This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2024.3382601

On Complexity Bounds for the Maximal Admissible Set of Linear Time-Invariant Systems

Hamid R. Ossareh, *Senior Member, IEEE,* Ilya Kolmanovsky, *Fellow, IEEE*

***Abstract*—Given a dynamical system with constrained outputs, the maximal admissible set (MAS) is defined as the set of all initial conditions such that the output constraints are satisfied for all time. It has been previously shown that for discrete-time, linear, time-invariant, stable, observable systems with polytopic constraints, this set is a polytope described by a finite number of inequalities (i.e., has finite complexity). However, it is not possible to know the number of inequalities *a priori* from problem data. To address this gap, this contribution presents two computationally efficient methods to obtain *upper bounds* on the complexity of the MAS. The first method is algebraic and is based on matrix power series, while the second is geometric and is based on Lyapunov analysis. The two methods are rigorously introduced, a detailed numerical comparison between the two is provided, and an extension to systems with constant inputs is presented.**

***Index Terms*—Maximal admissible set, admissibility index, finite determination, Cayley-Hamilton Theorem, Lyapunov analysis.**

I. Introduction

Consider a discrete-time linear time-invariant system

$$\begin{aligned} x(t+1) &= Ax(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $t \in \mathbb{Z}^+$ is the discrete time index, $x(t) \in \mathbb{R}^n$ is the state vector, and $y(t) \in \mathbb{R}^q$ is the output vector. The output is required to satisfy the constraint

$$y(t) \in \mathbb{Y} \tag{2}$$

where $\mathbb{Y}$ is a compact polytope with the origin in its interior. This paper is concerned with the set of all initial conditions for which (2) is satisfied for all time, that is:

$$O_\infty = \{x_0 : CA^t x_0 \in \mathbb{Y}, \forall t \geq 0\} \tag{3}$$

This set, which is referred to as the *maximal admissible set* (MAS) [1], is an invariant set that has been broadly employed in the control literature, for example, as a terminal constraint in the Model Predictive Control (MPC) optimization problem to guarantee closed-loop stability [2], [3], or in Reference Governors and Command Governors to guarantee infinite-horizon constraint satisfaction [4], [5]. This set also plays a major role in the analysis of constrained systems and in set-theoretic methods in control, see e.g., [6], [7]. The properties and computations of this set, as well as its extensions to other classes of systems, have also received much attention in the literature, see e.g., [8]–[15]. The topic of characterizing invariant sets is of continuing interest, see, e.g., the recent publications [16]–[19].

H. R. Ossareh is with the Electrical Engineering program at the University of Vermont, Burlington, VT USA 05405. I. Kolmanovsky is with the Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI USA 48105. e-mails: hossareh@uvm.edu, ilya@umich.edu This work was partially supported by NASA under cooperative agreement VT-80NSSC20M0213, NIST under cooperative agreement 70NANB21H162, and National Science Foundation award ECCS-1931738.

In the paper [1], it was shown that if (1) is asymptotically stable and the pair (A, C) is observable, then O_∞ is a compact polytope which is finitely determined, i.e., it can be described by a finite number of inequalities:

$$O_\infty = \{x_0 : CA^t x_0 \in \mathbb{Y}, t = 0, \ldots, t^*\} \tag{4}$$

where t^*, referred to as the *admissibility index* of MAS, is the last "prediction time-step" required to fully characterize the MAS. One difficulty, which the current paper seeks to overcome, is that t^* is not known *a priori* from problem data. To find it, one would need to construct the MAS iteratively by adding inequalities one time-step at a time and checking for redundancy of the newly added constraints. Once all the newly added constraints are redundant, t^* has been found. To carry out the redundancy check, Linear Programs (LPs) must be solved, which renders the construction of MAS computationally demanding for high dimensional systems, those with slow dynamics, those with constraint sets of high complexity, and in situations where O_∞ must be computed in real-time, e.g., to accommodate changing models or constraints.

To fill this gap, this paper provides two methods to obtain an *upper bound* on t^*. This allows one to replace t^* in (4) by its upper bound, thereby eliminating the need to solve LPs during the construction of MAS (at the expense of having potentially redundant inequalities in the set description). In addition to speeding up the computation of MAS, knowledge of such an upper bound is helpful for defining the memory and processing requirements to store and employ the MAS for the purpose of control.

The first method for finding an upper bound on t^* is algebraic and leverages matrix power series to express the output at a time t as a linear combination of outputs at previous times, which helps us determine the time-step after which the constraints become redundant. The second method is geometric and relies on the decay rate of a quadratic Lyapunov function towards a constraint-admissible ellipsoidal level set. Both methods are computationally efficient and do not rely on optimization solvers. To the best of our knowledge, the first method is new and is the key contribution of this work. The second method is inspired by the existing literature (see e.g., [5], [20]); however, it is presented here in complete details with explicit bounds, and several enhancements to it are proposed.

The upper bounds obtained from the two methods are compared against the true value of t^* using a Monte Carlo study. It is shown that the first method results in a tighter upper bound as compared with the second method for all the random systems considered.

Finally, the two methods are extended to systems with constant inputs, which have been studied extensively in the literature on reference and command governors:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu \\ y(t) &= Cx(t) + Du \end{aligned} \tag{5}$$

where $u \in \mathbb{R}^m$ is a constant input. The definition of MAS for (5) is similar to (3), but modified to account for the input:

$$O_\infty = \{(x_0, u) : y(t) \in \mathbb{Y}, \forall t \geq 0\} \quad (6)$$

It is shown in [1] that this set is generally *not* finitely determined (i.e., it cannot be described by a finite number of inequalities). However, a finitely-determined, positively-invariant inner approximation can be obtained by tightening the steady-state constraint:

$$\widetilde{O}_\infty = \{(x_0, u) : y(\infty) \in (1-\epsilon)\mathbb{Y}, y(t) \in \mathbb{Y}, t = 0, \dots, t^*\} \quad (7)$$

where $\epsilon \in (0, 1)$ is typically a small number. In (7), $y(\infty) = H_0 u$, where H_0 is the DC gain. Similar to the unforced case, the admissibility index, t^*, for this case is not known *a priori* from problem data. We thus extend the two methods described previously to find upper bounds on t^*. As we show, the upper bounds depend explicitly on the value of ϵ. A Monte Carlo study similar to the one described above is conducted to compare the two methods. Similar to the case of unforced systems, Method 1 results in tighter upper bounds for all random systems considered.

The outline of this paper is as follows. Section II presents the two methods described above for the unforced case and provides a numerical study to compare them. Section III extends the results to the case of systems with constant inputs. Conclusions and future works are provided in Section IV.

The notation in this paper is as follows: $\mathbb{Z}^+$, $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$, and $\mathbb{C}$ denote the set of non-negative integers, real numbers, n-dimensional vectors of real numbers, $n \times n$ matrices with real entries, and complex numbers, respectively. For a symmetric matrix $P = P^\mathsf{T}$, we say it is positive definite and write $P \succ 0$ if all the eigenvalues of P are strictly positive. We use the variables $t \in \mathbb{Z}^+$, $t^* \in \mathbb{Z}^+$, and $m \in \mathbb{Z}^+$ to denote the discrete time index, the admissibility index of MAS, and the upper bound on the admissibility index, respectively.

II. Main Results: Unforced Systems

Consider system (1) with constraint (2). Our goal is to obtain an upper bound on t^* in (4). This section presents two methods to obtain such an upper bound. The first method, which we refer to as "Method 1", is based on a matrix power series expansion and the second, which we refer to as "Method 2", is based on Lyapunov analysis. Throughout this paper, we assume the following:

Assumption 1. *System* (1) *is asymptotically stable and the pair* (A, C) *is observable. Furthermore, the constraint set in* (2) *is described by*

$$\mathbb{Y} := \{y : -y_j^l \leq y_j \leq y_j^u, \quad j = 1, \dots, q\} \quad (8)$$

where y_j *is the* j*-th element of* y*,* $y_j^l > 0$ *defines the lower limit for* y_j*, and* $y_j^u > 0$ *defines the upper limit for* y_j*.*

Assumption 1 reflects the commonly encountered requirements of asymptotic stability and observability within the MAS literature. For a comprehensive examination of the limitations and generalizability of these assumptions, please refer to [1]. The box-constraint assumption in (8) is introduced for the sake of clarity of our presentation. However, note that any compact polyhedral constraint set with origin in its interior defined by $\underline{s} \leq Sy \leq \overline{s}$ can be transformed into box constraints by redefining the output as $\bar{y} = Sy$, which effectively modifies the output equation in (5) to $\bar{y}(t) = \bar{C}x(t) + \bar{D}u$, where $\bar{C} = SC$ and $\bar{D} = SD$ (of course, for the results of the paper to still hold, the redefined output must satisfy the observability assumption in Assumption 1).

A. Method 1: Matrix Power Series

The general idea behind this method is to first expand A^t in terms of lower powers of A. This expansion allows us to express the output $y(t)$ in (1) as a linear combination of the outputs at previous times. We show that if there exists an integer m such that the sum of the coefficients in the expansion of A^{m+1} is "sufficiently small", then m is an upper bound on t^*. We then show that such an expansion always exists thanks to the Cayley-Hamilton Theorem, and provide an algorithm for finding such m.

We begin by stating the main result of this section.

Theorem 1. *Consider system* (1) *with constraint* (2)*, and suppose Assumption 1 holds. Suppose there exists an integer* m*,* $m \geq 0$*, such that* A^{m+1} *can be expanded as:*

$$A^{m+1} = \sum_{i=0}^{m} \alpha_i A^i \quad (9)$$

where $\alpha_i \in \mathbb{R}$*,* $i = 0, \dots, m$*, satisfy the following condition:*

$$\sum_{\alpha_i > 0} \alpha_i - \gamma \sum_{\alpha_i < 0} \alpha_i \leq 1 \quad (10)$$

where γ *is the largest asymmetry in the constraints, i.e.,*

$$\gamma = \max_j \Big\{ \max \Big\{ \frac{y_j^u}{y_j^l}, \frac{y_j^l}{y_j^u} \Big\} \Big\} \quad (11)$$

Then, m *is an upper bound on the admissibility index,* t^**, of the MAS for* (1)–(2)*; that is,* $t^* \leq m$*.*

Proof. To show that m, as defined in the theorem, is an upper bound on t^*, we must prove that $y(t) \in \mathbb{Y}$ for $t \leq m$ implies that $y(t) \in \mathbb{Y}$ for all $t \geq m+1$, i.e., the latter inequalities are implied by the former and, hence, redundant. We prove this assertion using mathematical induction.

Induction base case: Assume $y(t) \in \mathbb{Y}$ for $t \leq m$ and show that $y(m+1) \in \mathbb{Y}$. To show this, note that the j-th output, starting from an initial condition, x_0, can be expanded using Eq. (9): $y_j(m+1) = C_j A^{m+1} x_0 = \sum_{i=0}^{m} \alpha_i (C_j A^i x_0)$. The assumption $y(t) \in \mathbb{Y}$ for $t \leq m$ implies that $C_j A^i x_0$ in the above sum satisfies: $-y_j^l \leq C_j A^i x_0 \leq y_j^u$. Thus, if $\alpha_i > 0$, we have that $-\alpha_i y_j^l \leq \alpha_i C_j A^i x_0 \leq \alpha_i y_j^u$, and if $\alpha_i < 0$, we have that $\alpha_i y_j^u \leq \alpha_i C_j A^i x_0 \leq -\alpha_i y_j^l$. Thus, summation over i results in:

$$-y_j^l \sum_{\alpha_i > 0} \alpha_i + y_j^u \sum_{\alpha_i < 0} \alpha_i \leq y_j(m+1) \leq y_j^u \sum_{\alpha_i > 0} \alpha_i - y_j^l \sum_{\alpha_i < 0} \alpha_i$$

To ensure that $-y_j^l \leq y_j(m+1) \leq y_j^u$, it suffices for α_i to satisfy:

$$-y_j^l \leq -y_j^l \sum_{\alpha_i > 0} \alpha_i + y_j^u \sum_{\alpha_i < 0} \alpha_i \quad (12)$$

$$y_j^u \sum_{\alpha_i > 0} \alpha_i - y_j^l \sum_{\alpha_i < 0} \alpha_i \leq y_j^u \quad (13)$$

or if we divide both sides of (13) by $y_j^u > 0$, and both sides of (12) by $-y_j^l < 0$, it suffices that $\sum_{\alpha_i > 0} \alpha_i - \frac{y_j^u}{y_j^l} \sum_{\alpha_i < 0} \alpha_i \leq 1$ and

$\sum_{\alpha_i>0}\alpha_i - \frac{y_j^l}{y_j^u}\sum_{\alpha_i<0}\alpha_i \le 1$. Both of these inequalities hold as they are implied by (10). Thus, $-y_j^l \le y_j(m+1) \le y_j^u$. Since j was arbitrary, we have that $y(m+1)\in\mathbb{Y}$, as desired.

Induction main step: Assume $y(t)\in\mathbb{Y}$ for $t\le k$, where $k\ge m+1$, and show that $y(k+1)\in\mathbb{Y}$. We again write the j-th output: $y_j(k+1)=C_jA^{k+1}x_0$ but now decompose $A^{k+1}=A^{m+1}A^{k-m}$. We thus obtain: $y_j(k+1)=C_jA^{m+1}A^{k-m}x_0=\sum_{i=0}^m\alpha_i(C_jA^{i+k-m}x_0)$. The assumption $y(t)\in\mathbb{Y}$ for $t\le k$ together with $0\le i+k-m\le k$ imply that $C_jA^{i+k-m}x_0$ in the above sum satisfies: $-y_j^l\le C_jA^{i+k-m}x_0\le y_j^u$. The rest of the proof from this point on follows the same arguments as in the induction base case. This concludes the proof. □

Remark 1. *In the case of symmetric constraints, the expression in Theorem 1 can be further simplified. Specifically, suppose that $y_j^l=y_j^u,\forall j$ in* (8)*. Then, $\gamma=1$ and so condition* (10) *becomes:*

$$\sum_i|\alpha_i|\le 1 \tag{14}$$

Remark 2. *If all the coefficients in the expansion of A^{m+1} are positive, then* (10) *becomes $\sum_i\alpha_i\le 1$, which is completely independent of the constraint set (i.e., independent of the C matrix, y_j^l, and y_j^u).*

We now prove the existence of, and a develop a method to construct, the expansion in (9) satisfying condition (10). We first recall some facts from linear algebra. The characteristic polynomial of any square matrix $A\in\mathbb{R}^{n\times n}$ is defined as $\Delta(s):=\det(sI-A)$. It is an n-th degree polynomial whose roots are the eigenvalues, $\lambda_i\in\mathbb{C}$, of A. We can thus write:

$$\Delta(s)=(s-\lambda_1)\cdots(s-\lambda_n)=s^n+c_{n-1}s^{n-1}+\ldots+c_1s+c_0 \tag{15}$$

The Cayley-Hamilton theorem states that any square matrix satisfies its own characteristic polynomial:

Theorem 2 (see [21])**.** *Let $A\in\mathbb{R}^{n\times n}$ be a matrix and let $\Delta(s)$ be its characteristic polynomial. Then, $\Delta(A)=0$.*

This result allows us to express A^n as a finite power series in lower powers of A:

$$A^n=-c_0I-c_1A-\ldots-c_{n-1}A^{n-1} \tag{16}$$

where c_i are the coefficients in (15) and are uniquely defined. Similarly, A^{n+1} can be expanded in the same powers of A:

$$\begin{aligned}A^{n+1}=A(A^n)&=-c_0A-\ldots-c_{n-2}A^{n-1}-c_{n-1}A^n\\&=(c_0c_{n-1})I+(-c_0+c_1c_{n-1})A+\ldots+\\&\quad(-c_{n-2}+c_{n-1}c_{n-1})A^{n-1}\end{aligned}$$

Generalizing the above to any $t\ge n$, one can expand A^t as:

$$A^t=\sum_{i=0}^{n-1}\beta_i(t)A^i \tag{17}$$

where $\beta_i(t)$ denotes the i-th coefficient in the expansion of the t-th power of A. Note that expansion of A^t in lower powers of A is generally not unique, but $\beta_i(t)$ in (17) are, by construction, uniquely defined.

To simplify the presentation, we stack the coefficients of the t-th power into a vector and denote it by $\beta(t)$:

$$\beta(t)=[\beta_0(t)\ \cdots\ \beta_{n-1}(t)]^T\in\mathbb{R}^n$$

The following theorem characterizes $\beta(t)$ and its convergence properties as $t\to\infty$.

Theorem 3. *Let $A\in\mathbb{R}^{n\times n}$ be any square matrix and let $\beta(t),t\ge n$, be the vector of coefficients in the expansion of A^t, as defined above. Then, $\beta(t)$ satisfies the difference equation*

$$\beta(t+1)=\begin{bmatrix}0&0&\cdots&0&-c_0\\1&0&\cdots&0&-c_1\\0&1&\cdots&0&-c_2\\\vdots&\vdots&&\vdots&\\0&0&\cdots&1&-c_{n-1}\end{bmatrix}\beta(t) \tag{18}$$

with initial condition $\beta(n)=[-c_0\ \cdots\ -c_{n-1}]^T$. In addition, if A is asymptotically stable, then $\lim_{t\to\infty}\beta(t)=0$.

Proof. The initial condition is already shown in Eq. (16). To derive the recursion, suppose $A^t=\sum_{i=0}^{n-1}\beta_i(t)A^i$, where $\beta_i(t)$ are given. To find $\beta(t+1)$ in terms of $\beta(t)$, we expand A^{t+1} as follows:

$$\begin{aligned}A^{t+1}=A(A^t)=A\sum_{i=0}^{n-1}\beta_i(t)A^i&=\beta_{n-1}(t)A^n+\sum_{i=0}^{n-2}\beta_i(t)A^{i+1}\\&=-\beta_{n-1}(t)c_0I+\sum_{i=1}^{n-1}\left(-\beta_{n-1}(t)c_i+\beta_{i-1}(t)\right)A^i\end{aligned}$$

where, in the first line, A^n is replaced using (16). Thus, $\beta_0(t+1)=-\beta_{n-1}(t)c_0$ and $\beta_i(t+1)=-\beta_{n-1}(t)c_i+\beta_{i-1}(t)$ for $i=1,\ldots,n-1$. This coincides with recursion (18).

To prove that $\beta(t)\to 0$ as $t\to\infty$, note that the matrix in (18) is exactly the observable canonical form of matrix A, see [21] for details. Thus, it has the same eigenvalues as A. Therefore, the recursion in (18) corresponds to an asymptotically stable dynamical system and, hence, $\beta(t)$ must converge to 0. □

It can be shown that recursion (18) effectively performs Euclid's division and computes the remainder polynomial when the denominator is $\Delta(s)$. Thus, we could have arrived at the above recursion for β, as well as the proof that $\beta\to 0$, using polynomial and rational function models [22] as an alternative approach[1].

The above theorem guarantees the existence of an integer m such that the coefficients of the expansion of A^{m+1} as defined in (9) satisfy condition (10). To see this, compute $\beta(t)$ using the recursion in (18) for increasing t starting from $t=n$, and stop when

$$\sum_{\beta_i(t)>0}\beta_i(t)-\gamma\sum_{\beta_i(t)<0}\beta_i(t)\le 1 \tag{19}$$

Note that such t always exists, because according to Theorem 3, $\beta(t)\to 0$ as $t\to\infty$ and thus the left hand side of (19) can be made arbitrarily small. Such t corresponds to $m+1$ in Theorem 1, where the α_i in (10) are related to $\beta_i(t)$ in (19) as follows: $\alpha_i=\beta_i(t)$ for $i=0,\ldots,n-1$ and $\alpha_i=0$ for $i=n,\ldots,m$. This leads to Algorithm 1 for finding an upper bound for t^*.

The above results allow us to say more about the value of t^* itself in the case of first order systems.

Theorem 4. *Consider* (1) *with $A\in\mathbb{R}$ (i.e., a first order system) and assume Assumption 1 holds. If $A>-\frac{1}{\gamma}$, then $t^*=0$. In particular, if the constraints are symmetric (i.e., $\gamma=1$ in* (11)*), then $t^*=0$.*

[1]We acknowledge an anonymous reviewer for this suggestion.

Algorithm 1 Compute upper bound, m, on t^* using Method 1

Input: A, y_j^l, y_j^u

1: Compute the Cayley-Hamilton coefficients, c_i, using (15), and γ using (11).
2: Set $t = n$ and initialize $\beta(n)$ as in Theorem 3.
3: If $\beta(t)$ satisfies (19), then: $m = t - 1$, STOP.
4: Increment t by 1. Compute $\beta(t)$ using (18). Go to step 3.

Proof. From (16), it follows that $A = -c_0$ and from Theorem 3, $\beta(t) \in \mathbb{R}$ satisfies $\beta(1) = -c_0 = A$. Asymptotic stability of A and condition $A > -\frac{1}{\gamma}$ imply that $-\frac{1}{\gamma} < \beta(1) < 1$. Using this condition, it can be seen that, regardless of sign of $\beta(1)$, (19) is satisfied for $t = 1$. Per Algorithm 1, an upper bound on t^* is therefore $m = 0$, which implies that $t^* = 0$. □

This theorem suggests that the MAS for some first order systems is particularly straightforward to construct.

We conclude this section with a few remarks.

Remark 3. *The Cayley-Hamilton-based expansion in* (17) *provides only one possible expansion for A^{m+1} in Theorem 1. There may be other expansions that lead to smaller upper bounds for t^*.*

Remark 4. *The Cayley-Hamilton-based upper bound sheds light on the conditions under which t^* may be large. Specifically, as the recursion in Theorem 3 suggests, the upper bound on t^* depends on the eigenvalues of A. If the spectral radius of A is large (i.e., there is an eigenvalue close to the boundary of the unit disk), the upper bound on t^* (and likely t^* itself) will be large. On the other hand, if the spectral radius is small, then the upper bound will be small (and thus t^* must also be small). Thus, there is a relationship between t^* and the spectral radius of A, which we examine numerically in Section II-C. An interesting implication of this is the following: if A is obtained by discretizing a continuous-time model, the eigenvalues of A approach the origin as the sampling period increases, leading to smaller values for the upper bound on t^* and thus smaller values for t^*. Thus, there is also a relationship between t^* and the sampling rate used for the discretization.*

B. Method 2: Lyapunov Level Sets

The second method to find an upper bound on t^* relies on Lyapunov level sets. We begin by defining two sets:

$$\mathbb{X} = \{x : Cx \in \mathbb{Y}\} \tag{20}$$

which is the inverse image of $\mathbb{Y}$ in the x-space, and

$$O_{n-1} = \{x_0 : CA^t x_0 \in \mathbb{Y}, t = 0, \ldots, n-1\} \tag{21}$$

which is the set of all initial conditions such that the constraints are satisfied for the first n time-steps. From these definitions, we have:

$$O_\infty \subseteq O_{n-1} \subseteq \mathbb{X}$$

The set $\mathbb{X}$ is not generally compact, but it is shown in [1] that, under Assumption 1, O_∞ and O_{n-1} are. The compactness of O_{n-1} is the main reason why it is employed in the analysis that follows. If $\mathbb{X}$ itself is compact, then O_{n-1} may be replaced by $\mathbb{X}$ in the subsequent presentation.

Define the quadratic Lyapunov function

$$V(x) = x^T P x \tag{22}$$

where $P = P^\top \succ 0$ is the solution of the discrete Lyapunov equation

$$A^T P A - P = -Q \tag{23}$$

for a given $Q = Q^\top \succ 0$. For each real number $r > 0$, the r-th level set of $V(x)$, defined by

$$\Omega_r = \{x : V(x) \leq r\}, \tag{24}$$

is an ellipsoid and is positively invariant with respect to the dynamics of (1), see [21], [23].

The key idea behind Method 2 is to first find two level sets of $V(x)$: one that is inscribed in $\mathbb{X}$, denoted by Ω_{r_1}, and one that circumscribes O_{n-1}, denoted by Ω_{r_2}. One way to quantify an upper bound on t^* is to find the longest time it takes for any initial state within O_{n-1} to enter Ω_{r_1}. Indeed, if $x(t) \in \Omega_{r_1}$, then $x(t) \in \mathbb{X}$ and thus $y(t) \in \mathbb{Y}$ for all future times due to the invariance of Ω_{r_1}. However, instead of O_{n-1}, we consider initial states within $\Omega_{r_2} \supset O_{n-1}$, which allows for simple computations using the ellipsoidal mathematics at the expense of making the upper bound less tight. We now formally examine the above ideas, and then provide a method to find a suitable matrix Q for the problem at hand.

Theorem 5. *Consider system* (1) *with Lyapunov function* (22)–(23) *and constraint* (2), *and suppose Assumption 1 holds. Define $r_1, r_2 \in \mathbb{R}$ as follows:*

$$r_1 = \max\{r : \Omega_r \subset \mathbb{X}\}, \quad r_2 = \min\{r : O_{n-1} \subset \Omega_r\}$$

Then, $r_2 \geq r_1$ and an upper bound for t^* is given by:

$$m = \text{floor}\left(\frac{\log(\frac{r_1}{r_2})}{\log(\sigma)}\right) \tag{25}$$

where the floor *operator returns the previous largest integer,*

$$\sigma = 1 - \frac{\lambda_{min}(Q)}{\lambda_{max}(P)}, \tag{26}$$

and λ_{min} (λ_{max}) denotes the smallest (largest) eigenvalue.

Proof. First, note that r_1 exists because $\mathbb{X}$ is convex and non-empty and has the origin in its interior, and r_2 exists because O_{n-1} is compact. Second, note that $\Omega_{r_1} \subset O_\infty$ because Ω_{r_1} is an invariant, constraint-admissible set and O_∞ contains all such sets (see [1]). Thus, we have the following inclusions: $\Omega_{r_1} \subset O_\infty \subset O_{n-1} \subset \Omega_{r_2}$, which means that $r_2 \geq r_1$, as required. The rest of the proof leverages the following fact from linear algebra: for any $P = P^\top \succ 0$, we have that $\lambda_{min}(P)x^T x \leq x^T P x \leq \lambda_{max}(P)x^T x$, where λ_{min} and λ_{max} are well-defined because the eigenvalues of a symmetric, positive-definite matrix are all real. Given $V(x)$ in (22), we can thus write $-x^T x \leq -\frac{V(x)}{\lambda_{max}(P)}$, and show that the change in the Lyapunov function along the trajectories of the system satisfies:

$$\begin{aligned} V(x(t+1)) - V(x(t)) &= (Ax(t))^T P(Ax(t)) - x(t)^T P x(t) \\ &= x(t)^T (A^T P A - P) x(t) = -x(t)^T Q x(t) \\ &\leq -\lambda_{min}(Q) x(t)^T x(t) \leq -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} V(x(t)) \end{aligned}$$

The above can be rewritten as:

$$V(x(t+1)) \leq \sigma V(x(t)) \Rightarrow V(x(t)) \leq \sigma^t V(x(0)) \tag{27}$$

where σ is as defined in the Theorem.

We now determine the longest time it takes for any initial condition within Ω_{r_2} to enter Ω_{r_1}. Note that any $x(0) \in \Omega_{r_2}$ satisfies $V(x(0)) \leq r_2$. Therefore, $V(x(t)) \leq \sigma^t r_2$. Furthermore, to ensure $x(t) \in \Omega_{r_1}$, we must have $V(x(t)) \leq r_1$. Therefore, we set $V(x(t)) \leq \sigma^t r_2 \leq r_1$, which implies that $t > \frac{\log(\frac{r_1}{r_2})}{\log(\sigma)}$. Any integer t satisfying this inequality must be larger than t^*. Thus, to obtain the tightest upper bound, we apply the floor operator to the right hand side of this inequality, which completes the proof. $\square$

Procedures for computing r_1 and r_2 in the theorem are well-established, see, e.g., [24]. Specifically, r_1 can be found by

$$r_1 = \min_j \frac{(\min\{y_j^l, y_j^u\})^2}{c_j P^{-1} c_j^T} \tag{28}$$

To find r_2, O_{n-1} can first be converted from the H-representation (i.e., half-space description) to V-representation (i.e., vertex description) [25]. Let the vertices of O_{n-1} in the V-representation be denoted by v_j. Then, r_2 can be found by

$$r_2 = \max_j \{v_j^T P v_j\} \tag{29}$$

It remains to find a suitable Q to solve for P using (23). We approach this problem by analytically finding Q that results in the smallest σ in Theorem 5 and thus the fastest decay rate of the Lyapunov function along the system trajectories (see Eq. (27)). Note that this is not necessarily the Q that results in the globally minimal value for the upper bound on t^*. Other possible approaches for selecting Q include solving an optimization problem to find a Q that minimizes the upper bound; finding a Q such that Ω_{r_1} has the largest volume; or finding the Q such that Ω_{r_2} has the smallest volume. These approaches, however, require nonlinear program or second-order cone program solvers, which is what we seek to avoid. Furthermore, our numerical studies showed that choosing Q to minimize σ led to the best possible upper bound in most situations.

Theorem 6. *The scalar σ in Theorem 5 satisfies $0 \leq \sigma < 1$. Furthermore, the matrix Q that results in the smallest σ is $Q = I$, and the corresponding value of σ is $\sigma = \rho(A)^2$, where $\rho := \max_i |\lambda_i(A)|$ is the spectral radius of A.*

Proof. To prove the first part, note that $V(x(t+1)) \geq 0$ and $V(x(t)) \geq 0$ in Eq. (27). Thus, $\sigma \geq 0$. To show $\sigma < 1$, note that $\lambda_{min}(Q), \lambda_{max}(P) > 0$. Thus, $\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} > 0$, which implies that $\sigma = 1 - \frac{\lambda_{min}(Q)}{\lambda_{max}(P)} < 1$.

To prove the second part, we must find Q to maximize $\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}$. By linearity of the Lyapunov equation in (23), normalizing Q by any scalar normalizes P by the same scalar. Therefore, without loss of generality, one can normalize Q such that $\lambda_{min}(Q) = 1$, which implies that $Q \succeq I$ or $Q - I \succeq 0$. Since $\lambda_{min}(Q) = 1$, it now suffices to find a Q to minimize $\lambda_{max}(P)$.

It is known that the solution, P, of the Lyapunov equation (23) can be expressed as [21]:

$$P(Q) = \sum_{t=0}^{\infty} (A^T)^t Q A^t \tag{30}$$

We can thus write: $P(Q) - P(I) = \sum_{t=0}^{\infty} (A^T)^t (Q - I) A^t$. Since $Q - I \succeq 0$, we have that $P(Q) - P(I) \succeq 0$ or equivalently, $P(Q) \succeq P(I)$. Thus $\lambda_{max}(P(Q)) \geq \lambda_{max}(P(I))$ so to minimize the largest eigenvalue of P, one must take $Q = I$.

Finally, to show that the choice of $Q = I$ leads to $\sigma = \rho(A)^2$, we again leverage (30) and redefine $\bar{A} = A^T A$. We then apply the spectral mapping theorem from linear algebra to conclude that $\lambda_i(P) = \sum_{t=0}^{\infty} \lambda_i(\bar{A})^t = \frac{1}{1-\lambda_i(\bar{A})}$. Since $\lambda_i(\bar{A}) = (\lambda_i(A))^2$, we have that $\lambda_{max}(P) = \frac{1}{1-\rho(A)^2}$, which implies that $\sigma = \rho(A)^2$. $\square$

The above leads to Algorithm 2 for finding an upper bound for t^*.

Algorithm 2 Compute upper bound, m, on t^* using Method 2

Input: A, C, y_j^l, y_j^u

1: Compute P using (23) with $Q = I$. Compute $\sigma = \rho(A)^2$
2: Compute r_1 using (28).
3: Construct O_{n-1} as in (21), convert to V-representation, and compute r_2 using (29).
4: Compute m using expression (25).

C. Numerical Comparison

This section presents a comparative analysis of the upper bounds provided by Algorithm 1 for Method 1 (i.e., the power series-based method) and Algorithm 2 for Method 2 (i.e., the Lyapunov-based method). Since this comparison cannot be carried out analytically, we conduct a Monte Carlo study of randomly-generated systems using Matlab 2020b.

To generate each random system with diverse characteristics, we first randomly generate n, the order of the system, by sampling the uniform distribution between 1 and 8. We then generate a state-space model with that order by using Matlab's `drss` command, which returns Lyapunov stable systems with possibly repeated and/or complex poles. The `drss` function employs the following procedure. It first generates random pole locations, with a 5% probability of repeated poles and a 50% probability of complex poles until the number of poles equals the system order. The real poles and the magnitude of the complex poles are sampled from the standard uniform distribution while the phase of the complex poles is sampled from the uniform distribution between 0 and π. The `drss` function then creates a change of basis matrix, whose elements are obtained from the standard uniform distribution, and accordingly creates the A matrix through a similarity transformation. The elements of the B, C, and D matrices are finally generated by sampling the standard normal distribution. Some of these elements are forced to 0 afterwards with a probability of 0.25.

For each system computed by the `drss` function, we reject systems for which the spectral radius is greater than 0.999 and the smallest singular value of the observability matrix is less than 0.0001 to ensure that Assumption 1 is robustly satisfied. For simplicity, we assume a single output (i.e., $q = 1$) and symmetric constraints $y_1^u = y_1^l = 1$. Using the above methodology, we generate a total of 16,000 random systems. We assume that the input satisfies $u = 0$, which makes each system have the form (1). For each system, we compute t^* using the algorithm described in [1]. We also compute the upper bounds on t^* using Algorithms 1 and 2. We denote these upper

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2024.3382601

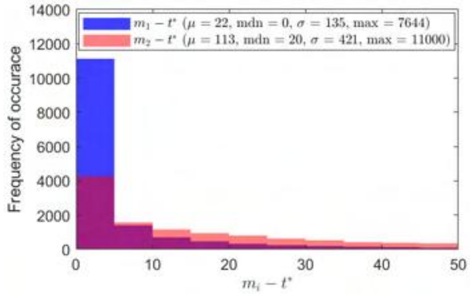


Fig. 1. Histograms of $m_1 - t^*$ and $m_2 - t^*$ (i.e., the tightness of each upper bound) obtained using our Monte Carlo study. In the legend, μ, σ, and mdn refer to the mean, standard deviation, and median, respectively.

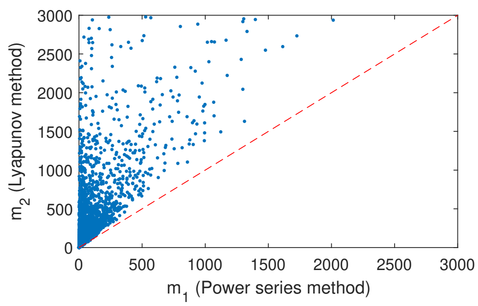
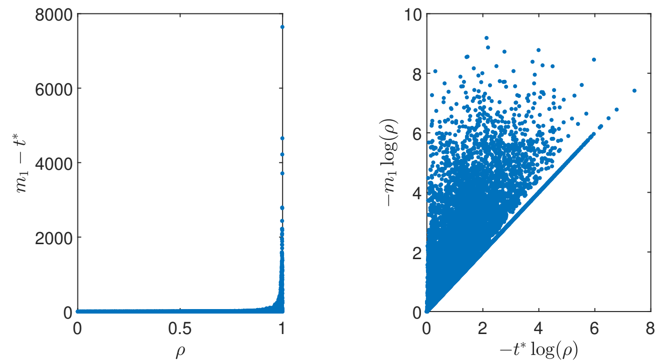


Fig. 2. Comparison between the upper bound provided by Methods 1, m_1, and by Method 2, m_2. Interestingly, $m_1 \leq m_2$ in all cases.

bounds by m_1 and m_2 respectively, where the subscript refers to the respective method. To compare the upper bounds against the true value of t^*, we construct the histograms of $m_i - t^*$, $i = 1, 2$, as seen in Fig. 1. In addition to the histograms, a point by point comparison between the two methods is provided in Fig. 2. As can be seen from the data, Method 1 performs well overall, with a median of 0 (i.e., for at least half of the random systems, the upper bound is tight). Furthermore, interestingly, Method 1 outperforms Method 2 in all cases. Investigation of this observation is an interesting topic for future research.

From these figures, it may appear that the upper bounds are too conservative for some systems, which, per Remark 4, could be attributed to the large spectral radius of those systems. This can be easily confirmed with our Monte Carlo study, as seen in Fig. 3a for Method 1. To investigate further, we normalize both t^* and its upper bound m_1 to allow for a fair comparison between the different systems. The normalization is achieved by scaling t^* and m_1 by $\log(\rho)$, where $\rho = \max_i(|\lambda_i(A)|)$ is the spectral radius. Taking logarithms is inspired by the fact that continuous-time poles and discrete-time poles are related through $z = e^{sT_s}$, where T_s is the sample time. Assuming $T_s = 1$ to allow for direct comparison between the systems, we obtain $s = \log(z)$. Thus, scaling by $\log(\rho)$ normalizes each t^* or m by the "continuous-time time constant" of the system. The results are reported in Fig. 3b. As can be seen, in the normalized coordinates, the spread is narrow and the upper bound is not as conservative as it appeared before. Similar plots can be generated for Method 2.

In the above Monte Carlo study, symmetric constraints were assumed. We have also examined asymmetric constraints and found that the conclusions are similar, namely m_1 is a tighter bound than m_2 even in the asymmetric case. We omit the full details due to space limitations.



(a) $m_1 - t^*$ vs. spectral radius, ρ. The larger the ρ, the more conservative the upper bound may be.

(b) Comparison of t^* vs. m_1, each scaled by the logarithm of the spectral radius of A.

Fig. 3. Analysis of the upper bounds obtained using Method 1.

As a final remark, we acknowledge that the quantitative results presented above depend on the distributions from which the random systems are sampled. However, the observation that Method 1 yields tighter bound as compared to Method 2 appears to be generically true and is also confirmed by other numerical experiments we performed.

III. Main Results: Systems with Constant Input

We now extend the results in the previous section to the forced system (5) with constraint (2). As explained in Section I, the MAS for this system may not be finitely determined. However, by tightening the steady-state constraint, a finitely-determined inner approximation, denoted by $\widetilde{O}_\infty$, can be obtained, see Eq. (7). For a given steady-state margin $\epsilon > 0$, our goal is to obtain upper bounds on t^* such that all constraints after time-step t^* are guaranteed to be redundant in (7). Similar to the unforced case, we assume that Assumption 1 holds. Furthermore, we assume that the input u is constant for all time.

A. Method 1: Matrix Power Series

Similar to the case of unforced systems, the general idea behind this method is finding an expansion of A^t, with "sufficiently small" coefficients, in terms of lower powers of A. The key difference with the unforced case is that the origin is no longer the equilibrium of the forced system, so we must perform a change of coordinates to shift the equilibrium to the origin. Furthermore, recall from (7) that the steady-state constraint is tightened by $(1 - \epsilon)$, which introduces additional complexities.

The equilibrium of (5) is given by $x(\infty) = (I - A)^{-1}Bu$ and $y(\infty) = H_0 u$, where $H_0 = C(I - A)^{-1}B + D$ is the DC gain from u to y. Note that the matrix inverse exists thanks to the asymptotic stability of A. We define a new state vector $z(t)$ to shift the equilibrium to the origin: $z(t) := x(t) - (I - A)^{-1}Bu$. In the new coordinate system, the dynamics are described by:

$$\begin{aligned} z(t+1) &= Az(t) \\ y(t) &= Cz(t) + H_0 u \end{aligned} \tag{31}$$

The output thus evolves as $y(t) = CA^t z(0) + H_0 u$, which is used below. We now state the main result of this section.

Theorem 7. *Consider system* (5) *with constraint* (2) *and constant input u, and suppose Assumption 1 holds. Suppose there exists an integer m, $m \geq 0$, such that A^{m+1} can be expanded as in* (9)*, where α_i satisfy:*

$$\Big(1+\gamma(1-\epsilon)\Big)\sum_{\alpha_i>0}\alpha_i - \Big(\gamma+(1-\epsilon)\Big)\sum_{\alpha_i<0}\alpha_i \leq \epsilon \tag{32}$$

and γ is defined in (11)*. Then, $t^* \leq m$.*

Proof. The proof is similar to that of Theorem 1 with some differences, which we highlight. As in Theorem 1, we use mathematical induction to prove that $y(t) \in \mathbb{Y}$ for $t \leq m$ implies that $y(t) \in \mathbb{Y}$ for $t \geq m+1$. For the sake of brevity, we only discuss the base case of the induction argument, as the proof of the induction step is similar. For the base case, we assume that $y(t) \in \mathbb{Y}$ for $t \leq m$ and show that $y(m+1) \in \mathbb{Y}$. To show this, we write the j-th output as: $y_j(m+1) = C_jA^{m+1}z_0 + H_0u = \sum_{i=0}^m \alpha_i(C_jA^ix_0) + H_0u$. We add and subtract $\sum_{i=0}^m \alpha_i H_0 u$ to this expression to obtain: $y_j(m+1) = \sum_{i=0}^m \alpha_i(C_jA^ix_0 + H_0u) + H_0u - \sum_{i=0}^m \alpha_i H_0u$. The assumption $y(t) \in \mathbb{Y}$ for $t \leq m$ implies that $C_jA^ix_0 + H_0u$ in the first sum satisfies: $-y_j^l \leq C_jA^ix_0 + H_0u \leq y_j^u$. Furthermore, the assumption $y(\infty) \in (1-\epsilon)\mathbb{Y}$ implies that $-(1-\epsilon)y_j^l \leq H_0u \leq (1-\epsilon)y_j^u$. Thus, breaking up the sum into positive and negative values of α_i as we did in the proof of Theorem 1, we obtain the following bounds on $y_j(m+1)$:

$$-y_j^l(1-\epsilon) - y_j^l\sum_{\alpha_i>0}\alpha_i + y_j^u\sum_{\alpha_i<0}\alpha_i - y_j^u(1-\epsilon)\sum_{\alpha_i>0}\alpha_i +$$
$$y_j^l(1-\epsilon)\sum_{\alpha_i<0}\alpha_i \leq \boxed{y_j(m+1)} \leq y_j^u(1-\epsilon) + y_j^u\sum_{\alpha_i>0}\alpha_i -$$
$$y_j^l\sum_{\alpha_i<0}\alpha_i + y_j^l(1-\epsilon)\sum_{\alpha_i>0}\alpha_i - y_j^u(1-\epsilon)\sum_{\alpha_i<0}\alpha_i$$

To ensure that $-y_j^l \leq y(m+1) \leq y_j^u$, we set the left inequality to be greater than $-y_j^l$ and the right inequality to be smaller than y_j^u. We then divide the left inequality by $-y_j^l$ and the right inequality by y_j^u and simplify terms to obtain the result. ☐

Remark 5. *In the case of symmetric constraints, i.e., $y_j^l = y_j^u, \forall j$ in* (8)*, we have that $\gamma = 1$ and so* (32) *becomes:*

$$\sum_i |\alpha_i| \leq \frac{\epsilon}{2-\epsilon} \tag{33}$$

Note that the right hand side tends to 0 and 1, as ϵ tends to 0 and 1, respectively.

As in the unforced case, the Cayley-Hamilton based expansion of Section II can be employed to obtain the expansion in Theorem 7 and thus obtain an upper bound on t^*. An algorithm similar to Algorithm 1 can be constructed for this purpose, wherein condition (19) is replaced with:

$$\Big(1+\gamma(1-\epsilon)\Big)\sum_{\beta_i(t)>0}\beta_i(t) - \Big(\gamma+(1-\epsilon)\Big)\sum_{\beta_i(t)<0}\beta_i(t) \leq \epsilon \tag{34}$$

We conclude this section with two remarks.

Remark 6. *In condition* (34)*, the smaller the ϵ (i.e., the steady-state tightening), the smaller the right hand side, and therefore the smaller the $\beta_i(t)$ must be to satisfy the condition. According to Theorem 3, smaller $\beta_i(t)$'s are achieved with larger t's. Therefore, the upper bound on t^* (and likely t^* itself) grows as ϵ becomes small. Furthermore, if $\epsilon \ll 1$ (which is typical in applications),* (34) *can be approximated by*

$$\sum_i |\beta_i(t)| \leq \frac{\epsilon}{1+\gamma}$$

which implies that $|\beta_i(t)| \ll 1$. Thus, for the same constraints y_j^l, y_j^u, and the same matrices A and C, $\beta_i(t)$'s that satisfy this condition are likely smaller than those that satisfy (19)*. Thus, the upper bound on t^* (and likely t^* itself) is larger in the forced case than the unforced case.*

Remark 7. *Note that condition* (19) *for the unforced case and* (34) *for the forced case become identical when $\epsilon = 1$ (this forces $u = 0$ due to $H_0u \in (1-\epsilon)\mathbb{Y}$, which makes intuitive sense). In this sense, Method 1 in the forced case is a proper extension of Method 1 in the unforced case.*

B. Method 2: Lyapunov Level Sets

The second method, which relies on Lyapunov level sets to find an upper bound on t^*, requires only minor modifications compared to the input-free case. We first extend the definition of O_{n-1} in (21) to account for the input in system (31), where we tighten the steady-state constraint similar to (7):

$$\widetilde{O}_{n-1} = \big\{(z_0, u) : CA^tz_0 + H_0u \in \mathbb{Y}, t = 0, \dots, n-1, \quad H_0u \in (1-\epsilon)\mathbb{Y}\big\}. \tag{35}$$

As in the case of O_{n-1}, this set is a compact polytope. We have the following result.

Theorem 8. *Consider system* (5) *with constant input u, Lyapunov function* (22)–(23)*, and constraint* (2)*, and suppose Assumption 1 holds. Define $r_1, r_2 \in \mathbb{R}$ as follows:*

$$r_1 = \max\big\{r : \ \Omega_r \subset \epsilon\mathbb{X}\big\} \tag{36}$$

$$r_2 = \min\big\{r : \ \texttt{Proj}_z\widetilde{O}_{n-1} \subset \Omega_r\big\}, \tag{37}$$

where $\texttt{Proj}_z$ denotes the projection onto z-coordinates. Then, an upper bound on t^* is given by expression (25)*.*

Proof. Suppose $(z(0), u) \in \widetilde{O}_{n-1}$. Then, by (35) and (37), $z(0) \in \Omega_{r_2}$ and $H_0u \in (1-\epsilon)\mathbb{Y}$. By the same arguments as in the proof of Theorem 5 applied to (31), $z(t)$ starting from such $z(0)$ satisfies $z(t) \in \Omega_{r_1}$ for all $t \geq m$, where m is given by (25) with r_1 and r_2 given by (36)–(37). This, together with (36), implies that $z(t) \in \epsilon\mathbb{X}$ or, equivalently, $Cz(t) \in \epsilon\mathbb{Y}$ for all $t \geq m$, which implies that $y(t)$ in (31) satisfies: $y(t) = Cz(t) + H_0u \in \epsilon\mathbb{Y} \oplus (1-\epsilon)\mathbb{Y} = \mathbb{Y}$, where $\oplus$ denotes the Minkowski set addition. To summarize, the n set inclusions, $CA^tz(0) + H_0u \in \mathbb{Y}$, $t = 0, \dots, n-1$, coupled with $H_0u \in (1-\epsilon)\mathbb{Y}$, make redundant the inequalities corresponding to $y(t) \in \mathbb{Y}$ for $t \geq m$. Thus $t^* \leq m$. ☐

Procedures for computing r_1 and r_2 in the theorem are similar to those in Section II. Specifically, given $\epsilon \in (0, 1)$, r_1 can be found by

$$r_1 = \min_j \frac{(\min\{\epsilon y_j^l, \epsilon y_j^u\})^2}{c_jP^{-1}c_j^T} \tag{38}$$

To find r_2, we first compute $\widetilde{O}_{n-1}$ using (35) and convert it into the V-representation. Let the vertices of $\widetilde{O}_{n-1}$ in the V-representation be denoted by $v_j \in \mathbb{R}^{n+m}$, where the first n components correspond to the z-coordinates and the next m components correspond to the u-coordinates. Then, r_2 can be found by

$$r_2 = \max_j \{\bar{v}_j^T P \bar{v}_j\} \tag{39}$$

where $\bar{v}_j \in \mathbb{R}^n$ is a vector consisting of the first n components of v_j.

An algorithm similar to Algorithm 2 can be constructed to find the upper bound using Theorem (8). To this end, steps 2 and 3 of the algorithm must be modified as follows: in step 2, replace Eq. (28) with (38) and, in step 3, replace O_{n-1} in (21) by $\widetilde{O}_{n-1}$ in (35) and Eq. (29) by (39).

Remark 8. *Note that r_1 and r_2 in (36)–(37) are smaller than those defined in Theorem 5 because $\epsilon < 1$. This means that the upper bound computed using Algorithm 2 for the forced case is generally larger than that for the unforced case. Furthermore, note that if $\epsilon = 1$, then the unforced case and the forced case become identical. This makes intuitive sense because if $\epsilon = 1$, then $u = 0$ to ensure $H_0 u \in (1-\epsilon)\mathbb{Y}$. In this sense, Method 2 in the forced case can be seen as the proper extension of Method 2 in the unforced case.*

C. Numerical Comparison

We performed a Monte Carlo study similar to the one presented in Section II-C to compare the upper bound obtained using Method 1 (Algorithm 1 modified as described in Section III-A) with that obtained using Method 2 (Algorithm 2 modified as described in Section III-B). For this purpose, we choose $\epsilon = 0.01$, and use the same 16,000 random systems described in Section II-C but this time allow $u \neq 0$. For each system, we compute t^* using the algorithm described in [1], as well as the two upper bounds, which we denote by m_1 for Method 1 and m_2 for Method 2. We made the following observations.

First, comparing the actual value of t^* in the forced case with that in the unforced case (Section II-C), we see that t^* in the forced case is larger than the t^* in the unforced case for all the random systems considered, which is an interesting observation. Second, similar to the true value of t^*, we find that the upper bounds in the forced case are always larger than those in the unforced case (presented in Section II-C), which is consistent with Remark 6. Third, narrowing our attention to the upper bounds for the forced case, we computed $m_1 - t^*$ and $m_2 - t^*$, whose median values are, respectively, 23 and 42. More importantly, we observe that, similar to the unforced case, Method 1 outperforms Method 2 in all the random systems considered.

IV. Conclusions and Future Work

This paper presented two computationally efficient methods to obtain *upper bounds* on the admissibility index of Maximal Admissible Sets for discrete-time LTI systems. The first method is algebraic and is based on matrix power series, while the second is geometric and is based on Lyapunov level sets. It was shown that Method 1 outperforms Method 2, and that the upper bounds (and likely the admissibility index itself) depend on the spectral radius of matrix A and also the steady-state tightening, ϵ, in the case of systems with constant inputs.

Future work will investigate the reason why Method 1 outperformed Method 2 in our numerical study. Another topic for future research is to find other power series expansions (beyond what is provided by the Cayley-Hamilton method) to further improve the upper bounds in Method 1. Upper bounds for the admissibility index of *robust* maximal admissible set for systems with disturbances is another avenue of future research.

References

[1] Elmer G Gilbert and K Tin Tan. Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Transactions on Automatic control*, 36(9):1008–1020, 1991.

[2] Eduardo F Camacho and Carlos Bordons Alba. *Model predictive control*. Springer Science & Business Media, 2013.

[3] James Blake Rawlings, David Q Mayne, and Moritz Diehl. *Model predictive control: Theory, computation, and design*, volume 2. Nob Hill Publishing Madison, WI, 2017.

[4] Elmer G Gilbert, Ilya Kolmanovsky, and Kok Tin Tan. Discrete-time reference governors and the nonlinear control of systems with state and control constraints. *International Journal of robust and nonlinear control*, 5(5):487–504, 1995.

[5] Emanuele Garone, Stefano Di Cairano, and Ilya Kolmanovsky. Reference and command governors for systems with constraints: A survey on theory and applications. *Automatica*, 75:306–328, 2017.

[6] Franco Blanchini. Set invariance in control. *Automatica*, 35(11):1747–1767, 1999.

[7] Franco Blanchini and Stefano Miani. *Set-theoretic methods in control*, volume 78. Springer, 2008.

[8] Ilya Kolmanovsky and Elmer G Gilbert. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical problems in engineering*, 4(4):317–367, 1998.

[9] B. Pluymers, J. A. Rossiter, J. A. K. Suykens, and B. De Moor. The efficient computation of polyhedral invariant sets for linear systems with polytopic uncertainty. In *Proceedings of the 2005, American Control Conference*, pages 804–809 vol. 2, June 2005.

[10] Saša V Raković and Mirko Fiacchini. Invariant approximations of the maximal invariant set or "encircling the square". *IFAC proceedings volumes*, 41(2):6377–6382, 2008.

[11] Kenji Hirata and Yoshito Ohta. Exact determinations of the maximal output admissible set for a class of nonlinear systems. *Automatica*, 44(2):526–533, 2008.

[12] Hamid R Ossareh. Reference governors and maximal output admissible sets for linear periodic systems. *International Journal of Control*, 93(1):113–125, 2020.

[13] Seyed Meisam Vaselnia, Shahram Aghaei, and Vicenç Puig. An inclusion-based approach for determination of a safe maximal output admissible set. In *2021 European Control Conference (ECC)*, pages 646–650. IEEE, 2021.

[14] Emilio Pérez, Carlos Arino, F Xavier Blasco, and Miguel A Martínez. Maximal closed loop admissible set for linear systems with non-convex polyhedral constraints. *Journal of Process Control*, 21(4):529–537, 2011.

[15] Joycer Osorio and Hamid R Ossareh. A stochastic approach to maximal output admissible sets and reference governors. In *2018 IEEE Conference on Control Technology and Applications (CCTA)*, pages 704–709. IEEE, 2018.

[16] Matthias Rungger and Paulo Tabuada. Computing robust controlled invariant sets of linear systems. *IEEE Transactions on Automatic Control*, 62(7):3665–3670, 2017.

[17] Dylan Janak and Behçet Açikmeşe. Maximal invariant set computation and design for markov chains. In *2019 American Control Conference (ACC)*, pages 1244–1249, 2019.

[18] Zheming Wang, Raphaël M. Jungers, and Chong Jin Ong. Computation of the maximal invariant set of discrete-time linear systems subject to a class of non-convex constraints. *Automatica*, 125:109463, 2021.

[19] Saša V Raković and Sixing Zhang. The implicit maximal positively invariant set. *IEEE Transactions on Automatic Control*, 2022.

[20] Emanuele Garone, Ilya Kolmanovsky, and Stefano DiCairano. Full-day workshop on reference supervision for constraint enforcement: theory and applications. Workshop at 2014 IEEE Conference on Decision and Control, url: https://saas.ulb.ac.be/CDC2014/, 2014.

[21] Chi-Tsong Chen. *Linear system theory and design, third edition*. Oxford University Press, Inc., 1998.

[22] PA Fuhrmann. Functional models in linear algebra. *Linear algebra and its applications*, 162:107–151, 1992.

[23] H.K. Khalil. *Nonlinear Systems*. Pearson Education. Prentice Hall, 2002.

[24] Stephen Boyd, Stephen P Boyd, and Lieven Vandenberghe. *Convex Optimization*. Cambridge university press, 2004.

[25] David Avis, Komei Fukuda, and Stefano Picozzi. On canonical representations of convex polyhedra. In *Mathematical Software*, pages 350–360. World Scientific, 2002.