# LONG TIME BEHAVIOR OF SOLUTIONS OF AN ELECTROCONVECTION MODEL IN $\mathbb{R}^2$

ELIE ABDO AND MIHAELA IGNATOVA

ABSTRACT. We consider a two dimensional electroconvection model which consists of a nonlinear and nonlocal system coupling the evolutions of a charge distribution and a fluid. We show that the solutions decay in time in $L^2(\mathbb{R}^2)$ at the same sharp rate as the linear uncoupled system. This is achieved by proving that the difference between the nonlinear and linear evolution decays at a faster rate than the linear evolution. In order to prove the sharp $L^2$ decay we establish bounds for decay in $H^2(\mathbb{R}^2)$ and a logarithmic growth in time of a quadratic moment of the charge density.

## 1. INTRODUCTION

We consider the electroconvection model

$$\partial_t q + u \cdot \nabla q + \Lambda q = 0, \tag{1}$$

$$\partial_t u + u \cdot \nabla u + \nabla p - \Delta u = -qRq, \tag{2}$$

$$\nabla \cdot u = 0 \tag{3}$$

in $\mathbb{R}^2$ describing the evolution of a surface charge density $q$ in a two-dimensional incompressible fluid flowing with a velocity $u$ and a pressure $p$. Here $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the square root of the two-dimensional Laplacian, and $R = \nabla\Lambda^{-1}$ is the two-dimensional Riesz transform. The initial data

$$u(\cdot, 0) = u_0 \tag{4}$$

and

$$q(\cdot, 0) = q_0 \tag{5}$$

are assumed to be regular enough and have good decay properties. The model is motivated by physical and numerical studies of electroconvection [9, 22, 23]. The nonlocal aspect of the evolution of the charge density and the nonlocal forcing on the Navier-Stokes equations in the model are due to the fact that the fluid and charges are confined to a thin two dimensional film. The global well-posedness of the system in bounded domains was obtained in [7] using commutator estimates and nonlocal nonlinear analysis. In [1], we investigated the long time dynamics of the model in two dimensions, with periodic boundary conditions and with applied voltage. When the fluid is forced by time-independent smooth mean zero body forces, we proved that the model (1)–(5) has a finite dimensional global attractor. In the absence of body forces, the charge density $q$ converges exponentially in time to a unique limit due to the applied voltage, and the velocity $u$ converges exponentially in time to zero. The rate of exponential decay depends on the periodic boundary conditions.

In this paper, we consider the time asymptotic behavior of solutions of (1)–(5) in $\mathbb{R}^2$, and adapt the Fourier splitting method [17, 18] of Schonbek to the present system. The method was initially used in [17] to prove decay of Leray weak solutions [14] of Navier-Stokes equations and to further decay studies for Navier-Stokes equations [3, 11, 18, 20, 24] and many other partial differential

*Date*: today.

equations (see for instance [4, 8, 10, 15, 25, 26]). Different approaches were employed as well to investigate the time decay [16] and space-time decay [2, 12, 13, 21] of higher-order derivatives of solutions to Navier-Stokes equations.

The electroconvection model (1)–(5) couples Navier-Stokes equations to a scalar equation for a surface charge density $q$, evolving via advection by $u$ and diffusion by $\Lambda$. We obtain in Theorem 1 of section 2 the long time $L^2$ decay of the type

$$\|q\|_{L^2} = O(t^{-1})$$

and

$$\|u\|_{L^2} = O(t^{-\frac{1}{2}}).$$

This rate of decay is sharp for the linear uncoupled system if the initial data have non vanishing finite $L^1$ norms, because functions of the form $Q(t) = e^{-t\Lambda^\alpha} q_0$ obey

$$\lim_{t\to\infty} t^{\frac{n}{\alpha}} \|Q(t)\|_{L^2(\mathbb{R}^n)}^2 = C_{n,\alpha} \left( \int_{\mathbb{R}^n} q_0 dx \right)^2$$

for any $\alpha > 0$ and $n \geq 1$. The fact that such a decay is sharp for the nonlinear evolution as well is a consequence of Theorem 4 of section 3 where we prove that $u - U$ with $U(t) = e^{t\Delta} u_0$ and $q - Q$ with $Q(t) = e^{-t\Lambda} q_0$ decay faster in $L^2$ than $u$ and $q$, respectively. Similar results were proved for the solutions to the critical SQG in [8] and their higher-order derivatives in [19]. The critical SQG velocity $u = R^\perp q$ decays in $L^2$ like $q$, that is at the rate $t^{-1}$, which helps lower the size of the nonlinear term $u \cdot \nabla q$ in that equation. In our case, the velocity has slower decay in $L^2$ due to the Navier-Stokes equation, namely of the order $t^{-\frac{1}{2}}$, and the nonlinear term is larger. The influence of the charge density $q$ is felt by the Navier-Stokes velocity via the electric force $-qRq$. In order to obtain a key fast enough decay at low wave numbers for the difference $v = u - U$, we need to control a moment of $q$, $\int_{\mathbb{R}^2} |x|^2 |q(x,t)|^2 dx = M^2(t)$, in view of the inequality

$$|\widehat{qRq}(\xi)| \leq C|\xi| \|q\|_{L^2} M(t)$$

(see Lemma 3). We prove that

$$M(t) = O(\sqrt{\log t})$$

for long time, by analyzing the evolution of the quantity $a(x)q(x,t)$ with $a(x) = \sqrt{|x|^2 + 1}$. This analysis uses the boundedness of the commutator between $\Lambda$ and multiplication by $a(x)$, which we establish in Lemma 1. In addition, in order to achieve the necessary sharp $L^2$ bounds we have to obtain bounds for the decay of higher norms of both $u$ and $q$. For instance, $H^1$ norms of $q$ are of the order

$$\|\nabla q\|_{L^2} = O(t^{-2}).$$

and are obtained by somewhat involved nonlinear and nonlocal analysis.

The paper is organized as follows. In section 2, we study the asymptotic behavior of solutions to the electroconvection model (1)–(5): we prove that the $L^2$ norm of the surface charge density $q$ decays in time to zero with a rate of order $t^{-1}$ whereas the velocity $u$ decays in time to zero with a rate of order $t^{-\frac{1}{2}}$. We also investigate the rate of decay of their higher-order derivatives, and we obtain decaying-in-time bounds in Hölder spaces $C^{0,\frac{1}{2}}$. In section 3, we prove that the differences $q - Q$ and $u - U$ decay to zero in $L^2$ faster than $q$ and $u$, with rates of order $t^{-1-\frac{5}{8}}$ and $t^{-\frac{1}{2}-\frac{1}{4}}$, respectively. In the Appendix, we present results on the existence and uniqueness of solutions to (1)–(5), based on the Banach fixed point theorem, the Aubin-Lions lemma and commutator estimates.

## 2. Long Time Behavior of Solutions

In this section, we consider the long-time behavior of solutions of the electroconvection model described by (1)–(5). We show that the charge density $q$ and the velocity $u$ converge to $0$ in the $H^2$ norm, and we investigate the rate of convergence.

For a function $f \in L^1(\mathbb{R}^2)$, we denote its Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} f(x) e^{-i\xi\cdot x} dx. \tag{6}$$

**Theorem 1.** *Let $u_0 \in L^2 \cap L^1$ be divergence-free and $q_0 \in L^2 \cap L^1$. There exist positive constants $\Gamma_0$ and $\Gamma_0'$ depending only on the initial data and some universal constants such that the global-in-time solution $(q, u)$ of (1)–(5) obeys*

$$\|q(t)\|_{L^2}^2 \le \frac{\Gamma_0}{(t+1)^2} \tag{7}$$

*and*

$$\|u(t)\|_{L^2}^2 \le \frac{\Gamma_0'}{t+1} \tag{8}$$

*for all $t \ge 0$.*

**Proof:** The proof is divided into several steps.

***Step 1*** *(Basic energy estimates)*. We take the $L^2$ inner product of equation (1) with $\Lambda^{-1} q$ and the $L^2$ inner product of equation (2) with $u$. Then we add the resulting energy equalities. Integrating by parts, we have the cancellations

$$(u \cdot \nabla u, u)_{L^2} = (\nabla p, u)_{L^2} = 0 \tag{9}$$

and

$$\begin{aligned} (u\cdot\nabla q, \Lambda^{-1}q)_{L^2} + (qRq, u)_{L^2} &= -(u\cdot\nabla\Lambda^{-1}q, q)_{L^2} + (qRq, u)_{L^2} \\ &= -(u\cdot Rq, q)_{L^2} + (qRq, u)_{L^2} = 0 \end{aligned} \tag{10}$$

due to the divergence-free condition (3). Thus, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\Lambda^{-\frac{1}{2}} q\|_{L^2}^2 + \|u\|_{L^2}^2\right) + \|q\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = 0. \tag{11}$$

We integrate in time from $0$ to $t$ and we take the supremum over all positive times $t \ge 0$. We get

$$\sup_{t\ge 0}\left\{\|\Lambda^{-\frac{1}{2}}q(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + \int_0^t 2\left(\|q(s)\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}^2\right) ds\right\} = \|\Lambda^{-\frac{1}{2}}q_0\|_{L^2}^2 + \|u_0\|_{L^2}^2 \tag{12}$$

ending the proof of Step 1.

***Step 2*** *(Pointwise bounds for the Fourier transform of the charge density $q$).* The Fourier transform of $q$ evolves according to

$$\partial_t \widehat{q}(\xi, t) + (\widehat{u\cdot\nabla q})(\xi, t) + \widehat{\Lambda q}(\xi, t) = 0. \tag{13}$$

The fractional Laplacian $\Lambda$ is a Fourier multiplier with symbol $|\xi|$, hence

$$\partial_t \widehat{q} + |\xi|\widehat{q} = -\widehat{u\cdot\nabla q}. \tag{14}$$

We estimate the Fourier transform of the nonlinear term

$$|\widehat{u\cdot\nabla q}| = |\widehat{\nabla\cdot(uq)}| \le C|\xi|\|u\|_{L^2}\|q\|_{L^2} \tag{15}$$

using the divergence-free condition (3), the boundedness of the Fourier transform of a function by its $L^1$ norm, and the Cauchy-Schwarz inequality. This yields the differential inequality

$$\partial_t \widehat{q} + |\xi| \widehat{q} \le C|\xi| \|u\|_{L^2} \|q\|_{L^2}. \tag{16}$$

We multiply both sides by the integrating factor $e^{|\xi|t}$ and integrate in time from $0$ to $t$. We obtain the bound

$$|\widehat{q}(\xi,t)| \le |\widehat{q_0}(\xi)| + C|\xi| \int_0^t \|u(s)\|_{L^2} \|q(s)\|_{L^2} ds. \tag{17}$$

As a consequence of Step 1 and the Cauchy-Schwarz inequality, we get the pointwise bound

$$|\widehat{q}(\xi,t)| \le \|q_0\|_{L^1} + C_0 |\xi| \sqrt{t} \tag{18}$$

where $C_0$ is a time-independent constant depending only on $\|u_0\|_{L^2}$ and $\|\Lambda^{-\frac{1}{2}} q_0\|_{L^2}$. This finishes the proof of Step 2.

***Step 3*** *(Decaying bound for the $L^2$ norm of the charge density).* The $L^2$ norm of $q$ evolves according to

$$\frac{1}{2} \frac{d}{dt} \|q\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} q\|_{L^2}^2 = 0. \tag{19}$$

In view of Parseval's identity and the fact that $\Lambda^{\frac{1}{2}}$ is a Fourier multiplier with symbol $|\xi|^{\frac{1}{2}}$, we have

$$\|\Lambda^{\frac{1}{2}} q\|_{L^2}^2 = \|\widehat{\Lambda^{\frac{1}{2}} q}\|_{L^2}^2 = \int_{\mathbb{R}^2} |\xi| |\widehat{q}(\xi,t)|^2 d\xi. \tag{20}$$

We bound the dissipation from below

$$\int_{\mathbb{R}^2} |\xi| |\widehat{q}(\xi,t)|^2 d\xi \ge \int_{|\xi| > \rho(t)} |\xi| |\widehat{q}(\xi,t)|^2 d\xi \tag{21}$$

where $\rho(t)$ is the function defined on $[0,\infty)$ by

$$\rho(t) = \frac{r}{2(t+1)} \tag{22}$$

for some positive constant $r$ to be determined later. We note that

$$\begin{aligned} \int_{|\xi|>\rho(t)} |\xi| |\widehat{q}(\xi,t)|^2 d\xi &\ge \rho(t) \int_{|\xi|>\rho(t)} |\widehat{q}(\xi,t)|^2 d\xi \\ &= \rho(t) \int_{\mathbb{R}^2} |\widehat{q}(\xi,t)|^2 d\xi - \rho(t) \int_{|\xi| \le \rho(t)} |\widehat{q}(\xi,t)|^2 d\xi \\ &= \rho(t) \|q\|_{L^2}^2 - \rho(t) \int_{|\xi| \le \rho(t)} |\widehat{q}(\xi,t)|^2 d\xi \end{aligned} \tag{23}$$

where we used Parseval's identity. Consequently, we obtain the energy inequality

$$\frac{d}{dt} \|q\|_{L^2}^2 + 2\rho(t) \|q\|_{L^2}^2 \le 2\rho(t) \int_{|\xi| \le \rho(t)} |\widehat{q}(\xi,t)|^2 d\xi. \tag{24}$$

By the pointwise bound (18) and Fubini's theorem for spherical coordinates, we estimate

$$\begin{aligned} \int_{|\xi| \le \rho(t)} |\widehat{q}(\xi,t)|^2 d\xi &\le \int_{|\xi| \le \rho(t)} \left( \|q_0\|_{L^1} + C_0 |\xi| \sqrt{t} \right)^2 d\xi = C \int_0^{\rho(t)} r \left( \|q_0\|_{L^1} + C_0 r \sqrt{t} \right)^2 dr \\ &\le C \int_0^{\rho(t)} r \left( \|q_0\|_{L^1}^2 + C_0^2 r^2 t \right) dr \le \Gamma_1 \left( \rho(t)^2 + t\rho(t)^4 \right) \end{aligned} \tag{25}$$

where $\Gamma_1$ depends only on the initial data. We obtain

$$\frac{d}{dt} \|q\|_{L^2}^2 + 2\rho(t) \|q\|_{L^2}^2 \le 2\Gamma_1 (\rho(t)^3 + t\rho(t)^5) \tag{26}$$

for all $t \geq 0$. We multiply both sides of the inequality by the integrating factor

$$e^{2\int_0^t \rho(s)ds} = e^{r\int_0^t \frac{1}{s+1}ds} = e^{r\ln(t+1)} = (t+1)^r \tag{27}$$

and then we integrate in time from $0$ to $t$. We get

$$\begin{aligned}\|q(t)\|_{L^2}^2 &\leq \frac{\|q_0\|_{L^2}^2}{(t+1)^r} + \frac{\Gamma_2}{(t+1)^r}\int_0^t \left(\frac{1}{(s+1)^3} + \frac{1}{(s+1)^4}\right)(s+1)^r ds \\ &\leq \frac{\|q_0\|_{L^2}^2}{(t+1)^r} + \frac{\Gamma_2}{(t+1)^r}\left(\frac{(t+1)^{r-2}}{r-2} - \frac{1}{r-2} + \frac{(t+1)^{r-3}}{r-3} - \frac{1}{r-3}\right) \\ &\leq \frac{\|q_0\|_{L^2}^2}{(t+1)^r} + \frac{\Gamma_2}{(r-2)(t+1)^2} + \frac{\Gamma_2}{(r-3)(t+1)^3}\end{aligned} \tag{28}$$

for any $r > 3$. Here $\Gamma_2$ depends on $r$ and the initial data. We choose $r = 4$ and we obtain the bound

$$\|q\|_{L^2}^2 \leq \frac{\Gamma_3}{(t+1)^2} \tag{29}$$

where $\Gamma_3$ is a positive constant depending only on the initial data. This completes the proof of (7) and Step 3.

***Step 4*** *(Pointwise bounds for the Fourier transform of the velocity $u$).* Applying the Leray Projector $\mathbb{P}$ to equation (2), we have

$$\partial_t u + \mathbb{P}(u \cdot \nabla u) - \Delta u = -\mathbb{P}(qRq), \tag{30}$$

where we used the incompressibility condition (3) and the fact that $\mathbb{P}$ and $-\Delta$ are Fourier multipliers so they commute. Hence the Fourier transform of $u$ obeys

$$\partial_t \widehat{u} + \mathbb{P}\widehat{(u \cdot \nabla u)} - \widehat{\Delta u} = -\widehat{\mathbb{P}(qRq)}. \tag{31}$$

We estimate

$$|\mathbb{P}\widehat{(u \cdot \nabla u)}(\xi, t)| \leq C|\xi||\widehat{u}(\xi, t)|^2 \leq C|\xi|\|u(t)\|_{L^2}^2 \tag{32}$$

and

$$|\widehat{\mathbb{P}(qRq)}(\xi, t)| \leq C\|(qRq)(t)\|_{L^1} \leq C\|q(t)\|_{L^2}^2 \tag{33}$$

in view of the boundedness of the Riesz transforms on $L^2(\mathbb{R}^2)$. We obtain

$$\partial_t \widehat{u} + |\xi|^2\widehat{u} \leq C|\xi|\|u\|_{L^2}^2 + C\|q\|_{L^2}^2 \tag{34}$$

and hence

$$|\widehat{u}(\xi, t)| \leq \|u_0\|_{L^1} + C|\xi|\int_0^t \|u(s)\|_{L^2}^2 ds + C\int_0^t \|q(s)\|_{L^2}^2 ds \tag{35}$$

for all $\xi \in \mathbb{R}^2$ and $t \geq 0$. In view of the bound (12), we get

$$|\widehat{u}(\xi, t)| \leq \Gamma_4 + C|\xi|\int_0^t \|u(s)\|_{L^2}^2 ds \tag{36}$$

where $\Gamma_4$ is a positive constant depending only on the initial data. This completes the proof of Step 4.

***Step 5*** *(Bounds for $\int_0^t (s+1)^\gamma \|\Lambda^{\frac{1}{2}} q(s)\|_{L^2}^2 ds$ where $\gamma \neq 2$ is a real number).* Let $\gamma \neq 2$ be a real number. The time evolution of $(t+1)^\gamma\|q\|_{L^2}^2$ is described by the energy equality

$$\frac{d}{dt}((t+1)^\gamma\|q\|_{L^2}^2) + 2(t+1)^\gamma\|\Lambda^{\frac{1}{2}} q\|_{L^2}^2 = \gamma(t+1)^{\gamma-1}\|q\|_{L^2}^2. \tag{37}$$

We integrate in time from $0$ to $t$, make use of the $L^2$ decay of the charge density $q$ given by (7), and obtain the bound

$$\int_0^t (s+1)^{\gamma} \|\Lambda^{\frac{1}{2}} q(s)\|_{L^2}^2 ds \le C \left( \|q_0\|_{L^2}^2 + \frac{\gamma \Gamma_0}{\gamma - 2}(t+1)^{\gamma - 2} - \frac{\gamma\Gamma_0}{\gamma - 2} \right). \tag{38}$$

***Step 6*** *(Decaying bound for the $L^2$ norm of the velocity).* The $L^2$ norm of the velocity evolves according to

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = -\int_{\mathbb{R}^2} qRq \cdot u dx. \tag{39}$$

In view of Hölder's inequality, the boundedness of the Riesz transforms on $L^4(\mathbb{R}^2)$, and Ladyzhenskaya's interpolation inequality, we bound

$$\left| \int_{\mathbb{R}^2} qRq \cdot u \right| \le C\|q\|_{L^2}\|q\|_{L^4}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} \tag{40}$$

yielding

$$\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \le C\|q\|_{L^2}^{\frac{4}{3}}\|q\|_{L^4}^{\frac{4}{3}}\|u\|_{L^2}^{\frac{2}{3}}. \tag{41}$$

In view of (12), this latter energy inequality reduces to

$$\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \le \Gamma_5\|q\|_{L^2}^{\frac{4}{3}}\|q\|_{L^4}^{\frac{4}{3}} \tag{42}$$

where $\Gamma_5$ is a positive constant depending only on the initial data. By Parseval's identity, we have

$$\frac{d}{dt}\|u\|_{L^2}^2 + \int_{\mathbb{R}^2} |\xi|^2 |\widehat{u}(\xi,t)|^2 d\xi \le \Gamma_5\|q\|_{L^2}^{\frac{4}{3}}\|q\|_{L^4}^{\frac{4}{3}}. \tag{43}$$

For a positive function $\rho_1(t)$ continuous on $[0,\infty)$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} |\xi|^2|\widehat{u}(\xi,t)|^2 d\xi &\ge \int_{|\xi|>\rho_1(t)} |\xi|^2|\widehat{u}(\xi,t)|^2 d\xi \ge \rho_1(t)^2 \int_{|\xi|>\rho_1(t)} |\widehat{u}(\xi,t)|^2 d\xi \\ &\ge \rho_1(t)^2 \left( \int_{\mathbb{R}^2} |\widehat{u}(\xi,t)|^2 d\xi - \int_{|\xi|\le\rho_1(t)} |\widehat{u}(\xi,t)|^2 d\xi \right) \\ &= \rho_1(t)^2\|u\|_{L^2}^2 - \rho_1(t)^2 \int_{|\xi|\le\rho_1(t)} |\widehat{u}(\xi,t)|^2 d\xi. \end{aligned} \tag{44}$$

Consequently, we obtain the energy inequality

$$\frac{d}{dt}\|u\|_{L^2}^2 + \rho_1(t)^2\|u\|_{L^2}^2 \le \Gamma_5\|q\|_{L^2}^{\frac{4}{3}}\|q\|_{L^4}^{\frac{4}{3}} + \rho_1(t)^2 \int_{|\xi|\le\rho_1(t)} |\widehat{u}(\xi,t)|^2 d\xi. \tag{45}$$

Using (36), we have

$$\begin{aligned} &\int_{|\xi|\le\rho_1(t)} |\widehat{u}(\xi,t)|^2 d\xi \le C\int_0^{\rho_1(t)} r \left( \Gamma_4^2 + Cr^2 \left\{ \int_0^t \|u(s)\|_{L^2}^2 ds \right\}^2 \right) dr \\ &\le \Gamma_6\rho_1(t)^2 + C\rho_1(t)^4 \left( \int_0^t \|u(s)\|_{L^2}^2 ds \right)^2 \end{aligned} \tag{46}$$

and thus

$$\frac{d}{dt}\|u\|_{L^2}^2 + \rho_1(t)^2\|u\|_{L^2}^2 \le \Gamma_6\rho_1(t)^4 + C\rho_1(t)^6 \left( \int_0^t \|u(s)\|_{L^2}^2 ds \right)^2 + \Gamma_5\|q\|_{L^2}^{\frac{4}{3}}\|q\|_{L^4}^{\frac{4}{3}}. \tag{47}$$

Multiplying by the integrating factor $e^{\int_0^t \rho_1(s)^2 ds}$, and integrating in time from $0$ to $t$, we obtain

$$\begin{aligned}\|u(t)\|_{L^2}^2 &\le \frac{\|u_0\|_{L^2}^2}{e^{\int_0^t \rho_1(s)^2 ds}} + \frac{\Gamma_6}{e^{\int_0^t \rho_1(s)^2 ds}} \int_0^t e^{\int_0^s \rho_1(\tau)^2 d\tau} \rho_1(s)^4 ds \\ &+ \frac{C}{e^{\int_0^t \rho_1(s)^2 ds}} \int_0^t \left(e^{\int_0^s \rho_1(\tau)^2 d\tau} \rho_1(s)^6\right) \left(\int_0^s \|u(\tau)\|_{L^2}^2 d\tau\right)^2 ds \\ &+ \frac{\Gamma_5}{e^{\int_0^t \rho_1(s)^2 ds}} \int_0^t \|q\|_{L^2}^{\frac{4}{3}} \|q\|_{L^4}^{\frac{4}{3}} e^{\int_0^s \rho_1(\tau)^2 d\tau} ds. \end{aligned} \tag{48}$$

In view of Young's inequality for products, the continuous embedding of $H^{\frac{1}{2}}$ in $L^4$, and the bound (29), we estimate

$$\begin{aligned}\int_0^t \|q\|_{L^2}^{\frac{4}{3}} \|q\|_{L^4}^{\frac{4}{3}} e^{\int_0^s \rho_1(\tau)^2 d\tau} ds &\le C \int_0^t \|q(s)\|_{L^2}^4 ds + C \int_0^t \|q(s)\|_{L^4}^2 e^{C\int_0^s \rho_1(\tau)^2 d\tau} ds \\ &\le C \int_0^t \|q(s)\|_{L^2}^4 ds + C \int_0^t (\|q(s)\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} q(s)\|_{L^2}^2) e^{C\int_0^s \rho_1(\tau)^2 d\tau} ds \\ &\le \Gamma_7 + C \int_0^t (\|q(s)\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} q(s)\|_{L^2}^2) e^{C\int_0^s \rho_1(\tau)^2 d\tau} ds \end{aligned} \tag{49}$$

for any $t \ge 0$, and so

$$\begin{aligned}\|u(t)\|_{L^2}^2 &\le \frac{\|u_0\|_{L^2}^2}{e^{\int_0^t \rho_1(s)^2 ds}} + \frac{\Gamma_6}{e^{\int_0^t \rho_1(s)^2 ds}} \int_0^t e^{\int_0^s \rho_1(\tau)^2 d\tau} \rho_1(s)^4 ds \\ &+ \frac{C}{e^{\int_0^t \rho_1(s)^2 ds}} \int_0^t \left(e^{\int_0^s \rho_1(\tau)^2 d\tau} \rho_1(s)^6\right) \left(\int_0^s \|u(\tau)\|_{L^2}^2 d\tau\right)^2 ds \\ &+ \frac{\Gamma_7}{e^{\int_0^t \rho_1(s)^2 ds}} + \frac{C}{e^{\int_0^t \rho_1(s)^2 ds}} \int_0^t (\|q(s)\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} q(s)\|_{L^2}^2) e^{C\int_0^s \rho_1(\tau)^2 d\tau} ds \end{aligned} \tag{50}$$

for any $t \ge 0$. In order to obtain the sharp decaying bound for the velocity $u$, we need the following three sub-steps:

***Step 6.1*** *(Logarithmic decaying bound for the $L^2$ norm of the velocity).* We take $\rho_1(t) = (e + t)^{-\frac{1}{2}} [\ln(e+t)]^{-\frac{1}{2}}$. In this case, the integrating factor is given by

$$e^{\int_0^t \rho_1(s)^2 ds} = e^{\int_0^t \frac{1}{(e+s)\ln(e+s)} ds} = e^{\ln[\ln(e+t)]} = \ln(e+t) \tag{51}$$

and so (50) becomes

$$\begin{aligned}\|u(t)\|_{L^2}^2 &\le \frac{\|u_0\|_{L^2}^2}{\ln(e+t)} + \frac{\Gamma_6}{\ln(e+t)} \int_0^t \frac{1}{(e+s)^2 \ln(e+s)} ds \\ &+ \frac{C\|u_0\|_{L^2}^2}{\ln(e+t)} \int_0^t \frac{s^2}{(e+s)^3 [\ln(e+s)]^2} ds + \frac{\Gamma_8}{\ln(e+t)} \end{aligned} \tag{52}$$

in view of the uniform boundedness of $\|u\|_{L^2}$ by $\|u_0\|_{L^2}$ and the bound (38) with $\gamma = 1$. Here $\Gamma_8$ is a positive constant depending only on the size of the initial charge density. We note that

$$\int_0^t \frac{s^2}{(e+s)^3 [\ln(e+s)]^2} ds \le \int_0^t \frac{1}{(e+s) [\ln(e+s)]^2} ds = 1 - \frac{1}{\ln(e+t)} \le 1 \tag{53}$$

for any $t \ge 0$. Therefore,

$$\|u(t)\|_{L^2}^2 \le \frac{\Gamma_9}{\ln(e+t)} \tag{54}$$

for all $t \ge 0$, where $\Gamma_9$ is a constant depending only on the initial data.

***Step 6.2*** *(Almost sharp decaying bound for the $L^2$ norm of the velocity).* In order to improve the logarithmic decay (54), we take $\rho_1(t) = r^{\frac{1}{2}}(t+1)^{-\frac{1}{2}}$ for some $r$ to be chosen later. In this case, the integrating factor is given by

$$e^{\int_0^t \rho_1(s)^2 ds} = e^{r\int_0^t \frac{1}{(s+1)}ds} = e^{r\ln(t+1)} = (t+1)^r \tag{55}$$

and so (50) becomes

$$\begin{aligned}\|u(t)\|_{L^2}^2 \le \frac{\|u_0\|_{L^2}^2}{(t+1)^r} + \frac{\Gamma_6}{(t+1)^r}\int_0^t \frac{(s+1)^r}{(s+1)^2}ds + \frac{C}{(t+1)^r}\int_0^t \frac{(s+1)^r}{(s+1)^3}\left(\int_0^s \|u(\tau)\|_{L^2}^2 d\tau\right)^2 ds \\ + \frac{C}{(t+1)^r}\int_0^t (s+1)^r\|\Lambda^{\frac{1}{2}}q(s)\|_{L^2}^2 ds\end{aligned} \tag{56}$$

for all $t \ge 0$. We have

$$\frac{\Gamma_6}{(t+1)^r}\int_0^t \frac{(s+1)^r}{(s+1)^2}ds = \frac{\Gamma_6}{(r-1)(t+1)^r}\left((t+1)^{r-1} - 1\right) \le \frac{\Gamma_6}{(r-1)(t+1)} \tag{57}$$

for any $r > 1$. Moreover, applying the Cauchy-Schwarz inequality in the time variable yields

$$\left(\int_0^s \|u(\tau)\|_{L^2}^2 d\tau\right)^2 \le s\int_0^s \|u(\tau)\|_{L^2}^4 d\tau, \tag{58}$$

so that

$$\begin{aligned}\frac{C}{(t+1)^r}\int_0^t \frac{(s+1)^r}{(s+1)^3}\left(\int_0^s \|u(\tau)\|_{L^2}^2 d\tau\right)^2 ds &\le \frac{C}{(t+1)^r}\left(\int_0^t (s+1)^{r-2}ds\right)\left(\int_0^t \|u(s)\|_{L^2}^4 ds\right) \\ &\le \frac{C}{(r-1)(t+1)}\left(\int_0^t \|u(s)\|_{L^2}^4 ds\right)\end{aligned} \tag{59}$$

for any $r > 1$. Taking $r = 3$ and using (54) and (38) give

$$\|u(t)\|_{L^2}^2 \le \frac{\Gamma_{10}}{t+1} + \frac{\Gamma_{10}}{t+1}\int_0^t \frac{\|u(s)\|_{L^2}^2}{\ln(e+s)}ds \tag{60}$$

and so

$$(t+1)\|u(t)\|_{L^2}^2 \le \Gamma_{10} + C'\Gamma_{10}\int_0^t \frac{(s+1)\|u(s)\|_{L^2}^2}{(s+e)\ln(e+s)}ds \tag{61}$$

for any $t \ge 0$. By Gronwall's inequality, we obtain

$$\begin{aligned}(t+1)\|u(t)\|_{L^2}^2 &\le \Gamma_{10} + C'\Gamma_{10}^2\int_0^t \frac{e^{\int_s^t \frac{1}{(e+\tau)\ln(e+\tau)}d\tau}}{(e+s)\ln(e+s)}ds \\ &= \Gamma_{10} + C'\Gamma_{10}^2\int_0^t \frac{\ln(e+t)}{(e+s)\left[\ln(e+s)\right]^2}ds \le \Gamma_{10} + C'\Gamma_{10}^2\ln(e+t).\end{aligned} \tag{62}$$

Therefore,

$$\|u(t)\|_{L^2}^2 \le \frac{\Gamma_{11}\ln(t+e)}{t+1} \tag{63}$$

for any $t \ge 0$, where $\Gamma_{11}$ is a constant depending only on the initial data.

***Step 6.3*** *(Sharp decaying bound for the $L^2$ norm of the velocity).* Finally, we prove (8). We take $\rho_1(t) = \sqrt{2}(t+1)^{-\frac{1}{2}}$ as in the previous sub-step, and we obtain the bound

$$\|u(t)\|_{L^2}^2 \le \frac{\|u_0\|_{L^2}^2}{(t+1)^3} + \frac{\Gamma_{12}}{t+1} + \frac{C}{(t+1)^3}\int_0^t \left(\int_0^s \|u(\tau)\|_{L^2}^2 d\tau\right)^2 ds \tag{64}$$

for all $t \geq 0$. We note that

$$\int_0^s \|u(\tau)\|_{L^2}^2 d\tau \leq \Gamma_{13} \int_0^s \frac{\ln(\tau+e)}{\tau+1} d\tau \leq C\Gamma_{13} \int_0^s \frac{\ln(\tau+e)}{\tau+e} d\tau \leq \Gamma_{14} \left[\ln(s+e)\right]^2 \tag{65}$$

and so

$$\int_0^t \left( \int_0^s \|u(\tau)\|_{L^2}^2 d\tau \right)^2 ds \leq \Gamma_{15} \int_0^t \sqrt{s+1} ds \tag{66}$$

for all $t \geq 0$. Therefore,

$$\|u(t)\|_{L^2}^2 \leq \frac{\Gamma_{16}}{t+1} \tag{67}$$

for all $t \geq 0$, where $\Gamma_{16}$ is a positive constant depending only on the initial data. This ends the proof of Theorem 1.

Now we study the rate of convergence of the gradients of the charge density and the velocity.

**Theorem 2.** *Let $u_0 \in H^1 \cap L^1$ be divergence-free and $q_0 \in H^1 \cap L^1$. There exist positive constants $K_0$ and $K_0'$ depending only on the initial data and some universal constants such that the unique global-in-time solution $(q, u)$ of (1)–(5) obeys*

$$\|\nabla u(t)\|_{L^2}^2 \leq \frac{K_0}{(t+1)^2} \tag{68}$$

*and*

$$\|\nabla q(t)\|_{L^2}^2 \leq \frac{K_0'}{(t+1)^4} \tag{69}$$

*for all $t \geq 0$.*

**Proof:** The proof is divided into 4 steps.

***Step 1*** *(Decaying bounds for the $L^4$ norm of $q$).* The evolution of the $L^4$ norm of $q$ is described by the energy equality

$$\frac{1}{4}\frac{d}{dt}\|q\|_{L^4}^4 + \int_{\mathbb{R}^2} q^3 \Lambda q dx = 0. \tag{70}$$

In view of the Córdoba-Córdoba inequality [6], the dissipation is bounded from below

$$\int_{\mathbb{R}^2} q^3 \Lambda q dx \geq \frac{1}{2}\|\Lambda^{\frac{1}{2}}(q^2)\|_{L^2}^2 \tag{71}$$

and thus

$$\int_{\mathbb{R}^2} q^3 \Lambda q dx \geq c\|q\|_{L^8}^4 \tag{72}$$

due to Gagliardo-Nirenberg inequalities. Using interpolation inequalities in $L^p$ spaces and the uniform boundedness of the $L^2$ norm of the charge density $q$ by $\|q_0\|_{L^2}$, we have the bound

$$\|q\|_{L^4} \leq \|q\|_{L^2}^{\frac{1}{3}} \|q\|_{L^8}^{\frac{2}{3}} \leq \|q_0\|_{L^2}^{\frac{1}{3}} \|q\|_{L^8}^{\frac{2}{3}} \tag{73}$$

from which we conclude that

$$\int_{\mathbb{R}^2} q^3 \Lambda q dx \geq C\|q_0\|_{L^2}^{-2}\|q\|_{L^4}^6 \tag{74}$$

and hence

$$\frac{d}{dt}\|q\|_{L^4}^4 + \frac{C}{\|q_0\|_{L^2}^2}\|q\|_{L^4}^6 \leq 0. \tag{75}$$

Letting $y = \|q\|_{L^4}^4$, we obtain the Bernouilli ordinary differential inequality

$$\frac{dy}{dt} + \frac{C}{\|q_0\|_{L^2}^2} y^{\frac{3}{2}} \leq 0. \tag{76}$$

We apply a change of variable given by $u = y^{-\frac{1}{2}}$ and we get

$$\frac{-2}{u^3}\frac{du}{dt} + \frac{C}{\|q_0\|_{L^2}^2}\frac{1}{u^3} \leq 0 \tag{77}$$

so

$$\frac{du}{dt} \geq \frac{C}{2\|q_0\|_{L^2}^2}. \tag{78}$$

Integrating in time from $0$ to $t$, we arrive at the bound

$$\|q\|_{L^4}^{-2} \geq \|q_0\|_{L^4}^{-2} + \frac{C}{\|q_0\|_{L^2}^2}t \geq K(1+t) \tag{79}$$

where $K$ is a constant depending on the initial data. Consequently, we obtain

$$\|q\|_{L^4} \leq \frac{1}{\sqrt{K}}\frac{1}{(1+t)^{\frac{1}{2}}} \tag{80}$$

for all $t \geq 0$.

***Step 2*** *(Decaying bound for the $L^2$ norm of $\nabla u$).* We take the $L^2$ inner product of equation (2) with $-\Delta u$ and we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 = \int_{\mathbb{R}^2} qRq \cdot \Delta u. \tag{81}$$

The nonlinear term $(u \cdot \nabla u, \Delta u)_{L^2}$ vanishes due to the fact that the matrix $M^tM^2$ has a zero trace where $M$ is the two-by-two traceless matrix whose entries are given by $M_{ij} = \frac{\partial u_i}{\partial x_j}$ and $M^t$ is its transpose. In view of Hölder's inequality with exponents 4,4,2, the boundedness of the Riesz transforms on $L^4(\mathbb{R}^2)$, the continuous embedding of $H^{\frac{1}{2}}$ in $L^4$ , and Young's inequality, we obtain

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \leq C\|q\|_{L^4}^4 \leq C\|q\|_{L^4}^2\left(\|q\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}}q\|_{L^2}^2\right) \tag{82}$$

Using the $L^4$ estimate (80) and the $L^2$ decay (7), we have

$$C\|q\|_{L^4}^2\left(\|q\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}}q\|_{L^2}^2\right) \leq K_1(1+t)^{-3} + K_2(1+t)^{-1}\|\Lambda^{\frac{1}{2}}q\|_{L^2}^2 \tag{83}$$

where $K_1$ and $K_2$ are positve constants depending on the initial data. We note that the initial charge density is assumed to be in $H^1$ and so it belongs to $L^4$ due to the Sobolev embedding of $H^1(\mathbb{R}^2)$ into $L^4(\mathbb{R}^2)$. Going back to (82), we have

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \leq K_1(1+t)^{-3} + K_2(1+t)^{-1}\|\Lambda^{\frac{1}{2}}q\|_{L^2}^2 \tag{84}$$

For $t \in [0,\infty)$, we let

$$\rho_2(t) = r^{\frac{1}{2}}(t+1)^{-\frac{1}{2}} \tag{85}$$

for some $r > 0$ to be chosen later. By Parseval's identity, we get

$$\begin{aligned}&\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \rho_2(t)^2\|\nabla u\|_{L^2}^2\\ &\quad\leq K_1(1+t)^{-3} + K_2(1+t)^{-1}\|\Lambda^{\frac{1}{2}}q\|_{L^2}^2 + \rho_2(t)^2\int_{|\xi|\leq\rho_2(t)}|\widehat{\nabla u}(\xi,t)|^2 d\xi.\end{aligned} \tag{86}$$

In view of the inequality $|\widehat{\nabla u}(\xi,t)|^2 \leq |\xi|^2|\widehat{u}(\xi,t)|^2$ that holds for all $\xi \in \mathbb{R}^2$ and all $t \geq 0$, Parseval's identity, and the $L^2$ decay of the velocity $u$ given by (8), we have

$$\int_{|\xi|\leq\rho_2(t)}|\widehat{\nabla u}(\xi,t)|^2 d\xi \leq \rho_2(t)^2\|u(t)\|_{L^2}^2 \leq \Gamma_0'\rho_2(t)^2(t+1)^{-1}, \tag{87}$$

and consequently, it holds that

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \rho_2(t)^2\|\nabla u\|_{L^2}^2 \le K_1(1+t)^{-3} + K_2(1+t)^{-1}\|\Lambda^{\frac{1}{2}}q\|_{L^2}^2 + \Gamma_0'\rho_2(t)^4(t+1)^{-1} \tag{88}$$

for $t \ge 0$. We multiply by the integrating factor $(t+1)^r$, integrate in time from $0$ to $t$, apply (38) with $\gamma = r-1$, and obtain

$$\begin{aligned}\|\nabla u\|_{L^2}^2 \le \frac{\|\nabla u_0\|_{L^2}^2}{(t+1)^r} + \frac{K_1}{(t+1)^r}\int_0^t \frac{(s+1)^r}{(s+1)^3}ds \\ + \frac{K_2}{(t+1)^r}\left(\|q_0\|_{L^2}^2 + \frac{r-1}{r-3}\Gamma_0(t+1)^{r-3} - \frac{r-1}{r-3}\Gamma_0\right) + \frac{\Gamma_0' r^2}{(t+1)^r}\int_0^t \frac{(s+1)^r}{(s+1)^3}ds.\end{aligned} \tag{89}$$

Now we chose any $r > 3$ and we obtain the bound (68).

***Step 3*** *(Bounds for $\int_0^t (s+1)^\gamma \|\Delta u(s)\|_{L^2}^2 ds$ where $\gamma \notin \{2,3\}$ is a real number).* Let $\gamma \notin \{2,3\}$. The differential inequality (82) yields

$$\begin{aligned}&\frac{d}{dt}((t+1)^\gamma\|\nabla u\|_{L^2}^2) - \gamma(t+1)^{\gamma-1}\|\nabla u\|_{L^2}^2 + (t+1)^\gamma\|\Delta u\|_{L^2}^2 \\ &\le C(t+1)^\gamma\|q\|_{L^4}^2\left(\|q\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}}q\|_{L^2}^2\right)\end{aligned} \tag{90}$$

for all $t \ge 0$. Integrating in time from $0$ to $t$ and using (80), (38), and (68), we obtain

$$\begin{aligned}\int_0^t (s+1)^\gamma\|\Delta u\|_{L^2}^2 ds &\le \|\nabla u_0\|_{L^2}^2 + (\gamma K_3 + K_4)\int_0^t (s+1)^{\gamma-3}ds \\ &\quad + K_5\left(\|q_0\|_{L^2}^2 + \frac{\gamma-1}{\gamma-3}\Gamma_0(t+1)^{\gamma-3} - \frac{\gamma-1}{\gamma-3}\Gamma_0\right) \\ &\le K_6 + \frac{\gamma K_3 + K_4}{\gamma-2}\left[(t+1)^{\gamma-2} - 1\right] + \frac{(\gamma-1)\Gamma_0 K_5}{\gamma-3}\left[(t+1)^{\gamma-3} - 1\right]\end{aligned} \tag{91}$$

for some positive constants $K_3, K_4, K_5, K_6$ depending on $\|\nabla u_0\|_{L^2}$ and $\|q_0\|_{L^4}$.

***Step 4*** *(Decaying bound for the $L^2$ norm of $\nabla q$).* The $L^2$ norm of the gradient of $q$ evolves according to the energy equality

$$\frac{1}{2}\frac{d}{dt}\|\nabla q\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}}q\|_{L^2}^2 = (u\cdot\nabla q, \Delta q)_{L^2}. \tag{92}$$

In view of the Ladyzhenskaya interpolation inequality

$$\|\nabla u\|_{L^4} \le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}} \tag{93}$$

and the interpolation inequality [1]

$$\|\nabla q\|_{L^{\frac{8}{3}}}^2 \le C\|q\|_{L^4}^{\frac{1}{2}}\|\Lambda^{\frac{3}{2}}q\|_{L^2}^{\frac{3}{2}}, \tag{94}$$

we estimate the nonlinear term

$$|(u\cdot\nabla q, \Delta q)_{L^2}| \le \|\nabla u\|_{L^4}\|\nabla q\|_{L^{\frac{8}{3}}}^2 \le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|q\|_{L^4}^{\frac{1}{2}}\|\Lambda^{\frac{3}{2}}q\|_{L^2}^{\frac{3}{2}}. \tag{95}$$

Applying Young's inequality, we obtain

$$\frac{d}{dt}\|\nabla q\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}}q\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2\|q\|_{L^4}^2. \tag{96}$$

In view of (68) and (83), we have

$$\frac{d}{dt}\|\nabla q\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}}\nabla q\|_{L^2}^2 \le \frac{K_7}{(t+1)^3}\|\Delta u\|_{L^2}^2 \tag{97}$$

for all $t \ge 0$. Here $K_7$ depends only the initial data. Letting

$$\rho_3(t) = r(t+1)^{-1}, \tag{98}$$

we split the dissipation term,

$$\|\Lambda^{\frac{3}{2}} q\|_{L^2}^2 \ge \rho_3(t)\|\nabla q\|_{L^2}^2 - \rho_3(t)\int_{|\xi|\le\rho_3(t)} |\widehat{\nabla q}(\xi,t)|^2 d\xi \tag{99}$$

yielding the differential inequality

$$\frac{d}{dt}\|\nabla q\|_{L^2}^2 + \rho_3(t)\|\nabla q\|_{L^2}^2 \le \frac{K_7}{(t+1)^3}\|\Delta u\|_{L^2}^2 + \rho_3(t)\int_{|\xi|\le\rho_3(t)} |\widehat{\nabla q}(\xi,t)|^2 d\xi \tag{100}$$

In view of the pointwise bound $|\widehat{\nabla q}(\xi,t)| \le |\xi||\widehat{q}(\xi,t)|$, we have

$$\int_{|\xi|\le\rho_3(t)} |\widehat{\nabla q}(\xi,t)|^2 d\xi \le \rho_3(t)^2\|q\|_{L^2}^2 \le \Gamma_0\rho_3(t)^2(t+1)^{-2}, \tag{101}$$

and so

$$\frac{d}{dt}\|\nabla q\|_{L^2}^2 + \rho_3(t)\|\nabla q\|_{L^2}^2 \le \frac{K_7}{(t+1)^3}\|\Delta u\|_{L^2}^2 + K_8\rho_3(t)^3(t+1)^{-2}. \tag{102}$$

We multiply both sides by $(t+1)^r$ and we integrate in time from $0$ to $t$. We obtain

$$\|\nabla q(t)\|_{L^2}^2 \le \frac{\|\nabla q_0\|_{L^2}^2}{(t+1)^r} + \frac{K_7}{(t+1)^r}\int_0^t (s+1)^{r-3}\|\Delta u(s)\|_{L^2}^2 ds \tag{103}$$

$$+ \frac{r^3 K_8}{(t+1)^r}\int_0^t (s+1)^{r-5} ds. \tag{104}$$

In view of (91) applied with $\gamma = r-3$, we have

$$\int_0^t (s+1)^{r-3}\|\Delta u(s)\|_{L^2}^2 ds \le K_6 + \frac{(r-3)K_3 + K_4}{r-5}\left[(t+1)^{r-5} - 1\right] + \frac{(r-4)\Gamma_0 K_5}{r-6}\left[(t+1)^{r-6} - 1\right] \tag{105}$$

for any $r \notin \{5,6\}$, and so

$$\frac{K_7}{(t+1)^r}\int_0^t (s+1)^{r-3}\|\Delta u(s)\|_{L^2}^2 ds \le \frac{K_9}{(t+1)^5} \tag{106}$$

for any $r > 6$. Putting (103) and (106) together and choosing $r = 7$ give the desired decay (69). This completes the proof of Theorem 2.

Now we establish decaying bounds for higher order derivatives. We need the following proposition.

**Proposition 1.** *Let $u_0 \in H^1 \cap L^1$ be divergence-free and $q_0 \in H^1 \cap L^1$ . Let $\beta > 6$. There exists a positive constant $C_\beta^0$ depending on $\beta$, the size of the initial data, and some universal constants such that the solution $q$ of* (1)–(5) *obeys*

$$\int_0^t (s+1)^{\beta}\|\Lambda^{\frac{3}{2}} q(s)\|_{L^2}^2 ds \le \|\nabla q_0\|_{L^2}^2 + C\|\nabla u_0\|_{L^2}^2 + C_\beta^0(t+1)^{\beta-4} \tag{107}$$

*for all $t \ge 0$.*

**Proof:** In view of the differential inequality (96), we have

$$\frac{d}{dt}(t+1)^{\beta}\|\nabla q\|_{L^2}^2 - \beta(t+1)^{\beta-1}\|\nabla q\|_{L^2}^2 + (t+1)^{\beta}\|\Lambda^{\frac{3}{2}}q\|_{L^2}^2 \le C(t+1)^{\beta}\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2\|q\|_{L^4}^2. \quad (108)$$

Integrating in time from $0$ to $t$, using the bounds (80) and (68) and applying (91) with $\gamma = \beta - 3$, we obtain (107).

**Theorem 3.** *Let $u_0 \in H^2 \cap L^1$ be divergence-free and $q_0 \in H^2 \cap L^1$. There exist positive constants $M_0$ and $M_0'$ depending only on the initial data and some universal constants such that the unique global-in-time solution $(q, u)$ of (1)–(5) obeys*

$$\|\Delta u(t)\|_{L^2}^2 \le \frac{M_0}{(t+1)^3} \quad (109)$$

*and*

$$\|\Delta q(t)\|_{L^2}^2 \le \frac{M_0'}{(t+1)^6} \quad (110)$$

*for all $t \ge 0$.*

**Proof:** The $L^2$ norm of $\Delta u$ evolves according to the energy equality

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \|\nabla\Delta u\|_{L^2}^2 = -\int_{\mathbb{R}^2}\Delta(qRq)\cdot\Delta u dx - \int_{\mathbb{R}^2}\Delta(u\cdot\nabla u)\cdot\Delta u dx. \quad (111)$$

Integrating by parts, using (3), and applying Ladyzhenskaya's interpolation inequality, we estimate the second term on the right hand side in (111) as

$$\left|\int_{\mathbb{R}^2}\Delta(u\cdot\nabla u)\cdot\Delta u dx\right| \le C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\|\nabla\Delta u\|_{L^2}. \quad (112)$$

In view of the boundedness of the Riesz transforms on $L^4$ and the continuous embedding of $\dot{H}^{\frac{1}{2}}$ in $L^4$, we obtain for the first term on the right hand side in (111)

$$\left|\int_{\mathbb{R}^2}\Delta(qRq)\cdot\Delta u dx\right| \le C\|q\|_{L^4}\|\Lambda^{\frac{3}{2}}q\|_{L^2}\|\nabla\Delta u\|_{L^2}. \quad (113)$$

From (111)–(113) and using Young's inequality, we obtain the energy inequality

$$\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \|\nabla\Delta u\|_{L^2}^2 \le C\|q\|_{L^4}^2\|\Lambda^{\frac{3}{2}}q\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2. \quad (114)$$

In view of Parseval's identity, we have

$$\begin{aligned}\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \rho_2(t)^2\|\Delta u\|_{L^2}^2 &\le C\|q\|_{L^4}^2\|\Lambda^{\frac{3}{2}}q\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2 \\ &\quad + \rho_2(t)^2\int_{|\xi|\le\rho_2(t)}|\widehat{\Delta u}(\xi,t)|^2 d\xi \quad (115)\end{aligned}$$

where $\rho_2$ is the function defined by (85). The decay bounds (7), (68), and (83) yield

$$\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \rho_2(t)^2\|\Delta u\|_{L^2}^2 \le \frac{M_1}{t+1}\|\Lambda^{\frac{3}{2}}q\|_{L^2}^2 + \frac{M_2}{(t+1)^2}\|\Delta u\|_{L^2}^2 + M_3\rho_2(t)^6(t+1)^{-1} \quad (116)$$

for all $t \geq 0$, where $M_1, M_2$ and $M_3$ are positive constants depending only on the initial data. Multiplying by the integrating factor and integrating in time from $0$ to $t$, we obtain

$$\|\Delta u\|_{L^2}^2 \leq \frac{\|\Delta u_0\|_{L^2}^2}{(t+1)^r} + \frac{M_1}{(t+1)^r} \int_0^t (s+1)^{r-1} \|\Lambda^{\frac{3}{2}} q\|_{L^2}^2 ds$$
$$+ \frac{M_2}{(t+1)^r} \int_0^t (s+1)^{r-2} \|\Delta u\|_{L^2}^2 ds + \frac{M_3}{(t+1)^r} \int_0^t (s+1)^{r-4} ds. \tag{117}$$

We choose $r = 8$. In view of the bound (91) applied with $\gamma = r - 2$ and Proposition 1 applied with $\beta = r - 1$, we obtain (109).

Now, we establish decaying estimate for $\|\Delta q\|_{L^2}^2$ which evolves according to

$$\frac{1}{2}\frac{d}{dt}\|\Delta q\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}} q\|_{L^2}^2 = 2\int_{\mathbb{R}^2} (\nabla u \cdot \nabla(\nabla q))\Delta q dx + \int_{\mathbb{R}^2} (\Delta u \cdot \nabla q)\Delta q dx. \tag{118}$$

In view of the Gagliardo-Nirenberg interpolation inequality

$$\|\Delta q\|_{L^2} \leq C\|\Lambda^{\frac{5}{2}} q\|_{L^2}^{\frac{4}{5}} \|q\|_{L^2}^{\frac{1}{5}}, \tag{119}$$

the Sobolev embedding inequality

$$\|\Delta q\|_{L^4} \leq C\|\Lambda^{\frac{5}{2}} q\|_{L^2}, \tag{120}$$

and the bound

$$\|\nabla\nabla q\|_{L^2} = \|\nabla\Lambda^{-1}\nabla\Lambda^{-1}\Delta q\|_{L^2} \leq C\|\Delta q\|_{L^2} \tag{121}$$

that follows from the boundedness of the Riesz transforms on $L^2$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta q\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}} q\|_{L^2}^2 &\leq C\|\nabla u\|_{L^4}\|\Delta q\|_{L^2}\|\Delta q\|_{L^4} + C\|\Delta u\|_{L^2}\|\nabla q\|_{L^4}\|\Delta q\|_{L^4} \\
&\leq C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\Lambda^{\frac{5}{2}} q\|_{L^2}^{\frac{9}{5}}\|q\|_{L^2}^{\frac{1}{5}} + C\|\Delta u\|_{L^2}\|\Lambda^{\frac{3}{2}} q\|_{L^2}\|\Delta q\|_{L^4} \\
&\leq \frac{1}{2}\|\Lambda^{\frac{5}{2}} q\|_{L^2}^2 + C\|\nabla u\|_{L^2}^5\|\Delta u\|_{L^2}^5\|q\|_{L^2}^2 + C\|\Delta u\|_{L^2}^2\|\Lambda^{\frac{3}{2}} q\|_{L^2}^2.
\end{aligned} \tag{122}$$

Consequently,

$$\begin{aligned}
\frac{d}{dt}\|\Delta q\|_{L^2}^2 + \rho(t)\|\Delta q\|_{L^2}^2 &\leq C\|\nabla u\|_{L^2}^5\|\Delta u\|_{L^2}^5\|q\|_{L^2}^2 \\
&+ C\|\Delta u\|_{L^2}^2\|\Lambda^{\frac{3}{2}} q\|_{L^2}^2 + \rho(t)\int_{|\xi|\leq\rho(t)} |\widehat{\Delta q}(\xi,t)|^2 d\xi
\end{aligned} \tag{123}$$

where $\rho$ is defined by (22). In view of the estimates (7), (68) and (109), and Proposition 1 applied with $\beta = r - 3$, we obtain (110). This ends the proof of Theorem 3.

Let $C^{0,\frac{1}{2}}$ be the space of bounded $1/2$-Hölder continuous functions on $\mathbb{R}^2$ with

$$\|f\|_{C^{0,\frac{1}{2}}} = \|f\|_{L^\infty} + \sup_{x,y\in\mathbb{R}^2, x\neq y} \frac{|f(x) - f(y)|}{|x-y|^{\frac{1}{2}}}. \tag{124}$$

In view of the continuous Sobolev embedding of $W^{1,4}$ into $C^{0,\frac{1}{2}}$, the Ladyzhenskaya interpolation inequality, and Theorems 1, 2, and 3, we obtain the following statement.

**Corollary 1.** *Let $u_0 \in H^2 \cap L^1$ be divergence-free and $q_0 \in H^2 \cap L^1$. There exist positive constants $A_0$ and $A_0'$ depending only on the initial data and some universal constants such that the unique global-in-time solution $(q, u)$ of (1)–(5) obeys*

$$\|u(t)\|^2_{C^{0,\frac{1}{2}}} \le \frac{A_0}{(t+1)^{\frac{3}{2}}} \tag{125}$$

*and*

$$\|q(t)\|^2_{C^{0,\frac{1}{2}}} \le \frac{A_0'}{(t+1)^3} \tag{126}$$

*for all $t \ge 0$.*

## 3. Decomposition of the solution

In this section, we decompose the charge density $q$ and the velocity $u$ solutions of (1)–(5) in the sum of solutions $Q$ and $U$ of the linear equations

$$\partial_t Q + \Lambda Q = 0 \tag{127}$$

and

$$\partial_t U - \Delta U = 0 \tag{128}$$

with initial datum $Q(0) = q_0$ and $U(0) = u_0$ and remainders. We study the decays of the remainders $q - Q$ and $u - U$ in $L^2$ and we show that they are faster than the decays of the $L^2$ norms of $q$ and $u$ respectively. The solutions of (127) and (128) are given explicitly by

$$Q(t) = \int_{\mathbb{R}^2} K_t^1(x-w) q_0(w) dw \tag{129}$$

and

$$U(t) = \int_{\mathbb{R}^2} K_t^2(x-w) u_0(w) dw \tag{130}$$

where $K_t^s$ is the kernel defined by its Fourier transform

$$\mathcal{F}(K_t^s)(\xi) = e^{-|\xi|^s t}. \tag{131}$$

We address the pointwise behavior of the Fourier transforms of the differences $q - Q$ and $u - U$. We need first the following lemmas.

**Lemma 1.** *For $f \in L^2(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, we let*

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{\sqrt{|y|^2+1} - \sqrt{|x|^2+1}}{|x-y|^3} f(y) dy. \tag{132}$$

*There exists a universal constant $C > 0$ (independent of $f$) such that*

$$\|Tf\|_{L^2} \le C \|f\|_{L^2}. \tag{133}$$

**Proof:** We write

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} (a(y) - a(x)) k(x-y) f(y) dy. \tag{134}$$

where $a(x)$ is the function defined on $\mathbb{R}^2$ by

$$a(x) = \sqrt{|x|^2+1} \tag{135}$$

and $k(x)$ is the function defined on $\mathbb{R}^2 \setminus \{0\}$ by

$$k(x) = \frac{1}{|x|^3}. \tag{136}$$

We note that $k$ is homogeneous of degree $-3$. Moreover, the gradient of $a$ is given by

$$\nabla a(x) = \left( \frac{x_1}{\sqrt{|x|^2+1}}, \frac{x_2}{\sqrt{|x|^2+1}} \right) \tag{137}$$

and satisfies $\|\nabla a\|_{L^\infty} \le 1$. Therefore, $T$ is a well-defined operator and bounded on $L^2$ (see page 435 in Section 2 of [5]).

Using Lemma 1, we study the evolution of $(\sqrt{|x|^2+1})q(x)$ in $L^2(\mathbb{R}^2)$.

**Lemma 2.** *Let $u_0 \in H^1 \cap L^1$ be divergence-free and $q_0 \in H^1 \cap L^1$. Furthermore, suppose that $\int_{\mathbb{R}^2} |x|^2 q_0(x)^2 dx < \infty$. Then there exists a positive constant $R_1 > 0$ depending only on the initial data such that*

$$\|(\sqrt{|\cdot|^2+1})q(\cdot,t)\|_{L^2} \le R_1 \ln(t+1) + \|(\sqrt{|\cdot|^2+1})q_0(\cdot)\|_{L^2} \tag{138}$$

*holds for all $t \ge 0$.*

**Proof:** Let $a(x) = \sqrt{|x|^2+1}$. The evolution of $aq$ is described by

$$\partial_t(aq) + au\cdot\nabla q + a\Lambda q = 0. \tag{139}$$

Multiplying by $aq$ and integrating in the space variable over $\mathbb{R}^2$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|aq\|_{L^2}^2 + \int_{\mathbb{R}^2} (a\Lambda q)aq = -\int_{\mathbb{R}^2} (au\cdot\nabla q)aq. \tag{140}$$

The cancellation

$$\int_{\mathbb{R}^2} (u\cdot\nabla(aq))aq = 0 \tag{141}$$

holds due to (3), so we can rewrite the nonlinear term as

$$-\int_{\mathbb{R}^2} (au\cdot\nabla q)aq = \int_{\mathbb{R}^2} (u\cdot\nabla a)q^2 a. \tag{142}$$

By Hölder's inequality, Ladyzhenskaya's interpolation inequality, and the decaying bounds for the $L^2$ norms of $q, u, \nabla u$ and $\nabla q$ given by (7), (8), (68) and (69), respectively, we estimate

$$\begin{aligned} \left| \int_{\mathbb{R}^2} (u\cdot\nabla a)q^2 a \right| &\le \|\nabla a\|_{L^\infty} \|q\|_{L^4} \|u\|_{L^4} \|aq\|_{L^2} \\ &\le C\|q\|_{L^2}^{\frac{1}{2}} \|\nabla q\|_{L^2}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|aq\|_{L^2} \le R_2 (t+1)^{-\frac{9}{4}} \|aq\|_{L^2} \end{aligned} \tag{143}$$

for some constant $R_2$ depending only on the initial data. Now we write the linear term as the sum

$$\begin{aligned} \int_{\mathbb{R}^2} (a\Lambda q)aq &= \int_{\mathbb{R}^2} aq\Lambda(aq) + \int_{\mathbb{R}^2} (aq)\left[a\Lambda q - \Lambda(aq)\right] \\ &= \|\Lambda^{\frac{1}{2}}(aq)\|_{L^2}^2 + \int_{\mathbb{R}^2} (aq)\left[a\Lambda q - \Lambda(aq)\right]. \end{aligned} \tag{144}$$

By the Cauchy-Schwarz inequality, we bound

$$\left| \int_{\mathbb{R}^2} (aq)\left[a\Lambda q - \Lambda(aq)\right] \right| \le \|aq\|_{L^2} \|a\Lambda q - \Lambda(aq)\|_{L^2}. \tag{145}$$

The pointwise formula for the fractional Laplacian of order 1 yields

$$(a\Lambda q - \Lambda(aq))(x) = C\int_{\mathbb{R}^2}\left[\frac{a(x)q(x)-a(x)q(y)}{|x-y|^3} - \frac{a(x)q(x)-a(y)q(y)}{|x-y|^3}\right]dy$$
$$= C\int_{\mathbb{R}^2}\frac{a(y)-a(x)}{|x-y|^3}q(y)dy \tag{146}$$

where $C$ is positive universal constant. As a consequence of Lemma 1 and (7), we obtain

$$\|a\Lambda q - \Lambda(aq)\|_{L^2} \le C\|q\|_{L^2} \le C(t+1)^{-1}. \tag{147}$$

Therefore, the $L^2$ norm of $aq$ obeys the energy inequality

$$\frac{1}{2}\frac{d}{dt}\|aq\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}}(aq)\|_{L^2} \le \left[R_2(t+1)^{-\frac{9}{4}} + C(t+1)^{-1}\right]\|aq\|_{L^2} \tag{148}$$

so

$$\frac{1}{2}\frac{d}{dt}\|aq\|_{L^2}^2 \le R_3(t+1)^{-1}\|aq\|_{L^2} \tag{149}$$

for some positive constant $R_3$ depending only on the initial data. Dividing both sides of the inequality by $\|aq\|_{L^2}$, we get

$$\frac{d}{dt}\|aq\|_{L^2} \le R_3(t+1)^{-1}. \tag{150}$$

Integrating in time from $0$ to $t$, we obtain (138).

The following lemma is needed to obtain a growth in $|\xi|$ for the Fourier transform of $\mathbb{P}(qRq)$.

**Lemma 3.** *Let $f \in L^2(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2}|x|^2 f(x)^2 dx < \infty$. Then*

$$|\mathbb{P}\widehat{(fRf)}(\xi)| \le C|\xi|\|f\|_{L^2}\left(\int_{\mathbb{R}^2}|x|^2|f(x)|^2dx\right)^{\frac{1}{2}}. \tag{151}$$

*where $\mathbb{P}$ is the Leray projector and $R = (R_1, R_2)$ is the Riesz transform vector on $\mathbb{R}^2$.*

**Proof:** The Leray projector is a Fourier multiplier with a symbol denoted by $m(\xi)$. We have

$$\mathbb{P}\widehat{(fRf)}(\xi) = m(\xi)\widehat{fRf}(\xi) \tag{152}$$

for all $\xi \in \mathbb{R}^2$. We note that $m(\xi)$ is bounded uniformly in $\xi$. Now, the Fourier transform of $fRf$ at $\xi$ is given by

$$\widehat{fRf}(\xi) = \int_{\mathbb{R}^2} f(x)Rf(x)e^{-i\xi\cdot x}dx \tag{153}$$

for $\xi \in \mathbb{R}^2$. Since the Riesz transform is antisymmetric, we have

$$\int_{\mathbb{R}^2} f(x)Rf(x)dx = 0 \tag{154}$$

and so we can write $\widehat{fRf}$ at $\xi$ as

$$\widehat{fRf}(\xi) = \int_{\mathbb{R}^2} f(x)Rf(x)\left(e^{-i\xi\cdot x}-1\right)dx. \tag{155}$$

Using the identity

$$|e^{-i\xi\cdot x} - 1| \le |\xi||x| \tag{156}$$

that holds for all $x, \xi \in \mathbb{R}^2$, we estimate

$$|\widehat{fRf}(\xi)| \le |\xi|\int_{\mathbb{R}^2}|x||f(x)||Rf(x)|dx \le |\xi|\|Rf\|_{L^2}\left(\int_{\mathbb{R}^2}|x|^2|f(x)|^2dx\right)^{\frac{1}{2}} \tag{157}$$

in view of the Cauchy-Schwarz inequality. This gives the pointwise estimate (151).

As a consequence of lemmas 2 and 3, we obtain the following statement.

**Proposition 2.** *Let $u_0 \in H^1 \cap L^1$ be divergence-free and $q_0 \in H^1 \cap L^1$ . Furthermore, suppose that $\int_{\mathbb{R}^2} |x|^2 q_0(x)^2 dx < \infty$. Let $(q,u)$ be the solution of (1)–(5). Let $\zeta = q - Q$ and $v = u - U$. Then there exist positive constants $R_4$, $R_5$ and $R_6$ depending only on the initial data such that the Fourier transforms of $\zeta$ and $v$ satisfy the pointwise bounds*

$$|\widehat{\zeta}(\xi,t)| \le R_4|\xi| \tag{158}$$

*and*

$$|\widehat{v}(\xi,t)| \le R_5|\xi|\ln(t+1) + R_6|\xi|\ln^2(t+1) \tag{159}$$

*for all $\xi \in \mathbb{R}^2$ and $t \ge 0$.*

**Proof:** The Fourier transform of $\zeta$ obeys

$$\partial_t\widehat{\zeta} + |\xi|\widehat{\zeta} = -\widehat{u\cdot\nabla q} \le |\xi|\|u\|_{L^2}\|q\|_{L^2}. \tag{160}$$

Consequently,

$$|\widehat{\zeta}(\xi,t)| \le \int_0^t |\xi|\|u\|_{L^2}\|q\|_{L^2} \le R_4|\xi| \tag{161}$$

in view of the decaying bounds (7) and (8). The Fourier transform of $v$ evolves according to

$$\partial_t\widehat{v} + |\xi|^2\widehat{v} = -\mathbb{P}\widehat{(u\cdot\nabla u)} - \mathbb{P}\widehat{(qRq)}. \tag{162}$$

Thus

$$|\widehat{v}(\xi,t)| \le C|\xi|\int_0^t \|u\|_{L^2}^2 ds + C|\xi|\int_0^t \|q\|_{L^2}\left(\int_{\mathbb{R}^2}|x|^2 q(x)^2 dx\right)^{\frac12} ds \tag{163}$$

by Lemma 3. In view of Lemma 2 and the decaying estimates (7) and (8), we obtain (159).

**Theorem 4.** *Let $u_0 \in H^1 \cap L^1$ be divergence-free and $q_0 \in H^1 \cap L^1$ . Furthermore, suppose that $\int_{\mathbb{R}^2} |x|^2 q_0(x)^2 dx < \infty$. Let $(q,u)$ be the solution of (1)–(5). Then there exist positive constants $R_7$ and $R_8$ depending only on the initial data such that the differences $q - Q$ and $u - U$ satisfy*

$$\|q(t)-Q(t)\|_{L^2}^2 \le \frac{R_7}{(t+1)^{2+\frac54}} \tag{164}$$

*and*

$$\|u(t)-U(t)\|_{L^2}^2 \le \frac{R_8}{(t+1)^{1+\frac12}} \tag{165}$$

*for all $t \ge 0$.*

**Proof:** Let $\zeta = q - Q$ and $v = u - U$. We have

$$\partial_t\zeta + \Lambda\zeta = -u\cdot\nabla q. \tag{166}$$

Taking the $L^2$ inner product of equation (166) with $\zeta$ and estimating the nonlinearity via interpolation, we obtain

$$\begin{aligned}\frac12\frac{d}{dt}\|\zeta\|_{L^2}^2 + \|\Lambda^{\frac12}\zeta\|_{L^2}^2 &= \int_{\mathbb{R}^2}(u\cdot\nabla q)Q dx \le \|u\|_{L^4}\|\nabla q\|_{L^2}\|Q\|_{L^4} \\ &\le C\|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\nabla q\|_{L^2}\|Q\|_{L^2}^{\frac12}\|\nabla Q\|_{L^2}^{\frac12}.\end{aligned} \tag{167}$$

As a consequence of Theorems 1 and 2, we obtain the energy inequality

$$\frac{d}{dt}\|\zeta\|_{L^2}^2 + \|\Lambda^{\frac12}\zeta\|_{L^2}^2 \le \frac{R_9}{(t+1)^{4+\frac14}} \tag{168}$$

where $R_9$ is a positive constant depending only on the initial data. For a fixed $r$, we let $\rho(t) = r(t+1)^{-1}$. Then

$$\frac{d}{dt}\|\zeta\|_{L^2}^2 + \rho(t)\|\zeta\|_{L^2}^2 \le \frac{R_9}{(t+1)^{4+\frac{1}{4}}} + \rho(t)\int_{|\xi|\le\rho(t)} |\widehat{\zeta}(\xi,t)|^2 d\xi. \tag{169}$$

Using (158), we estimate

$$\int_{|\xi|\le\rho(t)} |\widehat{\zeta}(\xi,t)|^2 d\xi \le R_{10}\rho(t)^4 \tag{170}$$

and we obtain

$$\frac{d}{dt}\|\zeta\|_{L^2}^2 + \rho(t)\|\zeta\|_{L^2}^2 \le \frac{R_9}{(t+1)^{4+\frac{1}{4}}} + R_{10}\rho(t)^5. \tag{171}$$

Multiplying by the factor $(s+1)^r$, integrating in the time variable $s$ from $0$ to $t$, and choosing any $r > 4$, we obtain the desired bound (164). Now, $v$ obeys

$$\partial_t v - \Delta v = -u\cdot\nabla u - qRq - \nabla p. \tag{172}$$

Taking the $L^2$ inner product of this latter equation with $v$ and using the fact that $v$ is divergence-free, we get the energy equation

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2} + \|\nabla v\|_{L^2}^2 = \int_{\mathbb{R}^2}(u\cdot\nabla u)\cdot U dx - \int_{\mathbb{R}^2}(qRq)\cdot v dx. \tag{173}$$

We estimate

$$\int_{\mathbb{R}^2}(u\cdot\nabla u)\cdot U dx \le \|u\|_{L^4}^2\|\nabla U\|_{L^2} \le C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla U\|_{L^2} \le \frac{R_{11}}{(t+1)^{2+\frac{1}{2}}} \tag{174}$$

in view of Theorems 1 and 2, and

$$\int_{\mathbb{R}^2}(qRq)\cdot v dx \le C\|q\|_{L^4}^2\|v\|_{L^2} \le C\|q\|_{L^2}\|\nabla q\|_{L^2}\|v\|_{L^2} \le \frac{R_{12}}{(t+1)^{2+\frac{3}{2}}} \tag{175}$$

in view of the decaying estimate (7), (8), and (69). This yields the energy inequality

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \rho_2(t)^2\|v\|_{L^2}^2 \le \frac{R_{13}}{(t+1)^{1+\frac{3}{2}}} + \rho_2(t)^2\int_{|\xi|\le\rho_2(t)} |\widehat{v}(\xi,t)|^2 d\xi \tag{176}$$

where $\rho_2(t)^2 = r(t+1)^{-1}$. Using the pointwise bound for the Fourier transform of $v$ given by (159), we have

$$\int_{|\xi|\le\rho_2(t)} |\widehat{v}(\xi,t)|^2 d\xi \le R_{14}\left[\ln^2(t+1) + \ln^4(t+1)\right]\rho_2(t)^4 \le R_{15}\sqrt{t+1}\rho_2(t)^4, \tag{177}$$

hence

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \rho_2(t)^2\|v\|_{L^2}^2 \le \frac{R_{13}}{(t+1)^{1+\frac{3}{2}}} + R_{15}\sqrt{t+1}\rho_2(t)^6. \tag{178}$$

We multiply both sides by $(s+1)^r$, we integrate from $0$ to $t$, we choose any $r > 3/2$, and we obtain (165).

## 4. Appendix: Existence and Uniqueness of Solutions

In this appendix, we prove the existence of weak and strong solutions for the electroconvection model (1)–(5).

**Definition 1.** *A solution $(q,u)$ of* (1)–(5) *is said to be a weak solution on $[0,T]$ if it solves* (1)–(5) *in the sense of distributions, $u$ is divergence-free in the sense of distributions,*

$$u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1) \tag{179}$$

*and*

$$q \in L^\infty(0,T;L^2) \cap L^2(0,T;H^{1/2}). \tag{180}$$

**Theorem 5.** *Let $u_0 \in L^2$ be divergence-free, let $q_0 \in L^2$. Let $T>0$ be arbitrary. There exists a weak solution $(q,u)$ of the system* (1)–(5) *on $[0,T]$.*

**Proof.** We briefly sketch the main ideas of the proof. For $0<\epsilon\le 1$, we consider a viscous approximation of (1)–(5) given by

$$\begin{cases} \partial_t q^\epsilon + u^\epsilon\cdot\nabla q^\epsilon + \Lambda q^\epsilon - \epsilon\Delta q^\epsilon = 0 \\ \partial_t u^\epsilon + u^\epsilon\cdot\nabla u^\epsilon - \Delta u^\epsilon + \nabla p^\epsilon = -q^\epsilon R q^\epsilon \\ \nabla\cdot u^\epsilon = 0 \end{cases} \tag{181}$$

with smoothed out initial data $u_0^\epsilon = J_\epsilon u_0$ and $q_0^\epsilon = J_\epsilon q_0$, where $J_\epsilon$ is a standard mollifier operator. For each $\epsilon>0$, we consider the map

$$(q(t),u(t)) \mapsto \Phi_\epsilon((q,u))(t) = (e^{\epsilon t\Delta}J_\epsilon q_0 - \mathcal{A}_t^\epsilon(q^\epsilon,u^\epsilon), e^{t\Delta}J_\epsilon u_0 - \mathcal{B}_t^\epsilon(q^\epsilon,u^\epsilon)) \tag{182}$$

where

$$\mathcal{A}_t^\epsilon(q^\epsilon,u^\epsilon) = \int_0^t e^{\epsilon(t-s)\Delta}(u^\epsilon\cdot\nabla q^\epsilon)(s)ds + \int_0^t e^{\epsilon(t-s)\Delta}\Lambda q^\epsilon(s)ds \tag{183}$$

and

$$\mathcal{B}_t^\epsilon(q^\epsilon,u^\epsilon) = \int_0^t e^{(t-s)\Delta}\mathbb{P}(u^\epsilon\cdot\nabla u^\epsilon)(s)ds + \int_0^t e^{(t-s)\Delta}\mathbb{P}(q^\epsilon R q^\epsilon)(s)ds. \tag{184}$$

There exists a time $T_\epsilon = T_\epsilon(\epsilon, \|u_0\|_{L^2}, \|q_0\|_{L^2})>0$ such that the map $\Phi_\epsilon$ is a contraction on the Banach space

$$X_T = L^\infty(0,T;\bar{B}_{L^2}(2\|q_0\|_{L^2}) \oplus L^\infty(0,T;\bar{B}_{L^2_\sigma}(2\|u_0\|_{L^2}) \tag{185}$$

where $\bar{B}_{L^2}(r)$ is the closed ball in $L^2$, and $\bar{B}_{L^2_\sigma}$ is the closed ball in the space of $L^2$ divergence-free vectors. Consequently, $\Phi_\epsilon$ has a fixed point $(q^\epsilon,u^\epsilon)\in X_{T_\epsilon}$ solving (181). This solution extends to the time interval $[0,T]$, and this can be obtained by establishing uniform-in-time bounds for $(q^\epsilon,u^\epsilon)$ on $[0,T]$. Indeed, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\Lambda^{-\frac12}q^\epsilon\|_{L^2}^2 + \|u^\epsilon\|_{L^2}^2\right) + \|q^\epsilon\|_{L^2}^2 + \|\nabla u^\epsilon\|_{L^2}^2 + \epsilon\|\Lambda^{\frac12}q^\epsilon\|_{L^2}^2 = 0 \tag{186}$$

as shown in (11). Hence the family of mollified velocities $(u^\epsilon)_\epsilon$ is uniformly bounded in $L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$. On the other hand, the $L^2$ norm of $q^\epsilon$ evolves according to

$$\frac{1}{2}\frac{d}{dt}\|q^\epsilon\|_{L^2}^2 + \|\Lambda^{\frac12}q^\epsilon\|_{L^2}^2 + \epsilon\|\nabla q^\epsilon\|_{L^2}^2 = 0, \tag{187}$$

and so the family of mollified charge densities $(q^\epsilon)_\epsilon$ is uniformly bounded in $L^\infty(0,T;L^2)\cap L^2(0,T;H^{\frac12})$. The $q^\epsilon$ and $u^\epsilon$ equations imply that the sequence of time derivatives $(\partial_t q^\epsilon)_\epsilon$ and

$(\partial_t u^\epsilon)_\epsilon$ are uniformly bounded in $L^2(0,T;H^{-\frac{3}{2}})$ and $L^2(0,T;H^{-1})$ respectively. By the Aubin-Lions lemma, the sequence $((q^\epsilon,u^\epsilon))_\epsilon$ has a subsequence that converges strongly in $L^2(0,T;L^2)$ to a weak solution $(q,u)$ of (1)–(5). We omit further details.

**Definition 2.** *A weak solution $(q,u)$ of (1)–(5) is said to be a strong solution on $[0,T]$ if*

$$u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2) \tag{188}$$

*and*

$$q \in L^\infty(0,T;L^4) \cap L^2(0,T;H^{1/2}). \tag{189}$$

**Theorem 6.** *Let $u_0 \in H^1$ be divergence-free and $q_0 \in L^4$. Let $T > 0$ be arbitrary. There exists a unique strong solution $(u,q)$ of the system (1)–(5) on $[0,T]$.*

**Proof.** We take the $L^2$ inner product of the equation satisfied by $q^\epsilon$ in (181) with $(q^\epsilon)^3$. In view of the divergence-free condition satisfied by $u^\epsilon$, the nonlinear term vanishes, that is

$$\int_{\mathbb{R}^2} u^\epsilon \cdot \nabla q^\epsilon (q^\epsilon)^3 dx = 0. \tag{190}$$

By the Córdoba-Córdoba inequality ([6]), we have

$$\int_{\mathbb{R}^2} (q^\epsilon)^3 \Lambda q^\epsilon dx \ge 0 \tag{191}$$

and

$$-\int_{\mathbb{R}^2} (q^\epsilon)^3 \Delta q^\epsilon dx \ge 0. \tag{192}$$

Consequently, we obtain

$$\frac{1}{4}\frac{d}{dt}\|q^\epsilon\|_{L^4}^4 \le 0 \tag{193}$$

which yields the boundedness of $q$ in $L^\infty(0,T;L^4(\mathbb{R}^2))$ by the Banach Alaoglu theorem and the lower semi-continuity of the norm. The $L^2$ norm of $\nabla u^\epsilon$ obeys the energy inequality

$$\frac{d}{dt}\|\nabla u^\epsilon\|_{L^2}^2 + \|\Delta u^\epsilon\|_{L^2}^2 \le C\|q^\epsilon\|_{L^4}^4 \tag{194}$$

as shown in (81), yielding the boundedness of $u$ in $L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$. Now we prove the uniqueness of strong solutions. Suppose $(q_1,u_1)$ and $(q_2,u_2)$ are strong solutions of (1)–(5) with same initial data. Let $q = q_1 - q_2, u = u_1 - u_2$ and $p = p_1 - p_2$. Then $q$ satisfies

$$\partial_t q + \Lambda q = -u_1 \cdot \nabla q - u \cdot \nabla q_2 \tag{195}$$

and $u$ satisfies

$$\partial_t u - \Delta u + \nabla p = -qRq_1 - q_2 Rq - u_1 \cdot \nabla u - u \cdot \nabla u_2. \tag{196}$$

We take the $L^2$ inner product of (195) with $\Lambda^{-1}q$ and the $L^2$ inner product of (196) with $u$. We add the resulting energy equalities. We have a cancellation

$$-\int_{\mathbb{R}^2} (u \cdot \nabla q_2)\Lambda^{-1} q dx - \int_{\mathbb{R}^2} (q_2 Rq) \cdot u dx = 0 \tag{197}$$

obtained from integration by parts. In view of the Ladyzhenskaya's interpolation inequality, we estimate

$$\left|\int_{\mathbb{R}^2} (qRq_1) \cdot u dx\right| \le C\|q\|_{L^2}\|q_1\|_{L^4}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} \le \frac{1}{4}\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|q\|_{L^2}^2 + C\|q_1\|_{L^4}^4\|u\|_{L^2}^2 \tag{198}$$

and

$$\left|\int_{\mathbb{R}^2} (u \cdot \nabla u_2) \cdot u dx\right| \le \|u\|_{L^4}^2 \|\nabla u_2\|_{L^2} \le \frac{1}{4}\|\nabla u\|_{L^2}^2 + C\|\nabla u_2\|_{L^2}^2\|u\|_{L^2}^2. \tag{199}$$

Now we write

$$\int_{\mathbb{R}^2} (u_1 \cdot \nabla q)\Lambda^{-1} q dx = \int_{\mathbb{R}^2} \left(\Lambda^{-\frac{1}{2}}(u_1 \cdot \nabla q) - u_1 \cdot \nabla \Lambda^{-\frac{1}{2}} q\right) \Lambda^{-\frac{1}{2}} q dx \tag{200}$$

via integration by parts, and we show below that

$$\left| \int_{\mathbb{R}^2} \left(\Lambda^{-\frac{1}{2}}(u_1 \cdot \nabla q) - u_1 \cdot \nabla \Lambda^{-\frac{1}{2}} q\right) \Lambda^{-\frac{1}{2}} q dx \right| \le C \|u_1\|_{H^2} \|q\|_{L^2} \|\Lambda^{-\frac{1}{2}} q\|_{L^2}. \tag{201}$$

Putting (197)-(201) together, we obtain the energy inequality

$$\frac{d}{dt}\left[\|\Lambda^{-\frac{1}{2}} q\|_{L^2}^2 + \|u\|_{L^2}^2\right] \le C\left[\|u_1\|_{H^2}^2 + \|\nabla u_2\|_{L^2}^2 + \|q_1\|_{L^4}^4\right]\left[\|\Lambda^{-\frac{1}{2}} q\|_{L^2}^2 + \|u\|_{L^2}^2\right] \tag{202}$$

from which we obtain uniqueness. Finally, we show that the estimate (201) holds by establishing the commutator estimate

$$\|\Lambda^{-\frac{1}{2}}(u_1 \cdot \nabla q) - u_1 \cdot \nabla \Lambda^{-\frac{1}{2}} q\|_{L^2} \le C \|u_1\|_{H^2} \|q\|_{L^2}. \tag{203}$$

Indeed, let $w \in L^2(\mathbb{R}^2)$. By Parseval's identity, we have

$$\int_{\mathbb{R}^2} (\Lambda^{-\frac{1}{2}}(u_1 \cdot \nabla q) - u_1 \cdot \nabla \Lambda^{-\frac{1}{2}} q)(x) w(x) dx = \int_{\mathbb{R}^2} \mathcal{F}(\Lambda^{-\frac{1}{2}}(u_1 \cdot \nabla q) - u_1 \cdot \nabla \Lambda^{-\frac{1}{2}} q)(\xi) \mathcal{F}w(\xi) d\xi. \tag{204}$$

But

$$\mathcal{F}(\Lambda^{-\frac{1}{2}}(u_1 \cdot \nabla q))(\xi) = \int_{\mathbb{R}^2} |\xi|^{-\frac{1}{2}} (\xi \cdot \mathcal{F}u_1(\xi - y)) \mathcal{F}q(y) dy \tag{205}$$

and

$$\mathcal{F}(u_1 \cdot \nabla \Lambda^{-\frac{1}{2}} q)(\xi) = \int_{\mathbb{R}^2} |y|^{-\frac{1}{2}} (\xi \cdot \mathcal{F}u_1(\xi - y)) \mathcal{F}q(y) dy. \tag{206}$$

Consequently,

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} (\Lambda^{-\frac{1}{2}}(u_1 \cdot \nabla q) - u_1 \cdot \nabla \Lambda^{-\frac{1}{2}} q)(x) w(x) dx \right| \\ &\le \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \min\{|\xi|, |y|\} \left| |\xi|^{-\frac{1}{2}} - |y|^{-\frac{1}{2}} \right| |\mathcal{F}u_1(\xi - y)| |\mathcal{F}q(y)| |\mathcal{F}w(\xi)| dy d\xi \end{aligned} \tag{207}$$

where we used

$$|\xi \cdot \mathcal{F}u_1(\xi - y)| \le \min\{|\xi|, |y|\} |\mathcal{F}u_1(\xi - y)| \tag{208}$$

which holds due to the fact that the velocity is divergence-free. We note that

$$\min\{|\xi|, |y|\} \left| |\xi|^{-\frac{1}{2}} - |y|^{-\frac{1}{2}} \right| \le \frac{\min\{|\xi|, |y|\}}{|\xi|^{\frac{1}{2}} |y|^{\frac{1}{2}}} |\xi - y|^{\frac{1}{2}} \le |\xi - y|^{\frac{1}{2}} \tag{209}$$

for all $\xi, y \in \mathbb{R}^2$. Therefore,

$$\begin{aligned} \left| \int_{\mathbb{R}^2} (\Lambda^{-\frac{1}{2}}(u_1 \cdot \nabla q) - u_1 \cdot \nabla \Lambda^{-\frac{1}{2}} q)(x) w(x) dx \right| &\le \||.|^{\frac{1}{2}} \mathcal{F}u_1(.)\|_{L^1} \|q\|_{L^2} \|w\|_{L^2} \\ &\le C \|u_1\|_{H^2} \|q\|_{L^2} \|w\|_{L^2} \end{aligned} \tag{210}$$

by Hölder's inequality and Young's convolution inequality. This gives (203) completing the proof of Theorem 6.

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DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PA 19122
*Email address*: elie.abdo@temple.edu

DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PA 19122
*Email address*: ignatova@temple.edu