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Equivariant $\underline{\mathbb{Z}/\ell}$-modules for the cyclic group C_2

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ABSTRACT

For the cyclic group C_2 we give a complete description of the derived category of perfect complexes of modules over the constant Mackey ring $\underline{\mathbb{Z}/\ell}$, for ℓ a prime. This is fairly simple for ℓ odd, but for $\ell = 2$ depends on a new splitting theorem. As corollaries of the splitting theorem we compute the associated Picard group and the Balmer spectrum for compact objects in the derived category, and we obtain a complete classification of finite modules over the C_2-equivariant Eilenberg–MacLane spectrum $H\underline{\mathbb{Z}/2}$. We also use the splitting theorem to give new and illuminating proofs of some facts about $RO(C_2)$-graded Bredon cohomology, namely Kronholm's freeness theorem [12,11] and the structure theorem of C. May [13].

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1. Introduction

Mackey functors were first defined by Dress and Green in the early 1970s [7–9]. They have proven to be important objects in equivariant homotopy theory and representation theory. For G a finite group, the

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category of G-Mackey functors is abelian and symmetric monoidal via the box product. We call a monoid R in this category a *Mackey ring* (note that commutative Mackey rings are also called Green functors in the literature). From R one obtains an abelian category of R-modules, a corresponding category of chain complexes of R-modules, and an associated derived category $\mathcal{D}(R)$.

The derived category of a Mackey ring is our main object of study in this paper, though our motivation comes from topology. In classical topology, if T is a ring then $\mathcal{D}(T)$ is the natural homotopical recipient for singular homology with T-coefficients. In the G-equivariant setting, if R is a Mackey ring then $\mathcal{D}(R)$ is the analogous recipient for ordinary Bredon homology with R-coefficients. For this reason it is natural to investigate $\mathcal{D}(R)$ and try to understand as much as we can about it.

This paper specializes to the case where $G = C_2$, the cyclic group of order two, and $R = \underline{\mathbb{Z}/\ell}$, the constant-coefficient Mackey functor for the ring $\mathbb{Z}/\ell$, with ℓ a prime. In this case we completely describe the full subcategory $\mathcal{D}(\underline{\mathbb{Z}/\ell})_{\mathrm{perf}}$ consisting of perfect complexes. Here a "perfect complex" is a bounded chain complex of finitely-generated projective modules. There are only two indecomposable, finitely-generated projectives, called H and F in the $\ell = 2$ case and H and S_Θ in the $\ell > 2$ case (see Section 2). Topologically, H and F correspond to fixed and free C_2-equivariant cells, respectively, and when $\ell > 2$ one has $F \cong H \oplus S_\Theta$.

When ℓ is odd, the category of $\underline{\mathbb{Z}/\ell}$-modules is semisimple and hence $\mathcal{D}(\underline{\mathbb{Z}/\ell})_{\mathrm{perf}}$ is easy to understand. Every perfect complex decomposes—as a chain complex, not just an object in the derived category—as a direct sum of suspensions of H and S_Θ (regarded as complexes concentrated in degree 0) as well as "disks" on H and S_Θ (chain complexes consisting of exactly one identity map)—see Proposition 3.5. But the case of real interest is $\ell = 2$. Here we are again able to describe all perfect complexes, but the result and techniques used are more complicated. For this case we define certain families of complexes A_k, B_r (for $k, r \geq 0$), and $H(n)$ (for $n \in \mathbb{Z}$). These are simple "linear strands" consisting of a single H or F in each degree; see Section 4 for the precise definitions. Our main classification result is the following:

Theorem 1.1. *Every perfect complex of $\underline{\mathbb{Z}/2}$-modules is isomorphic to a finite direct sum of suspensions of complexes of type A_k, B_r, and $H(n)$, together with suspensions of elementary contractible disk-complexes $\mathbb{D}(P)$, where P is a projective and $\mathbb{D}(P)$ denotes the complex $0 \longrightarrow P \xrightarrow{id} P \longrightarrow 0$.*

We actually prove more than is stated in the above theorem. We give an algorithm for taking any perfect complex and splitting it into the above form. We also prove a similar splitting theorem for bounded below complexes of finitely-generated modules. The bounded below case requires two more types of complexes (allowing $k = \infty$ and $r = \infty$); see Theorem 4.10.

We call each of the special complexes in the splitting theorems "strands" because they take a particularly nice form: they are nonzero for a finite string of degrees, and in each of these degrees they have a single summand that is either H or F. A similar splitting phenomenon occurs for the classical derived category of abelian groups: every perfect complex splits as a direct sum of shifts of the strands $0 \longrightarrow \mathbb{Z} \longrightarrow 0$ and $0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow 0$. In some ways our main argument is similar to what happens in this classical example, in that we can do concrete change-of-bases to produce the splitting—but the bookkeeping is more difficult and comes with several subtleties. Note that there is something special about $G = C_2$ here; for $G = C_p$ with p an odd prime, there are perfect complexes that cannot decompose into strands in $\mathcal{D}(\underline{\mathbb{Z}/p})$. We discuss this in Remark 4.11.

Returning to $G = C_2$, it follows from Theorem 1.1 that we can understand $\mathcal{D}(\underline{\mathbb{Z}/2})_{\mathrm{perf}}$ by studying all maps between the A_k, B_r, and $H(n)$ complexes. The derived category $\mathcal{D}(\underline{\mathbb{Z}/2})$ also has a monoidal product $\square$, analogous to the tensor product on the derived category of an ordinary commutative ring. We completely calculate all maps and products in $\mathcal{D}(\underline{\mathbb{Z}/2})_{\mathrm{perf}}$. We also study several other aspects of this derived category such as the Picard group, the Balmer spectrum, and a certain kind of duality. We find the Picard group of $\mathcal{D}(\underline{\mathbb{Z}/2})$ is $\mathbb{Z}^2$ (see Theorem 5.2) and the Balmer spectrum for $\mathcal{D}(\underline{\mathbb{Z}/2})_{\mathrm{perf}}$ consists of three points, two closed and one generic (see Theorem 5.4).

Remark 1.2. The reader will have noticed the different notation of indices for the H-family as opposed to the A- and B-families. In the derived category, the $H(n)$ are invertible with $H(n)\square H(m) \simeq H(n+m)$. The derived category $\mathcal{D}(\underline{\mathbb{Z}/2})$ in many ways behaves like the derived category of a projective curve having exactly two closed points x and y. The $H(n)$ are analogs of the canonical sheaves $\mathcal{O}(n)$, and the A_k and B_k are analogs of the kth order thickenings of the structure sheaves for the points x and y. For further analogies with algebraic geometry see Remark 4.4 and Theorem 4.21.

While the main work of this paper is entirely algebraic, our motivation comes from equivariant homotopy theory. For every Mackey functor M, there is a corresponding genuine equivariant spectrum HM called the Eilenberg–MacLane (or Bredon) spectrum. If A is an abelian group then we can take $M = \underline{A}$, the constant-coefficient Mackey functor, and if A is a ring then $H\underline{A}$ is a ring spectrum.

In the equivariant setting, homotopy and (co)homology are often graded on the representation ring $RO(G)$. For $G = C_2$, this gives us bigraded homotopy and (co)homology. Unlike the classical setting, the $RO(G)$-graded homotopy of $H\underline{A}$ is typically not concentrated in a single degree. This can make computations significantly more challenging. In general, the homotopy category of $H\underline{A}$-modules is not well understood. However, we can transform this into a problem in algebra.

The homotopy category of $H\underline{A}$-modules is equivalent to the derived category $\mathcal{D}(\underline{A})$ (for $A = \mathbb{Z}$ this was explained in [17]). We use this equivalence and our classification of perfect complexes in $\mathcal{D}(\underline{\mathbb{Z}/\ell})$ to give a classification of finite $H\underline{\mathbb{Z}/\ell}$-modules. For ℓ odd, every finite $H\underline{\mathbb{Z}/\ell}$-module decomposes as a wedge of suspensions of $H\underline{\mathbb{Z}/\ell}$. Here the suspensions are by (possibly virtual) representation spheres. Our description of perfect complexes is not actually required here: this decomposition follows from the fact that the $RO(C_2)$-graded homotopy of $H\underline{\mathbb{Z}/\ell}$ is a graded field, and graded modules over a graded field decompose nicely. As before, $\ell = 2$ is the much more interesting case. The $RO(C_2)$-graded homotopy of $H\underline{\mathbb{Z}/2}$ is a non-Noetherian ring and has a complicated module theory.

In [13] it was shown that if X is a finite C_2-CW spectrum then $H\underline{\mathbb{Z}/2} \wedge X$ splits into a wedge of suspensions of $H\underline{\mathbb{Z}/2}$ and $H\underline{\mathbb{Z}/2} \wedge (S^k_a)_+$, where S^k_a is the k-dimensional sphere with the antipodal action and suspensions are by (virtual) representation spheres. Using the splitting of perfect complexes in $\mathcal{D}(\underline{\mathbb{Z}/2})$, we prove a generalization. This generalization requires a third type of $H\underline{\mathbb{Z}/2}$-module given by "cofibers of powers of τ", where τ is a certain familiar element in the homotopy of $H\underline{\mathbb{Z}/2}$ (see Section 4.24 for a precise definition).

Theorem 1.3. *Every finite $H\underline{\mathbb{Z}/2}$-module is equivalent to a wedge of $RO(C_2)$-graded suspensions of $H\underline{\mathbb{Z}/2}$, $H\underline{\mathbb{Z}/2} \wedge (S^k_a)_+$, and $\mathrm{Cof}(\tau^r)$ for various $k \geq 0$ and $r \geq 1$.*

This follows from the splitting theorem by identifying the complexes $H(n)$ with weight n representation spheres (smashed with $H\underline{\mathbb{Z}/2}$), the complexes A_k with the antipodal spheres (again, smashed with $H\underline{\mathbb{Z}/2}$), and finally the complexes B_{r-1} with $\mathrm{Cof}(\tau^r)$.

While this splitting theorem might appear a bit asymmetric with the appearance of $\mathrm{Cof}(\tau^r)$, we can reinterpret the splitting by identifying $H\underline{\mathbb{Z}/2} \wedge (S^k_a)_+$ with a desuspension of $\mathrm{Cof}(\rho^{k+1})$. Here ρ is another familiar element in the homotopy of $H\underline{\mathbb{Z}/2}$. Then we observe every finite $H\underline{\mathbb{Z}/2}$-module splits as a wedge of suspensions of the unit $H\underline{\mathbb{Z}/2}$, $\mathrm{Cof}(\rho^k)$, and $\mathrm{Cof}(\tau^r)$ for $k, r \geq 1$.

The splitting result of Theorem 1.1 has some other nice consequences. One is that it leads to a new proof of the topological splitting theorem from [13]. We also use it to give a new perspective on an important result of Kronholm. Kronholm [12] observed that for a C_2-space built from a finite number of representation cells (called a finite $\mathrm{Rep}(C_2)$-complex), the $RO(C_2)$-graded Bredon cohomology is always free as a module over the Bredon cohomology of a point. The proof of this "freeness theorem" is subtle, and a small but significant gap was discovered and repaired by Hogle and C. May [11], who also extended the result to the finite-type case. We use Theorem 1.1 to give another proof of this freeness theorem, which again has some

subtleties but also offers some enlightening perspectives. In particular, one notable aspect of Kronholm's freeness theorem is the phenomenon now known as "Kronholm shifting", whereby cells from the original decomposition appear to shift weights in terms of how they contribute to the cohomology of the space. The algebraic splitting algorithm underlying Theorem 1.1 gives a concrete perspective on these Kronholm shifts.

Remark 1.4. In a private communication, Dylan Wilson shared a perspective on how the derived category versions of some of our results can be understood via the Tate square from equivariant homotopy theory. We briefly include a rough outline since this might serve as a useful guidepost to the reader, though some of the details are outside the scope of this paper.

The Tate square for $H\underline{\mathbb{Z}/2}$ takes the form

$$\begin{array}{ccc} H\underline{\mathbb{Z}/2} & \longrightarrow & H\underline{\mathbb{Z}/2}[\rho^{-1}] \\ \downarrow & & \downarrow \\ H\underline{\mathbb{Z}/2}[\tau^{-1}] & \longrightarrow & H\underline{\mathbb{Z}/2}[\tau^{-1},\rho^{-1}]. \end{array}$$

Isomorphism classes of modules over $H\underline{\mathbb{Z}/2}$ can be seen to correspond to isomorphism classes of triples (M, N, ϕ) where M is a module over $H\underline{\mathbb{Z}/2}[\tau^{-1}]$, N is a module over $H\underline{\mathbb{Z}/2}[\rho^{-1}]$, and ϕ is an isomorphism $M[\rho^{-1}] \to N[\tau^{-1}]$ (according to Wilson there is an appropriate ∞-categorical equivalence here).

The bigraded homotopy rings of the equivariant ring spectra $H\underline{\mathbb{Z}/2}[\tau^{-1}]$ and $H\underline{\mathbb{Z}/2}[\rho^{-1}]$ are very simple: they are $\mathbb{Z}/2[\tau^{\pm 1}, \rho]$ and $\mathbb{Z}/2[\tau, \rho^{\pm 1}]$, respectively. In particular, these are bigraded PIDs and this implies that the module theory of these ring spectra is formal—that is, the module theory is the same as the (bigraded) module theory of their bigraded homotopy rings (using that bigraded homotopy detects weak equivalences in the C_2-equivariant setting). By algebra, the graded module theory of these rings is the same as that of the singly-graded rings $\mathbb{Z}/2[x_1]$ and $\mathbb{Z}/2[x_2]$ where $x_1 = \frac{\rho}{\tau}$ and $x_2 = \frac{\tau}{\rho}$. One is therefore led to consider triples (M_1, M_2, ϕ) where M_i is a graded $\mathbb{Z}/2[x_i]$-module and ϕ is an isomorphism $M_1[x_1^{-1}] \to M_2[x_2^{-1}]$. Such tuples can be seen to break up into sums of three types: $(\mathbb{Z}/2[x_1]/(x_1^r), 0, 0)$, $(0, \mathbb{Z}/2[x_2]/(x_2^s), 0)$, and the free case $(\Sigma^{k_1}\mathbb{Z}/2[x_1], \Sigma^{k_2}\mathbb{Z}/2[x_2], 1 \mapsto x_2^{k_2-k_1})$. These are the analogs of our A-, B-, and $H(n)$-families, respectively (the third tuple listed above corresponds to $\Sigma^{k_1}H(k_1 - k_2)$).

One can find shades of the above argument sprinkled amongst our methods throughout the paper, though our perspective is more rigid and algebraic. But in particular we want to accentuate the viewpoint that the module theory of $H\underline{\mathbb{Z}/2}$-modules (or equivalently, the homological algebra of $\underline{\mathbb{Z}/2}$-modules) is almost like that of a PID. Note also that the above perspective shows the connection between finite-type $H\underline{\mathbb{Z}/2}$-modules and coherent sheaves over the projective line $\mathbb{P}^1_{\mathbb{F}_2}$, which was foreshadowed in Remark 1.2 and also appears later in Section 4.

1.5. Organization of the paper

Section 2 gives background on the category of Mackey functors and its closed symmetric monoidal structure, and sets up much of our notation. In Section 3 we focus on understanding $\underline{\mathbb{Z}/n}$-modules, and find that there are only a few finitely-generated indecomposables. The main results are in Section 4, which contains a thorough investigation of the derived category of $\underline{\mathbb{Z}/2}$-modules together with our main splitting theorem. Consequences of this work are then developed in Sections 5 and 6, whereas Section 7 gives the proof of the splitting theorem. Finally, Section 8 uses techniques from the proof of the splitting theorem to prove an algebraic version of Kronholm's freeness theorem.

1.6. Notation and terminology

If $\mathcal{C}$ is a category then we write $\mathcal{C}(A,B)$ for $\mathrm{Hom}_{\mathcal{C}}(A,B)$. The cyclic group with two elements is denoted C_2 when it is the background group acting on spaces, and denoted $\mathbb{Z}/2$ when it plays the role of a coefficient system.

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2. Background on Mackey functors

In this section we review the basic definitions and structures on Mackey functors. We then introduce the Mackey rings $\underline{\mathbb{Z}/n}$. The main objects of study throughout the paper will be $\underline{\mathbb{Z}/n}$-modules. More information about Mackey functors and their homological algebra can be found in papers such as [16], [10], [4], [5], and [17].

A **Mackey functor** for the group C_2 is a diagram of abelian groups

$$t \circlearrowright M_\Theta \underset{p^*}{\overset{p_*}{\rightleftarrows}} M_\bullet$$

where the maps satisfy the formulas $tp^* = p^*$, $p_*t = p_*$, $t^2 = id$, and $p^*p_* = 1 + t$. There are other ways of defining Mackey functors as certain additive functors defined on all finite C_2-sets. This 'coordinate-free' approach is more convenient for some purposes. If $\bullet$ denotes the trivial C_2-set with one element and Θ denotes C_2 regarded as a C_2-set via left multiplication, then every C_2-set is a disjoint union of copies of $\bullet$ and Θ. So an additive functor M on all C_2-sets is completely determined by the data in the above diagram.

Write $\mathcal{M}ack_{C_2}$ for the category of Mackey functors for the group C_2, and note that this is an abelian category.

Remark 2.1. It is often convenient to denote elements of M_Θ by x_Θ, and elements of $M_\bullet$ by $x_\bullet$. In short, we use subscripts to remind us what part of the Mackey functor elements live in.

2.2. Free Mackey functors

Consider the evaluation functors $ev_\bullet\colon \mathcal{M}ack_{C_2} \to \mathcal{A}b$ and $ev_\Theta\colon \mathcal{M}ack_{C_2} \to \mathcal{A}b$ sending $M \mapsto M_\bullet$ and $M \mapsto M_\Theta$, respectively. These have left adjoints, which will be denoted $F_\bullet$ and F_Θ, respectively. If A is an abelian group it is easy to compute that $F_\bullet(A)$ has $(F_\bullet(A))_\Theta = A$ and $(F_\bullet(A))_\bullet = A \oplus A$, where $t = id$, the map p_* is the inclusion of the right factor, and p^* is the identity on the left factor and multiplication-by-two on the right factor. We can use the symbols p_* and p^* as placekeepers and write

$$F_\bullet(A): \qquad id \circlearrowright p^*A \underset{\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]}{\overset{[0\ 1]}{\rightleftarrows}} A \oplus p_*p^*(A).$$

Similarly, $F_\Theta(A)$ can be computed to have $(F_\Theta(A))_\Theta = A \oplus A$ and $(F_\Theta(A))_\bullet = A$, where t switches the two summands, p_* is the fold map, and p^* is the diagonal. We can write

$$F_\Theta(A): \qquad \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \circlearrowright A \oplus tA \underset{\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]}{\overset{[1\ 1]}{\rightleftarrows}} p_*A.$$

Of particular importance are the Mackey functors $F_\Theta(\mathbb{Z})$ and $F_\bullet(\mathbb{Z})$, which play the role of free objects in our category. It can be useful to write these in terms of chosen generators g:

$$F_\bullet(\mathbb{Z}): \qquad id \circlearrowright \mathbb{Z}\langle p^* g_\bullet\rangle \underset{\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]}{\overset{[0\ 1]}{\rightleftarrows}} \mathbb{Z}\langle g_\bullet\rangle \oplus \mathbb{Z}\langle p_* p^* g_\bullet\rangle,$$

$$F_\Theta(\mathbb{Z}): \qquad \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \circlearrowright \mathbb{Z}\langle g_\Theta\rangle \oplus \mathbb{Z}\langle t g_\Theta\rangle \underset{\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]}{\overset{[1\ 1]}{\rightleftarrows}} \mathbb{Z}\langle p_* g_\Theta\rangle.$$

Note that giving a map $F_\Theta(\mathbb{Z}) \to M$ is equivalent to specifying the image of g_Θ in M_Θ and likewise giving a map $F_\bullet(\mathbb{Z}) \to M$ is equivalent to specifying the image of $g_\bullet$ in $M_\bullet$.

The Mackey functor $F_\bullet(\mathbb{Z})$ coincides with what is usually called the *Burnside Mackey functor* and is frequently denoted as $\mathcal{A}$.

2.3. Constant Mackey functors

Let A be an abelian group. There is a Mackey functor $\underline{A}$ where $\underline{A}_\Theta = \underline{A}_\bullet = A$, $p^* = t = id$, and p_* is multiplication by 2. This is called the **constant coefficient Mackey functor** with value A and has diagram:

$$\underline{A}: \qquad id \circlearrowright A \underset{id}{\overset{2}{\rightleftarrows}} A.$$

2.4. The box product

Given two Mackey functors M and N, their **box product** $M\square N$ is the Mackey functor given by

$$(M\square N)_\Theta = M_\Theta \otimes N_\Theta, \qquad (M\square N)_\bullet = [(M_\bullet \otimes N_\bullet) \oplus (M_\Theta \otimes N_\Theta)]/\sim$$

where the equivalence relation will be defined in a moment. The structure map p_* will send $M_\Theta \otimes N_\Theta$ to the image of the right summand in $(M\square N)_\bullet$, so we denote the element $m_\Theta \otimes n_\Theta$ in $(M\square N)_\bullet$ as $p_*(m_\Theta \otimes n_\Theta)$. The other structure maps in $M\square N$ are given by

- $t(m_\Theta \otimes n_\Theta) = t(m_\Theta) \otimes t(n_\Theta)$,
- $p^*(m_\bullet \otimes n_\bullet) = p^*(m_\bullet) \otimes p^*(n_\bullet)$,
- $p^*(p_*(m_\Theta \otimes n_\Theta)) = m_\Theta \otimes n_\Theta + t(m_\Theta) \otimes t(n_\Theta)$.

Finally, the equivalence relation that defines $(M\square N)_\bullet$ is generated by

- $p_*(p^*(m_\bullet) \otimes n_\Theta) = m_\bullet \otimes p_*(n_\Theta)$,
- $p_*(m_\Theta \otimes p^*(n_\bullet)) = p_*(m_\Theta) \otimes n_\bullet$, and
- $p_*(t(m_\Theta) \otimes t(n_\Theta)) = p_*(m_\Theta \otimes n_\Theta)$.

There is a natural unit isomorphism $\mathcal{A}\square M \to M$ that sends $g_\bullet \otimes m_\bullet \mapsto m_\bullet$ and $p^* g_\bullet \otimes m_\Theta \mapsto m_\Theta$. There is a similar right unital isomorphism $M\square\mathcal{A} \to M$.

The associativity isomorphism $(M\square N)\square Q \cong M\square(N\square Q)$ is the evident one. The twist isomorphism $M\square N \cong N\square M$ sends $m_\Theta \otimes n_\Theta \mapsto n_\Theta \otimes m_\Theta$ and $m_\bullet \otimes n_\bullet \mapsto n_\bullet \otimes m_\bullet$. With these structures, $(\mathcal{M}ack_{C_2}, \square, \mathcal{A})$ is a symmetric monoidal category.

Example 2.5. As an example we analyze $F_\Theta(\mathbb{Z})\square F_\Theta(\mathbb{Z})$. On the Θ side this is the free abelian group with generators $g_\Theta \otimes g_\Theta$, $g_\Theta \otimes tg_\Theta$, $tg_\Theta \otimes g_\Theta$, and $tg_\Theta \otimes tg_\Theta$, with the twist t acting diagonally. On the $\bullet$ side we have the free abelian group with generators $p_*(g_\Theta \otimes g_\Theta)$ and $p_*(g_\Theta \otimes tg_\Theta)$. A quick analysis of the p^* and p_* maps shows this is isomorphic to $F_\Theta(\mathbb{Z}) \oplus F_\Theta(\mathbb{Z})$, with corresponding generators $g_\Theta \otimes g_\Theta$ and $g_\Theta \otimes tg_\Theta$.

As a generalization of the above, one can check that if M is any Mackey functor then $M\square F_\Theta(\mathbb{Z}) \cong F_\Theta(M_\Theta)$.

2.6. Rings and modules

A ring structure on a given Mackey functor M consists of a unit map $\mathcal{A} \to M$ and a multiplication $M\square M \to M$ satisfying the usual axioms (the commutative version of this structure is also called a Green functor). This is equivalent to specifying a ring structure on $M_\bullet$ and a ring structure on M_Θ such that

(I) p^* and t are maps of rings, and
(II) $p_*\colon M_\Theta \to M_\bullet$ is a map of $M_\bullet$-$M_\bullet$-bimodules, where M_Θ is an $M_\bullet$-$M_\bullet$-bimodule via p^*.

Condition (II) is equivalent to the "projection formulas"

$$p_*(p^*(m_\bullet)\cdot m_\Theta) = m_\bullet \cdot p_*(m_\Theta), \qquad p_*(m_\Theta \cdot p^*(m_\bullet)) = p_*(m_\Theta)\cdot m_\bullet.$$

In particular, note that if R is a ring then the constant Mackey functor $\underline{R}$ is a Mackey ring in a natural way, by using the multiplication in R for the multiplication in both the Θ and $\bullet$ components.

If R is a Mackey ring and M is a Mackey functor, then equipping M with a (left) R-module structure is equivalent to giving an $R_\bullet$-module structure on $M_\bullet$ and an R_Θ-module structure on M_Θ such that

(i) $p^*(x_\bullet \cdot m_\bullet) = p^*(x_\bullet)\cdot p^*(m_\bullet)$
(ii) $t(x_\Theta \cdot m_\Theta) = t(x_\Theta)\cdot t(m_\Theta)$
(iii) $p_*(p^*(x_\bullet)\cdot m_\Theta) = x_\bullet \cdot p_*(m_\Theta)$
(iv) $p_*(x_\Theta \cdot p^*(m_\bullet)) = p_*(x_\Theta)\cdot m_\bullet$.

When R is the ring $\mathbb{Z}$ or $\mathbb{Z}/n$, we have the following description of $\underline{R}$-modules:

Proposition 2.7. *A Mackey functor M admits at most one structure of $\underline{\mathbb{Z}}$-module, and it admits such a structure if and only if $p_*p^* = 2$. Similarly, a Mackey functor M admits at most one structure of $\underline{\mathbb{Z}/n}$-module, and it admits such a structure if and only if $nM_\Theta = 0$, $nM_\bullet = 0$, and $p_*p^* = 2$.*

Proof. Because there is exactly one $\mathbb{Z}$-module structure on an abelian group, conditions (i)–(iii) above are trivial (they all involve products with a multiple of the identity). Only condition (iv) has content and it is the condition that $p_*p^* = 2$. The analysis for $\underline{\mathbb{Z}/n}$-modules is exactly the same. □

Remark 2.8. In the classical literature, $\underline{\mathbb{Z}}$-modules were originally called "cohomological Mackey functors". See [16], for example.

Remark 2.9. If R is a Mackey ring, M is a right R-module, and N is a left R-module, then we define $M\square_R N$ to be the coequalizer of the two evident arrows

$$M\square R\square N \rightrightarrows M\square N.$$

This is the usual definition that works in any monoidal category.

We will have special need for the "free" R-modules $F^R_\Theta = R\square F_\Theta(\mathbb{Z})$ and $F^R_\bullet = R\square F_\bullet(\mathbb{Z})$. Note that these have the adjunction properties

$$\mathrm{Hom}_R(F^R_\Theta, M) \cong M_\Theta, \qquad \mathrm{Hom}_R(F^R_\bullet, M) \cong M_\bullet.$$

Note as well that $F^R_\bullet$ is just another name for R, since $F_\bullet(\mathbb{Z}) = \mathcal{A}$ is the unit for the box product. It is best to think of F^R_Θ as the free R-module generated by an element on the Θ side, and $F^R_\bullet$ as the free R-module generated by an element on the $\bullet$ side. We leave it as an exercise for the reader to check that $F^R_\Theta \cong F_\Theta(R_\Theta)$. That is, F^R_Θ is isomorphic to the R-module

$$F^R_\Theta: \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \circlearrowright R_\Theta\langle g_\Theta\rangle \oplus R_\Theta\langle t g_\Theta\rangle \underset{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\overset{\begin{bmatrix} 1 & 1 \end{bmatrix}}{\rightleftarrows}} R_\Theta\langle p_* g_\Theta\rangle$$

where R_Θ acts diagonally on the Θ side and $R_\bullet$ acts via $p^*\colon R_\bullet \to R_\Theta$ on the $\bullet$ side.

It is easy to check that both F^R_Θ and $F^R_\bullet$ are projective R-modules. We say that an R-module is **free** if it is a direct sum of copies of F^R_Θ and $F^R_\bullet$. We have the expected notion of free basis:

Definition 2.10. Let R be a Mackey ring and let M be an R-module. Define a **basis** of M to be a collection of elements $b^\Theta_1, \dots, b^\Theta_m \in M_\Theta$ and $b^\bullet_{m+1}, \dots, b^\bullet_{m+n} \in M_\bullet$ such that the induced map

$$\left(\bigoplus_{i=1}^m F^R_\Theta\right) \oplus \left(\bigoplus_{j=1}^n F^R_\bullet\right) \longrightarrow M$$

is an isomorphism.

Not every R-module has a basis, of course, only the free modules.

2.11. The internal hom (cotensor)

Let M and N be Mackey functors, and write $\mathrm{Hom}(M, N)$ for the set of maps of Mackey functors from M to N. This is an abelian group in the natural way. We will next describe an internal hom $\underline{\mathrm{Hom}}(M, N)$, itself a Mackey functor, with the property that $\underline{\mathrm{Hom}}(M, N)_\bullet = \mathrm{Hom}(M, N)$.

Consider the diagram of Mackey functors

$$t \circlearrowright F_\Theta(\mathbb{Z}) \underset{p_*}{\overset{p^*}{\rightleftarrows}} F_\bullet(\mathbb{Z})$$

where here t is the map sending $g_\Theta \mapsto t g_\Theta$, p^* sends g_Θ to $p^*(g_\bullet)$, and p_* sends $g_\bullet$ to $p_*(g_\Theta)$. One readily checks that $p_* \circ p^* = 1 + t$. As a caution to the reader, note the maps in the above diagram go in the opposite direction from the similarly named maps of abelian groups within a Mackey functor. The names p^* and p_* are just used to remember how we defined these maps of Mackey functors.

Given our Mackey functors M and N, we first apply $(-)\square M$ to the above diagram to get

$$t \circlearrowright F_\Theta(\mathbb{Z})\square M \underset{p_*}{\overset{p^*}{\rightleftarrows}} F_\bullet(\mathbb{Z})\square M$$

(suppressing $\square id_M$ on the maps) and then we apply $\mathrm{Hom}(-,N)$ to get

$$\mathrm{Hom}(t,N)\circlearrowright \mathrm{Hom}(F_\Theta(\mathbb{Z})\square M,N) \underset{\mathrm{Hom}(p^*,N)}{\overset{\mathrm{Hom}(p_*,N)}{\rightleftarrows}} \mathrm{Hom}(F_\bullet(\mathbb{Z})\square M,N).$$

This is a Mackey functor, and this is our definition of $\underline{\mathrm{Hom}}(M,N)$. Note that $F_\bullet(\mathbb{Z})\square M \cong M$, and so $\underline{\mathrm{Hom}}(M,N)_\bullet \cong \mathrm{Hom}(M,N)$.

Proposition 2.12. *There are natural adjunctions*

$$\mathrm{Hom}(M,\underline{\mathrm{Hom}}(N,Q)) \cong \mathrm{Hom}(M\square N,Q)$$

and

$$\underline{\mathrm{Hom}}(M,\underline{\mathrm{Hom}}(N,Q)) \cong \underline{\mathrm{Hom}}(M\square N,Q).$$

Proof. This is a standard argument and left to the reader. □

Now suppose that R is a Mackey ring. If M and N are left R-modules we define $\mathrm{Hom}_R(M,N) \subseteq \mathrm{Hom}(M,N)$ to be the subset consisting of the R-linear maps.

If R is commutative we now define an R-module $\underline{\mathrm{Hom}}_R(M,N)$. The categorical definition is to define this to be the equalizer of

$$\underline{\mathrm{Hom}}(M,N) \rightrightarrows \underline{\mathrm{Hom}}(R\square M,N)$$

where one map is induced by $R\square M \to M$ and the other is the composition $\underline{\mathrm{Hom}}(M,N) \to \underline{\mathrm{Hom}}(R\square M, R\square N) \to \underline{\mathrm{Hom}}(R\square M,N)$. Alternatively, we can form the diagram

$$t\square id_M\circlearrowright F^R_\Theta\square_R M \underset{p_*\square id_M}{\overset{p^*\square id_M}{\rightleftarrows}} F^R_\bullet\square_R M$$

and then apply $\mathrm{Hom}_R(-,N)$. The resulting Mackey functor is $\underline{\mathrm{Hom}}_R(M,N)$.

The following result is again standard and left to the reader.

Proposition 2.13. *Suppose R is a commutative Mackey ring. There are natural adjunctions*

$$\mathrm{Hom}_R(M,\underline{\mathrm{Hom}}_R(N,Q)) \cong \mathrm{Hom}_R(M\square_R N,Q)$$

and

$$\underline{\mathrm{Hom}}_R(M,\underline{\mathrm{Hom}}_R(N,Q)) \cong \underline{\mathrm{Hom}}_R(M\square_R N,Q).$$

Proposition 2.14. *Let R be $\underline{\mathbb{Z}}$ or $\underline{\mathbb{Z}/n}$. If M and N are R-modules then the canonical map $M\square N \to M\square_R N$ is an isomorphism.*

Proof. Observe $\mathrm{Hom}_R(N,N') = \mathrm{Hom}(N,N')$ for any R-module N'. This fact and the above adjunctions give us that

$$\begin{aligned}\mathrm{Hom}(M\square N, N') \cong \mathrm{Hom}(M, \underline{\mathrm{Hom}}(N, N')) &= \mathrm{Hom}_R(M, \underline{\mathrm{Hom}}_R(N, N'))\\ &\cong \mathrm{Hom}_R(M\square_R N, N')\\ &= \mathrm{Hom}(M\square_R N, N').\end{aligned}$$

We see that $\mathrm{Hom}(M\square N, -)$ and $\mathrm{Hom}(M\square_R N, -)$ are the same functor, and the result follows. □

2.15. Some adjoint functors

Now we specialize to the case where R is a ring and $\underline{R}$ is the associated constant Mackey ring. An $\underline{R}$-module consists of a Mackey functor M together with R-module structures on M_Θ and $M_\bullet$ having the properties that p^*, p_*, and t^* are maps of R-modules and $p_*p^* = 2$.

The evaluation functor $ev_\Theta\colon \underline{R}\text{–Mod} \to R[C_2]\text{–Mod}$ has a left adjoint $\mathcal{K}_L$ and a right adjoint $\mathcal{K}_R$ given as follows:

$$\begin{aligned}\mathcal{K}_L(M)_\Theta = M, \quad \mathcal{K}_L(M)_\bullet = M/C_2, \quad p_* \text{ is projection}, \quad p^*(\bar{x}) = x + tx;\\ \mathcal{K}_R(M)_\Theta = M, \quad \mathcal{K}_R(M)_\bullet = M^{C_2}, \quad p_*(x) = x + tx, \quad p^* \text{ is inclusion}.\end{aligned}$$

Note that the adjunctions yield a natural transformation $\mathcal{K}_L \to \mathcal{K}_R$ that is the identity on the Θ-component.

The next result is an easy exercise:

Proposition 2.16. *If 2 is invertible in R then both pairs $(\mathcal{K}_L, ev_\Theta)$ and $(ev_\Theta, \mathcal{K}_R)$ are adjoint equivalences, and the natural transformation $\mathcal{K}_L \to \mathcal{K}_R$ is an isomorphism.*

3. $\underline{\mathbb{Z}/\ell}$-modules

In this section we investigate the category of $\underline{\mathbb{Z}/\ell}$-modules, where ℓ is a prime. We give a simple classification of all finitely-generated modules. Then we compute all box products and internal homs, and investigate their associated derived functors.

Recall from Proposition 2.7 that a $\underline{\mathbb{Z}/\ell}$-module is a pair of $\mathbb{Z}/\ell$-vector spaces V_Θ and $V_\bullet$ together with maps

$$t \circlearrowright V_\Theta \underset{p^*}{\overset{p_*}{\rightleftarrows}} V_\bullet$$

such that $p^*p_* = 1+t$ and $p_*p^* = 2$. Unsurprisingly, the structure of the category $\underline{\mathbb{Z}/\ell}$–Mod in the $\ell = 2$ case turns out to be very different from the $\ell \neq 2$ case. We investigate both cases in detail below. A $\underline{\mathbb{Z}/\ell}$-module M will be called **finitely-generated** if both M_Θ and $M_\bullet$ are finite-dimensional vector spaces over $\mathbb{Z}/\ell$.

3.1. Duality

If V is a $\mathbb{Z}/\ell$-vector space, let $V^\vee$ denote the dual vector space. If M is a $\underline{\mathbb{Z}/\ell}$-module, write M^{op} for the result of applying $(-)^\vee$ objectwise to M. That is,

$$(M^{op})_\Theta = (M_\Theta)^\vee \qquad \text{and} \qquad (M^{op})_\bullet = (M_\bullet)^\vee,$$

with the dual structure maps $t_{M^{op}} = (t_M)^\vee$, $p^*_{M^{op}} = (p^M_*)^\vee$, $p^{M^{op}}_* = (p^*_M)^\vee$. The contravariant functor $(-)^{op}$ is an anti-equivalence when restricted to the finitely-generated $\underline{\mathbb{Z}/\ell}$-modules. The following useful result is a consequence:

Proposition 3.2. *If P is a finitely-generated projective $\underline{\mathbb{Z}/\ell}$-module, then P^{op} is an injective $\underline{\mathbb{Z}/\ell}$-module.*

Proof. If R is a Mackey ring then an R-module M is injective if and only if it satisfies Baer's criterion. In this setting, that means M has the extension property with respect to submodule inclusions $I \hookrightarrow F^R_\bullet$ and $J \hookrightarrow F^R_\Theta$ (cf. [14, Theorem 2.4.1], but the proof is readily adapted from the classical proof of Baer's criterion as in [16, 2.3.1]). When $R = \underline{\mathbb{Z}/\ell}$, the modules $F^R_\bullet$ and F^R_Θ are finitely-generated, and so I and J will be as well. The result then follows immediately by applying duality. □

3.3. Classification

We begin by introducing some basic $\underline{\mathbb{Z}/\ell}$-modules. Define the two basic "free module" functors $F^{\underline{\mathbb{Z}/\ell}}_\Theta$, $F^{\underline{\mathbb{Z}/\ell}}_\bullet$: $\mathbb{Z}/\ell$–Mod $\to$ $\underline{\mathbb{Z}/\ell}$–Mod by

$$F^{\underline{\mathbb{Z}/\ell}}_\Theta(V) = \underline{\mathbb{Z}/\ell} \square F_\Theta(V), \qquad F^{\underline{\mathbb{Z}/\ell}}_\bullet(V) = \underline{\mathbb{Z}/\ell} \square F_\bullet(V).$$

We will shorten these to F^ℓ_Θ and $F^\ell_\bullet$, and also abbreviate $F^\ell_\Theta(\mathbb{Z}/\ell) = F$ and $F^\ell_\bullet(\mathbb{Z}/\ell) = H$. Once again, these are

$$H: \ \mathbb{Z}/\ell \underset{id}{\overset{2}{\rightleftarrows}} \mathbb{Z}/\ell \qquad \text{and} \qquad F: \ \mathbb{Z}/\ell \oplus \mathbb{Z}/\ell \underset{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}{\overset{[1\ 1]}{\rightleftarrows}} \mathbb{Z}/\ell.$$

If M is a $\underline{\mathbb{Z}/\ell}$-module then giving a $\underline{\mathbb{Z}/\ell}$-module map $H \to M$ is equivalent to specifying an element of $M_\bullet$, and giving a $\underline{\mathbb{Z}/\ell}$-module map $F \to M$ is equivalent to specifying an element in M_Θ.

Consider the duals of the basic free objects. We have

$$H^{op}: \ \mathbb{Z}/\ell \underset{2}{\overset{id}{\rightleftarrows}} \mathbb{Z}/\ell,$$

while $F^{op} \cong F$. The modules F and H are projective, and consequently their duals F and H^{op} are injective by Proposition 3.2. In particular, F is both projective and injective.

Let V be a $\mathbb{Z}/\ell$-vector space. We will also need the Mackey functor

$$S_\Theta(V): \quad V \underset{t}{\overset{}{\rightleftarrows}} 0$$

where t is multiplication by -1. Using the functors $\mathcal{K}_L$ and $\mathcal{K}_R$ from Section 2.15, observe that $S_\Theta(V)$ is $\mathcal{K}_L$ (or $\mathcal{K}_R$) applied to the sign representation. For convenience we will abbreviate $S_\Theta(\mathbb{Z}/\ell)$ as S_Θ.

3.4. Classification for ℓ odd

The classification of $\underline{\mathbb{Z}/\ell}$-modules for ℓ odd is straightforward: by Proposition 2.16 the category of $\underline{\mathbb{Z}/\ell}$-modules is equivalent to the category of $\mathbb{Z}/\ell[C_2]$-modules. The latter is semi-simple with two irreducibles: $\mathbb{Z}/\ell$ with the trivial action and $\mathbb{Z}/\ell$ with the sign action. Applying the functor $\mathcal{K}_R$ to these yields H and S_Θ. Thus, we obtain the following:

Proposition 3.5. *Assume ℓ is an odd prime. Then every $\underline{\mathbb{Z}/\ell}$-module is a direct sum of copies of H and S_Θ. Moreover, every $\underline{\mathbb{Z}/\ell}$-module is projective and (consequently) every short exact sequence splits.*

Note that when ℓ is odd, we have $H \cong H^{op}$ and $F \cong H \oplus S_\Theta$. Note also that if M is finitely-generated then $M \cong H^a \oplus S_\Theta^b$ where $a = \dim_{\mathbb{Z}/\ell}(M_\bullet)$ and $b = \dim_{\mathbb{Z}/\ell}(M_\Theta) - \dim_{\mathbb{Z}/\ell}(M_\bullet)$, since this can be detected using the above equivalence of categories.

Using that ev_Θ is strong monoidal lets us easily compute box products in $\underline{\mathbb{Z}/\ell}$-modules. For completeness we record the following nontrivial box products and internal homs (all other computations follow from H being the unit). From here on, we let $\underline{\mathrm{Hom}}$ and $\square$ denote $\underline{\mathrm{Hom}}_{\underline{\mathbb{Z}/\ell}}$ and $\square_{\underline{\mathbb{Z}/\ell}}$.

Proposition 3.6. *Assume ℓ is odd. Then we have*

$$S_\Theta \square S_\Theta \cong H, \qquad \underline{\mathrm{Hom}}(S_\Theta, S_\Theta) \cong H, \qquad \underline{\mathrm{Hom}}(S_\Theta, H) \cong S_\Theta.$$

Proof. Routine. □

3.7. Classification for $\ell = 2$

For the bulk of this paper we focus on the case $\ell = 2$. Note that when working mod 2 we can view t as the identity in S_Θ. We also pick up a new Mackey functor in this case: if V is a $\mathbb{Z}/2$-vector space then we have

$$S_\bullet(V): \quad \circlearrowright 0 \rightleftarrows V$$

As usual, we abbreviate $S_\bullet(\mathbb{Z}/2)$ as $S_\bullet$.

The following result gives a complete classification for finitely-generated $\underline{\mathbb{Z}/2}$-modules:

Proposition 3.8. *Every finitely-generated $\underline{\mathbb{Z}/2}$-module M is isomorphic to a direct sum of the Mackey functors H, H^{op}, F, $S_\bullet$, and S_Θ. The number of summands of each type is uniquely determined and is given by the following formulas. Let f be the rank of $1 + t\colon M_\Theta \to M_\Theta$. Then*

- *The number of F summands equals f;*
- *The number of H^{op} summands equals $\dim M_\Theta - f - \dim(\ker p_*)$;*
- *The number of H summands equals $\dim M_\bullet - \dim(\ker p^*) - f$;*
- *The number of $S_\bullet$ summands equals $\dim(\ker p^*) - \dim M_\Theta + f + \dim(\ker p_*)$;*
- *The number of S_Θ summands equals $\dim(\ker p_*) - \dim M_\bullet + \dim(\ker p^*)$.*

Note that the above decomposition of a module is not canonical. Also, we are not saying that the category of $\underline{\mathbb{Z}/2}$-modules is semisimple: for example, there is an evident exact sequence $0 \to S_\Theta \to H \to S_\bullet \to 0$ but $H \ncong S_\Theta \oplus S_\bullet$.

Proof of Proposition 3.8. Let M be a finitely-generated $\underline{\mathbb{Z}/2}$-module. If the map $t\colon M_\Theta \to M_\Theta$ is not the identity, pick $x \in M_\Theta$ such that $tx \neq x$. Since $p^*(p_*x) = x + tx \neq 0$, it follows that $p_*x \neq 0$. Then the map $F \to M$ sending the generator g_Θ to x is an injection. Since F is injective, we have $M \cong F \oplus M'$ for some M'. Proceeding inductively to keep splitting off copies of F, we reduce to the case where $t\colon M_\Theta \to M_\Theta$ is the identity.

If p_* is nonzero, pick $x \in M_\Theta$ such that $p_*(x) \neq 0$. Then $p^*(p_*x) = x + tx = x + x = 0$, and so we get an injection of $\underline{\mathbb{Z}/2}$-modules $H^{op} \hookrightarrow M$. Again, H^{op} is an injective $\underline{\mathbb{Z}/2}$-module and so we split $M \cong H^{op} \oplus M'$ for some M'. Continuing in this way, we reduce to the case where $t = id$ and $p_* = 0$.

Choose a vector space splitting $M_\bullet = (\ker p^*) \oplus U$. Likewise, choose a vector space slitting $M_\Theta = p^*(U) \oplus V$. One readily checks that $M \cong S_\Theta(V) \oplus S_\bullet(\ker p^*) \oplus H^n$ where $n = \dim U$.

The calculation of the number of summands of each type simply comes from going back through the above proof and counting. □

The only indecomposable objects are F, H, H^{op}, S_Θ, and $S_\bullet$. We next turn to understanding maps between these. Recall that maps $F \to M$ are in bijective correspondence with elements of M_Θ, and maps $H \to M$ are in bijective correspondence with elements of $M_\bullet$. Routine calculations show that, except for maps from F to itself, there is at most one nonzero map between any two of the indecomposables.

We will have particular need for understanding maps between the projective objects F and H. Maps $F \to H$ are determined by the image of the generator g_Θ, and since $H_\Theta = \mathbb{Z}/2$ there are exactly two of these: the zero map and the map that sends $g_\Theta \mapsto p^*(g_\bullet)$. We denote the nonzero map as p^*. Similarly, there is exactly one nonzero map $H \to F$, sending $g_\bullet \mapsto p_*(g_\Theta)$; we call this map p_*. The only nonzero map $H \to H$ is the identity, and there are three nonzero maps $F \to F$: 1, t, and $1+t$ (where t is the map sending $g_\Theta \mapsto t g_\Theta$). As a short summary, the following diagram depicts all of these nonzero maps:

$$1,t,1{+}t \circlearrowleft F \underset{p_*}{\overset{p^*}{\rightleftarrows}} H \circlearrowright 1$$

Observe that these satisfy the expected relations, e.g.

$$p^*t = p^*, \quad tp_* = p_*, \quad p_*p^* = 1 + t, \quad p^*p_* = 0, \quad t^2 = 1.$$

Again, a word of caution to the reader: the maps p^* and p_* in the diagram go in the opposite direction of those in a Mackey functor. Since there is only one nonzero map $F \to H$ and only one nonzero map $H \to F$, we sometimes just denote both by p since the precise map is clear from context.

For future reference we record how all these maps look on the Θ and $\bullet$ sides, with respect to our standard bases for F and H:

	t	$1+t$	p^*	p_*
Θ	$\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$
$\bullet$	1	0	0	1

3.9. Homological algebra of $\underline{\mathbb{Z}/2}$-modules

We now investigate the homological algebra of $\underline{\mathbb{Z}/2}$-modules, making use of the classification given in the previous section. We have the following projective resolutions of the nonfree indecomposable modules:

$$0 \longrightarrow H \longrightarrow F \longrightarrow S_\Theta \longrightarrow 0, \qquad 0 \longrightarrow H \longrightarrow F \longrightarrow H \longrightarrow S_\bullet \longrightarrow 0$$

and

$$0 \longrightarrow H \longrightarrow F \xrightarrow{1+t} F \longrightarrow H^{op} \longrightarrow 0.$$

We also have short exact sequences

$$0 \longrightarrow S_\Theta \longrightarrow H \longrightarrow S_\bullet \longrightarrow 0, \qquad 0 \longrightarrow S_\bullet \longrightarrow H^{op} \longrightarrow S_\Theta \longrightarrow 0.$$

The above sequences show that S_Θ, $S_\bullet$, and H^{op} all have projective resolutions with at most three nonzero terms. The following is an immediate consequence:

Proposition 3.10. *For any two $\underline{\mathbb{Z}/2}$-modules A and B where A is finitely-generated, we have $\underline{\mathrm{Ext}}^i(A,B) = 0$ for all $i \geq 3$.*

The following proposition lists the results of several routine calculations in the homological algebra of $\underline{\mathbb{Z}/2}$-modules. Recall that we use $\underline{\mathrm{Hom}}$ and $\square$ to mean $\underline{\mathrm{Hom}}_{\underline{\mathbb{Z}/2}}$ and $\square_{\underline{\mathbb{Z}/2}}$.

Proposition 3.11. *Let A and B be finitely-generated $\underline{\mathbb{Z}/2}$-modules. Then $\underline{\mathrm{Hom}}(A,B)$ and $A\square B$ are given by the following tables:*

$\underline{\mathrm{Hom}}(A,B)$:

$A\backslash B$	H	F	H^{op}	$S_\bullet$	S_Θ
H	H	F	H^{op}	$S_\bullet$	S_Θ
F	F	F^2	F	0	F
H^{op}	H	F	H	0	H
$S_\bullet$	0	0	$S_\bullet$	$S_\bullet$	0
S_Θ	H	F	S_Θ	0	H

$A\square B$:

$A\backslash B$	H	F	H^{op}	$S_\bullet$	S_Θ
H	H	F	H^{op}	$S_\bullet$	S_Θ
F		F^2	F	0	F
H^{op}			H^{op}	0	H^{op}
$S_\bullet$				$S_\bullet$	0
S_Θ					H^{op}

For the cases where A is not projective and B is not injective, the Mackey functors $\underline{\mathrm{Ext}}^i(A,B)$ are as follows:

$\underline{\mathrm{Ext}}^i(A,B)$:

$A,\ B$	$H^{op},\ H$	$H^{op},\ S_\Theta$	$H^{op},\ S_\bullet$	$S_\Theta,\ H$	$S_\Theta,\ S_\Theta$	$S_\Theta,\ S_\bullet$	$S_\bullet,\ S_\Theta$	$S_\bullet,\ S_\bullet$
2	$S_\bullet$	0	$S_\bullet$	0	0	0	$S_\bullet$	$S_\bullet$
1	$S_\bullet$	$S_\bullet$	0	$S_\bullet$	0	$S_\bullet$	$S_\bullet$	0
0	H	H	0	H	H	0	0	$S_\bullet$

Finally, for the cases where neither A nor B is free the Mackey functors $\underline{\mathrm{Tor}}_i(A,B)$ are as follows:

$\underline{\mathrm{Tor}}_i(A,B)$:

$A,\ B$	$H^{op},\ H^{op}$	$H^{op},\ S_\Theta$	$H^{op},\ S_\bullet$	$S_\Theta,\ S_\Theta$	$S_\Theta,\ S_\bullet$	$S_\bullet,\ S_\bullet$
2	$S_\bullet$	0	$S_\bullet$	0	0	$S_\bullet$
1	$S_\bullet$	$S_\bullet$	0	0	$S_\bullet$	0
0	H^{op}	H^{op}	0	H^{op}	0	$S_\bullet$

Proof. Lots of calculations, but completely routine. □

Remark 3.12. Notice in the above tables that $M\square F$, $\underline{\mathrm{Hom}}(M,F)$, and $\underline{\mathrm{Hom}}(F,M)$ are always sums of copies of F, no matter what M is. This can be explained by the observation that F can be given the structure of a Mackey field over $\underline{\mathbb{Z}/2}$: it is the Mackey field

$$t \circlearrowright \mathbb{F}_4 \underset{i}{\overset{\mathrm{tr}}{\rightleftarrows}} \mathbb{F}_2$$

where t is the nontrivial automorphism. The above constructions have natural structures of F-modules, and over a Mackey field all modules are free.

Likewise, one might notice that for $i > 0$ one always has $\underline{\mathrm{Ext}}^i(A,B)_\Theta = 0 = \underline{\mathrm{Tor}}_i(A,B)_\Theta$. This follows from the observation that for any Mackey functor M one has $M_\Theta = 0$ if and only if $M\square F = 0$ (because $M\square F = F_\Theta(M_\Theta)$ as in Example 2.5). Then one calculates that

$$\underline{\mathrm{Tor}}_i(A,B)\square F = \underline{\mathrm{Tor}}_i(A, B\square F), \qquad \underline{\mathrm{Ext}}^i(A,B)\square F = \underline{\mathrm{Ext}}^i(A, B\square F)$$

and uses the fact that $B\square F$ is a finite direct sum of copies of F, making it both projective (hence flat) and injective.

4. Complexes of $\underline{\mathbb{Z}/2}$-modules

If $\mathcal{A}$ is any abelian category we can consider the category of chain complexes $Ch(\mathcal{A})$ and the associated homotopy category $\mathcal{H}o(Ch(\mathcal{A}))$ obtained by inverting quasi-isomorphisms. This is also known as the derived category $\mathcal{D}(\mathcal{A})$. As usual, if R is a Mackey ring and $\mathcal{A} = R$–Mod this homotopy category will also be denoted $\mathcal{D}(R)$. Our goal is to understand what we can about the structure of $\mathcal{D}(\underline{\mathbb{Z}/2})$.

We will occasionally have to distinguish between constructions on $Ch(\underline{\mathbb{Z}/2})$ and their derived versions on $\mathcal{D}(\underline{\mathbb{Z}/2})$. The derived version of the box product will be denoted $\square$, and the derived version of the cotensor will be denoted $\boldsymbol{\mathcal{F}}(-,-)$.

A complex in $Ch(\underline{\mathbb{Z}/2})$ is called **perfect** if it is quasi-isomorphic to a bounded complex that in each degree is a direct sum of finitely-many copies of H and F. These are the compact objects in $\mathcal{D}(\underline{\mathbb{Z}/2})$, by the same argument as given in [3, Proposition 6.4]. Write $\mathcal{D}(\underline{\mathbb{Z}/2})_{\mathrm{perf}}$ for the full subcategory of $\mathcal{D}(\underline{\mathbb{Z}/2})$ whose objects are the perfect complexes. Our first goal is to completely classify these objects.

The full subcategory of $\underline{\mathbb{Z}/2}$–Mod whose objects are F and H is, by inspection, isomorphic to the full subcategory of $\mathbb{Z}/2[C_2]$-modules consisting of $\mathbb{Z}/2[C_2]$ and $\mathbb{Z}/2$ (with C_2 acting trivially). All finitely-generated $\mathbb{Z}/2[C_2]$-modules are direct sums of these two types, so the full subcategory of finitely-generated free $\underline{\mathbb{Z}/2}$-modules is isomorphic to the category of finitely-generated $\mathbb{Z}/2[C_2]$-modules. Comparing the box products of F and H from Proposition 3.11 with the tensor products of $\mathbb{Z}/2[C_2]$ and $\mathbb{Z}/2$, we see the symmetric monoidal structures agree. Finally, recall that $\mathcal{D}(\mathcal{R})_{\mathrm{perf}}$ for any ring (or Mackey-ring) $\mathcal{R}$ can be modeled by the category of bounded complexes of finitely-generated projectives and chain homotopy classes of maps. The following is an immediate consequence:

Proposition 4.1. *The map $M \mapsto M_\Theta$ gives an equivalence of symmetric monoidal categories $\mathcal{D}(\underline{\mathbb{Z}/2})_{\mathrm{perf}} \simeq K_{b,fg}(\mathbb{Z}/2[C_2])$, where K denotes the category of chain complexes and chain homotopy classes of maps and "b, fg" indicate bounded complexes of finitely-generated modules.*

The above proposition is not required for our classification, but it is informative to identify our main problem with a more classical problem from ordinary ring theory. We are grateful to Paul Balmer for pointing out this connection.

We now isolate the following special classes of perfect complexes. By a **strand**, we mean a perfect complex that in each degree is equal to either H, F, or 0, with all nonzero terms consecutive, and where all maps between nonzero terms are nonzero. It is not hard to determine all possible strands. In addition to the contractible complexes that have a single isomorphism and zero elsewhere (e.g. $t : F \to F$), we have the following complete list of strands.

For $k \geq 0$, let A_k be the complex

$$A_k: \qquad 0 \longrightarrow F \xrightarrow{1+t} F \xrightarrow{1+t} F \xrightarrow{1+t} \cdots \xrightarrow{1+t} F \longrightarrow 0,$$

where the nonzero groups are in degrees 0 through k (note that our complexes use homological grading, where the differentials decrease degree). For $n > 0$, let $H(-n)$ be the complex

$$H(-n)\colon \qquad 0 \longrightarrow H \xrightarrow{p} F \xrightarrow{1+t} F \xrightarrow{1+t} \cdots \xrightarrow{1+t} F \longrightarrow 0,$$

where H is in degree n and the rightmost F is in degree 0. Likewise (still for $n > 0$), let $H(n)$ be the complex

$$H(n)\colon \qquad 0 \longrightarrow F \xrightarrow{1+t} F \xrightarrow{1+t} \cdots \xrightarrow{1+t} F \xrightarrow{p} H \longrightarrow 0,$$

where the leftmost F is in degree 0 and the H is in degree $-n$. Let $H(0)$ be the complex $0 \longrightarrow H \longrightarrow 0$ with H in degree 0. Finally, let B_r be the complex

$$B_r\colon \qquad 0 \longrightarrow H \xrightarrow{p} F \xrightarrow{1+t} F \xrightarrow{1+t} \cdots \xrightarrow{1+t} F \xrightarrow{p} H \longrightarrow 0,$$

where the leftmost H is in degree 0 and the rightmost H is in degree $-(r+2)$.

We call A_k, B_r, and $H(n)$ (allowing $k, r \geq 0$ and $n \in \mathbb{Z}$) our **fundamental complexes**. There are evident homotopy cofiber sequences

$$A_0 \longrightarrow A_k \longrightarrow \Sigma A_{k-1}, \qquad B_0 \longrightarrow B_r \longrightarrow \Sigma^{-1} B_{r-1},$$
$$A_{n-1} \longrightarrow H(-n) \longrightarrow \Sigma^n H, \qquad \Sigma^{-(n+2)} H \longrightarrow B_n \longrightarrow \Sigma^{-(n+1)} H(-(n+1)),$$

as well as many others which the reader can readily identify.

Remark 4.2. Note that $S_\Theta \simeq H(-1)$, $S_\bullet \simeq \Sigma^2 B_0$, and $H^{op} \simeq H(-2)$, using the resolutions from Section 3.9.

It will turn out that for all $n \in \mathbb{Z}$, $H(n)$ is invertible in $\mathcal{D}(\underline{\mathbb{Z}/2})$, and in fact the objects $\Sigma^p H(n)$ constitute all the invertible objects in $\mathcal{D}(\underline{\mathbb{Z}/2})$. The inverse of $H(n)$ is $H(-n)$, and more generally one has $H(n) \square H(m) \simeq H(n+m)$. See Proposition 4.18 below for this and related facts, as well as Theorem 5.2.

For C any object in $\mathcal{D}(\underline{\mathbb{Z}/2})$, define

$$H^{p,q}(C) = \mathcal{D}(\underline{\mathbb{Z}/2})(C, \Sigma^p H(q)).$$

The derived box product of maps makes $\bigoplus_{p,q} H^{p,q}(H)$ into a bigraded ring, which we denote $\mathbb{M}_2$, and $\bigoplus_{p,q} H^{p,q}(C)$ is a bigraded $\mathbb{M}_2$-module. We record the bigraded module structure on a grid, where each dot indicates a $\mathbb{Z}/2$, and the vertical and diagonal lines indicate action by certain elements $\tau \in H^{0,1}(H)$ and $\rho \in H^{1,1}(H)$.

The ring $\mathbb{M}_2$ is shown in Fig. 1. The ring structure is completely determined by the picture (showing τ- and ρ-multiplications) together with the assertion that $\theta^2 = 0$, from which it follows that the product of any two classes in the "lower cone" descending from θ is also zero. This computation, as well as Proposition 4.3 below, is jumping ahead a bit, but it is useful to see these calculations right away to get a sense of the fundamental complexes. More information about these calculations and their consequences will be given in Section 4.24 below.

Proposition 4.3. *The cohomology groups $H^{p,q}(C)$ for $C \in \{A_k, H(n), B_r\}$ are given by the following pictures:*

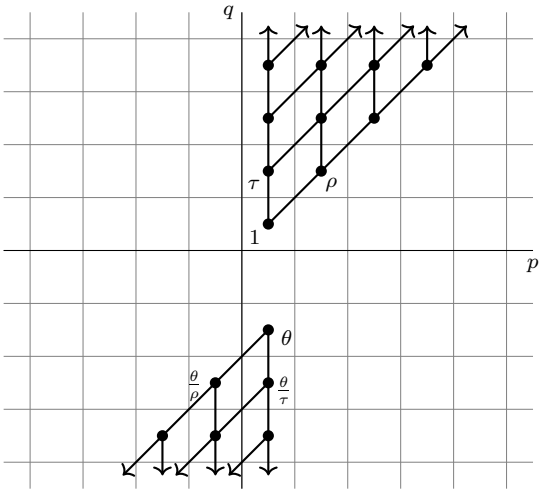


Fig. 1. The ring $\mathbb{M}_2 = H^{*,*}(H)$.

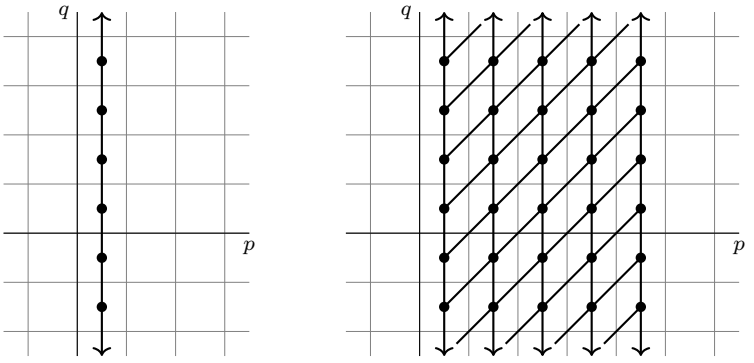


Fig. 2. The cohomology of A_0 and A_4.

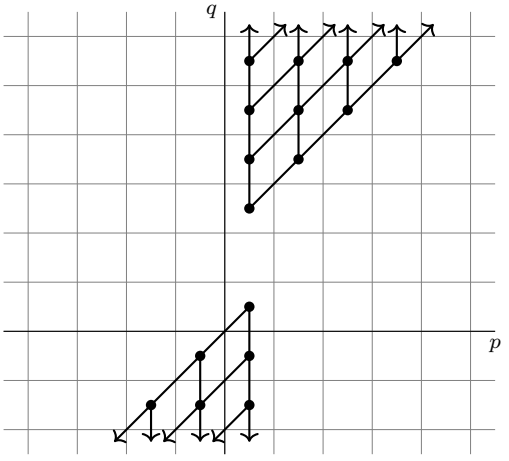


Fig. 3. The cohomology of $H(2)$.

(i) *A_k: Vertical strip stretching from $p = 0$ through $p = k$. See Fig. 2.*

(ii) *$H(n)$: Copy of $\mathbb{M}_2$ generated by a class in bidegree $(0, n)$. Here n can be positive or negative. See Fig. 3.*

(iii) *B_r: Diagonal strip occupying the diagonals $p - q = 0$ through $p - q = -r$. See Fig. 4.*

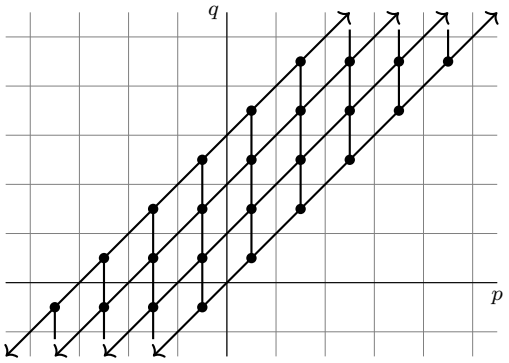


Fig. 4. The cohomology of B_3.

In more algebraic terms, we have

(i) $H^{*,*}(A_k) = \mathbb{M}_2[\tau^{-1}]/(\rho^{k+1})$,
(ii) $H^{*,*}(H(n)) = \mathbb{M}_2\langle e\rangle$ *where* e *has bidegree* $(0, n)$,
(iii) $H^{*,*}(B_r) = \mathbb{M}_2[\rho^{-1}]/(\tau^{r+1})$.

The proof of Proposition 4.3 appears in Section 4.24 below.

Remark 4.4. The ring $\mathbb{M}_2$ also appears in algebraic geometry, but with a different grading: it is the ring $\bigoplus_{p,q} H^p(\mathbb{P}^1, \mathcal{O}(q))$ (over the ground field $\mathbb{F}_2$). But in this context the "upper cone" is entirely concentrated in degree $p = 0$ whereas the "lower cone" is entirely concentrated in degree $p = 1$.

The following is our main algebraic classification theorem. In short, it says every bounded complex of projectives splits into strands. Before stating it we need one more piece of notation. If M is any $\underline{\mathbb{Z}/2}$-module, let $\mathbb{D}(M)$ denote the chain complex with M in degrees 0 and 1, zeros in all other degrees, and with the differential $\mathbb{D}(M)_1 \to \mathbb{D}(M)_0$ given by the identity map. Notice that $\mathbb{D}(M)$ is contractible (and here $\mathbb{D}$ stands for 'disk').

Theorem 4.5. *Let C be a bounded complex that in each degree is given by a finite direct sum of copies of F and H. Then C is isomorphic to a direct sum of shifts of the complexes A_k, B_r, $H(n)$, for various values of $k, r \geq 0$ and $n \in \mathbb{Z}$, and the contractible complexes $\mathbb{D}(F)$ and $\mathbb{D}(H)$.*

Proof. The proof is technical and we defer it to its own section. See Section 7. □

Corollary 4.6. *Any perfect complex is quasi-isomorphic to a direct sum of shifts of the fundamental complexes A_k, B_r, and $H(n)$, for various values of $k, r \geq 0$ and $n \in \mathbb{Z}$.*

Proof. Immediate. □

The following corollary is also very useful:

Corollary 4.7. *Let C be a bounded complex which in each degree is a finite direct sum of copies of F. Then C is isomorphic to a direct sum of shifts of the complexes A_k (for various $k \geq 0$) and the contractible complexes $\mathbb{D}(F)$.*

Proof. Theorem 4.5 implies that C is isomorphic to a direct sum of the desired types of complexes and also the B_r, $H(n)$, and $\mathbb{D}(H)$ complexes. None of these last three types can appear since H is not a direct

summand of any F^m. This last fact follows because in H the structure map p_* is zero, whereas in F^m it is surjective. □

4.8. Results for bounded below complexes

We can extend the above results to chain complexes that are bounded below by adding two more fundamental complexes to our list. Let B_∞ denote the complex that has an F in each positive degree and ends with an H in degree zero, and let A_∞ denote the complex that has an F in all nonnegative degrees. In both, the maps $F \to F$ are given by $1 + t$, and in B_∞ the map $F \to H$ is p^*.

The complex A_∞ clearly deserves its name, as it is the colimit of an evident sequence:

$$A_\infty \cong \operatorname{colim}\big[A_0 \to A_1 \to A_2 \to \cdots\big].$$

The situation is less clear for B_∞. On the one hand, B_∞ is the colimit of a sequence

$$H \to \Sigma H(1) \to \Sigma^2 H(2) \to \Sigma^2 H(3) \to \cdots.$$

But at the same time, B_∞ is also the inverse limit of an evident sequence involving B-complexes:

$$B_\infty \cong \lim\big[\cdots \to \Sigma^4 B_2 \to \Sigma^3 B_1 \to \Sigma^2 B_0\big].$$

The appropriateness of the name B_∞ really becomes clear in Proposition 4.18(vii) below, as well as in the following result (whose proof is again deferred until Section 4.24):

Proposition 4.9. *The bigraded cohomology of A_∞ is $H^{*,*}(A_\infty) = \mathbb{M}_2[\tau^{-1}]$. For B_∞, $H^{*,*}(B_\infty)$ is the $\Sigma^{1,0}$-suspension of the bigraded module quotient $\mathbb{M}_2[\tau^{-1}, \rho^{-1}]/\mathbb{M}_2[\rho^{-1}]$. The picture of $H^{*,*}(B_\infty)$ is similar to Fig. 4 except the diagonal strip extends to occupy the entire half-plane below the diagonal $p - q = 2$.*

Theorem 4.10. *Let C be a bounded below complex that in each degree is a finite direct sum of copies of F and H. Then C is isomorphic to a direct sum of shifts of the complexes A_k, B_r, $H(n)$, A_∞, B_∞, and the contractible complexes $\mathbb{D}(F)$ and $\mathbb{D}(H)$ (for various values of $k, r \geq 0$ and $n \in \mathbb{Z}$).*

Proof. Fix C as in the statement of the proposition. Without loss of generality we can assume that C vanishes below degree zero. Let $C[0, m]$ be the truncation of C consisting of degrees 0 through m. Recall the notion of basis from Definition 2.10 and let S_m be the set of all (homogeneous) bases for $C[0, m]$ that make the truncation split as a direct sum of shifts of the complexes A_k, B_r, $H(n)$, $\mathbb{D}(F)$, and $\mathbb{D}(H)$. There are evident maps $S_m \to S_{m-1}$ given by forgetting the basis in degree m. The proposition is then equivalent to the statement that the inverse limit $S = \varprojlim S_m$ is nonempty. By Theorem 4.5, the set S_m is nonempty for each m, and since C is finite-dimensional in each degree, S_m is finite for each m. A filtered inverse limit of any collection of finite, nonempty sets is nonempty [6, E III.58, Theorem 1], and thus S is nonempty. This completes the proof. □

Remark 4.11. When p is an odd prime and one considers C_p-Mackey functors, there are again basic projective $\underline{\mathbb{Z}/p}$-modules F and H. One could ask about a similar splitting theorem for complexes of $\underline{\mathbb{Z}/p}$-modules in this context. The situation here is much more complicated, however, and nothing as simple as Theorem 4.5 could work. Theorem 4.5 implies every perfect complex is quasi-isomorphic to a direct sum of strands, but in the C_p case there are perfect complexes that do not split this way.

For example, let $p = 3$ and consider $G = C_3$ with generator γ. Now let $H = \underline{\mathbb{Z}/3}$ and $F = F_\Theta(\mathbb{Z}/3)$ be the analogous Mackey functors in $\mathcal{M}ack_{C_3}$. The complex

$$0 \to F \xrightarrow{\begin{bmatrix} 1+\gamma+\gamma^2 \\ 1-\gamma \end{bmatrix}} F \oplus F \xrightarrow{\begin{bmatrix} 1-\gamma & 0 \end{bmatrix}} F \to 0$$

cannot be quasi-isomorphic to a direct sum of strands. In particular, the homology at the middle spot is a Mackey functor that is not found in the homology of any possible strand.

4.12. Duality for complexes

The functor $(-)^{op}\colon \underline{\mathbb{Z}/2}\text{–Mod} \to \underline{\mathbb{Z}/2}\text{–Mod}$ extends to a functor $Ch(\underline{\mathbb{Z}/2}) \to Ch(\underline{\mathbb{Z}/2})$. This functor preserves quasi-isomorphisms and so further extends to a functor $\mathcal{D}(\underline{\mathbb{Z}/2}) \to \mathcal{D}(\underline{\mathbb{Z}/2})$. We calculate this functor for the fundamental complexes:

Proposition 4.13. *The duals of the fundamental complexes are*

(a) $A_k^{op} \simeq \Sigma^{-k} A_k$,
(b) $H(n)^{op} \simeq H(-(n+2))$, *and*
(c) $B_r^{op} \simeq \Sigma^{r+4} B_r$.

Proof. Part (a) is a direct computation. For (b), when $n > 0$ the complex $H(n)^{op}$ is

$$0 \to H^{op} \to F \to F \to \cdots \to F \to 0$$

where the H^{op} is in degree n and the rightmost F is in degree 0. Patching the resolution $0 \to H \to F \to F \to H^{op}$ on to the end of this complex, we see that $H(n)^{op} \simeq H(-(n+2))$. Similarly, for the complex $H(-n)^{op}$ we use the quasi-isomorphism

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & F & \longrightarrow & F & \longrightarrow & \cdots & \longrightarrow & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & H^{op} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle p} & & \uparrow & & & & \\
0 & \longrightarrow & F & \longrightarrow & F & \longrightarrow & \cdots & \longrightarrow & F & \longrightarrow & H & \longrightarrow & 0 & & & &
\end{array}$$

to see that $H(-n)^{op} \simeq H(n-2)$. This proves (b).

Finally, the proof of (c) is a combination of the two types of arguments used for (b). The complex B_r^{op} has an H^{op} at both ends, in degrees $r+2$ and 0. The one in degree $r+2$ can be removed by patching in a resolution, and the one in degree 0 is removed via a quasi-isomorphism as we used in (b). The end result is a copy of B_r that lies in degrees $r+4$ down through 2, which is $\Sigma^{r+4} B_r$. □

Another kind of duality involves the functor $\mathcal{F}(-, H)$. There is a canonical map

$$\Gamma_{X,Z}\colon \mathcal{F}(X,H) \square Z \longrightarrow \mathcal{F}(X,Z)$$

obtained as the adjoint of the composite

$$\mathcal{F}(X,H) \square Z \square X \xrightarrow{id \otimes t_{Z,X}} \mathcal{F}(X,H) \square X \square Z \xrightarrow{ev \otimes id_Z} H \square Z \xrightarrow{\cong} Z.$$

The following is a standard argument:

Proposition 4.14. *If X is a perfect complex then $\Gamma_{X,Z}$ is an isomorphism in $\mathcal{D}(\underline{\mathbb{Z}/2})$, for any Z.*

Proof. One checks the result directly for the two special objects $X = H$ and $X = F$. Then use that the property persists under direct sums, extensions, and retracts. □

Remark 4.15. We will not need it, but the standard arguments also show that the canonical map X to $\mathcal{F}(\mathcal{F}(X,H),H)$ is an isomorphism in $\mathcal{D}(\underline{\mathbb{Z}/2})$ whenever X is perfect.

4.16. Calculations for fundamental complexes

Moving towards our goal of understanding everything we can about $\mathcal{D}(\underline{\mathbb{Z}/2})$, the next step is to calculate the effects of basic operations on the fundamental complexes. We start by computing the homology modules of the complexes:

Proposition 4.17. *For $k \geq 2$, $n \geq 1$, and $r \geq 0$ the homology of the different strands is given by the following formulas.*

(a) $H_i(A_0) = \begin{cases} F & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$

(b) $H_i(A_1) = \begin{cases} H & \text{if } i = 1, \\ H^{op} & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$

(c) $H_i(A_k) = \begin{cases} H & \text{if } i = k, \\ S^{\bullet} & \text{if } 0 < i < k, \\ H^{op} & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$

(d) $H_i(H(n)) = \begin{cases} H & \text{if } i = 0, \\ S^{\bullet} & \text{if } -n \leq i < 0, \\ 0 & \text{else.} \end{cases}$

(e) $H_i(H(0)) = \begin{cases} H & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$

(f) $H_i(H(-1)) = \begin{cases} S_{\Theta} & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$

(g) $H_i(H(-2)) = \begin{cases} H^{op} & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$

(h) For $n > 2$, $H_i(H(-n)) = \begin{cases} S^{\bullet} & \text{if } 1 \leq i \leq n-2, \\ H^{op} & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$

(i) $H_i(B_r) = \begin{cases} S^{\bullet} & \text{if } -r-2 \leq i \leq -2, \\ 0 & \text{else.} \end{cases}$

Proof. These are direct (and straightforward) computations with the complexes. □

Proposition 4.18. *Let $k, l, r \geq 0$ and $m, n \in \mathbb{Z}$. The (derived) box product of fundamental complexes is given by the following formulas:*

(i) $A_k \square A_l \simeq A_k \oplus \Sigma^l A_k$ *for* $k \leq l$.
(ii) $A_k \square H(n) \simeq A_k$.
(iii) $A_k \square B_r \simeq 0$.
(iv) $H(n) \square H(m) \simeq H(n+m)$.
(v) $H(n) \square B_r \simeq \Sigma^{-n} B_r$
(vi) $B_r \square B_l \simeq \Sigma^{-(l+2)} B_r \oplus B_r$ *for* $r \leq l$.
(vii) *Parts (i)–(iii) and (v)–(vi) also hold when* k, l, *or* r *is* ∞, *if we interpret any term with* Σ^∞ *or* $\Sigma^{-\infty}$ *as being zero. For example,* $A_k \square A_\infty \simeq A_k$ *and* $H(n) \square B_\infty \simeq \Sigma^{-n} B_\infty$ *for all* $k \leq \infty$ *and* $n < \infty$.

Proof. There are a variety of ways to do these computations using the standard machinery of homological algebra. The following is one route through this.

(1): We claim $F \square B_r \simeq 0$, $F \square H(n) \simeq F$, and $F \square A_k \simeq F \oplus \Sigma^k F$.

To see this, note since F is projective we have $H_i(F \square C) \cong F \square H_i(C)$ for any C. Then Proposition 4.17, together with Proposition 3.11, shows that $H_*(F \square B_r) = 0$, and $H_*(F \square H(n))$ is F concentrated entirely in degree 0. Now Corollary 4.7 implies that $F \square H(n) \simeq A_0 = F$. Similarly, $H_*(F \square A_k)$ is F in degrees 0 and k, and zero elsewhere. By Corollary 4.7, $F \square A_k$ must be quasi-isomorphic to a direct sum of A-strands, so the only possibility is $F \square A_k \simeq F \oplus \Sigma^k F$.

(2): $B_r \square A_k \simeq 0$, $B_r \square H(n) \simeq \Sigma^{-n} B_r$, and $A_k \square H(n) \simeq A_k$.

For this step, apply $B_r \square (-)$ to the cofiber sequence $A_0 \to A_k \to \Sigma A_{k-1}$ and use induction to conclude that $B_r \square A_k \simeq 0$ for all k. When $n > 0$, for $B_r \square H(n)$ use the cofiber sequence $A_{n-1} \to H \to \Sigma^n H(n)$ and apply $B_r \square (-)$. Similarly, for $B_r \square H(-n)$ use a cofiber sequence $\Sigma^{n-1} H \to A_{n-1} \to H(-n)$.

When $n \geq 0$, applying $A_k \square (-)$ to the cofiber sequence $\Sigma^{-(n+2)} H \to B_n \to \Sigma^{-(n+1)} H(-(n+1))$ and using what we have already proven immediately yields $A_k \square H(-(n+1)) \simeq A_k$. A similar argument applied to the sequence $\Sigma^{-1} H(n+1) \to B_n \to H$ yields $A_k \square H(n+1) \simeq A_k$.

(3): If $k \leq l$ then $A_k \square A_l \simeq A_k \oplus \Sigma^l A_k$ and if $r \leq l$ then $B_r \square B_l \simeq B_r \oplus \Sigma^{-(l+2)} B_r$.

Observe first that there is an evident cofiber sequence

$$\Sigma^{-(l+1)} H \longrightarrow H(l+1) \longrightarrow \Sigma^{-l} A_l.$$

Boxing with A_k and using $A_k \square H(l+1) \simeq A_k$, and for convenience applying Σ^l, gives a homotopy cofiber sequence in the derived category

$$\Sigma^{-1} A_k \longrightarrow \Sigma^l A_k \longrightarrow A_k \square A_l.$$

But it is immediately observed for degree reasons that there are no nonzero maps $\Sigma^{-1} A_k \to \Sigma^l A_k$ (since $k \leq l$), and so we conclude $A_k \square A_l \simeq \Sigma^l A_k \oplus A_k$. The same proof works for the B-strands, starting with the cofiber sequence

$$\Sigma^{-2} H(l) \longrightarrow B_l \longrightarrow \Sigma^{-1} H(-1)$$

and then using that there are no nonzero maps $\Sigma^{-1} B_r \to \Sigma^{-(l+2)} B_r$ for $r \leq l$.

(4): $H(n) \square H(m) \simeq H(n+m)$.

Corollary 4.6 says that $H(n) \square H(m)$ decomposes (up to weak equivalence) as a direct sum of shifts of A-, B-, and H-strands. Applying $A_0 \square (-)$ and using the previous parts immediately shows that no A-strands can appear, and that exactly one H-strand must appear. Similarly, applying $B_0 \square (-)$ shows that no B-strands can appear and also that the H-strand that appears must be $H(n+m)$. These four steps finish (i)–(vi).

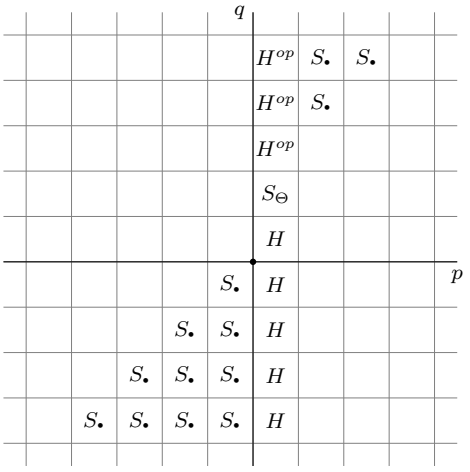


Fig. 5. The bigraded homology of H.

We only briefly sketch the proofs for (vii), leaving the details to the reader. For part (i), explicit hand calculation readily shows $F\square A_\infty \simeq F$. Pass to the general case of $A_k\square A_\infty$ by a double complex argument. For $B_r\square A_\infty$ in (iii) use that $B_r\square A_l \simeq 0$ and that A_∞ is the homotopy colimit of the A_l. Then for $H(n)\square A_\infty$ in (ii) use the cofiber sequences $\Sigma^{-2}H \to B_0 \to \Sigma^{-1}H(-1)$ and $H \to B_0 \to \Sigma^{-1}H(1)$ and induction.

For the remaining cases of (iii), first prove $F\square B_\infty \simeq 0$ by explicit calculation. Then get $A_k\square B_\infty \simeq 0$ by induction using the cofiber sequence $A_{k-1} \to A_k \to \Sigma^k F$. For $B_r\square B_\infty$ in (vi) use $H \to B_\infty \to \Sigma A_\infty$ together with (iii). Finally, for (v) use the cofiber sequence $\Sigma^{-n}H \to H(n) \to \Sigma^{-(n-1)}A_{n-1}$ and box with B_∞. $\square$

Corollary 4.19. *For all $n \in \mathbb{Z}$, $H(n)$ is invertible with inverse $H(-n)$.*

Just as we defined bigraded cohomology groups $H^{p,q}(X)$ for $X \in \mathcal{D}(\underline{\mathbb{Z}/2})$, we can also define bigraded homology groups. Here the definition is

$$H_{p,q}(X) = \mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^p H(q), X) = \mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^p H, X\square H(-q)) = H_p(X\square H(-q))_\bullet,$$

where we have used the invertibility of $H(q)$ for the second equality. The right-most term suggests an extension of this to Mackey-functor-valued homology, given simply by $H_{p,q}(X) = H_p(X\square H(-q))$. For example, a portion of the bigraded homology of H is shown in Fig. 5 (the bigraded homology extends infinitely in the vertical directions).

We next turn to the cotensor in $\mathcal{D}(\underline{\mathbb{Z}/2})$:

Proposition 4.20. *The cotensor between fundamental complexes is given by the following formulas for $k, r \geq 0$ and $m, n \in \mathbb{Z}$:*

(a) $\mathcal{F}(H(m), H(n)) \simeq H(n-m)$
(b) $\mathcal{F}(A_k, B_r) \simeq 0 \simeq \mathcal{F}(B_r, A_k)$
(c) $\mathcal{F}(H(n), B_r) \simeq \Sigma^n B_r$
(d) $\mathcal{F}(H(n), A_k) \simeq A_k$
(e) $\mathcal{F}(A_k, H(n)) \simeq \Sigma^{-k} A_k$
(f) $\mathcal{F}(B_r, H(n)) \simeq \Sigma^{r-n+2} B_r$
(g) $\mathcal{F}(A_k, A_l) \simeq \begin{cases} \Sigma^{-k}A_k \oplus \Sigma^{l-k}A_k & \text{if } k \leq l, \\ A_l \oplus \Sigma^{-k}A_l & \text{if } k \geq l. \end{cases}$

(h) $\mathbb{F}(B_r, B_l) \simeq \begin{cases} \Sigma^{r-l}B_r \oplus \Sigma^{r+2}B_r & \text{if } r \leq l, \\ \Sigma^{r+2}B_l \oplus B_l & \text{if } r \geq l. \end{cases}$

Proof. One readily computes by inspection that $\mathcal{F}(A_k, H) \cong \Sigma^{-k}A_k$, while $\mathcal{F}(H(n), H) \cong H(-n)$, and $\mathcal{F}(B_r, H) \cong \Sigma^{r+2}B_r$. The desired results are then immediate from Proposition 4.14 and Proposition 4.18. □

We include the following result as a curiosity. We have called this "Grothendieck-Serre duality" because of the evident analog to the similar statement in algebraic geometry. Note that the appearance of $H(-2)$ is really due to the equivalence $H^{op} \simeq H(-2)$.

Theorem 4.21 *(Grothendieck-Serre Duality). For perfect complexes X and Y one has*

$$\mathbb{F}(X, Y \square H(-2)) \simeq \mathbb{F}(Y, X)^{op}.$$

This theorem can be proven by brute force using Theorem 4.5 simply by checking it for the fundamental complexes, but this is clearly not the ideal method. As we have no need of the result elsewhere in the paper, we do not take the time here to develop a more satisfying proof.

We next look at morphisms between fundamental complexes. Recall that we write $\mathcal{D}(\underline{\mathbb{Z}/2})(X, Y)$ for maps in the derived category from X to Y. Note that since the $H(n)$ are invertible one always has

$$\mathcal{D}(\underline{\mathbb{Z}/2})(X, Y) \cong \mathcal{D}(\underline{\mathbb{Z}/2})(X \square H(n), Y \square H(n))$$

for every $n \in \mathbb{Z}$.

We start with the case $X = H$, which has already been done by virtue of the isomorphism $\mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^i H, Y) \cong H_i(Y)_\bullet$ (the $\bullet$ side of the Mackey functor $H_i(Y)$):

Proposition 4.22. *Let $k, n, r \geq 0$. The homology of the fundamental strands is nonzero only in the following cases:*

(a) $\mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^i H, A_k) = \mathbb{Z}/2$ *if* $0 \leq i \leq k$,
(b) $\mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^i H, H(n)) = \mathbb{Z}/2$ *if* $-n \leq i \leq 0$,
(c) $\mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^i H, H(-1)) = 0$ *for all* i,
(d) $\mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^i H, H(-n)) = \mathbb{Z}/2$ *if* $0 \leq i \leq n-2$ *(for* $n \geq 2$*)*,
(e) $\mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^i H, B_r) = \mathbb{Z}/2$ *if* $-(r+2) \leq i \leq -2$.

Proof. Immediate from Proposition 4.17. □

Using the previous results we can easily compute $\mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^i X, Y)$ for any two fundamental complexes X and Y. As one example, we compute $\mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^i A_2, A_5)$ for all i. We use the isomorphisms

$$\begin{aligned} \mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^i A_2, A_5) &\cong \mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^i H, \mathbb{F}(A_2, A_5)) \\ &\cong \mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^i H, \Sigma^{-2}A_2 \oplus \Sigma^3 A_2) \\ &\cong \mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^{i+2} H, A_2) \oplus \mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^{i-3} H, A_2) \\ &\cong \begin{cases} \mathbb{Z}/2 & \text{if } -2 \leq i \leq 0 \text{ or } 3 \leq i \leq 5, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The first isomorphism is just an adjunction, the second is by Proposition 4.20, and the fourth isomorphism is by Proposition 4.22.

The above technique allows one to compute $\mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^i X, Y)$ for any fundamental complexes X and Y. We state one specific result along these lines, which will be needed later:

Proposition 4.23. *One has $\mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^i A_k, B_r) = 0$ and $\mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^i B_r, A_k) = 0$ for $k, r \geq 0$ and all $i \in \mathbb{Z}$.*

Proof. This follows immediately from $\boldsymbol{\mathcal{F}}(A_k, B_r) \simeq 0 \simeq \boldsymbol{\mathcal{F}}(B_r, A_k)$, which is Proposition 4.20(b). □

4.24. Distinguishing complexes

The classification in 4.5 guarantees any perfect complex is quasi-isomorphic to a direct sum of fundamental complexes. It does not, however, guarantee there is a unique such direct sum. Our main goal in this subsection is to prove the following proposition, from which we can conclude every perfect complex is quasi-isomorphic to a direct sum of fundamental complexes in a unique way. This completes the classification of objects in $\mathcal{D}(\underline{\mathbb{Z}/2})_{\mathrm{perf}}$.

Proposition 4.25. *Let*

$$X = \bigoplus_{\alpha\in\mathcal{J}} \Sigma^{n_\alpha} A_\alpha \oplus \bigoplus_{\beta\in\mathcal{K}} \Sigma^{n_\beta} H(\beta) \oplus \bigoplus_{\gamma\in\mathcal{L}} \Sigma^{n_\gamma} B_\gamma$$

and similarly

$$Y = \bigoplus_{\alpha\in\mathcal{J}'} \Sigma^{n_\alpha} A_\alpha \oplus \bigoplus_{\beta\in\mathcal{K}'} \Sigma^{n_\beta} H(\beta) \oplus \bigoplus_{\gamma\in\mathcal{L}'} \Sigma^{n_\gamma} B_\gamma$$

where all the indexing sets are finite and allow for repetition. If $X \simeq Y$ then the sets $\mathcal{J}$ and $\mathcal{J}'$ are equal up to permutation, and likewise for $\mathcal{K}$, $\mathcal{K}'$ and $\mathcal{L}$, $\mathcal{L}'$. Moreover, the suspension factors for corresponding summands must be equal.

Note that we cannot prove this by simply looking at homology modules $H_*(-)$. For example, by Proposition 4.17 $H(1)$ and $H \oplus \Sigma B_0$ have isomorphic homology modules (H in degree 0 and $S_\bullet$ in degree -1), but we will see in this section that they are not quasi-isomorphic. In order to distinguish homotopy types in $\mathcal{D}(\underline{\mathbb{Z}/2})$ we therefore need to use a finer invariant. Recall the bigraded cohomology groups of a complex X given by

$$H^{p,q}(X) = \mathcal{D}(\underline{\mathbb{Z}/2})(X, \Sigma^p H(q))$$

and $H^{*,*}(X) = \bigoplus_{p,q} H^{p,q}(X)$. Recall also that $\mathbb{M}_2 = H^{*,*}(H)$. Note that the derived box product gives pairings

$$H^{p,q}(X) \otimes H^{r,s}(Y) \longrightarrow H^{p+r,q+s}(X \square Y)$$

where we are using the unique equivalence $H(q)\square H(s) \simeq H(q+s)$ in the derived category (such an equivalence exists by Proposition 4.18(iv), and is unique because $\mathcal{D}(\underline{\mathbb{Z}/2})(H(n), H(n)) \cong \mathbb{Z}/2$ for all n). In particular, $H^{*,*}(X)$ is an $\mathbb{M}_2$-bimodule, and one can readily check that the left and right module structures coincide.

Note that $\mathbb{M}_2^{p,q} = \mathcal{D}(\underline{\mathbb{Z}/2})(H, \Sigma^p H(q)) \cong H_{-p}(H(q))_\bullet$ and these groups are calculated by Proposition 4.22. This gives the picture of dots in Fig. 1. Let $\tau \in \mathbb{M}_2^{0,1}$, $\rho \in \mathbb{M}_2^{1,1}$, and $\theta \in \mathbb{M}_2^{0,-2}$ be the unique

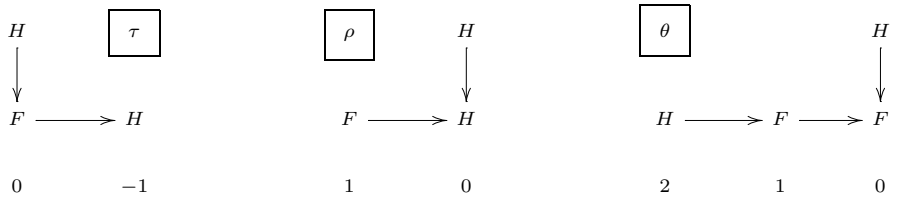


Fig. 6. The maps τ, ρ, and θ (complexes are drawn horizontally here).

nonzero classes. It is routine to check that these can be represented by the maps $H \to H(1)$, $H \to \Sigma H(1)$, and $H \to H(-2)$ shown in Fig. 6. By convention, unlabeled maps between H's and F's always denote the unique nonzero map, except in the case of maps $F \to F$ where an unlabeled map always denotes $1 + t$. To justify that the maps in Fig. 6 do represent the indicated classes, one only has to prove that the given maps are not chain homotopic to zero; this is routine.

Before giving the proof of Proposition 4.3 we need a simple lemma:

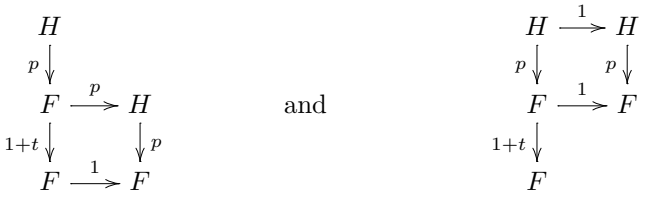
Lemma 4.26. *The cofiber of the map $\theta\colon H \to H(-2)$ is weakly equivalent to $H(-1) \oplus \Sigma H(-1)$, and under this weak equivalence the cofiber sequence is*

$$H \xrightarrow{\theta} H(-2) \xrightarrow{[\tau \quad \rho]} H(-1) \oplus \Sigma H(-1).$$

Proof. The cofiber of θ is the complex on the left of the diagram below (where the complexes are now drawn vertically):

Here we use the subscripts to denote basis elements. So $d(a) = p_*(b)$, $d(b) = (1+t)e$, and $d(c) = p_*(e)$. The change of basis $\{b, c\} \mapsto \{b + p^*(c), c\}$ gives the complex on the right, which is $H(-1) \oplus \Sigma H(-1)$.

Composing the inclusion of $H(-2)$ into the cofiber with the two projections for this direct sum, we get maps $H(-2) \to H(-1)$ and $H(-2) \to \Sigma H(-1)$. To see that these are τ and ρ we only need to check that they are nonzero in $\mathcal{D}(\underline{\mathbb{Z}/2})$, since there are unique nonzero maps in each case. The two compositions are



and in each case it is readily checked that null homotopies do not exist. □

Proof of Proposition 4.3 and Proposition 4.9. This is tedious but routine, and we only give a sketch. First note that the groups $H^{*,0}(F)$ are trivially computed, and then one gets a $(0,1)$-periodicity in $H^{*,*}(F)$ using that $F \square H(1) \simeq F$ (see Proposition 4.18). Similarly, $H^{*,0}(B_0)$ is readily computed, and then one gets a $(1,1)$-periodicity using that $B_0 \square H(1) \simeq \Sigma^{-1} B_0$.

Next use the cofiber sequence $H \xrightarrow{\rho} \Sigma^1 H(1) \longrightarrow \Sigma^1 F$ to deduce all the ρ-multiplications in $H^{*,*}(H)$. Similarly, the cofiber sequence $H \xrightarrow{\tau} H(1) \longrightarrow \Sigma B_0$ lets one deduce all the τ-multiplications in $H^{*,*}(H)$.

The cofiber sequences $A_{k-1} \hookrightarrow A_k \to \Sigma^k F$ and $F \hookrightarrow A_k \to \Sigma A_{k-1}$ allow one to inductively compute the $\mathbb{Z}/2[\tau,\rho]$-module structure on $H^{*,*}(A_k)$. Similarly, the cofiber sequences $B_{r-1} \to B_r \to \Sigma^{-r} B_0$ and $B_0 \to B_r \to \Sigma^{-1} B_{r-1}$ lead to the inductive computation of the $\mathbb{Z}/2[\tau,\rho]$-module structure on $H^{*,*}(B_r)$. Note also that the computations can be simplified by using the equivalences $A_k \square H(1) \simeq A_k$ and $B_r \square H(1) \simeq \Sigma^{-1} B_r$ from Proposition 4.18, which yield periodicities in the module structures.

Finally, by using Lemma 4.26 one readily computes all of the θ-multiplications in $\mathbb{M}_2 = H^{*,*}(H)$ (they are all zero, except for $1 \cdot \theta = \theta$). The rest of the ring structure can be deduced from simple algebraic arguments. (For example: $\frac{\theta}{\tau} \cdot \frac{\theta}{\rho}$ is a class that when multiplied by $\tau\rho$ gives θ^2—which is zero—and by our analysis of all the τ- and ρ-multiplications we know this forces the class to be zero.) Similarly, such algebraic arguments also readily yield the full $\mathbb{M}_2$-module structures on $H^{*,*}(A_k)$ and $H^{*,*}(B_r)$.

Only a bit more work is required to compute $H^{*,*}(A_\infty)$ and $H^{*,*}(B_\infty)$. For the first, the filtration by A_k's shows that $A_n \hookrightarrow A_\infty$ induces isomorphisms on $H^{p,*}$ for $p \leq n$ and the desired result readily follows from this. For the second, one first uses the cofiber sequence $H \to B_\infty \to \Sigma A_\infty$ to calculate all of the cohomology groups, but there is one set of unresolved extension problems. Then one considers the map $\Sigma B_\infty \to B_\infty$ that is $p : H \to F$ in degree one and the identity in higher degrees. The cofiber is readily seen to be quasi-isomorphic to $\Sigma^2 B_0$, and this cofiber sequence then resolves those extensions. □

Say that a bigraded $\mathbb{M}_2$-module is **perfect** if it is isomorphic to $H^{*,*}(X)$ for some perfect complex X. By Theorem 4.5 this is equivalent to saying that the module is a finite direct sum of bigraded shifts $\Sigma^{p,q}\mathbb{M}_2$ as well as shifts of $H^{*,*}(A_k)$ and $H^{*,*}(B_r)$ for various values of k and r. It is not *a priori* true that the constituent pieces of such a direct sum are uniquely determined, but they are:

Proposition 4.27. *Let M be a perfect $\mathbb{M}_2$-module. Then there exist unique integers (up to permutation) p_i, q_i, s_j, and t_r together with $n_j \geq 0$ and $k_r \geq 0$ such that*

$$M \cong \bigoplus_i \Sigma^{p_i,q_i}\mathbb{M}_2 \oplus \bigoplus_j \Sigma^{s_j} H^{*,*}(A_{n_j}) \oplus \bigoplus_r \Sigma^{t_r} H^{*,*}(B_{k_r}).$$

Proof. We know existence, so the only thing to be proven is uniqueness. Observe that $\mathrm{Ann}_M(\tau,\rho)$ is a finite-dimensional bigraded vector space over $\mathbb{Z}/2$ whose homogeneous basis is in bidegrees $(p_i - 2, q_i - 2)$, so this gives uniqueness of the p's and q's.

The operation $M \mapsto M[\rho^{-1}]$ kills the $H^{*,*}(A)$-summands and does nothing to the $H^{*,*}(B)$-summands. The construction

$$\mathrm{Ann}_{M[\rho^{-1}]}(\tau^\infty) = \{x \in M[\rho^{-1}] \,\big|\, \tau^n x = 0 \text{ for some } n > 0\}$$

exactly isolates the $H^{*,*}(B)$-summands of M. This construction is a module over $\mathbb{M}_2[\rho^{-1}] \cong \mathbb{Z}/2[\rho,\rho^{-1}][\tau]$ which is a graded PID, so the uniqueness of the t_r and k_r follow from the usual classification of modules over a PID.

Finally, the operation $M \mapsto M[\tau^{-1}]$ kills the $H^{*,*}(B)$-summands, and the construction $\mathrm{Ann}_{M[\tau^{-1}]}(\rho^\infty)$ exactly isolates the $H^{*,*}(A)$-summands. The analogous argument to the preceding paragraph shows that the s_j and n_j are uniquely determined, since $\mathbb{M}_2[\tau^{-1}] \cong \mathbb{Z}/2[\tau,\tau^{-1}][\rho]$ is again a graded PID. □

Proposition 4.25 is really just a corollary of the above:

Proof of Proposition 4.25. Let X and Y be as in the statement of the proposition and assume $X \simeq Y$. Let $M = H^{*,*}(X)$ and $N = H^{*,*}(Y)$, so that $M \cong N$ as bigraded $\mathbb{M}_2$-modules. By Proposition 4.27 it

follows that M and N have the same constituent summands, and these exactly correspond to the constituent summands of X and Y. $\square$

We also call attention to the following useful consequence:

Corollary 4.28. *Let X and Y be perfect complexes of $\underline{\mathbb{Z}/2}$-modules. Then $X \simeq Y$ if and only if $H^{*,*}(X) \cong H^{*,*}(Y)$ as bigraded $\mathbb{M}_2$-modules.*

Proof. Immediate from Proposition 4.27 and Theorem 4.5. $\square$

5. Algebraic consequences of the classification theorem

In this section we compute the Picard group of $\mathcal{D}(\underline{\mathbb{Z}/2})$ as well as the Balmer spectrum for $\mathcal{D}(\underline{\mathbb{Z}/2})_{\mathrm{perf}}$, deducing both as consequences of our work in Section 4.

5.1. The Picard group of $\mathcal{D}(\underline{\mathbb{Z}/2})$

Theorem 5.2. *The Picard group of $\mathcal{D}(\underline{\mathbb{Z}/2})$ is $\mathbb{Z} \oplus \mathbb{Z}$, with generators ΣH and $H(1)$.*

Proof. First note that an invertible object in $\mathcal{D}(\underline{\mathbb{Z}/2})$ is necessarily compact, hence perfect. To see this, observe that if X is invertible with inverse Y and $\{Z_\alpha\}$ is any collection of objects then there is a commutative diagram

$$\begin{array}{ccc} \bigoplus_\alpha \mathcal{D}(\underline{\mathbb{Z}/2})(X, Z_\alpha) & \longrightarrow & \mathcal{D}(\underline{\mathbb{Z}/2})(X, \bigoplus_\alpha Z_\alpha) \\ \cong \downarrow & & \downarrow \cong \\ \bigoplus_\alpha \mathcal{D}(\underline{\mathbb{Z}/2})(X\square Y, Z_\alpha \square Y) & \longrightarrow & \mathcal{D}(\underline{\mathbb{Z}/2})(X\square Y, (\bigoplus_\alpha Z_\alpha)\square Y) \end{array}$$

Now use the fact that $(-)\square Y$ commutes with direct sums, together with the fact that $X\square Y \simeq H$ is compact, to see that the bottom horizontal map is an isomorphism. Thus, the top horizontal map is an isomorphism as well.

Since X is compact, we know by Corollary 4.6 that X is quasi-isomorphic to a direct sum of shifts of complexes of type A_k, B_r, and $H(n)$. But it is easy to see that the direct sum must only involve *one* term. For suppose $X \simeq J \oplus K$, and let Y be the inverse of X. Then

$$H \simeq X\square Y \simeq (J\square Y) \oplus (K\square Y).$$

By taking homology and using that H is indecomposable as a $\underline{\mathbb{Z}/2}$-module, this can only happen if either $J\square Y \simeq 0$ or $K\square Y \simeq 0$. Without loss of generality, we assume the former. Then box with X to get $0 \simeq (J\square Y)\square X \simeq J$. This proves that a nontrivial direct sum can never be invertible.

The classification theorem then implies that the only possible invertible objects are suspensions of A_k, B_r, and $H(n)$. If A_k has an inverse W, then take $A_k \square B_r \simeq 0$ and box with W to get $B_r \simeq 0$; this is a contradiction (by Proposition 4.22(d), for example). So A_k is not invertible, and the same proof shows that B_r is not invertible. Of course we know that the $H(n)$ are invertible by Proposition 4.18(iv).

Consider the group homomorphism $\mathbb{Z}^2 \to \mathrm{Pic}(\mathcal{D}(\underline{\mathbb{Z}/2}))$ sending (m, n) to $\Sigma^m H \square H(n) \simeq \Sigma^m H(n)$. The first generator of $\mathbb{Z}^2$ maps to ΣH and the second to $H(1)$. We have just proven that this map is surjective. For injectivity just note that the homology calculations of Proposition 4.22(a,b) show that $\Sigma^m H(n) \simeq H$ can only happen if $m = n = 0$. $\square$

5.3. The Balmer spectrum of $\mathcal{D}(\underline{\mathbb{Z}/2})_{\mathrm{perf}}$

Recall that a tensor-triangulated category $(\mathcal{C}, \otimes)$ has an associated topological space $\operatorname{Spec}\mathcal{C}$ called the **Balmer spectrum** of $\mathcal{C}$ [1]. The elements of $\operatorname{Spec}\mathcal{C}$ are the primes of $\mathcal{C}$, i.e. proper thick subcategories $\mathcal{I}$ of $\mathcal{C}$ having the properties that

(i) If $X \in \mathcal{I}$ and $Y \in \mathcal{C}$ then $X \otimes Y \in \mathcal{I}$;
(ii) If $X, Y \in \mathcal{C}$ and $X \otimes Y \in \mathcal{I}$ then either $X \in \mathcal{I}$ or $Y \in \mathcal{I}$.

Recall that a subcategory is thick if it is full and closed under formations of suspensions and desuspensions, retracts, and extensions. Thick subcategories satisfying (i) are called **tensor ideals**.

The topology on $\operatorname{Spec}\mathcal{C}$ is an analog of the Zariski topology from algebraic geometry. For $X \in \mathcal{C}$ define $\operatorname{Supp} X = \{\mathcal{P} \in \operatorname{Spec}\mathcal{C} \mid X \notin \mathcal{P}\}$. Then $\{\operatorname{Supp} X \mid X \in \mathcal{C}\}$ is a basis for the closed sets in $\operatorname{Spec}\mathcal{C}$.

A convenient source of tensor ideals is via annihilators. If $\mathcal{S}$ is a set of objects in $\mathcal{C}$, let

$$\operatorname{Ann}\mathcal{S} = \{X \in \mathcal{C} \mid X \otimes Y \simeq 0 \text{ for all } Y \in \mathcal{S}\}.$$

Then $\operatorname{Ann}\mathcal{S}$ is a tensor ideal.

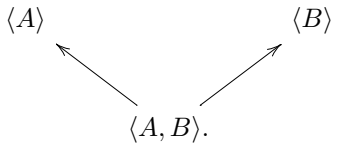
One more piece of notation: is $\mathcal{S}$ is a set of objects in $\mathcal{C}$, write $\overline{\Sigma}\mathcal{S}$ for the closure of $\mathcal{S}$ under suspension and desuspension.

By Proposition 4.1 the following result can also be interpreted as computing the Balmer spectrum for the triangulated category $(K_{b,fg}(\mathbb{Z}/2[C_2]), \otimes)$. In that context, the computation was independently done by Balmer and Gallauer [2].

Theorem 5.4. *There are only three prime ideals in* $(\mathcal{D}(\underline{\mathbb{Z}/2})_{\mathrm{perf}}, \square)$*:*

- *the full subcategory* $\langle A\rangle$ *whose objects are the finite direct sums made from the set* $\overline{\Sigma}\{A_k \mid k \geq 0\}$,
- *the full subcategory* $\langle B\rangle$ *whose objects are the finite direct sums made from the set* $\overline{\Sigma}\{B_r \mid r \geq 0\}$,
- *the full subcategory* $\langle A, B\rangle$ *whose objects are the finite direct sums made from the set* $\overline{\Sigma}\{A_k, B_r \mid k, r \geq 0\}$.

The first two are closed points of the Balmer spectrum, whereas the closure of the third point is the whole space. This is depicted via the diagram



Proof. First note that if a tensor ideal contains $H(n)$ then it also contains H, using that $H(n)$ is invertible. Therefore it contains every object, and so is not a proper ideal.

We make the following observations:

- Since $A_k \square B_r \simeq 0$ by Proposition 4.18(iii), any prime ideal must contain either A_k or B_r.
- By Proposition 4.18(i), if a tensor ideal contains A_k then it contains all A_i for $i \leq k$.
- Using the cofiber sequences $A_0 \to A_{n+1} \to \Sigma A_n$ and induction, any tensor ideal containing A_0 must contain all other A_n.
- By Proposition 4.18(vi), if a tensor ideal contains B_r then it contains all B_i for $i \leq r$.
- Using the cofiber sequences $B_0 \to B_{n+1} \to \Sigma^{-1} B_n$ and induction, any tensor ideal containing B_0 contains all other B_n.

It follows at once from Theorem 4.5 that the only possible prime ideals are $\langle A\rangle$, $\langle B\rangle$, and $\langle A, B\rangle$.

We must next check that each of these really is a prime ideal. Using Theorem 4.5 and Proposition 4.18 it follows immediately that $\langle A\rangle = \operatorname{Ann}\{B_r\}$, and so is a tensor ideal. Suppose X and Y are perfect complexes and $X \square Y \in \langle A\rangle$. By Theorem 4.5 we can write

$$X \simeq \bigoplus_{\alpha\in\mathcal{J}} A_\alpha \oplus \bigoplus_{\beta\in\mathcal{K}} H(\beta) \oplus \bigoplus_{\gamma\in\mathcal{L}} B_\gamma$$

where the formula should also have various suspensions on all the factors, which we have omitted to write. Similarly, we can write

$$Y \simeq \bigoplus_{\alpha'\in\mathcal{J}'} A_{\alpha'} \oplus \bigoplus_{\beta'\in\mathcal{K}'} H(\beta') \oplus \bigoplus_{\gamma'\in\mathcal{L}'} B_{\gamma'}.$$

Since $X \square Y \in \langle A\rangle$ we conclude immediately that either $\mathcal{K}$ or $\mathcal{K}'$ is empty; without loss of generality we assume $\mathcal{K} = \emptyset$.

If $\mathcal{K}' \neq \emptyset$ then we must have $\mathcal{L} = \emptyset$, else we have a B-type summand in $X \square Y$. But this yields $X \in \langle A\rangle$. So now assume $\mathcal{K}' = \emptyset$. If $\mathcal{L} \neq \emptyset$ and $\mathcal{L}' \neq \emptyset$ then we again get a B-type summand in $X \square Y$; so either $\mathcal{L} = \emptyset$ or $\mathcal{L}' = \emptyset$. But this precisely says that either $X \in \langle A\rangle$ or $Y \in \langle A\rangle$. This completes the proof that $\langle A\rangle$ is prime.

The same style of argument shows that $\langle B\rangle$ is prime.

It remains to prove that $\langle A, B\rangle$ is a prime ideal. This is largely similar to what we have already done, except the proof that the subcategory is closed under extensions. For this, assume that

$$X = \bigoplus_{\alpha\in\mathcal{J}} A_\alpha \oplus \bigoplus_{\beta\in\mathcal{K}} B_\beta, \qquad Y = \bigoplus_{\alpha'\in\mathcal{J}'} A_{\alpha'} \oplus \bigoplus_{\beta'\in\mathcal{K}'} B_{\beta'}$$

(again, with suspensions on all summands omitted for brevity). We must check that for any map $f\colon X \to Y$ the cofiber $\operatorname{Cof}(f)$ is still in $\langle A, B\rangle$. Here we use Proposition 4.23 to see that there are no maps from the A_k to the B_r and vice versa, so that our map f must split as $f_1 \oplus f_2$ where $f_1\colon \bigoplus_\alpha A_\alpha \to \bigoplus_{\alpha'} A_{\alpha'}$ and $f_2\colon \bigoplus_\beta B_\beta \to \bigoplus_{\beta'} B_{\beta'}$. Since we have already proven $\langle A\rangle$ and $\langle B\rangle$ are thick, we have $\operatorname{Cof}(f_1) \in \langle A\rangle$ and $\operatorname{Cof}(f_2) \in \langle B\rangle$. Since $\operatorname{Cof}(f) = \operatorname{Cof}(f_1) \oplus \operatorname{Cof}(f_2)$, we are done.

Finally, it remains to investigate the topology on $\operatorname{Spec} \mathcal{D}(\underline{\mathbb{Z}/2})$. For $k \geq 0$ one clearly has that $\operatorname{Supp}(B_k) = \{\langle A\rangle\}$ and $\operatorname{Supp}(A_k) = \{\langle B\rangle\}$, using Proposition 4.18. These generate the closed sets of $\operatorname{Spec} \mathcal{D}(\underline{\mathbb{Z}/2})$, so the topology is as described in the statement of the theorem. □

For good measure we also determine the Balmer spectrum of the category $\mathcal{D}(\underline{\mathbb{Z}/2})_{bb,fg}$ consisting of complexes which are bounded below and have finitely-generated projective modules in each degree. Let $\langle A, A_\infty\rangle$ denote the full subcategory whose objects are the direct sums made from the set $\overline{\Sigma}\{A_k \mid 0 \leq k \leq \infty\}$, with arbitrary indexing sets but where the direct sum must be bounded below and finitely-generated in each degree. Similarly, define the full subcategories $\langle B, B_\infty\rangle$, $\langle A, B, A_\infty\rangle$, and $\langle A, B, B_\infty\rangle$.

Theorem 5.5. *The Balmer spectrum for* $(\mathcal{D}(\underline{\mathbb{Z}/2})_{bb,fg}, \square)$ *consists of exactly four points, separated into two connected components as depicted here:*

The prime ideals in the top row are closed points of the Balmer spectrum, whereas

$$\overline{\langle A, B, A_\infty\rangle} = \{\langle A, B, A_\infty\rangle, \langle A, A_\infty\rangle\} \text{ and } \overline{\langle A, B, B_\infty\rangle} = \{\langle A, B, B_\infty\rangle, \langle B, B_\infty\rangle\}.$$

The inclusion of categories $\mathcal{D}(\underline{\mathbb{Z}/2})_{\mathrm{perf}} \hookrightarrow \mathcal{D}(\underline{\mathbb{Z}/2})_{bb,fg}$ *induces a map of the corresponding Balmer spectra in the other direction: this is the quotient map which sends* $\langle A, B, A_\infty\rangle$ *and* $\langle A, B, B_\infty\rangle$ *to the same point, namely* $\langle A, B\rangle$.

Proof. The cofiber sequence $H \to B_\infty \to \Sigma A_\infty$ shows that if a tensor ideal contains both A_∞ and B_∞ then it contains everything. So no prime ideal contains both A_∞ and B_∞, and since $A_\infty \square B_\infty \simeq 0$ it follows that every prime ideal must contain exactly one of A_∞ or B_∞.

If P is a prime ideal containing A_∞, then using $A_k \square A_\infty \simeq A_k$ it must also contain all the A_k. If it contains any B_r then just as in the proof of Theorem 5.4 it must contain all of the B_r. Since P cannot contain any of the invertible objects $H(n)$, this shows that P is either $\langle A, A_\infty\rangle$ or $\langle A, B, A_\infty\rangle$.

Similar reasoning for the case where P contains B_∞ shows that the only possible prime ideals are the four from the statement of the theorem. It remains to check that these are indeed prime.

One readily checks using Proposition 4.18 and Theorem 4.10 that $\mathrm{Ann}(B_\infty) = \langle A, A_\infty\rangle$, so this is a tensor ideal. Primality also readily follows from those two results. Similarly, $\mathrm{Ann}(A_\infty) = \langle B, B_\infty\rangle$ is a prime ideal.

Similar considerations show that $\langle A, B, A_\infty\rangle$ is a prime ideal, but here one must work a bit harder to check that it gives a triangulated subcategory. If X is a direct sum of shifts of copies of A_k, B_r, and A_∞, write $X(A)$ for the direct sum of the pieces of A-type, and $X(B)$ for the direct sum of pieces of B-type. If S is a single strand, of type A or B, we claim that any map $S \to X$ in $\mathcal{D}(\underline{\mathbb{Z}/2})$ must factor through $X(A)$ or $X(B)$, respectively. This is an easy computation. This claim then yields that $\langle A, B, A_\infty\rangle$ is triangulated by the argument from the proof of Theorem 5.4.

The topology on the Balmer spectrum is readily identified using Proposition 4.18, since one easily computes the support of any object. The computation of the map $\operatorname{Spec}\mathcal{D}(\underline{\mathbb{Z}/2})_{bb,fg} \to \operatorname{Spec}\mathcal{D}(\underline{\mathbb{Z}/2})_{\mathrm{perf}}$ is immediate. □

6. Topological consequences of the main theorem

In this section we explain how our description of $\mathcal{D}(\underline{\mathbb{Z}/2})$ leads to various topological results about equivariant $H\underline{\mathbb{Z}/2}$-modules and bigraded Bredon cohomology with coefficients in $\underline{\mathbb{Z}/2}$.

Let R be a Mackey ring. There is an associated equivariant Eilenberg–MacLane ring spectrum HR. As explained in [17, Corollary 5.2] for the special case of $R = \underline{\mathbb{Z}}$ (though it works in general), the general theory developed by Schwede–Shipley in [15] gives a Quillen equivalence between the algebraic model category of $Ch(R)$ and the topological category of HR-modules:

$$Ch(R) \underset{\Psi}{\overset{\Gamma}{\rightleftarrows}} HR - \mathrm{Mod}. \qquad (6.1)$$

In particular, the homotopy category of $Ch(\underline{\mathbb{Z}/2})$ is equivalent to the homotopy category of $H\underline{\mathbb{Z}/2} - \mathrm{Mod}$. The Quillen equivalence is set up so that $H\underline{\mathbb{Z}/2}$ and $H\underline{\mathbb{Z}/2} \wedge (C_2)_+$ correspond to H and F respectively. By examining cell structures for $S^{p,q}$ one finds that $H\underline{\mathbb{Z}/2} \wedge S^{p,q}$ corresponds to $\Sigma^p H(q)$ under the Quillen equivalence. It follows that if M is an $H\underline{\mathbb{Z}/2}$-module then its bigraded cohomology $\mathrm{Hom}_{H\underline{\mathbb{Z}/2}}(M, \Sigma^{*,*} H\underline{\mathbb{Z}/2})$, as an $\mathbb{M}_2$-module, is isomorphic to the bigraded cohomology of $\Psi(M)$. Here we may use either Mackey functor valued cohomology or consider only the $\bullet$ side. Since compact objects in $\mathcal{D}(\underline{\mathbb{Z}/2})$ are determined by

their bigraded cohomology (see Corollary 4.28), this implies that when M is a finite $H\underline{\mathbb{Z}/2}$-module $\Psi(M)$ is completely determined by the $\mathbb{M}_2$-module $\mathrm{Hom}_{H\underline{\mathbb{Z}/2}}(M, \Sigma^{*,*}H\underline{\mathbb{Z}/2})$.

If S^k_a denotes a k-sphere with the antipodal C_2-action, then one readily finds that $H\underline{\mathbb{Z}/2} \wedge (S^k_a)_+$ corresponds to the fundamental complex A_k: this can be done either by an analysis of explicit cell structure on S^k_a, or by computing the bigraded homology of S^k_a, and recognizing it as that of A_k.

It remains to identify the $H\underline{\mathbb{Z}/2}$-module corresponding to B_r. For this, recall that there is a unique nonzero homotopy class $\tau\colon H\underline{\mathbb{Z}/2} \to \Sigma^{0,1}H\underline{\mathbb{Z}/2}$. Write $\mathrm{Cof}(\tau)$ for the homotopy cofiber of this map, and $\mathrm{Cof}(\tau^r)$ for the homotopy cofiber of $\tau^r\colon H\underline{\mathbb{Z}/2} \to \Sigma^{0,r}H\underline{\mathbb{Z}/2}$. One readily computes the bigraded cohomology and observes that it exactly matches the cohomology of B_{r-1}.

In light of the Quillen equivalence from (6.1) we can thus reinterpret Corollary 4.6 as follows:

Theorem 6.2. *Let M be a finite $H\underline{\mathbb{Z}/2}$-module. Then up to weak equivalence M splits as a wedge of bigraded suspensions of $H\underline{\mathbb{Z}/2}$, $H\underline{\mathbb{Z}/2} \wedge (S^k_a)_+$, and $\mathrm{Cof}(\tau^r)$, for various $k \geq 0$ and $r \geq 1$.*

Remark 6.3. We have not proven that the Quillen equivalence from (6.1) is symmetric monoidal, but it is. There is a folklore proof using ∞-categorical techniques, and a proof is forthcoming in work in progress by Drew Heard and the third author. Therefore the results of Section 5.3 can be reinterpreted as computing the Balmer spectra for various homotopy categories of $H\underline{\mathbb{Z}/2}$-modules.

6.4. Structure theorem for C_2-spaces

The classification of finite $H\underline{\mathbb{Z}/2}$-modules in Theorem 6.2 implies the following structure theorem for $RO(C_2)$-graded cohomology of C_2-spaces from [13].

Theorem 6.5 *(C. May). Let X be a pointed finite C_2-CW complex. Then $H\underline{\mathbb{Z}/2} \wedge X$ splits as a wedge of bigraded suspensions of $H\underline{\mathbb{Z}/2}$ and $H\underline{\mathbb{Z}/2} \wedge (S^k_a)_+$ for various $k \geq 0$.*

Proof. From Theorem 6.2 we know that $H\underline{\mathbb{Z}/2} \wedge X$ splits as a wedge of suspensions of $H\underline{\mathbb{Z}/2}$, $H\underline{\mathbb{Z}/2} \wedge (S^k_a)_+$, and $\mathrm{Cof}(\tau^r)$, for various k and r. So it remains to show there cannot be any summands of the form $\mathrm{Cof}(\tau^r)$. Recall from Lemma 4.3 of [13] that ρ-localization of the cohomology of a finite C_2-CW complex is

$$\rho^{-1}H^{*,*}(X) \cong \rho^{-1}H^{*,*}(X^{C_2}) \cong H^*_{sing}(X^{C_2}) \otimes \rho^{-1}\mathbb{M}_2.$$

Notice $\rho^{-1}\mathbb{M}_2$ does not have any τ-torsion and thus neither does

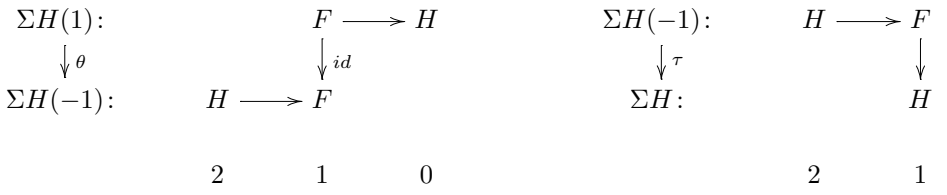
$$\rho^{-1}\tilde{H}^{*,*}(X) \cong \rho^{-1}\,\mathrm{Hom}_{H\underline{\mathbb{Z}/2}}(H\underline{\mathbb{Z}/2} \wedge X, \Sigma^{*,*}H\underline{\mathbb{Z}/2}),$$

since it is a free $\rho^{-1}\mathbb{M}_2$-module. However, ρ-localization preserves the cohomology of $\mathrm{Cof}(\tau^r)$, which has τ-torsion by construction. Thus $H\underline{\mathbb{Z}/2} \wedge X$ cannot have any wedge summands of the form $\mathrm{Cof}(\tau^r)$ for any r. □

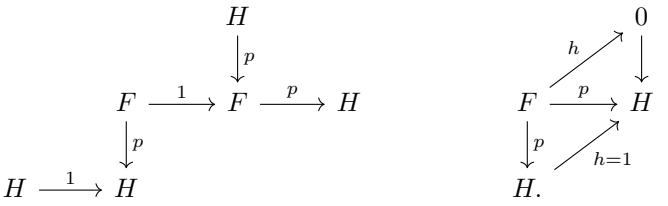
6.6. Toda bracket decomposition of 1

A key piece of the proof of the structure theorem for $RO(C_2)$-graded cohomology of C_2-spaces from [13] is the Toda bracket decomposition of 1 in $\mathbb{M}_2$ given by $\langle \tau, \theta, \rho \rangle = 1$ with zero indeterminacy. This Toda bracket can be witnessed in $Ch(\underline{\mathbb{Z}/2})$ as follows.

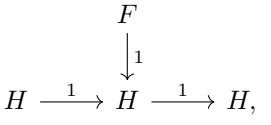
Recall the maps representing ρ, θ, and τ from Fig. 6. For our present purposes it will be better to represent θ as a map $\Sigma H(1) \to \Sigma H(-1)$ and τ as a map $\Sigma H(-1) \to \Sigma H$, namely the following (with complexes drawn horizontally):



The left diagram below depicts the composition $\tau \circ \theta \circ \rho$, where we now switch to drawing complexes vertically:



To construct an element of the Toda bracket $\langle \tau, \theta, \rho \rangle$ we start by choosing null-homotopies for $\tau\theta$ and $\theta\rho$. But $\theta \circ \rho = 0$ on the nose, so we can use the zero null-homotopy there. The second composition $\tau \circ \theta$ is not zero in degree 1, but it is null-homotopic via the null homotopy h shown in the right diagram above. Finally, forming $h \circ \rho$ gives



and the composite is the identity $H \to H$. Thus $1 \in \langle \tau, \theta, \rho \rangle$. The indeterminacy of the Toda bracket is zero since $\mathbb{M}_2^{0,0}$ contains no τ or ρ-multiples for degree reasons. So we may conclude the bracket identity $\langle \tau, \theta, \rho \rangle = 1$.

Remark 6.7. The splitting for finite $H\underline{\mathbb{Z}/2}$-modules in Theorem 6.2 is stated here in parallel with Theorem 6.5. In this form, there appears to be a lack of symmetry of the fundamental objects. However, we may reinterpret $H\underline{\mathbb{Z}/2} \wedge (S^k_a)_+$ as a desuspension of $\mathrm{Cof}(\rho^{k+1})$, where $\rho\colon H\underline{\mathbb{Z}/2} \to \Sigma^{1,1} H\underline{\mathbb{Z}/2}$. Thus Theorem 6.2 can be restated as: All finite $H\underline{\mathbb{Z}/2}$-modules split as a wedge of suspensions of $H\underline{\mathbb{Z}/2}$, $\mathrm{Cof}(\rho^k)$, and $\mathrm{Cof}(\tau^r)$ for various $k, r \geq 1$.

6.8. Classification of finite $H\underline{\mathbb{Z}/\ell}$-modules

Here we collect some of the analogous results for odd primes. In particular, we also obtain a classification of finite $H\underline{\mathbb{Z}/\ell}$-modules for ℓ an odd prime. The splitting theorem for $\mathcal{D}(\underline{\mathbb{Z}/\ell})_{\mathrm{perf}}$ and the computation of the Balmer spectrum follow immediately from Proposition 3.5.

Proposition 6.9. *Let ℓ be an odd prime. Every perfect complex of $\underline{\mathbb{Z}/\ell}$-modules decomposes as a direct sum of shifts of $0 \longrightarrow H \longrightarrow 0$, $0 \longrightarrow S_\Theta \longrightarrow 0$, and the contractible complexes $\mathbb{D}(H)$ and $\mathbb{D}(S_\Theta)$.*

Corollary 6.10. *Let ℓ be an odd prime. The Balmer spectrum of $\mathcal{D}(\underline{\mathbb{Z}/\ell})_{\mathrm{perf}}$ is a single point, the zero ideal.*

Proof. This follows immediately from Proposition 6.9 once one knows that S_Θ is invertible, which is true by Proposition 3.6. □

We also get a splitting at the spectrum level.

Proposition 6.11. *Let ℓ be an odd prime and M be a finite $H\underline{\mathbb{Z}/\ell}$-module. Then up to weak equivalence M splits as a wedge of bigraded suspensions of $H\underline{\mathbb{Z}/\ell}$.*

Proof. This follows from Proposition 6.9 and the observation that $0 \to S_\Theta \to 0$ corresponds under the Quillen equivalence to $\Sigma^{0,1}H\underline{\mathbb{Z}/\ell}$. To see the latter claim, note that ΣS_Θ is quasi-isomorphic to the complex $F \to H$ (concentrated in degrees 0 and 1), which is $H\underline{\mathbb{Z}/\ell} \wedge S^{1,1}$ under the Quillen equivalence.

Alternatively, one may compute the bigraded cohomology on the $\bullet$ side of $H\underline{\mathbb{Z}/\ell}$. The result is a graded field $\mathbb{Z}/\ell[x, x^{-1}]$ with x in degree $(0, 2)$. □

The computation of the Balmer spectrum in this context is then immediate, as every nonzero $H\underline{\mathbb{Z}/\ell}$-module is invertible.

Corollary 6.12. *Let ℓ be an odd prime. The Balmer spectrum of compact objects in $H\underline{\mathbb{Z}/\ell}$–Mod is a single point, the zero ideal.*

7. Proof of the classification theorem for chain complexes

In this section we will prove Theorem 4.5. That is, we will show that any perfect complex in $Ch(\underline{\mathbb{Z}/2})$ is isomorphic to a direct sum of "strands". Recall from Section 4 that complete lists of strands are the fundamental complexes A_k, B_r, and $H(n)$, for various values of $k, r \geq 0$ and $n \in \mathbb{Z}$, together with the contractible disk complexes $\mathbb{D}(H)$ and $\mathbb{D}(F)$.

Our proof of the splitting decomposition involves changing the basis at various levels in the complex in an algorithmic fashion. We work our way from the bottom of the complex up, but at each stage the algorithm involves possibly changing the basis at some or all of the lower levels. So although the proof is constructive, the algorithm can be time-intensive to implement in practice (at least by hand), and there is no simple way in general to look at a complex and know what strands will come out at the end. This differs from Proposition 3.8, for example, where the algorithm in that proof yielded formulas for the number of summands based on properties of the given H-module.

7.1. Proof outline

Our approach is roughly to proceed by induction on the length of the complex. Suppose the complex is nonzero only in degrees 0 through m. For the inductive step, we assume the portion of the complex in degrees 0 to $m-1$ has been split into strands. We consider summands of H and F in the top degree m mapping via the differential to the strands below. Then, through a series of steps, we split off all subcomplexes having top degree m.

We begin by considering any isomorphisms at the top and use these to split off copies of the contractible complexes $\mathbb{D}(H)$ and $\mathbb{D}(F)$. Next we consider the case where copies of H in degree m map nontrivially to various strands below. We use these maps to split off copies of B_r. Then we split off any summands of $H(-n)$ for $n > 0$. Turning to the case where we have copies of F in degree m mapping to strands below, we split off any summands of the form $H(n)$. Finally, we split off summands of A_k. All other summands of H and F in degree m support trivial maps and thus split off. Once we have dealt with all the terms in degree m we will be done, since by induction the rest of the complex in lower degrees was already split.

At each stage of the proof, we choose a particular type of strand mapped to by a particular copy of either H or F in degree m. We change the basis of the complex at level m so no other summands in degree m hit the chosen strand. Then we change the basis of each term in the chosen strand to form a new strand that

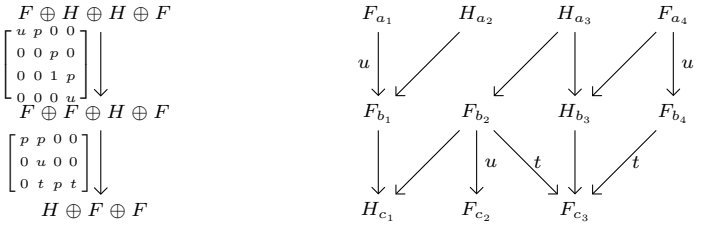


Fig. 7. An example using matrix notation and using basis labels.

splits off from the rest. The general rule for the choice of strand at each stage is this: if we are splitting off a type of strand that ends in an H then we choose the *shortest* strand of that type, whereas if we are splitting off a type of strand that ends in an F then we choose the *longest* strand of that type. As the reader will see, these choices guarantee that the evident change of basis does what we need in order to split off.

Recall the notion of a basis for a free $\underline{\mathbb{Z}/2}$-module from Definition 2.10. The following lemma states that certain adjustments (analogous to the usual column operations of linear algebra) give a change of basis. Notice we can use p^* and p_* to mix basis elements from the $\bullet$ and Θ sides.

Lemma 7.2. *Let M be a free $\underline{\mathbb{Z}/2}$-module with $\gamma = \{b_1^\Theta, \dots, b_m^\Theta, b_{m+1}^\bullet, \dots, b_{m+n}^\bullet\}$ a chosen basis M. For a fixed choice of i and j with $i \neq j$, each of the following modifications to γ yields a new basis:*

(i) Replace b_i^Θ with $\tilde{b}_i^\Theta = b_i^\Theta + b_j^\Theta$.
(ii) Replace b_i^Θ with $\tilde{b}_i^\Theta = b_i^\Theta + tb_j^\Theta$.
(iii) Replace b_i^Θ with $\tilde{b}_i^\Theta = b_i^\Theta + (1+t)b_j^\Theta$.
(iv) Replace b_i^Θ with $\tilde{b}_i^\Theta = b_i^\Theta + p^* b_j^\bullet$.
(v) Replace $b_i^\bullet$ with $\tilde{b}_i^\bullet = b_i^\bullet + b_j^\bullet$.
(vi) Replace $b_i^\bullet$ with $\tilde{b}_i^\bullet = b_i^\bullet + p_* b_j^\Theta$.

Proof. By inspection. □

In the following proof we draw chain complexes vertically, with basis elements appearing as subscripts (so that F_a is a copy of F with basis a^Θ for example). To simplify notation, we omit the superscripts Θ and $\bullet$ when it is clear from context where each basis element lives. Our convention is to only draw arrows for nonzero maps. We also omit labels of maps whenever there is only one possible nonzero map, such as $H_a \to H_b$. For convenience, we denote the map $1 + t\colon F \to F$ by u. This map appears frequently in chain complexes and so whenever we omit the label of a map $F \to F$, we mean that it is u.

An example of a complex with this notation is given in Fig. 7 on the right. On the left in Fig. 7, the same complex is drawn using matrix notation (where matrices act on the left). In the main proof, we will refer to individual arrows. For example, in Fig. 7 there is a map $u\colon F_{a_1} \to F_{b_1}$ from the top level to the level below.

We now give the proof of our main splitting theorem for complexes.

Proof of Theorem 4.5. Let C be a bounded complex that has a finite direct sum of copies of H and F in each degree. We aim to show that C is isomorphic to a direct sum of shifts of copies of A_k, B_r, $H(n)$, for various values of $k, r \geq 0$ and $n \in \mathbb{Z}$, and the contractible complexes $\mathbb{D}(H)$ and $\mathbb{D}(F)$.

If $C = 0$ we are done, so assume $C \neq 0$. By shifting as needed, we can assume C is concentrated in degrees 0 through d with $C_0 \neq 0$. Let $T_i(C)$ be the truncated complex given by $(T_i(C))_j = C_j$ if $j \leq i$ and $(T_i(C))_j = 0$ if $j > i$. Observe that any choice of basis for $T_0(C)$ is trivially a decomposition into strands (just direct sums of H and $A_0 = F$). Assume for $m > 1$ that there exists a basis for $T_{m-1}(C)$ decomposing it as a direct sum of shifts of fundamental complexes and contractible complexes. For the inductive step, we

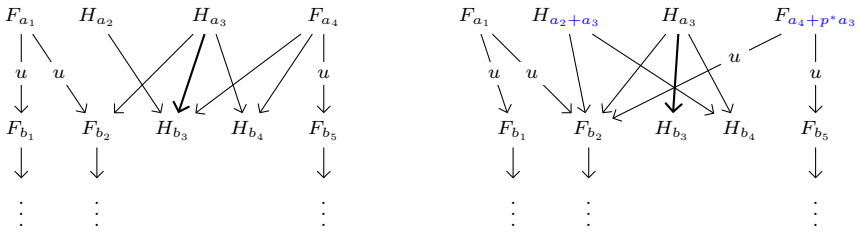


Fig. 8. An example of Step 1.

will find a basis for $T_m(C)$ decomposing it into such a direct sum. In the process, we will possibly change the basis at level m as well as (potentially all) levels below.

Fix a basis for $T_{m-1}(C)$ so that the truncated complex is a direct sum of fundamental complexes together with contractible ones. Choose any basis for C_m. One can visualize $T_{m-1}(C)$ written in strands with the basis elements in C_m mapping to some combination of those strands (see the left side of Fig. 8 for an example). We provide an algorithm to adjust the bases at each level so that $T_m(C)$ is written as a direct sum of fundamental complexes and contractible ones. In each step, we explain what to do in general and give an example. To distinguish these, we use Greek letters for the basis elements in the general case and Roman letters for the examples.

Step 1: Split off disks. In this step, we will split off disks in C with nonzero terms in degrees m and $m-1$. Suppose an isomorphism of summands appears in the map from C_m to C_{m-1}. See for example the left side of Fig. 8 (which has three such isomorphisms, all from H to H). For concreteness, assume the isomorphism is $id\colon H \to H$. We will demonstrate how to change bases to split off a shifted copy of $\mathbb{D}(H)$. The other cases of isomorphisms $id, t\colon F \to F$ can be handled similarly and are addressed at the end of this step. If there are no isomorphisms of summands between degree m and degree $m-1$, proceed to Step 2.

Fix one of the identity maps $H_\alpha \to H_\beta$ between levels m and $m-1$. We first adjust the basis at level m so that no other summand in degree m maps to H_β. This change of basis proceeds as follows. Any other basis element α' that maps nontrivially to the submodule generated by β is either on the $\bullet$ side or the Θ side. Replace α' with $\alpha' + \alpha$ if α' is on the $\bullet$ side, and with $\alpha' + p^*\alpha$ if α' is on the Θ side. The result is again a basis (see parts (v) and (iv) of Lemma 7.2). After changing all such α' in this way, no basis element other than α will map nontrivially to the submodule generated by β.

For example, in Fig. 8 we choose the identity map $H_{a_3} \to H_{b_3}$ sending the element a_3 to b_3. As there is also a nonzero map $H_{a_2} \to H_{b_3}$, the basis element a_2 at the top is replaced with $a_2 + a_3$. Similarly, the element a_4 is replaced with $a_4 + p^*a_3$. These new basis elements are depicted on the right of Fig. 8 in blue. Notice after the change of basis $H_{a_2+a_3}$ maps to the sum of two strands, as does $F_{a_4+p^*a_3}$, but neither maps to H_{b_3} since we are working over $\mathbb{Z}/2$. At this point, no other summands at level m map nontrivially to H_{b_3}. However, a_3 maps nontrivially to the summands generated by b_2 and b_4, so we are not yet able to split off a copy of $\mathbb{D}(H)$. Thus we consider the basis below degree m.

In general (as in the example), it is possible α maps nontrivially to some other summand. In this situation, we change the basis at level $m-1$, adjusting the target basis element β so that this is no longer the case. Suppose β' is another basis element at level $m-1$ such that α maps nontrivially to the summand generated by β'. Replace β with $\beta + \beta'$ if β' is on the $\bullet$ side. Replace β with $\beta + p_*\beta'$ if β' is on the Θ side. Repeat this process until α only maps to a single summand, and call this new basis element $\tilde{\beta}$. We can now split off the complex $H_\alpha \to H_{\tilde{\beta}}$.

In our example, this change is shown in Fig. 9. We first replace b_3 with $b_3 + p_*b_2$, and then replace that with $\tilde{\beta} = b_3 + p_*b_2 + b_4$. Now $H_{a_3} \to H_{b_3+p_*b_2+b_4}$ splits off. Observe that we have effectively chosen the diagonal basis element hit by a_3. That is, we replace b_3 with the sum of the elements that are hit by a_3 (using p_* as necessary, since all these elements must lie on the $\bullet$ side).

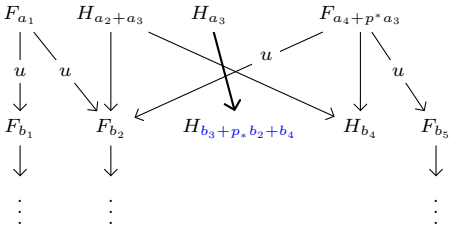


Fig. 9. The example for Step 1, continued.

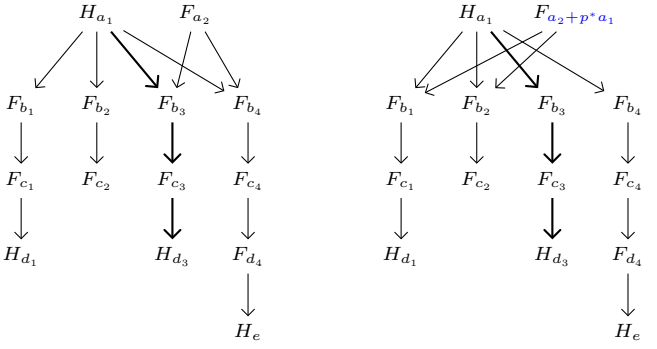


Fig. 10. The example for Step 2.

Note the change of basis at level $m-1$ has not destroyed the decomposition into strands in lower degrees. The fact that $H_\alpha \to H_\beta$ was the identity and the criteria that $d^2 = 0$ in the chain complex, means the original strand involving H_β could not have had any lower degree terms. The same is true of $H_{\tilde{\beta}}$. Thus we have indeed split off a copy of $\mathbb{D}(H)$.

If initially we had instead chosen the identity map $F_\alpha \to F_\beta$ the process to split off a copy of $\mathbb{D}(F)$ would be analogous, except we would replace basis elements α' in level m with either $\alpha' + \alpha$, $\alpha' + t\alpha$, $\alpha' + u\alpha$, or $\alpha' + p_*\alpha$. At level $m-1$, we similarly would add β', $t\beta'$, $u\beta'$, or $p^*\beta'$ to β. If we had instead chosen $t\colon F_\alpha \to F_\beta$, we could simply replace α with $t\alpha$ and then apply the steps for $id\colon F_{t\alpha} \to F_\beta$.

Continue this process until all isomorphisms from level m to level $m-1$ have been split off as disks.

Now we turn to decomposing the remaining complex where these top-level disks have been split off. We abuse notation and again refer to this complex as C. Notice C may still have contractible complexes as summands in lower degrees, but these can essentially be ignored.

Having completed Step 1, we may assume C has no isomorphisms from degree m to degree $m-1$. We consider nonzero maps out of a copy of H in degree m, and since there are no maps of the form $H \to H$, we suppose there is a map of the form $p\colon H \to F$. This type of map appears at the top of both B_r strands and $H(-n)$ strands, for $n > 0$. In general, to split off a strand ending in H we choose the shortest strand, and to split off a strand ending in F we choose the longest. In the next two steps, we will split off the B_r summands, shortest strands first, and then the $H(-n)$ summands, longest first. If C has no maps of the form $p\colon H \to F$ at this level, proceed to Step 4.

Step 2: Split off B_r summands. Assume there exists a map $p\colon H_\alpha \to F_\beta$ from level m to level $m-1$, and further assume that the strand in $T_{m-1}(C)$ with F_β at the top is isomorphic to a shift of $H(i)$ for some $i > 0$. If there is no such strand, proceed to Step 3. If there is such a strand, amongst all such α, β pairs, select one so the complex beginning with F_β is isomorphic to a shifted copy of $H(i)$ for the smallest possible i. That is, choose the shortest strand that ends in an H. An example is shown on the left in Fig. 10 with $\alpha = a_1$ and $\beta = b_3$, where the chosen strand is a shifted copy of $H(2)$.

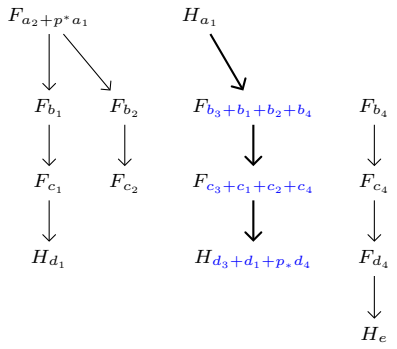


Fig. 11. The example for Step 2, continued.

Our goal is to change the bases in order to split off a shifted copy of B_r with $r = i-1$. To do so, we first adjust the basis at level m so that H_α is the only term hitting our chosen strand. If α' is some other basis element that maps nontrivially to the summand generated by β, replace α' with either $\alpha' + \alpha$ or $\alpha' + p^*\alpha$. Continue in this way until H_α is the only term with a nontrivial map to the chosen strand. This change is made in the example on the right in Fig. 10.

Next we change the basis of our chosen strand at every level below, starting with level $m-1$. Recall the element β generates a copy of F. Replace β with the diagonal element, i.e. the sum of all basis elements whose summands are mapped to nontrivially by H_α. Note that because there are no longer any isomorphisms from degree m to $m-1$, each of these basis elements must generate a copy of F. Thus all these elements are on the Θ side and the sum makes sense (there is no need to involve p^* or p_*). Call this new diagonal basis element $\tilde{\beta}$. Observe that H_α now maps nontrivially only to $F_{\tilde{\beta}}$. We adjust the basis at level $m-2$ in a similar fashion, again replacing the basis element in the chosen strand with a sum of basis elements. Eventually we will encounter a summand of the form H_δ at the bottom of the chosen strand. At this level, we replace the basis element δ with the diagonal element, but adjust as necessary. We take the sum of all the relevant copies of basis elements on the $\bullet$ side (including δ) and p_* applied to relevant basis elements on the Θ side. See Fig. 11 for clarification.

In the example, note that $d_3 + d_1 + p_*d_4$ maps to zero because $p\colon F_{d_4} \to H_e$ is zero on the $\bullet$ side, so we can now split off a complex of the form B_1. The general case works as in the example: after replacing the basis element δ with the sum, we can split off a shift of B_r with $r = i-1$.

Continue this process until there are no more maps $p\colon H_\alpha \to F_\beta$ from level m to level $m-1$, where F_β is the top of a shifted copy of $H(i)$ for any $i > 0$. Afterward, if there are no more maps $p\colon H \to F$ at this level then skip to Step 4. Otherwise proceed to Step 3 to split off shifted copies of $H(-n)$.

Step 3: Split off $H(-n)$ summands. Having completed Steps 1 and 2, and again abusing notation, we may assume that any maps $p\colon H \to F$ from level m to level $m-1$ in C must hit strands made entirely of copies of F. So assume there is a map of the form $p\colon H_\alpha \to F_\beta$ from level m to level $m-1$ where in the truncated complex $T_{m-1}(C)$ the summand with F_β at the top is of the form A_i for some $i \geq 0$. In this step, choose the pair α, β such that the summand A_i beginning with F_β has the *largest* possible value of i (in other words, F_β begins the longest possible strand). Our goal is to use this strand to split off a shifted copy of $H(-n)$ with $n = i+1$.

We can adjust the basis at level m as in Step 2. We do not repeat these details and instead begin by assuming there are no other basis elements from level m that map nontrivially to F_β. We next change the basis at level $m-1$ as in Step 2. That is, replace β with the sum of all basis elements whose summands are mapped to nontrivially by α. If $i > 0$, make the same adjustment at level $m-2$ and continue making this change until reaching level $m-(i+1)$. An example is provided in Fig. 12 for clarification.

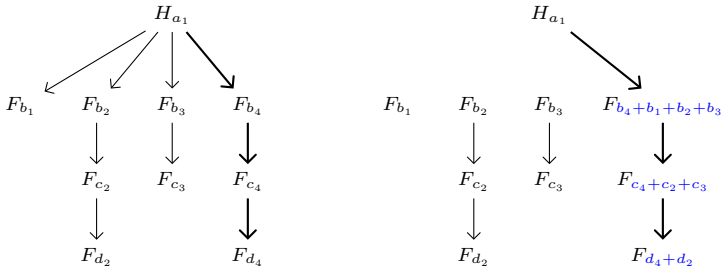


Fig. 12. The example for Step 3.

Now split off a complex that is a shift of $H(-(i+1))$. Continue this process until there are no longer any maps of the form $p\colon H \to F$ from degree m to $m-1$.

Step 4: Split off $H(n)$ summands. Suppose we have completed Steps 1, 2, and 3. If there are any nonzero maps remaining from level m to level $m-1$, then they must be of the form $u : F \to F$. We now consider basis pairs α, β such that $u : F_\alpha \to F_\beta$ appears in C from level m to $m-1$, and the summand beginning with F_β in $T_{m-1}(C)$ ends in H. That is, the summand beginning with F_β is isomorphic to $H(i)$ for some $i > 0$. As in Step 2, choose the pair α, β such that F_β begins a copy of $H(i)$ for the smallest possible i. Change bases as in Step 2 to split off a copy of $H(n)$ with $n = i+1$ from $T_m(C)$.

Repeat the above until there are no such α, β pairs remaining.

Step 5: Split off A_k summands. Assume we have completed Steps 1 through 4. Again, if there are any nonzero maps remaining from level m to level $m-1$, they must be of the form $u : F_\alpha \to F_\beta$. Having completed Step 4, it must be that the summand in $T_{m-1}(C)$ that begins with F_β is of the form A_i for some $i \geq 0$. As in Step 3, choose the α, β pair such that F_β begins the longest possible A_i. Change bases as in Step 3 to split off a copy of A_k with $k = i+1$ from $T_m(C)$.

Repeat the above until there are no such α, β pairs remaining.

After completing these steps, all remaining maps from level m to level $m-1$ will be zero. Thus any remaining H or F summands split off as chain complexes with zero differentials. We have successfully decomposed $T_m(C)$ as a sum of fundamental chain complexes. We can now repeat the above steps to split $T_{m+1}(C)$, and so on. Since C is a bounded complex, there is a large enough d so that $T_d(C) = C$. Thus this process will eventually terminate to give a splitting of C. □

8. An algebraic version of Kronholm's theorem

Kronholm's freeness theorem, which first appeared in [12], states the $RO(C_2)$-graded Bredon cohomology with $\underline{\mathbb{Z}/2}$-coefficients of any finite $\mathrm{Rep}(C_2)$-complex is free as a module over the cohomology of a point $\mathbb{M}_2$. A mistake in the original proof was fixed in [11], which also expanded the result to $\mathrm{Rep}(C_2)$-complexes of finite type. The proof of the freeness theorem involved delicate arguments about extensions of $\mathbb{M}_2$-modules. In this section, we give two alternate proofs. The first uses τ-localization to quickly deduce freeness as suggested by the referee. The second uses the splitting algorithm from Section 7 to describe the cohomology explicitly. The first has the advantage of being short and clear, while the second perspective helps clarify an important phenomenon observed by Kronholm now known as "Kronholm shifts". This is a phenomenon in which the representation cell structure of a $\mathrm{Rep}(C_2)$-complex determines the bidegrees of the free generators in cohomology, up to some shifting of bidegrees.

We begin by defining an analog of $\mathrm{Rep}(C_2)$-cells and $\mathrm{Rep}(C_2)$-complexes in $Ch(\underline{\mathbb{Z}/2})$. This is somewhat complicated by the fact that complexes in $Ch(\underline{\mathbb{Z}/2})$ correspond to general $H\underline{\mathbb{Z}/2}$-modules, not just those of the form $H\underline{\mathbb{Z}/2}$ smashed with a pointed C_2-space. Recall that a $\mathrm{Rep}(G)$-complex is a particular type of

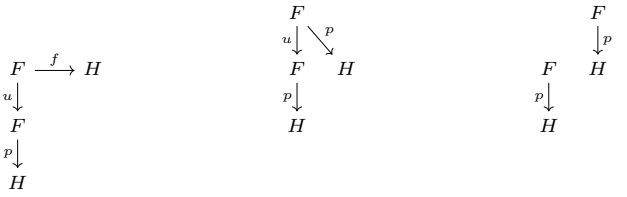


Fig. 13. Freeness theorem and shifts for $H\underline{\mathbb{Z}/2} \wedge \mathbb{R}P^2_{tw}$.

G-space built by attaching $\mathrm{Rep}(G)$-cells of the form $D(V)$ so that each filtration quotient looks like a wedge of representation spheres S^V. In this section, we will focus on $\mathrm{Rep}(G)$-complexes and not consider G-CW complexes, which are built by attaching orbit cells $G/H \times D^m$.

We define a **representation cell** to be a fundamental complex of the form $\Sigma^m H(q)$ where $0 \leq q \leq m$. As discussed in Section 6, $\Sigma^m H(q)$ corresponds to $H\underline{\mathbb{Z}/2} \wedge S^{m,q}$. We require $0 \leq q \leq m$ so that $S^{m,q}$ is an actual (rather than virtual) representation sphere. Recall $\Sigma^m H(q)$ is the complex

$$\Sigma^m H(q)\colon \qquad 0 \longrightarrow F \xrightarrow{u} F \xrightarrow{u} \cdots \xrightarrow{u} F \xrightarrow{p} H \longrightarrow 0,$$

where the leftmost F is in degree m and the H is in degree $m-q$. We refer to m as the **topological dimension**, q as the **weight**, and $m-q$ as the **coweight** of the representation cell. For a representation cell W we write $\mathrm{top}(W)$, $\mathrm{wt}(W)$, and $\mathrm{cowt}(W)$ for the topological dimension, weight, and coweight, respectively.

The complex for a representation cell W is zero in degrees above $\mathrm{top}(W)$ and in degrees below $\mathrm{cowt}(W)$. Colloquially, $\mathrm{top}(W)$ is the degree of the "top F" and $\mathrm{cowt}(W)$ is the degree of the "bottom H". The weight $\mathrm{wt}(W)$ is the "length" of the fundamental complex, where length is one less than the number of nonzero terms. The weight is also the number of F's in the strand. It is useful to keep these interpretations of the three invariants in mind while reading the arguments in this section.

We next want to define $\mathrm{Rep}(C_2)$-chain complexes so that under the equivalence (6.1) we have

$$\mathrm{Rep}(C_2)\text{-chain complexes} \longleftrightarrow H\underline{\mathbb{Z}/2} \wedge \mathrm{Rep}(C_2)\text{-complexes}.$$

If a space K is formed by attaching an (m,q)-cell $D(\mathbb{R}^{m,q})$ to the space L, then we have a cofiber sequence $L \to K \to S^{m,q}$. Desuspending in the homotopy category then gives $K \simeq \mathrm{Cof}(S^{m-1,q} \to L)$. Thus we are led to define **attaching an (m,q)-representation cell** to a chain complex of $\underline{\mathbb{Z}/2}$-modules Y to be taking the cofiber of a map $\Sigma^{m-1}H(q) \to Y$ in $Ch(\underline{\mathbb{Z}/2})$. We will show that (under well-controlled circumstances) the cofiber splits as a direct sum of representation cells, and thus the bigraded cohomology is a free $\mathbb{M}_2$-module by Proposition 4.3.

The following example illustrates the freeness theorem and the shifting phenomenon in the context of chain complexes. The reader is invited to compare it with the extension of $\mathbb{M}_2$-modules from Example 3.2 in [11]. In the chain complex setting, a change of basis immediately solves the extension problem.

Example 8.1. The projective space $\mathbb{R}P^2_{tw} = \mathbb{P}(\mathbb{R}^{3,1})$ sits in a cofiber sequence $S^{1,0} \to \mathbb{R}P^2_{tw} \to S^{2,2}$. Desuspending in the stable homotopy category, we may view $\mathbb{R}P^2_{tw}$ as $\mathrm{Cof}(S^{1,2} \to S^{1,0})$. Smashing with $H\underline{\mathbb{Z}/2}$ and translating to chain complexes, we consider the corresponding complex $X = \mathrm{Cof}(\Sigma^1 H(2) \to \Sigma^1 H)$. The map $f\colon \Sigma^1 H(2) \to \Sigma^1 H$ is pictured on the left in Fig. 13, where complexes are drawn vertically. If f is not null-homotopic (as turns out to be the case here), it must be given by $p\colon F \to H$ in degree 1. The cofiber X is then pictured in the middle of Fig. 13. Applying the change of basis algorithm from Section 7, Step 4 of the algorithm splits the complex into the two summands as pictured on the right. Thus we find $X \simeq \Sigma^1 H(1) \oplus \Sigma^2 H(1)$ (this can also be deduced from Lemma 4.26 by identifying $f \wedge id_{H(-2)}$ with the map $\Sigma\theta$).

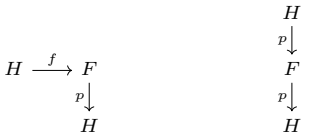


Fig. 14. Freeness theorem fails for non-spaces.

Notice the splitting algorithm decomposes X as a direct sum of representation cells $\Sigma^1 H(1)$ and $\Sigma^2 H(1)$. So the (reduced) bigraded cohomology of X is a free $\mathbb{M}_2$-module generated in bidegrees $(1,1)$ and $(2,1)$. Translating to topology we have

$$H\underline{\mathbb{Z}/2} \wedge \mathbb{R}P^2_{tw} \simeq H\underline{\mathbb{Z}/2} \wedge (S^{1,1} \vee S^{2,1}).$$

Moreover, back in chain complexes, the splitting algorithm has effectively transferred a copy of F onto the second chain complex. This is precisely how the shifting phenomenon observed by Kronholm manifests in chain complexes. The original spheres $S^{1,0}$ and $S^{2,2}$ give rise to free $\mathbb{M}_2$ generators in the same topological dimensions but with shifted weights: the first generator shifts up one in weight and the second generator shifts down one. These weight shifts occur because a single F "moved" to the second complex during the change of basis. Notice that no copies of F appeared or disappeared, so the total weight is preserved.

As in the previous example, more general Kronholm shifts will be determined by copies of F moving onto different strands and the total weight will be preserved.

Before proceeding to the proof in the general case, we need one more restriction on the complexes. Without this restriction, it is easy to construct a chain complex out of representation cells that cannot correspond to a space and hence not a $\mathrm{Rep}(C_2)$-complex.

Example 8.2. Consider the nontrivial map $f\colon \Sigma^1 H \to \Sigma^1 H(1)$. We can compute $\mathrm{Cof}(\Sigma^1 H \to \Sigma^1 H(1)) \simeq \Sigma^2 B_0$ as depicted in Fig. 14. However, by the structure theorem (Theorem 6.5), the bigraded cohomology of $\Sigma^2 B_0$ cannot be the bigraded cohomology of a C_2-space.

To exclude these sorts of chain complexes, we make the following definition.

Definition 8.3. A map $X \to Y$ in $\mathcal{D}(\underline{\mathbb{Z}/2})$ is **spacelike** if the cofiber does not have any B-type summands in its decomposition (see Theorem 4.5).

We are now ready to define the appropriate analog of $\mathrm{Rep}(C_2)$-complexes.

Definition 8.4. A **$\mathrm{Rep}(C_2)$-chain complex** is a chain complex X with a filtration $X_0 \subseteq X_1 \subseteq \cdots \subseteq X$, where X_0 is either 0 or a direct sum of copies of H, and X_m is formed by attaching m-cells to X_{m-1}. That is,

$$X_m = \mathrm{Cof}\left(\bigoplus_i \Sigma^{m-1} H(q_i) \to X_{m-1}\right)$$

where $0 \leq q_i \leq m$ for each i. Furthermore, we require that all of the attaching maps are spacelike.

We are now ready to state a version of Kronholm's theorem in $\mathcal{D}(\underline{\mathbb{Z}/2})$.

Theorem 8.5. *Suppose X is a $\mathrm{Rep}(C_2)$-chain complex built from finitely-many cells. Then X is quasi-isomorphic to a direct sum of complexes of the form $\Sigma^{k_i} H(r_i)$ where $0 \leq r_i \leq k_i$ for all i.*

Before giving the proof, we note the topological version of Kronholm's theorem is an immediate corollary.

Corollary 8.6 *(Kronholm; Hogle–May). Let L be a pointed C_2-space with the structure of a finite $Rep(C_2)$-complex. Then $\tilde{H}^{*,*}(L;\underline{\mathbb{Z}/2})$ is free as an $\mathbb{M}_2$-module.*

Proof. The object of $\mathcal{D}(\underline{\mathbb{Z}/2})$ corresponding to $H\underline{\mathbb{Z}/2} \wedge \Sigma^\infty L$ can be represented by a $\mathrm{Rep}(C_2)$-chain complex X satisfying Definition 8.4. So Theorem 8.5 applies and X is quasi-isomorphic to a finite direct sum $\oplus_i \Sigma^{k_i} H(r_i)$ where $0 \leq r_i \leq k_i$ for all i. By Proposition 4.3, the bigraded cohomology of X is a free $\mathbb{M}_2$-module and thus so is the bigraded cohomology of the C_2-space L. □

Now we turn to the proofs of Theorem 8.5. We first outline a short proof of the freeness theorem via localization. We thank the referee for suggesting this argument.

Proof of freeness in Theorem 8.5 via localization. Let X be a finite $\mathrm{Rep}(C_2)$-chain complex. Theorem 4.5 implies that up to quasi-isomorphism X is the sum of fundamental complexes. The spacelike assumptions rules out the appearance of any B-type summands. To rule out A-type summands, it is enough to rule out any ρ-torsion in the τ-localization of the cohomology. This can be done by inducting on the dimension of the complex and using that $\mathbb{M}_2[\tau^{-1}]$ is a graded PID. We outline this argument in the paragraph below.

The base case is immediate because X_0 is just a direct sum of copies of H. In the inductive step we build X_m from X_{m-1} by attaching m-cells. The attaching map induces a map on the τ-localized cohomology, which leads us to analyze the map of free $\mathbb{M}_2[\tau^{-1}]$-modules

$$\tau^{-1}H^{*,*}(X_{m-1}) \to \bigoplus_{\#\text{ of } m\text{-cells}} \Sigma^{m-1}\mathbb{M}_2[\tau^{-1}].$$

The module $\tau^{-1}H^{*,*}(X_{m-1})$ will be a direct sum of free $\mathbb{M}_2[\tau^{-1}]$-modules with generators in topological degrees strictly less than m. For degree reasons, the only summands on which this map can be nonzero are the ones generated in topological degree $m-1$. On these summands, analyzing the map of $\mathbb{M}_2[\tau^{-1}]$-modules reduces to analyzing a map of $\mathbb{F}_2$-vector spaces.

By making a change of basis if necessary, there are only two options for a degree $m-1$ summand in the domain: the summand is in the kernel, or it is mapped isomorphically onto a degree $m-1$ summand in the codomain. Thus, using the long exact sequence on cohomology, $\tau^{-1}H^{*,*}(X_m)$ will again be a direct sum of free $\mathbb{M}_2[\tau^{-1}]$-modules whose generators are in topological degrees m or less. In particular, $\tau^{-1}H^{*,*}(X_m)$ has no ρ-torsion and so neither does $H^{*,*}(X_m)$. By induction, X has no A-type summands and is thus quasi-isomorphic to a sum of terms of the form $\Sigma^{k_i}H(r_i)$. □

Remark 8.7. The above argument has the advantage of being short and clear. However, it obscures how the chain complexes are glued together and how the weights shift. That is, while the localization argument guarantees the result will be a direct sum of strands of the form $\Sigma^{k_i}H(r_i)$, it does not give us a way to predict the values of k_i or r_i based on our starting cells.

We thus give a second proof that sheds some light on the nature of the Kronholm shifts, by which the bigradings in the original cell structure shift around to become the bigradings in the splitting decomposition. We certainly do not claim the following proof is easier or clearer, but rather that it illustrates explicitly how the weights of the attached cells get shifted. One can see these shifts are a consequence of the splitting algorithm in the proof below.

We now embark on proving Theorem 8.5 using our splitting algorithm. We again aim to induct on the dimension of a $\mathrm{Rep}(C_2)$-chain complex X. However, it will be simpler to consider the attaching map for a single representation cell V. We thus consider a map $f\colon \Sigma^{-1}V \to Y$ where V is a single representation cell

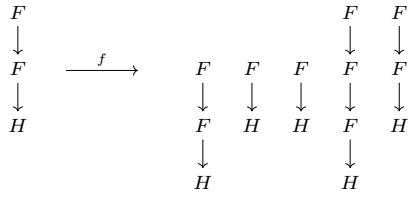


Fig. 15. Attaching a single representation cell.

and by induction we assume Y is split as a direct sum of representation cells. An example is depicted in Fig. 15, where there is a single representation cell $\Sigma^{-1}V$ shown on the left with a (potentially complicated) attaching map f to the direct sum of representation cells on the right. Notice on the right-hand side, the cells are ordered by topological dimension, the degree of the top F.

If we take the cofiber of f, then we get a complicated complex to which we can apply the splitting algorithm from Section 7. According to the algorithm, the cofiber will split into strands, but not necessarily the same vertical strands we started with. One can picture various copies of F and H breaking and reattaching to each other. However, at the end of the process, there are the same number of summands of F and H.

Since we have assumed the attaching map is spacelike, there will be no B-type strands in the final decomposition. Our aim is to show that no strands of the form A_k or $H(-n)$ appear, so there are only strands of the form $H(n)$ for $n \geq 0$. For a nontrivial attaching map, we will find either several disks split off or some number of copies of F from V will transfer to other strands. The latter will decrease the weight of the newly attached cell and increase the weights of the others.

In the course of this, we will need to consider maps between representation cells of particular dimensions. Up to homotopy, we may choose nice representatives for the attaching map on each summand. In the following lemma, we determine the possible maps for our setting. From now on we draw chain complexes horizontally.

Lemma 8.8. *Let $V = \Sigma^a H(b)$ and $W = \Sigma^s H(t)$ be representation cells with $a \geq s$, and if $a = s$ then $b \geq t$. Then for any map $\Sigma^{-1}V \to W$, exactly one of the following holds:*

(1) the cells satisfy $\operatorname{top}(\Sigma^{-1}V) = \operatorname{top}(W)$ and $\operatorname{cowt}(\Sigma^{-1}V) = \operatorname{cowt}(W)$, and the map is homotopic to the identity,

$$\begin{array}{ccccccccc} \Sigma^{-1}V & & F & \longrightarrow & F & \longrightarrow \cdots \longrightarrow & F & \longrightarrow & H \\ \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ W & & F & \longrightarrow & F & \longrightarrow \cdots \longrightarrow & F & \longrightarrow & H \end{array} \quad ;$$

(2) the cells satisfy $\operatorname{top}(\Sigma^{-1}V) = \operatorname{top}(W)$ and $\operatorname{cowt}(\Sigma^{-1}V) > \operatorname{cowt}(W)$, and the map is homotopic to one of the form

$$\begin{array}{ccccccccccccc} \Sigma^{-1}V & & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & H & & \\ \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle p} & & \\ W & & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & H \end{array} \quad ;$$

(3) the cells satisfy $\operatorname{top}(\Sigma^{-1}V) \geq \operatorname{top}(W)$ and $\operatorname{cowt}(\Sigma^{-1}V) \leq \operatorname{cowt}(W) - 2$, and the map is homotopic to one of the form

$$\begin{array}{ccccccccccc} \Sigma^{-1}V & & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & H \\ \downarrow & & & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\ W & & & & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & H & & \end{array} \quad ;$$

(4) the map is null-homotopic.

Proof. To justify these are the only cases, we start by computing homotopy classes of maps $\mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^{-1}V,W)$. Using that $H(b)$ is invertible with inverse $H(-b)$, we find that

$$\begin{aligned} \mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^{-1}V, W) &\cong \mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^{a-1}H(b), \Sigma^{s}H(t)) \\ &\cong \mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^{a-1-s}H(b), H(t)) \\ &\cong \mathcal{D}(\underline{\mathbb{Z}/2})(\Sigma^{a-1-s}H, H(t-b)) \\ &\cong \begin{cases} \mathbb{Z}/2 & \text{if } t-b \geq 0 \text{ and } -(t-b) \leq a-1-s \leq 0 \\ \mathbb{Z}/2 & \text{if } t-b \leq -2 \text{ and } 0 \leq a-1-s \leq -(t-b)-2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The last isomorphism follows from cases *(b)–(d)* of Proposition 4.22. That the only nonzero value is $\mathbb{Z}/2$ means there is at most one nontrivial homotopy class of maps between representation cells. One can readily check the three maps described in parts *(1)–(3)* of the lemma are non-null. It remains to show these are the only possibilities that satisfy the given constraints on V and W. We have two cases for non-null maps: $t-b \geq 0$ or $t-b \leq -2$.

Observe in the context of representation cells V and W, the value $t-b$ measures how many more copies of F the cell W has compared to V, i.e. how much longer the second strand is than the first. We now consider the two cases.

Case 1 ($t-b \geq 0$): When $t-b \geq 0$, the strand V is the same length or shorter than W. Under these circumstances, we show the constraints placed on the dimensions of the cells lead to either the identity map or the map in part *(2)*.

To get a non-null map, we must also have $a-1-s \leq 0$ or equivalently $a-1 \leq s$. This means the top F in W is in the same degree or higher than (i.e. to the left of) the top F in $\Sigma^{-1}V$. By hypothesis, we also have $a \geq s$ or equivalently $a-1-s \geq -1$. We see there are only two possibilities: $a-1-s=0$ and the strands begin at the same place, or $a-1-s=-1$ and there is a single F in W above the start of $\Sigma^{-1}V$.

Suppose $a-1-s=0$, so the strands begin in the same degree. If the strands are the same length, we have an easy choice of non-null map: the identity. Otherwise, W is longer than $\Sigma^{-1}V$. In that case, we may assume the map is of the form in part *(2)* of the lemma, having identity maps between copies of F and $p\colon H \to F$ in the degree of $\mathrm{cowt}(V)$, as below:

$$\begin{array}{ccccccccccc} \Sigma^{-1}V & & F & \longrightarrow & F & \longrightarrow & \cdots & \longrightarrow & F & \longrightarrow & H & & \\ \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle p} & & \\ W & & F & \longrightarrow & F & \longrightarrow & \cdots & \longrightarrow & F & \longrightarrow & F & \longrightarrow & H. \end{array}$$

This map is not null and thus represents the nonzero homotopy class. Notice here $\mathrm{top}(\Sigma^{-1}V) = \mathrm{top}(W)$ and the restrictions on the lengths of the strands mean $\mathrm{cowt}(\Sigma^{-1}V) > \mathrm{cowt}(W)$, as desired.

Lastly, consider when $a-1-s=-1$. We show there are no maps that satisfy the hypotheses of the lemma. In this case we have $a=s$, for which the lemma specifies that we must have $b \geq t$. On the other hand, $t \geq b$ in this case. Thus $t=b$, but this contradicts that $-(t-b) \leq a-1-s$ since 0 is not less than -1.

Case 2 ($t-b \leq -2$): This inequality implies $\Sigma^{-1}V$ has at least two more nonzero terms than W. Rearranging the two inequalities from the homotopy class calculation, $0 \leq a-1-s$ and $a-1-s \leq -(t-b)-2$, we immediately find $s \leq a-1$ and $(a-1)-b \leq s-t-2$. That is, W and V have $\operatorname{top}(W) \leq \operatorname{top}(\Sigma^{-1}V)$ and $\operatorname{cowt}(\Sigma^{-1}V) \leq \operatorname{cowt}(W)-2$. A map of the form in part *(3)* of the lemma satisfies these constraints and is not null, and thus is a representative of the unique nontrivial homotopy class. □

Remark 8.9. Observe that each of the non-null maps in Lemma 8.8 has a distinct lowest degree (right-most) nonzero component: either id, p, or u, corresponding to the cases (1)–(3). This will be important in our application below.

We are now ready to prove Kronholm's theorem in the chain complex setting using the splitting algorithm.

Proof of Theorem 8.5 via splitting. We induct on the number of representation cells in the $\operatorname{Rep}(C_2)$-chain complex X. To build X, instead of attaching all the m-cells at once, attach representation cells one at a time in order of increasing topological dimension, and within each topological dimension in order of increasing weight. We attach the cells in order of increasing weight to have better control of the attaching maps. This will allow us to reduce to studying the attaching maps from Lemma 8.8.

In order to maintain control of the attaching maps, our induction will prove a slightly stronger result than in the statement of the theorem. We prove that if X is obtained by attaching an (m,q)-representation cell V to a $\operatorname{Rep}(C_2)$-chain complex Y where $Y \simeq \oplus_i \Sigma^{k_i} H(r_i)$ with $0 \leq r_i \leq k_i$ **and also satisfying**

$$k_i \leq m, \text{ and if } k_i = m \text{ then } r_i \leq q \tag{**}$$

for all i, then X is quasi-isomorphic to a direct sum of representation cells satisfying the same inequalities as in (**). In particular, any cells of dimension m have weight no more than q. The base case where Y has no cells is trivial.

We need to analyze the attaching map $f\colon \Sigma^{m-1}H(q) \to Y$. Since Y is quasi-isomorphic to $\oplus_i \Sigma^{k_i} H(r_i)$ and Y is cofibrant, while $\oplus_i \Sigma^{k_i} H(r_i)$ is fibrant, in the projective model structure on $Ch(\underline{\mathbb{Z}/2}\text{–Mod})$, there must be a quasi-isomorphism $g\colon Y \to \oplus_i \Sigma^{k_i} H(r_i)$. So we can instead consider the attaching map gf

$$\Sigma^{m-1}H(q) \to \oplus_i \Sigma^{k_i} H(r_i),$$

whose cofiber will still be quasi-isomorphic to X. Thus, we can just replace Y with the direct sum in the codomain. For convenience, let W_i denote the summand $\Sigma^{k_i} H(r_i)$, so X is quasi-isomorphic to the cofiber of $\Sigma^{-1}V \to \oplus_i W_i$. We will apply the splitting algorithm from Section 7 to the cofiber. Recall that the algorithm proceeds from lowest degree nonzero term to highest, or from right to left when we draw complexes horizontally.

If the map $\Sigma^{-1}V \to \oplus_i W_i$ is null-homotopic, then the cofiber immediately splits as the direct sum $V \oplus (\oplus_i W_i)$. The condition (**) on the cells is immediate.

The more interesting situation arises from a non-null map. Projecting onto each summand, we get maps of the form $\Sigma^{-1}V \to W_i$. Since the cells W_i satisfy condition (**) by assumption, we can apply Lemma 8.8 letting $W = W_i$. Thus there are only three non-null possibilities (up to homotopy) for the map $\Sigma^{-1}V \to W_i$; as in Lemma 8.8 we label these possibilities (1)–(3). Choose these nice representatives for each non-null map and the zero map for any null portion. As the splitting algorithm proceeds from bottom to top, we will want to consider the lowest degrees first. Recall the lowest degree nonzero map for each representative is: $id\colon H \to H$ for case *(1)*, $p\colon H \to F$ in case *(2)*, and $u\colon F \to F$ in case *(3)*.

Now consider $\operatorname{Cof}(\Sigma^{-1}V \to \oplus_i W_i)$ and apply the splitting algorithm. There are no maps between the W_i, so reading from right to left the complex is split into strands until degree $\operatorname{cowt}(V)$. In degree $\operatorname{cowt}(V)$,

the bottom H from V supports either the identity map, p, or the zero map into the various strands. If that copy of H supports a nonzero map, we begin to apply the splitting algorithm here.

The splitting algorithm considers isomorphisms first and uses them to split off disks. So if the bottom H in V supports the identity map to a copy of H in any of the summands, then for at least one i there is a map of the form $id : V \to W_i$. An example of the cofiber (omitting other strands for brevity) is pictured below:

$$\begin{array}{lccccccccccc} V & F & \xrightarrow{u} & F & \xrightarrow{u} & \cdots & \xrightarrow{u} & F & \xrightarrow{p} & H & & \\ & & \searrow^{1} & & \searrow^{1} & & & & \searrow^{1} & & \searrow^{1} & \\ W_i & & & F & \xrightarrow[u]{} & F & \xrightarrow[u]{} & \cdots & \xrightarrow[u]{} & F & \xrightarrow[p]{} & H. \end{array}$$

Regardless of whether the terms from V support other nonzero maps, in each degree the algorithm prioritizes the identity maps between terms and we can choose to use the identity maps to W_i. The algorithm will split off many disks, as many as the length of V. After the change of bases, no terms from V will support nonzero maps, so the remainder of the complex will be split as it was in Y. Thus X will be quasi-isomorphic to Y but with one strand (W_i) removed. As before, none of the remaining strands have changed so the inequalities from (**) still hold.

Now suppose there are no identity maps supported by the bottom H from V. If that H supports a nonzero map, it must be $p\colon H \to F$. We will ultimately see this is not possible. If it were, there would be at least one summand W_i with a map $V \to W_i$ of the form in part *(2)* of Lemma 8.8. An example of the cofiber (again omitting other strands) is pictured below:

$$\begin{array}{lccccccccccc} V & F & \xrightarrow{u} & F & \xrightarrow{u} & F & \xrightarrow{u} & F & \xrightarrow{u} & H & & \\ & & \searrow^{1} & & \searrow^{1} & & \searrow^{1} & & \searrow^{1} & & \searrow^{p} & \\ W_i & & & F & \xrightarrow[u]{} & F & \xrightarrow[u]{} & F & \xrightarrow[u]{} & F & \xrightarrow[u]{} & F & \xrightarrow[p]{} & H. \end{array}$$

In the degree of $\mathrm{cowt}(V)$, the algorithm prioritizes the map $p\colon H \to F$. Applying the algorithm would split off a B-type summand (B_1 in the example pictured). By assumption, all the attaching maps in our complex are spacelike, contradicting that there are any B-type summands. Thus the bottom H from V cannot support any nonzero maps to other summands other than the identity, and hence there are no maps of the form in part *(2)* of Lemma 8.8.

Having dealt with these possibilities, we may now assume the bottom H from V does not support any nonzero maps, and any remaining non-null maps supported by this strand are of the form in part *(3)* of Lemma 8.8. Reading from bottom to top (i.e. right to left) in the cofiber, the strands will be split until we encounter a copy of F from V supporting the map $u\colon F \to F$. Here Step 4 of the algorithm splits the strands by attaching the F from V to the shortest strand ending in H.

An example of the cofiber of such a map $\Sigma^{-1}V \to W_i$ (omitting other strands as usual) is depicted below:

$$\begin{array}{lccccccccccc} V & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & H \\ & & & & \searrow^{u} & & & & & & & \\ W_i & & & & & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & H \end{array} .$$

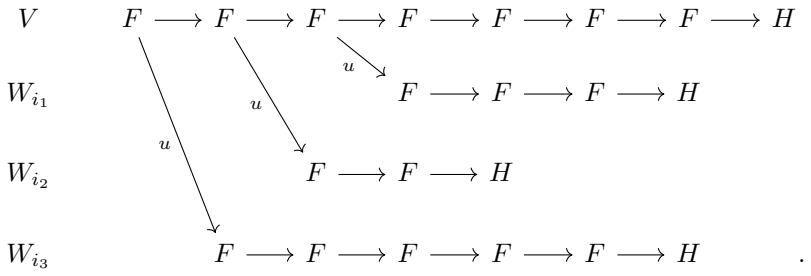
After the splitting, the resulting strands for this example are

$$\begin{array}{ccccccccccc} F & \longrightarrow & F & & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & H \\ & & \searrow & & & & & & & & \\ & & & & F & \longrightarrow & F & \longrightarrow & F & \longrightarrow & H \end{array} .$$

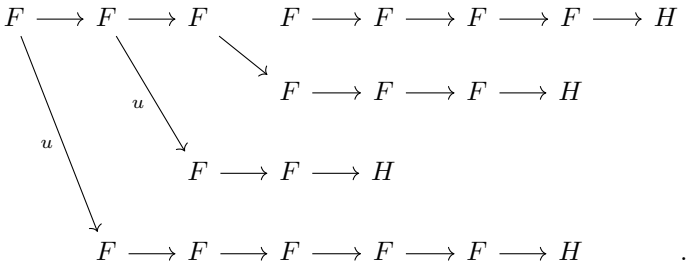
Observe that after the splitting, there are again two representation cells with topological dimensions $\mathrm{top}(V)$ and $\mathrm{top}(W_i)$. Moreover, attaching the F to the second strand changes the weights. The weight of the first strand decreases by the same amount the weight of the second strand increases.

Of course, there may be many non-null maps of the form in part *(3)*. Continue reading from right to left until all strands have been split according to the algorithm. The strand V is finite so this process eventually terminates.

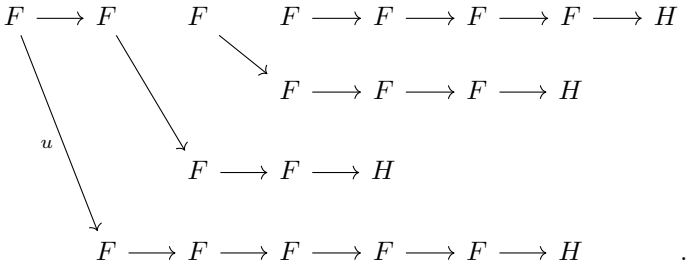
We give an example to illustrate these last steps in more detail. In the example, we only show the shortest strands at each stage, since the algorithm will effectively ignore any longer ones. Alternatively, using a diagonal change of basis for the cofiber, one may assume the copies of F from V support a nonzero map to at most one strand of each topological dimension. The example is shown below:



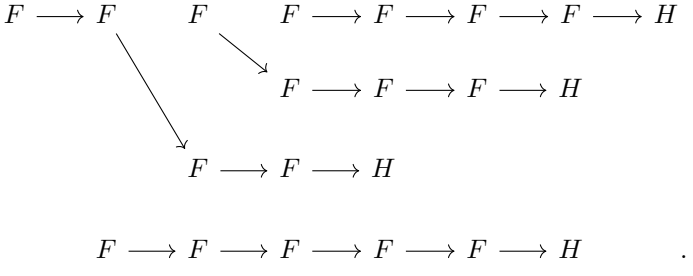
At the first stage, the fifth copy of F from V (reading right to left) moves to the second strand, as in the previous example:



Then the algorithm attaches the next F from V onto the shortest strand, which is now the third strand pictured:



Finally, the remaining F from V attaches to the third strand since it is shorter than the fourth:



At the end of this process we have split the cofiber into a number of strands, though they may have different lengths than the original strands. In any case, all of the strands are of the form $\Sigma^k H(r)$ for various k and r satisfying $0 \leq r \leq k$. Thus X is quasi-isomorphic to a direct sum of representation cells as desired.

It only remains to verify that the inequalities from (**) are satisfied by X. In fact, since the copies of F from V have simply detached and reattached, the collection of topological dimensions of the strands is preserved (as well as the total weight). So all the topological dimensions are still less than or equal to m. Note that by Lemma 8.8, $\Sigma^{-1}V$ does not admit a non-null map to any of the summands in Y having topological dimension m; so these cells are unaffected in the cofiber X and still have weights less than or equal to q by (**). The only other cell in X of topological dimension m is the one onto which the top F from V gets attached. According to Step 4 of the algorithm, which at each iteration chooses the shortest strand ending in H to split off, this top F from V must have attached to a strand no longer than the original V. So that strand contributes a representation cell with weight at most q, and thus (**) is still satisfied. This completes the induction. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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