On Electroconvection in Porous Media

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ABSTRACT. We consider the evolution of a surface charge density interacting with a two-dimensional fluid in a porous medium. In the momentum equation, Stokes's law is replaced by Darcy's law balanced by the electrical forces. This results in an active scalar equation, in which the transport velocity is computed from the scalar charge density via a nonlinear and nonlocal relation. We address the model in the whole space $\mathbb{R}^2$ and in the periodic setting on $\mathbb{T}^2$. We prove the global existence and uniqueness of solutions in Besov spaces $\dot{B}^{2/p}_{p,1}$ for small initial data. We also obtain the analyticity, regularity, and long-time behavior of solutions.

1. INTRODUCTION

Electroconvection, the evolution of charge distributions in fluids, was investigated experimentally and numerically in situations in which the fluid and charges are confined to thin films [13, 19, 20]. The charge distribution is carried by the fluid and diffuses because of the parallel component of the electrical field. This results in a nonlocal transport equation for the charge density ρ:

$$\partial_t \rho + u \cdot \nabla \rho + \Lambda \rho = 0 \tag{1.1}$$

where $\Lambda = (-\Delta)^{1/2}$ is the square root of the two-dimensional Laplacian and u is the fluid velocity. The fluid is incompressible and is forced by electrical forces $F = \rho E$, where E is the parallel component of the electrical field $E = -\nabla\Phi$, with ∇ the gradient in $\mathbb{R}^2$. The relationship between the electrical potential Φ and the charge distribution confined to a two-dimensional region is

$$\Phi = \Lambda^{-1}\rho,$$

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and we thus have

$$F = -\rho R\rho, \tag{1.2}$$

with $R = \nabla\Lambda^{-1}$ the Riesz transforms. In general, the fluid obeys Navier-Stokes or related equations driven by the forces F. The derivation of this system for the physical setup in bounded domains was obtained in [7], where global regularity and uniqueness of solutions were obtained for the coupling with Navier-Stokes equations.

In this paper we consider flow through a porous medium, in which the dominant dissipation mechanism is due not to the viscosity of the fluid, but rather to an effective damping caused by flow through pores. The Stokes operator is then replaced by $u + \nabla p$. We consider a system in which the fluid equilibrates rapidly and the Reynolds number is low, so that forces are balanced by damping,

$$u + \nabla p = F.$$

This balance, together with (1.2) and the requirement of incompressibility,

$$\nabla \cdot u = 0,$$

leads to

$$u = -\mathbb{P}(\rho R\rho) \tag{1.3}$$

where $\mathbb{P}$ is the Leray-Hodge projector on divergence-free vector fields. The electroconvection situation described above leads to the active scalar equation (1.1) with constitutive law (1.3), which is the equation we study in this work. In comparison to the work [7], the nonlinear advection is missing, but also there is no viscosity, and because of the nonlinearity in the electrical force, the velocity's dependence of the charge density is more singular. The equation is L^∞-critical, and resembles critical SQG [8–10, 15] except for the constitutive law (1.3) which in this case is nonlinear and doubly nonlocal. Global regularity of critical SQG was originally proved by different methods in [4, 18] and was subsequently extensively studied. In [16], the balance law (1.3) was used to describe the solvent in a Nernst-Planck-Darcy system of ionic diffusion in 2D and 3D. An active scalar equation describing flow through porous media with fractional dissipation and linear nonlocal constitutive law was studied in [5], and global regularity was obtained.

In this paper we show that the equation (1.1), (1.3) has global weak solutions. We describe local existence and uniqueness results for strong solutions. We also show that solutions with small initial data in Besov spaces slightly smaller than L^∞ exist globally and are Gevrey regular.

This paper is organized as follows. In Section 2, we recall results about Besov spaces and Littlewood-Paley decomposition. In Section 3, we prove existence of global-in-time weak solutions of (1.1), (1.3) for initial data in $L^{2+\delta}(\mathbb{R}^2)$ for some

$\delta > 0$. If the initial data is in $L^p(\mathbb{R}^2)$ for $p \in (2, \infty]$, then the L^p norm of any solution of (1.1), (1.3) remains bounded in time. If the initial data is $H^2(\mathbb{R}^2)$ regular, then we obtain a unique local strong solution. In Section 4, we show that a global-in-time solution exists provided that the initial data is sufficiently small in Besov spaces that are slightly smaller than $L^\infty(\mathbb{R}^2)$. In Section 5 we prove that solutions are Gevrey regular under a smallness condition imposed on the initial data. In Section 6, we study the regularity and long-time behavior of solutions for small initial data, whereas in Section 7, we show that Hölder continuity of the charge distribution is a sufficient condition for the smoothness of solutions for arbitrary initial data, a result that is similar to the situation for SQG [11]. In Section 8, we treat the periodic case, and we prove that the solution of the problem (1.1), (1.3) posed on the two-dimensional torus converges exponentially in time to zero. Finally, we consider in Section 9 the subcritical Darcy's law electroconvection, and show existence of global smooth solutions for arbitrary initial data.

2. Preliminaries

For $f \in S'(\mathbb{R}^2)$, we denote the Fourier transform of f by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x)e^{-i\xi\cdot x}\,\mathrm{d}x,$$

and its inverse by $\mathcal{F}^{-1}$.

Let Φ be a nonnegative, nonincreasing, infinitely differentiable, radial function such that $\Phi(r) = 1$ for $r \in [0, \frac{1}{2}]$ and $\Phi(r) = 0$ for $r \in [\frac{5}{8}, \infty]$. Let

$$\Psi(r) = \Phi\left(\frac{r}{2}\right) - \Phi(r).$$

For each $j \in \mathbb{Z}$, let

$$\Psi_j(r) = \Psi(2^{-j}r).$$

We have

$$\Phi(|\xi|) + \sum_{j=0}^{\infty} \Psi_j(|\xi|) = 1 \quad \text{for all } \xi \in \mathbb{R}^2$$

and

$$\sum_{j=-\infty}^{\infty} \Psi_j(|\xi|) = 1 \quad \text{for all } \xi \in \mathbb{R}^2 \setminus \{0\}.$$

We define the homogeneous dyadic blocks

$$\Delta_j f(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \Psi_j(|\xi|)\hat{f}(\xi)e^{i\xi\cdot x}\,\mathrm{d}\xi = \mathcal{F}^{-1}[\Psi_j(|\cdot|)\hat{f}(\cdot)](x)$$

and the lower frequency cutoff functions

$$S_j f = \sum_{k \le j-1} \Delta_k f.$$

We note that the Fourier transform of each dyadic block is compactly supported. More precisely, we have

$$\operatorname{supp} \mathcal{F}(\Delta_j f) \subset 2^j \left[\frac{1}{2}, \frac{5}{4}\right] \quad \text{for all } j \in \mathbb{Z}. \tag{2.1}$$

Let $S_h'(\mathbb{R}^2)$ be the set of all tempered distributions $u \in S'(\mathbb{R}^2)$ such that

$$\lim_{j \to -\infty} S_j u = 0 \quad \text{in } S'(\mathbb{R}^2).$$

For $f \in S_h'(\mathbb{R}^2)$, we denote the homogeneous Littlewood-Paley decomposition of f by

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f.$$

For $s \in \mathbb{R}, 1 \le p, q \le \infty$, we denote the homogeneous Besov space

$$\dot{B}^s_{p,q}(\mathbb{R}^2) = \{f \in S_h'(\mathbb{R}^2) : \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^2)} < \infty\}$$

where

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^2)} = \Big(\sum_{j \in \mathbb{Z}} 2^{jsq} \|\Delta_j f\|^q_{L^p(\mathbb{R}^2)}\Big)^{1/q}$$

and the inhomogeneous Besov space

$$B^s_{p,q}(\mathbb{R}^2) = \{f \in S'(\mathbb{R}^2) : \|f\|_{B^s_{p,q}(\mathbb{R}^2)} < \infty\}$$

where

$$\|f\|_{B^s_{p,q}(\mathbb{R}^2)} = \Big(2^{-sq}\|\tilde{\Delta}_{-1} f\|^q_{L^p(\mathbb{R}^2)} + \sum_{j=0}^{\infty} 2^{jsq} \|\Delta_j f\|^q_{L^p(\mathbb{R}^2)}\Big)^{1/q}$$

with the usual modification when $q = \infty$. Here,

$$\tilde{\Delta}_{-1} f = \frac{1}{2\pi} \int_{\mathbb{R}^2} \Phi(|\xi|) \hat{f}(\xi) e^{i\xi \cdot x} \, \mathrm{d}\xi = \mathcal{F}^{-1}[\Phi(|\cdot|)\hat{f}(\cdot)](x).$$

We note that the definition of the space $\dot{B}^s_{p,q}$ is independent of the function Φ which defines the dyadic blocks. Indeed, any other dyadic partition yields an equivalent norm.

If $s > 0$, $1 \le p, q \le \infty$, then

$$B^s_{p,q}(\mathbb{R}^2) = \dot{B}^s_{p,q}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2). \tag{2.2}$$

Moreover, the norms $\|f\|_{B^s_{p,q}(\mathbb{R}^2)}$ and $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^2)} + \|f\|_{L^p(\mathbb{R}^2)}$ are equivalent.

We also consider the following time-dependent homogeneous Besov spaces:

$$\begin{aligned} L^r(0,T;\dot{B}^s_{p,q}(\mathbb{R}^2)) = \Big\{ f(t) \in S'_h(\mathbb{R}^2) : \|f\|_{L^r(0,T;\dot{B}^s_{p,q}(\mathbb{R}^2))} \\ = \big\| \|f(\cdot,t)\|_{\dot{B}^s_{p,q}(\mathbb{R}^2)} \big\|_{L^r(0,T)} < \infty \Big\} \end{aligned}$$

and

$$\tilde{L}^r(0,T;\dot{B}^s_{p,q}(\mathbb{R}^2)) = \{ f(t) \in S'_h(\mathbb{R}^2) : \|f\|_{\tilde{L}^r(0,T;\dot{B}^s_{p,q}(\mathbb{R}^2))} < \infty \},$$

where

$$\|f\|_{\tilde{L}^r(0,T;\dot{B}^s_{p,q}(\mathbb{R}^2))} = \Big(\sum_{j\in\mathbb{Z}} 2^{jsq} \|\Delta_j f\|^q_{L^r(0,T;L^p(\mathbb{R}^2))} \Big)^{1/q}.$$

We recall inequalities that are used in the paper (see, e.g., [3, 14, 21]).

Proposition 2.1. *Let $f \in S'_h(\mathbb{R}^2)$.*

(1) (*Bernstein's inequality*). *Let $1 \le p \le \infty$. Let k be a nonnegative integer. Then,*

$$\sup_{|\alpha|=k} \|\partial^\alpha \Delta_j f\|_{L^p(\mathbb{R}^2)} \le C_k 2^{jk} \|\Delta_j f\|_{L^p(\mathbb{R}^2)} \tag{2.3}$$

holds for all $j \in \mathbb{Z}$.

(2) *Let $1 \le p \le q \le \infty$. Then,*

$$\|\Delta_j f\|_{L^q(\mathbb{R}^2)} \le C 2^{2j(1/p-1/q)} \|\Delta_j f\|_{L^p(\mathbb{R}^2)} \tag{2.4}$$

holds for all $j \in \mathbb{Z}$. Moreover, the continuous Besov embedding

$$\dot{B}^s_{p_1,q_1}(\mathbb{R}^2) \hookrightarrow \dot{B}^{s-2(1/p_1-1/p_2)}_{p_2,q_2}(\mathbb{R}^2) \tag{2.5}$$

holds for $1 \le p_1 \le p_2 \le \infty$, $1 \le q_1 \le q_2 \le \infty$, and $s \in \mathbb{R}$.

(3) *Let $1 \le p \le \infty$, $t \ge 0$, $\alpha > 0$. Then,*

$$\|e^{-t\Lambda^\alpha} \Delta_j f\|_{L^p(\mathbb{R}^2)} \le C e^{-C^{-1} t 2^{j\alpha}} \|\Delta_j f\|_{L^p(\mathbb{R}^2)} \tag{2.6}$$

holds for all $j \in \mathbb{Z}$. Here, Λ^α is the fractional Laplacian of order α defined as a Fourier multiplier with symbol $|\xi|^\alpha$.

(4) *Let $R = (R_1, R_2)$ be the Riesz transform, that is, for $k \in \{1,2\}$, $R_k = \partial_k \Lambda^{-1}$. For each $p \in [1,\infty]$, there is a positive constant $C > 0$ depending only on p (independent of j) such that*

$$\|\Delta_j R f\|_{L^p(\mathbb{R}^2)} \le C \|\Delta_j f\|_{L^p(\mathbb{R}^2)} \tag{2.7}$$

holds for all $j \in \mathbb{Z}$. Hence, for $s \in \mathbb{R}$ and $1 \le p,q \le \infty$, R is bounded from $\dot{B}^s_{p,q}(\mathbb{R}^2)$ to itself.

The following decomposition formula holds.

Proposition 2.2. *Let $f, g \in S'_h(\mathbb{R}^2)$. Then,*

$$\begin{aligned} \Delta_j(fg) &= \sum_{k \ge j-2} \Delta_j(S_{k+1} f \Delta_k g) + \sum_{k \ge j-2} \Delta_j(S_k g \Delta_k f) \\ &= \sum_{k \ge j-2} \Delta_j(S_{k+1} g \Delta_k f) + \sum_{k \ge j-2} \Delta_j(S_k f \Delta_k g) \end{aligned} \tag{2.8}$$

holds for any $j \in \mathbb{Z}$.

The proof is based on Bony's paraproduct, and is presented in Appendix A.

Throughout this paper C (or C_i, $i = 1, 2, \dots$) denotes a positive constant that may change from line to line in the proofs.

3. WELL-POSEDNESS IN LEBESGUE SPACES

We consider the transport and nonlocal diffusion equation

$$\partial_t \rho + u \cdot \nabla \rho + \Lambda \rho = 0 \tag{3.1}$$

in the whole space $\mathbb{R}^2$, where

$$u = -\mathbb{P}(\rho R \rho). \tag{3.2}$$

The initial data are

$$\rho(x,0) = \rho_0(x). \tag{3.3}$$

Here, $\mathbb{P}$ is the Leray-Hodge projector, $\Lambda = (-\Delta)^{1/2}$ is the fractional Laplacian, and $R = \nabla \Lambda^{-1}$ is the 2D vector of Riesz transforms.

Definition 3.1. A solution ρ of the initial value problem (3.1)–(3.3) is said to be a weak solution on $[0,T]$ if

$$\rho \in L^\infty(0,T; L^2(\mathbb{R}^2)) \cap L^2(0,T; \dot{H}^{1/2}(\mathbb{R}^2))$$

and ρ obeys

$$(\rho(t), \Phi)_{L^2} - (\rho_0, \Phi)_{L^2} - \int_0^t (\rho, u \cdot \nabla \Phi)_{L^2} \, ds + \int_0^t (\Lambda^{1/2}\rho, \Lambda^{1/2}\Phi)_{L^2} \, ds = 0$$

for all time-independent test functions $\Phi \in H^{5/2}(\mathbb{R}^2)$ and almost every $t \in [0,T]$.

For $\varepsilon \in (0, 1]$, let J_ε be the standard mollifier operator $J_\varepsilon f = J_\varepsilon * f$, and let ρ^ε be the solution of

$$\partial_t \rho^\varepsilon + \tilde{u}^\varepsilon \cdot \nabla \rho^\varepsilon + \Lambda \rho^\varepsilon - \varepsilon \Delta \rho^\varepsilon = 0, \tag{3.4}$$

where $\tilde{u}^\varepsilon = -J_\varepsilon \mathbb{P}(\rho^\varepsilon R \rho^\varepsilon)$ with smoothed-out initial data

$$\rho_0^\varepsilon = J_\varepsilon \rho_0. \tag{3.5}$$

Remark 3.2. We note that $\mathbb{P}$ and J_ε commute, so $\tilde{u}^\varepsilon$ is divergence free.

Theorem 3.3. *Let $T > 0$ be arbitrary. Let $\rho_0 \in L^2(\mathbb{R}^2)$. Then, for each $\varepsilon \in (0, 1]$, the mollified initial value problem* (3.4)–(3.5) *has a solution ρ^ε on $[0, T]$ satisfying*

$$\frac{1}{2}||\rho^\varepsilon(t)||_{L^2}^2 + \int_0^t ||\Lambda^{1/2}\rho^\varepsilon(s)||_{L^2}^2 \, \mathrm{d}s \le \frac{1}{2}||\rho_0||_{L^2}^2 \tag{3.6}$$

for all $t \in [0, T]$. Moreover, the sequence $\{\rho^{1/n}\}_{n=1}^\infty$ has a subsequence that converges strongly in $L^2(0, T; L^2(\mathbb{R}^2))$ and weakly in $L^2(0, T; H^{1/2}(\mathbb{R}^2))$ to a function ρ obeying

$$\frac{1}{2}||\rho(t)||_{L^2}^2 + \int_0^t ||\Lambda^{1/2}\rho(s)||_{L^2}^2 \, \mathrm{d}s \le \frac{1}{2}||\rho_0||_{L^2}^2 \tag{3.7}$$

for almost every $t \in [0, T]$. If $\rho_0 \in L^{2+\delta}(\mathbb{R}^2)$ for some $\delta > 0$, then ρ is a weak solution of (3.1)–(3.3) *on $[0, T]$.*

The proof is found in Appendix B.

As a consequence of the Córdoba-Córdoba inequality [12], the L^p norm of any solution of the equation (3.1)–(3.2) is bounded by the L^p norm of the initial data for any $p \in (2, \infty]$.

Proposition 3.4. *Let $p > 2$ and $\rho_0 \in L^p(\mathbb{R}^2)$. Suppose ρ is a smooth solution of* (3.1)–(3.3) *on $[0, T]$. Then,*

$$\|\rho(t)\|_{L^p} \le \|\rho_0\|_{L^p} \tag{3.8}$$

holds for all $t \in [0, T]$.

Proof. We multiply (3.1) by $\rho|\rho|^{p-2}$ and we integrate in the space variable. We obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^p} \le 0.$$

This gives (3.8). ∎

Remark 3.5. Weak solutions also obey

$$\|\rho(\cdot,t)\|_{L^\infty} \le \|\rho_0\|_{L^\infty}, \tag{3.9}$$

a fact that can be proved by using a De Giorgi methodology [4].

Definition 3.6. A weak solution ρ of (3.1)–(3.3) is said to be a strong solution on $[0,T]$ if it obeys

$$\rho \in L^\infty(0,T;\dot{H}^2(\mathbb{R}^2)) \cap L^2(0,T;\dot{H}^{5/2}(\mathbb{R}^2)).$$

Theorem 3.7. *Let $\rho_0 \in H^2(\mathbb{R}^2)$. Then, there exists $T_0 > 0$ depending only on $\|\rho_0\|_{H^2}$ such that a unique strong solution of (3.1)–(3.3) exists on $[0,T_0]$.*

The proof is found in Appendix C.

4. EXISTENCE OF GLOBAL SOLUTIONS IN BESOV SPACES

In this section, we show the existence of a global-in-time solution in Besov spaces for sufficiently small initial data. The proof uses methods of [2, 6].

Theorem 4.1. *Let $1 \le p < \infty$. Let $\rho_0 \in \dot{B}^{2/p}_{p,1}(\mathbb{R}^2)$ be sufficiently small. We consider the functional space E_p defined by*

$$E_p = \{f(t) \in S'_h(\mathbb{R}^2) : \|f\|_{E_p} = \|f\|_{\tilde{L}^\infty_t \dot{B}^{2/p}_{p,1}} + \|f\|_{\tilde{L}^1_t \dot{B}^{2/p+1}_{p,1}} < \infty\}.$$

Then, (3.1)–(3.3) has a unique global-in-time solution $\rho \in E_p$.

Proof. Let $\rho^{(0)} = 0$. For each positive integer n, let $\rho^{(n)}$ be the solution of

$$\partial_t \rho^{(n)} + \Lambda \rho^{(n)} = -u^{(n-1)} \cdot \nabla \rho^{(n-1)} \quad \text{in } \mathbb{R}^2, \tag{4.1}$$

where

$$u^{(n-1)} = -\mathbb{P}(\rho^{(n-1)} R \rho^{(n-1)}),$$

with initial data

$$\rho_0^{(n)} = \rho^{(n)}(\cdot,0) = \rho_0. \tag{4.2}$$

We write $\rho^{(n)}$ in the integral form

$$\begin{aligned}\rho^{(n)}(t) &= e^{-t\Lambda}\rho_0 - \int_0^t e^{-(t-s)\Lambda}\nabla \cdot (u^{(n-1)}\rho^{(n-1)})(s)\,ds \\ &= e^{-t\Lambda}\rho_0 - \mathcal{B}(u^{n-1},\rho^{n-1}),\end{aligned}$$

where $\mathcal{B}$ is the bilinear form defined by

$$\mathcal{B}(v,\theta) = \int_0^t e^{-(t-s)\Lambda}\nabla \cdot (v\theta)(s)\,ds.$$

(See [6] for a similar approach.)

Step 1. Fix a positive integer n. We show that

$$\|\rho^{(n)}\|_{E_p} \le C_1\|\rho_0\|_{\dot{B}^{2/p}_{p,1}} + C_2\|\rho^{(n-1)}\|^3_{E_p}. \tag{4.3}$$

We start by estimating $e^{-t\Lambda}\rho_0$ in E_p. We apply Δ_j and take the L^p norm. In view of the bound (2.6), we have

$$\|e^{-t\Lambda}\Delta_j\rho_0\|_{L^p} \le Ce^{-C^{-1}t2^j}\|\Delta_j\rho_0\|_{L^p},$$

and so

$$\|e^{-t\Lambda}\rho_0\|_{E_p} = \|e^{-t\Lambda}\rho_0\|_{\tilde{L}^\infty_t\dot{B}^{2/p}_{p,1}} + \|e^{-t\Lambda}\rho_0\|_{\tilde{L}^1_t\dot{B}^{2/p+1}_{p,1}} \le C\|\rho_0\|_{\dot{B}^{2/p}_{p,1}}.$$

Now, we estimate the term $\mathcal{B}(u^{(n-1)},\rho^{(n-1)})$ in E_p. First, we note that

$$\|\mathcal{B}(u^{(n-1)},\rho^{(n-1)})\|_{E^p} \le C\|u^{(n-1)}\rho^{(n-1)}\|_{\tilde{L}^1_t\dot{B}^{2/p+1}_{p,1}}. \tag{4.4}$$

Indeed, we apply Δ_j to $\mathcal{B}(u^{(n-1)},\rho^{(n-1)})$ and we estimate. On one hand,

$$\begin{aligned}
&\|\Delta_j\mathcal{B}(u^{(n-1)},\rho^{(n-1)})\|_{L^\infty_tL^p}\\
&\quad\le C2^j\left\|\int_0^t e^{-c^{-1}(t-s)2^j}\|\Delta_j(u^{(n-1)}\rho^{(n-1)})(s)\|_{L^p}\,\mathrm{d}s\right\|_{L^\infty_t}\\
&\quad\le C2^j\|\Delta_j(u^{(n-1)}\rho^{(n-1)})\|_{L^1_tL^p}
\end{aligned}$$

in view of Bernstein's inequality (2.3) and the bound (2.6). We multiply by $2^{j(2/p)}$ and take the ℓ^1 norm. We obtain the bound

$$\|\mathcal{B}(u^{(n-1)},\rho^{(n-1)})\|_{\tilde{L}^\infty_t\dot{B}^{2/p}_{p,1}} \le C\|u^{(n-1)}\rho^{(n-1)}\|_{\tilde{L}^1_t\dot{B}^{2/p+1}_{p,1}}. \tag{4.5}$$

On the other hand,

$$\begin{aligned}
&\|\Delta_j\mathcal{B}(u^{(n-1)},\rho^{(n-1)})\|_{L^1_tL^p}\\
&\quad\le C\left\|\int_0^t 2^je^{-c^{-1}(t-s)2^j}\|\Delta_j(u^{(n-1)}\rho^{(n-1)})(s)\|_{L^p}\,\mathrm{d}s\right\|_{L^1_t}\\
&\quad\le C\int_0^\infty\left(\int_0^\infty 2^je^{-c^{-1}(t-s)2^j}\chi_{[0,t]}(s)\,\mathrm{d}t\right)\|\Delta_j(u^{(n-1)}\rho^{(n-1)})(s)\|_{L^p}\,\mathrm{d}s\\
&\quad\le C\|\Delta_j(u^{(n-1)}\rho^{(n-1)})\|_{L^1_tL^p}
\end{aligned}$$

where χ_E denotes the characteristic function of the set E. Multiplying by $2^{j(2/p+1)}$ and taking the ℓ^1 norm yields the bound

$$\|\mathcal{B}(u^{(n-1)},\rho^{(n-1)})\|_{\tilde{L}^1_t\dot{B}^{2/p+1}_{p,1}} \le C\|u^{(n-1)}\rho^{(n-1)}\|_{\tilde{L}^1_t\dot{B}^{2/p+1}_{p,1}}. \tag{4.6}$$

Combining (4.5) and (4.6), we obtain (4.4). Accordingly, our next goal is to show that

$$\|u^{(n-1)}\rho^{(n-1)}\|_{\tilde{L}_t^1 \dot{B}_{p,1}^{2/p+1}} \le C\|\rho^{(n-1)}\|_{E^p}^3, \tag{4.7}$$

which gives (4.3). To establish the bound (4.7), we use the decomposition (2.8)

$$\begin{aligned}\Delta_j(u^{(n-1)}\rho^{(n-1)}) = &\sum_{k\ge j-2} \Delta_j(S_k u^{(n-1)} \Delta_k \rho^{(n-1)}) \\ &+ \sum_{k\ge j-2} \Delta_j(S_{k+1}\rho^{(n-1)}\Delta_k u^{(n-1)}).\end{aligned}$$

We apply the $L_t^1 L^p$ norm, and use the bound

$$\|\Delta_j f\|_{L^p} \le C\|f\|_{L^p}$$

that holds for any $f \in S_h'$ where C is a positive universal constant independent of j; we obtain

$$\begin{aligned}&\|\Delta_j(u^{(n-1)}\rho^{(n-1)})\|_{L_t^1 L^p} \\ &\quad\le C\sum_{k\ge j-2} \|S_k u^{(n-1)}\|_{L_t^\infty L^\infty} \|\Delta_k \rho^{(n-1)}\|_{L_t^1 L^p} \\ &\qquad + C\sum_{k\ge j-2} \|S_{k+1}\rho^{(n-1)}\|_{L_t^\infty L^\infty} \|\Delta_k u^{(n-1)}\|_{L_t^1 L^p}.\end{aligned}$$

In view of Bernstein's inequality (2.4), we have

$$\begin{aligned}\|S_{k+1}\rho^{(n-1)}\|_{L_t^\infty L^\infty} &\le \sum_{l\le k} \|\Delta_l \rho^{(n-1)}\|_{L_t^\infty L^\infty} \\ &\le C\sum_{l\le k} 2^{l(2/p)}\|\Delta_l\rho^{(n-1)}\|_{L_t^\infty L^p} \\ &\le C\|\rho^{(n-1)}\|_{\tilde{L}_t^\infty \dot{B}_{p,1}^{2/p}}.\end{aligned} \tag{4.8}$$

We show below that

$$\|S_k u^{(n-1)}\|_{L_t^\infty L^\infty} \le C\|\rho^{(n-1)}\|_{\tilde{L}_t^\infty \dot{B}_{p,1}^{2/p}}^2 \tag{4.9}$$

and

$$\|\Delta_k u^{(n-1)}\|_{L_t^1 L^p} \le C\|\rho^{(n-1)}\|_{\tilde{L}_t^\infty \dot{B}_{p,1}^{2/p}} \Big(\sum_{m\ge k-2} \|\Delta_m \rho^{(n-1)}\|_{L_t^1 L^p}\Big). \tag{4.10}$$

Using the bounds (4.9) and (4.10), we obtain

$$\begin{aligned}(4.11)\qquad &\|\Delta_j(u^{(n-1)}\rho^{(n-1)})\|_{L_t^1L^p}\\ &\quad\le C\|\rho^{(n-1)}\|^2_{\tilde L_t^\infty \dot B^{2/p}_{p,1}}\Big\{\sum_{k\ge j-2}\|\Delta_k\rho^{(n-1)}\|_{L_t^1L^p}\\ &\qquad+\sum_{k\ge j-2}\sum_{m\ge k-2}\|\Delta_m\rho^{(n-1)}\|_{L_t^1L^p}\Big\}.\end{aligned}$$

We multiply (4.11) by $2^{j(2/p+1)}$ and take the ℓ^1 norm. In view of Young's convolution inequality, we have in the first term

$$\begin{aligned}(4.12)\qquad &\sum_{j\in\mathbb{Z}}\sum_{k\ge j-2}2^{j(2/p+1)}\|\Delta_k\rho^{(n-1)}\|_{L_t^1L^p}\\ &\quad=\sum_{j\in\mathbb{Z}}\sum_{k\ge j-2}2^{-(k-j)(2/p+1)}2^{k(2/p+1)}\|\Delta_k\rho^{(n-1)}\|_{L_t^1L^p}\\ &\quad\le\Big(\sum_{j\ge -2}2^{-j(2/p+1)}\Big)\Big(\sum_{j\in\mathbb{Z}}2^{j(2/p+1)}\|\Delta_j\rho^{(n-1)}\|_{L_t^1L^p}\Big)\\ &\quad\le C\|\rho^{(n-1)}\|_{\tilde L_t^1\dot B^{2/p+1}_{p,1}}.\end{aligned}$$

For the second summation on the righthand side of (4.11), we apply Fubini's theorem and then estimate as in (4.12). Thus, we have

$$\begin{aligned}(4.13)\qquad &\sum_{j\in\mathbb{Z}}\sum_{k\ge j-2}\sum_{m\ge k-2}2^{j(2/p+1)}\|\Delta_m\rho^{(n-1)}\|_{L_t^1L^p}\\ &=\sum_{j\in\mathbb{Z}}\sum_{m\ge j-4}\sum_{j-2\le k\le m+2}2^{-(m-j)(2/p+1)}2^{m(2/p+1)}\|\Delta_m\rho^{(n-1)}\|_{L_t^1L^p}\\ &=\sum_{j\in\mathbb{Z}}\sum_{m\ge j-4}(m-j+5)2^{-(m-j)(2/p+1)}2^{m(2/p+1)}\|\Delta_m\rho^{(n-1)}\|_{L_t^1L^p}\\ &\le C\sum_{j\in\mathbb{Z}}\sum_{m\ge j-4}2^{-(m-j)(1/p+1/2)}2^{m(2/p+1)}\|\Delta_m\rho^{(n-1)}\|_{L_t^1L^p}\\ &\qquad+5\sum_{j\in\mathbb{Z}}\sum_{m\ge j-4}2^{-(m-j)(2/p+1)}2^{m(2/p+1)}\|\Delta_m\rho^{(n-1)}\|_{L_t^1L^p}\\ &\le C\|\rho^{(n-1)}\|_{\tilde L_t^1\dot B^{2/p+1}_{p,1}}.\end{aligned}$$

Here, we have used the fact that

$$x2^{-x}\le C2^{-x/2}$$

for all $x\in\mathbb{R}$. Putting (4.12) and (4.13) together, we obtain (4.7).

We end the proof of *Step 1* by showing the estimates (4.9) and (4.10). For each $l \in \mathbb{Z}$, we use again paraproducts to decompose $\Delta_l(\rho^{(n-1)} R\rho^{(n-1)})$ as

$$\begin{aligned} \Delta_l(\rho^{(n-1)} R\rho^{(n-1)}) = &\sum_{m \geq l-2} \Delta_l(S_{m+1}\rho^{(n-1)} \Delta_m R\rho^{(n-1)}) \\ &+ \sum_{m \geq l-2} \Delta_l(S_m R\rho^{(n-1)} \Delta_m \rho^{(n-1)}). \end{aligned} \tag{4.14}$$

In view of the boundedness of the Riesz transform (2.7) and the definition of the Leray projector as $\mathbb{P} = I + R \otimes R$, we bound

$$\begin{aligned} \|S_k u^{(n-1)}\|_{L_t^\infty L^\infty} &\leq \sum_{l \leq k-1} \|\Delta_l u^{(n-1)}\|_{L_t^\infty L^\infty} \\ &\leq C \sum_{l \leq k-1} 2^{l(2/p)} \|\Delta_l u^{(n-1)}\|_{L_t^\infty L^p} \\ &\leq C \sum_{l \leq k-1} 2^{l(2/p)} \|\Delta_l(\rho^{(n-1)} R\rho^{(n-1)})\|_{L_t^\infty L^p} \end{aligned}$$

for any $p \in [1, \infty]$; using the paraproduct decomposition (4.14), we obtain

$$\begin{aligned} &\|S_k u^{(n-1)}\|_{L_t^\infty L^\infty} \\ &\quad \leq C \sum_{l \leq k-1} 2^{l(2/p)} \sum_{m \geq l-2} \|S_{m+1}\rho^{(n-1)}\|_{L_t^\infty L^\infty} \|\Delta_m R\rho^{(n-1)}\|_{L_t^\infty L^p} \\ &\qquad + C \sum_{l \leq k-1} 2^{l(2/p)} \sum_{m \geq l-2} \|S_m R\rho^{(n-1)}\|_{L_t^\infty L^\infty} \|\Delta_m \rho^{(n-1)}\|_{L_t^\infty L^p}. \end{aligned}$$

We note that

$$\|S_{m+1}\rho^{(n-1)}\|_{L_t^\infty L^\infty} \leq C\|\rho^{(n-1)}\|_{\tilde{L}_t^\infty \dot{B}_{p,1}^{2/p}}$$

as shown in (4.8). Moreover, in view of (2.7), we have

$$\begin{aligned} \|S_m R\rho^{(n-1)}\|_{L_t^\infty L^\infty} &\leq \sum_{z \leq m-1} \|\Delta_z R\rho^{(n-1)}\|_{L_t^\infty L^\infty} \\ &\leq C \sum_{z \leq m-1} 2^{z(2/p)} \|\Delta_z R\rho^{(n-1)}\|_{L_t^\infty L^p} \\ &\leq C \sum_{z \leq m-1} 2^{z(2/p)} \|\Delta_z \rho^{(n-1)}\|_{L_t^\infty L^p} \\ &\leq C\|\rho^{(n-1)}\|_{\tilde{L}_t^\infty \dot{B}_{p,1}^{2/p}}. \end{aligned} \tag{4.15}$$

Now we use the assumption that $p < \infty$, which implies $2/p > 0$, so we can apply Young's convolution inequality to obtain

$$
\begin{aligned}
(4.16)\quad & \|S_k u^{(n-1)}\|_{L_t^\infty L^\infty} \\
&\le C\|\rho^{(n-1)}\|_{\tilde{L}_t^\infty \dot{B}_{p,1}^{2/p}} \Big\{ \sum_{l\le k-1} 2^{l(2/p)} \sum_{m\ge l-2} \|\Delta_m \rho^{(n-1)}\|_{L_t^\infty L^p} \Big\} \\
&= C\|\rho^{(n-1)}\|_{\tilde{L}_t^\infty \dot{B}_{p,1}^{2/p}} \Big\{ \sum_{l\le k-1} \sum_{m\ge l-2} 2^{-(m-l)(2/p)} 2^{m(2/p)} \|\Delta_m \rho^{(n-1)}\|_{L_t^\infty L^p} \Big\} \\
&\le C\|\rho^{(n-1)}\|^2_{\tilde{L}_t^\infty \dot{B}_{p,1}^{2/p}}
\end{aligned}
$$

which proves (4.9). We proceed to show (4.10). Using the paraproduct decomposition (4.14) and the bound (2.7), we have

$$
\begin{aligned}
\|\Delta_k u^{(n-1)}\|_{L_t^1 L^p} &\le C\|\Delta_k(\rho^{(n-1)} R\rho^{(n-1)})\|_{L_t^1 L^p} \\
&\le C \sum_{m\ge k-2} \|S_{m+1}\rho^{(n-1)}\|_{L_t^\infty L^\infty} \|\Delta_m R\rho^{(n-1)}\|_{L_t^1 L^p} \\
&\quad + C \sum_{m\ge k-2} \|S_m R\rho^{(n-1)}\|_{L_t^\infty L^\infty} \|\Delta_m \rho^{(n-1)}\|_{L_t^1 L^p} \\
&\le C\|\rho^{(n-1)}\|_{\tilde{L}_t^\infty \dot{B}_{p,1}^{2/p}} \Big(\sum_{m\ge k-2} \|\Delta_m \rho^{(n-1)}\|_{L_t^1 L^p} \Big),
\end{aligned}
$$

yielding (4.10). This ends the proof of *Step 1*.

Step 2. We show there is an $\varepsilon > 0$ sufficiently small such that if $C_1\|\rho_0\|_{\dot{B}_{p,1}^{2/p}} < \varepsilon$, then the sequence $\{\rho^{(n)}\}_{n=1}^\infty$ converges to a unique solution ρ of (3.1)–(3.3) that obeys $\|\rho\|_{E_p} < 2\varepsilon$.

First, choose an $\varepsilon > 0$ such that $C_2(2\varepsilon)^3 < \varepsilon$, where C_2 is the constant in (4.3), and suppose that $C_1\|\rho_0\|_{\dot{B}_{p,1}^{2/p}} < \varepsilon$. Then, an inductive argument yields

$$
(4.17)\qquad \|\rho^{(n)}\|_{E_p} < 2\varepsilon \quad \text{for all } n \ge 1.
$$

Indeed,

$$
\|\rho^{(1)}\|_{E_p} \le C_1\|\rho_0\|_{\dot{B}_{p,1}^{2/p}} < \varepsilon < 2\varepsilon
$$

in view of (4.3). Suppose that $\|\rho^{(n-1)}\|_{E_p} < 2\varepsilon$ Then,

$$
\|\rho^{(n)}\|_{E_p} < \varepsilon + C_2(2\varepsilon)^3 < \varepsilon + \varepsilon = 2\varepsilon.
$$

Therefore, we obtain (4.17).

Now, we show that the sequence $\{\rho^{(n)}\}_{n=1}^\infty$ is Cauchy. Indeed, the difference $\rho^{(n)} - \rho^{(n-1)}$ obeys

$$
\begin{aligned}
&(\rho^{(n)} - \rho^{(n-1)})(t) \\
&\quad = \int_0^t e^{-(t-s)\Lambda} \nabla \cdot [u^{(n)}(\rho^{(n)} - \rho^{(n-1)}) - (u^{(n-1)} - u^{(n)})\rho^{(n-1)}](s)\, ds \\
&\quad = \mathcal{B}(u^{(n)}, \rho^{(n)} - \rho^{(n-1)}) - \mathcal{B}(u^{(n-1)} - u^{(n)}, \rho^{(n-1)}).
\end{aligned}
$$

As in *Step 1*, and by using (4.17), it can be shown that

$$
\begin{aligned}
(4.18)\quad \|\rho^{(n)} - \rho^{(n-1)}\|_{E_p}
&\le \|\mathcal{B}(u^{(n)}, \rho^{(n)} - \rho^{(n-1)})\|_{E_p} + \|\mathcal{B}(u^{(n-1)} - u^{(n)}, \rho^{(n-1)})\|_{E_p} \\
&\le C(\varepsilon)\|\rho^{(n-1)} - \rho^{(n-2)}\|_{E_p}
\end{aligned}
$$

where $C(\varepsilon)$ is a constant depending on ε obeying $C(\varepsilon) < 1$ for a sufficiently small ε. Therefore, the sequence $\{\rho^{(n)}\}_{n=1}^{\infty}$ is Cauchy in E_p, and converges to a solution ρ of (3.1)–(3.3). Uniqueness follows from a similar estimate to (4.18). This finishes the proof of *Step 2*. Therefore, the proof of Theorem 4.1 is complete. ◻

5. ANALYTICITY OF SOLUTIONS IN BESOV SPACES

In this section, we prove that solutions of (3.1)–(3.3) are analytic in Besov spaces.

Theorem 5.1. *Let $p \in (1, \infty)$. Let $\alpha \in (0, \frac{1}{2})$. Let $\rho_0 \in \dot{B}_{p,1}^{2/p}(\mathbb{R}^2)$ be sufficiently small. Then, the unique solution $\rho \in E_p$ of* (3.1)–(3.3)*, obtained in Theorem 4.1, obeys $e^{\alpha t \Lambda_1}\rho \in E_p$ for all $t > 0$, where Λ_1 is the Fourier multiplier with symbol $|\xi|_1 = |\xi_1| + |\xi_2|$.*

Proof. The main step in the proof is to show that if

$$
\rho(t) = e^{-t\Lambda}\rho_0 - \int_0^t e^{-(t-s)\Lambda}\nabla \cdot (u\rho)(s)\, ds,
$$

then

$$
\|e^{\alpha t\Lambda_1}\rho\|_{E_p} \le C_3\|\rho_0\|_{\dot{B}_{p,1}^{2/p}} + C_4\|e^{\alpha t\Lambda_1}\rho\|_{E_p}^3.
$$

First, we note that the operator $e^{\alpha t\Lambda_1 - 2\alpha t\Lambda}$ is a Fourier multiplier that is bounded on L^p spaces for $p \in (1, \infty)$. The proof of this latter statement is similar to the proof of Lemma 2 in [2], and this is based on the fact that $e^{\alpha t\Lambda_1 - 2\alpha t\Lambda}$ is uniformly bounded by 1.

Accordingly, for $j \in \mathbb{Z}$, we have

$$
\|e^{\alpha t\Lambda_1}e^{-t\Lambda}\Delta_j\rho_0\|_{L^p} = \|e^{\alpha t\Lambda_1 - 2\alpha t\Lambda}e^{(2\alpha-1)t\Lambda}\Delta_j\rho_0\|_{L^p} \le Ce^{-ct2^j}\|\Delta_j\rho_0\|_{L^p},
$$

and so we have $\|e^{\alpha t\Lambda_1}e^{-t\Lambda}\rho_0\|_{E_p} \le C\|\rho_0\|_{\dot{B}_{p,1}^{2/p}}$.

Now, we estimate

$$
e^{\alpha t\Lambda_1}\mathcal{B}(u, \rho) = e^{\alpha t\Lambda_1}\int_0^t e^{-(t-s)\Lambda}\nabla \cdot (u\rho)(s)\, ds \quad \text{in } E_p.
$$

We start by writing $e^{\alpha t\Lambda_1}\mathcal{B}(u, \rho)$ as

$$
\begin{aligned}
e^{\alpha t\Lambda_1}B(u, v) = \int_0^t & e^{\alpha(t-s)\Lambda_1}e^{-2\alpha(t-s)\Lambda}e^{(2\alpha-1)(t-s)\Lambda} \\
&\times e^{\alpha s\Lambda_1}\nabla \cdot (e^{-\alpha s\Lambda_1}\tilde{u}e^{-\alpha s\Lambda_1}\tilde{\rho})(s)\, ds
\end{aligned}
$$

where $\tilde{u}(s) = e^{\alpha s\Lambda_1}u(s)$ and $\tilde{\rho}(s) = e^{\alpha s\Lambda_1}\rho(s)$.

Using the uniform boundedness of the operator $e^{\alpha t\Lambda_1 - 2\alpha t\Lambda}$ on L^p spaces for $p \in (1,\infty)$, Bernstein's inequality, and the bound (2.6), we get

$$\|e^{\alpha t\Lambda_1}\mathcal{B}(u,v)\|_{E_p} \le C\|e^{\alpha s\Lambda_1}(e^{-\alpha s\Lambda_1}\tilde{u}e^{-\alpha s\Lambda_1}\tilde{\rho})\|_{\tilde{L}^1_t\dot{B}^{2/p+1}_{p,1}}.$$

Decomposing $\Delta_j(e^{-\alpha s\Lambda_1}\tilde{u}e^{-\alpha s\Lambda_1}\tilde{\rho})$ as

$$\begin{aligned}\Delta_j(e^{-\alpha s\Lambda_1}\tilde{u}e^{-\alpha s\Lambda_1}\tilde{\rho}) = &\sum_{k\ge j-2}\Delta_j[(e^{-\alpha s\Lambda_1}S_k\tilde{u})(e^{-\alpha s\Lambda_1}\Delta_k\tilde{\rho})]\\ &+\sum_{k\ge j-2}\Delta_j[(e^{-\alpha s\Lambda_1}S_{k+1}\tilde{\rho})(e^{-\alpha s\Lambda_1}\Delta_k\tilde{u})],\end{aligned}$$

we have

$$\begin{aligned}&\|e^{\alpha s\Lambda_1}\Delta_j(e^{-\alpha s\Lambda_1}\tilde{u}e^{-\alpha s\Lambda_1}\tilde{\rho})\|_{L^1_tL^p}\\ &\quad\le C\sum_{k\ge j-2}\|e^{\alpha s\Lambda_1}(e^{-\alpha s\Lambda_1}S_k\tilde{u})(e^{-\alpha s\Lambda_1}\Delta_k\tilde{\rho})\|_{L^1_tL^p}\\ &\qquad+C\sum_{k\ge j-2}\|e^{\alpha s\Lambda_1}(e^{-\alpha s\Lambda_1}S_{k+1}\tilde{\rho})(e^{-\alpha s\Lambda_1}\Delta_k\tilde{u})\|_{L^1_tL^p}.\end{aligned}$$

It is shown in [2] that the bilinear operator $B_w(f,g)$ defined by

$$B_w(f,g) = e^{w\Lambda_1}(e^{-w\Lambda_1}e^{-\Lambda_1}g) \tag{5.1}$$

obeys

$$\|B_w(f,g)\|_{L^p} \le C\|Z^1_w f Z^2_w g\|_{L^p}$$

where $C > 0$ is a positive constant depending only on p, and Z^1_w and Z^2_w are bounded linear operators on L^p for $p \in (1,\infty)$ such that their norms are independent of w. For simplicity, we drop the index w, and we write Z^1 for Z^1_w and Z^2 for Z^2_w.

Consequently,

$$\begin{aligned}&\|e^{\alpha s\Lambda_1}\Delta_j(e^{-\alpha s\Lambda_1}\tilde{u}e^{-\alpha s\Lambda_1}\tilde{\rho})\|_{L^1_tL^p}\\ &\quad\le C\sum_{k\ge j-2}\|Z^1S_k\tilde{u}Z^2\Delta_k\tilde{\rho}\|_{L^1_tL^p}+C\sum_{k\ge j-2}\|Z^1S_{k+1}\tilde{\rho}Z^2\Delta_k\tilde{u}\|_{L^1_tL^p}.\end{aligned}$$

Now we proceed as in *Step 1* of the proof of Theorem 4.1. Indeed,

$$\begin{aligned}\|Z^1S_{k+1}\tilde{\rho}\|_{L^\infty_tL^\infty} &\le \sum_{l\le k}2^{l(2/p)}\|Z^1\Delta_l\tilde{\rho}\|_{L^\infty_tL^p}\\ &\le C\sum_{l\le k}2^{l(2/p)}\|\Delta_l\tilde{\rho}\|_{L^\infty_tL^p} \le C\|\tilde{\rho}\|_{\tilde{L}^\infty_t\dot{B}^{2/p}_{p,1}}.\end{aligned}$$

If we show that

$$\|\Delta_k \tilde{u}\|_{L_t^1 L^p} \le C \|\tilde{\rho}\|_{L_t^\infty \dot{B}_{p,1}^{2/p}} \Big(\sum_{m \ge k-2} \|\Delta_m \tilde{\rho}\|_{L_t^1 L^p} \Big) \tag{5.2}$$

and

$$\|Z^1 S_k \tilde{u}\|_{L_t^\infty L^\infty} \le C ||\tilde{\rho}||^2_{L_t^\infty \dot{B}_{p,1}^{2/p}}, \tag{5.3}$$

then the rest follows as in *Step 1* of Theorem 4.1.

Hence, we proceed to prove the bounds (5.2) and (5.3). We note that

$$\tilde{u} = e^{\alpha s \Lambda_1} u = e^{\alpha s \Lambda_1} \mathbb{P}(\rho R \rho) = e^{\alpha s \Lambda_1} \mathbb{P}(e^{-\alpha s \Lambda_1} \tilde{\rho} e^{-\alpha s \Lambda_1} R \tilde{\rho}).$$

We decompose $\Delta_l(e^{-\alpha s \Lambda_1} \tilde{\rho} e^{-\alpha s \Lambda_1} R \tilde{\rho})$ as

$$\begin{aligned}\Delta_l(e^{-\alpha s \Lambda_1} \tilde{\rho} e^{-\alpha s \Lambda_1} R \tilde{\rho}) = \sum_{m \ge l-2} & \Delta_l[(e^{-\alpha s \Lambda_1} S_{m+1} \tilde{\rho})(e^{-\alpha s \Lambda_1} \Delta_m R \tilde{\rho})] \\ & + \sum_{m \ge l-2} \Delta_l[(e^{-\alpha s \Lambda_1} \Delta_m \tilde{\rho})(e^{-\alpha s \Lambda_1} S_m R \tilde{\rho})].\end{aligned}$$

In view of the boundedness of the operators Z^1 and $\mathbb{P}$, we estimate

$$\begin{aligned}\|Z^1 S_k \tilde{u}\|_{L_t^\infty L^\infty} &\le C \sum_{l \le k-1} 2^{l(2/p)} \|\Delta_l \tilde{u}\|_{L_t^\infty L^p} \\ &\le C \sum_{l \le k-1} 2^{l(2/p)} ||\Delta_l [e^{\alpha s \Lambda_1} (e^{-\alpha s \Lambda_1} \tilde{\rho} e^{-\alpha s \Lambda_1} R \tilde{\rho})]||_{L_t^\infty L^p} \\ &\le C \sum_{l \le k-1} \sum_{m \ge l-2} 2^{l(2/p)} \|B_{\alpha s}(S_{m+1} \tilde{\rho}, \Delta_m R \tilde{\rho})\|_{L_t^\infty L^p} \\ &\quad + C \sum_{l \le k-1} \sum_{m \ge l-2} 2^{l(2/p)} \|B_{\alpha s}(S_m R \tilde{\rho}, \Delta_m \tilde{\rho})\|_{L_t^\infty L^p}\end{aligned}$$

where $B_w(f, g)$ is defined in (5.1). This implies that

$$\begin{aligned}\|Z^1 S_k \tilde{u}\|_{L_t^\infty L^\infty} &\le C \sum_{l \le k-1} \sum_{m \ge l-2} 2^{i(2/p)} \|(Z^1 S_{m+1} \tilde{\rho})(Z^2 \Delta_m R \tilde{\rho})\|_{L_t^\infty L^p} \\ &\quad + C \sum_{l \le k-1} \sum_{m \ge l-2} 2^{l(2/p)} \|(Z^1 S_m R \tilde{\rho})(Z^2 \Delta_m \tilde{\rho})\|_{L_t^\infty L^p}.\end{aligned}$$

Now we proceed as in (4.15) and (4.16), and we obtain (5.3). Finally, we estimate

$$\begin{aligned}\|\Delta_k \tilde{u}\|_{L_t^1 L_p} &\le C||\Delta_k e^{\alpha s \Lambda_1} [(e^{-\alpha s \Lambda_1} \tilde{\rho})(e^{-\alpha s \Lambda_1} R \tilde{\rho})]||_{L_t^1 L^p} \\ &\le C \sum_{m \ge k-2} ||e^{\alpha s \Lambda_1} [(e^{-\alpha s \Lambda_1} S_{m+1} \tilde{\rho})(e^{-\alpha s \Lambda_1} \Delta_m R \tilde{\rho})]||_{L_t^1 L^p} \\ &\quad + C \sum_{m \ge k-2} ||e^{\alpha s \Lambda_1} [(e^{-\alpha s \Lambda_1} S_m R \tilde{\rho})(e^{-\alpha s \Lambda_1} \Delta_m \tilde{\rho})]||_{L_t^1 L^p}\end{aligned}$$

$$
\begin{aligned}
&= C \sum_{m\geq k-2} \|B_{\alpha s}(S_{m+1}\tilde{\rho}, \Delta_m R\tilde{\rho})\|_{L^1_t L^p} \\
&\qquad + C \sum_{m\geq k-2} \|B_{\alpha s}(S_m R\tilde{\rho}, \Delta_m \tilde{\rho})\|_{L^1_t L^p} \\
&\leq C \sum_{m\geq k-2} \|Z^1 S_{m+1}\tilde{\rho}\|_{L^\infty_t L^\infty} \|Z^2 \Delta_m R\tilde{\rho}\|_{L^1_t L^p} \\
&\qquad + C \sum_{m\geq k-2} \|Z^1 S_m R\tilde{\rho}\|_{L^\infty_t L^\infty} \|Z^2 \Delta_m \tilde{\rho}\|_{L^1_t L^p} \\
&\leq C \|\tilde{\rho}\|_{\tilde{L}^\infty_t \dot{B}^{2/p}_{p,1}} \Big(\sum_{m\geq k-2} \|\Delta_m \tilde{\rho}\|_{L^1_t L^p} \Big),
\end{aligned}
$$

which proves (5.2). This ends the proof of Theorem 5.1. ◻

6. Regularity of Solutions for Small Initial Data

Here, we consider the regularity of solutions of (3.1)–(3.3) for small initial data.

We use the following lemma.

Lemma 6.1. *Let $j \in \mathbb{Z}$, $t > 0$, $\alpha \in [0,1)$, $c > 0$. Then, there is a positive constant $C > 0$ that depends on α but does not depend on j or t such that the estimate*

$$
\int_0^t 2^j e^{-c(t-s)2^j} s^{-\alpha}\, \mathrm{d}s \leq C t^{-\alpha} \tag{6.1}
$$

holds.

Proof. We split the given integral into the sum

$$
\int_0^t 2^j e^{-c(t-s)2^j} s^{-\alpha}\, \mathrm{d}s = \int_0^{t/2} 2^j e^{-c(t-s)2^j} s^{-\alpha}\, \mathrm{d}s + \int_{t/2}^t 2^j e^{-c(t-s)2^j} s^{-\alpha}\, \mathrm{d}s.
$$

Using the fact that $2^j e^{-c(t-s)2^j} \leq C(t-s)^{-1}$ for all $s \in [0, t/2]$, we estimate

$$
\begin{aligned}
\int_0^{t/2} 2^j e^{-c(t-s)2^j} s^{-\alpha}\, \mathrm{d}s &\leq C \int_0^{t/2} (t-s)^{-1} s^{-\alpha}\, \mathrm{d}s \\
&\leq C t^{-1} \int_0^{t/2} s^{-\alpha}\, \mathrm{d}s \leq C_\alpha t^{-\alpha}.
\end{aligned} \tag{6.2}
$$

Using the fact that $s^{-\alpha} \leq 2^\alpha t^{-\alpha}$ for all $s \in [t/2, t]$, we estimate

$$
\begin{aligned}
\int_{t/2}^t 2^j e^{-c(t-s)2^j} s^{-\alpha}\, \mathrm{d}s &\leq C_\alpha t^{-\alpha} \int_{t/2}^t 2^j e^{-c(t-s)2^j}\, \mathrm{d}s \\
&= C_\alpha t^{-\alpha} [1 - e^{-2^{j-1}ct}] \leq C_\alpha t^{-\alpha}.
\end{aligned} \tag{6.3}
$$

Adding (6.2) and (6.3), we obtain (6.1). ◻

Theorem 6.2. *Let $\alpha \in [0,1)$, $\beta > 0$. Let $\rho_0 \in \dot{B}^1_{2,1}(\mathbb{R}^2) \cap \dot{B}^{\beta-\alpha}_{\infty,\infty}(\mathbb{R}^2)$ be sufficiently small. Then, there is a positive constant $C > 0$ depending on the initial data so that the unique solution ρ of (3.1)–(3.3) satisfies $\sup_{t>0} t^\alpha \|\rho(t)\|_{\dot{B}^\beta_{\infty,\infty}} \le C$.*

Proof. We consider the approximating initial value problem (4.1)–(4.2) whose solution is given by

$$\rho^{(n)}(t) = e^{-t\Lambda}\rho_0 - \mathcal{B}(u^{(n-1)}, \rho^{(n-1)}).$$

First, we estimate

$$\begin{aligned} t^\alpha 2^{j\beta}\|e^{-t\Lambda}\Delta_j\rho_0\|_{L^\infty} &\le Ct^\alpha e^{-ct2^j}2^{j\beta}\|\Delta_j\rho_0\|_{L^\infty} \\ &\le C2^{-j\alpha}2^{j\beta}\|\Delta_j\rho_0\|_{L^\infty} \le C\|\rho_0\|_{\dot{B}^{\beta-\alpha}_{\infty,\infty}} \end{aligned}$$

in view of (2.6) and the bound $x^\alpha e^{-x} \le C$ that holds for all $x \ge 0$.

We show that

$$\begin{aligned} \sup_{t>0}\{t^\alpha\|\mathcal{B}(u^{(n-1)}, \rho^{(n-1)})(t)\|_{\dot{B}^\beta_{\infty,\infty}}\} \\ \le C\|\rho^{(n-1)}\|^2_{\tilde{L}^\infty_t\dot{B}^{2/p}_{p,1}} \sup_{t>0}\{t^\alpha\|\rho^{(n-1)}(t)\|_{\dot{B}^\beta_{\infty,\infty}}\}. \end{aligned} \tag{6.4}$$

First, we apply Δ_j to $\mathcal{B}(u^{(n-1)}, \rho^{n-1)})$, using the paraproduct decomposition

$$\begin{aligned} \Delta_j(u^{(n-1)}\rho^{(n-1)}) = &\sum_{k\ge j-2}\Delta_j(S_k u^{(n-1)}\Delta_k\rho^{(n-1)}) \\ &+ \sum_{k\ge j-2}\Delta_j(S_{k+1}\rho^{(n-1)}\Delta_k u^{(n-1)}), \end{aligned}$$

and we obtain

$$\Delta_j\mathcal{B}(u^{(n-1)}, \rho^{(n-1)}) = \mathcal{B}_{1,j}(u^{(n-1)}, \rho^{(n-1)}) + \mathcal{B}_{2,j}(u^{(n-1)}, \rho^{(n-1)})$$

where

$$\begin{aligned} &\mathcal{B}_{1,j}(u^{(n-1)}, \rho^{(n-1)}) \\ &\quad = \int_0^t e^{-(t-s)\Lambda}\nabla\cdot\Big[\sum_{k\ge j-2}\Delta_j(S_k u^{(n-1)}\Delta_k\rho^{(n-1)})\Big](s)\,\mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} &\mathcal{B}_{2,j}(u^{(n-1)}, \rho^{(n-1)}) \\ &\quad = \int_0^t e^{-(t-s)\Lambda}\nabla\cdot\Big[\sum_{k\ge j-2}\Delta_j(S_{k+1}\rho^{(n-1)}\Delta_k u^{(n-1)})\Big](s)\,\mathrm{d}s. \end{aligned}$$

In view of Bernstein's inequality (2.3), the bounds (2.6) and (4.9), and Lemma 6.1, we estimate

$$
\begin{aligned}
&2^{j\beta}\|\mathcal{B}_{1,j}(u^{(n-1)},\rho^{(n-1)})\|_{L^\infty}\\
&\quad\le C2^{j\beta}2^j\int_0^t e^{-c(t-s)2^j}\Big[\sum_{k\ge j-2}\|S_k u^{(n-1)}\|_{L^\infty}\,\|\Delta_k\rho^{(n-1)}\|_{L^\infty}\Big]\,\mathrm{d}s\\
&\quad\le C\|\rho^{(n-1)}\|^2_{\tilde L^\infty_t\dot B^1_{2,1}}\\
&\qquad\quad\times\Big\{\int_0^t(2^je^{-c(t-s)2^j}s^{-\alpha})\Big(\sum_{k\ge j-2}2^{-(k-j)\beta}s^\alpha 2^{k\beta}\|\Delta_k\rho^{(n-1)}\|_{L^\infty}\Big)\,\mathrm{d}s\Big\}\\
&\quad\le C\|\rho^{(n-1)}\|^2_{\tilde L^\infty_t\dot B^1_{2,1}}\sup_{t>0}\{t^\alpha\|\rho^{(n-1)}\|_{\dot B^\beta_{\infty,\infty}}\}\int_0^t 2^je^{-c(t-s)2^j}s^{-\alpha}\,\mathrm{d}s\\
&\quad\le Ct^{-\alpha}\|\rho^{(n-1)}\|^2_{\tilde L^\infty_t\dot B^1_{2,1}}\sup_{t>0}\{t^\alpha\|\rho^{(n-1)}\|_{\dot B^\beta_{\infty,\infty}}\},
\end{aligned}
$$

and so,

$$
\begin{aligned}
t^\alpha 2^{j\beta}\|\mathcal{B}_{1,j}(u^{(n-1)},\rho^{(n-1)})\|_{L^\infty}&\\
\le C\|\rho^{(n-1)}\|^2_{\tilde L^\infty_t\dot B^1_{2,1}}&\sup_{t>0}\{t^\alpha\|\rho^{(n-1)}\|_{\dot B^\beta_{\infty,\infty}}\}.
\end{aligned}
\tag{6.5}
$$

Now, we estimate $2^{j\beta}\|\mathcal{B}_{2,j}(u^{(n-1)},\rho^{(n-1)})\|_{L^\infty}$. We note first that

$$
\begin{aligned}
&\sum_{k\ge j-2}s^\alpha 2^{j\beta}\|\Delta_k u^{(n-1)}\|_{L^\infty}\\
&\quad\le C\sum_{k\ge j-2}s^\alpha 2^{j\beta}\|\Delta_k(\rho^{(n-1)}R\rho^{(n-1)})\|_{L^\infty}\\
&\quad\le C\sum_{k\ge j-2}s^\alpha 2^{j\beta}\Big(\sum_{l\ge k-2}\|S_{l+1}\rho^{(n-1)}\|_{L^\infty}\,\|\Delta_l R\rho^{(n-1)}\|_{L^\infty}\Big)\\
&\qquad+C\sum_{k\ge j-2}s^\alpha 2^{j\beta}\Big(\sum_{l\ge k-2}\|S_l R\rho^{(n-1)}\|_{L^\infty}\|\Delta_l\rho^{(n-1)}\|_{L^\infty}\Big)\\
&\quad\le C\|\rho^{(n-1)}\|_{\tilde L^\infty_t\dot B^1_{2,1}}\sum_{k\ge j-2}\sum_{l\ge k-2}2^{-(l-j)\beta}s^\alpha 2^{l\beta}\|\Delta_l\rho^{(n-1)}\|_{L^\infty}\\
&\quad= C\|\rho^{(n-1)}\|_{\tilde L^\infty_t\dot B^1_{2,1}}\sum_{l\ge j-4}\sum_{j-2\le k\le l+2}2^{-(l-j)\beta}s^\alpha 2^{l\beta}\|\Delta_l\rho^{(n-1)}\|_{L^\infty}\\
&\quad= C\|\rho^{(n-1)}\|_{\tilde L^\infty_t\dot B^1_{2,1}}\sum_{l\ge j-4}(l-j+5)2^{-(l-j)\beta}s^\alpha 2^{l\beta}\|\Delta_l\rho^{(n-1)}\|_{L^\infty}\\
&\quad\le C\|\rho^{(n-1)}\|_{\tilde L^\infty_t\dot B^1_{2,1}}\sup_{t>0}\{t^\alpha\|\rho^{(n-1)}\|_{\dot B^\beta_{\infty,\infty}}\}.
\end{aligned}
$$

Here, we have used the boundedness of the Leray projector in Besov spaces, the paraproduct decomposition (4.14), the bound (4.8), Fubini's theorem, and Young's convolution inequality. This implies

$$\begin{aligned}
&2^{j\beta}\|\mathcal{B}_{2,j}(u^{(n-1)},\rho^{(n-1)})\|_{L^\infty}\\
&\quad\le C2^{j\beta}2^j\int_0^t e^{-c(t-s)2^j}\Big[\sum_{k\ge j-2}\|S_{k+1}\rho^{(n-1)}\|_{L^\infty}\|\Delta_k u^{(n-1)}\|_{L^\infty}\Big]\,\mathrm{d}s\\
&\quad\le C\|\rho^{(n-1)}\|^2_{\tilde{L}^\infty_t\dot{B}^1_{2,1}}\sup_{t>0}\{t^\alpha\|\rho^{(n-1)}\|_{\dot{B}^\beta_{\infty,\infty}}\}\int_0^t 2^j e^{-c(t-s)2^j}s^{-\alpha}\,\mathrm{d}s\\
&\quad\le Ct^{-\alpha}\|\rho^{(n-1)}\|^2_{\tilde{L}^\infty_t\dot{B}^1_{2,1}}\sup_{t>0}\{t^\alpha\|\rho^{(n-1)}\|_{\dot{B}^\beta_{\infty,\infty}}\},
\end{aligned}$$

and hence

$$\begin{aligned}
t^\alpha 2^{j\beta}&\|\mathcal{B}_{2,j}(u^{(n-1)},\rho^{(n-1)})\|_{L^\infty}\\
&\le C\|\rho^{(n-1)}\|^2_{\tilde{L}^\infty_t\dot{B}^1_{2,1}}\sup_{t>0}\{t^\alpha\|\rho^{(n-1)}\|_{\dot{B}^\beta_{\infty,\infty}}\}.
\end{aligned}\tag{6.6}$$

Putting (6.5) and (6.6) together, we obtain (6.4). Therefore,

$$\begin{aligned}
&\sup_{t>0} t^\alpha\|\rho^{(n)}(t)\|_{\dot{B}^\beta_{\infty,\infty}}\\
&\quad\le C_3\|\rho_0\|_{\dot{B}^{\beta-\alpha}_{\infty,\infty}}+C_4\|\rho^{(n-1)}\|^2_{\tilde{L}^\infty_t\dot{B}^1_{2,1}}\sup_{t>0}\{t^\alpha\|\rho^{(n-1)}(t)\|_{\dot{B}^\beta_{\infty,\infty}}\}.
\end{aligned}$$

Now, we use the smallness of the initial data and proceed as in Theorem 4.1. We omit further details. ◻

We recall the following relationship between the inhomogeneous Besov space $B^s_{\infty,\infty}$ and Hölder spaces.

Remark 6.3. For $s\in\mathbb{R}^+\setminus\mathbb{N}$, the inhomogeneous Besov space $B^s_{\infty,\infty}(\mathbb{R}^2)$ coincides with the Hölder space $C^{[s],s-[s]}(\mathbb{R}^2)$ of bounded functions f whose derivatives of order $|\alpha|\le s$ are bounded and satisfy

$$|\partial^\alpha f(x)-\partial^\alpha f(y)|\le C|x-y|^{s-[s]}$$

for $|x-y|\le 1$ (see [3]).

As a consequence, we obtain the following regularity result.

Corollary 6.4. *Let $s\ge 1$ be an integer, and let*

$$\rho_0\in L^\infty(\mathbb{R}^2)\cap\dot{B}^1_{2,1}(\mathbb{R}^2)\cap\dot{B}^{s+1/2}_{\infty,\infty}(\mathbb{R}^2)$$

be sufficiently small. Further, let ρ be the unique solution of (3.1)–(3.3). Then, $\mathrm{D}^\gamma\rho\in L^\infty(\mathbb{R}^2)$ for $|\gamma|\le s$, and its L^∞ norm is uniformly bounded in time. Moreover, for $|\gamma|\le s$, $\mathrm{D}^\gamma\rho$ is Hölder continuous with a uniform-in-time Hölder bound.

Proof. In view of (2.2), the bound (3.9), and Theorem 6.2 applied with $\alpha = 0$ and $\beta = s + \frac{1}{2}$, we have

$$\|\rho(t)\|_{B^{s+1/2}_{\infty,\infty}} \leq C\{\|\rho(t)\|_{\dot{B}^{s+1/2}_{\infty,\infty}} + \|\rho(t)\|_{L^\infty}\} \leq C(1 + \|\rho_0\|_{L^\infty}),$$

where C is a constant depending only on the initial data. Remark 6.3 completes the proof of Corollary 6.4. ◻

We now consider the long-time behavior of derivatives of solutions of (3.1)–(3.3) for sufficiently small initial data in Besov spaces.

Corollary 6.5. *Let $s \geq 1$ be an integer, and $\delta \in (0, 1)$. Let also*

$$\rho_0 \in L^\infty(\mathbb{R}^2) \cap \dot{B}^1_{2,1}(\mathbb{R}^2) \cap \dot{B}^{s+\delta}_{\infty,\infty}(\mathbb{R}^2)$$

be sufficiently small. Let ρ be the unique solution of (3.1)–(3.3)*. Then,*

$$\lim_{t\to\infty}\{\|\mathrm{D}^\gamma\rho(t)\|_{L^\infty} + [\mathrm{D}^\gamma\rho(t)]_\delta\} = 0 \quad \text{for all } |\gamma| \leq s, \tag{6.7}$$

where

$$[\mathrm{D}^\gamma\rho(t)]_\delta = \sup_{0<|x-y|\leq 1} \frac{|\mathrm{D}^\gamma\rho(y) - \mathrm{D}^\gamma\rho(x)|}{|x-y|^\delta}.$$

Proof. We show that $\rho(\cdot, t) \in H^2(\mathbb{R}^2)$ in order to apply Remark 3.5. Indeed,

$$\begin{aligned}\int_0^\infty \|\rho(t)\|_{\dot{B}^2_{2,2}}\,\mathrm{d}t &\leq C\int_0^\infty \|\rho(t)\|_{\dot{B}^2_{2,1}}\,\mathrm{d}t = C\int_0^\infty \sum_{j\in\mathbb{Z}} 2^{2j}\|\Delta_j\rho(t)\|_{L^2}\,\mathrm{d}t \\ &= C\sum_{j\in\mathbb{Z}} 2^{2j}\|\Delta_j\rho(t)\|_{L^1_tL^2} = C\|\rho\|_{\tilde{L}^1_t\dot{B}^2_{2,1}} < \infty\end{aligned}$$

in view of the continuous Besov embedding (2.5) and the monotone convergence theorem. But $B^2_{2,2}$ coincides with the Sobolev space H^2. Thus, we have $\|\rho(t)\|_{H^2} < \infty$ for almost every $t \in (0, \infty)$, and so $\rho(\cdot, t) \in H^2$ for almost every $t \in [0, \infty)$.

In view of Remark 6.3 and Theorem 6.2 applied with $\alpha = \frac{1}{2}$ and $\beta = s + \delta$, we obtain

$$\begin{aligned}\|\mathrm{D}^\gamma\rho(t)\|_{L^\infty} + [\mathrm{D}^\gamma\rho(t)]_\delta &\leq C\|\rho(t)\|_{B^{s+\delta}_{\infty,\infty}} \\ &\leq C\{\|\rho(t)\|_{\dot{B}^{s+\delta}_{\infty,\infty}} + \|\rho(t)\|_{L^\infty}\} \leq C\left(\frac{1}{\sqrt{t}} + \frac{\|\rho_0\|_{L^\infty}}{1 + Ct\|\rho_0\|_{L^\infty}}\right),\end{aligned}$$

where in the last inequality we used a time-decay estimate [12]. Letting $t \to \infty$, we obtain (6.7). ◻

7. REGULARITY OF SOLUTIONS FOR ARBITRARY INITIAL DATA

In this section, we prove that any solution of (3.1)–(3.3) is smooth for arbitrary initial data, provided it satisfies a certain regularity condition.

Theorem 7.1. *Let ρ be a weak solution of* (3.1)–(3.3) *on* $[0,\infty)$. *Let* $0 < t_0 < t < \infty$. *If*

$$\rho \in L^\infty([t_0,t]; C^\delta(\mathbb{R}^2)) \quad \text{for some } \delta \in (0,1), \tag{7.1}$$

then

$$\rho \in C^\infty((t_0,t] \times \mathbb{R}^2). \tag{7.2}$$

Proof. We sketch the main ideas. Let us note first that

$$u \in L^\infty([t_0,t]; C^\delta(\mathbb{R}^2)). \tag{7.3}$$

where $u = -\mathbb{P}(\rho R\rho)$. Indeed, for any $s \in [t_0,t]$, we have

$$\begin{aligned}
\|u(s)\|_{C^\delta} &\le C\|\rho(s)R\rho(s)\|_{C^\delta} \\
&\le C\|\rho(s)\|_{L^\infty}\|R\rho(s)\|_{L^\infty} + C\|\rho(s)\|_{L^\infty}\|R\rho(s)\|_{C^\delta} \\
&\qquad + C\|R\rho(s)\|_{L^\infty}\|\rho(s)\|_{C^\delta} \\
&\le C\|\rho(s)\|_{C^\delta}^2
\end{aligned}$$

in view of the boundedness of the Leray projector and Riesz transforms on the Hölder space C^δ. Consequently, the Hölder regularity of ρ imposed in (7.1) gives (7.3).

Next, we show that

$$\rho \in L^\infty([t_0,t]; \dot{B}^{\delta_1}_{p,\infty}(\mathbb{R}^2) \cap C^{\delta_1}(\mathbb{R}^2)) \tag{7.4}$$

and

$$u \in L^\infty([t_0,t]; \dot{B}^{\delta_1}_{p,\infty}(\mathbb{R}^2) \cap C^{\delta_1}(\mathbb{R}^2)) \tag{7.5}$$

for any $p \ge 2$ and $\delta_1 = \delta(1-2/p)$. Indeed, for any $s \in [t_0,t]$, we have

$$\begin{aligned}
\|\rho(s)\|_{\dot{B}^{\delta_1}_{p,\infty}} &= \sup_{j\in\mathbb{Z}}(2^{\delta_1 j}\|\Delta_j\rho(s)\|_{L^p}) \\
&\le \sup_{j\in\mathbb{Z}}(2^{\delta_1 j}\|\Delta_j\rho(s)\|_{L^\infty}^{1-2/p}\|\Delta_j\rho(s)\|_{L^2}^{2/p}) \\
&\le C(\|\rho(s)\|_{\dot{B}^{\delta}_{\infty,\infty}})^{1-2/p}\|\rho(s)\|_{L^2}^{2/p} \\
&\le C(\|\rho(s)\|_{C^\delta})^{1-2/p}\|\rho(s)\|_{L^2}^{2/p},
\end{aligned}$$

and similarly

$$\begin{aligned}\|u(s)\|_{\dot B^{\delta_1}_{p,\infty}} &\le C(\|u(s)\|_{\dot B^{\delta}_{\infty,\infty}})^{1-2/p}\|u(s)\|_{L^2}^{2/p}\\ &\le C(\|u(s)\|_{C^\delta})^{1-2/p}\|\rho(s)\|_{L^4}^{4/p}.\end{aligned}$$

The last inequality holds in view of the boundedness of the Leray projector on L^2 followed by an application of Hölder's inequality with exponents $4, 4$. The interpolation inequality $\|\rho(s)\|_{L^4} \le \|\rho(s)\|_{L^\infty}^{1/2}\|\rho(s)\|_{L^2}^{1/2}$ together with (7.3) and (7.1) gives (7.4) and (7.5).

Now, we proceed as in [11]. We apply Δ_j to (3.1), we multiply the resulting equation by $p|\Delta_j\rho|^{p-2}\Delta_j\rho$, and we integrate first in the space variable $x \in \mathbb{R}^2$ and then in time from t_0 to t. We obtain the bound

$$\begin{aligned}\|\Delta_j\rho(t)\|_{L^p} \le Ce^{-c2^j(t-t_0)}\|\Delta_j\rho(t_0)\|_{L^p} + C\int_{t_0}^t e^{-c2^j(t-s)}2^{(1-2\delta_1)j}\\ \times(\|\rho(s)\|_{C^{\delta_1}}\|u(s)\|_{\dot B^{\delta_1}_{p,\infty}} + \|u(s)\|_{C^{\delta_1}}\|\rho(s)\|_{\dot B^{\delta_1}_{p,\infty}})\,ds\end{aligned} \tag{7.6}$$

(see [11] for details). We multiply by $2^{2\delta_1 j}$ and take the ℓ^∞ norm in j. This yields the bound

$$\begin{aligned}\|\rho(t)\|_{\dot B^{2\delta_1}_{p,\infty}} \le C\sup_{j\in\mathbb{Z}}\{2^{\delta_1 j}e^{-c2^j(t-t_0)}\}\|\rho(t_0)\|_{\dot B^{\delta_1}_{p,\infty}} + C\sup_{j\in\mathbb{Z}}\{1-e^{-c2^j(t-t_0)}\}\\ \times\sup_{s\in[t_0,t]}\{\|\rho(s)\|_{C^{\delta_1}}\|u(s)\|_{\dot B^{\delta_1}_{p,\infty}} + \|u(s)\|_{C^{\delta_1}}\|\rho(s)\|_{\dot B^{\delta_1}_{p,\infty}}\}.\end{aligned}$$

Therefore,

$$\rho(\cdot,t) \in \dot B^{2\delta_1}_{p,\infty}(\mathbb{R}^2), \quad \text{for any } p \ge 2.$$

In view of the continuous Besov embedding (2.5), we have the continuous inclusion

$$\dot B^{2\delta_1}_{p,\infty}(\mathbb{R}^2) \hookrightarrow \dot B^{2\delta_1-2/p}_{\infty,\infty}(\mathbb{R}^2) \quad \text{for any } p \ge 2.$$

We choose $p > (2+2\delta)/\delta$ so that $2\delta_1 - 2/p > \delta_1$, so

$$\rho(\cdot,t) \in \dot B^{\delta_2}_{p,\infty}(\mathbb{R}^2) \cap C^{\delta_2}(\mathbb{R}^2) \quad \text{where } \delta_2 > \delta_1. \tag{7.7}$$

In fact, the spacial regularity (7.7) holds at any s in $[t_0, t]$ because the pointwise-in-time estimate (7.6) holds at those times as well. Now we iterate the above process infinitely many times to upgrade the spacial regularity of the solution, and we simultaneously use the PDE (3.1) to upgrade their time regularity. This yields the desired smoothness (7.2), completing the proof of Theorem 7.1. ◻

8. PERIODIC CASE

In this section, we consider the initial value problem (3.1)–(3.3) posed on the torus $\mathbb{T}^2$ with periodic boundary conditions. We assume the initial data ρ_0 have zero mean. We prove existence and regularity of solutions.

Theorem 8.1. *Let $1 \le p < \infty$. Let $\rho_0 \in \dot{B}^{2/p}_{p,1}(\mathbb{T}^2)$ be sufficiently small. We consider the functional space E_p defined by*

$$E_p(\mathbb{T}^2) = \Big\{ f(t) \in \mathcal{D}_0'(\mathbb{T}^2) : \\ \|f\|_{E_p(\mathbb{T}^2)} = \|f\|_{\tilde{L}^\infty_t \dot{B}^{2/p}_{p,1}(\mathbb{T}^2)} + \|f\|_{\tilde{L}^1_t \dot{B}^{2/p+1}_{p,1}(\mathbb{T}^2)} < \infty \Big\}$$

where $\mathcal{D}_0'(\mathbb{T}^2)$ is the dual space of

$$\mathcal{D}_0(\mathbb{T}^2) = \Big\{ f \in C^\infty(\mathbb{T}^2) : \int_{\mathbb{T}^2} f(x)\,\mathrm{d}x = 0 \Big\}.$$

Then, (3.1)–(3.3) *has a unique global-in-time solution $\rho \in E_p(\mathbb{T}^2)$.*

The proof of Theorem 8.1 follows from the proof of Theorem 4.1.

In view of the Besov embedding and Theorem 8.1, we can conclude that if $\rho_0 \in \dot{B}^1_{2,1}(\mathbb{T}^2)$ is sufficiently small, then there is a constant $C > 0$ depending only on the initial data such that the unique solution ρ of (3.1)–(3.3) obeys

$$\sup_{t>0} \|\nabla \rho(t)\|_{L^2(\mathbb{T}^2)} + \int_0^\infty \|\Delta \rho(t)\|_{L^2(\mathbb{T}^2)}\,\mathrm{d}t \le C. \tag{8.1}$$

Using this latter estimate, we end this section by showing that the $L^2(\mathbb{T}^2)$ norm of $\Lambda^{1/2}\rho$ converges exponentially in time to zero.

We use the following uniform Gronwall lemma [1].

Lemma 8.2. *Let $y(t) \ge 0$ obey a differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} y + c_1 y \le F_1 + F(t)$$

with initial datum $y(0) = y_0$ with F_1 a nonnegative constant, and $F(t) \ge 0$ obeying

$$\int_t^{t+1} F(s)\,\mathrm{d}s \le g_0 e^{-c_2 t} + F_2$$

where c_1, c_2, g_0 are positive constants and F_2 is a nonnegative constant. Then,

$$y(t) \le y_0 e^{-c_1 t} + g_0 e^{c_1 + c}(t+1)e^{-ct} + \frac{1}{c_1}F_1 + \frac{e^{c_1}}{1 - e^{-c_1}} F_2$$

holds with $c = \min\{c_1, c_2\}$.

Corollary 8.3. *Let $\rho_0 \in \dot{B}^1_{2,1}(\mathbb{T}^2)$ be sufficiently small. Then, there is a constant $C > 0$ depending only on the initial data such that the unique solution ρ of* (3.1)–(3.3) *obeys*

$$\|\Lambda^{1/2}\rho(t)\|^2_{L^2(\mathbb{T}^2)} \le Ce^{-t} \quad \text{for all } t \ge 0. \tag{8.2}$$

Proof. We take the inner product in $L^2(\mathbb{T}^2)$ of (3.1) with $\Lambda\rho$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{1/2}\rho(t)\|^2_{L^2(\mathbb{T}^2)} + \|\Lambda\rho(t)\|^2_{L^2(\mathbb{T}^2)} = -\int_{\mathbb{T}^2}(u\cdot\nabla\rho)\Lambda\rho\,\mathrm{d}x.$$

We estimate the nonlinear term

$$\begin{aligned}\left|\int_{\mathbb{T}^2}(u\cdot\nabla\rho)\Lambda\rho\,\mathrm{d}x\right| &\le C\|\rho\|_{L^\infty(\mathbb{T}^2)}\|\rho\|_{L^4(\mathbb{T}^2)}\|\nabla\rho\|_{L^4(\mathbb{T}^2)}\|\nabla\rho\|_{L^2(\mathbb{T}^2)}\\ &\le C\|\rho\|_{L^4(\mathbb{T}^2)}\|\nabla\rho\|^2_{L^4(\mathbb{T}^2)}\|\nabla\rho\|_{L^2(\mathbb{T}^2)}\\ &\le C\|\rho\|^{1/2}_{L^2(\mathbb{T}^2)}\|\nabla\rho\|^{5/2}_{L^2(\mathbb{T}^2)}\|\Delta\rho\|_{L^2(\mathbb{T}^2)}\end{aligned}$$

in view of the boundedness of the Leray projector and Riesz transforms on $L^4(\mathbb{T}^2)$, the continuous embedding $W^{1,4}(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$, and the Ladyzhenskaya interpolation inequality.

Since $H^1(\mathbb{T}^2)$ is continuously embedded in $H^{1/2}(\mathbb{T}^2)$, we have

$$\|\Lambda^{1/2}\rho\|_{L^2(\mathbb{T}^2)} \le C\|\Lambda\rho\|_{L^2(\mathbb{T}^2)},$$

yielding the differential inequality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{1/2}\rho\|_{L^2(\mathbb{T}^2)} &+ C_1\|\Lambda^{1/2}\rho\|_{L^2(\mathbb{T}^2)}\\ &\le C_2\|\rho\|^{1/2}_{L^2(\mathbb{T}^2)}\|\nabla\rho\|^{5/2}_{L^2(\mathbb{T}^2)}\|\Delta\rho\|_{L^2(\mathbb{T}^2)}.\end{aligned} \tag{8.3}$$

We note that

$$\|\rho(t)\|_{L^2(\mathbb{T}^2)} \le C\|\rho_0\|_{L^2(\mathbb{T}^2)}e^{-ct} \quad \text{for all } t \ge 0. \tag{8.4}$$

Indeed, we multiply (3.1) by ρ and we integrate in the space variable. Then, we use the cancellation of the nonlinear term and the continuous embedding of $H^{1/2}(\mathbb{T}^2)$ in $L^2(\mathbb{T}^2)$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho(t)\|_{L^2(\mathbb{T}^2)} + C\|\rho(t)\|_{L^2(\mathbb{T}^2)} \le 0,$$

which gives (8.4).

Now we go back to the differential inequality (8.3). Using the bounds (8.1) and (8.4) together with Lemma 8.2, we obtain (8.2). □

9. SUBCRITICAL PERIODIC CASE

In this section, we consider the subcritical case where the dissipation is given by Λ^α for $\alpha \in (1, 2]$; that is, we consider the equation

$$\partial_t \rho + u \cdot \nabla \rho + \Lambda^\alpha \rho = 0 \tag{9.1}$$

posed on $\mathbb{T}^2$, where

$$u = -\mathbb{P}(\rho R \rho). \tag{9.2}$$

The initial data are given by

$$\rho(x, 0) = \rho_0(x) \tag{9.3}$$

and have zero mean.

Global weak solutions exist, as follows.

Theorem 9.1. *Let $\alpha \in (1, 2]$, and $T > 0$ be arbitrary. Let also $\rho_0 \in L^2(\mathbb{T}^2)$. Then,* (9.1)–(9.3) *has a weak solution ρ on $[0, T]$ obeying*

$$\frac{1}{2}\|\rho(t)\|_{L^2(\mathbb{T}^2)}^2 + \int_0^t \|\Lambda^{\alpha/2}\rho(s)\|_{L^2(\mathbb{T}^2)}^2 \, ds \le \frac{1}{2}\|\rho_0\|_{L^2(\mathbb{T}^2)}^2 \quad \text{for } t \in [0, T].$$

The proof is similar to that of Theorem 3.3, and we omit the details.

We note that the regularity of the initial data imposed in the critical case $(\alpha = 1)$, namely $\rho_0 \in L^{2+\delta}$ for some $\delta > 0$, is not required in the subcritical case in view of the fact that ρ obeys

$$\rho \in L^2(0, T; H^{\alpha/2}(\mathbb{T}^2)).$$

The following proposition is the analogue of Proposition 3.4.

Proposition 9.2. *Let $\alpha \in (1, 2]$, and let $p > 2$ and $\rho_0 \in L^p(\mathbb{T}^2)$. Suppose ρ is a smooth solution of* (9.1)–(9.3) *on $[0, T]$. Then,*

$$\|\rho(t)\|_{L^p(\mathbb{T}^2)} \le \|\rho_0\|_{L^p(\mathbb{T}^2)} \quad \text{holds for all } t \in [0, T].$$

Moreover, if $\rho_0 \in L^\infty(\mathbb{T}^2)$, then

$$\|\rho(t)\|_{L^\infty(\mathbb{T}^2)} \le \|\rho_0\|_{L^\infty(\mathbb{T}^2)} \quad \text{holds for all } t \in [0, T].$$

The solutions of the initial value problem (9.1)–(9.3) with large smooth data are globally regular.

Theorem 9.3. *Let $\alpha \in (1,2]$, $s > 0$, and $T > 0$ be arbitrary. Furthermore, let $\rho_0 \in H^s(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$. Then, there are positive constants C_1, C_2, and C_3 depending only on $\|\rho_0\|_{L^\infty(\mathbb{T}^2)}$ such that the solution of (9.1)–(9.2) with initial data ρ_0 exists and satisfies*

$$\|\Lambda^s \rho(t)\|_{L^2(\mathbb{T}^2)} \leq \|\Lambda^s \rho_0\|_{L^2(\mathbb{T}^2)} e^{C_1 t} \tag{9.4}$$

and

$$\begin{aligned} \int_0^t ||\Lambda^{s+\alpha/2}\rho(\tau)||^2_{L^2(\mathbb{T}^2)}\,\mathrm{d}\tau \\ \leq ||\Lambda^s \rho_0||^2_{L^2(\mathbb{T}^2)} + C_2 ||\Lambda^s \rho_0||^2_{L^2(\mathbb{T}^2)}(e^{C_3 t} - 1) \quad \text{for } t \in [0,T]. \end{aligned} \tag{9.5}$$

Proof. Fix a small $\varepsilon \in (0,1)$ such that $\alpha \geq \varepsilon + 1$. We multiply (9.1) by $\Lambda^{2s}\rho$, and we integrate in the space variable over $\mathbb{T}^2$. We obtain the equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\Lambda^s \rho||^2_{L^2(\mathbb{T}^2)} + ||\Lambda^{s+\alpha/2}\rho||^2_{L^2(\mathbb{T}^2)} = -\int_{\mathbb{T}^2} (u \cdot \nabla\rho)\Lambda^{2s}\rho\,\mathrm{d}x.$$

We estimate the nonlinear term. Integrating by parts and using Hölder's inequality, we have

$$\begin{aligned} \left| \int_{\mathbb{T}^2} (u \cdot \nabla \rho)\Lambda^{2s}\rho\,\mathrm{d}x \right| &= \left| \int_{\mathbb{T}^2} \Lambda^{s-\alpha/2}\nabla \cdot (u\rho)\Lambda^{s+\alpha/2}\rho\,\mathrm{d}x \right| \\ &\leq \|\Lambda^{s-\alpha/2+1}(u\rho)\|_{L^2(\mathbb{T}^2)}\,\|\Lambda^{s+\alpha/2}\rho\|_{L^2(\mathbb{T}^2)}. \end{aligned}$$

In view of the commutator estimate

$$\begin{aligned} \|\Lambda^s(fg)\|_{L^p(\mathbb{T}^2)} \leq C\|g\|_{L^{p_1}(\mathbb{T}^2)}\,\|\Lambda^s f\|_{L^{p_2}(\mathbb{T}^2)} \\ + C\|\Lambda^s g\|_{L^{p_3}(\mathbb{T}^2)}\,\|f\|_{L^{p_4}(\mathbb{T}^2)} \end{aligned} \tag{9.6}$$

that holds for any mean zero functions $f, g \in C^\infty(\mathbb{T}^2)$, $s > 0$, $p \in (1,\infty)$, with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \qquad p_2, p_3 \in (1,\infty)$$

(see [10]), we estimate

$$\begin{aligned} \|\Lambda^{s-\alpha/2+1}(u\rho)\|_{L^2(\mathbb{T}^2)} \leq C\|u\|_{L^{2/\varepsilon}(\mathbb{T}^2)}\,\|\Lambda^{s-\alpha/2+1}\rho\|_{L^{2/(1-\varepsilon)}(\mathbb{T}^2)} \\ + C\|\rho\|_{L^\infty(\mathbb{T}^2)}\,\|\Lambda^{s-\alpha/2+1}u\|_{L^2(\mathbb{T}^2)}. \end{aligned}$$

In view of the boundedness of the Riesz transforms (and hence the Leray projector) on $L^p(\mathbb{T}^2)$ for $p \in (1,\infty)$ and Proposition 9.2, we bound

$$\begin{aligned} \|u\|_{L^{2/\varepsilon}(\mathbb{T}^2)} &\leq C\|\rho R\rho\|_{L^{2/\varepsilon}(\mathbb{T}^2)} \leq C\|\rho\|_{L^\infty(\mathbb{T}^2)}\,\|\rho\|_{L^{2/\varepsilon}(\mathbb{T}^2)} \\ &\leq C||\rho||^2_{L^\infty(\mathbb{T}^2)} \leq C||\rho_0||^2_{L^\infty(\mathbb{T}^2)}. \end{aligned}$$

By the commutator estimate (9.6), we have

$$
\begin{aligned}
\|\Lambda^{s-\alpha/2+1}u\|_{L^2(\mathbb{T}^2)} &\le C\|\Lambda^{s-\alpha/2+1}(\rho R\rho)\|_{L^2(\mathbb{T}^2)} \\
&\le C\|\rho\|_{L^\infty(\mathbb{T}^2)}\|\Lambda^{s-\alpha/2+1}R\rho\|_{L^2(\mathbb{T}^2)} \\
&\qquad + C\|R\rho\|_{L^{2/\varepsilon}(\mathbb{T}^2)}\|\Lambda^{s-\alpha/2+1}\rho\|_{L^{2/(1-\varepsilon)}(\mathbb{T}^2)} \\
&\le C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}\|\Lambda^{s-\alpha/2+1}\rho\|_{L^2(\mathbb{T}^2)} \\
&\qquad + C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}\|\Lambda^{s-\alpha/2+1}\rho\|_{L^{2/(1-\varepsilon)}(\mathbb{T}^2)}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|\Lambda^{s-\alpha/2+1}(u\rho)\|_{L^2(\mathbb{T}^2)} &\le C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}^2\|\Lambda^{s-\alpha/2+1}\rho\|_{L^{2/(1-\varepsilon)}(\mathbb{T}^2)} \\
&\qquad + C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}^2\|\Lambda^{s-\alpha/2+1}\rho\|_{L^2(\mathbb{T}^2)}.
\end{aligned}
$$

In view of the continuous Sobolev embedding $H^\varepsilon(\mathbb{T}^2)\hookrightarrow L^{2/(1-\varepsilon)}(\mathbb{T}^2)$, we obtain the bound

$$
\begin{aligned}
\|\Lambda^{s-\alpha/2+1}(u\rho)\|_{L^2(\mathbb{T}^2)} &\le C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}^2\|\Lambda^{s-\alpha/2+1+\varepsilon}\rho\|_{L^2(\mathbb{T}^2)} \\
&\qquad + C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}^2\|\Lambda^{s-\alpha/2+1}\rho\|_{L^2(\mathbb{T}^2)}.
\end{aligned}
$$

Using the Sobolev interpolation inequality

$$
\|\Lambda^{s_1}f\|_{L^2(\mathbb{T}^2)} \le C\|\Lambda^{s_0}f\|_{L^2(\mathbb{T}^2)}^{1-\sigma}\|\Lambda^{s_2}f\|_{L^2(\mathbb{T}^2)}^{\sigma} \tag{9.7}
$$

that holds for any mean zero function $f\in H^{s_2}(\mathbb{T}^2)$ and $s_1=(1-\sigma)s_0+\sigma s_2$, $\sigma\in[0,1]$, we estimate

$$
\begin{aligned}
\|\Lambda^{s-\alpha/2+1}\rho\|_{L^2(\mathbb{T}^2)} &\le C(\|\Lambda^s\rho\|_{L^2(\mathbb{T}^2)})^{2(\alpha-1)/\alpha} \\
&\qquad \times(\|\Lambda^{s+\alpha/2}\rho\|_{L^2(\mathbb{T}^2)})^{2/\alpha-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\Lambda^{s-\alpha/2+1+\varepsilon}\rho\|_{L^2(\mathbb{T}^2)} &\le C(\|\Lambda^s\rho\|_{L^2(\mathbb{T}^2)})^{2(\alpha-\varepsilon-1)/\alpha} \\
&\qquad \times(\|\Lambda^{s+\alpha/2}\rho\|_{L^2(\mathbb{T}^2)})^{2(\varepsilon+1)/\alpha-1}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\|\Lambda^{s-\alpha/2+1}(u\rho)\|_{L^2(\mathbb{T}^2)}\|\Lambda^{s+\alpha/2}\rho\|_{L^2(\mathbb{T}^2)} \\
&\quad\le C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}^2(\|\Lambda^s\rho\|_{L^2(\mathbb{T}^2)})^{2(\alpha-\varepsilon-1)/\alpha}(\|\Lambda^{s+\alpha/2}\rho\|_{L^2(\mathbb{T}^2)})^{2(\varepsilon+1)/\alpha} \\
&\qquad + C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}^2(\|\Lambda^s\rho\|_{L^2(\mathbb{T}^2)})^{2(\alpha-1)/\alpha}(\|\Lambda^{s+\alpha/2}\rho\|_{L^2(\mathbb{T}^2)})^{2/\alpha}.
\end{aligned}
$$

By Young's inequality, we end up with

$$\left| \int_{\mathbb{T}^2} (u \cdot \nabla \rho) \Lambda^{2s} \rho \, \mathrm{d}x \right| \leq C_{\rho_0} \|\Lambda^s \rho\|_{L^2(\mathbb{T}^2)}^2 + \frac{1}{2} \|\Lambda^{s+\alpha/2} \rho\|_{L^2(\mathbb{T}^2)}^2$$

where C_{ρ_0} is a constant depending on the L^∞ norm of the initial data ρ_0.

Therefore, we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\Lambda^s \rho\|_{L^2(\mathbb{T}^2)}^2 + \|\Lambda^{s+\alpha/2} \rho\|_{L^2(\mathbb{T}^2)}^2 \leq 2C_{\rho_0} \|\Lambda^s \rho\|_{L^2(\mathbb{T}^2)}^2, \tag{9.8}$$

which gives (9.4) and (9.5). ◻

We have shown existence of global smooth solutions in the subcritical case, provided that the initial data is smooth enough. No smallness condition is imposed on the size of the initial data.

Remark 9.4. The solutions in the subcritical case are unique. This is obtained by following the same argument as for the uniqueness of local strong solutions in the critical case (see the proof of Theorem 3.7 in Appendix C).

Remark 9.5. The results obtained in Theorem 9.3 hold as well in the whole space $\mathbb{R}^2$ when the initial data is smooth. The proof of Theorem 9.3 is mainly based on commutator estimates (9.6) which hold in the whole space (see [17]), the uniform boundedness of the L^p norms of solutions to the subcritical equation which is obtained in $\mathbb{R}^2$ (see Proposition 3.4 and Remark 3.5), and periodic Sobolev interpolation inequalities given by (9.7) which, in the whole space setting, becomes

$$\|f\|_{H^{s_1}(\mathbb{R}^2)} \leq C \|f\|_{H^{s_0}(\mathbb{R}^2)}^{1-\sigma} \|f\|_{H^{s_1}(\mathbb{R}^2)}^{\sigma}$$

for $f \in H^{s_2}(\mathbb{R}^2)$ and $s_1 = (1-\sigma)s_0 + \sigma s_2$, $\sigma \in [0,1]$. Therefore, the differential inequality (9.8) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\Lambda^s \rho\|_{L^2(\mathbb{R}^2)}^2 + \|\Lambda^{s+\alpha/2} \rho\|_{L^2(\mathbb{R}^2)}^2 \leq C_1^0 \|\Lambda^s \rho\|_{L^2(\mathbb{T}^2)}^2 + C_2^0,$$

where C_1^0 and C_2^0 are constants depending only on the initial data, yielding the desired bounds.

Appendix A. Proof of Proposition 2.2

In this appendix, we prove Proposition 2.2.

Proof. Let $f, g \in S_h'$. Bony's paraproduct gives the decomposition

$$fg = \sum_{j \in \mathbb{Z}} S_{j-1} f \Delta_j g + \sum_{j \in \mathbb{Z}} S_{j-1} g \Delta_j f + \sum_{|j-j'| \leq 1} \Delta_j f \Delta_{j'} g.$$

We note that

$$\begin{aligned}\sum_{|j-j'|\le 1}\Delta_j f\Delta_{j'}g &= \sum_{j\in\mathbb{Z}}\Delta_j f\Delta_j g+\sum_{j\in\mathbb{Z}}\Delta_j f\Delta_{j-1}g+\sum_{j\in\mathbb{Z}}\Delta_j f\Delta_{j+1}g\\ &= \sum_{j\in\mathbb{Z}}\Delta_j f\Delta_j g+\sum_{j\in\mathbb{Z}}\Delta_j f\Delta_{j-1}g+\sum_{j\in\mathbb{Z}}\Delta_{j-1} f\Delta_{j}g\\ &= \sum_{j\in\mathbb{Z}}(\Delta_{j-1}f+\Delta_j f)\Delta_j g+\sum_{j\in\mathbb{Z}}\Delta_j f\Delta_{j-1}g.\end{aligned}$$

This implies that

$$fg=\sum_{j\in\mathbb{Z}}S_{j+1}f\Delta_j g+\sum_{j\in\mathbb{Z}}S_j g\Delta_j f.$$

Now we apply Δ_j. In view of (2.1), we have

$$k\le j-2\Rightarrow \Delta_j(S_k g\Delta_k f)=0 \tag{A.1}$$

and

$$k\le j-3\Rightarrow \Delta_j(S_{k+1} f\Delta_k g)=0. \tag{A.2}$$

Indeed,

$$\begin{aligned}&\mathcal{F}(\Delta_j(S_k g\Delta_k f)(\xi)=\Psi_j(|\xi|)\mathcal{F}(S_k g\Delta_k f)(\xi)\\ &\quad=\Psi_j(|\xi|)\Big\{\sum_{l\le k-1}\int_{\mathbb{R}^2}\Psi_l(|\xi-y|)\mathcal{F}g(\xi-y)\Psi_k(|y|)\mathcal{F}f(y)\,\mathrm{d}y\Big\}\\ &\quad=\Psi_j(|\xi|)\Big\{\sum_{l\le k-1}\int_{2^k/2\le|y|\le 2^k5/4}\Psi_l(|\xi-y|)\mathcal{F}g(\xi-y)\Psi_k(|y|)\mathcal{F}f(y)\,\mathrm{d}y\Big\}\\ &\quad=\Psi_j(|\xi|)\tilde{\Psi}_k(\xi),\end{aligned}$$

where

$$\tilde{\Psi}_k(\xi)=\sum_{l\le k-1}\int_{2^k/2\le|y|\le 2^k5/4}\Psi_l(|\xi-y|)\mathcal{F}g(\xi-y)\Psi_k(|y|)\mathcal{F}f(y)\,\mathrm{d}y.$$

Fix $l\le k-1$. Let $y\in\mathbb{R}^2$ be such that $2^k/2\le|y|\le 2^k5/4$ and $\Psi_l(|\xi-y|)\ne 0$. This implies that $|\xi-y|\le 2^l5/4$, and so

$$|\xi|\le|\xi-y|+|y|\le\frac{2^l5}{4}+\frac{2^k5}{4}\le\frac{2^{k-1}5}{4}+\frac{2^k5}{4}=2^{k-3}15.$$

Consequently, if $|\xi|>2^{k-3}15$, then $\Psi_l(|\xi-y|)=0$ for all $l\le k-1$ and for all y satisfying $2^k/2\le|y|\le 2^k5/4$, and so $\tilde{\Psi}_k(\xi)=0$. We conclude that the support of $\tilde{\Psi}_k$ is included in the closed ball centered at 0 with radius $2^{k-3}15$. But the support of $\Psi_j(|\cdot|)$ is included in the closed annulus centered at 0 with radii $2^j/2$ and $2^j5/4$. Therefore, if $k+1 \le j-1$, then $2^{k-3}15 < 2^{k+1} \le 2^{j-1}$, and so $\mathcal{F}(\Delta_j(S_k g \Delta_k f)) = 0$, which gives (A.1). The property (A.2) follows from a similar argument. Therefore, we obtain the decomposition

$$\Delta_j(fg) = \sum_{k \ge j-2} \Delta_j(S_{k+1} f \Delta_k g) + \sum_{k \ge j-2} \Delta_j(S_k g \Delta_k f).$$

This ends the proof of Proposition 2.2. □

Appendix B. Proof of Theorem 3.3

Proof. We take the L^2 inner product of (3.4) with ρ^ε, and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\rho^\varepsilon||_{L^2}^2 + ||\Lambda^{1/2}\rho^\varepsilon||_{L^2}^2 + \varepsilon||\nabla\rho^\varepsilon||_{L^2}^2 = 0. \tag{B.1}$$

Here, we used the fact that $\tilde{u}^\varepsilon$ is divergence free, which implies that

$$(\tilde{u}^\varepsilon \cdot \nabla\rho^\varepsilon, \rho^\varepsilon)_{L^2} = 0.$$

Integrating (B.1) in time from 0 to t, we obtain (3.6). Therefore, the family $\{\rho^\varepsilon : \varepsilon \in (0,1]\}$ is uniformly bounded in $L^2(0,T;H^{1/2})$. Moreover, we have

$$\begin{aligned} |(\Lambda\rho^\varepsilon, \Phi)_{L^2}| &= |(\Lambda^{1/2}\rho^\varepsilon, \Lambda^{1/2}\Phi)_{L^2}| \le \|\Lambda^{1/2}\rho^\varepsilon\|_{L^2}\,\|\Lambda^{/12}\Phi\|_{L^2} \\ &\le C\|\Lambda^{1/2}\rho^\varepsilon\|_{L^2}\,\|\Phi\|_{H^{5/2}}, \\ \varepsilon|(-\Delta\rho^\varepsilon, \Phi)_{L^2}| &= \varepsilon|(\rho^\varepsilon, -\Delta\Phi)_{L^2}| \le C\|\rho^\varepsilon\|_{L^2}\,\|\Phi\|_{H^{5/2}}, \end{aligned}$$

and

$$\begin{aligned} |(\tilde{u}^\varepsilon \cdot \nabla\rho^\varepsilon, \Phi)_{L^2}| &= |(\tilde{u}^\varepsilon\rho^\varepsilon, \nabla\Phi)_{L^2}| \le \|\tilde{u}^\varepsilon\|_{L^2}\,\|\rho^\varepsilon\|_{L^2}\,\|\nabla\Phi\|_{L^\infty} \\ &\le C||\rho^\varepsilon||_{L^4}^2\,\|\rho^\varepsilon\|_{L^2}\,\|\Phi\|_{H^{5/2}} \end{aligned}$$

for all $\Phi \in H^{5/2}$. Here, we used the boundedness of the Riesz operator on L^4, and the continuous Sobolev embedding $H^{3/2} \hookrightarrow L^\infty$. Therefore, we obtain the bound

$$\begin{aligned} &\|\tilde{u}^\varepsilon \cdot \nabla\rho^\varepsilon\|_{H^{-5/2}} + \|\Lambda\rho^\varepsilon\|_{H^{-5/2}} + \varepsilon\|\Delta\rho^\varepsilon\|_{H^{-5/2}} \\ &\quad \le C(||\rho^\varepsilon||_{L^4}^2\,\|\rho^\varepsilon\|_{L^2} + \|\rho^\varepsilon\|_{L^2} + \|\Lambda^{1/2}\rho^\varepsilon\|_{L^2}). \end{aligned}$$

In view of the continuous embedding $H^{1/2} \hookrightarrow L^4$, we conclude that the family $\{\partial_t\rho^\varepsilon : \varepsilon \in (0,1]\}$ is uniformly bounded in $L^1(0,T;H^{-5/2})$. Now, we note that the inclusion $H^{1/2} \hookrightarrow L^2$ is compact whereas the inclusion $L^2 \hookrightarrow H^{-5/2}$ is continuous. Let ε_n be a decreasing sequence in $(0,1]$ converging to 0. By the Aubin-Lions lemma and (3.6), the sequence $\{\rho^{\varepsilon_n}\}_{n=1}^\infty$ has a subsequence that converges strongly in $L^2(0,T;L^2)$ and weakly in $L^2(0,T;H^{1/2})$ to some function ρ. By the lower semi-continuity of the norms, we obtain (3.7).

For simplicity of notation, assume ρ^ε converges to ρ strongly in $L^2(0,T;L^2)$ and weakly in $L^2(0,T;H^{1/2})$. We note that

$$\begin{aligned}(\rho^\varepsilon(t),\Phi)_{L^2} - (\rho_0,\Phi)_{L^2} &+ \int_0^t (\tilde{u}^\varepsilon\cdot\nabla\rho^\varepsilon,\Phi)_{L^2}\,\mathrm{d}s\\ &+ \int_0^t (\Lambda^{1/2}\rho^\varepsilon,\Lambda^{1/2}\Phi)_{L^2}\,\mathrm{d}s + \varepsilon\int_0^t (\nabla\rho^\varepsilon,\nabla\Phi)_{L^2}\,\mathrm{d}s = 0\end{aligned}$$

holds for all $\Phi\in H^{5/2}$ and $t\in[0,T]$. Without loss of generality, we may assume that ρ^ε converges to ρ in L^2 for almost every $t\in[0,T]$, and so

$$|(\rho^\varepsilon(t),\Phi)_{L^2} - (\rho(t),\Phi)_{L^2}| \le \|\rho^\varepsilon-\rho\|_{L^2}\,\|\Phi\|_{L^2}\to 0$$

for all $\Phi\in H^{5/2}$ and almost every $t\in[0,T]$. By the weak convergence in $L^2(0,T;H^{1/2})$, we obtain

$$\left|\int_0^t (\Lambda^{1/2}\rho^\varepsilon,\Lambda^{1/2}\Phi)_{L^2}\,\mathrm{d}s - \int_0^t (\Lambda^{1/2}\rho,\Lambda^{1/2}\Phi)_{L^2}\,\mathrm{d}s\right|\to 0$$

for all $\Phi\in H^{5/2}$ and all $t\in[0,T]$. For the nonlinear term, we let $\Phi\in H^{5/2}$, $t\in[0,T]$, and write

$$\begin{aligned}\int_0^t (\tilde{u}^\varepsilon\cdot\nabla\rho^\varepsilon,\Phi)_{L^2}\,\mathrm{d}s &- \int_0^t (u\cdot\nabla\rho,\Phi)_{L^2}\,\mathrm{d}s\\ &= -\int_0^t ((\rho^\varepsilon-\rho)u,\nabla\Phi)_{L^2}\,\mathrm{d}s - \int_0^t ((\tilde{u}^\varepsilon-u)\rho^\varepsilon,\nabla\Phi)_{L^2}\,\mathrm{d}s\\ &= I_1+I_2.\end{aligned}$$

We note that

$$|I_1|\le C\|\Phi\|_{H^{5/2}}\int_0^t ||\rho||_{L^4}^2\,\|\rho^\varepsilon-\rho\|_{L^2}\,\mathrm{d}s\to 0$$

by the Lebesgue Dominated Convergence theorem. For I_2, we split it as

$$\begin{aligned}I_2 &= \int_0^t ((J_\varepsilon\mathbb{P}(\rho(R\rho^\varepsilon-R\rho)))\rho^\varepsilon,\nabla\Phi)_{L^2}\,\mathrm{d}s\\ &\qquad + \int_0^t ((J_\varepsilon\mathbb{P}((\rho^\varepsilon-\rho)R\rho^\varepsilon))\rho^\varepsilon,\nabla\Phi)_{L^2}\,\mathrm{d}s\\ &= I_{2,1}+I_{2,2}.\end{aligned}$$

In view of the boundedness of the Riesz transform on L^2 and the boundedness of the Leray operator on $L^{4/3}$, we have

$$
\begin{aligned}
|I_{2,1}| &\le C\|\Phi\|_{H^{5/2}} \int_0^t \|\rho^\varepsilon\|_{L^4} \, \|\mathbb{P}(\rho R(\rho^\varepsilon - \rho))\|_{L^{4/3}} \, \mathrm{d}s \\
&\le C\|\Phi\|_{H^{5/2}} \int_0^t \|\rho^\varepsilon\|_{L^4} \, \|\rho\|_{L^4} \, \|\rho^\varepsilon - \rho\|_{L^2} \, \mathrm{d}s \\
&\le C\|\Phi\|_{H^{5/2}} \left(\int_0^t \|\rho^\varepsilon\|_{L^4}^2 \, \mathrm{d}s \right)^{1/2} \left(\int_0^t \|\rho\|_{L^4}^2 \, \|\rho^\varepsilon - \rho\|_{L^2}^2 \, \mathrm{d}s \right)^{1/2} \to 0
\end{aligned}
$$

by the Lebesgue Dominated Convergence theorem.

We note that we have not yet used the assumption that $\rho_0 \in L^{2+\delta}$. This will be needed to estimate $|I_{2,2}|$. Indeed, we multiply equation (3.4) by $\rho^\varepsilon |\rho^\varepsilon|^\delta$, and we integrate in the space variable. We use the Córdoba-Córdoba inequality [12]

$$
\int_{\mathbb{R}^2} |\rho^\varepsilon|^\delta (\rho^\varepsilon \Lambda \rho^\varepsilon) \, \mathrm{d}x \ge 0,
$$

and we obtain the differential inequality

$$
\frac{\mathrm{d}}{\mathrm{d}t} \|\rho^\varepsilon(t)\|_{L^{2+\delta}} \le 0.
$$

Integrating in time from 0 to t, we end up having the bound

$$
\|\rho^\varepsilon(t)\|_{L^{2+\delta}} \le \|\rho_0\|_{L^{2+\delta}} \quad \text{for all } t \in [0, T].
$$

As a consequence,

$$
\begin{aligned}
|I_{2,2}| &\le C\|\Phi\|_{H^{5/2}} \int_0^t \|\rho^\varepsilon\|_{L^4} \, \|\rho^\varepsilon\|_{L^{2+\delta}} \, \|\rho^\varepsilon - \rho\|_{L^{(8+4\delta)/(2+3\delta)}} \, \mathrm{d}s \\
&\le C\|\Phi\|_{H^{5/2}} \, \|\rho_0\|_{L^{2+\delta}} \int_0^t \|\rho^\varepsilon\|_{L^4} \, \|\rho^\varepsilon - \rho\|_{L^2}^{(2\delta)/(2+\delta)} \, \|\rho^\varepsilon - \rho\|_{L^4}^{(2-\delta)/(2+\delta)} \, \mathrm{d}s \\
&\le C\|\Phi\|_{H^{5/2}} \, \|\rho_0\|_{L^{2+\delta}} \left(\int_0^t \|\rho^\varepsilon\|_{L^4}^2 \right)^{2/(2+\delta)} \left(\int_0^t \|\rho^\varepsilon - \rho\|_{L^2}^2 \, \mathrm{d}s \right)^{\delta/(2+\delta)} \\
&\quad + C\|\Phi\|_{H^{5/2}} \, \|\rho_0\|_{L^{2+\delta}} \left(\int_0^t \|\rho^\varepsilon\|_{L^4}^2 \right)^{1/2} \\
&\quad \times \left(\int_0^t \|\rho\|_{L^4}^2 \, \mathrm{d}s \right)^{(2-\delta)/(4+2\delta)} \left(\int_0^t \|\rho^\varepsilon - \rho\|_{L^2}^2 \, \mathrm{d}s \right)^{\delta/(2+\delta)} \to 0.
\end{aligned}
$$

Here, we used the interpolation inequality

$$
\|f\|_{L^{(8+4\delta)/(2+3\delta)}} \le C\|f\|_{L^2}^{(2\delta)/(2+\delta)} \|f\|_{L^4}^{(2-\delta)/(2+\delta)},
$$

which holds for any $f \in L^4$.

Thus, ρ is a weak solution of (3.1). This ends the proof of Theorem 3.3. ◻

APPENDIX C. PROOF OF THEOREM 3.7

Proof. We apply $-\Delta = \Lambda^2$ to (3.4), and obtain

$$-\partial_t \Delta\rho^\varepsilon - \tilde{u}^\varepsilon \cdot \nabla\Delta\rho^\varepsilon - 2\nabla\tilde{u}^\varepsilon\nabla\nabla\rho^\varepsilon - \Delta\tilde{u}^\varepsilon \cdot \nabla\rho^\varepsilon + \Lambda^3\rho^\varepsilon + \varepsilon\Delta\Delta\rho^\varepsilon = 0. \tag{C.1}$$

Multiply (C.1) by $-\Delta\rho^\varepsilon$, and integrate over $\mathbb{R}^2$. Given $(\tilde{u}^\varepsilon \cdot \nabla\Delta\rho^\varepsilon, \Delta\rho^\varepsilon)_{L^2} = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\Delta\rho^\varepsilon||_{L^2}^2 + ||\Lambda^{5/2}\rho^\varepsilon||_{L^2}^2 + \varepsilon||\Lambda^3\rho^\varepsilon||_{L^2}^2 \\
&\qquad = -2(\nabla\tilde{u}^\varepsilon\nabla\nabla\rho^\varepsilon, \Delta\rho^\varepsilon)_{L^2} - (\Delta\tilde{u}^\varepsilon \cdot \nabla\rho^\varepsilon, \Delta\rho^\varepsilon)_{L^2}.
\end{aligned}$$

Using the product rule $\|fg\|_{H^s} \le C\|f\|_{H^s}\|g\|_{L^\infty} + C\|g\|_{H^s}\|f\|_{L^\infty}$ that holds for any $f, g \in H^s$, $s > 0$, we estimate

$$\begin{aligned}
\|\nabla\tilde{u}^\varepsilon\|_{L^4} &\le C\|\tilde{u}^\varepsilon\|_{H^{3/2}} \le C\|\rho^\varepsilon R\rho^\varepsilon\|_{H^{3/2}} \\
&\le C\|\rho^\varepsilon\|_{L^\infty}\|R\rho^\varepsilon\|_{H^{3/2}} + C\|R\rho^\varepsilon\|_{L^\infty}\|\rho^\varepsilon\|_{H^{3/2}} \\
&\le C||\rho^\varepsilon||_{H^{3/2}}^2.
\end{aligned}$$

Here, we have used the continuous embedding $H^{1/2} \hookrightarrow L^4$, the fact that the Leray projector is bounded on $H^{3/2}$, and the boundedness of the Riesz transforms as operators from $H^{3/2}$ into L^∞. Similarly, we bound

$$\begin{aligned}
\|\Delta\tilde{u}^\varepsilon\|_{L^4} &\le C\|\rho^\varepsilon R\rho^\varepsilon\|_{H^{5/2}} \\
&\le C\|\rho^\varepsilon\|_{L^\infty}\|R\rho^\varepsilon\|_{H^{5/2}} + C\|R\rho^\varepsilon\|_{L^\infty}\|R\rho^\varepsilon\|_{H^{5/2}} \\
&\le C\|\rho^\varepsilon\|_{H^{3/2}}\|\rho^\varepsilon\|_{H^{5/2}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\Delta\rho^\varepsilon||_{L^2}^2 + ||\Lambda^{5/2}\rho^\varepsilon||_{L^2}^2 \\
&\qquad \le 2\|\nabla\tilde{u}^\varepsilon\|_{L^4}\|\nabla\nabla\rho^\varepsilon\|_{L^4}\|\Delta\rho^\varepsilon\|_{L^2} + \|\Delta\tilde{u}^\varepsilon\|_{L^4}\|\nabla\rho^\varepsilon\|_{L^4}\|\Delta\rho^\varepsilon\|_{L^2} \\
&\qquad \le C||\rho^\varepsilon||_{H^{3/2}}^2\|\rho^\varepsilon\|_{H^{5/2}}\|\Delta\rho^\varepsilon\|_{L^2},
\end{aligned}$$

and, by Young's inequality, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}||\Delta\rho^\varepsilon||_{L^2}^2 + ||\Lambda^{5/2}\rho^\varepsilon||_{L^2}^2 \\
&\qquad \le C||\rho^\varepsilon||_{H^{3/2}}^4||\Delta\rho^\varepsilon||_{L^2}^2 + C||\rho^\varepsilon||_{H^{3/2}}^2\|\rho^\varepsilon\|_{L^2}\|\Delta\rho^\varepsilon\|_{L^2} \\
&\qquad \le C(||\rho^\varepsilon||_{H^2}^6 + ||\rho^\varepsilon||_{H^2}^4).
\end{aligned}$$

We note that

$$\begin{aligned}\|\rho^\varepsilon\|_{H^2} &= \|(1+|.|^2)\mathcal{F}(\rho^\varepsilon)(.)\|_{L^2} \le C\|\mathcal{F}\rho^\varepsilon\|_{L^2} + C\|\Delta\rho^\varepsilon\|_{L^2}\\ &= C\|\rho^\varepsilon\|_{L^2} + C\|\Delta\rho^\varepsilon\|_{L^2} \le C\|\rho_0\|_{L^2} + C\|\Delta\rho^\varepsilon\|_{L^2}\end{aligned}$$

in view of Plancherel's theorem and the uniform boundedness of ρ^ε in L^2 described by (3.7). Therefore, we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}||\Delta\rho^\varepsilon||_{L^2}^2 + ||\Lambda^{5/2}\rho^\varepsilon||_{L^2}^2 \le C||\Delta\rho^\varepsilon||_{L^2}^6 + C_{\rho_0} \tag{C.2}$$

where C_{ρ_0} is a positive constant depending only on ρ_0 and some universal constants. This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(||\Delta\rho^\varepsilon||_{L^2}^2 + 1) \le C_0(||\Delta\rho^\varepsilon||_{L^2}^2 + 1)^3$$

for some constant C_0 depending only on the initial data. Dividing both sides by $(\|\Delta\rho^\varepsilon\|_{L^2}^2 + 1)^3$ and integrating in time from 0 to t, we get

$$\frac{1}{2(||\Delta\rho^\varepsilon(t)||_{L^2}^2 + 1)^2} \ge \frac{1}{2(||\Delta\rho_0||_{L^2}^2 + 1)^2} - C_0T_0 \quad \text{for all } t\in[0,T_0].$$

We choose a positive time $T_0 > 0$ such that

$$T_0 < \frac{1}{2C_0(||\Delta\rho_0||_{L^2}^2 + 1)^2},$$

and we conclude that

$$||\Delta\rho^\varepsilon(t)||_{L^2}^2 \le \frac{||\Delta\rho_0||_{L^2}^2 + 1}{\sqrt{1 - 2C_0T_0(||\Delta\rho_0||_{L^2}^2 + 1)^2}} \quad \text{for all } t\in[0,T_0].$$

In view of the energy inequality (C.2), we also have that

$$\int_0^{T_0} ||\Lambda^{5/2}\rho^\varepsilon(t)||_{L^2}^2\,\mathrm{d}t \le \Gamma(\rho_0, T_0),$$

where $\Gamma(\rho_0, T_0)$ is a positive constant depending only on the initial data and T_0. This shows that $\{\rho^\varepsilon : \varepsilon\in(0,1]\}$ is uniformly bounded in

$$L^\infty(0,T;\dot H^2(\mathbb{R}^2)) \cap L^2(0,T;\dot H^{5/2}(\mathbb{R}^2)).$$

Passing to the limit on a subsequence and using the lower semicontinuity of norms, we conclude that the weak solution ρ, obtained in Theorem 3.3, is strong.

For uniqueness, suppose that ρ_1 and ρ_2 are two strong solutions of (3.1) on $[0, T_0]$ with the same initial condition. Let $\rho = \rho_1 - \rho_2$ and $u = u_1 - u_2$. Then, ρ obeys the equation

$$\partial_t \rho + u \cdot \nabla \rho_1 + u_2 \cdot \nabla \rho + \Lambda \rho = 0.$$

We take the L^2 inner product with ρ, and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\rho||_{L^2}^2 + ||\Lambda^{1/2}\rho||_{L^2}^2 = -(u \cdot \nabla \rho_1, \rho)_{L^2}.$$

In view of the boundedness of the Riesz transforms on L^4, we have

$$\begin{aligned} \|u\|_{L^4} &\le \|\mathbb{P}(\rho R \rho_1)\|_{L^4} + \|\mathbb{P}(\rho_2 R \rho)\|_{L^4} \\ &\le C\|\rho\|_{L^4}\, \|R\rho_1\|_{L^\infty} + \|\rho_2\|_{L^\infty}\, \|R\rho\|_{L^4} \\ &\le C\|\rho\|_{L^4}(\|\rho_1\|_{H^{3/2}} + \|\rho_2\|_{H^{3/2}}). \end{aligned}$$

Hence,

$$\begin{aligned} |(u \cdot \nabla \rho_1, \rho)_{L^2}| &\le \|u\|_{L^4}\, \|\nabla \rho_1\|_{L^4}\, \|\rho\|_{L^2} \\ &\le \frac{1}{2}||\rho||_{H^{1/2}}^2 + C(||\rho_1||_{H^{3/2}}^2 + ||\rho_2||_{H^{3/2}}^2)||\rho_1||_{H^{3/2}}^2\, ||\rho||_{L^2}^2. \end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}||\rho||_{L^2}^2 \le K(t)||\rho||_{L^2}^2$$

where

$$K(t) = C(||\rho_1||_{H^{3/2}}^2 + ||\rho_2||_{H^{3/2}}^2)||\rho_1||_{H^{3/2}}^2.$$

We note that $K(t)$ is time integrable on $[0, T_0]$ since ρ_1 and ρ_2 belong to the space $L^\infty(0, T_0; H^2(\mathbb{R}^2))$. This shows that for each $t \ge 0$, $\rho_1(\cdot, t) = \rho_2(\cdot, t)$ almost everywhere in $\mathbb{R}^2$, and so we obtain uniqueness. This completes the proof of Theorem 3.7. □

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REFERENCES

[1] E. ABDO and M. IGNATOVA, *Long time dynamics of a model of electroconvection*, Trans. Amer. Math. Soc. **374** (2021), no. 8, 5849–5875. https://dx.doi.org/10.1090/tran/8394. MR4293790

[2] H. BAE, A. BISWAS, and E. TADMOR, *Analyticity and decay estimates of the Navier-Stokes equations in critical Besov spaces*, Arch. Ration. Mech. Anal. **205** (2012), no. 3, 963–991. https://dx.doi.org/10.1007/s00205-012-0532-5. MR2960037

[3] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 343, Springer, Heidelberg, 2011. https://dx.doi.org/10.1007/978-3-642-16830-7. MR2768550

[4] L. A. Caffarelli and A. Vasseur, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. of Math. (2) **171** (2010), no. 3, 1903–1930. https://dx.doi.org/10.4007/annals.2010.171.1903. MR2680400

[5] Á. Castro, D. Córdoba, F. Gancedo, and R. Orive, *Incompressible flow in porous media with fractional diffusion*, Nonlinearity **22** (2009), no. 8, 1791–1815. https://dx.doi.org/10.1088/0951-7715/22/8/002. MR2525811

[6] J.-Y. Chemin, *Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel*, J. Anal. Math. **77** (1999), 27–50 (French). https://dx.doi.org/10.1007/BF02791256. MR1753481

[7] P. Constantin, T. Elgindi, M. Ignatova, and V. Vicol, *On some electroconvection models*, J. Nonlinear Sci. **27** (2017), no. 1, 197–211. https://dx.doi.org/10.1007/s00332-016-9329-2. MR3600829

[8] P. Constantin and M. Ignatova, *Critical SQG in bounded domains*, Ann. PDE **2** (2016), no. 2, Art. 8, 42. https://dx.doi.org/10.1007/s40818-016-0017-1. MR3595454

[9] ———, *Estimates near the boundary for critical SQG*, Ann. PDE **6** (2020), no. 1, Paper No. 3, 30. https://dx.doi.org/10.1007/s40818-020-00079-7. MR4093852

[10] P. Constantin, A. Tarfulea, and V. Vicol, *Long time dynamics of forced critical SQG*, Comm. Math. Phys. **335** (2015), no. 1, 93–141. https://dx.doi.org/10.1007/s00220-014-2129-3. MR3314501

[11] P. Constantin and J. Wu, *Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **25** (2008), no. 6, 1103–1110. https://dx.doi.org/10.1016/j.anihpc.2007.10.001. MR2466323

[12] A. Córdoba and D. Córdoba, *A maximum principle applied to quasi-geostrophic equations*, Comm. Math. Phys. **249** (2004), no. 3, 511–528. https://dx.doi.org/10.1007/s00220-004-1055-1. MR2084005

[13] Z. A. Daya, V. B. Deyirmenjian, S. W. Morris, and J. R. de Bruyn, *Annular electroconvection with shear*, Phys. Rev. Lett. **80** (1998), no. 5, 964–967. https://dx.doi.org/10.1103/PhysRevLett.80.964.

[14] T. Hmidi and S. Keraani, *Global solutions of the super-critical 2D quasi-geostrophic equation in Besov spaces*, Adv. Math. **214** (2007), no. 2, 618–638. https://dx.doi.org/10.1016/j.aim.2007.02.013. MR2349714

[15] M. Ignatova, *Construction of solutions of the critical SQG equation in bounded domains*, Adv. Math. **351** (2019), 1000–1023. https://dx.doi.org/10.1016/j.aim.2019.05.034. MR3956764

[16] M. Ignatova and J. Shu, *Global smooth solutions of the Nernst-Planck-Darcy system*, J. Math. Fluid Mech. **24** (2022), no. 1, Paper No. 26, 21. https://dx.doi.org/10.1007/s00021-022-00666-7. MR4379110

[17] N. Ju, *Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space*, Comm. Math. Phys. **251** (2004), no. 2, 365–376. https://dx.doi.org/10.1007/s00220-004-1062-2. MR2100059

[18] A. Kiselev, F. Nazarov, and A. Volberg, *Global well-posedness for the critical 2D dissipative quasi-geostrophic equation*, Invent. Math. **167** (2007), no. 3, 445–453. https://dx.doi.org/10.1007/s00222-006-0020-3. MR2276260

[19] P. Tsai, Z. A. Daya, and S. W. Morris, *Charge transport scaling in turbulent electroconvection*, Phys. Rev E **72** (2005), no. 4, 046311, 1–12. https://dx.doi.org/10.1103/PhysRevE.72.046311.

[20] P. Tsai, Z. A. Daya, V. B. Deyirmenjian, and S. W. Morris, *Direct numerical simulation of supercritical annular electroconvection*, Phys. Rev E **76** (2007), no. 2, 026305, 1–11. https://dx.doi.org/10.1103/PhysRevE.76.026305.

[21] H. TSURUMI, *Well-posedness and ill-posedness of the stationary Navier-Stokes equations in toroidal Besov spaces*, Nonlinearity **32** (2019), no. 10, 3798–3819. https://dx.doi.org/10.1088/1361-6544/ab22d3. MR4002400

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