}_1$ is 0.002867. We identify that the data during August 14, 1997 and August 13, 1998 seem highly abnormal. We find out that both winning numbers (3, 7, 5) and (4, 4, 8) appeared 4 times, and each of the other 11 numbers appeared 3 times.

Our second test procedure consists of 8 individual tests. In order to control the overall type I error at the 5% level, we use the Bonferroni inequality for multiple comparisons, that is, we reject the null hypothesis that $p_1 = \dots = p_{1000} = 1/1000$ at level 0.05 if any of the 8 individuals tests is rejected at level $0.05/8 = 0.00625$. Since both tests $\mathcal{R}$ and $\overline{\mathcal{R}}_1$ have a p-value smaller than 0.00625 for Period 8, we conclude that the hypothesis of equiprobability can be rejected at level 0.05.

Table 6: **Estimated Sizes and Powers of Tests under Sparsity (Nominal Level $\alpha = 0.05$):** Probability distributions under null hypotheses and the true distributions that are used to generate samples are from family (3.3) with different values for parameter r

Sample Size(n)	k_n	Distribution under H_0	True distribution	Size or power	Probability of Rejecting H_0 for Tests				
					$\mathcal{R}$	$\mathcal{R}_0$	$\mathcal{R}_1$	$\mathcal{R}_3$	$\mathcal{R}_5$
100	100	$r = 2$	$r = 2$	size	0.0623	0.0607	0.0610	0.0547	0.0509
100	100	2	1	power	0.0000	0.0416	0.7327	0.8545	0.8613
100	100	2	3	power	0.6877	0.2145	0.6008	0.7410	0.7559
100	200	$r = 2$	$r = 2$	size	0.0609	0.0632	0.0597	0.0516	0.0520
100	200	2	1	power	0.0000	0.0331	0.7878	0.8529	0.8688
100	200	2	3	power	0.7450	0.1899	0.6616	0.7526	0.7626
100	400	2	2	size	0.0584	0.0617	0.0517	0.0476	0.0468
100	400	2	1	power	0.0000	0.0246	0.8237	0.8714	0.8712
100	400	2	3	power	0.7837	0.1690	0.6973	0.7543	0.7611
1000	300	$r = 2$	$r = 2$	size	0.0598	0.0559	0.0601	0.0547	0.0518
1000	300	2	1	power	0.0000	0.9779	1.0000	1.0000	1.0000
1000	300	2	3	power	0.9999	0.8164	0.9997	1.0000	1.0000
1000	1000	$r = 2$	$r = 2$	size	0.0582	0.0602	0.0579	0.0540	0.0544
1000	1000	2	1	power	0.0000	0.5604	1.0000	1.0000	1.0000
1000	1000	2	3	power	1.0000	0.5143	1.0000	1.0000	1.0000
1000	3000	$r = 2$	$r = 2$	size	0.0551	0.0597	0.0536	0.0515	0.0516
1000	3000	2	1	power	0.0000	0.1361	1.0000	1.0000	1.0000
1000	3000	2	3	power	1.0000	0.3219	1.0000	1.0000	1.0000
1000	10000	$r = 2$	$r = 2$	size	0.0511	0.0633	0.0520	0.0488	0.0487
1000	10000	2	1	power	0.0000	0.0384	1.0000	1.0000	1.0000
1000	10000	2	3	power	1.0000	0.2211	1.0000	1.0000	1.0000

5 Concluding remarks

In this paper, we investigate the performance of Pearson's goodness-of-fit test when both the number of cells and the sample size go to infinity. Under quite general conditions, we have obtained the limiting distributions for Pearson's goodness-of-fit test statistic and Zelterman's D^2 test statistic. By decomposing Pearson's test statistic, we propose new test statistics so as to overcome the bias problem of Pearson's chi-squared test. Our simulation study indicates that test $\mathcal{R}_1$ is superior to Pearson's goodness-of-fit test $\mathcal{R}$ in general. $\mathcal{R}_1$ gains a much larger power than $\mathcal{R}$ in most cases and is no longer biased. Our new test $\overline{\mathcal{R}}_1$ is also much more powerful than $\mathcal{R}$ for testing the equiprobable cells. For small and moderate sample sizes, as pointed out by a referee, one can apply permutation procedures (Drikvandi et al. [5]) or bootstrap procedures

(Nordhausen et al. [14]) to conduct the tests.

Supplementary material

Proofs of the main results in the paper are given in the Supplement.

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Pearson's goodness-of-fit tests for sparse distributions

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Appendix: Proofs of the Main Results

For each $j \in \{1, \dots, n\}$, define random variable $X_j = i$ if E_i occurs in the j -th trial of the experiment. Then X_j , $1 \leq j \leq n$, are independent and identically distributed random variables with $P(X_j = i) = p_i$ for $1 \leq i \leq k_n$, $1 \leq j \leq n$. Define

$$\delta_{i,j} = I(X_j = i) - p_i \quad \text{for } 1 \leq i \leq k_n, \ 1 \leq j \leq n$$

and set

$$\Delta_{i,\ell} = \sum_{j=1}^{\ell} \delta_{i,j} \quad \text{for } 1 \leq i \leq k_n, \ 1 \leq \ell \leq n.$$

For convenience, set $\Delta_{i,0} = 0$ for any $1 \leq i \leq k_n$.

From now on, we use $\mathbb{E}(\cdot)$ to denote the expectation under the null hypothesis that $P(E_i) = p_i$ for $1 \leq i \leq k_n$. When an alternative is specified as $H_1: P(E_i) = p'_i$ for $1 \leq i \leq k_n$, where $(p'_1, \dots, p'_{k_n}) \neq (p_1, \dots, p_{k_n})$, $\mathbb{E}(\cdot|H_1)$ denotes the conditional expectation under H_1 .

We can easily verify the following equations:

$$\mathbb{E}(\delta_{i,j}) = 0, \quad \mathbb{E}(\delta_{i,j}^2) = p_i(1 - p_i); \quad (\text{A.1})$$

$$\mathbb{E}(\delta_{i,j}\delta_{i',j'}) = 0 \quad \text{if } j \neq j'; \quad (\text{A.2})$$

$$\mathbb{E}(\delta_{i,j}\delta_{i',j}) = \begin{cases} p_i(1 - p_i), & \text{if } i = i'; \\ -p_i p_{i'}, & \text{if } i \neq i', \end{cases} \quad (\text{A.3})$$

$$\sum_{i=1}^{k_n} \delta_{i,j} = 0 \quad \text{and} \quad \sum_{i=1}^{k_n} \Delta_{i,j} = 0, \quad (\text{A.4})$$

where $1 \leq i, i' \leq k_n$, $1 \leq j, j' \leq n$ in the above equations.

Since $\sum_{j=1}^{\ell} I(X_j = i)$ is the sum of ℓ independent Bernoulli random variables, its distribution is binomial. We have

$$\mathbb{E}(\Delta_{i,\ell}^2) = \ell p_i(1 - p_i), \quad 1 \leq \ell \leq n. \quad (\text{A.5})$$

We can also verify that

$$\mathbb{E}(\Delta_{i_1,\ell}\Delta_{i_2,\ell}) = -\ell p_{i_1}p_{i_2}, \quad 1 \leq i_1 \neq i_2 \leq k_n, \quad 1 \leq \ell \leq n. \quad (\text{A.6})$$

For each $i \in \{1, \dots, k_n\}$, we have

$$o_i = \sum_{j=1}^n I(X_j = i) \quad \text{and} \quad o_i - e_i = \sum_{j=1}^n \delta_{i,j} = \Delta_{i,n}. \quad (\text{A.7})$$

We also need the following expectations under the alternative H_1

$$\mathbb{E}(\delta_{i,j}|H_1) = p'_i - p_i \quad \text{and} \quad \mathbb{E}(\delta_{i,j}\delta_{i,j'}|H_1) = (p'_i - p_i)^2 \quad (\text{A.8})$$

for $1 \leq i \leq k_n$, $1 \leq j \neq j' \leq n$.

Lemma A.1. *Let S_{n1} and S_{n2} be defined in (1.2). Then*

$$S_{n1} = \frac{1}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} \sum_{1 \leq j_1 \neq j_2 \leq n} \delta_{i,j_1}\delta_{i,j_2} + k_n - 1 \quad (\text{A.9})$$

and

$$S_{n2} = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^{k_n} \frac{\delta_{i,j}}{p_i}. \quad (\text{A.10})$$

Under the null hypothesis that $P(X_1 = i) = p_i$ for $1 \leq i \leq k_n$, we have

$$\mathbb{E}(S_{n1} - (k_n - 1)) = 0, \quad \mathbb{E}(S_{n2}) = 0; \quad (\text{A.11})$$

Under the alternative H_1 : $P(X_1 = i) = p'_i$ for $1 \leq i \leq k_n$, we have

$$\mathbb{E}(S_{n1} - (k_n - 1)|H_1) = (n - 1) \sum_{i=1}^{k_n} \frac{(p'_i - p_i)^2}{p_i}, \quad \mathbb{E}(S_{n2}|H_1) = \sum_{i=1}^{k_n} \frac{p'_i - p_i}{p_i}. \quad (\text{A.12})$$

Proof. With the notations in the beginning of the section, (A.10) can be verified easily by using (A.7). To show (A.9), notice that

$$\begin{aligned} \mathcal{X}_n^2 &= \sum_{i=1}^{k_n} \frac{(\sum_{j=1}^n \delta_{i,j})^2}{np_i} \\ &= \frac{1}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} \sum_{j_1=1}^n \sum_{j_2=1}^n \delta_{i,j_1} \delta_{i,j_2} \\ &= \frac{1}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} \sum_{1 \leq j_1 \neq j_2 \leq n} \delta_{i,j_1} \delta_{i,j_2} + \frac{1}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} \sum_{j=1}^n \delta_{i,j}^2. \end{aligned} \quad (\text{A.13})$$

Since $\delta_{i,j}^2 = I(X_j = i) - 2I(X_j = i)p_i + p_i^2$, $\sum_{i=1}^{k_n} I(X_j = i) = 1$, and $\sum_{i=1}^{k_n} p_i = 1$, we have

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} \sum_{j=1}^n \delta_{i,j}^2 &= \frac{1}{n} \sum_{i=1}^{k_n} \sum_{j=1}^n \left(\frac{I(X_j = i)}{p_i} - 2I(X_j = i) + p_i \right) \\ &= \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^{k_n} \left(\frac{I(X_j = i)}{p_i} - 2I(X_j = i) + p_i \right) \\ &= \frac{1}{n} \sum_{j=1}^n \left(\sum_{i=1}^{k_n} \frac{I(X_j = i)}{p_i} - 1 \right) \\ &= \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^{k_n} \frac{I(X_j = i) - p_i}{p_i} + (k_n - 1) \\ &= \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^{k_n} \frac{\delta_{i,j}}{p_i} + (k_n - 1) \\ &= S_{n2} + k_n - 1, \end{aligned}$$

which, together with (A.13), yields (A.9).

Equations (A.11) and (A.12) follow from (A.9), (A.10), (A.2) and (A.8). This completes the proof. $\square$

Lemma A.2. Let c_i , $1 \leq i \leq k_n$, be non-negative numbers with $\sum_{i=1}^{k_n} c_i = k_n$. Then for any $j \geq 1$

$$\beta_{nj} := \sum_{i=1}^{k_n} \frac{c_i^{j+1}}{p_i^j} - k_n^{j+1} \geq 0, \quad (\text{A.14})$$

and the equality holds only if for some $c > 0$, $c_i = cp_i$ for $1 \leq i \leq k_n$.

Proof. We will prove (A.14) by induction. By using the Cauchy-Schwarz inequality we get

$$\sum_{i=1}^{k_n} \frac{c_i^2}{p_i} = \sum_{i=1}^{k_n} \left(\frac{c_i}{p_i^{1/2}} \right)^2 \sum_{i=1}^{k_n} (p_i^{1/2})^2 \geq \left(\sum_{i=1}^{k_n} \frac{c_i}{p_i^{1/2}} p_i^{1/2} \right)^2 = k_n^2$$

and the equality holds only if $(\frac{c_1}{\sqrt{p_1}}, \dots, \frac{c_{k_n}}{\sqrt{p_{k_n}}}) = c(\sqrt{p_1}, \dots, \sqrt{p_{k_n}})$ for some $c > 0$, and the latter is equivalent to $c_i = cp_i$ for $1 \leq i \leq k_n$. This implies (A.14) holds with $j = 1$. Now assume (A.14) holds for all $j \leq j_0$ for some $j_0 \geq 1$. We need to show (A.14) holds with $j = j_0 + 1$. If $j_0 + 1 = 2k$ is an even number where $k \geq 1$, then it follows from the Cauchy-Schwarz inequality that

$$\sum_{i=1}^{k_n} \frac{c_i^{j_0+2}}{p_i^{j_0+1}} = \frac{1}{k_n} \sum_{i=1}^{k_n} \left(\frac{c_i^{k+1/2}}{p_i^k} \right)^2 \sum_{i=1}^{k_n} (c_i^{1/2})^2 \geq \frac{1}{k_n} \left(\sum_{i=1}^{k_n} \frac{c_i^{k+1}}{p_i^k} \right)^2 \geq \frac{1}{k_n} (k_n^{k+1})^2 = k_n^{j_0+2},$$

and the equality holds only if $c_i = cp_i$ for $1 \leq i \leq k_n$ for some $c > 0$, proving (A.14) with $j = j_0 + 1$. If $j_0 + 1 = 2k + 1$ is an odd number with $k \geq 1$, then again from the Cauchy-Schwarz inequality

$$\begin{aligned} \sum_{i=1}^{k_n} \frac{c_i^{j_0+2}}{p_i^{j_0+1}} &= \sum_{i=1}^{k_n} \left(\frac{c_i^{k+1}}{p_i^{(2k+1)/2}} \right)^2 \sum_{i=1}^{k_n} (p_i^{1/2})^2 \geq \left(\sum_{i=1}^{k_n} \frac{c_i^{k+1} p_i^{1/2}}{p_i^{(2k+1)/2}} \right)^2 \\ &\geq \left(\sum_{i=1}^{k_n} \frac{c_i^{k+1}}{p_i^k} \right)^2 \geq (k_n^{k+1})^2 = k_n^{j_0+2}, \end{aligned}$$

i.e. (A.14) holds with $j = j_0 + 1$. Similarly, we have the equality only if $c_i = cp_i$ for $1 \leq i \leq k_n$ for some $c > 0$. This completes the proof. $\square$

Lemma A.3. Assume $\{n_r, r \geq 1\}$ is an increasing sequence of positive integers. If (2.2) holds with $n = n_r$ as $r \rightarrow \infty$, then (2.4) holds with $n = n_r$ as $r \rightarrow \infty$.

Proof. For brevity, we will drop the subscript r and write n_r as n in the proof.

To prove (2.4), we will employ a martingale technique. To this end, we first rewrite S_{n1} as

$$\begin{aligned}
S_{n1} - (k_n - 1) &= \frac{1}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} \sum_{1 \leq j_1 \neq j_2 \leq n} \delta_{i,j_1} \delta_{i,j_2} \\
&= \frac{2}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} \sum_{1 \leq j_1 < j_2 \leq n} \delta_{i,j_1} \delta_{i,j_2} \\
&= \frac{2}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} \sum_{\ell=2}^n \sum_{j=1}^{\ell-1} \delta_{i,j} \delta_{i,\ell} \\
&= \sum_{\ell=2}^n \left(\frac{2}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} \sum_{j=1}^{\ell-1} \delta_{i,j} \delta_{i,\ell} \right) \\
&= \sum_{\ell=2}^n \left(\frac{2}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} \Delta_{i,\ell-1} \delta_{i,\ell} \right).
\end{aligned}$$

Let $\mathcal{F}_{n\ell} = \sigma(X_1, X_2, \dots, X_\ell)$ denote the σ -algebra generated by $\{X_1, X_2, \dots, X_\ell\}$ for $1 \leq \ell \leq n$, and $\mathcal{F}_{n0} = \{\phi, \Omega\}$ is the trivial σ -algebra. Now set

$$z_{n\ell} = \frac{2}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} \Delta_{i,\ell-1} \delta_{i,\ell}, \quad 1 \leq \ell \leq n. \quad (\text{A.15})$$

Note that $z_{n1} = 0$. By the independence of $\delta_{i,\ell}$ and $\mathcal{F}_{n(\ell-1)}$, we have

$$\mathbb{E}(z_{n\ell} | \mathcal{F}_{n(\ell-1)}) = \frac{2}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} \Delta_{i,\ell-1} \mathbb{E}(\delta_{i,\ell} | \mathcal{F}_{n(\ell-1)}) = 0 \quad \text{for } 1 \leq \ell \leq n.$$

Therefore, $\{z_{n\ell}, \mathcal{F}_{n\ell}, 1 \leq \ell \leq n, n \geq 2\}$ form an array of martingale differences. Since $S_{n1} = \sum_{\ell=1}^n z_{n\ell}$, it is sufficient to show that

$$\frac{\sum_{\ell=1}^{p_n} z_{n\ell}}{\sigma_{n1}} \xrightarrow{d} N(0, 1). \quad (\text{A.16})$$

In view of Corollary 3.1 in Hall and Heyde [1], the martingale central limit theorem (A.16) holds if the following two conditions hold:

$$\frac{1}{\sigma_{n1}^2} \sum_{\ell=1}^n \mathbb{E}(z_{n\ell}^2 I(|z_{n\ell}| \geq \varepsilon \sigma_{n1}) | \mathcal{F}_{n(\ell-1)}) \rightarrow 0 \quad \text{in probability} \quad (\text{A.17})$$

for every $\varepsilon > 0$, and

$$\frac{1}{\sigma_{n1}^2} \sum_{\ell=1}^n \mathbb{E}(z_{n\ell}^2 | \mathcal{F}_{n(\ell-1)}) \rightarrow 1 \quad \text{in probability.} \quad (\text{A.18})$$

Recall $z_{n\ell}$ is defined in (A.15). We have

$$z_{n\ell}^2 = \frac{4}{n^2} \sum_{1 \leq i_1, i_2 \leq k_n} \frac{\Delta_{i_1, \ell-1} \Delta_{i_2, \ell-1}}{p_{i_1} p_{i_2}} \delta_{i_1, \ell} \delta_{i_2, \ell}. \quad (\text{A.19})$$

By taking conditional expectations on $\mathcal{F}_{n(\ell-1)}$, using the independence of $\delta_{i_1, \ell} \delta_{i_2, \ell}$ and $\mathcal{F}_{n(\ell-1)}$ we get

$$\begin{aligned} & \mathbb{E}(z_{n\ell}^2 | \mathcal{F}_{n(\ell-1)}) \\ &= \frac{4}{n^2} \sum_{1 \leq i_1, i_2 \leq k_n} \frac{\Delta_{i_1, \ell-1} \Delta_{i_2, \ell-1}}{p_{i_1} p_{i_2}} \mathbb{E}(\delta_{i_1, \ell} \delta_{i_2, \ell} | \mathcal{F}_{n(\ell-1)}) \\ &= \frac{4}{n^2} \sum_{1 \leq i_1, i_2 \leq k_n} \frac{\Delta_{i_1, \ell-1} \Delta_{i_2, \ell-1}}{p_{i_1} p_{i_2}} \mathbb{E}(\delta_{i_1, \ell} \delta_{i_2, \ell}) \\ &= \frac{4}{n^2} \left(\sum_{1 \leq i_1 = i_2 \leq k_n} \frac{\Delta_{i_1, \ell-1} \Delta_{i_2, \ell-1}}{p_{i_1} p_{i_2}} \mathbb{E}(\delta_{i_1, \ell} \delta_{i_2, \ell}) + \sum_{1 \leq i_1 \neq i_2 \leq k_n} \frac{\Delta_{i_1, \ell-1} \Delta_{i_2, \ell-1}}{p_{i_1} p_{i_2}} \mathbb{E}(\delta_{i_1, \ell} \delta_{i_2, \ell}) \right). \end{aligned}$$

In view of (A.3) and (A.4), we get for $2 \leq \ell \leq n$

$$\begin{aligned} & \mathbb{E}(z_{n\ell}^2 | \mathcal{F}_{n(\ell-1)}) \\ &= \frac{4}{n^2} \left(\sum_{1 \leq i \leq k_n} \frac{\Delta_{i, \ell-1}^2}{p_i} (1 - p_i) - \sum_{1 \leq i_1 \neq i_2 \leq k_n} \Delta_{i_1, \ell-1} \Delta_{i_2, \ell-1} \right) \\ &= \frac{4}{n^2} \left(\sum_{1 \leq i \leq k_n} \frac{\Delta_{i, \ell-1}^2}{p_i} (1 - p_i) + \sum_{1 \leq i_1 = i_2 \leq k_n} \Delta_{i_1, \ell-1} \Delta_{i_2, \ell-1} - \sum_{1 \leq i_1, i_2 \leq k_n} \Delta_{i_1, \ell-1} \Delta_{i_2, \ell-1} \right) \\ &= \frac{4}{n^2} \left(\sum_{1 \leq i \leq k_n} \frac{\Delta_{i, \ell-1}^2}{p_i} (1 - p_i) + \sum_{1 \leq i \leq k_n} \Delta_{i, \ell-1}^2 - \left(\sum_{1 \leq i \leq n} \Delta_{i, \ell-1} \right)^2 \right) \\ &= \frac{4}{n^2} \sum_{1 \leq i \leq k_n} \frac{\Delta_{i, \ell-1}^2}{p_i}. \end{aligned}$$

Therefore, we get the conditional variance for the martingale differences $\{z_{n\ell}, \mathcal{F}_{n\ell}, 1 \leq \ell \leq n, n \geq 2\}$

$$\sigma_{n|c}^2 := \sum_{\ell=2}^n \mathbb{E}(z_{n\ell}^2 | \mathcal{F}_{n(\ell-1)}) = \frac{4}{n^2} \sum_{\ell=2}^n \sum_{1 \leq i \leq k_n} \frac{\Delta_{i, \ell-1}^2}{p_i}, \quad (\text{A.20})$$

and from (A.5)

$$\begin{aligned}
\mathbb{E}(\sigma_{n|c}^2) &= \frac{4}{n^2} \sum_{\ell=2}^n \sum_{1 \leq i \leq k_n} \frac{(\ell-1)p_i(1-p_i)}{p_i} \\
&= \frac{4}{n^2} \sum_{\ell=2}^n (\ell-1) \sum_{1 \leq i \leq k_n} (1-p_i) \\
&= \frac{4}{n^2} \frac{n(n-1)}{2} (k_n-1) \\
&= \frac{2(n-1)(k_n-1)}{n} \\
&= \sigma_{n1}^2,
\end{aligned} \tag{A.21}$$

where σ_{n1}^2 is the variance defined in Theorem 2.1.

Taking into account the above computation, (A.17) and (A.18) follow if we can verify the following equations

$$\sum_{\ell=1}^n \mathbb{E}(z_{n\ell}^4) = o(\sigma_{n1}^4) \quad \text{as } n \rightarrow \infty \tag{A.22}$$

and

$$\mathbb{E}(\sigma_{n|c}^2 - \sigma_{n1}^2)^2 = o(\sigma_{n1}^4) \quad \text{as } n \rightarrow \infty. \tag{A.23}$$

We will prove (A.23) first. Rewrite

$$\begin{aligned}
\sigma_{n|c}^2 &= \frac{4}{n^2} \sum_{\ell=2}^n \sum_{1 \leq i \leq k_n} \frac{\Delta_{i,\ell-1}^2}{p_i} \\
&= \frac{4}{n^2} \sum_{\ell=2}^n \sum_{i=1}^{k_n} \frac{1}{p_i} \sum_{1 \leq j_1, j_2 \leq \ell-1} \delta_{i,j_1} \delta_{i,j_2} \\
&= \frac{4}{n^2} \sum_{\ell=2}^n \sum_{i=1}^{k_n} \frac{1}{p_i} \left(\sum_{j=1}^{k_n} \delta_{i,j}^2 + 2 \sum_{1 \leq j_1 < j_2 \leq \ell-1} \delta_{i,j_1} \delta_{i,j_2} \right) \\
&= \frac{4}{n^2} \sum_{\ell=2}^n \sum_{i=1}^{k_n} \frac{1}{p_i} \sum_{j=1}^{\ell-1} \delta_{i,j}^2 + \frac{8}{n^2} \sum_{\ell=3}^n \sum_{i=1}^{k_n} \frac{1}{p_i} \sum_{1 \leq j_1 < j_2 \leq \ell-1} \delta_{i,j_1} \delta_{i,j_2} \\
&= : I_{n1} + I_{n2}.
\end{aligned}$$

Then (A.23) follows if

$$\mathbb{E}(I_{n1} - \sigma_{n1}^2)^2 = o(\sigma_{n1}^4) \quad \text{and} \quad \mathbb{E}(I_{n2}^2) = o(\sigma_{n1}^4). \tag{A.24}$$

Note that

$$\begin{aligned}
I_{n1} &= \frac{4}{n^2} \sum_{\ell=2}^n \sum_{j=1}^{\ell-1} \sum_{i=1}^{k_n} \frac{1}{p_i} \delta_{i,j}^2 \\
&= \frac{4}{n^2} \sum_{\ell=2}^n \sum_{j=1}^{\ell-1} \sum_{i=1}^{k_n} \left(\frac{I(X_j = i)}{p_i} + p_i - 2I(X_j = i) \right) \\
&= \frac{4}{n^2} \sum_{\ell=2}^n \sum_{j=1}^{\ell-1} \left(\sum_{i=1}^{k_n} \frac{I(X_j = i)}{p_i} - 1 \right) \\
&= \frac{4}{n^2} \sum_{\ell=2}^n \sum_{j=1}^{\ell-1} \left(\sum_{i=1}^{k_n} \frac{I(X_j = i) - p_i}{p_i} + k_n - 1 \right) \\
&= \frac{4}{n^2} \sum_{\ell=2}^n \sum_{j=1}^{\ell-1} \left(\sum_{i=1}^{k_n} \frac{\delta_{i,j}}{p_i} + k_n - 1 \right) \\
&= \frac{4}{n^2} \sum_{\ell=2}^n \sum_{j=1}^{\ell-1} \sum_{i=1}^{k_n} \frac{\delta_{i,j}}{p_i} + \frac{4}{n^2} \sum_{\ell=2}^n \sum_{j=1}^{\ell-1} (k_n - 1) \\
&= \frac{4}{n^2} \sum_{\ell=2}^n \sum_{j=1}^{\ell-1} \sum_{i=1}^{k_n} \frac{\delta_{i,j}}{p_i} + \sigma_{n1}^2 \\
&= \frac{4}{n^2} \sum_{j=1}^{n-1} \sum_{i=1}^{k_n} \frac{(n-j)\delta_{i,j}}{p_i} + \sigma_{n1}^2.
\end{aligned}$$

The last step is obtained from the previous one by taking summation over ℓ first. In view of

(A.2) and (A.3) we get

$$\begin{aligned}
\mathbb{E}(I_{n1} - \sigma_{n1}^2)^2 &= \frac{16}{n^4} \mathbb{E} \left(\sum_{j=1}^{n-1} \sum_{i=1}^{k_n} \frac{(n-j)\delta_{i,j}}{p_i} \right)^2 \\
&= \frac{16}{n^4} \sum_{1 \leq j_1, j_2 \leq n-1} \sum_{1 \leq i_1, i_2 \leq k_n} \frac{(n-j_1)(n-j_2) \mathbb{E}(\delta_{i_1, j_1} \delta_{i_2, j_2})}{p_{i_1} p_{i_2}} \\
&= \frac{16}{n^4} \sum_{j=1}^{n-1} \sum_{1 \leq i_1, i_2 \leq k_n} \frac{(n-j)^2 \mathbb{E}(\delta_{i_1, j} \delta_{i_2, j})}{p_{i_1} p_{i_2}} \\
&= \frac{16}{n^4} \sum_{j=1}^{n-1} (n-j)^2 \left(\sum_{1 \leq i \leq k_n} \frac{p_i(1-p_i)}{p_i^2} + \sum_{1 \leq i_1 \neq i_2 \leq k_n} \frac{-p_{i_1} p_{i_2}}{p_{i_1} p_{i_2}} \right) \\
&= \frac{16}{n^4} \sum_{j=1}^{n-1} (n-j)^2 \left(\sum_{i=1}^{k_n} \frac{1}{p_i} - k_n^2 \right) \\
&= \frac{O(k_n^2)}{n k_n^2} \left(\sum_{i=1}^{k_n} \frac{1}{p_i} - k_n^2 \right) \\
&= o(\sigma_{n1}^4).
\end{aligned}$$

We have used the fact that $\lim_{n \rightarrow \infty} \frac{1}{n k_n^2} (\sum_{i=1}^{k_n} \frac{1}{p_i} - k_n^2) = 0$, which follows from condition (2.2) since

$$\sum_{i=1}^{k_n} \frac{1}{p_i} \leq \sqrt{\sum_{i=1}^{k_n} \frac{1}{p_i^2} \sum_{i=1}^{k_n} 1} = o(n k_n) k_n^{1/2} = o(n k_n^2) \quad (\text{A.25})$$

from the Cauchy-Schwarz inequality. The first part of (A.24) is obtained.

To prove the second part of (A.24), we can take summation over ℓ first. Then we have

$$\begin{aligned}
I_{n2} &= \frac{8}{n^2} \sum_{\ell=3}^n \sum_{i=1}^{k_n} \frac{1}{p_i} \sum_{1 \leq j_1 < j_2 \leq \ell-1} \delta_{i, j_1} \delta_{i, j_2} \\
&= \frac{8}{n^2} \sum_{j=2}^{k_n} \sum_{i=1}^{k_n} \frac{n-j}{p_i} \Delta_{i, j-1} \delta_{i, j}.
\end{aligned}$$

We note that $\mathbb{E}(\Delta_{i_1, j_1-1} \delta_{i_1, j_1} \Delta_{i_2, j_2-1} \delta_{i_2, j_2}) = 0$ if $j_1 \neq j_2$ for any $1 \leq i_1, i_2 \leq k_n$. We thus have

from equations (A.1), (A.3), (A.5) and (A.6) that

$$\begin{aligned}
\mathbb{E}(I_{n2}^2) &= \frac{64}{n^4} \sum_{2 \leq j_1, j_2 \leq n} \sum_{1 \leq i_1, i_2 \leq k_n} \frac{(n-j_1)(n-j_2)}{p_{i_1} p_{i_2}} \mathbb{E}(\Delta_{i_1, j_1-1} \delta_{i_1, j_1} \Delta_{i_2, j_2-1} \delta_{i_2, j_2}) \\
&= \frac{64}{n^4} \sum_{2 \leq j \leq n} \sum_{1 \leq i_1, i_2 \leq k_n} \frac{(n-j)^2}{p_{i_1} p_{i_2}} \mathbb{E}(\Delta_{i_1, j-1} \delta_{i_1, j} \Delta_{i_2, j-1} \delta_{i_2, j}) \\
&= \frac{64}{n^4} \sum_{2 \leq j \leq n} \sum_{1 \leq i_1, i_2 \leq k_n} \frac{(n-j)^2}{p_{i_1} p_{i_2}} \mathbb{E}(\Delta_{i_1, j-1} \Delta_{i_2, j-1}) \mathbb{E}(\delta_{i_1, j} \delta_{i_2, j}) \\
&= \frac{64}{n^4} \sum_{2 \leq j \leq n} (n-j)^2 \left(\sum_{1 \leq i \leq k_n} \frac{(j-1)p_i^2(1-p_i)^2}{p_i^2} + \sum_{1 \leq i_1 \neq i_2 \leq k_n} \frac{(j-1)p_{i_1}^2 p_{i_2}^2}{p_{i_1} p_{i_2}} \right) \\
&= \frac{64}{n^4} \sum_{2 \leq j \leq n} (n-j)^2 (j-1) \left(\sum_{1 \leq i \leq k_n} (1-p_i)^2 + \sum_{1 \leq i_1 \neq i_2 \leq k_n} p_{i_1} p_{i_2} \right) \\
&= \frac{64(k_n-1)}{n^4} \sum_{2 \leq j \leq n} (n-j)^2 (j-1) \\
&= O(k_n-1) \\
&= o(\sigma_{n1}^4),
\end{aligned}$$

proving the second part of (A.24).

Finally, we show (A.22). To estimate $\mathbb{E}(z_{n\ell}^4)$, we need the following calculations which are straightforward:

$$\begin{aligned}
d_4(i) : &= \mathbb{E}(\delta_{i,\ell}^4) = p_i(1-p_i)^4 + p_i^4(1-p_i), \\
d_{3,1}(i, j) : &= \mathbb{E}(\delta_{i,\ell}^3 \delta_{j,\ell}) = p_i p_j (1-p_i-p_j) - (1-p_i)^3 p_i p_j - p_i^3 (1-p_j) p_j, \\
d_{2,2}(i, j) : &= \mathbb{E}(\delta_{i,\ell}^2 \delta_{j,\ell}^2) = p_i p_j (1-p_i-p_j) + p_i p_j^2 (1-p_i)^2 + p_i^2 p_j (1-p_j)^2, \\
d_{2,1,1}(i, j, m) : &= \mathbb{E}(\delta_{i,\ell}^2 \delta_{j,\ell} \delta_{m,\ell}) = p_i^2 p_j p_m (1-p_i-p_j-p_m) + (1-p_i)^2 p_i p_j p_m \\
&\quad - p_i^2 p_j (1-p_j) p_m - p_i^2 p_j p_m (1-p_m), \\
d_{1,1,1,1}(i, j, m, r) : &= \mathbb{E}(\delta_{i,\ell} \delta_{j,\ell} \delta_{m,\ell} \delta_{r,\ell}) \\
&= p_i p_j p_m p_r (1-p_i-p_j-p_m-p_r) - p_i p_j p_m p_r (1-p_i) \\
&\quad - p_i p_j p_m p_r (1-p_j) - p_i p_j p_m p_r (1-p_m) - p_i p_j p_m p_r (1-p_r) \\
&= -3p_i p_j p_m p_r,
\end{aligned}$$

where integers $i, j, m, r \in \{1, \dots, k_n\}$ assume different values if they appear in the same equa-

tions. Then it follows from the above equations that

$$d_4(i) \leq 2p_i, \quad |d_{3,1}(i, j)| \leq 3p_i p_j, \quad d_{2,2}(i, j) \leq 3p_i p_j, \quad |d_{2,1,1}(i, j, m)| \leq 4p_i p_j p_m$$

and

$$d_{1,1,1,1}(i, j, m, r) = -3p_i p_j p_m p_r.$$

Now we estimate $\mathbb{E}(\Delta_{i_1, \ell} \Delta_{i_2, \ell} \Delta_{i_3, \ell} \Delta_{i_4, \ell})$. Note that

$$\begin{aligned} \mathbb{E}(\Delta_{i_1, \ell} \Delta_{i_2, \ell} \Delta_{i_3, \ell} \Delta_{i_4, \ell}) &= \mathbb{E}\left(\sum_{\ell_1=1}^{\ell} \delta_{i_1, \ell_1} \sum_{\ell_2=1}^{\ell} \delta_{i_2, \ell_2} \sum_{\ell_3=1}^{\ell} \delta_{i_3, \ell_3} \sum_{\ell_4=1}^{\ell} \delta_{i_4, \ell_4}\right) \\ &= \mathbb{E}\left(\sum_{1 \leq \ell_1, \ell_2, \ell_3, \ell_4 \leq \ell} \delta_{i_1, \ell_1} \delta_{i_2, \ell_2} \delta_{i_3, \ell_3} \delta_{i_4, \ell_4}\right) \\ &= \sum_{1 \leq \ell_1, \ell_2, \ell_3, \ell_4 \leq \ell} \mathbb{E}(\delta_{i_1, \ell_1} \delta_{i_2, \ell_2} \delta_{i_3, \ell_3} \delta_{i_4, \ell_4}). \end{aligned}$$

It is easy to see that $\mathbb{E}(\delta_{i_1, \ell_1} \delta_{i_2, \ell_2} \delta_{i_3, \ell_3} \delta_{i_4, \ell_4}) \neq 0$ only if $\ell_1 = \ell_2 = \ell_3 = \ell_4$, or $\ell_1, \ell_2, \ell_3, \ell_4$ form two distinct matching pairs such as $\ell_1 = \ell_2 \neq \ell_3 = \ell_4$. Therefore, we have

$$\begin{aligned} \mathbb{E}(\Delta_{i_1, \ell} \Delta_{i_2, \ell} \Delta_{i_3, \ell} \Delta_{i_4, \ell}) &= \sum_{1 \leq j \leq \ell} \mathbb{E}(\delta_{i_1, j} \delta_{i_2, j} \delta_{i_3, j} \delta_{i_4, j}) + \sum_{1 \leq m \neq r \leq \ell} \mathbb{E}(\delta_{i_1, m} \delta_{i_2, m}) \mathbb{E}(\delta_{i_3, r} \delta_{i_4, r}) \\ &\quad + \sum_{1 \leq m \neq r \leq j} \mathbb{E}(\delta_{i_1, m} \delta_{i_3, m}) \mathbb{E}(\delta_{i_2, r} \delta_{i_4, r}) + \sum_{1 \leq m \neq r \leq j} \mathbb{E}(\delta_{i_1, m} \delta_{i_4, m}) \mathbb{E}(\delta_{i_2, r} \delta_{i_3, r}). \end{aligned}$$

This, together with (A.3), yields

$$\begin{aligned} D_4^{(\ell)}(i) &: = \mathbb{E}(\Delta_{i, \ell}^4) = \ell d_4(i) + 3\ell(\ell-1)p_i^2(1-p_i)^2, \\ D_{3,1}^{(\ell)}(i, j) &: = \mathbb{E}(\Delta_{i, \ell}^3 \Delta_{j, \ell}) = \ell d_{3,1}(i, j) - 3\ell(\ell-1)p_i^2(1-p_i)p_j, \\ D_{2,2}^{(\ell)}(i, j) &: = \mathbb{E}(\Delta_{i, \ell}^2 \Delta_{j, \ell}^2) = \ell d_{2,2}(i, j) + \ell(\ell-1)(p_i p_j (1-p_i)(1-p_j) + 2p_i^2 p_j^2), \\ D_{2,1,1}^{(\ell)}(i, j, m) &: = \mathbb{E}(\Delta_{i, \ell}^2 \Delta_{j, \ell} \Delta_{m, \ell}) = \ell d_{2,1,1}(i, j, m) + \ell(\ell-1)(3p_i^2 p_j p_m - p_i p_j p_m), \\ D_{1,1,1,1}^{(\ell)}(i, j, m, r) &: = \mathbb{E}(\Delta_{i, \ell} \Delta_{j, \ell} \Delta_{m, \ell} \Delta_{r, \ell}) = \ell d_{1,1,1,1}(i, j, m, r) + 3\ell(\ell-1)p_i p_j p_m p_r, \end{aligned}$$

where i, j, m, r are different integers if they appear in the same equation. Therefore, we get

$$2 \leq \ell \leq n$$

$$\begin{aligned} D_4^{(\ell-1)}(i)d_4(i) &\leq 4\ell p_i^2 + 6\ell^2 p_i^3, \\ D_{3,1}^{(\ell-1)}(i,j)d_{3,1}(i,j) &\leq 9\ell p_i^2 p_j^2 + 9\ell^2 p_i^3 p_j^2, \\ D_{2,2}^{(\ell-1)}(i,j)d_{2,2}(i,j) &\leq 9\ell^2 p_i^2 p_j^2, \\ D_{2,1,1}^{(\ell-1)}(i,j,m)d_{2,1,1}(i,j,m) &\leq 16\ell^2 p_i^2 p_j^2 p_m^2, \\ D_{1,1,1,1}^{(\ell-1)}(i,j,m,r)d_{1,1,1,1}(i,j,m,r) &\leq 9\ell^2 p_i^2 p_j^2 p_m^2 p_r^2. \end{aligned}$$

It follows from (A.19) that for $2 \leq \ell \leq n$

$$z_{n\ell}^4 = \frac{4}{n^4} \sum_{1 \leq i_1, i_2, i_3, i_4 \leq k_n} \frac{\Delta_{i_1, \ell-1} \Delta_{i_2, \ell-1} \Delta_{i_3, \ell-1} \Delta_{i_4, \ell-1}}{p_{i_1} p_{i_2} p_{i_3} p_{i_4}} \delta_{i_1, \ell} \delta_{i_2, \ell} \delta_{i_3, \ell} \delta_{i_4, \ell},$$

and thus

$$\mathbb{E}(z_{n\ell}^4) = \frac{4}{n^4} \sum_{1 \leq i_1, i_2, i_3, i_4 \leq k_n} \frac{\mathbb{E}(\Delta_{i_1, \ell-1} \Delta_{i_2, \ell-1} \Delta_{i_3, \ell-1} \Delta_{i_4, \ell-1})}{p_{i_1} p_{i_2} p_{i_3} p_{i_4}} \mathbb{E}(\delta_{i_1, \ell} \delta_{i_2, \ell} \delta_{i_3, \ell} \delta_{i_4, \ell}). \quad (\text{A.26})$$

We will divide $\{(i_1, i_2, i_3, i_4) : 1 \leq i_1, i_2, i_3, i_4 \leq n\}$ into several subsets, and classify these subsets into groups. The contributions to $\mathbb{E}(z_{n\ell}^4)$ from subsets within each group are the same, and we will list only one representative subset within each group. The above inequalities will be used in the following estimations.

- Group 1: i_1, i_2, i_3, i_4 are the same, that is, $\{(i_1, i_2, i_3, i_4) : 1 \leq i_1 = i_2 = i_3 = i_4 \leq n\} =: G_1$.

The sum of the summands over G_1 on the right-hand side of (A.26) is equal to

$$\sigma_1^{(\ell)} := \sum_{1 \leq i \leq n} \frac{1}{p_i^4} D_4^{(\ell-1)}(i)d_4(i) \leq 4(\ell-1) \sum_{i=1}^{k_n} \frac{1}{p_i^2} + 6(\ell-1)^2 \sum_{i=1}^{k_n} \frac{1}{p_i}.$$

- Group 2: Exactly three of i_1, i_2, i_3, i_4 are the same. A representative is $\{(i_1, i_2, i_3, i_4) : 1 \leq i_1 = i_2 = i_3 \neq i_4 \leq n\} =: G_2$. There are 4 such subsets. The sum of the summands over G_2 on the right-hand side of (A.26) is equal to

$$\sigma_2^{(\ell)} := \sum_{1 \leq i, j \leq n} \frac{1}{p_i^3 p_j} D_{3,1}^{(\ell-1)}(i,j)d_{3,1}(i,j) \leq 9(\ell-1) \sum_{i=1}^{k_n} \frac{1}{p_i} + 9(\ell-1)^2 k_n.$$

- Group 3: i_1, i_2, i_3, i_4 form two distinct matching pairs. A representative is $\{(i_1, i_2, i_3, i_4) : 1 \leq i_1 = i_2 \neq i_3 = i_4 \leq n\} =: G_3$. There are 3 such subsets within this group. The sum of the summands over G_3 on the right-hand side of (A.26) is equal to

$$\sigma_3^{(\ell)} := \sum_{1 \leq i \neq j \leq n} \frac{1}{p_i^2 p_j^2} D_{2,2}^{(\ell-1)}(i,j)d_{2,2}(i,j) \leq 9(\ell-1)^2 k_n^2.$$

- Group 4: Exactly two of i_1, i_2, i_3, i_4 are the same and there is only one matching pair. A representative is $\{(i_1, i_2, i_3, i_4) : 1 \leq i_1 = i_2 \neq i_3 \neq i_4 \leq n\} = G_4$. There are 6 subsets within this group. The sum of the summands over G_4 on the right-hand side of (A.26) is equal to

$$\sigma_4^{(\ell)} := \sum_{1 \leq i \neq j \neq m \leq n} \frac{1}{p_i^2 p_j p_m} D_{2,1,1}^{(\ell-1)}(i, j, m) d_{2,1,1}(i, j, m) \leq 16(\ell-1)^2 k_n.$$

- Group 5: i_1, i_2, i_3, i_4 are distinct, that is, $\{(i_1, i_2, i_3, i_4) : 1 \leq i_1 \neq i_2 \neq i_3 \neq i_4 \leq n\} =: G_5$. The sum of the summands over G_5 on the right-hand side of (A.26) is equal to

$$\sigma_5^{(\ell)} := \sum_{1 \leq i \neq j \neq m \neq r \leq n} \frac{1}{p_i p_j p_m p_r} D_{1,1,1,1}^{(\ell-1)}(i, j, m, r) d_{1,1,1,1}(i, j, m, r) \leq 9(\ell-1)^2.$$

Therefore, we have for $2 \leq \ell \leq n$

$$\mathbb{E}(z_{n\ell}^4) \leq \frac{4}{n^4} \left(\sigma_1^{(\ell)} + 4\sigma_2^{(\ell)} + 3\sigma_3^{(\ell)} + 6\sigma_4^{(\ell)} + \sigma_5^{(\ell)} \right).$$

By summing up on both sides of the above inequality we have

$$\begin{aligned} \sum_{\ell=2}^n \mathbb{E}(z_{n\ell}^4) &\leq \frac{4}{n^4} \left(\sum_{\ell=2}^n \sigma_1^{(\ell)} + 4 \sum_{\ell=2}^n \sigma_2^{(\ell)} + 3 \sum_{\ell=2}^n \sigma_3^{(\ell)} + 6 \sum_{\ell=2}^n \sigma_4^{(\ell)} + \sum_{\ell=2}^n \sigma_5^{(\ell)} \right) \\ &\leq \left(\frac{2}{n^2} \sum_{i=1}^{k_n} \frac{1}{p_i^2} + \frac{8}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} \right) + \left(\frac{72}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} + \frac{48k_n}{n} \right) \\ &\quad + \frac{36k_n^2}{n} + \frac{72k_n}{n} + \frac{12}{n} \\ &\leq \frac{2}{n^2} \sum_{i=1}^{k_n} \frac{1}{p_i^2} + \frac{80}{n} \sum_{i=1}^{k_n} \frac{1}{p_i} + \frac{36(k_n+2)^2}{n}. \end{aligned}$$

Since $\sigma_{n1}^2 \sim 2k_n$, equation (A.22) follows immediately from (A.25) and condition (2.2). This completes the proof. $\square$

Lemma A.4. *Let $c_i \geq 0$, $1 \leq i \leq k_n$, be given weights such that $\sum_{i=1}^{k_n} c_i = k_n$. Assume $\{n_r, r \geq 1\}$ is an increasing sequence of positive integers. If*

$$\frac{\beta_{n_r 3}}{n_r \beta_{n_r 1}^2} \rightarrow 0 \quad \text{and} \quad \frac{k_{n_r}^2}{n_r \beta_{n_r 1}} \rightarrow 0 \quad \text{as } r \rightarrow \infty, \quad (\text{A.27})$$

where $\beta_{n_r j}$'s are defined in (A.14), then we have

$$\frac{\bar{S}_{n_r 2}}{\bar{\sigma}_{n_r 2}} \xrightarrow{d} N(0, 1) \quad \text{as } r \rightarrow \infty, \quad (\text{A.28})$$

where $\bar{\sigma}_{n2}$ is defined in (2.12). If, additionally, (2.2) holds with $n = n_r$ as $r \rightarrow \infty$, we have

$$\left(\frac{S_{n_r1} - (k_{n_r} - 1)}{\sigma_{n_r1}}, \frac{\bar{S}_{n_r2}}{\bar{\sigma}_{n_r2}} \right) \xrightarrow{d} (Z_1, Z_2), \quad (\text{A.29})$$

where Z_1 and Z_2 are i.i.d. standard normal random variables.

Proof. As in the proof of Lemma A.3, we denote n_r as n for brevity.

It follows from (2.11) and (A.7) that

$$\begin{aligned} \bar{S}_{n2} &= \sum_{i=1}^{k_n} c_i \left(\frac{o_i}{e_i} - 1 \right) \\ &= \frac{1}{n} \sum_{i=1}^{k_n} \left(\frac{c_i}{p_i} \sum_{j=1}^n I(X_j = i) - c_i n \right) \\ &= \frac{1}{n} \sum_{j=1}^n \left(\sum_{i=1}^{k_n} \frac{c_i I(X_j = i)}{p_i} - k_n \right). \end{aligned}$$

Set

$$y_{nj} = \sum_{i=1}^{k_n} \frac{c_i I(X_j = i)}{p_i} - k_n, \quad 1 \leq j \leq n. \quad (\text{A.30})$$

Then $\bar{S}_{n2} = \frac{1}{n} \sum_{j=1}^n y_{nj}$. Note that $y_{n1}, \dots, y_{nn}$ are n i.i.d. random variables with mean 0. Since for any integer $r \geq 2$

$$\mathbb{E} \left(\sum_{i=1}^{k_n} \frac{c_i I(X_j = i)}{p_i} \right)^r = \mathbb{E} \left(\sum_{i=1}^{k_n} \frac{c_i^r I(X_j = i)}{p_i^r} \right) = \sum_{i=1}^{k_n} \frac{c_i^r}{p_i^{r-1}},$$

we have

$$\mathbb{E}(y_{n1}^2) = \mathbb{E} \left(\sum_{i=1}^{k_n} \frac{c_i I(X_j = i)}{p_i} \right)^2 - k_n^2 = \sum_{i=1}^{k_n} \frac{c_i^2}{p_i} - k_n^2 = \beta_{n1},$$

which implies

$$\bar{\sigma}_{n2}^2 = \text{Var}(\bar{S}_{n2}) = \frac{\beta_{n1}}{n}. \quad (\text{A.31})$$

Furthermore, we have

$$\begin{aligned}
\mathbb{E}(y_{n1}^4) &= \mathbb{E}\left(\sum_{i=1}^{k_n} \frac{c_i I(X_j = i)}{p_i} - k_n\right)^4 \\
&= \mathbb{E}\left(\sum_{i=1}^{k_n} \frac{c_i I(X_j = i)}{p_i}\right)^4 - 4k_n \mathbb{E}\left(\sum_{i=1}^{k_n} \frac{c_i I(X_j = i)}{p_i}\right)^3 \\
&\quad + 6k_n^2 \mathbb{E}\left(\sum_{i=1}^{k_n} \frac{c_i I(X_j = i)}{p_i}\right)^2 - 4k_n^3 \mathbb{E}\left(\sum_{i=1}^{k_n} \frac{c_i I(X_j = i)}{p_i}\right) + k_n^4 \\
&= \sum_{i=1}^{k_n} \frac{c_i^4}{p_i^3} - k_n^4 - 4k_n \left(\sum_{i=1}^{k_n} \frac{c_i^3}{p_i^2} - k_n^3\right) + 6k_n^2 \left(\sum_{i=1}^{k_n} \frac{c_i^2}{p_i} - k_n^2\right) \\
&\leq \sum_{i=1}^{k_n} \frac{c_i^4}{p_i^3} - k_n^4 + 6k_n^2 \left(\sum_{i=1}^{k_n} \frac{c_i^2}{p_i} - k_n^2\right) \\
&= \beta_{n3} + 6k_n^2 \beta_{n1}
\end{aligned}$$

from (A.14).

Note that as $n \rightarrow \infty$

$$\frac{1}{(\sqrt{n}\beta_{n1})^4} \sum_{j=1}^n \mathbb{E}(y_{nj}^4) = \frac{\mathbb{E}(y_{n1}^4)}{n\beta_{n1}^2} \leq \frac{\beta_{n3}}{n\beta_{n1}^2} + \frac{6k_n^2}{n\beta_{n1}} \rightarrow 0 \quad (\text{A.32})$$

from (A.27). This is Lyapunov's condition for the central limit theorem

$$\frac{\bar{S}_{n2}}{\bar{\sigma}_{n2}} = \frac{\sum_{j=1}^n y_{nj}}{\sqrt{n}\beta_{n1}} \xrightarrow{d} N(0, 1).$$

Therefore, we have proved (A.28).

Since both $\frac{S_{n1} - (k_n - 1)}{\sigma_{n1}}$ and $\frac{\bar{S}_{n2}}{\bar{\sigma}_{n2}}$ converge in distribution to the standard normal, to show (A.29), it suffices to show that for any $s, t \in \mathbb{R}$

$$s \frac{S_{n1} - (k_n - 1)}{\sigma_{n1}} + t \frac{\bar{S}_{n2}}{\bar{\sigma}_{n2}} \xrightarrow{d} N(0, s^2 + t^2),$$

or equivalently

$$T_n(s, t) := \frac{s}{\sqrt{s^2 + t^2}} \frac{S_{n1} - (k_n - 1)}{\sigma_{n1}} + \frac{t}{\sqrt{s^2 + t^2}} \frac{\bar{S}_{n2}}{\bar{\sigma}_{n2}} \xrightarrow{d} N(0, 1). \quad (\text{A.33})$$

Now fix $s, t \in \mathbb{R}$. Set

$$a_n = \frac{s}{\sqrt{s^2 + t^2}} \frac{1}{\sigma_{n1}}, \quad b_n = \frac{t}{\sqrt{s^2 + t^2}} \frac{1}{\sqrt{n}\beta_{n1}}.$$

Note that $y_{n\ell} = \sum_{i=1}^{k_n} \frac{c_i \delta_{i,\ell}}{p_i}$. Define

$$x_{n\ell} = a_n z_{n\ell} + b_n y_{n\ell},$$

where $z_{n\ell}$'s are defined in (A.15) in the proof of Lemma A.3. Then we have

$$T_n(s, t) = \sum_{\ell=1}^n x_{n\ell}.$$

Obviously, $\{x_{n\ell}, \mathcal{F}_{n\ell}, 1 \leq \ell \leq n, n \geq 1\}$ is an array of martingale differences.

In view of (A.3) and (A.4), we have $\mathbb{E}(\delta_{i,\ell} \delta_{i',\ell}) = p_i I(i = i') - p_i p_{i'}$ and $\sum_{i=1}^{k_n} \Delta_{i,\ell-1} = 0$, which imply

$$\begin{aligned} \mathbb{E}(y_{n\ell} z_{n\ell} | \mathcal{F}_{n(\ell-1)}) &= \frac{2}{n} \sum_{i=1}^{k_n} \sum_{i'=1}^{k_n} \frac{c_{i'}}{p_i p_{i'}} \Delta_{i,\ell-1} \mathbb{E}(\delta_{i,\ell} \delta_{i',\ell}) \\ &= \frac{2}{n} \sum_{1 \leq i=i' \leq k_n} \frac{c_{i'}}{p_i p_{i'}} \Delta_{i,\ell-1} p_i - \frac{2}{n} \sum_{i=1}^{k_n} \sum_{i'=1}^{k_n} \frac{c_{i'}}{p_i p_{i'}} \Delta_{i,\ell-1} p_i p_{i'} \\ &= \frac{2}{n} \sum_{i=1}^{k_n} \frac{c_i \Delta_{i,\ell-1}}{p_i} - \frac{2}{n} \sum_{i'=1}^{k_n} c_{i'} \sum_{i=1}^{k_n} \Delta_{i,\ell-1} \\ &= \frac{2}{n} \sum_{i=1}^{k_n} \frac{c_i \Delta_{i,\ell-1}}{p_i}. \end{aligned}$$

Therefore, we have

$$\mathbb{E}(\mathbb{E}(y_{n\ell} z_{n\ell} | \mathcal{F}_{n(\ell-1)})) = 0.$$

Define

$$\tau_n = \sum_{\ell=1}^n \mathbb{E}(y_{n\ell} z_{n\ell} | \mathcal{F}_{n(\ell-1)}).$$

Then τ_n can be written as

$$\tau_n = \frac{2}{n} \sum_{j=1}^{n-1} (n-j) \sum_{i=1}^{k_n} \frac{c_i \delta_{i,j}}{p_i} = \frac{2}{n} \sum_{j=1}^{n-1} (n-j) y_{nj},$$

where y_{nj} 's, as defined in (A.30), are iid random variables with mean 0 and variance β_{n1} . We conclude that

$$\mathbb{E}(\tau_n^2) \leq 4n\beta_{n1}. \quad (\text{A.34})$$

By using the formula

$$x_{n\ell}^2 = (a_n z_{n\ell} + b_n y_{n\ell})^2 = a_n^2 z_{n\ell}^2 + b_n^2 y_{n\ell}^2 + 2a_n b_n z_{n\ell} y_{n\ell},$$

we get the conditional variance for the martingale differences $\{x_{n\ell}\}$

$$\begin{aligned} \lambda_{n|c}^2 &= \sum_{\ell=1}^n \mathbb{E}(x_{n\ell}^2 | \mathcal{F}_{n(\ell-1)}) \\ &= a_n^2 \sum_{\ell=1}^n \mathbb{E}(z_{n\ell}^2 | \mathcal{F}_{n(\ell-1)}) + b_n^2 \sum_{\ell=1}^n \mathbb{E}(y_{n\ell}^2 | \mathcal{F}_{n(\ell-1)}) + 2a_n b_n \sum_{\ell=1}^n \mathbb{E}(z_{n\ell} y_{n\ell} | \mathcal{F}_{n(\ell-1)}) \\ &= a_n^2 \sigma_{n|c}^2 + n b_n^2 \beta_{n1} + 2a_n b_n \tau_n. \end{aligned}$$

Therefore, we have from (A.20) and (A.21) that

$$\mathbb{E}(\lambda_{n|c}^2) = a_n^2 \sigma_{n1}^2 + n b_n^2 \beta_{n1} + 0 = 1.$$

By using the same argument as that in the proof of Lemma A.3, if we can show

$$\sum_{\ell=1}^n \mathbb{E}(x_{n\ell}^4) = o(1) \tag{A.35}$$

and

$$\mathbb{E}(\lambda_{n|c}^2 - 1)^2 = o(1), \tag{A.36}$$

then we can apply the martingale central limit theorem to obtain (A.33). In fact, since

$$\lambda_{n|c}^2 - 1 = a_n^2 (\sigma_{n|c}^2 - \sigma_{n1}^2) + 2a_n b_n \tau_n,$$

we have from the c_r -inequality

$$\begin{aligned} \mathbb{E}(\lambda_{n|c}^2 - 1)^2 &\leq 2 \left(a_n^4 \mathbb{E}(\sigma_{n|c}^2 - \sigma_{n1}^2)^2 + 4a_n^2 b_n^2 \mathbb{E}(\tau_n^2) \right) \\ &\leq 2 \left(\frac{\mathbb{E}(\sigma_{n|c}^2 - \sigma_{n1}^2)^2}{\sigma_{n1}^4} + \frac{4\mathbb{E}(\tau_n^2)}{\sigma_{n1}^2 n \beta_{n1}} \right) \\ &\rightarrow 0 \end{aligned}$$

in view of (A.23) and (A.34). This proves (A.36). Again, by using the c_r -inequality we have that as $n \rightarrow \infty$

$$\begin{aligned} \sum_{\ell=1}^n \mathbb{E}(x_{n\ell}^4) &\leq 8 \sum_{\ell=1}^n \left(\mathbb{E}(a_n^4 z_{n\ell}^4) + b_n^4 \mathbb{E}(y_{n\ell}^4) \right) \\ &\leq \frac{8}{\sigma_{n1}^4} \sum_{\ell=1}^n \mathbb{E}(z_{n\ell}^4) + \frac{1}{(\sqrt{n\beta_{n1}})^4} \sum_{\ell=1}^n \mathbb{E}(y_{n\ell}^4) \\ &\rightarrow 0 \end{aligned}$$

from (A.22) and (A.32), proving (A.35). This completes the proof of the lemma. $\square$

Proof of Theorem 2.1. Theorem 2.1 is a direct consequence of Lemma A.3 when n_r is the entire sequence of all positive integers. $\square$

Proof of Theorem 2.2. We will employ subsequence arguments, that is, (2.5) holds if and only if for any increasing sequence of positive integers, there exists its further subsequence, say, $\{n_r, r \geq 1\}$ such that (2.5) holds along $n = n_r$ as $r \rightarrow \infty$. Since $\sigma_{n2}^2/\sigma_n^2 \in [0, 1]$, n_r can be selected in a way that $\sigma_{n_r2}^2/\sigma_{n_r}^2$ has a limit in $[0, 1]$. Therefore, it suffices to show that (2.5) holds for $n = n_r$ for any increasing sequence of integers $\{n_r\}$ as long as

$$\lim_{r \rightarrow \infty} \frac{\sigma_{n_r2}^2}{\sigma_{n_r}^2} = v \text{ for some } v \in [0, 1].$$

First, consider the case $v = 0$. From Chebyshev's inequality, we have that for every $\delta > 0$

$$P\left(\frac{|S_{n_r2}|}{\sigma_{n_r}} > \delta\right) \leq \frac{1}{\delta^2} \mathbb{E}\left(\frac{|S_{n_r2}|}{\sigma_{n_r}}\right)^2 = \frac{1}{\delta^2} \frac{\sigma_{n_r2}^2}{\sigma_{n_r}^2} \rightarrow 0$$

as $r \rightarrow \infty$. This implies S_{n_r2}/σ_{n_r} converges to zero in probability as $r \rightarrow \infty$. Since (2.2) implies (2.4) from Theorem 2.1, we obtain

$$\begin{aligned} \frac{\mathcal{X}_{n_r}^2 - (k_{n_r} - 1)}{\sigma_{n_r}} &= \frac{\sigma_{n_r1}}{\sigma_{n_r}} \frac{S_{n_r1} - (k_{n_r} - 1)}{\sigma_{n_r1}} + \frac{S_{n_r2}}{\sigma_{n_r}} \\ &= (1 + o(1)) \frac{S_{n_r1} - (k_{n_r} - 1)}{\sigma_{n_r1}} + o_p(1) \\ &\xrightarrow{d} N(0, 1) \end{aligned}$$

as $r \rightarrow \infty$, i.e. (2.5) holds with $n = n_r$.

Now consider the case $v \in (0, 1]$. We can use Lemma A.4 for $c_1 = \dots = c_{k_n} = 1$. In this case, $\bar{S}_{n2} = S_{n2}$, $\bar{\sigma}_{n2} = \sigma_{n2}$, and conditions (2.13) and (2.3) are the same.

Since $\sigma_{n1}^2 \sim 2k_n$, we have

$$\frac{\sigma_{n_r2}^2}{k_{n_r}} \sim \frac{2\sigma_{n_r2}^2}{\sigma_{n_r1}^2} = \frac{2\sigma_{n_r2}^2}{\sigma_{n_r}^2 - \sigma_{n_r2}^2} \rightarrow \frac{2v}{1-v} > 0.$$

The above limit is interpreted as infinity if $v = 1$. This, together with (2.3), implies that the first term within the parentheses in (2.3) must tend to zero as $n = n_r$ goes to infinity, that is,

$$\frac{\beta_{n_r3}}{n_r \beta_{n_r1}^2} = \frac{\sum_{i=1}^{k_{n_r}} \frac{1}{p_i^3} - k_{n_r}^4}{n_r^3 \sigma_{n_r2}^4} \rightarrow 0$$

as $r \rightarrow \infty$. We have used (A.31) here. Furthermore, (2.2) and (A.14) with $j = 2$ imply that $\frac{k_n}{n^2} \rightarrow 0$ as $n \rightarrow \infty$, and thus

$$\frac{k_{n_r}^2}{n_r \beta_{n_r 1}} = \frac{k_{n_r}}{n_r^2} \frac{k_{n_r}}{\sigma_{n_r 1}^2} \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

Therefore, (A.27) is satisfied. In view of (A.29) we have

$$\frac{\mathcal{X}_{n_r}^2 - (k_{n_r} - 1)}{\sigma_{n_r}} = \frac{\sigma_{n_r 1}}{\sigma_{n_r}} \frac{S_{n_r 1} - (k_{n_r} - 1)}{\sigma_{n_r 1}} + \frac{\sigma_{n_r 2}}{\sigma_{n_r}} \frac{S_{n_r 2}}{\sigma_{n_r 2}} \xrightarrow{d} \sqrt{v} Z_1 + \sqrt{1-v} Z_2.$$

The above limit is a standard normal random variable. Thus, we have proved (2.5) with $n = n_r$. $\square$

Proof of Theorem 2.3. Theorem 2.3 is a special case of Theorem 2.4. $\square$

Proof of Theorem 2.4. When $c = 0$, the test statistic $S_{n1} + c|\bar{S}_{n2}|$ is the same as S_{n1} , and Theorem 2.1 ensures Theorem 2.4. Therefore, we focus on the case $c > 0$. We note that

$$\begin{aligned} & \sup_x \left| P\left(\frac{S_{n1} + c|\bar{S}_{n2}|}{\sigma_{n1}} \leq x\right) - P\left(Z_1 + \frac{c\bar{\sigma}_{n2}}{\sigma_{n1}}|Z_2| \leq x\right) \right| \\ &= \sup_x \left| P\left(\frac{1}{1 + \frac{c\bar{\sigma}_{n2}}{\sigma_{n1}}} \frac{S_{n1} + c|\bar{S}_{n2}|}{\sigma_{n1}} \leq x\right) - P\left(\frac{1}{1 + \frac{c\bar{\sigma}_{n2}}{\sigma_{n1}}} (Z_1 + \frac{c\bar{\sigma}_{n2}}{\sigma_{n1}}|Z_2|) \leq x\right) \right| =: \Theta_n. \end{aligned}$$

We will also use subsequence arguments as those in the proof of Theorem 2.2. To show that Θ_n converges to zero, it suffices to prove that $\Theta_{n_r} \rightarrow 0$ as $r \rightarrow \infty$ for every increasing sequence of integers $\{n_r\}$ such that

$$\frac{\frac{c\bar{\sigma}_{n_r 2}}{\sigma_{n_r 1}}}{1 + \frac{c\bar{\sigma}_{n_r 2}}{\sigma_{n_r 1}}} \rightarrow v \quad \text{for some } v \in [0, 1].$$

The proof is similar to that in the proof of Theorem 2.2.

When $v = 0$, we have $\frac{c\bar{\sigma}_{n_r 2}}{\sigma_{n_r 1}} \rightarrow 0$ as $r \rightarrow \infty$. By using Chebyshev's inequality we can show that

$$\frac{c\bar{S}_{n_r 2}}{\sigma_{n_r 1}} = \frac{c\bar{\sigma}_{n_r 2}}{\sigma_{n_r 1}} \frac{\bar{S}_{n_r 2}}{\bar{\sigma}_{n_r 2}} \quad \text{converges to zero in probability,}$$

which, coupled with Lemma A.3, yields that

$$\frac{1}{1 + \frac{c\bar{\sigma}_{n_r 2}}{\sigma_{n_r 1}}} \frac{S_{n_r 1} + c|\bar{S}_{n_r 2}|}{\sigma_{n_r 1}} = \frac{1}{1 + \frac{c\bar{\sigma}_{n_r 2}}{\sigma_{n_r 1}}} \frac{S_{n_r 1}}{\sigma_{n_r 1}} + \frac{c\bar{S}_{n_r 2}}{\sigma_{n_r 1}} = (1 + o(1)) \frac{S_{n_r 1}}{\sigma_{n_r 1}} + o_p(1) \xrightarrow{d} N(0, 1).$$

Obviously, we have $\frac{1}{1+\frac{c\bar{\sigma}_{nr2}}{\sigma_{nr1}}}(Z_1 + \frac{c\bar{\sigma}_{nr2}}{\sigma_{nr1}}|Z_2|) \rightarrow Z_1$. Therefore, we get

$$\sup_x |P\left(\frac{1}{1+\frac{c\bar{\sigma}_{nr2}}{\sigma_{nr1}}}\frac{S_{nr1}+c|\bar{S}_{nr2}|}{\sigma_{nr1}} \leq x\right) - \Phi(x)| \rightarrow 0 \quad (\text{A.37})$$

and

$$\sup_x |P\left(\frac{1}{1+\frac{c\bar{\sigma}_{nr2}}{\sigma_{nr1}}}(Z_1 + \frac{c\bar{\sigma}_{nr2}}{\sigma_{nr1}}|Z_2|) \leq x\right) - \Phi(x)| \rightarrow 0 \quad (\text{A.38})$$

as $r \rightarrow \infty$. By using the triangle inequality, Θ_{nr} is dominated by the sum of the two suprema above and thus converges to zero.

When $v \in (0, 1]$, by following the same arguments in the proof of Theorem 2.2, we can show (A.27) is satisfied. Hence, we can have (A.29), and both $\frac{1}{1+\frac{c\bar{\sigma}_{nr2}}{\sigma_{nr1}}}\frac{S_{nr1}+c|\bar{S}_{nr2}|}{\sigma_{nr1}}$ and $\frac{1}{1+\frac{c\bar{\sigma}_{nr2}}{\sigma_{nr1}}}(Z_1 + \frac{c\bar{\sigma}_{nr2}}{\sigma_{nr1}}|Z_2|)$ converge in distribution to $(1-v)Z_1 + v|Z_2|$ which is a continuous random variable. Denote the cumulative distribution of this limit as Φ_v . Then (A.37) and (A.38) hold if Φ is replaced by Φ_v . Again, by using the triangle inequality we get that Θ_{nr} converges to zero as $r \rightarrow \infty$. $\square$

References

- [1] Hall, P. and Heyde, C. C. (1980). Martingale Limit Theory and its Applications. Academic Press, New York.