Limiting Spectral Radii for Products of Ginibre Matrices and Their Inverses

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Abstract

Consider the product of m independent n by n Ginibre matrices and their inverses, where $m = p + q$, p is the number of Ginibre matrices, and q is the number of inverses of Ginibre matrices. The maximum absolute value of the eigenvalues of the product matrices is known as the spectral radius. In this paper, we explore the limiting spectral radii of the product matrices as n tends to infinity and m varies with n. Specifically, when $q \geq 1$ is a fixed integer, we demonstrate that the limiting spectral radii display a transition phenomenon when the limit of p/n changes from zero to infinity. When $q = 0$, the limiting spectral radii for Ginibre matrices have been obtained by Jiang and Qi [J. Theor. Probab. 30, 326–364 (2017)]. When q diverges to infinity as n approaches infinity, we prove that the logarithmic spectral radii exhibit a normal limit, which reduces to the limiting distribution for spectral radii for the spherical ensemble obtained by Chang et al. [J. Math. Anal. Appl. 461, 1165–1176 (2018)] when $p = q$.

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1 Introduction

The study of the spectral properties of eigenvalues has been a focus in random matrix theory. For example, Wigner [36], Jaynes [22], and Dyson [16] investigated the statistical behaviors of energy levels in quantum mechanics for a system represented by a Hermitian operator. In particular, Wigner proved the semicircle law, which states that the density of eigenvalues for a large class of random Hermitian matrices is approximately semicircular in shape. This law is a fundamental result in random matrix theory and has found numerous applications in physics, statistics, and engineering. Furthermore, the statistical properties of the largest eigenvalues of random matrices are also of great interest in random matrix theory. Tracy and Widom [33, 34] investigated three important classes of Hermitian matrices, including the Gaussian orthogonal ensemble, Gaussian unitary ensemble, and Gaussian symplectic ensemble. Their work demonstrated that the largest eigenvalues of these three ensembles converge weakly to the Tracy-Widom distributions. This result has had important implications for a variety of applications, including the study of the stability of dynamical systems, combinatorial optimization, and statistical physics. Johnstone [26] showed that the limiting distribution of the largest eigenvalue of Wishart matrices has the Tracey-Widom law of order 1. Dieng [14] generalized the results of Tracy and Widom [33, 34] to the mth largest eigenvalues of Gaussian orthogonal and Gaussian symplectic matrices. Further discussion on this topic can be found in, for instance, Baik et al. [5], Jiang [23], and Ramirez et al. [32]. Meanwhile, for non-Hermitian random matrices, Ginibre was the first to study the empirical spectral distribution for some non-Hermitian random matrices which are known as Ginibre ensembles. He established the circle law for these ensembles, including real, complex, or quaternionic Gaussian random matrices [19].

In addition to the study of individual random matrices, the study of products of random matrices is also of great interest. Bellman [6], who first investigated the behavior of systems under non-commutative effects, pioneered the research in this area. Later on, Furstenberg and Kesten [17] strengthened his result. A wide range of applications of products of random matrices can be found in the literature, including the stability of chaotic dynamical systems, the free energy of disordered systems, wireless communications, symplectic maps, Hamiltonian mechanics, and quantum chromodynamics at non-zero chemical potential; see, e.g., Tulino and Verdú [35], Crisanti et al. [13], Ipsen [21], Akemann and Ipsen [3]. Products of Hermitian matrices are, in general, non-Hermitian, and they have been applied to various fields such as chaotic quantum systems, growth processes, and dissipative systems; see, e.g., Akemann et al. [1], Haake [20], Di Francesco et al. [15].

Götze and Tikhomirov [18] considered products of independent random matrices with independent and identically distributed entries having a mean of zero and a variance of one when the number of multiplicands is fixed. They obtained the limit of the expected empirical distribution. Under similar settings, O'Rourke and Soshnikov [28] obtained the limiting empirical spectral distribution. See also the work by Bordenave [7]. Later on, Burda et al. [8] and

O'Rourke et al. [29] demonstrated that the limiting empirical spectral distribution remains unchanged even if the entries of multiplicands in the product are not identically distributed. Kösters and Tikhomirov [27] considered sums of products of independent non-Hermitian random matrices and their inverses and established the universality of the limiting singular value and eigenvalue distributions. Akemann and Burda [2] obtained the correlation functions for products of independent Ginibre matrices.

In recent years, the study of product of non-Hermitian matrices allows the number of the multiplicands in the product to change with the dimension n of the product matrix. For example, Jiang and Qi [25] investigated the limiting empirical spectral distribution for products of Ginibre matrices, Zeng [37] and Chang and Qi [12] extended this work to the product of spherical ensembles. Qi and Zhao [31] generalized the results for products of squared matrices to products of rectangular matrices.

For non-Hermitian matrices, the largest absolute values of their eigenvalues are referred to as spectral radii. Jiang and Qi [24] obtained the limiting spectral radius for product of m complex Ginibre matrices. They showed that the limiting distributions have a transition phenomenon according to the limit of the ratio m/n when n tends to infinity and m changes with n. Chang et al. [11] investigated the limiting spectral radius for products of m spherical ensembles. They demonstrated that, when m is a fixed integer, the spectral radius converges weakly to distributions that depend on independent Gamma random variables, and when m diverges with n as n tends to infinity, the logarithmic spectral radius has a normal limit. Qi and Xie [30] investigated products of m independent rectangular matrices and obtained the limiting spectral radius which depends on the limit of the ratio m/n, where n tends to infinity and m changes with n.

In this paper, we are interested in products of Ginibre matrices and their inverses, and our objective is to obtain the limiting spectral radius for the product matrices. The study of this type of products of random matrices was initially discussed in Adhikari et al. [4], and the joint density functions for eigenvalues of both products of Ginibre matrices and their inverses and product of truncated Haar unitary matrices and their inverses were obtained in the paper. Very recently, Chang et al. [10] obtained the limiting spectral distributions for the products.

More specifically, let $A_1, A_2, \cdots, A_m$ be independent $n \times n$ random matrices with independent and identically distributed (i.i.d.) standard complex Gaussian entries. Define the product matrix $A_n^{(m)} = A_1^{\epsilon_1} \cdots A_m^{\epsilon_m}$, where $\{\epsilon_k; 1 \leq k \leq m\}$ are either 1 or -1. We will investigate the limiting spectral radius of products matrices by allowing m to change with n. When $\epsilon_k = 1$ for $1 \leq k \leq m$, Jiang and Qi [24] showed that the limiting spectral radii exhibit a transition phenomenon depending on the limit of the ratio: $m/n \to 0$, $m/n \to \alpha \in (0, \infty)$ and $m/n \to \infty$, respectively. In the present paper, we obtain similar results depending on the ratio p/n when $q \geq 1$ is fixed integer, where $p = \#\{k : \epsilon_k = 1, 1 \leq k \leq m\}$ and $q = m - p$. When $p = q$ varies with n and $\lim_{n\to\infty} q = \infty$, our result can reduce to the conclusion obtained by Chang et al. [11] for the product of spherical ensembles, that is, the logarithmic spectral radii have a normal limit.

The rest of the paper is organized as follows. In Section 2, we introduce the main results of the paper. In Section 3, we present some preliminary lemmas and provide the proofs for the main results.

2 Main Results

Let $A_1, A_2, \cdots, A_m$ be independent $n \times n$ random matrices with i.i.d. complex standard Gaussian entries whose real and imaginary parts are two i.i.d. normal random variables with mean 0 and variance 1/2. Define $A_n^{(m)}$ as the product of the m matrices $A_k^{\epsilon_k}$'s, that is,

$$A_n^{(m)} = A_1^{\epsilon_1} \cdots A_m^{\epsilon_m},$$

where $\{\epsilon_k; 1 \le k \le m\}$ are 1 or -1. Let $\mathbf{z}_1, \cdots, \mathbf{z}_n$ be the eigenvalues of $A_n^{(m)}$. Note that $\{\mathbf{z}_j; 1 \le j \le n\}$ are complex random variables. The joint density function for $\mathbf{z}_1, \cdots, \mathbf{z}_n$, obtained in Theorem 1 of Adhikari et al. [4], is given as follows:

$$C \prod_{l=1}^{n} \omega(z_l) \prod_{i<j}^{n} |z_i - z_j|^2 \tag{2.1}$$

with respect to the Lebesgue measure on $\mathbb{C}^n$, where C is a normalizing constant, and function $\omega(z)$ is a weight function with

$$\omega(z)dz = \int_{x_1^{\epsilon_1}\cdots x_m^{\epsilon_m}=z} e^{-\sum_{k=1}^m |x_k|^2} \prod_{k=1}^{m} |x_k|^{(1-\epsilon_k)(n-1)} \prod_{k=1}^{m} dx_k, \tag{2.2}$$

and dz is Lebesgue measure on the complex plane and equivalent to $dz = dxdy$ with $z = x+yi$ on $\mathbb{R}^2$ space. Note that $\omega(z)dz$ is rotation-invariant and $\omega(z) = \omega(|z|)$. Lemma 3.4 of Chang et al. [10] proved that

$$\int_0^\infty r^t \omega(r) dr = \frac{\pi^{m-1}}{2} \prod_{k=1}^{m} \Gamma\left(\frac{1}{2}\Big(n + 1 + \epsilon_k(t-n)\Big)\right), \quad t > 0, \tag{2.3}$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ for $x > 0$ is the Gamma function. Moreover, the weight function $\omega(z)$ has a representation in terms of the so-called Meijer's G-function from the formula (2.42) of Akemann and Ipsen [3], and can be written as

$$\omega(z) = (2\pi)^{m-1} G^{p,m-p}_{m-p,p}\left[\begin{array}{c} (-n,-n,\cdots,-n)_{m-p} \\ (0,0,\cdots,0)_p \end{array} \middle| |z|^2 \right],$$

where $p = \#\{k : \epsilon_k = 1, 1 \le k \le m\}$, $q = m - p$, and the symbol $G_{p,q}^{m,n}\left[\cdots \middle| z\right]$ denotes Meijer's G-function.

The spectral radius of $A_n^{(m)}$ is defined as the maximal absolute value of the n eigenvalues $\mathbf{z}_1, \cdots, \mathbf{z}_n$, i.e.

$$M_n := \max_{1 \le j \le n} |\mathbf{z}_j|. \tag{2.4}$$

The goal of this paper is to investigate the limiting distribution of M_n as n tends to infinity.

Before the introduction of the main results, we will define some notations. Let ψ be the digamma function defined by

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \Gamma'(x)/\Gamma(x), \quad x > 0.$$

Then, we have

$$\frac{\Gamma(b)}{\Gamma(a)} = \exp\big(\log \Gamma(b) - \log \Gamma(a)\big) = \exp\Big(\int_a^b \psi(x)dx\Big) \quad \text{for } a > 0, b > 0. \tag{2.5}$$

Moreover, we have from Lemma 3.2 in Chang et al. [11]

$$\psi(x) = \log(x) - \frac{1}{2x} + O(\frac{1}{x^2}), \quad \psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + O(\frac{1}{x^3}) \text{ as } x \to \infty; \tag{2.6}$$

$$\psi(1) = -\gamma, \quad \psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \quad \text{for } n \ge 2; \tag{2.7}$$

$$\psi'(x) = \sum_{k=0}^{\infty} \frac{1}{(k+x)^2}, \quad x > 0 \quad \text{and} \quad \psi'(1) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \tag{2.8}$$

where $\gamma = 0.57721\cdots$ is the Euler constant.

We need some notations for our main results and their proofs. Notations $\xrightarrow{\text{d}}$ and $\xrightarrow{\text{p}}$ denote convergence in distribution and convergence in probability and $\stackrel{\text{d}}{=}$ denotes the equality in distribution. $N(\mu, \sigma^2)$ denotes a normal distribution or normal random variable with mean μ and variance σ^2.

Assume $\{X_n; n \ge 1\}$ is a sequence of random variables and $\{a_n; n \ge 1\}$ is any sequence of positive constants. Denote $X_n = o_p(a_n)$ if $X_n/a_n \xrightarrow{\text{p}} 0$ as $n \to \infty$. If $\lim_{c \to \infty} \limsup_{n \to \infty} P(|X_n/a_n| > c) = 0$, denote $X_n = O_p(a_n)$. In particular, if $X_n \xrightarrow{\text{p}} 0$, then $X_n = o_p(1)$. If X_n converges in distribution, we have $X_n = O_p(1)$.

We will first give the general results on the limiting distributions for the spectral radius, and denote $m = m_n$, $p = p_n$, and $q_n = m_n - p_n$ when they change with n.

Theorem 2.1 *Assume that $m = m_n$ depends on n, and $q_n \to \infty$ as $n \to \infty$. Then we have*

$$\frac{\log M_n - \mu_n}{\sigma_n} \xrightarrow{\mathrm{d}} N(0,1) \text{ as } n \to \infty, \tag{2.9}$$

where $\mu_n = p_n\psi(n)/2 - q_n\psi(1)/2$ and $\sigma_n^2 = p_n\psi'(n)/4 + q_n\psi'(1)/4$.

Theorem 2.2 *Assume $\{M_{j,k};\ j \geq 1,\ k \geq 1\}$ are independent random variables such that for each $j \geq 1$ and $k \geq 1$, $M_{j,k}$ has a Gamma(j) distribution; that is, $M_{j,k}$ has a density function $y^{j-1}e^{-y}I(y > 0)/\Gamma(j)$. Suppose that $m = m_n$ depends on n and $q \geq 1$ is a fixed integer. Set $\alpha_n = p_n/n$.*

(a). If $\lim_{n\to\infty} \alpha_n = 0$, then

$$\frac{M_n}{n^{p_n/2}} \xrightarrow{\mathrm{d}} \max_{1\leq j<\infty} \frac{1}{\prod_{k=1}^{q} M_{j,k}^{1/2}} \quad \text{as } n \to \infty. \tag{2.10}$$

(b). If $\lim_{n\to\infty} \alpha_n =: \alpha \in (0,\infty)$, then for each $\alpha \in (0,\infty)$

$$\frac{M_n}{n^{p_n/2}} \xrightarrow{\mathrm{d}} \max_{1\leq j<\infty} \frac{\Theta_{j,\alpha}}{\prod_{k=1}^{q} M_{j,k}^{1/2}} \text{ as } n \to \infty, \tag{2.11}$$

where random variables $\{\Theta_{j,\alpha}; j \geq 1\}$ are independent random variables and $\Theta_{j,\alpha}$ has a log-normal distribution with parameters $-\frac{1}{2}(j-\frac{1}{2})\alpha$ and $\alpha^{1/2}/2$ with the cumulative distribution

$$\Phi\left(\alpha^{1/2}/2 + 2\alpha^{-1/2}\log y + (j-1)\alpha^{1/2}\right) \text{ for } y \in (0,\infty),$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$ is the cumulative distribution function of the standard normal distribution, that is, $\log\Theta_{j,\alpha}$ has a $N(-\frac{1}{2}(j-\frac{1}{2})\alpha, \frac{\alpha}{4})$ distribution, and $\{\Theta_{j,\alpha}; j \geq 1\}$ are independent of $\{M_{j,k};\ j \geq 1,\ k \geq 1\}$.

(c). If $\lim_{n\to\infty} \alpha_n = \infty$, then

$$\frac{\log M_n - p_n\psi(n)/2}{\sqrt{p_n/n}/2} \xrightarrow{\mathrm{d}} N(0,1) \text{ as } n \to \infty. \tag{2.12}$$

Remark 1. The spectral radius for the product of Ginibre matrices and inverse Ginibre matrices does not depend on their order but relies on the number of Ginibre matrices or inverse Ginibre matrices from the above theorems.

Remark 2. The existence for the random variable on the right-hand side of (2.10) in part (a) of Theorem 2.2 can be found in Lemma 3.7. The limiting distributions in part (b) of Theorem 2.2 is well defined. Note that $\max_{1\leq j<n} \frac{\Theta_{j,\alpha}}{\prod_{k=1}^{q} M_{j,k}^{1/2}}$ is non-decreasing in n with probability

one, the limit exists and is larger than 0. We show that $P(\max_{1\le j<\infty} \frac{\Theta_{j,\alpha}}{\prod_{k=1}^q M_{j,k}^{1/2}} < \infty) = 1$. The details are as follows. First, we have $\mathbb{E}(1/M_{j,k}^2) \le j^{-1.5}$ from the Lemma 3.3 in Chang et al. [11] when $j \ge j_0$ for some $j_0 \ge 3$. Note that $\{\Theta_{j,\alpha}; j \ge 1\}$ and $\{M_{j,k}; j \ge 1, k \ge 1\}$ are independent for every $\alpha \in (0,\infty)$, and $\mathbb{E}(\Theta_{j,\alpha}^4) = \exp\big(-\frac{4}{2}(j-\frac{1}{2})\alpha + 4^2(\alpha^{1/2}/2)^2/2\big) = \exp(-2j+3\alpha)$ since $\Theta_{j,\alpha}$ has a log-normal distribution with parameters $-\frac{1}{2}(j-\frac{1}{2})\alpha$ and $\alpha^{1/2}/2$. So we have

$$\mathbb{E}\Big(\sum_{j=j_0}^{\infty} \frac{\Theta_{j,\alpha}^4}{\prod_{k=1}^q M_{j,k}^2}\Big) = \sum_{j=j_0}^{\infty} \mathbb{E}(\Theta_{j,\alpha}^4) \prod_{k=1}^{q} \mathbb{E}(1/M_{j,k}^2) \le \sum_{j=j_0}^{\infty} \exp(-2j+3\alpha) j^{-1.5q} < \infty,$$

yielding that $P(\sum_{j=1}^{\infty} \frac{\Theta_{j,\alpha}^4}{\prod_{k=1}^q M_{j,k}^2} < \infty) = 1$, which implies that $P(\max_{1\le j<\infty} \frac{\Theta_{j,\alpha}}{\prod_{k=1}^q M_{j,k}^{1/2}} < \infty) = 1$ because

$$\max_{1\le j<\infty} \frac{\Theta_{j,\alpha}^4}{\prod_{k=1}^q M_{j,k}^2} \le \sum_{j=1}^{\infty} \frac{\Theta_{j,\alpha}^4}{\prod_{k=1}^q M_{j,k}^2}.$$

Remark 3. Theorem 2.2 covers three different types of the limiting spectral distributions when $q = \#\{k : \varepsilon_k = -1,\ 1 \le k \le m_n\} \ge 1$ is a fixed integer. When $q = 0$, the product matrix is the same as the production of m_n independent Ginibre matrices, and the corresponding limiting spectral radii were obtained by Jiang and Qi [24]. Although Theorem 2.2 does not cover the case when $q = 0$, the expressions of the limiting spectral distributions when $q = 0$ in parts (b) and (c) in Theorem 2.2 actually reduce to the limiting spectral distributions for the production of Ginibre matrices; see Theorem 3 (b, c) in Jiang and Qi [24]. One should interpret $\prod_{k=1}^q M_{j,k}^{1/2}$ as 1 when $q = 0$ in Theorem 2.2. We also notice that the limit on the right-hand side of (2.10) in part (a) of Theorem 2.2 for $q = 0$ is degenerate at 1. After renormalization, M_n converges in distribution to Gumbel distribution. In fact, it follows from Theorem 3 (a) in Jiang and Qi [24] that

$$\sqrt{\frac{n}{p_n}\log\frac{n}{p_n}}\Big(\frac{M_n}{n^{p_n/2}} - 1\Big) - \Big(\log\frac{n}{p_n} - \log\log\frac{n}{p_n} - \frac{1}{2}\log(2\pi)\Big) \xrightarrow{\mathrm{d}} \Lambda,$$

where the cumulative distribution function $\Lambda(x) = \exp(-e^{-x})$ is the Gumbel distribution.

3 Proofs

In this section, we prove the main results given in Section 2. We will use the distributional representations for the spectral radii in our proofs and then develop limit theorems for sums of independent random variables. Some approaches are classical such as validating the central limit theorem via the method of the moment generating functions. Some strategies for the proofs are also close to those used in Chang et al. [11] and Jiang and Qi [24].

Before proceeding with the proofs, we will introduce several lemmas.

3.1 Some Preliminary Lemmas

Lemma 3.1 *(Lemma 3.1 in Chang et al. [11]) Let random variable Y have a Gamma(α) distribution and $X = \log(Y)$. Then the moment generating function of X is given by*

$$\mathbb{E}(e^{tX}) = \frac{\Gamma(\alpha + t)}{\Gamma(\alpha)}, \quad t > -\alpha.$$

Moreover, $\mathbb{E}(X) = \psi(\alpha)$ and $\mathrm{Var}(X) = \psi'(\alpha)$.

Lemma 3.2 *(Lemma 1.1 in Jiang and Qi [24]) Let $Y_1, \cdots, Y_n$ be independent random variables such that the density of Y_j is proportional to $y^{2j-1}\omega(y)I(y \geq 0)$ for each $1 \leq j \leq n$. Assume $\mathbf{z}_1, \cdots, \mathbf{z}_n \in \mathbb{C}$ have joint density $f(z_1, \cdots, z_n) = C \prod_{l=1}^{n} \omega(z_l) \prod_{i<j}^{n} |z_i - z_j|^2$, where $\omega(x) \geq 0$ for all $x \geq 0$, and C is a normalizing constant. Then $g(|\mathbf{z}_1|, \cdots, |\mathbf{z}_n|)$ and $g(Y_1, \cdots, Y_n)$ have the same distribution for any symmetric function $g(y_1, \cdots, y_n)$.*

Lemma 3.3 *(Lemma 3.5 in Chang et al. [10]) Let $Y_1, \cdots, Y_n$ be independent random variables defined in Lemma 3.2. Assume $\{s_{j,k}; 1 \leq k \leq m, 1 \leq j \leq n\}$ are independent random variables such that $s_{j,k}$ has a Gamma($\alpha_{j,k}$) distribution with density function $y^{\alpha_{j,k}-1}e^{-y}I(y \geq 0)/\Gamma(\alpha_{j,k})$, where*

$$\alpha_{j,k} = \frac{1}{2}\big(n + 1 + \epsilon_k(2j - 1 - n)\big), \quad 1 \leq k \leq m, 1 \leq j \leq n. \tag{3.1}$$

Then Y_j has the same distribution as $\prod_{k=1}^{m} s_{j,k}^{\epsilon_k/2}$ for each $1 \leq j \leq n$.

Lemma 3.4 *Let $\{s_{j,k}; 1 \leq k \leq m, 1 \leq j \leq n\}$ be independent random variables defined in Lemma 3.3. Then M_n, $\max_{1\leq j\leq n} Y_j$ and $\max_{1\leq j\leq n} \prod_{k=1}^{m} s_{j,k}^{\epsilon_k/2}$ have the same distribution.*

Proof. It follows from Lemma 3.2 that M_n and $\max_{1\leq j\leq n} Y_j$ have the same distribution. Furthermore, from Lemma 3.3, $\{Y_j; 1 \leq j \leq n\}$ and $\{\prod_{k=1}^{m} s_{j,k}^{\epsilon_k/2}; 1 \leq j \leq n\}$ have the same distribution function. Therefore, we conclude that M_n, $\max_{1\leq j\leq n} Y_j$ and $\max_{1\leq j\leq n} \prod_{k=1}^{m} s_{j,k}^{\epsilon_k/2}$ have the same distribution. □

Lemma 3.5 *Let $Y_1, \cdots, Y_n$ be independent random variables defined in Lemma 3.2. Then for any $y \in \mathbb{R}$, $P(Y_j > y)$ is non-decreasing in $1 \leq j \leq n$.*

Proof. Denote f_j as the density function of Y_j for $1 \leq j \leq n$. Then all f_j's have a common support, say $D \subset \mathbb{R}$. Recall that $f_j(y) = c_j y^{2j-1}\omega(y) > 0$ for all $y \in D$, where $c_j > 0$ are constants. For every $2 \leq j \leq n$, $f_j(y)/f_{j-1}(y) = c_j y/c_{j-1}$ is increasing in $y \in D$. According to Lemma 2.4 in Chang et al. [9], $P(Y_j > y) \geq P(Y_{j-1} > y)$ for all $y \in \mathbb{R}$. This proves that $P(Y_j > y)$ is non-decreasing in j for any $y \in \mathbb{R}$. □

Lemma 3.6 *(Lemma 2.1 in Jiang and Qi [24]) Let $c_{nj} \in [0, 1)$ be constants for $j \ge 1, n \ge 1$ and $\sup_{n\ge1,j\ge1} c_{nj} < 1$. For each $j \ge 1$, $c_j = \lim_{n\to\infty} c_{nj}$. Assume $c_n = \sum_{j=1}^{\infty} c_{nj} < \infty$ for each $n \ge 1$, $c := \sum_{j=1}^{\infty} c_j < \infty$ and $\lim_{n\to\infty} c_n = c$. Then*

$$\lim_{n\to\infty} \prod_{j=1}^{\infty}(1 - c_{nj}) = \prod_{j=1}^{\infty}(1 - c_j).$$

Review that random variables $\{M_{j,k};\ j \ge 1,\ k \ge 1\}$ are defined in Theorem 2.2. For the rest of the paper, let $\{\widetilde{M}_{j,k};\ j \ge 1,\ k \ge 1\}$ be an independent copy of $\{M_{j,k};\ j \ge 1,\ k \ge 1\}$. With the above lemmas, we can develop a distributional representation for $M_n = \max_{1\le j\le n} |\mathbf{z}_j|$ as in (2.4).

For the $\alpha_{j,k}$ given in (3.1), we have

$$\alpha_{j,k} = \frac{1}{2}\big(n + 1 + \epsilon_k(2j - 1 - n)\big) = \begin{cases} j, & \text{if } \ \epsilon_k = 1; \\ n + 1 - j, & \text{if } \ \epsilon_k = -1. \end{cases}$$

Also recall $p_n = \#\{k : \epsilon_k = 1, 1 \le k \le m\}$ and $q_n = m_n - p_n$. We have from Lemma 3.3 that $Y_j \stackrel{d}{=} \prod_{k=1}^{m_n} s_{j,k}^{\epsilon_k/2} \stackrel{d}{=} \frac{\prod_{k=1}^{p_n} \widetilde{M}_{j,k}^{1/2}}{\prod_{k=1}^{q_n} M_{n-j+1,k}^{1/2}}$ for $1 \le j \le n$, or

$$Y_{n+1-j} \stackrel{d}{=} \frac{\prod_{k=1}^{p_n} \widetilde{M}_{n+1-j,k}^{1/2}}{\prod_{k=1}^{q_n} M_{j,k}^{1/2}}, \quad j = 1, \cdots, n. \tag{3.2}$$

Then it follows from Lemma 3.4 that

$$M_n \stackrel{\mathrm{d}}{=} \max_{1\le j\le n} \frac{\prod_{k=1}^{p_n} \widetilde{M}_{n+1-j,k}^{1/2}}{\prod_{k=1}^{q_n} M_{j,k}^{1/2}}. \tag{3.3}$$

The above representation will greatly simplify our proofs in this paper. In particular, when $q_n = 0$, $\max\limits_{1\le j\le n} \prod_{k=1}^{p_n} \widetilde{M}_{n+1-j,k}^{1/2}$ has the same distribution as that of the largest absolute value of the n eigenvalues of $A_1 \cdots A_{p_n}$, the product of p_n independent $n \times n$ Ginibre ensembles. This is the special case of $A_n^{(p_n)}$ when all $\epsilon_k = 1$ and the corresponding limiting spectral distributions were obtained by Jiang and Qi [24]. This fact will be used in the proofs for Theorem 2.2 and Lemma 3.9.

Lemma 3.7 *(Lemma 3.3 in Chang et al. [11]) For each fixed integer $q \ge 1$, the random variable*

$$T_q := \lim_{n\to\infty} \max_{1\le j\le n} \frac{1}{\prod_{k=1}^{q} M_{j,k}^{1/2}}$$

is well defined, and $P(0 < T_q < \infty) = 1$.

Lemma 3.8 *For every $j \geq 1$, we have*

$$N_{1,j} := \frac{\sum_{k=1}^{p_n} \log \widetilde{M}_{n+1-j,k} - p_n \psi(n+1-j)}{\sqrt{p_n \psi'(n+1-j)}} \xrightarrow{\mathrm{d}} N(0,1) \tag{3.4}$$

as $n \to \infty$, where p_n is a sequence of positive integers.

Proof. Denote the moment generating function of $N_{1,j}$ as $M_{N_{1,j}}(t)$. It suffices to show that

$$\lim_{n\to\infty} M_{N_{1,j}}(t) = \exp\Big(\frac{t^2}{2}\Big), \quad |t| \leq 1. \tag{3.5}$$

It follows from Lemma 3.1 that the moment generating function of $\log(\widetilde{M}_{n+1-j,k}) - \psi(n+1-j)$ is given by

$$\exp\big(-\psi(n+1-j)t\big)\frac{\Gamma(n+1-j+t)}{\Gamma(n+1-j)}, \quad t > -n-1+j.$$

Then we have from (2.5) that

$$\begin{aligned}
M_{N_{1,j}}(t) &= \left(\exp\Big(-\frac{\psi(n+1-j)t}{\sqrt{p_n\psi'(n+1-j)}}\Big)\frac{\Gamma\big(n+1-j+\frac{t}{\sqrt{p_n\psi'(n+1-j)}}\big)}{\Gamma(n+1-j)}\right)^{p_n} \\
&= \left(\exp\Big(-\frac{\psi(n+1-j)t}{\sqrt{p_n\psi'(n+1-j)}}\Big)\exp\Big(\int_0^{\frac{t}{\sqrt{p_n\psi'(n+1-j)}}}\psi(n+1-j+s)ds\Big)\right)^{p_n} \\
&= \left(\exp\Big(-\frac{\psi(n+1-j)t}{\sqrt{p_n\psi'(n+1-j)}} + \int_0^{\frac{t}{\sqrt{p_n\psi'(n+1-j)}}}\psi(n+1-j+s)ds\Big)\right)^{p_n} \\
&= \exp\Big(p_n\int_0^{\frac{t}{\sqrt{p_n\psi'(n+1-j)}}}\big(\psi(n+1-j+s)-\psi(n+1-j)\big)ds\Big) \\
&= \exp\Big(p_n\int_0^{\frac{t}{\sqrt{p_n\psi'(n+1-j)}}} s\int_0^1 \psi'(n+1-j+sv)dvds\Big) \\
&= \exp\Big(\frac{t^2}{\psi'(n+1-j)}\int_0^1 u\int_0^1 \psi'\big(n+1-j+\frac{tuv}{\sqrt{p_n\psi'(n+1-j)}}\big)dvdu\Big).
\end{aligned}$$

In the last step, we changed the variable $s = tu/\sqrt{p_n\psi'(n+1-j)}$.

Fix t within $|t| \leq 1$. In view of (2.6), we have $\psi'(n+1-j) \geq \frac{1}{n+1-j}$ for $j \geq 1$, which implies

$$\frac{|tuv|}{\sqrt{p_n\psi'(n+1-j)}} \leq \sqrt{n+1-j}$$

uniformly over $|u| \le 1$ and $|v| \le 1$. Then we have

$$\psi'\big(n+1-j+\frac{tuv}{\sqrt{p_n\psi'(n+1-j)}}\big) = \frac{1+o(1)}{n+1-j}$$

holds uniformly over $|u| \le 1$ and $|v| \le 1$ as $n \to \infty$. Thus, we get

$$\begin{aligned} M_{N_{1,j}}(t) &= \exp\Big(\frac{t^2\big(1+o(1)\big)}{\psi'(n+1-j)}\int_0^1\int_0^1 \frac{u}{n+1-j}dvdu\Big) \\ &= \exp\Big(\frac{t^2\big(1+o(1)\big)}{\psi'(n+1-j)(n+1-j)}\int_0^1\int_0^1 udvdu\Big) \\ &= \exp\Big(\frac{t^2\big(1+o(1)\big)}{2\psi'(n+1-j)(n+1-j)}\Big) \\ &\to \exp\Big(\frac{t^2}{2}\Big) \end{aligned}$$

as $n \to \infty$, that is, (3.5) holds. Therefore, (3.4) is proved. $\square$

Lemma 3.9 *Let $p = p_n$ be a sequence of positive integers. If $\lim_{n\to\infty}\frac{p_n}{n} = 0$, particularly for $p_n \equiv p$, then*

$$\max_{1\le j\le n}\prod_{k=1}^{p_n}\big(M_{n+1-j,k}/n\big)^{1/2} \xrightarrow{\text{p}} 1 \ \textit{as}\ n \to \infty.$$

Proof. Recall that $\max\limits_{1\le j\le n}\prod\limits_{k=1}^{p_n} M_{n+1-j,k}^{1/2}$ and $\max\limits_{1\le j\le n}\prod\limits_{k=1}^{p_n} \widetilde{M}_{n+1-j,k}^{1/2}$ are identically distributed. From the statement in the paragraph after equation (3.3), $\max\limits_{1\le j\le n}\prod\limits_{k=1}^{p_n} M_{n+1-j,k}^{1/2}$ has the same distribution as that of the largest absolute value of the n eigenvalues of $A_1\cdots A_{p_n}$, the product of p_n independent $n\times n$ random matrices with i.i.d. complex standard Gaussian entries. Then it follows from Theorem 3(a) of Jiang and Qi [24] that

$$\Lambda_n := a_n\Big(\max_{1\le j\le n}\prod_{k=1}^{p_n}\big(M_{n+1-j,k}/n\big)^{1/2}-1\Big)-b_n = a_n\Big(\frac{\max\limits_{1\le j\le n}\prod\limits_{k=1}^{p_n} M_{n+1-j,k}^{1/2}}{n^{p_n/2}}-1\Big)-b_n \xrightarrow{\text{d}} \Lambda, \quad (3.6)$$

which gives the asymptotic distribution for the largest spectral radius of the product of independent Ginibre ensembles when $\lim_{n\to\infty}\frac{p_n}{n} = 0$, where $\Lambda(x) = \exp\left(-e^{-x}\right)$ is the standard Gumbel distribution, and

$$a_n = \Big(\frac{n}{p_n}\log\frac{n}{p_n}\Big)^{1/2}, \quad b_n = \log\frac{n}{p_n} - \log\log\frac{n}{p_n} - \frac{1}{2}\log(2\pi).$$

Therefore, we have from (3.6)

$$\max_{1\le j\le n}\prod_{k=1}^{p_n}\big(M_{n+1-j,k}/n\big)^{1/2}-1=\frac{b_n}{a_n}+\frac{\Lambda_n}{a_n}\xrightarrow{\text{p}}0,$$

since

$$a_n\to\infty\ \text{ and }\ \frac{b_n}{a_n}\to\infty\quad \text{ as } n\to\infty.$$

This completes the proof of the lemma. □

Lemma 3.10 *Define for every integer* $r\in\{1,2,\cdots,n\}$

$$W_{n,r}=\min_{1\le j\le r}\prod_{k=1}^{p_n}\big(M_{n+1-j,k}/n\big)^{1/2}.$$

Assume $\lim_{n\to\infty}p_n/n=0$*. Then there exists a sequence of integer* $\{r_n\}$ *with* $1\le r_n\le n$ *and* $r_n\to\infty$ *as* $n\to\infty$ *such that* $W_{n,r_n}\xrightarrow{\text{p}}1$ *as* $n\to\infty$*.*

Proof. Using the notation defined in (3.4), we have for every $j\ge 1$

$$\prod_{k=1}^{p_n}M_{n+1-j,k}^{1/2}=\exp\big(\frac{1}{2}p_n\psi(n+1-j)+\frac{1}{2}\sqrt{p_n\psi'(n+1-j)}N_{1,j}\big)$$

from Lemma 3.8. From equations (2.6) and (2.7), we have $\log(n)=\psi(n)+\frac{1}{2n}+O(\frac{1}{n^2})$, $\psi'(n+1-j)=\frac{1}{n+1-j}+\frac{1}{2(n+1-j)^2}+O(\frac{1}{(n+1-j)^3})$, and $\psi(n)-\psi(n+1-j)=\frac{j-1}{n}\big(1+O(\frac{j}{n})\big)$ as $n\to\infty$. Then for every $j\ge 1$

$$\begin{aligned}
&\prod_{k=1}^{p_n}\big(M_{n+1-j,k}/n\big)^{1/2}\\
=\;&\exp\Big(\frac{1}{2}p_n\psi(n+1-j)+\frac{1}{2}\sqrt{p_n\psi'(n+1-j)}N_{1,j}\Big)\exp\big(-\frac{1}{2}p_n\log n\big)\\
=\;&\exp\Big(\frac{1}{2}p_n\big(\psi(n+1-j)-\psi(n)-\frac{1}{2n}+O(\frac{1}{n^2})\big)+\frac{1}{2}\sqrt{p_n\psi'(n+1-j)}N_{1,j}\Big)\\
=\;&\exp\Big(\frac{1}{2}\Big(-\frac{p_n(j-1)}{n}(1+O(\frac{j}{n}))-\frac{p_n}{2n}+O(\frac{p_n}{n^2})\Big)\Big)\\
&\times\exp\Big(\frac{1}{2}\Big(\frac{p_n}{n+1-j}+\frac{p_n}{2(n+1-j)^2}+O(\frac{p_n}{(n+1-j)^3})\Big)^{1/2}N_{1,j}\Big)\\
\xrightarrow{\text{p}}\;&1\ \text{ as } n\to\infty
\end{aligned}$$

since $\lim_{n\to\infty} p_n/n = 0$ and $N_{1,j} \xrightarrow{d} N(0,1)$ from Lemma 3.8. This implies that for every $r \ge 1$

$$W_{n,r} \xrightarrow{\mathrm{p}} 1 \quad \text{as } n \to \infty.$$

Hence, for every $r \ge 1$, there exists a positive integer n_r such that

$$P(|W_{n,r} - 1| > \frac{1}{r}) < \frac{1}{r} \quad \text{for all } n \ge n_r,$$

where n_r can be selected in such a way that $n_1 < n_2 < n_3 < \cdots$. Now for each $n \ge n_1$, there exists a unique r such that $n_r \le n < n_{r+1}$, and we define $r_n = r$ for $n_r \le n < n_{r+1}$. We see that $1 \le r_n < n$, and $r_n \to \infty$ as $n \to \infty$, and

$$P(|W_{n,r_n} - 1| > \frac{1}{r_n}) < \frac{1}{r_n} \quad \text{for } n \ge n_r,$$

which yields $W_{n,r_n} \xrightarrow{p} 1$ as $n \to \infty$. This completes the proof. $\square$

3.2 Proofs of Theorems 2.1 and 2.2

Recall that we have assumed that $\{M_{j,k};\ j \ge 1,\ k \ge 1\}$ are independent random variables and $M_{j,k}$ has a Gamma(j) distribution, and $\{\widetilde{M}_{j,k};\ j \ge 1,\ k \ge 1\}$ is an independent copy of $\{M_{j,k};\ j \ge 1,\ k \ge 1\}$.

Proof of Theorem 2.1. Define

$$V_j = \frac{\prod_{k=1}^{p_n} \widetilde{M}_{n+1-j,k}^{1/2}}{\prod_{k=1}^{q_n} M_{j,k}^{1/2}}, \quad 1 \le j \le n,$$

which are independent random variables. Note that $V_j \overset{\mathrm{d}}{=} Y_{n+1-j}, 1 \le j \le n$ from equation (3.2). In view of (3.3) we have

$$P(\max_{1\le j\le n} \log |\mathbf{z}_j| \le \mu_n + \sigma_n x) = P(\log M_n \le \mu_n + \sigma_n x) = \prod_{j=1}^{n} a_{n,j}(x) \quad \text{for every } x \in \mathbb{R},$$

where $a_{n,j}(x) = P(\log V_j \le \mu_n + \sigma_n x)$ for $1 \le j \le n$, $\mu_n = p_n\psi(n)/2 - q_n\psi(1)/2$ and $\sigma_n^2 = p_n\psi'(n)/4 + q_n\psi'(1)/4$. Therefore, Theorem 2.1 is true if we can show

$$\lim_{n\to\infty} a_{n,1}(x) = \Phi(x) \tag{3.7}$$

and

$$\lim_{n\to\infty}\prod_{j=2}^{n} a_{n,j}(x) = 1 \tag{3.8}$$

for every $x \in \mathbb{R}$, where $\Phi(x)$ is defined as the cumulative distribution of a standard normal random variable.

Let us first prove (3.7). Since

$$\log V_j = \frac{1}{2}\Big(\sum_{k=1}^{p_n}\log \widetilde{M}_{n+1-j,k} - \sum_{k=1}^{q_n}\log M_{j,k}\Big), \tag{3.9}$$

(3.7) is equivalent to

$$\frac{\log V_1 - \mu_n}{\sigma_n} = \frac{\sum_{k=1}^{p_n}\log \widetilde{M}_{n,k} - p_n\psi(n)}{\sqrt{p_n\psi'(n)+q_n\psi'(1)}} - \frac{\sum_{k=1}^{q_n}\log M_{1,k} - q_n\psi(1)}{\sqrt{p_n\psi'(n)+q_n\psi'(1)}} \xrightarrow{\mathrm{d}} N(0,1) \tag{3.10}$$

as $n \to \infty$.

For each integer $j \geq 1$, $\{\log M_{j,k};\ k = 1, \cdots, q_n\}$ are i.i.d. random variables, and their means and variances are equal to $\psi(j)$ and $\psi'(j)$, respectively, from Lemma 3.1. Then we have

$$N_{2,j} := \frac{\sum_{k=1}^{q_n}\log M_{j,k} - q_n\psi(j)}{\sqrt{q_n\psi'(j)}} \xrightarrow{\mathrm{d}} N(0,1) \text{ as } n\to\infty \tag{3.11}$$

by the central limit theorem for each $j \geq 1$ when $q_n \to \infty$ as $n \to \infty$.

When $j = 1$, it follows from (3.11) and Lemma 3.8 with the independence of $\{M_{j,k}\}$ and $\{\widetilde{M}_{n+1-j,k}\}$ that

$$\frac{\log V_1 - \mu_n}{\sigma_n} = \frac{\sqrt{p_n\psi'(n)}N_{1,1}}{\sqrt{p_n\psi'(n)+q_n\psi'(1)}} - \frac{\sqrt{q_n\psi'(1)}N_{2,1}}{\sqrt{p_n\psi'(n)+q_n\psi'(1)}} \xrightarrow{\mathrm{d}} N(0,1),$$

which implies (3.7).

Now, we prove (3.8). Similarly, we have when $j = 2$

$$N_3 := \frac{\sum_{k=1}^{p_n}\log \widetilde{M}_{n-1,k} - p_n\psi(n-1)}{\sqrt{p_n\psi'(n-1)+q_n\psi'(2)}} - \frac{\sum_{k=1}^{q_n}\log M_{2,k} - q_n\psi(2)}{\sqrt{p_n\psi'(n-1)+q_n\psi'(2)}} \xrightarrow{\mathrm{d}} N(0,1) \tag{3.12}$$

as $n \to \infty$. It follows from (2.6) and (2.7) that $\psi(n) - \psi(n-1) = \frac{1}{n-1}$, $\psi(2) - \psi(1) = 1$, and

$\psi'(n) = \frac{1+o(1)}{n-1}$ as $n \to \infty$. Then we have from (3.12)

$$\begin{aligned}
\frac{\log V_2 - \mu_n}{\sigma_n} &= \frac{\sum_{k=1}^{p_n} \log \widetilde{M}_{n-1,k} - p_n\psi(n)}{\sqrt{p_n\psi'(n) + q_n\psi'(1)}} - \frac{\sum_{k=1}^{q_n} \log M_{2,k} - q_n\psi(1)}{\sqrt{p_n\psi'(n) + q_n\psi'(1)}} \\
&= -\frac{p_n\psi(n) - p_n\psi(n-1) + q_n\psi(2) - q_n\psi(1)}{\sqrt{p_n\psi'(n) + q_n\psi'(1)}} + \frac{\sqrt{p_n\psi'(n-1) + q_n\psi'(2)}}{\sqrt{p_n\psi'(n) + q_n\psi'(1)}} N_3 \\
&= -\frac{p_n\frac{1}{n-1} + q_n}{\sqrt{p_n\psi'(n) + q_n\psi'(1)}} + O_p(1) \\
&= -\sqrt{p_n\psi'(n) + q_n}\big(1 + o(1)\big) + O_p(1) \\
&\overset{\text{p}}{\to} -\infty \text{ as } n \to \infty,
\end{aligned}$$

which implies $1 - a_{n,2}(x) = P(\log V_2 > \mu_n + \sigma_n x) \to 0$ as $n \to \infty$ for any $x \in \mathbb{R}$.

Since $V_j \overset{\text{d}}{=} Y_{n+1-j}$, we have from Lemma 3.5 that $P(V_j \ge x)$ is non-increasing in $j \in \{1, \cdots, n\}$. Then $\max_{2\le j\le n}\big(1 - a_{n,j}(x)\big) = 1 - a_{n,2}(x) \to 0$ as $n \to \infty$. Moreover,

$$1 - \sum_{j=2}^{n}\big(1 - a_{n,j}(x)\big) \le \prod_{j=2}^{n}\big(1 - (1 - a_{n,j}(x))\big) = \prod_{j=2}^{n} a_{n,j}(x) \le 1.$$

Thus, in order to prove (3.8), it suffices to show

$$\lim_{n\to\infty} \sum_{j=2}^{n}\big(1 - a_{n,j}(x)\big) = 0, \quad x \in (-\infty, \infty). \tag{3.13}$$

By applying inequality $P(X > 0) \le \mathbb{E}(e^X)$ and noting that all the summands on right-hand side of (3.9) are independent, we have from Lemma 3.1 and (2.5) that for any fixed

$x \in (-\infty, \infty)$

$$\begin{aligned}
& 1 - a_{n,j}(x) \\
= \ & P(\log V_j > \mu_n + \sigma_n x) \\
= \ & P\left(\sum_{k=1}^{p_n} \log \widetilde{M}_{n+1-j,k} - \sum_{k=1}^{q_n} \log M_{j,k} - p_n\psi(n) + q_n\psi(1) - \sqrt{p_n\psi'(n) + q_n\psi'(1)}x > 0\right) \\
\le \ & \mathbb{E}\left(\exp\Big(\sum_{k=1}^{p_n} \log \widetilde{M}_{n+1-j,k} - \sum_{k=1}^{q_n} \log M_{j,k} - p_n\psi(n) + q_n\psi(1) - \sqrt{p_n\psi'(n) + q_n\psi'(1)}x\Big)\right) \\
= \ & \Big(\frac{\Gamma(n+2-j)}{\Gamma(n+1-j)}\Big)^{p_n}\Big(\frac{\Gamma(j-1)}{\Gamma(j)}\Big)^{q_n} \exp\Big(-p_n\psi(n) + q_n\psi(1) - \sqrt{p_n\psi'(n) + q_n\psi'(1)}x\Big) \\
= \ & \exp\Big(p_n\int_0^1 \psi(n+1-j+t)dt\Big) \exp\Big(-q_n\int_0^1 \psi(j-1+t)dt\Big) \\
& \times \exp\Big(-p_n\psi(n) + q_n\psi(1) - \sqrt{p_n\psi'(n) + q_n\psi'(1)}x\Big) \\
= \ & \exp\Big(-p_n\int_0^1 \big(\psi(n) - \psi(n+1-j+t)\big)dt\Big) \exp\Big(-q_n\int_0^1 \big(\psi(j-1+t) - \psi(1)\big)dt\Big) \\
& \times \exp\Big(-\sqrt{p_n\psi'(n) + q_n\psi'(1)}x\Big).
\end{aligned}$$

Note that $\psi(y)$ is increasing in $y > 0$ since $\psi'(y) > 0$ from (2.8). Therefore,

$$0 \le \psi(n) - \psi(n+2-j) \le \psi(n) - \psi(n+1-j+t)$$

and

$$0 \le \psi(j-1) - \psi(1) \le \psi(j-1+t) - \psi(1)$$

for $t \in [0,1]$ and $j \ge 3$. By using the trivial inequality $\sqrt{p_n\psi'(n) + q_n\psi'(1)} \le \sqrt{p_n\psi'(n)} + \sqrt{q_n\psi'(1)}$, we have for any $3 \le j \le n$

$$\begin{aligned}
& 1 - a_{n,j}(x) \\
\le \ & \exp\Big(-p_n\int_0^1 \big(\psi(n) - \psi(n+2-j)\big)dt\Big) \exp\Big(-q_n\int_0^1 \big(\psi(j-1) - \psi(1)\big)dt\Big) \\
& \quad \times \exp\Big(\sqrt{p_n\psi'(n)}|x| + \sqrt{q_n\psi'(1)}|x|\Big) \\
= \ & \exp\Big(-p_n\big(\psi(n) - \psi(n+2-j)\big) + \sqrt{p_n\psi'(n)}|x|\Big) \\
& \times \exp\Big(-q_n\big(\psi(j-1) - \psi(1)\big) + \sqrt{q_n\psi'(1)}|x|\Big).
\end{aligned}$$

Define j_0 as the smallest integer larger than $3 + 3|x|$. In view of (2.7), for all $j_0 \le j \le n$

$$\psi(n) - \psi(n+2-j) = \sum_{k=n+1-j}^{n-1} \frac{1}{k} \ge \frac{j-2}{n} \ge \frac{j_0 - 2}{n} \tag{3.14}$$

and

$$\psi(j-1)-\psi(1)=\sum_{k=1}^{j-1}\frac{1}{k}\geq \log j. \tag{3.15}$$

If $p_n\psi'(n)\leq 1$, then

$$\exp\Big(-p_n\big(\psi(n)-\psi(n+2-j)\big)+\sqrt{p_n\psi'(n)}|x|\Big)\leq e^{|x|}.$$

If $p_n\psi'(n)\geq 1$, $\sqrt{p_n\psi'(n)}\leq p_n\psi'(n)\leq 2p_n/n$ for all large n from (2.6), and we have from (3.14)

$$\begin{aligned}
& \exp\Big(-p_n\big(\psi(n)-\psi(n+2-j)\big)+\sqrt{p_n\psi'(n)}|x|\Big)\\
\leq\ & \exp\Big(-\frac{p_n(j_0-2)}{n}+\frac{p_n 2|x|}{n}\Big)\\
=\ & \exp\Big(-\frac{p_n(j_0-2-2|x|)}{n}\Big)\\
\leq\ & e^{|x|}
\end{aligned}$$

for all large n. Therefore, by virtue of (3.15), we obtain that for $j_0\leq j\leq n$

$$\begin{aligned}
1-a_{n,j}(x) &\leq e^{|x|}\exp\Big(-q_n\big(\psi(j-1)-\psi(1)\big)+\sqrt{q_n\psi'(1)}|x|\Big)\\
&\leq e^{|x|}\exp\big(-q_n\log j+\sqrt{q_n\psi'(1)}|x|\big)
\end{aligned}$$

for all large n. Since $q_n\to\infty$ as $n\to\infty$, we have $\sqrt{q_n\psi'(1)}|x|\leq q_n\log j_0/2$ for all large n, and hence we conclude that

$$1-a_{n,j}(x)\leq e^{|x|}\exp\big(-q_n\log j+\frac{1}{2}q_n\log j_0\big)\leq e^{|x|}\exp\big(-\frac{1}{2}q_n\log j\big)$$

for all $j_0\leq j\leq n$ when large n is large. Therefore, we have

$$\begin{aligned}
\sum_{j=2}^{n}\big(1-a_{n,j}(x)\big) &\leq (j_0-2)(1-a_{n,2}(x))+e^{|x|}\sum_{j=j_0}^{n}\exp\big(-\frac{1}{2}q_n\log j\big)\\
&= (j_0-2)(1-a_{n,2}(x))+e^{|x|}\sum_{j=j_0}^{n}j^{-\frac{q_n}{2}}\\
&\leq (j_0-2)(1-a_{n,2}(x))+e^{|x|}\int_1^{n+1}t^{-\frac{q_n}{2}}\,dx\\
&\leq (j_0-2)(1-a_{n,2}(x))+\frac{2e^{|x|}}{q_n-2}\\
&\to 0
\end{aligned}$$

as $n \to \infty$, which yields (3.13). The proof of Theorem 2.1 is complete. $\square$

Proof of Theorem 2.2. We show part (a) first. Assume $\lim_{n\to\infty} \alpha_n = 0$.

Define for $1 \le r \le n$

$$Z_r = \max_{1\le j\le r} \frac{1}{\prod_{k=1}^q M_{j,k}^{1/2}}.$$

Then we have

$$Z_{r_n} W_{n,r_n} \le \max_{1\le j\le n} \frac{\prod_{k=1}^{p_n} \big(\widetilde{M}_{n+1-j,k}/n\big)^{1/2}}{\prod_{k=1}^q M_{j,k}^{1/2}} \le Z_n \max_{1\le j\le n} \prod_{k=1}^{p_n} \big(\widetilde{M}_{n+1-j,k}/n\big)^{1/2}, \tag{3.16}$$

where $W_{n,r}$ is defined in Lemma 3.10 and r_n satisfies the conditions in Lemma 3.10 such that $r_n \to \infty$ and $W_{n,r_n} \xrightarrow{p} 1$ as $n \to \infty$. From Lemma 3.7, we have $Z_n \to T_q = \max\limits_{1\le j<\infty} \frac{1}{\prod_{k=1}^q M_{j,k}^{1/2}}$ and $Z_{r_n} \to T_q$ as $n \to \infty$ with probability one. From Lemma 3.9, we have that $\max\limits_{1\le j\le n} \prod_{k=1}^{p_n} \big(\widetilde{M}_{n+1-j,k}/n\big)^{1/2} \xrightarrow{p} 1$ as $n \to \infty$. Therefore, we conclude that $Z_{r_n} W_{n,r_n} \xrightarrow{p} T_q$ and $Z_n \max\limits_{1\le j\le n} \prod_{k=1}^{p_n} \big(\widetilde{M}_{n+1-j,k}/n\big)^{1/2} \xrightarrow{p} T_q$ as $n \to \infty$, which together with (3.16) and (3.3) yield

$$\frac{M_n}{n^{p_n/2}} \stackrel{d}{=} \max_{1\le j\le n} \frac{\prod_{k=1}^{p_n} \big(\widetilde{M}_{n+1-j,k}/n\big)^{1/2}}{\prod_{k=1}^q M_{j,k}^{1/2}} \xrightarrow{p} T_q,$$

proving (2.10).

Now we show the part (b) of Theorem 2.2 under condition $\lim_{n\to\infty} \alpha_n =: \alpha \in (0, \infty)$.

We have from (3.3) that for any fixed $x \in (0, \infty)$

$$P\left(\frac{M_n}{n^{p_n/2}} \le x\right) = \prod_{j=1}^n P\Big(\frac{\prod_{k=1}^{p_n} \widetilde{M}_{n+1-j,k}^{1/2}/n^{p_n/2}}{\prod_{k=1}^q M_{j,k}^{1/2}} \le x\Big).$$

Hence, to prove (2.11), it suffices to show

$$\lim_{n\to\infty} \prod_{j=1}^n P\Big(\frac{\prod_{k=1}^{p_n} \widetilde{M}_{n+1-j,k}^{1/2}/n^{p_n/2}}{\prod_{k=1}^q M_{j,k}^{1/2}} \le x\Big) = \prod_{j=1}^\infty P\Big(\frac{\Theta_{j,\alpha}}{\prod_{k=1}^q M_{j,k}^{1/2}} \le x\Big) \tag{3.17}$$

for $x \in (0, \infty)$, where $\Theta_{j,\alpha}$'s are independent random variables as defined in Theorem 2.2. We see that $\Theta_{j,\alpha}$ has a cumulative distribution function $\Phi\big(\alpha^{1/2}/2 + 2\alpha^{-1/2} \log y + (j-1)\alpha^{1/2}\big)$; that is, $\Theta_{j,\alpha} \stackrel{d}{=} \exp\Big(-\frac{1}{2}\big((j-1)\alpha + \alpha/2)\big) + \frac{1}{2}\alpha^{1/2} Z\Big)$, where Z is a standard normal random variable.

Now fix $x > 0$ and we prove (3.17). Note that (3.17) is equivalent to

$$\lim_{n\to\infty}\prod_{j=1}^{\infty}(1-c_{nj})=\prod_{j=1}^{\infty}(1-c_j), \tag{3.18}$$

where

$$c_{nj}=1-P\Big(n^{-p_n/2}\prod_{k=1}^{p_n}\widetilde{M}_{n+1-j,k}^{1/2}/\prod_{k=1}^{q}M_{j,k}^{1/2}\le x\Big),\quad 1\le j\le n,$$

and $c_{nj}=0$ for $j>n$, and $c_j=1-P\big(\Theta_{j,\alpha}/\prod_{k=1}^{q}M_{j,k}^{1/2}\le x\big)$, $j\ge 1$.

We apply Lemma 3.6 to verify (3.18).

Let us first prove that $\lim_{n\to\infty}c_{nj}=c_j$ for each $j\ge 1$. It follows from Lemma 3.8 that

$$\prod_{k=1}^{p_n}\widetilde{M}_{n+1-j,k}^{1/2}=\exp\big(\frac{1}{2}p_n\psi(n+1-j)+\frac{1}{2}\sqrt{p_n\psi'(n+1-j)}N_{1,j}\big)$$

for every $j\ge 1$. It follows from (2.6) and (2.7) that $\log n=\psi(n)+\frac{1}{2n}+O(\frac{1}{n^2})$, $\psi'(n+1-j)=\frac{1}{n+1-j}+\frac{1}{2(n+1-j)^2}+O(\frac{1}{(n+1-j)^3})$, and $\psi(n)-\psi(n+1-j)=\frac{j-1}{n}(1+O(\frac{j}{n}))$ as $n\to\infty$ for every $j\ge 1$. Then we have

$$\begin{aligned}
& n^{-p_n/2}\prod_{k=1}^{p_n}\widetilde{M}_{n+1-j,k}^{1/2}\\
=\ & \exp\big(\frac{1}{2}p_n\psi(n+1-j)+\frac{1}{2}\sqrt{p_n\psi'(n+1-j)}N_{1,j}\big)\exp\big(-\frac{1}{2}p_n\log n\big)\\
=\ & \exp\Big(\frac{1}{2}p_n\big(\psi(n+1-j)-\psi(n)-\frac{1}{2n}+O(\frac{1}{n^2})\big)+\frac{1}{2}\sqrt{p_n\psi'(n+1-j)}N_{1,j}\Big)\\
=\ & \exp\Big(\frac{1}{2}\big(-\frac{p_n(j-1)}{n}(1+O(\frac{j}{n}))-\frac{p_n}{2n}\big)+\frac{1}{2}\big(\frac{p_n}{n+1-j}+\frac{p_n}{2(n+1-j)^2}\big)^{1/2}N_{1,j}\Big)\\
=\ & \exp\Big(-\frac{1}{2}\big((j-1)(1+o(1))\alpha_n+\alpha_n/2\big)+\frac{1}{2}(\alpha_n^{1/2}+o(1))N_{1,j}\Big),
\end{aligned}$$

which implies $\prod_{k=1}^{p_n}\widetilde{M}_{n+1-j,k}^{1/2}/n^{p_n/2}\xrightarrow{d}\Theta_{j,\alpha}$ as $n\to\infty$ for each $j\ge 1$ by noting that $N_{1,j}\xrightarrow{\mathrm{d}}N(0,1)$ as $n\to\infty$. Since $\prod_{k=1}^{q}M_{j,k}^{1/2}$ is independent of $\prod_{k=1}^{p_n}\widetilde{M}_{n+1-j,k}^{1/2}/n^{p_n/2}$ and its distribution is free of n, we further have

$$\Big(\prod_{k=1}^{p_n}\widetilde{M}_{n+1-j,k}^{1/2}/n^{p_n/2},\prod_{k=1}^{q}M_{j,k}^{1/2}\Big)\xrightarrow{d}\Big(\Theta_{j,\alpha},\prod_{k=1}^{q}M_{j,k}^{1/2}\Big).$$

Then by the continuous mapping theorem, we conclude that for each $j\ge 1$

$$\frac{\prod_{k=1}^{p_n}\widetilde{M}_{n+1-j,k}^{1/2}/n^{p_n/2}}{\prod_{k=1}^{q}M_{j,k}^{1/2}}\xrightarrow{d}\frac{\Theta_{j,\alpha}}{\prod_{k=1}^{q}M_{j,k}^{1/2}}\quad\text{as } n\to\infty.$$

Since all limits on the right-hand side above are continuous random variables, we have $\lim_{n\to\infty} c_{nj} = c_j$ for each $j \geq 1$.

Since $n^{-p_n/2} \prod_{k=1}^{p_n} \widetilde{M}_{n+1-j,k}^{1/2} / \prod_{k=1}^{q} M_{j,k}^{1/2}$ and $\Theta_{j,\alpha}/\prod_{k=1}^{q} M_{j,k}^{1/2}$ are continuous random variables having density functions with a common support $(0, \infty)$, $c_{nj} < 1$ and $c_j < 1$ for all $j \geq 1$ and $n \geq 1$.

Now we are ready to show $\sup_{n\geq 1, j\geq 1} c_{nj} < 1$. It follows from Lemma 3.5 that $c_{nj} \leq c_{n1}$ for all $1 \leq j \leq n$ and $n \geq 1$, implying $\sup_{n\geq 1, j\geq 1} c_{nj} = \sup_{n\geq 1} c_{n1}$. Since $c_{n1} \to c_1 < 1$ as $n \to \infty$, we have $c_{nj} < (1 + c_1)/2 < 1$ for all $n \geq n_0$, where $n_0 > 1$ is an integer. Furthermore, since $c_{nj} < 1$, we obtain that

$$\sup_{n\geq 1, j\geq 1} c_{nj} = \sup_{n\geq 1} c_{n1} \leq \max(\max_{1\leq n<n_0} c_{n1}, (1 + c_1)/2) < 1.$$

Recall that $0 \leq c_{nj} < 1$ for $1 \leq j \leq n$ and $c_{nk} = 0$ for $j > n$. Trivially, we obtain

$$\sum_{j=1}^{r+1} c_{nj} = \sum_{j=1}^{\min(n,r+1)} c_{nj} < \min(n, r + 1), \quad r \geq 1, \quad n \geq 1. \tag{3.19}$$

By using Markov's inequality $P(X > 0) = P(e^{2X} \geq 1) \leq \mathbb{E}(e^{2X})$ and the independence of $\{M_{j,k}; j \geq 1, k \geq 1\}$ and $\{\widetilde{M}_{n+1-j}; j \geq 1, k \geq 1\}$, we have from Lemma 3.1 and (2.5) that for any fixed x

$$\begin{aligned}
& c_{nj} \\
= \;& P\Big(\frac{\prod_{k=1}^{p_n} \widetilde{M}_{n+1-j,k}^{1/2} / n^{p_n/2}}{\prod_{k=1}^{q} M_{j,k}^{1/2}} \geq x\Big) \\
= \;& P\Big(\log \frac{\prod_{k=1}^{p_n} \widetilde{M}_{n+1-j,k}^{1/2} / n^{p_n/2}}{\prod_{k=1}^{q} M_{j,k}^{1/2}} \geq \log x\Big) \\
= \;& P\Big(\sum_{k=1}^{p_n} \log \widetilde{M}_{n+1-j,k} - \sum_{k=1}^{q} \log M_{j,k} - p_n \log n - 2\log x > 0\Big) \\
\leq \;& \mathbb{E}\Big(\exp\big(2\sum_{k=1}^{p_n} \log \widetilde{M}_{n+1-j,k} - 2\sum_{k=1}^{q} \log M_{j,k} - 2p_n \log n - 4\log x\big)\Big) \\
= \;& \big(\frac{\Gamma(n+3-j)}{\Gamma(n+1-j)}\big)^{p_n} \big(\frac{\Gamma(j-2)}{\Gamma(j)}\big)^{q} \exp\big(-2p_n \log n - 4\log x\big) \\
= \;& \exp\big(p_n \int_0^2 \psi(n+1-j+t)dt\big) \exp\big(-q\int_0^2 \psi(j-2+t)dt\big) \exp\big(-2p_n \log n - 4\log x\big) \\
= \;& \exp\Big(-p_n \int_0^2 \big(\log n - \psi(n+1-j+t)\big)dt - q\int_0^2 \psi(j-2+t)dt - 4\log x\Big)
\end{aligned}$$

for $2 \le j \le n$. Note that $\psi(y)$ is increasing in $y > 0$ since $\psi'(y) > 0$ from (2.8). Therefore, $0 \le \log n - \psi(n+3-j) \le \log n - \psi(n+1-j+t)$ and $0 \le \psi(j-2) \le \psi(j-2+t)$ from (2.6) for $t \in [0,2]$ and $3 \le j \le n$. Then we have

$$\begin{aligned}
& c_{nj} \\
\le\ & \exp\Big(- p_n \int_0^2 \big(\log n - \psi(n+3-j)\big)dt\Big) \exp\big(- q \int_0^2 \psi(j-2)dt\big) \exp\big(- 4\log x\big) \\
\le\ & \exp\Big(- 2p_n\big(\log n - \psi(n+3-j)\big) - 2q\psi(j-2) - 4\log x\big)\Big) \\
\le\ & \exp\big(- 2q\psi(j-2) - 4\log x\big).
\end{aligned}$$

Moreover, we have from (2.6) that $\psi(j-2) \ge \log(j-2) - \frac{1}{2(j-2)}$ for all $r+2 \le j \le n$ for large r. Hence

$$\begin{aligned}
\sup_{n\ge r+2} \sum_{j=r+2}^{\infty} c_{n,j} &\le \sup_{n\ge r+2} \sum_{j=r+2}^{n} c_{n,j} \\
&\le \sup_{n\ge r+2} \sum_{j=r+2}^{n} \exp\big(- 2q\psi(j-2) - 4\log x\big) \\
&\le \sup_{n\ge r+2} \sum_{j=r+2}^{n} \exp\big(- 2q\log(j-2) + \frac{q}{(j-2)} - 4\log x\big) \\
&\le e^{q/r}x^{-4} \sum_{j=r+2}^{\infty} (j-2)^{-2q} \\
&\le e^{q/r}x^{-4} \int_r^{\infty} (x-2)^{-2q}dx \\
&\le \frac{e^q x^{-4}}{2q-1}(r-2)^{1-2q},
\end{aligned}$$

which is of order $1/r$ for all large r. This, coupled with (3.19), yields that

$$C := \sup_{n\ge 1} \sum_{j=1}^{\infty} c_{n,j} < \infty.$$

Because $\lim_{n\to\infty} c_{nj} = c_j$ for $j \ge 1$, it follows that $\sum_{j=1}^{r+1} c_j = \lim\limits_{n\to\infty} \sum_{j=1}^{r+1} c_{nj} \le C$ for all $r \ge 1$. Therefore $\sum_{j=1}^{r+1} c_j$ is a bounded and monotone increasing sequence, and $\sum_{j=1}^{\infty} c_j \le C < \infty$ as it is convergent by the monotone convergence theorem. Consequently, we have

$\lim_{n\to\infty} \sum_{j=1}^{\infty} c_{nj} = \sum_{j=1}^{\infty} c_j$ since for all large r

$$
\begin{aligned}
\lim_{n\to\infty} |\sum_{j=1}^{\infty} c_{nj} - \sum_{j=1}^{\infty} c_j| &\leq \lim_{n\to\infty} \Big(|\sum_{j=1}^{r+1} c_{nj} - \sum_{j=1}^{r+1} c_j| + |\sum_{j=r+2}^{\infty} c_{nj} - \sum_{j=r+2}^{\infty} c_j| \Big) \\
&= \lim_{n\to\infty} |\sum_{j=r+2}^{\infty} c_{nj} - \sum_{j=r+2}^{\infty} c_j| \\
&\leq \sup_{n\geq r+2} \sum_{j=r+2}^{\infty} c_{nj} + \sum_{j=r+2}^{\infty} c_j \\
&\leq \frac{e^q x^{-4}}{2q-1}(r-2)^{1-2q} + \sum_{j=r+2}^{\infty} c_j
\end{aligned}
$$

which tends to zero by letting r tend to infinity.

Thus, we have proven (3.18) by using Lemma 3.6.

Now we show part (c) of Theorem 2.2 under condition $\lim_{n\to\infty} \alpha_n = \infty$. We have

$$
\begin{aligned}
\log M_n &= \max_{1\leq j\leq n} \left(\log \prod_{k=1}^{p_n} \widetilde{M}_{n+1-j,k}^{1/2} + \log \frac{1}{\prod_{k=1}^{q} M_{j,k}^{1/2}} \right) \\
&\leq \max_{1\leq j\leq n} \log \prod_{k=1}^{p_n} \widetilde{M}_{n+1-j,k}^{1/2} + \max_{1\leq j\leq n} \log \frac{1}{\prod_{k=1}^{q} M_{j,k}^{1/2}} =: R_n
\end{aligned}
$$

and

$$
\log M_n \geq \log \prod_{k=1}^{p_n} \widetilde{M}_{n,k}^{1/2} - \log \prod_{k=1}^{q} M_{1,k}^{1/2} =: L_n.
$$

By using the two inequalities above we have

$$
\frac{L_n - p_n\psi(n)/2}{\sqrt{p_n/n}/2} \leq \frac{\log M_n - p_n\psi(n)/2}{\sqrt{p_n/n}/2} \leq \frac{R_n - p_n\psi(n)/2}{\sqrt{p_n/n}/2}. \tag{3.20}
$$

To prove (2.12), it suffices to show that both the left-hand side and the right-hand side in (3.20) converge in distribution to the standard normal.

As we have pointed out in the paragraph after equation (3.3) that $\max_{1\leq j\leq n} \prod_{k=1}^{p_n} \widetilde{M}_{n+1-j,k}^{1/2}$ has the same distribution as that of the largest absolute value of the n eigenvalues for product of p_n independent $n \times n$ random matrices with i.i.d. complex standard Gaussian entries. From the part (c) in Theorem 3 of Jiang and Qi [24], we have

$$\frac{\max\limits_{1\le j\le n}\log\prod_{k=1}^{p_n}\widetilde{M}_{n+1-j,k}^{1/2}-p_n\psi(n)/2}{\sqrt{p_n/n}/2}\xrightarrow{\mathrm{d}}N(0,1)\quad\text{as } n\to\infty.$$

Meanwhile, from Lemma 3.7, $\lim\limits_{n\to\infty}\max\limits_{1\le j\le n}\log\frac{1}{\prod_{k=1}^{q}M_{j,k}^{1/2}}=\log T_q$ is well defined. Therefore

$$\frac{\max\limits_{1\le j\le n}\log\frac{1}{\prod_{k=1}^{q}M_{j,k}^{1/2}}}{\sqrt{p_n/n}/2}\xrightarrow{\mathrm{p}}0\quad\text{as } n\to\infty.$$

Then it follows that as $n\to\infty$

$$\frac{R_n-p_n\psi(n)/2}{\sqrt{p_n/n}/2}=\frac{\max\limits_{1\le j\le n}\log\prod\limits_{k=1}^{p_n}\widetilde{M}_{n+1-j,k}^{1/2}-p_n\psi(n)/2-\min\limits_{1\le j\le n}\log\prod\limits_{k=1}^{q}M_{j,k}^{1/2}}{\sqrt{p_n/n}/2}\xrightarrow{d}N(0,1).\quad(3.21)$$

Now we apply Lemma 3.8 with $j=1$ to get

$$\frac{\sum_{k=1}^{p_n}\log\widetilde{M}_{n,k}^{1/2}-p_n\psi(n)/2}{\sqrt{p_n/n}/2}\xrightarrow{\mathrm{d}}N(0,1)\text{ as } n\to\infty,$$

by noting that $\psi'(n)=\frac{1+o(1)}{n}$ from (2.6). Trivially, we also have

$$\frac{\log\prod_{k=1}^{q}M_{1,k}^{1/2}}{\sqrt{p_n/n}/2}\xrightarrow{\mathrm{p}}0\text{ as } n\to\infty,$$

and conclude that

$$\frac{L_n-p_n\psi(n)/2}{\sqrt{p_n/n}/2}=\frac{\log\prod_{k=1}^{p_n}\widetilde{M}_{n,k}^{1/2}-p_n\psi(n)/2-\log\prod_{k=1}^{q}M_{1,k}^{1/2}}{\sqrt{p_n/n}/2}\xrightarrow{\mathrm{d}}N(0,1).\qquad(3.22)$$

By combining (3.20), (3.21) and (3.22), we have shown (2.12), that is, part (c) of Theorem 2.2 is true. The proof of Theorem 2.2 is completed. □

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Conflict of Interest
The authors declare that they have no conflict of interest.

Data Availability
Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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