

ON THE CONNES-KASPAROV ISOMORPHISM, II: THE VOGAN CLASSIFICATION OF ESSENTIAL COMPONENTS IN THE TEMPERED DUAL

PIERRE CLARE, NIGEL HIGSON, AND YANLI SONG

ABSTRACT. This is the second of two papers dedicated to the computation of the reduced C^* -algebra of a connected, linear, real reductive group up to C^* -algebraic Morita equivalence, and the verification of the Connes-Kasparov conjecture in operator K-theory for these groups. In Part I we presented the Morita equivalence and the Connes-Kasparov morphism. In this part we shall compute the morphism using David Vogan's description of the tempered dual. In fact we shall go further by giving a complete representation-theoretic description and parametrization, in Vogan's terms, of the essential components of the tempered dual, which carry the K-theory of the tempered dual.

1. INTRODUCTION

This is the second and concluding part of a work whose objectives are to determine the reduced C^* -algebra of a connected, linear, real reductive group, up to Morita equivalence, compute its K-theory, and verify that the Connes-Kasparov index homomorphism in K-theory is an isomorphism.

For further background on this problem we refer the reader to the introduction to Part I of this work [CHST22]. We showed there if G is a real reductive group, then:

- (i) The reduced C^* -algebra of G may be described, up to C^* -algebra isomorphism, in a way that neatly encapsulates the major results of tempered representation theory, as developed by Harish-Chandra, Langlands, Knapp, Stein and others.
- (ii) Using the Knapp-Stein theory of the R-group and following Wassermann [Was87], the reduced group C^* -algebra may be described in a more streamlined way, up to Morita equivalence, and its K-theory may be computed. Only certain components of the tempered dual contribute to C^* -algebra K-theory; these are the *essential* components, and they may be characterized using the Knapp-Stein theory.
- (iii) Contingent on a *matching theorem* that pairs Clifford algebra data associated to a maximal compact subgroup of G with the above essential components, the C^* -algebraic indexes of indecomposable Dirac operators on the symmetric space associated to G , which are elements

in the K-theory of the reduced group C^* -algebra, freely generate the K-theory. This is the Connes-Kasparov isomorphism.

In this paper we shall complete our account of the Connes-Kasparov isomorphism in [CHST22] by proving the matching theorem.¹ The main results are:

- (iv) A uniform construction of all the essential components of the tempered dual.
- (v) A parametrization of the essential components of the tempered dual in terms of dominant weights for a maximal compact subgroup of G .

See Theorems 3.10 and 6.3. Our proofs will make very extensive use of David Vogan's approach to the representation theory of G [Vog79, Vog81], which involves minimal K-types and cohomological induction. This is in contrast to previous works on $C_r^*(G)$, which have used Harish-Chandra's approach, via discrete series representations and cuspidal parabolic induction.

The parameters that we use to list the essential components will turn out to be *exactly the same* as the parameters used to construct indecomposable Dirac operators. This puts the matching theorem, stated in this paper as Theorem 7.4, within easy reach, as we shall explain in Section 7.

Notes on Terminology. We shall generally follow the terminology and notation of Vogan's monograph [Vog81]. Thus throughout the paper, G will be a linear real reductive group in the sense of [Vog81, Sec. 0.1], except that we shall in addition assume that G is *connected as a real Lie group*, which Vogan does not.

Throughout the paper we shall work with a fixed Cartan involution θ of G and denote by $K \subseteq G$ the (necessarily connected) maximal compact subgroup that is fixed by θ . We shall also work with a fixed maximal torus $T^c \subseteq K$.

We shall denote real Lie algebras of real groups by fraktur letters decorated with the subscript 0, as in $\mathfrak{k}_0 \subseteq \mathfrak{g}_0$. The same fraktur letters, but without the subscript 0, will be used to denote complexified Lie algebras. Other fraktur letters without the subscript 0 will be used to denote other complex Lie algebras.

We shall write the Lie algebraic Cartan decomposition as $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{s}_0$ (which is a slight departure from Vogan's notation) and we shall fix, once and for all, a nondegenerate, G -invariant symmetric bilinear form

$$(1.1) \quad \langle \cdot, \cdot \rangle: \mathfrak{g}_0 \times \mathfrak{g}_0 \longrightarrow \mathbb{R}$$

that is positive-definite on \mathfrak{s}_0 and negative-definite on \mathfrak{k}_0 . We shall use various forms induced from (1.1), most notably the associated positive-definite inner product on the space $\mathrm{it}_0^{c*} = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}_0^c, i\mathbb{R})$.

¹Actually, there are some other, lesser, issues outstanding from [CHST22]. We shall resolve those too, in Section 8.

Finally we shall denote by $\Delta(\mathfrak{k}, \mathfrak{t}^c)$ the set of roots for $(\mathfrak{k}, \mathfrak{t}^c)$. We shall fix, once and for all, a system of positive roots $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$. The term *dominant* (which means $\langle \lambda, \alpha \rangle \geq 0$ for all positive roots α) will always mean *with respect to this fixed system of positive roots*. We shall use the standard notation $\rho(\Delta^+(\mathfrak{k}, \mathfrak{t}^c))$ for the half-sum of positive roots, and we shall use similar notation in other contexts, where we shall always count roots with multiplicities. This is all as in [Vog81].

2. THETA-STABLE PARABOLIC SUBALGEBRAS

Vogan's monograph [Vog81] uses Zuckerman's method of cohomological induction to construct the entire admissible dual of a real reductive group, and in particular the tempered dual. In this section and the next we shall quickly review some of the basic concepts that are involved.

Definition 2.1 (See [Vog81, Def. 5.2.1]). A θ -stable parabolic subalgebra of \mathfrak{g} is any Lie subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$ defined by a weight $\lambda \in \mathfrak{it}_0^{c*}$ using the formula

$$\mathfrak{q} = \bigoplus \{ \mathfrak{g}_\beta : \beta \in \mathfrak{it}_0^{c*} \text{ and } \langle \lambda, \beta \rangle \geq 0 \},$$

where $\mathfrak{g}_\beta \subseteq \mathfrak{g}$ is the β -weight space for the adjoint action of \mathfrak{t}_0^c on \mathfrak{g} . We shall say that \mathfrak{q} is *defined* by λ .

The subalgebra \mathfrak{q} decomposes as semidirect product $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, where

$$(2.2) \quad \mathfrak{l} = \bigoplus \{ \mathfrak{g}_\beta : \beta \in \mathfrak{it}_0^{c*} \text{ and } \langle \lambda, \beta \rangle = 0 \}$$

and

$$(2.3) \quad \mathfrak{u} = \bigoplus \{ \mathfrak{g}_\beta : \beta \in \mathfrak{it}_0^{c*} \text{ and } \langle \lambda, \beta \rangle > 0 \}.$$

One has $\mathfrak{l} = \mathfrak{q} \cap \bar{\mathfrak{q}}$, so that \mathfrak{l} is the complexification of a real algebra \mathfrak{l}_0 .

Throughout the paper, given a θ -stable parabolic subalgebra \mathfrak{q} , we shall follow Vogan and denote by L the normalizer of \mathfrak{q} in G . This is a connected Lie subgroup of G and a reductive group in its own right, stable under the Cartan involution of G . Its Lie algebra is \mathfrak{l}_0 .

We shall now present the examples of θ -stable parabolic subalgebras that will be of concern to us in this paper.

Proposition 2.4. *If the parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$ is defined by a strictly dominant weight $\lambda \in \mathfrak{it}_0^{c*}$, meaning that $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Delta^+(\mathfrak{k}, \mathfrak{t}^c)$, then there is an isomorphism of Lie algebras*

$$\mathfrak{l}_0 \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \cdots \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{z}_0$$

with the following properties:

- (i) The summand \mathfrak{z}_0 is the center of \mathfrak{l}_0 , and the isomorphism is the identity on \mathfrak{z}_0 .
- (ii) The isomorphism is compatible with the Cartan involution on \mathfrak{l}_0 that is obtained by restriction from \mathfrak{g}_0 and the standard Cartan involutions on $\mathfrak{sl}(2, \mathbb{R})$ -summands.

(iii) *The summands on the right are orthogonal for the nondegenerate bilinear form on \mathfrak{l}_0 that is obtained by restriction from \mathfrak{g}_0 .*

Remark 2.5. The $\mathfrak{sl}(2, \mathbb{R})$ -summands on the right-hand side correspond to simple ideals in \mathfrak{l}_0 , so the isomorphism is canonical up to permutation of the $\mathfrak{sl}(2, \mathbb{R})$ summands and automorphisms of $\mathfrak{sl}(2, \mathbb{R})$ preserving the Cartan involution.

Remark 2.6. If λ is strictly dominant not just for $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$ but indeed for some system of positive roots for the action of \mathfrak{t}^c on \mathfrak{g} , then \mathfrak{l}_0 is in fact abelian; that is, it has no $\mathfrak{sl}(2, \mathbb{R})$ -summands. This is in fact the generic case, and it is very easy to analyze from the point of view of this paper. But it is not quite general enough for our purposes.

Proof of Proposition 2.4. The hypothesis that λ is *strictly* dominant and the definition of \mathfrak{l} imply that $\mathfrak{l} \cap \mathfrak{k} = \mathfrak{t}^c$. It follows that if we decompose \mathfrak{l} into weight spaces for the adjoint action of \mathfrak{t}^c ,

$$\mathfrak{l} = \mathfrak{t}^c \oplus \bigoplus_{\beta} \mathfrak{l}_{\beta},$$

then $\mathfrak{l}_{\beta} \subseteq \mathfrak{s}$ for all β . Hence the Cartan involution θ is -1 on each \mathfrak{l}_{β} .

Let us examine the weight spaces \mathfrak{l}_{β} for $\beta \neq 0$. Minor variations on the standard arguments for semisimple Lie algebras (as in for example [Ser87, Sec. VI.2]) show that

$$\dim(\mathfrak{l}_{\beta}) = \dim(\mathfrak{l}_{-\beta}) = \dim([\mathfrak{l}_{\beta}, \mathfrak{l}_{-\beta}]) = 1.$$

So $\mathfrak{l}_{\beta}, \mathfrak{l}_{-\beta}$ and $[\mathfrak{l}_{\beta}, \mathfrak{l}_{-\beta}]$ span a 3-dimensional Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. It is stable under both the Cartan involution and complex conjugation, which switches \mathfrak{l}_{β} and $\mathfrak{l}_{-\beta}$. So the Lie algebra is isomorphic to the complexification of $\mathfrak{sl}(2, \mathbb{R})$ with its standard Cartan involution.

If $\beta_1 \neq -\beta_2$, then $[\mathfrak{l}_{\beta_1}, \mathfrak{l}_{\beta_2}] = 0$ since

$$[\mathfrak{l}_{\beta_1}, \mathfrak{l}_{\beta_2}] \subseteq \mathfrak{k} \quad \text{and} \quad [\mathfrak{l}_{\beta_1}, \mathfrak{l}_{\beta_2}] \subset \mathfrak{l}_{\beta_1 + \beta_2}.$$

Moreover $[\mathfrak{l}_{\beta_1}, \mathfrak{l}_{-\beta_1}]$ and $[\mathfrak{l}_{\beta_2}, \mathfrak{l}_{-\beta_2}]$ are orthogonal subspaces of \mathfrak{t}^c . The proposition follows from these observations. \square

At the Lie group level, we obtain from the isomorphism of Lie algebras in Proposition 2.4 a morphism

$$(2.7) \quad \mathrm{SL}(2, \mathbb{R}) \times \cdots \times \mathrm{SL}(2, \mathbb{R}) \times Z(L)^0 \longrightarrow L$$

(the superscript denotes the connected component of the identity) that is surjective and a local isomorphism. The kernel is central, as it is for any local isomorphism, and its projection onto the product of $\mathrm{SL}(2, \mathbb{R})$ -factors is injective because $Z(L) \subseteq L$. So the kernel identifies via this projection with a subgroup of the (finite) group generated by the matrices $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ in each $\mathrm{SL}(2, \mathbb{R})$ -factor. The maximal torus T^c , which is a maximal compact subgroup of L , is the image of the corresponding morphism

$$(2.8) \quad \mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2) \times (T^c \cap Z(L)^0) \longrightarrow L$$

in which the final factor is the maximal compact subgroup of $Z(L)^0$, and is therefore connected.

3. VOGAN DATA

The following definition will be used in the next section to construct representations of G using a two-step process that goes from characters of a Cartan subgroup $H \subseteq G$ to representations of an intermediate group L by means of parabolic induction, and then from these representations of L to representations of G by means of cohomological induction.

Definition 3.1 (See [Vog81, Def. 6.5.1]). A triple (q, H, δ) is a set of *Vogan data* for G (Vogan uses the term *discrete, θ -stable data*) if

- (i) $q = \mathfrak{l} + \mathfrak{u}$ is a θ -stable parabolic subalgebra of \mathfrak{g} .
- (ii) H is a θ -stable Cartan subgroup of G (it is necessarily abelian under our assumptions on G , but not necessarily connected) and a subgroup of $L = N_G(q)$.
- (iii) L is *quasi-split* (see [Vog81, Def. 4.3.5]) and H is a *maximally split Cartan subgroup* of L . Altogether, this means that if we write $H = TA$, where $T = H \cap K$ and $A = \exp[\mathfrak{h}_0 \cap \mathfrak{s}_0]$, then $H = TA$ is the Levi factor of a minimal (real) parabolic subgroup $P = TAN$ of L .
- (iv) $\delta: T \rightarrow U(1)$ is a *fine* representation of T with respect to L , in the sense of [Vog81, Definition 4.3.8]. This means that δ is trivial on the intersection of T with the connected component of the identity in the semisimple part of L .
- (v) If $\lambda^L \in \mathfrak{it}_0^* = \text{Hom}_{\mathbb{R}}(\mathfrak{t}_0, i\mathbb{R})$ is the differential of δ , and if

$$\lambda^G = \lambda^L + \rho(u, \mathfrak{t}),$$

then it is required that $\langle \lambda^G, \alpha \rangle > 0$ for all weights α for the adjoint action of \mathfrak{t} on u . (The positive-definite inner product here is on \mathfrak{it}_0^* , and it is again obtained from the bilinear form on \mathfrak{g} . In addition $\rho(u, \mathfrak{t})$ is the half-sum of the weights of the action of \mathfrak{t} on u , multiplicities included.)

Examples of Vogan data may be constructed using the following development of the computation in Proposition 2.4. They represent all the examples that we shall need to study in detail in this paper.

Proposition 3.2. Let $\kappa \in \mathfrak{it}_0^{c*}$ be a dominant weight, and let $q = \mathfrak{l} + \mathfrak{u}$ be the θ -stable parabolic subalgebra defined by the strictly dominant weight

$$\lambda^G = \kappa + \rho(\Delta^+(\mathfrak{k}, \mathfrak{t}^c)).$$

From the isomorphism in Proposition 2.4, the nonzero weights for the action of \mathfrak{t}^c on \mathfrak{l} consist of a collection of pairs $\pm\beta$, one for each $\mathfrak{sl}(2, \mathbb{R})$ -summand in \mathfrak{l}_0 . Denote by $\Delta^+(\mathfrak{l}, \mathfrak{t}^c)$ be any set consisting of one member from each pair. Let $H = TA$ be a maximally split Cartan subalgebra of $L = N_G(q)$. If the weight

$$\mu = \kappa - \rho(\mathfrak{s} \cap \mathfrak{u}, \mathfrak{t}^c) - \rho(\Delta^+(\mathfrak{l}, \mathfrak{t}^c))$$

is analytically integral, and if δ is the restriction to T of the character $\exp(\mu)$ of T^c associated to μ , then (q, H, δ) is a set of Vogan data for G . Whether or not μ is integral is independent of the choice of $\Delta^+(l, t^c)$, and, assuming that μ is integral, the character δ is independent of the choice of $\Delta^+(l, t^c)$, too.

Remark 3.3. Taken together, the weights in $\Delta(u, t^c)$ and $\Delta^+(l, t^c)$ form a system of positive roots for (g, t^c) that includes $\Delta^+(t, t^c)$, and if $\Delta^+(s, t^c)$ is the subset of noncompact roots in $\Delta^+(g, t^c)$, then

$$\Delta^+(s, t^c) = \rho(s \cap u, t^c) + \rho(\Delta^+(l, t^c)).$$

The weight λ^G is dominant for $\Delta^+(g, t^c)$. Conversely, all of the instances of the construction in this remark, using all possible $\Delta^+(l, t^c)$, yield all the systems of positive roots for (g, t^c) for which λ^G is dominant.

Remark 3.4. Before starting the proof we should clear up a notational ambiguity. In the statement of the proposition, λ^G is defined as a weight in t_0^* in terms of the dominant weight $\kappa \in it_0^{c*}$, whereas in Definition 3.1, λ^G is defined as a weight in it_0^* in terms of the differential of δ . Let us check that the two definitions are related in the following way: if the weight $\lambda^G \in it_0^*$ of Definition 3.1 is extended by 0 on the orthogonal complement of t_0 in t_0^c , then one obtains the weight $\lambda^G \in it_0^*$ in the statement of the proposition.

First, for the weight λ^G in the statement of the proposition we have $\lambda^G|_{t^\perp} = 0$. To see this, use the orthogonal direct sum decomposition

$$(3.5) \quad t_0^\perp \cong \mathfrak{so}(2) \oplus \cdots \oplus \mathfrak{so}(2)$$

resulting from the isomorphism in Proposition 2.4. Let $\beta \in it_0^{c*}$ be a nonzero weight for the action of t^c on (the complexification of) one of the $\mathfrak{sl}(2, \mathbb{R})$ -summands of t_0 in Proposition 2.4. The restriction of β to the orthogonal complement of the corresponding $\mathfrak{so}(2)$ -summand in t_0^\perp is zero. Hence

$$\langle \lambda^G, \beta \rangle = \langle \lambda^G|_{\mathfrak{so}(2)}, \beta|_{\mathfrak{so}(2)} \rangle_{\mathfrak{so}(2)}$$

But according to the definition of l in (2.2), $\langle \lambda^G, \beta \rangle = 0$, and so it follows that $\lambda^G|_{\mathfrak{so}(2)} = 0$ for each $\mathfrak{so}(2)$ -summand in (3.5). Hence $\lambda^G|_{t^\perp} = 0$.

Second, if λ^G is again as in the statement of the proposition, then

$$\lambda^G = \kappa + \rho(t \cap u, t^c) = \kappa - \rho(s \cap u, t^c) + \rho(u, t^c),$$

where $\rho(t \cap u, t^c)$, etc, denote the half-sum of the weights of t^c in $t \cap u$, etc. Meanwhile

$$\lambda^L = \mu|_t = \kappa|_t - \rho(s \cap u, t^c)|_t - \rho(\Delta^+(l, t^c))|_t,$$

and therefore

$$\lambda^L = \kappa|_t - \rho(s \cap u, t^c)|_t,$$

since $\rho(\Delta^+(l, t^c))|_t = 0$. This gives

$$\lambda^G|_t = \kappa|_t - \rho(s \cap u, t^c)|_t + \rho(u, t^c)|_t = \lambda^L + \rho(u, t),$$

as required.

In a somewhat similar vein, we shall use the following fact in the proof of Proposition 3.2:

Lemma 3.6. *Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the θ -stable parabolic subalgebra defined by a strictly dominant weight in \mathfrak{it}_0^* . If $H = TA$ is a maximally split Cartan subalgebra of $L = N_G(\mathfrak{q})$, then $\rho(\mathfrak{u}, \mathfrak{t}^c)|_{\mathfrak{t}_0^\perp} = 0$.*

Proof. The orthogonal complement of \mathfrak{t}_0 in \mathfrak{t}_0^c is spanned by the images of the one-dimensional Lie algebras $\mathfrak{so}(2)$ in the $\mathfrak{sl}(2, \mathbb{R})$ -summands of \mathfrak{l}_0 , so it suffices to show that $\rho(\mathfrak{u}, \mathfrak{t}^c)$ restricts to zero on each $\mathfrak{so}(2)$. But the ideal \mathfrak{u} is a representation, under the adjoint action, not just of $\mathfrak{so}(2)$ but of $\mathfrak{sl}(2, \mathbb{R})$, and the sum of the weights of any finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{R})$ is zero. \square

Proof of Proposition 3.2. The group $A \subseteq H$ is generated by images under the Lie group morphism (2.7) of A -subgroups in each of the $SL(2, \mathbb{R})$ -factors that map to L . Because of this, the group $T \subseteq H$ is generated by the torus $T^c \cap Z(L)^0$ and the images $m \in L$ of the matrices $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ in each $SL(2, \mathbb{R})$ factors in (2.7). The character δ is therefore fine, because the intersection of T with the semisimple part of L is the finite group generated by the elements m alone.

Next, let us show that if $\lambda^L \in \mathfrak{it}_0^*$ is the differential of δ , and if $\lambda^G = \lambda^L + \rho(\mathfrak{u}, \mathfrak{t})$ (this is the version of λ^G that is a weight on \mathfrak{t} , as in Definition 3.1), then $\langle \lambda^G, \alpha \rangle > 0$ for every $\alpha \in \mathfrak{it}^*$ that belongs to the set $\Delta(\mathfrak{u}, \mathfrak{t})$ of weights for the adjoint action of \mathfrak{t} on \mathfrak{u} . Given $\alpha \in \Delta(\mathfrak{u}, \mathfrak{t})$, there exists $\gamma \in \Delta(\mathfrak{u}, \mathfrak{t}^c)$ with $\gamma|_{\mathfrak{t}} = \alpha$. It follows from the definition (2.3) of \mathfrak{u} and from the fact that $\lambda^G|_{\mathfrak{t}^\perp} = 0$ that

$$\langle \lambda^G, \alpha \rangle = \langle \lambda^G, \gamma \rangle > 0,$$

as required.

The above proves that $(\mathfrak{q}, H, \delta)$ is a set of Vogan data. It is clear that the difference of any two choices of $\Delta^+(\mathfrak{l}, \mathfrak{t}^c)$ is analytically integral, so it remains to show that δ is independent of the choice of $\Delta^+(\mathfrak{l}, \mathfrak{t}^c)$.

We shall show that $\delta(m) = -1$ for all of the images in L of the elements $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ in the $SL(2, \mathbb{R})$ -factors in (2.7), assuming of course that the weight

$$\mu = \kappa - \rho(\mathfrak{s} \cap \mathfrak{u}, \mathfrak{t}^c) - \rho(\Delta^+(\mathfrak{l}, \mathfrak{t}^c))$$

is an integral weight, so that δ is defined. This will suffice, since δ is determined by its differential on the torus $T^c \cap Z(L)^0$, and the differential is independent of the choice of $\Delta^+(\mathfrak{l}, \mathfrak{t}^c)$.

Denote by $\pm\beta_j$ two nonzero weights for the action of \mathfrak{t}_0^c on the j 'th $\mathfrak{sl}(2)$ -summand in \mathfrak{l} , and arrange the signs so that

$$\rho(\Delta^+(\mathfrak{l}, \mathfrak{t}^c)) = \frac{1}{2}(\beta_1 + \cdots + \beta_N).$$

It follows from the definition (2.2) of \mathfrak{l} that

$$\langle \beta_j, \kappa \rangle + \langle \beta_j, \rho(\mathfrak{k} \cap \mathfrak{u}, \mathfrak{t}^c) \rangle = \langle \beta_j, \kappa \rangle + \langle \beta_j, \rho(\Delta^+(\mathfrak{k}, \mathfrak{t}^c)) \rangle = \langle \beta_j, \lambda^G \rangle = 0$$

(all of the inner products above and below are taken in \mathfrak{it}_0^{c*}) and so

$$\begin{aligned}\langle \beta_j, \kappa \rangle - \langle \beta_j, \rho(\mathfrak{s} \cap \mathfrak{u}, \mathfrak{t}^c) \rangle &= -\langle \beta_j, \rho(\mathfrak{k} \cap \mathfrak{u}, \mathfrak{t}^c) \rangle - \langle \beta_j, \rho(\mathfrak{s} \cap \mathfrak{u}, \mathfrak{t}^c) \rangle \\ &= -\langle \beta_j, \rho(\mathfrak{u}, \mathfrak{t}^c) \rangle.\end{aligned}$$

But this is zero, since by Lemma 3.6 $\rho(\mathfrak{u}, \mathfrak{t}^c)$ is supported on \mathfrak{t}_0 , while β_j is supported on a single $\mathfrak{so}(2)$ -summand in \mathfrak{t}_0^\perp . As a result,

$$\langle \beta_j, \mu \rangle = -\langle \beta_j, \rho(\Delta^+(\mathfrak{l}, \mathfrak{t}^c)) \rangle = -\frac{1}{2}\langle \beta_j, \beta_j \rangle.$$

It follows that the restriction of μ to the j 'th $\mathfrak{so}(2)$ -summand is equal to $-\frac{1}{2}\beta_j$. Now if β is either nonzero weight for the action of $\mathfrak{so}(2)$ on $\mathfrak{sl}(2, \mathbb{R})$, then

$$\beta\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = \pm 2i,$$

and so if $\varphi: \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{l}_0$ is the inclusion of the j 'th $\mathfrak{sl}(2, \mathbb{R})$ -summand, then

$$\mu\left(\varphi\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)\right) = \pm i.$$

Using the fact that δ is the restriction to T of the character $\exp(\mu)$ of T^c , we therefore find that if Φ the Lie group morphism corresponding to φ , then

$$\begin{aligned}\delta(\mathfrak{m}) &= \exp(\mu)\left(\Phi\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\right) \\ &= \exp(\mu)\left(\Phi\left(\exp\left(\pi \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)\right)\right) \\ &= \exp(\mu)\left(\exp\left(\pi \cdot \varphi\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)\right)\right) \\ &= \exp\left(\pi \cdot \mu\left(\varphi\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)\right)\right) = \exp(\pm \pi i) = -1,\end{aligned}$$

as required. \square

To summarize, Proposition 3.2 associates to each dominant weight $\mu \in \mathfrak{it}_0^{c*}$ a unique set of Vogan data, as long as μ satisfies a certain integrality condition. We are now going to characterize the sets of Vogan data that are obtained in this way.

Definition 3.7. A set (q, H, δ) of Vogan data is *essential* if

(i) There is an isomorphism

$$\mathfrak{l}_0 \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \cdots \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{z}_0$$

with the properties listed in Proposition 2.4.

(ii) The character δ takes the value -1 on the images of each of the elements $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$ under the associated morphism of Lie groups

$$\mathrm{SL}(2, \mathbb{R}) \times \cdots \times \mathrm{SL}(2, \mathbb{R}) \times Z(L) \longrightarrow L.$$

Remark 3.8. The set of images in (ii) does not depend on the choice of isomorphism in (i).

The proof that we have just completed shows that:

Proposition 3.9. *Each of the sets of Vogan data provided by Proposition 3.2 is essential.* \square

Theorem 3.10. *The construction in Proposition 3.2 determines a bijection from the dominant weights $\kappa \in \mathfrak{it}_0^{c*}$ for which the weight*

$$\mu = \kappa - \rho(\mathfrak{s} \cap \mathfrak{u}, \mathfrak{t}^c) - \rho(\Delta^+(\mathfrak{l}, \mathfrak{t}^c))$$

is analytically integral to the K -conjugacy classes of sets of essential Vogan data.

Once again, we need a preliminary computation:

Lemma 3.11. *Let (q, H, δ) be a set of essential Vogan data, and write $H = TA$, as usual, so that δ is a character of T . There is an analytically integral weight $\mu \in \mathfrak{it}_0^{c*}$ for which*

- (i) *the corresponding global character $\exp(\mu)$ of T^c restricts to δ on T , and*
- (ii) *if $\beta \in \Delta^+(\mathfrak{l}, \mathfrak{t}^c)$ is the positive root associated to a given $\mathfrak{sl}(2, \mathbb{R})$ summand in Definition 3.7, then the restriction of μ to the corresponding $\mathfrak{so}(2)$ -summand of \mathfrak{t}^c is*

$$\mu|_{\mathfrak{so}(2)} = -\frac{1}{2}\beta|_{\mathfrak{so}(2)}.$$

Proof. This is a special case of one of the most important technical results in Vogan's monograph, [Vog81, Thm. 4.3.16]. But the special case is easy to handle directly, as follows.

We shall use the surjective morphism of Lie groups (2.7). As indicated in (2.8), the torus T^c is generated by the image of $SO(2) \times \cdots \times SO(2)$ and the compact part of the center of L . Because of this we need only define a unitary character on $SO(2) \times \cdots \times SO(2)$, namely a product of generating characters on each factor, and a unitary character on the compact part of the center, namely the restriction of δ , and then check that the product factors through

$$(SO(2) \times \cdots \times SO(2)) \times (T^c \cap Z(L)) \longrightarrow T^c$$

using the property of δ in the definition of essential Vogan data. This is straightforward. \square

Proof of Theorem 3.10. Let (q, H, δ) be a set of essential Vogan data. After conjugating by an element of K we may, and shall, assume that $\mathfrak{u} \cap \mathfrak{k}$ is the direct sum of the already-fixed positive weight spaces in \mathfrak{k} .

Let λ^L be the differential of δ and define the weight $\lambda^G \in \mathfrak{it}_0^*$ by

$$\lambda^G = \lambda^L + \rho(\mathfrak{u}, \mathfrak{t}).$$

If we extend λ^G to a weight in \mathfrak{it}_0^{c*} by defining it to be zero on the orthogonal complement of $\mathfrak{t}_0 \subseteq \mathfrak{t}^c$, then it follows from part (v) of Definition 3.1 that the extension satisfies

$$\langle \lambda^G, \alpha \rangle > 0 \quad \forall \alpha \in \Delta(\mathfrak{u}, \mathfrak{t}^c);$$

compare the proof of Proposition 3.2. In addition,

$$\langle \lambda^G, \beta \rangle = 0 \quad \forall \beta \in \Delta(\mathfrak{l}, \mathfrak{t}^c),$$

since λ^G vanishes on \mathfrak{t}_0^\perp while β vanishes on \mathfrak{t}_0 . So λ^G defines q .

Because the maximal compact subgroup of L is the torus T^c , the maximally split Cartan subgroup $H = TA$ of L is unique up to conjugacy by an element of T^c , and T is uniquely determined, and a subgroup of T^c .

Choose an integral weight μ of \mathfrak{t}^c as in Lemma 3.11, so that

$$\mu + \rho(\Delta^+(\mathfrak{l}, \mathfrak{t}^c)) = \lambda^L;$$

here we extend λ^L by zero on \mathfrak{t}_0^\perp . The unitary character of T^c associated to μ restricts to δ on $T \subseteq T^c$. Indeed the differential of this character is equal to λ^L on \mathfrak{t} , so the character agrees with δ on the connected component of the identity in T . But in addition both the character and δ are equal to -1 on the elements m that generate the component group of T .

Next, define a weight $\kappa \in \mathfrak{it}_0^{c*}$ by

$$\mu = \kappa - \rho(\mathfrak{s} \cap \mathfrak{u}, \mathfrak{t}^c) - \rho(\Delta^+(\mathfrak{l}, \mathfrak{t}^c))$$

and observe that, thanks to Lemma 3.6,

$$\kappa + \rho(\Delta^+(\mathfrak{k}, \mathfrak{t}^c)) = \mu + \rho(\Delta^+(\mathfrak{l}, \mathfrak{t}^c)) + \rho(\mathfrak{u}, \mathfrak{t}^c) = \lambda^L + \rho(\mathfrak{u}, \mathfrak{t}^c) = \lambda^G,$$

and so $\kappa + \rho(\Delta^+(\mathfrak{k}, \mathfrak{t}^c))$ defines q .

It follows from the above that $\kappa + \rho(\Delta^+(\mathfrak{k}, \mathfrak{t}^c))$ is strictly dominant. If κ was integral, then it would follow from this that κ is dominant. But κ is integral for a certain double cover \tilde{K} of K , namely the one defined in Section 7 below; see the discussion following Proposition 7.5. So κ is dominant whether or not it is integral.

We have now shown that (q, H, δ) is one of the sets of essential Vogan data defined in Proposition 3.9, up to K -conjugacy. Since there are no inner automorphisms of K that globally preserve both T^c and the system of positive roots $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$, the map from weights to K -conjugacy classes of essential Vogan data is not only surjective, but injective too. \square

4. COMPONENTS OF THE TEMPERED DUAL

We shall now review Vogan's construction of the tempered dual² of G , and compare it with Harish-Chandra's construction, which we used in [CHST22] to describe the reduced group C^* -algebra.

Vogan's construction is carried out in the context of (\mathfrak{g}, K) -modules rather than that of unitary Hilbert space representations. But we shall take advantage of the fact, due to Harish-Chandra, that every unitary admissible (\mathfrak{g}, K) -module may be completed to a unitary representation of G , and usually make no distinction between the two contexts. But whenever it is helpful to do so we shall denote (\mathfrak{g}, K) -modules using the letter X , and Hilbert space representations using the letter \mathcal{H} .

The following definition formalizes the construction of representations from Vogan data.

²Vogan constructs the full admissible dual in [Vog81]; the fact that we are interested only in the tempered dual simplifies matters for us considerably.

Definition 4.1. Let (q, H, δ) be a set of Vogan data for G , and let $L = N_G(q)$. Let $T = MA$ as in Definition 3.1, and let $\nu \in \mathfrak{a}_0^*$.

- (i) We shall denote by $X^L(\delta, \nu)$ the unitary principal series representation of L that is obtained by unitary parabolic induction from the character $\delta \otimes \exp(i\nu)$ of H . When $\nu = 0$ we shall abbreviate the notation to $X^L(\delta)$ or even $X(\delta)$.
- (ii) We shall denote by $X^G(q, H, \delta, \nu)$ the representation of G that is obtained by cohomologically inducing $X^L(\delta, \nu)$ from L to G via the parabolic subalgebra q . See [Vog81, Def. 6.5.2]; this is the *standard representation with parameters* (q, H, δ, ν) . Again, when $\nu = 0$ we shall abbreviate the notation to $X^G(q, H, \delta)$ or $X(q, H, \delta)$.

Up to equivalence, the representation $X^G(q, H, \delta, \nu)$ depends only on the K -conjugacy class of (q, H, δ, ν) . It is unitarizable, tempered and a finite direct sum of irreducible representations (generically $X^G(q, H, \delta, \nu)$ is itself irreducible). This follows, for example, from Theorem 4.2 below.

The representations $X^G(q, H, \delta, \nu)$ fit into the Harish-Chandra picture of the tempered dual as follows. According to Harish-Chandra, the components of the tempered dual of G are parametrized by *associate classes* of pairs (P, σ) consisting of a parabolic subgroup P of G and an irreducible, square-integrable representation σ of the compactly generated part M in the Langlands decomposition $P = MAN$. See for instance [CHST22] for a summary that is tailored to our viewpoint.

Let $[P, \sigma]$ be an associate class, with $P = MAN$. We shall construct from it a set of Vogan data.

First, choose a Cartan subgroup T (centralizer of a maximal torus) in $K \cap M$, and set $H = TA$.

The compact factor T is a compact Cartan subgroup of M , and the Vogan-Zuckerman method attaches to the discrete series representation σ of M a character γ of T from which σ may be obtained by cohomological induction using an appropriate θ -stable parabolic subalgebra of \mathfrak{m} ; see [Vog81, Lem. 6.6.12].

The character γ , in turn, determines a θ -stable parabolic subalgebra $q \subseteq \mathfrak{g}$, as well as a character δ of T , which is a ρ -shift of γ , and we obtain a set of Vogan data (q, H, δ) ; see [Vog81, Prop. 6.6.2].

The K -conjugacy class $[q, H, \delta]$ depends only on the associate class $[P, \sigma]$, and the representations of G associated to the two sets of data are related as follows:

Theorem 4.2 ([Vog81, Thm. 6.6.15]). *Suppose that the K -conjugacy class of Vogan data $[q, H, \delta]$ corresponds to the associate class $[P, \sigma]$, as above. For every $\nu \in \mathfrak{a}_0^*$, the unitary (P, σ) -principal series representation $\pi_{\sigma, \nu}$ on $\text{Ind}_P^G \mathcal{H}_\sigma \otimes \mathbb{C}_\nu$ is equivalent to the representation $X^G(q, H, \delta, \nu)$ described in Definition 4.1. \square*

Putting this together with known facts about how associate classes organize the tempered dual, as summarized in [CHST22] for instance, or alternatively working directly with Vogan data, we obtain:

Theorem 4.3. *Let (q, H, δ) be a set of Vogan data for G , let $H = TA$ as in Definition 3.1, and let $\nu \in \mathfrak{a}_0^*$.*

- (i) *Every tempered irreducible representation of G arises as one of the summands of some $X^G(q, H, \delta, \nu)$, and it does so from a unique set of Vogan data, up to K -conjugacy.*
- (ii) *As ν varies, with (q, H, δ) fixed, the irreducible summands of all the representations $X^G(q, H, \delta, \nu)$ belong to a single component of the tempered dual, and exhaust it.* \square

In short, the components of the tempered dual of G are parametrized by the K -conjugacy classes of sets of Vogan data, and this parametrization determines a bijection

$$\{\text{Associate classes } [P, \sigma]\} \xrightarrow{\cong} \{K\text{-conjugacy classes } [q, H, \delta]\}.$$

5. THE KNAPP-STEIN AND VOGAN INTERTWINING GROUPS

Let $[P, \sigma]$ be an associate class, as in the previous section. Harish-Chandra showed that the decomposition of the (P, σ) -principal series representation $\pi_{\sigma, \nu}$ into irreducible constituents is governed by the intertwining group

$$W_\sigma = \{w \in N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0) : \text{Ad}_w^* \sigma \simeq \sigma\},$$

and more specifically by the subgroup

$$W_{\sigma, \nu} = \{w \in W_\sigma : \text{Ad}_w^* \nu = \nu\}.$$

Each element of $W_{\sigma, \nu}$ corresponds to a unitary self-intertwiner of the representation $\pi_{\sigma, \nu}$ on $\text{Ind}_P^G \mathcal{H}_\sigma \otimes \mathbb{C}_\nu$, and together these unitaries span the intertwining algebra.

In addition, Langlands proved that the unitary principal series representations $\pi_{\sigma, \nu}$ and $\pi_{\sigma', \nu'}$ on $\text{Ind}_P^G \mathcal{H}_\sigma \otimes \mathbb{C}_\nu$ and $\text{Ind}_P^G \mathcal{H}_{\sigma'} \otimes \mathbb{C}_{\nu'}$ are equivalent if and only if the data (A, σ, ν) and (A', σ', ν') are conjugate by an element of K , and that otherwise the two representations are disjoint. See [Lan89, p.142ff].

Now let (q, H, δ) be a set of Vogan data. As usual let $L = N_G(q)$ and let $H = TA$. If we set

$$W = N_{K \cap L}(\mathfrak{a}_0)/Z_{K \cap L}(\mathfrak{a}_0),$$

then the intertwining groups in the Vogan-Zuckerman theory are

$$W_\delta = \{w \in W : \text{Ad}_w^* \delta = \delta\}$$

and

$$W_{\delta, \nu} = \{w \in W_\delta : \text{Ad}_w^* \nu = \nu\}.$$

See [Vog81, Defs. 4.4.1 and 4.4.9].

Theorem 5.1 ([Vog81, Thm. 6.5.12]). *Let (q', H', δ') be another set of Vogan data and let $\nu' \in \mathfrak{a}_0^*$. If the representations $X^G(q, H, \delta, \nu)$ and $X^G(q', H', \delta', \nu')$ have equivalent subrepresentations, then (q, H, δ, ν) and (q', H', δ', ν') are conjugate by an element of K .*

Let (q, H, δ) be the set of Vogan data that is constructed from (P, σ) as in the discussion prior to the statement of Theorem 4.2. Since both the Vogan and Knapp-Stein intertwining groups may be faithfully represented as groups of linear automorphisms of the same vector space \mathfrak{a}_0 , they may be compared with one another. In fact they are equal:

Lemma 5.2. *If (q, H, δ) is the set of Vogan data constructed from the pair (P, σ) , as in the discussion prior to Theorem 4.2, then $W_\sigma = W_\delta$.*

Proof. Suppose that w is an element in W_δ , acting on \mathfrak{a}_0^* through $w(\nu) = \text{Ad}_w^* \nu$. Then the representations $X^G(q, H, \delta, \nu)$ and $X^G(q, H, \delta, w(\nu))$ are equivalent, for all ν , and therefore by Theorem 4.2 the representations $\pi_{\sigma, \nu}$ and $\pi_{\sigma, w(\nu)}$ are equivalent, for all ν . So by Langlands' aforementioned result, for each ν there is an element $w'_\nu \in W_\sigma$ such that $w'_\nu(\nu) = w(\nu)$.

Since W_σ is finite, the elements w'_ν cannot all be distinct from one another. Indeed there must exist a single $w' \in W_\sigma$ such that

$$w'(\nu) = w'_\nu(\nu) = w(\nu)$$

for all ν in a spanning set for \mathfrak{a}_0^* . This means that $w' = w$, and we have proved that $W_\delta \subseteq W_\sigma$. The argument can be reversed to prove the reverse inclusion. \square

From now on we shall focus on the special case $\nu = 0$. The Knapp-Stein theory [KS71, KS80] considerably refines Harish-Chandra's completeness result by decomposing W_σ as a semidirect product

$$(5.3) \quad W_\sigma = W'_\sigma \rtimes R_\sigma,$$

in which intertwiners corresponding to the elements of W'_σ all act as multiples of the identity on the Hilbert space $\text{Ind}_P^G \mathcal{H}_\sigma$ of the representation $\pi_{\sigma, 0}$, while the intertwiners corresponding to the elements of R_σ act as a linearly independent set of intertwining operators on $\text{Ind}_P^G \mathcal{H}_\sigma$. Every element in R_σ has order 2. It follows from this and Harish-Chandra's theorem that the (P, σ) -principal series representation $\pi_{\sigma, 0}$ decomposes as a direct sum of a finite set of mutually inequivalent irreducible representations that may be parametrized by the elements of the dual group \widehat{R}_σ .

We shall now present some of the corresponding results in the Vogan-Zuckerman theory.

Definition 5.4 ([Vog81, Def. 4.1.1]). Let \mathfrak{q} be a θ -stable parabolic subgroup of \mathfrak{g} . Let $L = N_G(\mathfrak{q})$, as usual, and let $H = TA$ be a maximally split Cartan subgroup of L .

- (i) Denote by $\Delta(L, \mathfrak{a}) \subseteq \mathfrak{a}_0^*$ the set of (non-zero) roots associated to the adjoint action of \mathfrak{a} on L . These are the *restricted roots* for (q, H, δ) .

- (ii) Denote by $\overline{\Delta}(\mathfrak{l}, \mathfrak{a}) \subseteq \mathfrak{a}_0^*$ set of those roots $\alpha \in \Delta(\mathfrak{l}, \mathfrak{a})$ for which $\frac{\alpha}{2} \notin \Delta(\mathfrak{l}, \mathfrak{a})$. These are the *reduced roots* for $(\mathfrak{q}, H, \delta)$.

The following lemma is a standard fact about the set of restricted roots associated to the Iwasawa decomposition of any reductive group [Kna02, Thm. 6.57].

Lemma 5.5. *If $(\mathfrak{q}, H, \delta)$ is a set of Vogan data, and if $H = TA$ and $L = N_G(\mathfrak{q})$, as in Section 4, then*

$$W(\overline{\Delta}(\mathfrak{k}, \mathfrak{l})) = N_{K \cap L}(\mathfrak{a}_0) / Z_{K \cap L}(\mathfrak{a}_0)$$

if both groups are viewed as groups of linear automorphisms of \mathfrak{a}_0 . \square

Definition 5.6 ([Vog81, Def. 4.3.6]). A root $\alpha \in \overline{\Delta}(\mathfrak{l}, \mathfrak{a})$ is said to be *real* if it is the restriction to \mathfrak{a} of a root for the adjoint action of $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ on \mathfrak{l} that vanishes on \mathfrak{t} . Otherwise, α is said to be *complex*.

Associated to each real root $\alpha \in \overline{\Delta}(\mathfrak{l}, \mathfrak{a})$, there exists a morphism ([Vog81, Notation 4.3.6])

$$(5.7) \quad \varphi_\alpha: \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{l}_0.$$

that is compatible with θ and the standard Cartan involution on $\mathfrak{sl}(2, \mathbb{R})$, and that maps the strictly upper triangular matrices into the α -root space in \mathfrak{l}_0 . See [Vog81, Def. 4.3.6]. Since the group L is linear, the Lie algebra morphism φ_α exponentiates to a Lie group morphism

$$(5.8) \quad \Phi_\alpha: \mathrm{SL}(2, \mathbb{R}) \rightarrow L.$$

Using this we put

$$(5.9) \quad m_\alpha = \Phi_\alpha \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \in L.$$

This is in fact an element of T , and it is independent of the choice of φ_α [Vog81, Lem. 4.3.7].

Of course, in the case of an essential set of Vogan data, all these structures match those introduced in Section 3.

Definition 5.10 ([Vog81, Notation 4.3.6]). Let $(\mathfrak{q}, H, \delta)$ be a set of Vogan data. The set of *good roots with respect to δ* , denoted $\Delta_\delta(\mathfrak{l}, \mathfrak{a}) \subseteq \overline{\Delta}(\mathfrak{l}, \mathfrak{a})$, is the set of those roots $\alpha \in \overline{\Delta}(\mathfrak{l}, \mathfrak{a})$ such that either

- (i) α is real and $\delta(m_\alpha) = 1$, or
- (ii) α is complex.

Vogan shows in [Vog81, Lem. 4.3.12] that $\Delta_\delta(\mathfrak{l}, \mathfrak{a})$ is a root system, and makes the following definition:

Definition 5.11 ([Vog81, Def. 4.3.13]). The group W_δ^0 is the subgroup of $W(\overline{\Delta}(\mathfrak{l}, \mathfrak{a}))$ generated by the reflections associated to good roots.

Definition 5.12 ([Vog81, Def. 4.3.13]). Let $(\mathfrak{q}, H, \delta)$ be a set of Vogan data. We define $R_\delta = W_\delta / W_\delta^0$. (This is actually a group since W_δ^0 is a normal subgroup of W_δ [Vog81, Lems. 4.3.12 and 4.3.14]. But the group structure will not be used here.)

Theorem 5.13 ([Vog81, Cor. 4.4.11 and Cor. 6.5.14]). *The representation $X^L(\delta)$ of L in Definition 4.1 is a direct sum of $|R_\delta|$ inequivalent irreducible subrepresentations, and the same is true of the representation $X^G(q, H, \delta)$ of G .*

Corollary 5.14. *If (q, H, δ) is a set of Vogan data, and if $[P, \sigma]$ is the corresponding associate class, as in Theorem 4.2, then $|R_\sigma| = |R_\delta|$.* \square

Proof. According to the theorem above, and what we have noted about the Knapp-Stein theory, both $|R_\delta|$ and $|R_\sigma|$ equal the number of irreducible summands of the representation $X^G(q, H, \sigma) \cong \text{Ind}_P^G \mathcal{H}_\sigma$. \square

6. ESSENTIAL COMPONENTS OF THE TEMPERED DUAL

The following is Definition 4.1 in [CHST22]. Its significance is that the essential components of the tempered dual of G are precisely those that contribute to C^* -algebra K -theory; see [CHST22, Theorem 4.9].

Definition 6.1. A component of the tempered dual of G corresponding to the associate class $[P, \sigma]$ is *essential* if the Knapp-Stein subgroup $W'_\sigma \triangleleft W_\sigma$ is trivial.

Lemma 6.2. *A K -conjugacy class $[q, H, \delta]$ corresponds to an essential component of the tempered dual, as in Section 4, if and only if W_δ^0 is the trivial subgroup of W_δ .*

Proof. Suppose that a component of the tempered dual corresponds to an associate class $[P, \sigma]$. It follows from Lemma 5.2 and Corollary 5.14 that

$$|W'_\sigma| = 1 \quad \Leftrightarrow \quad |W_\delta^0| = 1$$

The lemma now follows directly from the definition above. \square

Theorem 6.3. *The component of the tempered dual that is labeled by the set of Vogan data (q, H, δ) is essential if and only if (q, H, δ) is essential in the sense of Definition 3.7.*

This is a special case of [Vog81, Lem. 4.3.31]. But since our result somewhat hidden there, we shall give some of the details here.

Proof. If (q, H, δ) is essential, then it is immediate from Definitions 3.7, 5.10 and 5.11 that W_δ^0 is trivial. So suppose, conversely, that W_δ^0 is trivial, so that, according to the definitions, every root in $\overline{\Delta}(l, a)$ is real, and moreover

$$(6.4) \quad \delta(m_\alpha) = -1 \quad \forall \alpha \in \overline{\Delta}(l, a).$$

Write $L = N_G(q)$ and $H = TA$, as usual.

Our first task is to show that the reduced root system $\overline{\Delta}(l, a)$ matches one of those obtained from an essential (q, H, δ) . These are the systems of type $A_1 \times \cdots \times A_1$.

Now $\overline{\Delta}(l, a)$ is of type $A_1 \times \cdots \times A_1$ if and only if for every $\alpha, \beta \in \overline{\Delta}(l, a)$ with $\alpha \neq \pm\beta$ the roots in the plane spanned by α and β span a system of type $A_1 \times A_1$. But apart from $A_1 \times A_1$, the only other possibilities are A_2, B_2

and G_2 , and in all these cases there exist roots $\gamma_1, \gamma_2, \gamma_3$ for which the dual roots (which also constitute a root system of rank 2) satisfy

$$(6.5) \quad \check{\gamma}_1 = \check{\gamma}_2 + \check{\gamma}_3.$$

But it is shown in [Vog81, Cor. 4.3.20], again by considering the various cases, that (6.5) implies

$$\delta(m_{\gamma_1}) = \delta(m_{\gamma_2})\delta(m_{\gamma_3}),$$

and this is obviously inconsistent with (6.4).

Now let $\alpha \in \overline{\Delta}(\mathfrak{l}, \mathfrak{a})$. The root α is real, and is therefore the restriction to $\mathfrak{a} \subseteq \mathfrak{h}$ of a root $\gamma \in \Delta(\mathfrak{l}, \mathfrak{h})$ that vanishes on $\mathfrak{t} \subseteq \mathfrak{h}$. Standard arguments in Lie theory (see again [Ser87, Sec. VI.2] for instance) show that the spaces $\mathfrak{l}_\gamma, \mathfrak{l}_{-\gamma}$ and $[\mathfrak{l}_\gamma, \mathfrak{l}_{-\gamma}]$ are each one-dimensional and span a copy of $\mathfrak{sl}(2, \mathbb{C}) \subseteq \mathfrak{l}$. The reality condition implies that $[\mathfrak{l}_\gamma, \mathfrak{l}_{-\gamma}]$ is included in $\mathfrak{a} \subseteq \mathfrak{h}$. In fact if $H_\gamma \in \mathfrak{a}$ is chosen so that $\langle H, H_\gamma \rangle = \gamma(H)$ for all $H \in \mathfrak{h}$, then $[\mathfrak{l}_\gamma, \mathfrak{l}_{-\gamma}] = \mathbb{C} \cdot H_\gamma$.

Now form the vector subspace

$$(6.6) \quad \mathfrak{l}_{-2\alpha} \oplus \mathfrak{l}_{-\alpha} \oplus [\mathfrak{l}_\alpha, \mathfrak{l}_{-\alpha}] \oplus \mathfrak{l}_\alpha \oplus \mathfrak{l}_{2\alpha}$$

within \mathfrak{l} . This includes our copy of $\mathfrak{sl}(2, \mathbb{C})$ since $\mathfrak{l}_{\pm\gamma} \subseteq \mathfrak{l}_{\pm\alpha}$, and so the direct sum is a module over $\mathfrak{sl}(2, \mathbb{C})$. Because $H_\gamma \in \mathfrak{a}$ and because $\gamma|_{\mathfrak{a}} = \alpha$, the individual summands are the weight spaces for this module action. It therefore follows from the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ that if $X \in \mathfrak{l}_\gamma$ is nonzero, then the morphism

$$\text{ad}_X: \mathfrak{l}_{-\alpha} \longrightarrow [\mathfrak{l}_\alpha, \mathfrak{l}_{-\alpha}]$$

is injective. But if $Y \in \mathfrak{l}_{-\alpha}$ and $H \in \mathfrak{h}$, then

$$\langle [X, Y], H \rangle = \langle Y, [H, X] \rangle = \langle Y, \gamma(H)X \rangle = \langle Y, X \rangle \langle H_\gamma, H \rangle.$$

As a result, $\text{ad}_X(Y) \in \mathbb{C} \cdot H_\gamma$, and therefore $\dim(\mathfrak{l}_{-\alpha})=1$, which means that $\mathfrak{l}_{-\alpha} = \mathfrak{l}_{-\gamma}$. Similarly $\mathfrak{l}_\alpha = \mathfrak{l}_\gamma$. It follows that the middle summand in (6.6), which is the weight zero space for the $\mathfrak{sl}(2, \mathbb{C})$ -representation, is one-dimensional, and so it follows from the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ that

$$\mathfrak{l}_{-2\alpha} \oplus \mathfrak{l}_{-\alpha} \oplus [\mathfrak{l}_\alpha, \mathfrak{l}_{-\alpha}] \oplus \mathfrak{l}_\alpha \oplus \mathfrak{l}_{2\alpha} = \mathfrak{l}_{-\gamma} \oplus \mathbb{C} \cdot H_\gamma \oplus \mathfrak{l}_\gamma.$$

As a result, the above is the complexification of $\varphi_\alpha[\mathfrak{sl}(2, \mathbb{R})] \subseteq \mathfrak{l}_0$.

Now if $\alpha \neq \pm\beta$, then the subalgebras $\varphi_\alpha[\mathfrak{sl}(2, \mathbb{R})]$ and $\varphi_\beta[\mathfrak{sl}(2, \mathbb{R})]$ commute with one another and are orthogonal to one another, so it follows from the above that \mathfrak{l}_0 is a direct sum

$$\mathfrak{l}_0 = \varphi_{\alpha_1}[\mathfrak{sl}(2, \mathbb{R})] \oplus \cdots \oplus \varphi_{\alpha_N}[\mathfrak{sl}(2, \mathbb{R})] \oplus \mathfrak{z}_0,$$

where the direct sum is over an enumeration of a system of positive roots, and this together with (6.4) shows that $(\mathfrak{q}, H, \delta)$ is essential. \square

7. DIRAC COHOMOLOGY

In [CHST22] we approached the Dirac operator from a direction congenial to K-theory, but in representation theory it is convenient to follow a different but equivalent approach, which we shall review here. For full details see the monograph [HP06].

The algebraic Dirac operator. Pick an irreducible representation of associative algebras

$$\text{Cliff}(\mathfrak{s}_0) \longrightarrow \text{End}(\mathcal{S}).$$

When $\dim(\mathfrak{s}_0)$ is even there is a unique irreducible representation, up to equivalence. When $\dim(\mathfrak{s}_0)$ is odd there are two, but the main construction below (of Dirac cohomology) is independent of the choice.

Definition 7.1. The *Dirac operator* associated to an admissible (for instance irreducible) unitary (\mathfrak{g}, K) -module X is the linear operator

$$\mathcal{D}_S: X \otimes \mathcal{S} \longrightarrow X \otimes \mathcal{S}$$

given by the formula

$$\mathcal{D} = \sum X_a \otimes c(X_a),$$

in which the sum is over an orthonormal basis for \mathfrak{s}_0 . See [HP06]. The *Dirac cohomology* X is

$$H_{\text{Dirac}}(X) = \text{Kernel}(\mathcal{D}: X \otimes \mathcal{S} \longrightarrow X \otimes \mathcal{S}).$$

Occasionally we shall write $H_{\text{Dirac}}^G(X)$, if it is helpful to emphasize the group involved.

The Dirac cohomology of X is a finite-dimensional vector space. It is also a representation of the *spin double cover* of K , which is the (not necessarily connected) double cover \tilde{K} of K defined by the pullback diagram

$$(7.2) \quad \begin{array}{ccc} \tilde{K} & \longrightarrow & \text{Spin}(\mathfrak{s}_0) \\ \downarrow & & \downarrow \\ K & \longrightarrow & \text{SO}(\mathfrak{s}). \end{array}$$

This follows from the existence of a natural Lie algebra homomorphism

$$\mathfrak{k}_0 \longrightarrow \text{Cliff}(\mathfrak{s}_0),$$

see [HP06, Sec. 2.3], which allows us to equip $X \otimes \mathcal{S}$ with a diagonal \mathfrak{k}_0 -action that exponentiates to a \tilde{K} -action, with respect to which the Dirac operator is equivariant.

In [CHST22] in effect we used the spaces $S = \mathcal{S} \otimes V^*$, where V is an irreducible representation of \tilde{K} , and formed the Dirac operator \mathcal{D}_S acting on $[X \otimes S]^K$ (the diagonal action of \tilde{K} descends to K). This approach is related to the definition of Dirac cohomology by

$$\dim(\text{kernel}(\mathcal{D}_S)) = \text{multiplicity of } V \text{ in } H_{\text{Dirac}}(X).$$

In what follows we shall translate statements from the paper [CHST22] about $\text{kernel}(\mathcal{D}_S)$ into the language of Dirac cohomology without further comment.

The matching theorem.

Definition 7.3. A genuine, irreducible representation of \tilde{K} and a component $[q, H, \delta]$ of the tempered dual are *matched* if π occurs in the Dirac cohomology space $H_{\text{Dirac}}(X^G(q, H, \delta))$.

The result that we need to establish the Connes-Kasparov isomorphism is as follows (it is Theorem 6.2 in [CHST22]):

Theorem 7.4 (Matching Theorem). *Let G be a connected linear real reductive group. Let K be a maximal compact subgroup of G , and let \tilde{K} be its spin double cover, as in (7.2).*

- (i) *For every essential component of the tempered dual of G there is a unique genuine, irreducible representation of \tilde{K} to which it is matched.*
- (ii) *For every genuine, irreducible representation of \tilde{K} there is a unique essential component of the tempered dual of G which it is matched.*

As for the genuine, irreducible representation of \tilde{K} , they may be classified by highest weight theory, as follows. Pick any system of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{t}^c)$ for the action of \mathfrak{t}^c on \mathfrak{g} . We can do so in a way that includes our choice of $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$, and then there is a partition

$$\Delta^+(\mathfrak{g}, \mathfrak{t}^c) = \Delta^+(\mathfrak{k}, \mathfrak{t}^c) \cup \Delta^+(\mathfrak{s}, \mathfrak{t}^c)$$

into compact and noncompact positive roots. The weight $\rho(\Delta^+(\mathfrak{s}, \mathfrak{t}^c)) \in \mathfrak{t}_0^{c*}$ is the highest weight of the spin representation \mathcal{S} , and because of this we have:

Proposition 7.5. *Each genuine, irreducible representation of \tilde{K} has a unique highest weight $\kappa \in \mathfrak{it}_0^{c*}$. The weight*

$$\mu = \kappa - \rho(\Delta^+(\mathfrak{s}, \mathfrak{t}^c)) \in \mathfrak{it}_0^{c*}$$

is analytically integral for T^c . Conversely, if $\kappa \in \mathfrak{it}_0^{c}$ is a dominant weight, and if the weight μ above is integral, then κ is the highest weight of a unique genuine irreducible representation of \tilde{K} . \square*

As usual, the integrality condition in the proposition is independent of the choice of positive system $\Delta^+(\mathfrak{g}, \mathfrak{t}^c)$. But given a dominant weight $\kappa \in \mathfrak{it}_0^{c*}$, we could form the θ -stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ in Proposition 3.2, then choose a positive system $\Delta^+(\mathfrak{l}, \mathfrak{t}^c)$ as in Proposition 3.2, and then define

$$\rho(\Delta^+(\mathfrak{s}, \mathfrak{t}^c)) = \rho(\mathfrak{s} \cap \mathfrak{u}) + \rho(\Delta^+(\mathfrak{l}, \mathfrak{t}^c)).$$

Compare Remark 3.3. It follows that the integrality condition in Proposition 7.5 above is exactly the same as the integrality condition in Proposition 3.2. Therefore, thanks to Theorem 3.10, this means that:

Theorem 7.6. *There is a bijection from the equivalence classes of genuine irreducible representations of \tilde{K} to the K -conjugacy classes of essential sets of Vogan data that maps the representation with highest weight κ to the set of Vogan data generated by κ , as in Proposition 3.2. \square*

It only remains to describe this bijection in terms of Dirac cohomology.

Dirac cohomology for essential theta-stable data. Let (q, H, δ) be a set of essential Vogan data, let $L = N_G(q)$ and let $H = TA$. Write

$$l_0 \cong \underbrace{\mathfrak{sl}(2, \mathbb{R}) \oplus \cdots \oplus \mathfrak{sl}(2, \mathbb{R})}_{N \text{ times}} \oplus \mathfrak{z}_0$$

where \mathfrak{z}_0 is the center of l_0 .

Lemma 7.7. *Let $\kappa^L \in \mathfrak{t}_0^{c*}$ be the extension by 0 on the orthogonal complement of \mathfrak{t}_0 of the differential of δ . The Dirac cohomology space $H_{\text{Dirac}}^L(X(\delta))$ is a direct sum of 2^N copies of the unitary character with differential κ^L .*

Proof. This is a simple explicit computation using the odd principal series for $SL(2, \mathbb{R})$. The only subtle point is to construct enough vectors in the principal series representation $X(\delta)$ to account for $H_{\text{Dirac}}^L(X(\delta))$. These vectors are functions on T^c (which is the θ -fixed maximal compact subgroup of L) that transform on the right according to the character $\delta : T \rightarrow U(1)$, and the particular functions needed are those that are supplied by Lemma 3.11. \square

Dirac cohomology and cohomological induction. As explained in Section 4, the representations $X(q, H, \delta)$ are obtained from the representations $X(\delta)$ by means of cohomological induction. Fortunately Dirac cohomology and cohomological induction are very easily related to one another, as follows:

Theorem 7.8. *Let G be a linear, connected, real reductive group with maximal compact subgroup K and let (q, H, δ) be a set of essential Vogan data for G . Write $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ and $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{q}_0$ and let $\kappa^L \in \mathfrak{t}_0^{c*}$ be the extension by zero of the differential of δ . Let*

$$\kappa^G = \kappa^L + \rho(\mathfrak{u} \cap \mathfrak{s}, \mathfrak{t}^c).$$

The Dirac cohomology space $H_{\text{Dirac}}^G(X(q, H, \delta))$ is a direct sum of 2^N copies the irreducible, genuine representation of \tilde{K} with highest weight κ^G .

Proof. This is mostly proved in [DH15, Theorem 5.7], which gives a general formula for the Dirac cohomology of a cohomologically induced representation in terms of the Dirac cohomology of the initial representation, assuming that infinitesimal character of the initial representation is weakly good for q . In our case, the initial representation $X^L(\delta)$ has infinitesimal character

$$\Lambda = \lambda^L + \rho(\Delta^+(\mathfrak{l}, \mathfrak{t})) = \lambda^L,$$

where λ^L is the differential of δ . See [Vog81, Lem. 4.1.8]; λ^L is of course a weight of \mathfrak{t} , and to identify it with Λ we extend it by zero to become a weight of the Cartan subalgebra $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$. The weakly good condition is in this case that

$$\langle \Lambda + \rho(\mathfrak{u}, \mathfrak{t}), \alpha \rangle \geq 0 \quad \forall \alpha \in \Delta(\mathfrak{u}, \mathfrak{t}),$$

and this follows from the definition of \mathfrak{q} , since $\Lambda + \rho(\mathfrak{u}, \mathfrak{t}) = \lambda^G$.

So [DH15, Theorem 5.7] applies. Given Lemma 7.7, the conclusion is as indicated in the statement of our theorem. \square

Proof of the matching theorem. The proof of the matching theorem is now within easy reach:

Proof of Theorem 7.4. Lemma 7.7 and Theorem 7.8 imply that if the set of essential Vogan data $(\mathfrak{q}, H, \delta)$ is constructed from the dominant weight κ , as in Proposition 3.2, then the Dirac cohomology of $X(\mathfrak{q}, H, \delta)$ is a nonzero multiple of the irreducible representation of \tilde{K} with highest weight κ . The matching theorem is therefore a consequence of Theorems 3.10, 4.3 and 6.3, and the classification of the irreducible representations of \tilde{K} by highest weight. \square

8. MORE ON DIRAC COHOMOLOGY

In this concluding section we shall do two things. First we shall make note of a detail from Section 5 of this paper that is used in the C^* -algebra K -theory computation in [CHST22]; see Theorem 3.7 there.

Theorem 8.1. *If $(\mathfrak{q}, H, \delta)$ is a set of essential Vogan data, and if $H = TA$, then $\log_2 |\mathcal{R}_\delta| = \dim(\mathfrak{a}) - \text{rank}(G) + \text{rank}(K)$.*

Proof. Since H is a Cartan subgroup of G ,

$$\text{rank}(G) = \dim(\mathfrak{t}) + \dim(\mathfrak{a}),$$

and since T^c is a maximal torus in K ,

$$\text{rank}(K) = \dim(\mathfrak{t}^c).$$

If $(\mathfrak{q}, H, \delta)$ is essential, then $|\mathcal{R}_\delta| = 2^N$, where N is the number of $\mathfrak{sl}(2, \mathbb{R})$ summands in \mathfrak{l}_0 . But this number N is also the difference in dimension between \mathfrak{t}^c and \mathfrak{t} , and so

$$\log_2 |\mathcal{R}_\delta| = \dim(\mathfrak{t}^c) - \dim(\mathfrak{t}).$$

The formula in the statement of the theorem follows from the three displayed identities. \square

Our second task is to prove a theorem that describes the Dirac cohomology of an essential component of the tempered dual in a bit more detail. This is used in a second proof of the Connes-Kasparov isomorphism in [CHST22], which, in comparison to the first proof, uses less K -theory but more representation theory. The following is the translation of Theorem 8.4 in [CHST22] into the language of this paper:

Theorem 8.2. *Let $(\mathfrak{q}, H, \delta)$ be a set of essential Vogan data.*

- (i) *Each minimal K-type of $X(\mathfrak{q}, H, \delta)$ has multiplicity one, and each irreducible direct summand of $X(\mathfrak{q}, H, \delta)$ includes precisely one of these minimal K-types.*
- (ii) *If τ is the irreducible representation of \tilde{K} to which $X(\mathfrak{q}, H, \delta)$ is matched, then the \tilde{K} -module $H_{\text{Dirac}}(X(\mathfrak{q}, H, \delta))$ is the direct sum of $|\mathbb{R}_\delta|$ copies of τ .*
- (iii) *If X is an irreducible summand of $X(\mathfrak{q}, H, \delta)$ and if $V \subseteq X$ is its minimal K-type, then*

$$V \otimes \mathbb{S} \subseteq H_{\text{Dirac}}(X)$$

and the inclusion is a vector space isomorphism.

Proof. Statement (i) is contained in [Vog81, Thm. 6.5.9]. Statement (ii) is Theorem 7.8. Statement (iii) is proved in [DD16, Sec. 4] (see the final paragraph of the proof of Theorem 1.2 there; our statement (iii) appears in the proof, but not in the formulation, of that theorem). \square

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