Multiparticle correlations, cumulants, and moments sensitive to fluctuations in rare-probe azimuthal anisotropy in heavy ion collisions

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Correlations of two or more particles have been an essential tool for understanding the hydrodynamic behavior of the quark-gluon plasma created in ultrarelativistic nuclear collisions. In this paper, we extend that framework to introduce a mathematical construction of multiparticle correlators that utilize correlations between arbitrary numbers of particles of interest (e.g., particles selected for their strangeness, heavy flavor, and conserved charges) and inclusive reference particles to estimate the azimuthal anisotropies of rare probes. To estimate the fluctuations and correlations in the azimuthal anisotropies of these particle of interest, we use these correlators in a system of cumulants, raw moments, and central moments. Finally, we introduce two classes of observables that can compare the fluctuations in the azimuthal anisotropies of particles of interest with reference particles at each order.

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I. INTRODUCTION

The quark-gluon plasma (QGP) is a state of hot nuclear matter created in the collisions of nuclei at the Large Hadron Collider (LHC) and the Relativistic Heavy Ion Collider (RHIC); for a recent review see Ref. [1]. The fluid nature of this matter has been confirmed using collective flow measurements at both colliders [2–4]. The early stages of heavy ion collisions produce predominately elliptical shapes due to the nature of the geometry of these collisions but other geometrical shapes are possible due to quantum-mechanical fluctuations of the quarks and gluons within the nucleus. Due to the asymmetric pressure gradients caused by these geometrical shapes, the fluid nature of the QGP converts these geometrical shapes into collective flow patterns in momentum space. These collective properties manifest as azimuthal anisotropies in the distribution of particle angles ϕ and are quantified by v_n, the Fourier harmonics for the distribution of particle yields around Ψ_n, the nth order event-plane:

$$\frac{dN}{d\phi} \propto 1 + 2\sum_{n=1}^{\infty} v_n \cos\left[n(\phi - \Psi_n)\right]. \tag{1}$$

where nonzero harmonics v_n attest to collectivity within the strongly interacting, nearly perfect fluid nature of the QGP. The cumulant framework [5–8] has also been used extensively to measure v_n[9–16]. One particular advantage of using cumulants is that these measurements can provide sensitivity to both the root mean square (rms) values of the anisotropies, $(\langle v_n{}^2\rangle)^{1/2}$, over some collection of events, but also to higher-order fluctuations present in the distribution of event-by-event v_n. Measurements of v_n are essential for constraining transport coefficients within relativistic viscous fluid dynamics using Bayesian analyses [17–21] and are one of the standard benchmarks that models must reproduce [22,23]. Previous work has demonstrated that experimental measurements of flow fluctuations [16,24–27] can play a crucial role in constraining initial-state models [28–30].

Up until this point we have discussed generic collective flow harmonics that are measured using a nearly inclusive set of particles dominated by those at low transverse momenta (p_T) *reference particles*. However, a number of useful signals of the QGP come from particles of interest (POIs), a class of either identified particles (e.g., protons) or high-p_T particles that generally do not significantly overlap with reference particles. Typically these differential classes of particles have insufficient statistics from just POI angles to simply measure v_n' using the same techniques as measurements of v_n using charged particles. Thus, the azimuthal anisotropies that characterize the distributions of exclusively POIs are measured using "differential" correlators and cumulants which rely on correlations between POI and reference particles [5,7].

Quantum-chromodynamics (QCD) and thereby the QGP is required to locally conserve quantities such as electric charge, baryon number, and strangeness. Realistic handling of these quantities is necessary to accurately compare theoretical models to experimental data. The study of fluctuations of v_n' for conserved quantities is still in its infancy; however, recent work that includes baryon stopping in the initial conditions [31] and other work that includes gluon splittings into quark antiquark pairs [32,33] would allow one to study the fluctuations of conserved charges in the azimuthal anisotropies; these studies may shed light on the effects of event-by-event fluctuations of baryon stopping and/or charge diffusion transport coefficients.

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Additionally, jets are of great interest in heavy ion collisions [34]. They are created in large momentum transfer processes in the very early stages of the collision prior to the fluid formation; thus, they experience the same collision evolution as the fluid but they are not equilibrated with it because the associated momentum scale is much larger than the temperature of the fluid. Jets are sensitive to the short-length-scale properties of the QGP and the average suppression of jets in the QGP can be used to constrain the strength of jet quenching [35–38].

In the case of jets, the value of v_n' is understood to be sensitive to the path-length dependence of the interaction between the jets and the QGP [39,40] and measurements have been made which show positive values for these v_n' quantities [41–43]. Theoretically, hydrodynamical models have been used to elucidate decorrelations between the event plane angles Ψ_n and Ψ_n' for reference particles and for POI respectively, as well as for event plane angles between two different harmonics [44–46]. This decorrelation of POI event planes from the reference particle event plane is important when considering event-by-event fluctuations in v_n'. However, current techniques and observables do not provide a comprehensive way to study these jet-by-jet fluctuations in energy loss [47].

Measurements of mesons containing heavy, charm, and bottom quarks may provide interesting insight to various phenomena unique to heavy flavor particles. At high momentum, heavy quarks come from jet production and suffer energy loss [48,49]. At intermediate momenta, heavy quarks are understood to undergo Langevin-like diffusion as they move through the QGP [48,50]. Additionally, hadronization of heavy quarks is thought to be modified in heavy ion collisions with recombination processes playing an important role [51–53]. Due to all these effects, it is of great interest to measure the azimuthal anisotropies and their fluctuations of hadrons containing heavy quarks to constrain theoretical models [54–57].

Finally, both jets and heavy flavor lead to ambiguous signals in proton-nucleus collisions where positive v_n' values for high-p_T particles are measured [58] but, there is no significant suppression [59–62]. Thus, studies of the fluctuations of v_n' could provide new information in these small systems to understand the origin of the observed v_n'.

In this work, we focus on the development of observables that utilize angular correlations between arbitrarily many reference particles with one or more POI. The POI selection varies with the physics of interest and could include high-p_T jets, heavy-flavor hadrons, or some other object classification. Our explicit intent is to derive observables that are sensitive to not just the rms of the v_n Fourier coefficients for POI, $(\langle v_n'^2 \rangle)^{1/2}$ but also higher-order fluctuations displayed by v_n.

Generally, we expect the fluctuations of v_n' in POI to be affected by both the initial geometry of the collision, as for the reference particles, and by additional process specific fluctuations. Measurements of these quantities could provide unique information about the mechanism of jet energy loss [63]. Therefore, we qualitatively expect the fluctuations in v_n' to be larger than those observed in the soft sector. This is illustrated in Fig. 1, which shows a typical example of a differential two-particle estimate for v_n': $v_2'\{2\}$ as a function of p_T for hadrons. The plotted $v_2'\{2\}$ values are the average over the events in that particular event selection, typically a centrality bin in heavy ion collisions. The insets show example distributions of event-by-event v_2 values relative to the mean. The goal of this work is to suggest experimental observables using multiparticle correlations to provide experimental access to these underlying fluctuation distributions.

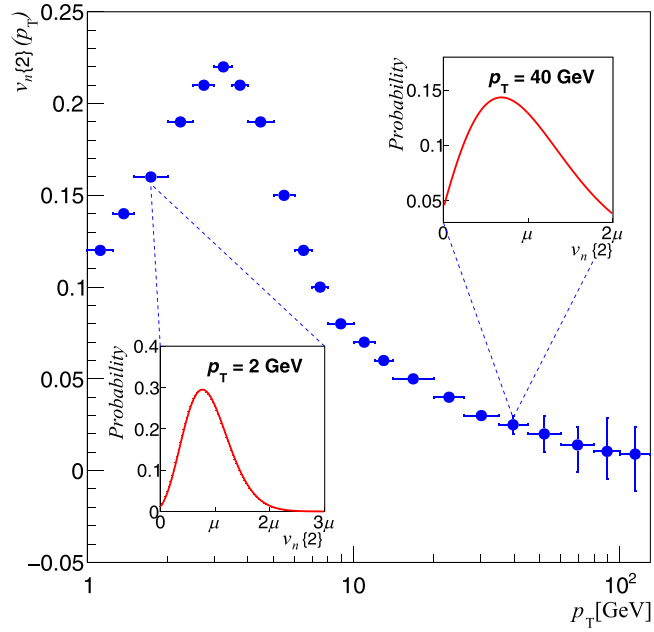

FIG. 1. Cartoon of $v_n\{2\}(p_T)$ as a function of p_T. Insets show examples of the distributions of $v_n\{2\}(p_T)$ for low-p_T particles (dominated by reference particles) and high p_T (particles of interest).

Given the impact of these types of measurements in the soft sector, it is very likely that fluctuations of v_n' can provide new insight into the processes discussed above. The very large data samples at the LHC [64], the forthcoming upgrades at the LHC, and the very large data samples projected from sPHENIX [65], provide new opportunities to make these measurements. Up until today, only a handful of experimental papers have explored multiparticle cumulants that contain one POI [66–71]. We expect this field to expand significantly with upcoming data.

In this paper we derive the formalism to study different types of multiparticle cumulants with an arbitrary number of POI in azimuthal anisotropy measurements for jets, and other rare probes. We generalize the following observables to include one or more POI: differential cumulants developed in Refs. [5,6], higher-order moments of v_n', multiharmonic cumulants developed by Moravcova *et al.* [72], and asymmetric cumulants developed by Bilandzic *et al.* [73]. Finally, we propose central moments of arbitrary order in POI angle dependence, and two observables to estimate the contribution to a cumulant or moment for an arbitrary number of particles. Finally, we summarize our work by discussing features of each specific observable and what types of fluctuations and POIs they are most ideal for measuring.

II. AZIMUTHAL ANISOTROPY AND CORRELATORS

The event-by-event azimuthal anisotropies of particles in heavy ion collisions are studied using a Fourier expansion for the distribution of particle angles around the beam axis, where v_n represents the contribution of the nth harmonic to the net particle yield:

$$\frac{dN}{d\phi} \propto 1 + 2\sum_{n=1}^{\infty} v_n \cos\left[n(\phi - \Psi_n)\right], \tag{2}$$

where the detector angle ϕ_i for each of the N particles is compared with the nth order event plane; an azimuthal angle about which the distribution of $n\phi_i$ is symmetric [74]. When using the symbol v_n, we specifically refer to the azimuthal anisotropy coefficients for reference particles, typically all measured charged particles.

Likewise, we can define a related quantity, v_n', as the Fourier coefficient for the distribution of POI angles. This quantity is an analog to v_n but generally relies on different approximation methods than v_n due to properties of the probe in question:

$$\frac{dN'}{d\psi} \propto 1 + 2\sum_{n=1}^{\infty} v_n' \cos[n(\psi - \Psi_n')], \tag{3}$$

where the multiplicity of POI is labeled by N' for a single event, and each POI is denoted with angle ψ_i with respect to their theoretical event plane Ψ_n', about which the angles $n\psi_i$ are symmetric.

Using a Fourier expansion, we rewrite the explicit definition of both v_n and v_n' [5,6]:

$$v_n \equiv \langle e^{in(\phi - \Psi_n)} \rangle = \langle \cos\left[n(\phi - \Psi_n)\right] \rangle, \tag{4}$$

$$v_n' \equiv \langle e^{in(\psi - \Psi_n')} \rangle = \langle \cos[n(\psi - \Psi_n')] \rangle, \tag{5}$$

where angle brackets $\langle\rangle$ indicate an average taken over all particle angles within a single event. Flow harmonic vectors that contain information about v_n, and the nth-order event plane and can be defined as

$$V_n \equiv v_n e^{in\Psi_n}, \tag{6}$$

$$V_n' \equiv v_n' e^{in\Psi_n'} \tag{7}$$

within a single event. The definition of these vectors allows for a more concise way to express the evaluation of v_n and v_n' to different powers using multiparticle angular correlations, as detailed in the next section.

A. Correlators

Multiparticle correlations can determine v_n and v_n' to different powers, a more accurate and computationally effective process than approximating the event plane, and measuring v_n and v_n' as they are defined in Eqs. (4) and (5) [75,76]. To measure correlations, we rely on the assumption that reference particles and POI are distributed symmetrically around Ψ_n and Ψ_n', respectively, which is accurate in the limits of $N, N' \gg 1$. If this assumption holds for the reference particles, it has been shown that we can correlate n-tuples of particle angles to approximate powers of v_n for different harmonics [8,77,78]. The correlation among reference particles can be written as a product of v_n coefficients with an exponential dependence on event planes, or as a product of unconjugated and conjugated complex flow vectors V_n, V_n^*:

$$\begin{aligned}
&\langle k+m \rangle_{n_1,\ldots,n_k|n_{k+1},\ldots,n_{m+k}} \\
&\quad \equiv \langle e^{i(n_1\phi_1 + \ldots + n_k\phi_k - n_{k+1}\phi_{k+1} - \ldots - n_{m+k}\phi_{m+k})} \rangle &(8)\\
&\quad = v_{n_1}\cdots v_{n_m} e^{i\left(n_1\Psi_{n_1} + \ldots + n_k\Psi_{n_k} - n_{k+1}\Psi_{n_{k+1}} - \ldots - n_{k+m}\Psi_{n_{k+m}}\right)} &(9)\\
&\quad = V_{n_1}\cdots V_{n_k} V^*_{n_{k+1}}\cdots V^*_{n_m}, &(10)
\end{aligned}$$

for an arbitrary number $k+m$ of correlated particles. Each reference particle is labeled with angle ϕ_p, numbered one through N, and the average on the right-hand side (RHS) is taken over all $k+m$ tuples of particles. The indices $n_1, \ldots, n_k$ indicate that the particles are "positively correlated;" their azimuthal angles have a positive sign in the exponential in the above equation. The indices $n_{k+1}, \ldots, n_{k+m}$ that follow the vertical bar | indicate "negatively correlated" harmonics, which are subtracted in the above equation's exponential. We require the sum of the positively and negatively correlated harmonics to be zero: for any $\langle k+m \rangle$ we must have

$$\sum_{i=1}^{k} n_i = \sum_{j=1}^{m} n_{k+j}, \tag{11}$$

otherwise the resulting quantity is isotropic and will average to zero due to symmetry of particles around each event plane.

To measure v_n', and correlations between v_n' and v_n within an event, we apply the same assumption about event plane symmetry for POIs, and incorporate their angles into correlations in the same way as in Eq. (8), generalizing to correlators that use $k'+m'$ POIs and $k+m$ reference particles:

$$\begin{aligned}
&\langle k'+m'+k+m \rangle_{n_1,\ldots,n_k,n'_{k+1},\ldots n'_{k+k'}|n_{k+k'+1},\ldots,n_{k+k'+m},n'_{k+k'+m+1}\ldots n'_{k+k'+m+m'}} \\
&\quad \equiv \left\langle \exp i\left[\sum_{\alpha=1}^{k} n_\alpha\phi_\alpha + \sum_{\beta=k+1}^{k+k'} n'_\beta\psi_\beta - \sum_{\gamma=k+k'+1}^{k+k'+m} n_\gamma\phi_\gamma - \sum_{\delta=k+k'+m+1}^{k+k'+m+m'} n'_\delta\psi_\delta\right]\right\rangle, \qquad (12)
\end{aligned}$$

where we have in particular k' positively correlated POI harmonics, m' negatively correlated POI harmonics, k positively correlated reference particle harmonics, and m negatively correlated reference particle harmonics. Each reference particle index

iterates through all N reference particles in the event with angle ϕ_p, and likewise each of the N' POI angles ψ_p is iterated through by POI indices.

Plugging Eqs. (2) and (3) into Eq. (12), we can convert the correlator with $k'+m'$ arbitrary POI angles and $k+m$ reference particle angles into an approximation for v_n and v'_n at arbitrary power and harmonic number:

$$\begin{aligned}
&\langle k'+m'+k+m\rangle_{n_1,\ldots,n_k,n'_{k+1},\ldots n'_{k+k'}|n_{k+k'+1},\ldots,n_{k+k'+m},n'_{k+k'+m+1}\ldots n'_{k+k'+m+m'}} \\
&= \prod_{\alpha=1}^{k} v_{n_\alpha} e^{in_\alpha\Psi_\alpha} \prod_{\beta=k}^{k+k'} \left(v'_{n_\beta}\right) e^{in'_\beta\Psi'_\beta} \prod_{\gamma=k+k'+1}^{k+k'+m} v_{n_\gamma} e^{-in_\gamma\Psi_\gamma} \prod_{\delta=k+k'+m+1}^{k+k'+m+m'} \left(v'_{n_\delta}\right) e^{-in'_\delta\Psi'_\delta} \\
&= V_{n_1}\cdots V_{n_k} V'_{n_{k+1}}\cdots V'_{n_{k+k'}} V^*_{n_k+k'+1}\cdots V^*_{n_k+k'+m} V'^*_{n_{k+k'+m+1}}\cdots V'^*_{n_{k+k'+m'+m}}. \qquad (13)
\end{aligned}$$

Similar to the condition in Eq. (11), we require that the sum of harmonics produces an anisotropic quantity:

$$\sum_{i=1}^{k} n_i + \sum_{j=k+1}^{k+k'} n'_j = \sum_{g=k+k'+1}^{k+k'+m} n_g + \sum_{h=k+k'+m+1}^{k+k'+m+m'} n'_h \qquad (14)$$

for Eq. (13) to be used. Otherwise, Eq. (13) will yield values consistent with zero due to the symmetry planes.

Evaluating $\langle k'+m'+k+m\rangle$ will generally produce some dependence on the angle between the event planes for v_n and v'_n at each of the different harmonics n_i, defined by the event plane angle in the exponential. When considering any two flow vectors V_n and V^*_m, for either POI or reference particles, we can understand that this angle dependence is simply the dot product of the various vectors. An example of a multiharmonic correlation between v'_3 and the product of v_2 and v_4 is demonstrated below:

$$\begin{aligned}
\langle 2'+2\rangle_{2,4|3',3'} &= v_2 v_4 (v'_3)^2 \cos\left(2\Psi_2 + 4\Psi_4 - 6\Psi'_3\right) \\
&= \langle e^{i(2\phi_1+4\phi_2-3\psi_3-3\psi_4)}\rangle \\
&= V_2 V_4 (V'^*_3)^2, \qquad (15)
\end{aligned}$$

where the harmonics $n_1 \ldots n_4$ obey the condition in Eq. (14). Also, we remind the reader that the subscripts in $\phi_1 \ldots \phi_4$ indicate the indices for the particles being correlated, not the harmonics.

To evaluate Eq. (8) using experimental data, an average is taken over all $k+m$ tuples with unique particle angles at each index—indicated by a primed summation $\sum'$ [8,78]:

$$\begin{aligned}
\langle k+m\rangle = \frac{(N-m-k)!}{N!} \sum_{1,\ldots k+m}{}' \\
\times e^{i(n_1\phi_1+\ldots+n_k\phi_k-n_{k+1}\phi_{k+1}-\ldots-n_{k+m}\phi_{k+m})} \qquad (16)
\end{aligned}$$

where the normalization factor $(N-k-m)!/N!$ is the reciprocal of the number of $k+m$ tuples of unique particles in an event with N total particles. Likewise, in keeping with the evaluation of correlators using one POI index in Ref. [78], we can construct the same summations with arbitrary dependence on particles of interest, using N reference particles and N' POI by writing Eq. (12) as a primed summation:

$$\begin{aligned}
&\langle k'+m'+k+m\rangle \\
&= \frac{(N-k-m)!}{N!}\frac{(N'-k'-m')!}{N'!} \sum{}' \exp i\left[\sum_{\alpha=1}^{k} n_\alpha\phi_\alpha + \sum_{\beta=k+1}^{k+k'} n'_\beta\psi_\beta - \sum_{\gamma=k+k'+1}^{k+k'+m} n_\gamma\phi_\gamma - \sum_{\delta=k+k'+m+1}^{k+k'+m+m'} n'_\delta\psi_\delta\right], \qquad (17)
\end{aligned}$$

where in this case we assume that there are no POI that are also considered reference particles, which we expect to be a reasonable definition in most analyses. For a more general treatment with non-negligible overlap between reference particles and POI, see Appendix A.

Calculating a $(k'+m'+k+m)$-particle correlation by iterating over $k'+m'+k+m$ tuples can be computationally challenging for higher-order correlations. However, in Refs. [7,8] a much faster method to calculate these correlators using polynomials of the event Q_n vector is introduced, which makes the calculations of these correlators much easier, but more complex to write analytically. Traditionally, a Q_n vector is defined by averaging all of the particle angles within an event:

$$Q_n = \sum_{p=0}^{N} e^{in\phi_p}, \qquad (18)$$

where each reference particle has angle ϕ_p with respect to the event plane Ψ_n. In Appendix B we introduce an algorithm to recursively write a correlation of the form $\langle k'+m'+k+m\rangle$ using Q_n vectors, analogous to the method described in Ref. [8], but incorporating arbitrary POI dependence.

While mathematically the correlators defined in this section evaluate products of v_n as described in Eq. (13), their values can be biased by sensitivity to nonflow fluctuations that contribute to v_n in the form of correlations due to small subsets of particles produced in heavy ion collision such as jets or resonance decays. These highly correlated particles fluctuate event-by-event and do not necessarily reflect the collectivity induced by initial-state geometry in the QGP. Additionally, because single event measurements fluctuate due to finite statistics, and the varying initial geometry of the QGP, it is more common to measure event by event averages of v_n

and associated quantities. In the next section, we express the event-by-event averages of the correlations developed above, as the raw moments of a multivariate distribution over events, describe the stochastic variables that comprise the multivariate distribution, and relate them to existing observables.

B. Raw moments

Due to quantum-mechanical fluctuations of the nucleons (or quarks and gluons) within the nucleus, as well as impact-parameter fluctuations, there are fluctuating v_n values on an event-by-event basis for a fixed centrality class [28,74,79–83]. The distribution of v_n values for reference particles in a fixed centrality class has already been measured experimentally [11]. The use of various underlying probability distributions to fit experimentally determined distributions for v_n have been studied [79], and used to parametrize these experimental measurements, as well as simulation data, such as Gaussian [84], Bessel Gaussian [68,75], elliptic Gaussian, and elliptic power distributions [85,86]. As of yet, no attempts have been made to experimentally measure the underlying distribution for v_n'. Since a distribution and its fluctuations can be constrained by its moments or cumulants, we focus on methods to obtain moments and cumulants of v_n' to attempt to extract the underlying distribution.

To better understand these distributions and their event-by-event fluctuations, we first consider the raw moments of v_n. Raw moments provide significant insight into various properties of the distribution's symmetry and tail behavior, and, more importantly, can always be used to express more conventional measures of fluctuations like central moments, or cumulants, which we will define later in the paper.

The most natural way to define the raw moments in terms of the quantities already defined in the previous section would be to simply take event-by-event averages of $\langle k' + m' + k + m\rangle$. Since $\langle k' + m' + k + m\rangle$ measures a product of $V_{n_1}\cdots V'_{n_{m+k}}$, we understand the stochastic variables measured in these moments will also be a product of some collection of r vectors: $V_{n_{i_1}}\cdots V'_{n_{i_r}}$. The exact determination of each stochastic variable represented in such a raw moment is addressed in depth in the next section. Given an arbitrary collection of stochastic variables $X_1,\ldots,X_n$, the raw moment μ of order ν_i in each stochastic variable X_i is defined as

$$\mu_{\nu_1,\ldots,\nu_m}(X_1\ldots X_m) \equiv \left\langle \prod_{i=0}^{n} X_i^{\nu_i} \right\rangle, \tag{19}$$

where an average over all events is shown.

Earlier, we have defined multiparticle correlations, which specifically correlate $k+m$ reference particle angles at varying harmonics, and $k'+m'$ POI at varying harmonics in Eq. (13) [8]. The expectation value of the correlator from Eq. (13)—a raw moment—is then obtained by averaging over an ensemble of events with index i, and weighting W_i, which is often uniform, based on total p_T, or total number of particles produced. This is indicated by the double brackets $\langle\langle\rangle\rangle$:

$$\langle\langle k' + m' + k + m\rangle\rangle = \frac{\sum_i W_i \langle k' + m' + k + m\rangle_i}{\sum_i W_i}, \tag{20}$$

and on the RHS, we also consider the *single bracket* $\langle\rangle$ to be a weighted average taken over events:

$$\langle\langle k' + m' + k + m\rangle\rangle_{n_1,\ldots,n_k,n'_{k+1},\ldots n'_{k+k'}|n_{k+k'+1},\ldots,n_{k+k'+m},n'_{k+k'+m+1}\ldots n'_{k+k'+m+m'}}$$
$$= \left\langle \prod_{\alpha=1}^{k} v_{n_\alpha} e^{in_\alpha\Psi_\alpha} \prod_{\beta=k}^{k+k'} \left(v'_{n_\beta}\right) e^{in'_\beta\Psi'_\beta} \prod_{\gamma=k+k'+1}^{k+k'+m} v_{n_\gamma} e^{in_\gamma\Psi_\gamma} \prod_{\delta=k+k'+m+1}^{k+k'+m+m'} \left(v'_{n_\delta}\right) e^{in'_\delta\Psi'_\delta} \right\rangle, \tag{21}$$

where we have shown the full subindex of the correlator in Eq. (21) but suppressed it in Eq. (20) for simplicity's sake.

As a special case of this method, we note that correlators for exclusively reference particles, and correlators using one POI angle index have been defined explicitly in Refs. [6,7], using angle ψ_p for POI angles. The one POI differential correlator approximates the joint moment of order $2k-2$ in v_n and first order in $V_n'V_n^*$. These correlators are used extensively in differential flow analyses with generating function cumulants to estimate values of v_n' [9,14,87]:

$$\langle 1' + 2k - 1\rangle = \langle e^{in(\psi_1+\phi_1..\phi_{k-1}-\phi_k-\ldots-\phi_{2k-1})}\rangle$$
$$= \frac{(N-2k+1)!}{N'(N!)} {\sum}' e^{in(\psi_1+\phi_1..\phi_{k-1}-\phi_k-\ldots-\phi_{2k-1})}, \tag{22}$$

where in the above equation we assume there is no overlap between POI and reference particles. After averaging this correlator over an ensemble of events,

$$\mu_{k-1,1}\left(v_n^2, V_n'V_n\right) = \langle\langle 1' + 2k - 1\rangle\rangle, \tag{23}$$

we can write it as a raw moment of the variabes v_n^2, and $V_n'V_n$.

As another example, we create a raw moment from the example in Eq. (15) by averaging the correlator $\langle 2' + 2\rangle_{2,4|3',3'}$:

$$\langle\langle 2' + 2\rangle\rangle_{2,4|3',3'} = \langle (v_2 e^{i2\Psi_2})(v_4 e^{i4\Psi_4})(v_3' e^{-i3\Psi_3'})^2\rangle$$
$$= \langle v_2 v_4 (v_3')^2 \cos(2\Psi_2 + 4\Psi_4 - 6\Psi_3')\rangle, \tag{24}$$

where we find that the condition in Eq. (14) is satisfied just as before. This raw moment identifies one of the contributions of V_2, V_4, and V_3' to a multivariate distribution. We address how to determine the variables in the multivariate distribution and address how to determine what correlations the above types of raw moments actually measured in the next section.

C. Stochastic variables and normalization

Raw moments of a distribution often scale with the mean value of their distributions, $\langle v_n^m \rangle \sim \langle v_n \rangle^m$, making their comparisons difficult. For example, if stochastic variables X and Y are distributed normally with $\mu_2(X) = \mu_2(Y) = 5$, but $\mu_1(X) = 0.05$ and $\mu_1(Y) = 3$, the magnitude of their "fluctuations" relative to their mean are very different, despite both distributions having the same second raw moment. We define a normalized moment for a collection of stochastic variables $X_1, \ldots, X_n$, denoted $N\mu_{\nu_1,\ldots,\nu_n}(X_1, \ldots, X_n)$:

$$N\mu_{\nu_1,\ldots,\nu_n}(X_1, \ldots, X_n) = \frac{\mu_{\nu_1,\ldots,\nu_n}(X_1, \ldots, X_n)}{\prod_i^n \mu_1(X_i)^{\nu_i}}, \tag{25}$$

where the scaling of $\mu_{\nu_1,\ldots,\nu_n}(X_1, \ldots, X_n)$ that comes from $\langle X_i \rangle^{\nu_i}$ for each i is canceled by explicitly dividing by $\langle X_i \rangle^{\nu_i}$. If each stochastic variable $X_1, \ldots, X_n$ is statistically independent of the others, then normalizing the raw moment will give unity; $N\mu_{\nu_1,\ldots,\nu_n}(X_1, \ldots, X_n) = 1$, and normalized moments that differ significantly from one indicate correlation or anticorrelation between the variables $X_1, \ldots, X_n$.

Normalizing in this way is consistent with the existing normalization technique for symmetric cumulants (SCs), which use the stochastic variables $X_1 = v_n^2$, and $X_2 = v_m^2$ to correlate the Fourier coefficients of two harmonics [8,88,89]:

$$SC\left(v_n^2, v_m^2\right) = \left\langle v_n^2 v_m^2 \right\rangle - \left\langle v_n^2 \right\rangle \left\langle v_m^2 \right\rangle, \tag{26}$$

$$NSC\left(v_n^2, v_m^2\right) = \frac{\left\langle v_n^2 v_m^2 \right\rangle - \left\langle v_n^2 \right\rangle \left\langle v_m^2 \right\rangle}{\left\langle v_n^2 \right\rangle \left\langle v_m^2 \right\rangle}. \tag{27}$$

It is clear that SC is normalized by dividing SC by each stochastic variable that it takes as an argument. It becomes intuitive to normalize raw moments of other stochastic variables the same way. We can now write $NSC(v_n^2, v_m^2)$ as a sum of a normalized raw moment and 1:

$$NSC\left(v_n^2, v_m^2\right) = \frac{\left\langle v_n^2 v_m^2 \right\rangle}{\left\langle v_n^2 \right\rangle \left\langle v_m^2 \right\rangle} - 1 = N\mu_{1,1}\left(v_n^2, v_m^2\right) - 1, \tag{28}$$

where, when considering quantities like $\langle v_n^2, v_m^2 \rangle = \langle\langle 2 + 2 \rangle\rangle_{m,n|m,n}$, it is clear that $X_1 = v_n^2$ and $X_2 = v_m^2$ allows for consistency with the normalization for SC, as further discussed in Ref. [73]. However, for quantities like the correlator defined in Eq. (15), it is less clear how to select the correct stochastic variables. Additionally, for a correlator such as $\langle\langle 2' + 1 \rangle\rangle_{2',2'|4}$ with odd numbers of particles, it is not well defined how to select the variables of interest in a manner consistent with Eqs. (26)–(28).

As shown in Eq. (21), a raw moment of the form $\langle\langle k' + m' + k + m \rangle\rangle$ evaluates the event-by-event averages of products of flow vectors $V_n = v_n e^{in\Psi_n}$ and $V_n' = e^{in\Psi_n'}$. It is tempting to use these flow vectors as stochastic variables, but they cannot work within the normalization scheme, because $\mu_1(V_n)$ and $\mu_1(V_n')$ are isotropic quantities and will average to zero, leaving the normalized raw moment in Eq. (25) undefined.

Instead, we use "nontrivial" stochastic variables that are selected by factoring out subsets of vectors for which the harmonics of vectors in the subset add to zero. If $\langle k' + m' + k + m \rangle$ evaluates to a product of flow vectors,

$$\begin{aligned}\langle k' + m' + k + m \rangle &= V_{n_1} \ldots V_{n_k} V'_{n_{k+1}} \ldots V'_{n_{k+k'}} V^*_{n_{k+k'+1}} V^*_{n_{k+k'+m}} V'^*_{n_{k+k'+m+1}} V'^*_{k+k'+m+m'},\end{aligned} \tag{29}$$

we consider any disjoint subset of those vectors that produce a nonisotropic quantity to be a viable stochastic variable for our formalism. These groups can be represented by different correlators with fewer indices, $X_i = \langle k_i' + m_i' + k_i + m_i \rangle$, and since they are disjoint, the product of each stochastic variable yields: $X_1 \cdot \ldots \cdot X_n = \langle k' + m' + k + m \rangle$. To validate that the correlators $\langle\langle k' + m' + k + m \rangle\rangle$ are actually expectation values of the raw variables $X_i = \langle k_i' + m_i' + k_i + m_i \rangle$, we relate $\langle\langle k' + m' + k + m \rangle\rangle$, X_i, and $\mu_{\nu_1,\ldots,\nu_n}(X_1, \ldots X_n)$ in the equations

$$\langle\langle k' + m' + k + m \rangle\rangle = \left\langle \prod_i \langle k_i' + m_i' + k_i + m_i \rangle \right\rangle, \tag{30}$$

$$X_i \equiv \langle k_i' + m_i' + k_i + m_i \rangle, \tag{31}$$

$$\mu_{\nu_1,\ldots,\nu_n}(X_1, \ldots X_n) = \left\langle \prod_i X_i^{\nu_i} \right\rangle = \langle\langle k' + m' + k + m \rangle\rangle, \tag{32}$$

where subscripts with each harmonic, $n_1, n_2, \ldots$, are dropped for clarity. Clearly, v_n^2 and v_m^2 both remain consistent with this definition, since $\langle 2 \rangle_{n|n} = v_n^2$ and $\langle 2 \rangle_{m|m} = v_m^2$ are both isotropic quantities.

When only considering nontrivial stochastic variables, there exist many $\langle\langle k' + m' + k + m \rangle\rangle$ that cannot be further normalized if no smaller group of indices within $\langle\langle k' + m' + k + m \rangle\rangle$ can be isolated that will add to zero. Most notably, $\langle\langle 2 \rangle\rangle_{n|n}$ cannot be normalized further; it functions well as a stochastic variable, but it does not contain any "smaller" nontrivial stochastic variables. Additionally, there exist correlators with more than one way to normalize, or select stochastic variables. The actual selection of stochastic variables is arbitrary, and can be changed for each individual analysis. For example, we can find two possible selections of stochastic variables for $\langle\langle 2' + 3 \rangle\rangle_{2',2,1|3',2}$, although the raw moment will return the same value for each, the meaning of $\langle\langle 2' + 3 \rangle\rangle$ will change based on normalization:

$$\frac{\langle\langle 2' + 3 \rangle\rangle_{1,2',2|2,3'}}{\langle\langle 1' + 1 \rangle\rangle_{2'|2} \langle\langle 1' + 2 \rangle\rangle_{1,2|3'}} = N\mu_{1,1}((V_2' V_2^*), (V'^*_3 V_2 V_1)), \tag{33}$$

$$\frac{\langle\langle 2' + 3 \rangle\rangle_{1,2',2|2,3'}}{\langle\langle 2 \rangle\rangle_{2|2} \langle\langle 2' + 1 \rangle\rangle_{1,2'|3'}} = N\mu_{1,1}\left(v_2^2, (V'^*_3 V_2' V_1)\right), \tag{34}$$

where we see that the raw moment is valid according to Eq. (14), and likewise $2 = 2$, $2 + 1 = 3$ ensuring that the stochastic variables are nontrivial. In the top equation, stochastic variables are selected to examine the correlation between $V_2' V_2^*$ and $V'^*_3 V_2 V_1$, which has greater sensitivity to the difference between Ψ_2 and Ψ_2' than the quantities in the bottom equation: v_2^2, and $V_3' V_2' V_1$. A more intentional choice of stochastic variables and harmonics can prove useful for explicitly isolating correlation and decorrelation between

event planes. This can be seen clearly when considering four-particle correlators that share the same harmonic:

$$\langle\langle 2'+2\rangle\rangle_{2',2|2',2} = \mu_{1,1}\left(v'^2_2, v^2_2\right), \tag{35}$$

$$\langle\langle 2'+2\rangle\rangle_{2',2'|2,2} = \mu_2[v'_2 v_2 \cos 4(\Psi'_2 - \Psi_2)], \tag{36}$$

where, in the first case, the stochastic variables must be $v_2{}^2$ and $v'_2{}^2$ because in order to satisfy Eq. (14) the POI and reference harmonics (n', n) add to zero separately: $2' + (-2') = 0 = 2 + (-2)$. Likewise, in the second case, the stochastic variable must be $v'_2 v_2 \cos 2(\Psi'_2 - \Psi_2)$ because the only groups of harmonics that add to zero are $2' + (-2) = 0$, which is included twice. It is clear from the above equation that changing the position of 2 and $2'$ relative to the bar allows us either to create a moment sensitive to the covariance in the magnitudes of V_2 and V'_2, or to create a moment sensitive to the variance of the dot product of these vectors., $V'_2 V^*_2$, a quantity that is strongly sensitive to $\Psi'_2 - \Psi_2$, the difference in angle between the differential event plane, and the reference particle event plane.[1] Choices of this nature have been instrumental in isolating dependence on the angular fluctuations between event-plane angles for reference particles and POI [44,46,73,90,91] or to exclude event-plane dependence and analyze exclusively the magnitude fluctuations between POIs and reference v_n and v'_n coefficients [68].

D. Relation to existing observables

Common observables measured in heavy ion collisions are two-particle correlations used to estimate v_n and v'_n. These can be easily recovered within the framework of POI correlators and raw moments defined in this section. The two-particle correlation to measure v_n is defined in Ref. [6] as

$$(v_n\{2\})^2 = \langle\langle 2\rangle\rangle_{n|n} = \langle v^2_n\rangle, \tag{37}$$

where $\{2\}$ indicates a two-particle estimate for v_n. Likewise, a two-particle estimate for v'_n can be defined using a similar two-particle correlation, but one reference particle index with a POI index:

$$v'_n\{2\} = \frac{\langle\langle 1'+1\rangle\rangle_{n'|n}}{\sqrt{\langle\langle 2\rangle\rangle_{n|n}}} = \frac{\langle V'_n V^*_n\rangle}{\sqrt{\langle v^2_n\rangle}} \approx \langle v'_n\rangle, \tag{38}$$

$$v'_n\{2\}v_n\{2\} = \langle\langle 1'+1\rangle\rangle_{n'|n} = \langle V'_n V^*_n\rangle, \tag{39}$$

where the approximation in Eq. (38) is valid in the limit that $\Psi'_n = \Psi_n$. More complicated v'_n and v_n cumulants require sums of more specific correlations and will be used to motivate more general cumulants in Sec. III B.

[1]Exploiting this subtlety has given rise to the observable $A^f_n = \langle v^2_n v'^2_n \cos 2n(\Psi'_n - \Psi_n)\rangle / \langle v^2_2 v'^2_2\rangle$ in Ref. [68]. Generalizing this practice can be a useful alternative normalization scheme when trying to isolate decorrelations between reference and differential event planes.

III. DIFFERENTIAL MOMENTS AND CUMULANTS

A. Fluctuations

An important reason for studying fluctuations in v_n and related observables in heavy ion collisions is to measure event-by-event fluctuations in the initial geometry of the collision [75] and in the shape of the resulting QGP. Measuring the fluctuations of v'_n for various probes, and relating them to fluctuations in v_n will provide information about the sensitivity of these probes to the path length they travel through the QGP—and how fluctuations in their path length produce fluctuations in their own abundance, angular distribution, and energy.

If there were no statistical limitations, one could simply analyze the similarities in the probability distributions $p(v_n)$ and $p(v'_n)$. For the azimuthal anisotropy of reference particles, the distribution $p(v_n)$ has been studied and compared with existing parametrizations for initial-state anisotropy [79]. In general, we expect the event-by-event distribution for differential azimuthal anisotropies, $p(v'_n)$ to be different from the event-by-event distribution of reference Fourier coefficients $p(v_n)$: we expect both the means and relative fluctuations to be different between $p(v_n)$ and $p(v'_n)$. While there may not be sufficient particles of interest to approximate an event-by-event distribution for $p(v'_n)$, measuring raw moments constrain the distribution $p(v'_n)$ and can also measure angle and magnitude correlations between v'_n and v_n [68]. Both of these correlations can be used to understand the path-length dependence of energy loss.

While the raw moments as described above can help approximate correlations and constrain distributions, they can be combined into functions with more interpretive value. A measurement of the raw moment $\mu_{\nu_1,\ldots,\nu_n}(X_1,\ldots,X_n)$ still retains some dependence on smaller moments $\mu_{\tilde{\nu}_1,\ldots,\tilde{\nu}_n}(X_1,\ldots,X_n)$, where $\tilde{\nu}_i \leqslant \nu_i$. To isolate the genuine contribution of order ν_i for each stochastic variables X_i to a joint distribution $F(X_1,\ldots,X_n)$, we use functions of the raw moments to construct cumulants according to the generating functions used in Refs. [6,72], and the cumulant formalism introduced in Ref. [73]. Additionally, we introduce central moments that can be used to discern correlations between the "spread" or distance from the mean between different stochastic variables.

B. Generating function cumulants

In Refs. [5,6], a set of cumulants for estimating v_n and its fluctuations are defined using a unique generating function. Aside from the prevalence of these cumulants in literature, and well understood measurements [16,66,90], a benefit of using these "generating function cumulants" is that they estimate the mean and fluctuations for various powers of v_n, and v'_n, while suppressing sensitivity to nonflow contributions. These cumulants are derived from a generating function following the formalism outlined by Kubo in Ref. [92], although they do not meet some of the convenient properties for cumulants specified in Ref. [92] (specifically the properties of statistical independence, reduction, semi-invariance, and homogeneity [8,73]). However, cumulants derived from these generating functions have been used effectively for the estimation of

v_n, v_n', and their fluctuations in experimental and theoretical contexts [66,75,78]. Moreover, in Ref. [73], the authors recommend the continued study of generating function cumulants, under the understanding that these quantities are not true statistical cumulants according to the formalism introduced in Ref. [92]. The evaluation of generating functions using a higher number of POI angles as defined in this section can allow for an easy comparison with existing results for reference particles. Finally, the generating function cumulants defined here are the primary quantities which we discuss that can be interchanged to approximate v_n' directly, or approximate correlations between v_n' and v_n.

The cumulants of any distribution are defined as coefficients in the Taylor expansion of the multivariate moment generating function around $z = 0$, for some complex variable z. For azimuthal angles in heavy ion collisions, the generating function below [93] was used as a moment-generating function to obtain cumulants:

$$G_n(z) = \left\langle \prod_{j=1}^{M} (1 + z^* e^{in\phi_j} + z e^{-in\phi_j}) \right\rangle, \tag{40}$$

where ϕ_j corresponds to the jth reference particle angle of M reference particles produced in an event. Taking the event-by-event average $\langle G_n(z)\rangle$:

$$\langle G_n(z)\rangle = \sum_{k=0}^{M/2} \frac{|z|^{2k}}{M^{2k}} \binom{M}{k} \binom{M-k}{k} \times \langle\langle e^{in(\phi_1+\cdots+\phi_k-\phi_{k+1}-\cdots-\phi_{2k})}\rangle\rangle, \tag{41}$$

generates every possible combination of correlations between reference particle angles of a given order when expanded.

The cumulant-generating function $\mathcal{C}_n(z)$ and differential cumulant-generating function $\mathcal{D}_{p/n}(z)$ are defined in reference to $G_n(z)$ [5,6]. $\mathcal{C}_n(z)$ is used to estimate univariate generating function cumulants for v_n at different powers:

$$C_n(z) = \ln\left(\langle G_n(z)\rangle\right), \tag{42}$$

where, for fixed multiplicity M in the limit of $M \gg 1$, we see that the above yields approximately $\mathcal{C}_n(z) \equiv M[\langle\langle G_n(z)\rangle\rangle^{1/M} - 1]$, as described in Refs. [6,85]. The cumulant-generating function approximating the natural log of the moment-generating function is a result fundamental to the definition of multivariate cumulants. In the differential regime, $\mathcal{D}_{p/n}(z)$ is used to estimate v_p' for particles of interest:

$$\mathcal{D}_{p/n}(z) \equiv \frac{\langle e^{ip\psi} G_n(z)\rangle}{\langle G_n(z)\rangle}, \tag{43}$$

where we see that $\mathcal{D}_{p/n}(z)$ represents the event-by-event correlation between POI angles and $G_n(z)$. To calculate the cumulants, the generating functions can be decomposed into a power series of correlators and powers of z and z^*, as shown below:

$$\mathcal{C}_n(z) = \sum_{k,m} \frac{z^{*k} z^m}{k!m!} \langle\langle k+m\rangle\rangle_{\underbrace{n,\ldots,n}_{k}|\underbrace{n,\ldots,n}_{m}}. \tag{44}$$

Likewise, differential cumulants can be expressed as a power series in z and z^*, with slightly different angular correlations:

$$\mathcal{D}_{p/n}(z) = \sum_{k,m} \frac{z^{*k} z^m}{k!m!} \langle\langle 1' + k + m\rangle\rangle_{p',\underbrace{n,\ldots,n}_{k}|\underbrace{n,\ldots,n}_{m}}, \tag{45}$$

where the averaged quantities $\langle\langle\rangle\rangle$ are simply raw moments defined in Eq. (21). Correlators where $k \neq m$ are isotropic and average to zero for $\mathcal{C}_n(z)$, so we only consider the case when $m = k$. Matching orders in $|z|^2$ between Eqs. (42) and (44) give the expression for the cumulant $c_n\{2k\}$ in terms of the correlators defined in the previous section. For example, we can see easily from Eq. (44) that the correlator of first order in $|z|^2$ is simply

$$c_n\{2\} = \langle\langle e^{in(\phi_1-\phi_2)}\rangle\rangle = \langle v_n^2\rangle, \tag{46}$$

which is the two-particle correlation from Eq. (37). The representations of generating function cumulants for up to 14 particles ($k = m = 7$) can be found in Ref. [72]. Likewise, differential correlators where $m - k \neq p$ also average to zero. Again, by matching orders of $|z|^2$ between Eqs. (43) and (45) it is possible to construct the "differential cumulants" $d_n\{m + k + 1\}$.

Here, we propose cumulant generating functions for differential cumulants with two, and arbitrary POI dependence, respectively. These generating functions produce cumulants with a dependence on angle tuples with a set number of POI angles and an arbitrary number of reference particle angles. The two POI differential cumulant $\mathcal{F}_{p/n}(z)$ is defined similarly to $\mathcal{D}_{p/n}(z)$ but correlates $G_n(z)$ with a pair of POI angles,

$$\mathcal{F}_{p/n}(z) = \frac{\langle e^{ip(\psi_1-\psi_2)} G_n(z)\rangle}{\langle G_n(z)\rangle}, \tag{47}$$

where ψ_1, ψ_2 are POI angles with index 1 and 2, similar to ϕ_1, ϕ_2 from Eq. (41). For generating function cumulants that correlate arbitrary numbers of POI and reference angles both positively k' and negatively m', we can define a more general differential cumulant generating function:

$$\mathcal{H}_{p/n}^{k',m'}(z) = \frac{\langle e^{ip(\psi_1+\ldots+\psi_{k'}'-\psi_{k'+1}-\ldots-\psi_{k'+m'})} G_n(z)\rangle}{\langle G_n(z)\rangle}, \tag{48}$$

where k' and m' continue to indicate the number of separate positive and negative indices for POI angles. Note that $\mathcal{F}_{p/n}(z)$ is just a special case of $\mathcal{H}_{p/n}^{1,1}(z)$. These multidifferential cumulants can once again be expanded in terms of joint differential moments, just as was seen for $\mathcal{D}_n(z)$ and $\mathcal{C}_n(z)$:

$$\mathcal{F}_{p/n}(z) \equiv \sum_{k,m} \frac{z^{*k} z^m}{k!m!} \langle\langle 2' + k + m\rangle\rangle_{p',\underbrace{n,\ldots,n}_{k}|p',\underbrace{n,\ldots,n}_{m}}, \tag{49}$$

and

$$\mathcal{H}_{p/n}^{k',m'}(z) \equiv \sum_{k,m} \frac{z^{*k} z^m}{k!m!} \langle\langle k' + m' + k + m\rangle\rangle_{\underbrace{p',\ldots,p'}_{k'},\underbrace{n,\ldots,n}_{k}|\underbrace{p',\ldots,p'}_{m'},\underbrace{n,\ldots,n}_{m}}, \tag{50}$$

where we can see each raw moment used in the definitions of $\mathcal{F}_{p/n}(z)$ correlates pairs of POI angles, and the raw moments defining $\mathcal{H}^{k',m'}_{p/n}(z)$ rely on $k'+m'$ POI angles. By matching orders in $|z^2|$, just as was done for $\mathcal{C}_n(z)$ and $\mathcal{D}_{p/n}(z)$, we can obtain an expression for the cumulants $f_{p/n}\{k'+m'+k+m\}$ and $h^{k',m'}_{p/n}\{k'+m'+k+m\}$ generated by $\mathcal{F}_n(z)$ and $\mathcal{H}^{k',m'}_{p/n}(z)$, using correlators. Again, for $\mathcal{F}_n(z)$ we require that $k=m$. For $\mathcal{H}^{k',m'}_{n}(z)$ we require that $k-m=m'-k'$ to ensure that the quantity is not isotropic, because in general we will not have the same number of positively and negatively correlated POI angles. The first few orders are shown below for $\mathcal{F}_{n/n}(z)$:

$$f_n\{2\} = \langle\langle 2'\rangle\rangle_{n'|n'}, \tag{51}$$

where the two-particle generating function cumulant is simply a two-particle correlation using only POI. The higher orders are more complicated:

$$f_n\{4\} = \langle\langle 2'+2\rangle\rangle_{n,n'|n,n'} - \langle\langle 1'+1'\rangle\rangle_{n'|n'}\langle\langle 1+1\rangle\rangle_{n|n}. \tag{52}$$

and

$$\begin{aligned} f_n\{6\} = {}& \langle\langle 2'+4\rangle\rangle_{n',n,n,|n,n,n'} - 4\langle\langle 2'+2\rangle\rangle_{n',n|n,n'}\langle\langle 1+1\rangle\rangle_{n|n} \\ & + 4\langle\langle 1'+1'\rangle\rangle_{n'|n'}\langle\langle 1+1\rangle\rangle^2_{n|n} \\ & - \langle\langle 1'+1'\rangle\rangle_{n'|n'}\langle\langle 2+2\rangle\rangle_{n,n|n,n}. \end{aligned} \tag{53}$$

We can see that, in each equation, each term depends on a correlator with two POI angles. Also note that for each term, the number of particles correlated in each correlator sum to the order: for $f_n\{6\}$, each term has either a six particle correlator, a product of a two-particle correlator and four-particle correlator, or a product of three two-particle correlators. This is consistent with existing cumulants for reference flow and differential flow using up to one POI [6]. Given k' and m' positive and negative separate indices for POI angles, we can evaluate $\mathcal{H}^{k',m'}_{p/n}(z)$ in much the same way, by matching terms in the power series expansion of Eqs. (48) and (50).

The coefficients found when evaluating $f_n\{k'+m'+k+m\}$ at different orders in Eq. (51) correspond exactly to coefficients in the multivariate harmonic cumulants in Ref. [72], which use generating function cumulants to correlate v_n and v_m. An area for further study would be to construct generating function cumulants to measure arbitrary powers of v_n and v'_n in different harmonics, using the full generality of the correlators defined in Eq. (21).

Now that the evaluation of generating function cumulants $f_{p/n}\{k'+m'+k+m\}$ or $h^{k',m'}_{p/n}\{k'+m'+k+m\}$ at each order in terms of raw moments is clear, we show how $f_{p/n}\{k'+m'+k+m\}$ and $h^{k',m'}_{p/n}\{k'+m'+k+m\}$ relate to v_n and v'_n, and their fluctuations. Using the method in Refs. [6,94], we calculate the contribution from v_n and v'_p to the values of the $f_n\{k'+m'+k+m\}$ cumulants. First, we define notation, in which $\langle x\rangle|_{\Phi_n}$ indicates the average of x in all events with reaction plane angle Φ_n. While the plane cannot be measured, we simply used it as a placeholder, varying between zero and 2π:

$$\langle x\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi/n}\langle x\rangle\Big|_{\Phi_n} d\Phi_n, \tag{54}$$

indicating the average value of quantity x at angle Φ_n before integrating around the entire transverse plane [24]. We now use this convention to express v_n and $v'_{p/n}$ in the equation below:[2]

$$\langle e^{in\phi}\rangle|_{\Phi_n} = \langle e^{in(\phi-\Phi_n)}\rangle|_{\Phi_n} e^{in\Phi_n} = v_n e^{in\Phi_n}, \tag{55}$$

$$\langle e^{ip\psi}\rangle|_{\Phi_n} = \langle e^{ip(\psi-\Phi_n)}\rangle|_{\Phi_n} e^{in\Phi_n} = v'_n e^{in\Phi_n}, \tag{56}$$

where we consider integrating around the event plane to be approximately equivalent to an average taken over all particle angles. Using this notation, we examine $\langle e^{ip(\psi_1-\psi_2)}G_n(z)\rangle$, and $\langle G_n(z)\rangle$ to evaluate $\mathcal{F}_{p/n}(z)$ with the same method as [94], by decomposing $\langle\rangle$ into an integral:

$$\begin{aligned} \mathcal{F}_{p/n}(z) &= \frac{\langle e^{ip(\psi_1-\psi_2)}G_n(z)\rangle}{\langle G_n(z)\rangle} \\ &= \frac{\frac{1}{2\pi}\int_0^{2\pi}\langle e^{ip(\psi_1-\psi_2)}G_n(z)\rangle|_{\Phi_n} d\Phi_n}{\langle G_n(z)\rangle}. \end{aligned} \tag{57}$$

We now use Eq. (54) to substitute in $\pm e^{ip\Psi_n}$, which immediately cancels, allowing us to remove $|v'_{p/n}|^2$ from the numerator. This shows that the power series expansion in z^2 of $\mathcal{F}_{p/n}$ approximates $|v'_{p/n}|^2$:

$$\begin{aligned} & \frac{\frac{1}{2\pi}\int_0^{2\pi}\langle e^{ip(\psi_1-\psi_2)}G_n(z)\rangle|_{\Phi_n} d\Phi_n}{\langle G_n(z)\rangle} \\ &= \frac{\frac{1}{2\pi}\int_0^{2\pi}\langle e^{ip(\psi_1-\Phi_n+\Phi_n-\psi_2)}G_n(z)\rangle|_{\Phi_n} d\Phi_n}{\langle G_n(z)\rangle}, \end{aligned} \tag{58}$$

$$\begin{aligned} \mathcal{F}_{p/n}(z) &= \frac{\frac{v'_{p/n}v'^*_{p/n}}{2\pi}\int_0^{2\pi}\langle G_n(z)\rangle|_{\Phi'_n} d\Phi'_n}{\langle G_n(z)\rangle} \\ &= \frac{|v'_{p/n}|^2\langle G_n(z)\rangle}{\langle G_n(z)\rangle} = {v'_{p/n}}^2, \end{aligned} \tag{59}$$

where the above displays consistency with cumulant estimates from $\mathcal{H}^{k',m'}_{p/n}(z)$, which will give values of $|{v'}_{p/n}|^{k'+m'}|v_n|^{k+m}$, with varying accuracy depending on the values of k', m', k, m and the multiplicity of particles in the events being used. An explicit calculation of the contribution is performed in Appendix C.

Now that we have established $\mathcal{F}(z) = |v'_{p/n}|^2$, we can see it has no z dependence, indicating that, aside from $f\{2\}$ (the zeroth order z contribution), the higher-order cumulants $f\{2'+2k\}$ *do not scale with* ${v'_n}^2$, and instead measure corrections to $\mathcal{F}_{p/n}$ that come from the correlations between ${v'_n}^2$ and v_n^2. This is not surprising, when considering that $f_{p/n}\{4\}$ is a SC. While $f_{p/n}\{2'+2k\}$ does not scale with ${v'_n}^2$ beyond $f\{2\}$, different scaling behavior can be selected with different values

[2]For this derivation, we use the assumption that $\Psi_n \approx \Psi'_p \approx \Phi_n$: both event planes lie more or less on the reaction plane of the event. This assumption is reasonable for high multiplicity collisions and nontrivial v_n and v'_n values, and one required in the derivation for one POI differential flow cumulants. When this assumption fails, and event planes decorrelate, we will only measure the real part of v'_n projected onto Ψ_n

for k', m', k, and m for $\mathcal{H}_{k,m}^{k',m'}$, which is further explained in Appendix C.

C. Symmetric and asymmetric cumulants

Symmetric and asymmetric cumulants (ASCs) are statistical quantities used to correlate different even powers of v_n and v_m, the azimuthal anisotropies of reference particles at different orders [8,73]. These quantities obey a number of fundamental properties as defined in Refs. [73,92], and, thus, meet the rigorous mathematical definition of a cumulant for the variables v_n^2 and v_m^2. The simplest quantity of this form, the SC, is already described in Eq. (26).

Cumulants of a set of stochastic variables $X_1, \ldots, X_n$ isolate the "genuine" correlation between the variables, subtracting correlations between each subset of the stochastic variables, as well as the product of the variables. SC is widely used to measure the correlation between v_n^2 and v_m^2, and ASC extends the framework of SC to measure the correlations between larger sets of Fourier harmonics $v_{n_1}^2 \ldots v_{n_l}^2$, or higher orders of dependence, measuring, for example, the correlations between $(v_n^2)^3$ and v_m^2.

We generalize the framework of ASC and SC to define cumulants for a selection of arbitrary variables $X_1, \ldots, X_n$ as in Sec. II C, which each may contain arbitrarily many POI. We show these cumulants are consistent with the existing ASC and SC and demonstrate how they can be used to interpret correlations and fluctuations in v_n'.

A multivariate cumulant can be expressed as a set of derivatives of the cumulant generating function of a multivariate distribution with probability density $P(X_1, \ldots, X_n)$:

$$C(\xi_1, \ldots, \xi_n) = \ln\left(\int e^{(\xi_1 X_1 + \ldots + \xi_n X_n)} P(X_1, \ldots, X_n) dx_1 \ldots dx_n\right), \quad (60)$$

where $C(\xi_1, \ldots, \xi_n)$ is the generating function of a multivariate distribution, with dummy variables $\xi_1, \ldots, \xi_n$, and stochastic variables $X_1, \ldots, X_n$. Taylor expanding the cumulant generating function around $\xi_1 = \ldots = \xi_n = 0$ yields the cumulants

$$\kappa_{\nu_1,\ldots,\nu_m} = \frac{\partial^{\nu_1}}{\partial \xi_1^{\nu_1}} \cdots \frac{\partial^{\nu_m}}{\partial \xi_m^{\nu_m}} \ln\left\langle \exp\left(\sum_j \xi_j X_j\right)\right\rangle\Bigg|_{\xi_1=\xi_2=\cdots=\xi_m=0}, \quad (61)$$

where the integral in Eq. (60) is replaced with an expectation value.

Moreover, it is shown in Ref. [92] that cumulants can be written as a function of raw moments as follows:

$$\kappa_{1,\ldots,1}(X_1, \ldots, X_m) = \sum_{l=1}^{P_m} (|l|-1)!(-1)^{|l|-1} \sum_{B\in P_m, |B|=l} \left\langle \prod_{i\in B} X_i \right\rangle, \quad (62)$$

where P_m is the set of all partitions[3] of a subset of integers $\{1, \ldots, m\}$. While it is hard to grasp intuitively from the above equation, the cumulant $k_{\nu_1,\ldots,\nu_n}(X_1, \ldots, X_n)$ represents the subtraction of every "smaller" correlation between subsets of variables $X_{i_1}, \ldots X_{i_{\tilde{n}}}$ from the raw moment $\mu_{\nu_1,\ldots,\nu_n}(X_1, \ldots, X_n)$.

Using Eq. (62), the first few cumulants can be calculated easily, although it becomes significantly more difficult at higher orders. We start with $\{1, 2\}$:

$$P_2 = \left\{\underbrace{\{\{1\}, \{2\}\}}_{B_1}, \underbrace{\{1, 2\}}_{B_2}\right\}, \quad (63)$$

$$\kappa_{1,1}(X, Y) = \langle XY\rangle - \langle X\rangle\langle Y\rangle, \quad (64)$$

and then form $\{1, 2, 3\}$:

$$P_3 = \left\{\underbrace{\{\{1\}, \{2\}, \{3\}\}}_{B_1}, \underbrace{\{1, 2\}\{3\}}_{B_2}, \underbrace{\{1, 3\}\{2\}}_{B_3}, \underbrace{\{2, 3\}\{1\}}_{B_4}, \underbrace{\{1, 2, 3\}}_{B_5}\right\}, \quad (65)$$

$$\kappa_{1,1,1}(X, Y, Z) = \langle XYZ\rangle - \langle XY\rangle\langle Z\rangle - \langle XZ\rangle\langle Y\rangle - \langle ZY\rangle\langle X\rangle + 2\langle X\rangle\langle Y\rangle\langle Z\rangle, \quad (66)$$

where $\kappa_{1,1}$ is an example of a SC and $\kappa_{1,1,1}$ is an example of a ASC. In these equations we first compute the possible partitions that can be created with the sets $\{1, 2\}$ and $\{1, 2, 3\}$. Then we evaluate the summation in Eq. (62). The above relations can quickly be used to create cumulants of higher orders for specific variables because cumulants demonstrate reduction, allowing the variables in Eq. (62) to be interchanged and duplicated, e.g.,

$$\kappa_{1\ldots1}\left(\underbrace{X, \ldots, X}_{j}, \underbrace{Y, \ldots, Y}_{k}\right) = \kappa_{j,k}(X, Y), \quad (67)$$

as was shown in Refs. [73,92].

While Eq. (62) expresses how to define a cumulant with first-order dependence on n unique variables, we can see from this equation that if they are set equal $X_1 = X_2 = \ldots = X_n$, we will obtain an nth order cumulant in X_n. This is demonstrated for the cumulant $k_2(X)$:

$$\kappa_{1,1}(X, X) = \kappa_2(X) = \langle X^2\rangle - \langle X\rangle^2, \quad (68)$$

which can be compared with the result in Eq. (63). The coefficients obtained from the method for evaluating cumulants detailed in Eqs. (62)–(68) replicate the coefficients in the

[3]A partition is a way to divide a set of m unique elements $\{1, \ldots, m\}$ into subsets whose union contains the entire set $\{1, \ldots, m\}$ but which does not use the same element twice. Then index i is one "block" of the partition $B \in P_m$, a subset of $\{1, \ldots, m\}$. For example, given the set $\{1, 2, 3, 4\}$, we have $\{\{2, 3\}, \{1\}, \{4\}\}$ and $\{\{1, 3, 4\}, \{2\}\}$ are partitions, but $\{\{2\}, \{2, 3, 4\}\}$ is not a partition because 1 is not included in any subset, and 2 is used twice. For the partition $\{\{1, 3, 4\}, \{2\}\}$, the blocks $i \in B$ a partition are simply the elements: $\{2\}$ and $\{1, 3, 4\}$. The expression $|P_m|$ refers to the cardinality of P_m; the total number of partitions for the set $\{1, \ldots, m\}$, while $|B|$ indicates the number of blocks in each partition $B \in P_m$. The summation $\sum B \in P_m, |B| = l$ means that the summation is over all blocks in the partition which have l elements.

asymmetric cumulants defined in Ref. [73]. We can simply replace X, Y, Z with the same stochastic variables we could include in raw moments as defined in Sec. II C. It has been shown [92] that multivariate cumulants like those defined here can always be written as a function of raw multivariate moments. Furthermore, we can see that $\kappa_{v_1,\ldots,v_m}$ can be written as a function of raw moments of the same stochastic variables $\mu_{v_1,\ldots,v_m}$. To demonstrate how each cumulant can be written as a sum of raw moments, we present some lower order bivariate cumulants of v_n^2 and $v_n'^2$, and a trivariate cumulant in Eqs. (69)–(71), and Eq. (72), respectively:

$$\kappa_{1,1}\left(v_n^2, v_n'^2\right) = \left\langle v_n^2 v_n'^2\right\rangle - \left\langle v_n^2\right\rangle\left\langle v_n'^2\right\rangle, \tag{69}$$

$$\begin{aligned}\kappa_{2,1}\left(v_n^2, v_n'^2\right) =& \left\langle v_n^4 v_n'^2\right\rangle - \left\langle v_n^4\right\rangle\left\langle v_n'^2\right\rangle \\ &- 2\left\langle v_n^2 v_n'^2\right\rangle\left\langle v_n^2\right\rangle + 2\left\langle v_n^2\right\rangle^2\left\langle v_n'^2\right\rangle,\end{aligned} \tag{70}$$

$$\begin{aligned}\kappa_{3,1}\left(v_n^2, v_n'^2\right) =& \left\langle v_n^6 v_n'^2\right\rangle - \left\langle v_n^6\right\rangle\left\langle v_n'^2\right\rangle - 3\left\langle v_n^2 v_n'^2\right\rangle\left\langle v_n^4\right\rangle \\ &- 3\left\langle v_n^4 v_n'^2\right\rangle\left\langle v_n^2\right\rangle + 6\left\langle v_n^4\right\rangle\left\langle v_n^2\right\rangle\left\langle v_n'^2\right\rangle \\ &+ 6\left\langle v_n^2 v_n'^2\right\rangle\left\langle v_n^2\right\rangle^2 - 6\left\langle v_n^2\right\rangle^3\left\langle v_n'^2\right\rangle,\end{aligned} \tag{71}$$

$$\begin{aligned}\kappa_{1,1,1}\left(v_n^2, v_n'^2, v_m^2\right) =& \left\langle v_n^2 v_m^2 v_n'^2\right\rangle - \left\langle v_n^2 v_m^2\right\rangle\left\langle v_n'^2\right\rangle - \left\langle v_m^2 v_n'^2\right\rangle\left\langle v_n^2\right\rangle \\ &- \left\langle v_n^2 v_n'^2\right\rangle\left\langle v_m^2\right\rangle + 2\left\langle v_n^2\right\rangle\left\langle v_m^2\right\rangle\left\langle v_n'^2\right\rangle,\end{aligned} \tag{72}$$

where, using Eq. (21), we can rewrite each average as expressed above $\langle v_n^{2a}(v'_n)^{2b}v_m^{2c}\rangle = \langle\langle 2a' + 2b + 2c\rangle\rangle_{n',n,m|n',n,m}$. Unsurprisingly, the bivariate asymmetric cumulant of first order in v_n^2 and $v_n'^2$, $\kappa_{1,1}(v_n^2, v_n'^2)$, forms a symmetric cumulant, which has been well studied in Ref. [8].

Since cumulants can be written as sums of products of raw moments, they can also be be normalized according to the scheme introduced in Sec. II C:

$$N\kappa_{v_1,\ldots,v_n}(X_1\ldots X_n) = \frac{\kappa_{v_1,\ldots,v_n}(X_1\ldots X_n)}{\prod_i \langle X_i\rangle_i^v}, \tag{73}$$

and as a result, normalized cumulants $N\kappa_{v_1,\ldots,v_m}$ can be written a sum of normalized moments $N\mu_{v_1,\ldots,v_m}$. We show the normalization of $\kappa_{2,1}(v_n^2, v_n'^2)$ using the stochastic variables $\langle v_n^2\rangle$ and $\langle v_n'^2\rangle$, canceling redundant terms in each fraction:

$$\begin{aligned}N\kappa_{2,1}\left(v_n^2, v_n'^2\right) &= \frac{\left\langle v_n^4 v_n'^2\right\rangle}{\left\langle v_n^2\right\rangle^2\left\langle v_n'^2\right\rangle} - \frac{\left\langle v_n^4\right\rangle}{\left\langle v_n^2\right\rangle^2} - 2\frac{\left\langle v_n^2 v_n'^2\right\rangle}{\left\langle v_n^2\right\rangle\left\langle v_n'^2\right\rangle} + 2, \qquad (74)\\ &= N\mu_{2,1}\left(v_n^2, v_n'^2\right) - N\mu_2\left(v_n^2\right) \\ &\quad - 2N\mu_{1,1}\left(v_n^2, v_n'^2\right) + 2, \qquad (75)\end{aligned}$$

where it can be seen that we obtain a function of normalized moments, up to a constant +2.

D. Central moments

Like cumulants, central moments also define measures of fluctuation of the distributions of X_i. Central moments in general do not obey reduction, a property of cumulants, but they are more straightforward to interpret: they correlate the distance from the mean ("spread") in a set of stochastic variables with the spread in each other stochastic variable. Their interpretation has been studied often, as kurtosis, variance, and skewness are all central moments which are often used to describe properties of probability distributions.

Given nontrivial stochastic variables $X_1, \ldots, X_n$ as described in Sec. II C, requiring each X_i to be some correlator $\langle k' + m' + k + m\rangle$ for which the harmonics that are correlated and anticorrelated cancel, the central moments for $X_1 \ldots X_n$, are defined as follows:

$$\tilde{\mu}_{v_1\ldots,v_n}(X_1\ldots X_n) \equiv \left\langle \prod_{i=1}^{n} (X_i - \langle X_i\rangle)^{v_i}\right\rangle. \tag{76}$$

In general, these central moments can be expanded to functions of raw moments by using the multinomial theorem and the fact that $\langle X + c\rangle = \langle X\rangle + \langle c\rangle$. Applying the procedure to Eq. (76) gives the following:

$$\begin{aligned}\tilde{\mu}_{v_1,\ldots,v_n} &= \langle (X_1 - \langle X_1\rangle)^{v_1}\cdots(X_n - \langle X_n\rangle)^{v_n}\rangle \qquad (77)\\ &= \sum_{s=0}^{v}\left(\sum_{s_1+\ldots+s_n=s}\binom{s}{s_1,\ldots,s_n}\langle X_1\rangle^{s_1}\right. \\ &\quad \left.\times\cdots\langle X_n\rangle^{s_n}\left\langle X_1^{v_1-s_1}\cdots X_n^{v_n-s_n}\right\rangle\right), \qquad (78)\end{aligned}$$

where we define $\sum_i v_i = v$. The summation $\sum_{s_1+\ldots+s_n=s}$ is taken over every combination of positive integers $s_1, \ldots, s_n$ such that $s_1 + \cdots + s_n = s$, and $s_i \leqslant v_i$ for all $1 \leqslant i \leqslant n$. The result in Eq. (78) is obtained by expanding $\prod_i (X_i - \langle X_i\rangle)^{v_i}$ using multinomial theorem, before averaging the resulting quantity. In Sec. II, we explicitly write $\tilde{\mu}$ for $v = 2$, 3, and 4, for arbitrary random variables X, Y, Z, W. Since Eq. (78) allows us to express $\tilde{\mu}$ as a function of raw moments $\mu(X_i)$. Following the same methods used for raw moments and cumulants, we seek to evaluate correlations by normalizing these central moments. Since each central moment can be written as a function of raw moments, we can use the normalization scheme from Eq. (25) and normalize a central moment by dividing by the average of each stochastic variable X_i that it describes,[4]

$$N\tilde{\mu}_{v_1\ldots,v_n}(X_1\ldots X_n) = \frac{\tilde{\mu}_{v_1\ldots,v_n}(X_1\ldots X_n)}{\prod_{i=1}^{n}\langle X_i\rangle^{v_i}}, \tag{79}$$

[4]Note that, conventionally, central moments are normalized not by the product of their stochastic variables, but by the standard deviation of their stochastic variables:

$$N\tilde{\mu}_{v_1,\ldots,v_n}(X_1,\ldots,X_n) \equiv \frac{\mu_{v_1,\ldots,v_n}(X_1,\ldots,X_n)}{\prod_{i=1}^{n}\left(\sqrt{\langle X_i^2\rangle - \langle X_i\rangle^2}\right)^{v_i}}.$$

While this practice is well motivated statistically, we prefer to introduce normalization schemes that ensure that even $\tilde{\mu}_{1,1}(v_n^2, v_m^2) = SC(v_n^2, v_m^2)$ can be normalized in a way that is consistent with existing normalizations for SC and related variables. Additionally, the possibility of measuring a raw moment including stochastic variable $X_i : \mu_{\ldots v_i\ldots}(\ldots X_i \ldots)$ does not guarantee the capacity to accurately measure its standard deviation, a quantity that generally has greater dependence on each stochastic variable $\langle X_i\rangle^2$.

where the stochastic variables X_i and their associated dependencies ν_i are the same as in the definition of central moments in Eq. (76).

As an example of central moments, we express the multivariate central moments of order two in $V_n'V_n^*$, and order $r = 0, 1, 2$ in v_n^2:

$$\tilde{\mu}_{2,r}\big(V_n'V_n^*, v_n^2\big) = \big\langle (V_n'V_n^* - \langle V_n'V_n^*\rangle)^2 \big(v_n^2 - \langle v_n^2\rangle\big)^r \big\rangle, \quad (80)$$

$$\begin{aligned}
\tilde{\mu}_2(V_n'V_n^*) &= \langle (V_n'V_n^*)\rangle - \langle V_n'V_n^*\rangle^2, \\
\tilde{\mu}_{2,1}\big(V_n'V_n^*, v_n^2\big) &= \big\langle (V_n'V_n^*)^2 v_n^2\big\rangle - \big\langle v_n^2\big\rangle \langle (V_n'V_n^*)^2\rangle \\
&\quad - 2\langle V_n'V_n^*\rangle \big\langle V_n'V_n^* v_n^2\big\rangle + 2\big\langle v_n^2\big\rangle \langle V_n'V_n^*\rangle^2, \\
\tilde{\mu}_{2,2}\big(V_n'V_n^*, v_n^2\big) &= \big\langle (V_n'V_n^*)^2 v_n^4\big\rangle - 2\big\langle v_n^2\big\rangle \big\langle (V_n'V_n^*)^2 v_n^2\big\rangle \\
&\quad - 2\langle V_n'V_n^*\rangle \big\langle (V_n'V_n^*) v_n^4\big\rangle + \big\langle v_n^2\big\rangle^2 \big\langle (V_n'V_n^*)^2\big\rangle \\
&\quad + 4\langle V_n'V_n^*\rangle \big\langle v_n^2\big\rangle \big\langle (V_n'V_n^*) v_n^2\big\rangle + \langle V_n'V_n^*\rangle^2 \big\langle v_n^4\big\rangle \\
&\quad - 3\langle V_n'V_n^*\rangle^2 \big\langle v_n^2\big\rangle^2, \qquad (81)
\end{aligned}$$

where the above quantities correspond to the variance of $V_n'V_n^*$ in the $r = 0$ case, the extent to which a deviation from the mean of v_n^2 is generally accompanied by a "squared" deviation from the mean of $V_n'V_n^*$, in the $r = 1$ case. Finally, in the $r = 2$ case, $\tilde{\mu}(V_n'V_n^*, v_n^2)_{2,2}$ represents the extent to which a squared deviation from the mean of v_n^2 is accompanied by a squared deviation from the mean in $V_n'V_n^*$.

While a measurement of a cumulant provides useful information about the various correlations between each stochastic variable, a measurement of $\tilde{\mu}$ by definition expresses how strongly a departure from the mean $\langle X_i\rangle$ is correlated to departures from the mean for other stochastic variables. This can be seen in the example above with $\tilde{\mu}_{2,1}$ and is a more traditional measure of fluctuation. While cumulants satisfy more mathematical properties, central moments also by definition include some mathematical properties that we outline in Appendix D. Additionally, in the univariate case, central moments like skewness and variance have already been used to study fluctuations in the distribution of v_n [29,63].

IV. FUNCTIONS OF THE MOMENTS (Γ, ζ)

A. Motivation for Γ, ζ

When studying fluctuations in differential azimuthal anisotropy coefficients v_n', it can be helpful to make a comparison to reference azimuthal anisotropy fluctuations in v_n to understand their relative magnitudes. Since each central moment, cumulant, and generating function cumulant defined in this paper can be represented as a sum of products of raw moments, the difference between any cumulants or central moments that require the same number of particles in their largest correlator can be rewritten as a function of differences between raw moments that require the same number of particles. Central moments, and cumulants with order $\nu \leqslant 3$ are unique in that they can always be expressed as a product of correlations and the averages of stochastic variables, as expressed in Eqs. (77) and (78). This means, when normalized, they can always be written exclusively as a sum of normalized moments and constants. The difference between any two central moments that require the same number of particles can likewise be decomposed into the differences of normalized raw moments, and possibly constants.

One example of this result comes from the comparison of two normalized six-particle central moments, of which one relies on six reference particles, and the other relies on four reference particles and two POIs:

$$N\tilde{\mu}_3\big(v_n^2\big) = \frac{\langle v_n^6\rangle}{\langle v_n^2\rangle^3} - 3\frac{\langle v_n^4\rangle}{\langle v_n^2\rangle^2} - 2, \quad (82)$$

where the above equation is the normalized third central moment for v_n^2, and the below equation has dependence on $V_n'V_n^*$:

$$\begin{aligned}
N\tilde{\mu}_{2,1}\big(V_n'V_n^*, v_n^2\big) &= \frac{\big\langle v_n^2 (V_n'V_n^*)^2\big\rangle}{\big\langle v_n^2\big\rangle \langle V_n'V_n^*\rangle^2} - 2\frac{\big\langle v_n^2 V_n'V_n^*\big\rangle}{\big\langle v_n^2\big\rangle \langle V_n'V_n\rangle} \\
&\quad - \frac{\langle (V_n'V_n^*)^2\rangle}{\langle V_n'V_n^*\rangle^2} - 2. \qquad (83)
\end{aligned}$$

Clearly, both of the above central moments correlate the n order harmonics of six particles within an event, but $N\tilde{\mu}_{2,1}(V_n'V_n^*, v_n^2)$ correlates deviations in v_n^2 with squared deviations in $V_n'V_n^*$, whereas $N\tilde{\mu}_3(v_n^2)$ simply measures a quantity similar to the skewness of v_n^2. A direct subtraction of these quantities yields the following:

$$\begin{aligned}
N\tilde{\mu}_3(v_n^2) - N\tilde{\mu}_{2,1}(V_n'V_n^*, v_n^2) &= \left(\frac{\langle v_n^6\rangle}{\langle v_n^2\rangle^3} - \frac{\big\langle v_n^2 (V_n'V_n^*)^2\big\rangle}{\langle v_n^2\rangle \langle V_n'V_n^*\rangle^2}\right) \\
&\quad -2\left(\frac{\langle v_n^4\rangle}{\langle v_n^2\rangle^2} - \frac{\big\langle v_n^2 (V_n'V_n^*)\big\rangle}{\langle v_n^2\rangle \langle V_n'V_n^*\rangle}\right) \\
&\quad -\left(\frac{\langle v_n^4\rangle}{\langle v_n^2\rangle^2} - \frac{\langle (V_n'V_n^*)^2\rangle}{\langle V_n'V_n^*\rangle^2}\right) \qquad (84)
\end{aligned}$$

which we can then write as a difference in normalized *raw* moments:

$$\begin{aligned}
&N\tilde{\mu}_3\big(v_n^2\big) - N\tilde{\mu}_{2,1}\big(V_n'V_n^*, v_n^2\big) \\
&\quad = \big[N\mu_3\big(v_n^2\big) - N\mu_{2,1}\big(V_n'V_n^*, v_n^2\big)\big] \\
&\qquad - 2\big[N\mu_2\big(v_n^2\big) - N\mu_{1,1}\big(V_n'V_n^*, v_n^2\big)\big] \\
&\qquad - \big[N\mu_2\big(v_n^2\big) - N\mu_2(V_n'V_n^*)\big]. \qquad (85)
\end{aligned}$$

We find that the difference in the normalized raw moments can easily be grouped by the number of particles required for each raw moment. Parentheses in the above equation separate pairs of raw moments containing six, four, and two particles.

Given two raw moments $\mu_{\nu_1,\ldots,\nu_n}(X_1, \ldots, X_n)$ and $\mu_{\nu_1,\ldots,\nu_n}(Y_1, \ldots, Y_n)$ where $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are two sets of nontrivial stochastic variables described in Sec. II C, with the same dependence (coefficient ν_i) for each stochastic variable X_i or Y_i, and differing dependence on v_n', we can determine the correlations between $X_1, \ldots, X_n$, and the correlations between $Y_1, \ldots, Y_n$, as well as remove scaling with $X_i^{\nu_i}$ by simply normalizing the moments. To compare the

normalized raw moments of $X_1, \ldots, X_n$ with $Y_1, \ldots, Y_n$, we introduce Γ and ζ:

$$\Gamma_{v_1 \ldots v_n}(X_1, \ldots X_n; Y_1, \ldots Y_n) \equiv N\mu_{v_1 \ldots v_n}(X_1 \ldots X_n) - N\mu_{v_1 \ldots v_n}(Y_1 \ldots Y_n), \tag{86}$$

$$\zeta_{v_1 \ldots v_n}(X_1, \ldots X_n; Y_1, \ldots Y_n) \equiv \frac{N\mu_{v_1 \ldots v_n}(X_1 \ldots X_n)}{N\mu_{v_1 \ldots v_n}(Y_1 \ldots Y_n)}, \tag{87}$$

where Γ is the *difference* between normalized moments with the same dependence on $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$, and ζ is the *ratio* between normalized moments with the same dependence on $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$. Since fluctuations in stochastic variables $X_1, \ldots X_n$ and $Y_1, \ldots, Y_n$ are measured by the normalized central moments, evaluating differences in central moments can describe the relative magnitude of fluctuations in $Y_1, \ldots, Y_n$ relative to $X_1, \ldots, X_n$. By decomposing the central moments using the methods above, we can determine how much of the difference in fluctuations between $Y_1, \ldots, Y_n$ and $X_1, \ldots, X_n$ comes from correlations involving a specific number of particles. In the example in Eqs. (82)–(85) we can compare how two-particle correlations, four-particle correlations, and six-particle correlations all contribute to the difference between fluctuations isolated by $N\tilde{\mu}_{2,1}(V_n'V_n^*, v_n^2)$ and $N\tilde{\mu}_3(v_n^2)$.

B. Applications of Γ and ζ

Already, there have been cases where Γ and ζ quantities have been studied. In Ref. [63], the quantity Δ^{sh} was derived and is equivalent to $\Delta^{sh} = \Gamma_{1,1}(v_n^2, v_n^2; V_n'V_n^*, v_n^2)$ in our notation. It was used to evaluate decorrelation and between v_n and v'_n in hydrodynamical models to describe jet energy loss. This quantity can be expressed as

$$\Gamma_{1,1}\left(v_n^2, v_n^2; V_n'V_n^*, v_n^2\right) = \frac{\langle v_n^4 \rangle}{\langle v_n^2 \rangle^2} - \frac{\langle v_n^2 V_n'V_n^* \rangle}{\langle v_n^2 \rangle \langle V_n'V_n^* \rangle}, \tag{88}$$

where we can see that a positive value for $\Gamma_{2;1,1}(v_n^2; V_n'V_n^*, v_n^2)$ obtained in this context indicates that the correlations in magnitude between v_n^2 and $V_n'V_n^*$ are not as great as the contribution of v_n^4 to the distribution $P(v_n^4)$, or essentially that $V_n'V_n^*$ displays suppressed fluctuations around its mean compared with v_n^2. In this instance, using Γ to obtain a *difference* between these quantities allowed the authors to establish an absolute scale to compare the fluctuations in v'_n and v_n for POIs at different values of p_T and using hydrodynamical models with different parameters.

Additionally, ALICE [68] developed the correlation M_n^f that is equivalent to $\zeta_{1,1}(v_n^2, v_n^2; v'^2_n, v_n^2)$ in our notation. It was used to measure correlations between the magnitudes of $v_n(p_T)$ and v_n, and is expressed below. Writing out $\zeta_{1,1}(v_n^2, v_n^2; v'^2_n, v_n^2)$,

$$\zeta_{1,1}\left(v_n^2, v_n^2; v_n'^2, v_n^2\right) = \frac{\langle v_n^2 v_n'^2 \rangle}{\langle v_n^2 \rangle \langle v_n'^2 \rangle} \frac{\langle v_n^2 \rangle^2}{\langle v_n^4 \rangle} \tag{89}$$

where $\zeta_{1,1}(v_n^2, v_n^2; v_n'^2, v_n^2)$ describes the magnitude of correlation between v_n^2 and $v_n'^2$ in reference to the magnitude of the correlation of v_n^2 with itself, comparing the two quantities with a *ratio* in this instance allowed for a cleaner comparison of a wide variety of behavior between theoretical models, as well as the value of the observable over large regions of centrality and p_T.

The authors of Ref. [68] also developed the correlation A_n^f that is equivalent to

$$A_n^f = \frac{\langle\langle 2' + 2 \rangle\rangle_{n',n'|n,n}}{\langle\langle 2' + 2 \rangle\rangle_{n',n|n',n}}$$

in our notation. In a manner similar to that detailed in Eqs. (35) and (36), A_n^f allowed for them to monitor the extent to which the Ψ_2' and Ψ_2 symmetry planes differ for different p_T selections of POIs.

While the motivation for each of these correlations was not the same, they all fit into the formalism presented into this paper and its associated interpretive context. By understanding M_n^f, A_n^f, and Δ^{sh} as variations on the same quantity, we understand more from a comparison of their values. Likewise, we can understand each quantity as a difference between NSC, a normalized covariance, or alternately measuring the four-particle contribution to arbitrary differences in central moments between v_n^2 and $V_n'V_n^*$ and v'^2_n, respectively.

In general, a measurement of Γ corresponds more closely to the difference between central moments, as detailed in Eq. (85), while ζ as a ratio may be more experimentally feasible, providing a more stark numerical difference, with smaller error between the magnitudes of the raw moments it compares. Additionally ζ is useful for understanding the relative magnitude of two raw moments, whereas Γ on its own is more useful for understanding the absolute difference in magnitude between two central moments. Regardless, both observables achieve the same purpose: a direct comparison between the relative fluctuations of $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ for a fixed number of particles.

V. DISCUSSION

Each of the observables we have discussed in this paper has its own unique benefits, which may help to answer a broad class of questions related to measuring fluctuations in v'_n, correlations between powers of v'_n, and other azimuthal anisotropy measurements with rare probes. The study of each rare probe or identified particle imposes different statistical constraints, and motivations such that it is not guaranteed that observables that successfully measure fluctuations in heavy flavor quark v'_n are suitable for the study of p_T-dependent POIs. To help discern between their features, we summarize all the observables we developed here into Table I that discusses their potential uses as well as caveats due to available statistics.

First, $\langle k' + m' + k + m \rangle$, as detailed in Sec. II A can be used in a given event to evaluate the products of V_n and V'_n to different powers, and at different harmonics, by using correlations of azimuthal angles between harmonics. These estimations for the products of V_n and V'_n provide the basis for evaluating event-by-event fluctuations in V_n and V'_n, so they are averaged over events to evaluate raw moments.

TABLE I. A summary of observable quantities defined in this paper, and their prospects for measuring fluctuations in v_n' in heavy ion collisions.

Observables	Explanation
Correlators $\langle k'+m'+k+m\rangle$	These correlators evaluate multiharmonic products of event flow vectors with arbitrary dependence on POI angles. They can be calculated using Q_n vectors.
Raw Moments $\mu_{\nu_1,\ldots,\nu_n}(X_1,\ldots,X_n)$ $\langle\langle k'+m'+k+m\rangle\rangle$	Raw moments evaluate the expectation value of a product of flow vectors $\langle V_{n_1}\cdots V_{n_k+m}V'_{n_{k+m+1}}\cdots V'_{n_{k'+m'+m+k}}\rangle$ by taking a weighted average over many events. Selection of stochastic variables X_i can be done by considering smaller groups of $V_{n_{1_i}}\cdots V'_{n_{ik_i}}$ for which $\sum_j n_{j_i}=0$. This selection of stochastic variables X_i allows for a normalization scheme.
Generating function cumulant $f_n\{k'+m'+k+m\}$, $h_n^{k',m'}\{k'+m'+k+m\}$	Generating function cumulants use $\langle\langle k'+m'+k+m\rangle\rangle$ to approximate $v_n'^{k'+m'}$ using generating functions. These observables are analogs of $c_n\{2k\}$ and $d_{p/n}\{2k\}$, with higher dependence on POIs.
Multivariate cumulants $\kappa_{\nu_1,\ldots,\nu_n}(X_1,\ldots,X_n)$	A cumulant of arbitrary order in each variable. Multivariate cumulants generalize the framework of multiharmonic cumulants introduced in Ref. [73] to a larger set of stochastic variables and POIs. These observables in general obey more statistical properties than generating function cumulants and represent the genuine contribution of a moment at each order to fluctuations by subtracting autocorrelations.
Multivariate central moment $\tilde{\mu}_{\nu_1,\ldots,\nu_n}(X_1,\ldots,X_n)$	Central moments explicitly describe the correlation between many different variables and are traditionally used to calculate the higher-order fluctuations in a distribution. They directly measure correlations of spread around the mean between stochastic variables.
Differences of normalized moments $\Gamma_{\nu_1,\ldots,\nu_n}(X_1,\ldots,X_n;Y_1,\ldots Y_n)$	The differences between two normalized raw moments can be used to decompose the difference between two central moments of the same order, but in different variables X_i and Y_i. Using Γ, we can discern the contribution at each order to a difference between central moments.
Ratios of normalized moments $\zeta_{\nu_1,\ldots,\nu_n}(X_1,\ldots,X_n;Y_1,\ldots Y_n)$	Taking the ratio of two normalized raw moments of different variables at the same order allows a better understanding of the relative correlations of each raw moment and its constituent stochastic variables. These quantities are analogous to Γ but may be easier to experimentally determine because they constitute a ratio.

The joint raw moments with dependence on v_n', $\langle\langle k'+m'+k+m\rangle\rangle$ are necessary to calculate any of the more complex observables described in this paper. In general, a measurement of these quantities, as shown in Sec. II B will provide enough information to additionally calculate and normalize any of the generating function cumulants, asymmetric cumulants, central moments, and by extension Γ and ζ. However, each one of the above observables requires values for different $\langle\langle k'+m'+k+m\rangle\rangle$ moments; a choice of a more complex observable will determine which joint raw moments must be measured.

Generating function cumulants, defined in Sec. III B are unique from the other observables introduced here in that they can both estimate v_n' to different powers, or evaluate correlations between v_n' and v_n using different orders of dependence on azimuthal angles from POIs. Naturally, these quantities are suitable for comparison to existing generating function cumulant measurements of reference v_n using the same number of reference particles $v_n\{2k\}$ and differential cumulant estimates for v_n using only one particle of interest. Understanding how the contribution from the correlation of two or more POI azimuthal angles to a measurement of v_n' differs from the contribution of one POI will provide unique and novel information about the azimuthal anisotropy of POIs.

Asymmetric cumulants do not directly estimate v_n' or any differential Fourier harmonic. However, as shown in Sec. III C they can isolate the genuine correlations between v_n' with v_n, and compare with experimental results, which have recently been studied in a multiharmonic context [95]. Additionally, since the cumulants $k_\nu(X)$, $\nu>2$ of a Gaussian distribution are always zero, asymmetric cumulants can be used to evaluate "deviations from Gaussianity" in univariate distributions, [14].

The central moments we have introduced in Sec. III D bear many similarities to the asymmetric cumulants and are identical at low order: $\kappa_{\nu_1,\ldots,\nu_n}(X_1,\ldots,X_n)=\tilde{\mu}_{\nu_1,\ldots,\nu_n}(X_1,\ldots,X_n)$, for $\sum_i \nu_i \leqslant 3$. Measuring a central moment is useful for comparisons with skewness measured in Refs. [29,96] or variance defined in Refs. [11,63,75]. Additionally, the measurement of a central moment provides knowledge about the correlations between v_n and v_n', and their dispersion around their means to different powers.

In Sec. IV, we showed the difference between the normalized raw moments of two collections of variables, $\Gamma_{\nu_1,\ldots,\nu_n}(X_1,\ldots,X_n;Y_1,\ldots Y_n)$, is more useful for determining the relative contribution at each order from v_n' to a large multivariate distribution of v_n and v_n' at different powers and harmonics. Additionally, we have shown that the difference between any two normalized central moments, including the traditional symmetric cumulant can be written as a linear combination of Γ. This means Γ can determine the differences in correlations between stochastic variables including v_n' and stochastic variables relying only on v_n as a way of comparing the fluctuations in v_n' and v_n.

While ζ, also defined in Sec. IV, cannot decompose the differences between central moments or cumulants (since it is a ratio rather than a difference), ζ is perhaps more easily measurable experimentally because it is normalized to one, whereas Γ can be both negative and positive. Additionally, $\zeta_{\nu_1,\ldots,\nu_n}(X_1,\ldots,X_n;Y_1,\ldots,Y_n)$ can identify the *relative*

magnitude of fluctuations in $Y_1, \ldots, Y_n$ versus fluctuations in $X_1, \ldots, X_n$, whereas $\Gamma_{\nu_1,\ldots,\nu_n}(X_1, \ldots, X_n; Y_1, \ldots, Y_n)$ can only describe their absolute differences.

We anticipate the use of these variables to explore differential phenomena like the event-by-event fluctuations in energy loss for jets traversing the QGP medium and other related questions. By evaluating ζ for sets of stochastic variables involving the azimuthal anisotropy of jets, we can compare the fluctuations of jet energy loss—probed by fluctuations and correlations between the suppression of v_2' and v_2.

VI. CONCLUSION

In this paper we introduce a way to apply existing statistical correlations and observables of azimuthal anisotropies to the study of rare probes and identified particles in heavy ion collisions. To achieve this goal, we allowed for the inclusion of arbitrarily many unique particle of interest (POI) indices in differential correlators to provide an arbitrary dependence on differential azimuthal anisotropies. We lay out the methods to use these multi-POI differential correlators to construct raw and central moments of their underlying distribution(s), generating functions for cumulants, asymmetric cumulants, and ratios or differences of correlations. We also outline various methods for properly normalizing these new observables. Examples are also provided for certain harmonics and POI dependence to guide the reader on how to construct these new observables.

The purpose of these observables is to study rare probes such as jets and/or heavy mesons as well as identified particles like strange hadrons. These new observables will provide a method to constrain the fluctuations in the underlying distribution that is only now possible experimentally in the high-luminosity era of the LHC and sPHENIX [64,65,97,98]. In another work, we studied [99] the feasibility of extracting different type of underlying distributions with these new observables, depending on the type of distribution and the magnitude of the observable calculated. While future studies are warranted to determine the statistics required to obtain precise measurements of these observables, we anticipate that experimental measurements and theoretical calculations of these observables will prove useful in understanding properties of the QGP with identified particles.

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APPENDIX A: NONTRIVIAL OVERLAP BETWEEN POIS AND REFERENCE PARTICLES

In the analysis of the azimuthal anisotropies of jets and high-p_T particles, the high-p_T POIs are rarely included in the set of reference particles. Likewise for the azimuthal anisotropies of charm and bottom mesons, the overlap between charm mesons and reference particles is suppressed simply due to the scarcity of these mesons. However, in general, there are some cases in which it is plausible that scenarios with overlap between POI and reference particles are of interest. In this instance, the actual computation of $\langle k' + m + k + m\rangle$ becomes somewhat more complex. In this Appendix we discuss these cases.

Using a summation over all $m' + k' + m + k$ tuples consisting of $m + k$ reference particle angles, and $m' + k'$ POI angles, we can see that considering the existence of N_q overlapping POI and reference particles, we obtain a different weighting for the value of $\langle k' + m' + k + m\rangle$ than seen in Eq. (17). The summation for a correlator weighs each $m + k$ tuple of POIs and reference particle angles the same but only considers tuples in which each angle is distinct from the others to omit autocorrelations. To calculate the average, we simply complete the primed summation and divide by the number of $m + k$ tuples in which each angle is unique. Since there are N reference particles, N' POIs, and N_q reference particles that are also considered to be POI, we can evaluate the following:

(1) For $k + m$ reference particle indices, the number of unique tuples is simply the number of ways to arrange N particles in $k + m$ slots: $N!/(N - m - k)!$.

(2) Having selected $k + m$ reference particles, we fill $k' + m'$ more slots with POI. If there is no overlap between reference particles and POI, we receive the same result as before; the number of tuples is now $N'!/(N' - k' - m')!$. If there is N_q overlap between POIs and reference particles, then the number of choices is reduced: the computation in step (1) is the same, but now there are only $N' - (k + m)N_q$ POI to "choose" from. Like always, they are placed into $k' + m'$ slots, and, thus, we have

$$\frac{N'! - (k + m)N_q}{(N' - (k + m)N_q - k' - m')!}$$

choices.

Since the total number of unique n-tuples is multiplicative, we obtain the following equation to calculate the correlator assuming a uniform weighting for each particle:

$$\langle k' + m' + k + m\rangle_{n_1,\ldots,n_k,n'_{k+1},\ldots n'_{k+k'}|n_{k+k'+1},\ldots,n_{k+k'+m},n'_{k+k'+m+1}\ldots n'_{k+k'+m+m'}}$$
$$= \frac{(N - m - k)!}{N!}\frac{[N' - (m + k)N_q - k' - m']!}{[N' - (m + k)N_q]!}\sum{}'\left(\prod_{\alpha=1}^{k} e^{in_\alpha\phi_\alpha}\prod_{\beta=k}^{k+k'} e^{in'_\beta\psi_\beta}\prod_{\gamma=k+k'+1}^{k+k'+m} e^{-in_\gamma\phi_\gamma}\prod_{\delta=k+k'+m+1}^{k+k'+m+m'} e^{-in'_\delta\psi_\delta}\right), \quad \text{(A1)}$$

where we obtain the average by dividing by the total number of $m' + k' + m + k$-tuples with unique particles. This result is corroborated by the computations done in Ref. [78] for $\langle 1' + 1\rangle_{n'|n}$ and $\langle 1' + 3\rangle_{n',n|n,n}$.

APPENDIX B: EVALUATING $\langle k' + m\rangle$ USING Q_n VECTORS

When defining $\langle k' + m' + k + m\rangle$ using a Q_n vector method, we obtain a more complex recurrence relation but also can weight each event differently in the calculation. More importantly, we are able to use far less computation time than it would take to make and iterate through $k' + m' + k + m$ tuples of particle angles. The event $\mathrm{Q}_{n,t}$ vector is calculated by summing over all reference particle angles within the event:

$$\mathrm{Q}_{n,t} = \sum_{j=1}^{N} \omega_j^t e^{in\phi_j}, \tag{B1}$$

with a weight ω_j^t for each event. Currently t is just a placeholder, but it gains significance when considering autocorrelations within an event and, in most instances, it is unity. A $p_{n,t}$ vector, the POI analog to a Q_n vector, can be defined as

$$\mathrm{p}_{n,t} = \sum_{k=1}^{N'} \omega_k^t e^{in\psi_k}, \tag{B2}$$

with the weighted sum over POI angles ψ. The overlap vector $q_{n,t}$ can similarly be defined as

$$\mathrm{q}_{n,t} = \sum_{l=1}^{N_q} \omega_l^t e^{in\theta_l}, \tag{B3}$$

with the weighted sum over the angles θ labeled as both POIs and reference particles. Having defined the three different Q_n vectors we will be using, we follow a recursive procedure of the same nature as outlined in Ref. [8].

(1) We begin by writing the $\langle k' + m' + k + m\rangle$ correlator as a ratio of its numerator and denominator, where D is a normalization factor and N contains the actual correlations.

$$\langle k' + m' + k + m\rangle \equiv \frac{N\langle k' + m' + k + m\rangle}{D\langle k' + m' + k + m\rangle}. \tag{B4}$$

(2) We obtain the value of $\langle k' + m' + k + m\rangle$ first by considering an average over all $k' + m' + k + m$ tuples of POIs and reference particle angles in which each index references a unique particle. Additionally, we add a weight to each event, in keeping with Ref. [8]:

$$N\langle k' + m' + k + m\rangle \equiv \sum_{\substack{v_1,v_2,\ldots,v_{k'+m'+k+m}=1 \\ v_1\neq v_2\neq\ldots\neq v_{k'+m'+k+m}}}^{N,N'} \left(\prod_{i=1}^{k} \omega_{v_i} e^{in_i\phi_{v_i}} \prod_{j=k}^{m+k} \omega_{v_j} e^{-in_j\phi_{v_j}} \prod_{h=m+k}^{k'+m+k} \omega_{v_h} e^{in_h\psi_{v_h}} \prod_{l=k'+m+k}^{k'+m'+k+m} \omega_{v_l} e^{-in_l\psi_{v_l}}\right), \tag{B5}$$

$$D\langle k' + m' + k + m\rangle \equiv \sum_{\substack{v_1,v_2,\ldots,v_{k'+m'+k+m}=1 \\ v_1\neq v_2\neq\ldots\neq v_{k'+m'+k+m}}}^{N,N'} \left(\prod_{i=1}^{k} \omega_{v_i} \prod_{j=k}^{m+k} \omega_{v_j} \prod_{h=m+k}^{k'+m+k} \omega_{v_h} \prod_{l=k'+m+k}^{k'+m'+k+m} \omega_{v_l}\right), \tag{B6}$$

where the summation in both equations is over all groups of $k' + m' + k + m$ particles (of which $k' + m'$ are POIs and $k + m$ are reference particles), each with unique index $v_1, \ldots, v_{k'+m'+k'+m}$. For each of the $k' + m'$ indices for POI particles, the index is allowed to run from one to N', and for each of the $m + k$ indices for reference particles, the index is allowed to run from one to N.

(3) We can expand the previous equations and substitute in the various event Q_n vectors. We use the idea that the sum over each permutation of the product of weighted particle angles can be written as a product of the sums over each particle angle:

$$\sum_{r_1,\ldots,r_{k'+m'+k+m}}^{N,N'} \omega_{r_1} e^{in\phi_{r_1}} \cdots \omega_{r_{k'+m'+k+m}} e^{in\phi_{r_{k'+m'+k+m}}} = \prod_{h=1}^{k'+m'} \left(\sum_{i=1}^{N'} \omega_i e^{in_h\psi_i}\right) \prod_{j=1}^{k+m} \left(\sum_{l=1}^{N} \omega_l e^{in_j\phi_l}\right), \tag{B7}$$

where the summation contains *all* permutations of $r_1, \ldots, r_{k'+m'+k+m}$, so some terms must be subtracted, accounting for the cases in which $r_j = r_k$ for two-particle indices. Then we substitute the Q_n vectors introduced in Eqs. (B1)–(B3):

$$\frac{N\langle k' + m' + k + m\rangle}{D\langle k' + m' + k + m\rangle} = \frac{\prod_{i=1}^{k+m} \mathrm{Q}_{n_i,1} \prod_{j=1}^{k'+m'} \mathrm{p}_{n_j,1}}{\prod_{i=1}^{k+m} \mathrm{Q}_{0,1} \prod_{j=1}^{k'+m'} \mathrm{p}_{0,1}} - \text{autocorrelations}, \tag{B8}$$

where the autocorrelations must be subtracted.

(4) To account for autocorrelations, we consider the cases in which two indices reference the same particle angle. To do this, we recursively iterate through all r "subcorrelators" with $k' + m' + k + m - 1$ harmonics, from which $n_{k'+m'+k+m}$ is eliminated, and "added" onto the rth harmonic:

$$\langle k' + m' + k + m - 1\rangle_{n_1,\ldots,(n_r+n_{k'+m'+k+m})+\cdots+n_{k'+m'+k+m-1}}, \tag{B9}$$

where this correlator considers the cases in which the angles from particles in the n_r and $n_{k'+m'+k+m}$ harmonic overlap. The recursive aspect to this algorithm means that we can apply the same procedure to calculate $\langle k'+m'+k+m-1\rangle_{n_1,\ldots,(n_r+n_{k'+m'+k+m})+\cdots+n_{k'+m'+k+m-1}}$ by calculating $\langle k'+m'+k+m-2\rangle$, all the way until there are no more coinciding harmonics to consider, in the same manner as described in Ref. [8].

Using this recursive process, we can represent $\langle k'+m'+k+m\rangle$ using Q_n vectors. However, in step 4, it is not trivial to determine which Q_n vector must be selected when representing the "overlapped" index $n_r = n_{k'+m'+k+m}$. In general, we consider four cases, regarding whether each harmonic is for POIs or for reference particles. Note that in this paper, we have generally written the reference harmonics n first, and the POI harmonics n' second, but this ordering is arbitrary, and it is possible that any of these four cases might apply:

(a) n_r and $n_{k'+m'+k+m}$ correspond to reference particle harmonics. In this case $\mathrm{Q}_{n_r,1}$ is replaced by $\mathrm{Q}_{n_r+n_{k'+m'+k+m},2}$ when evaluating $\langle k'+m'+k+m-1\rangle_{n_1,\ldots,(n_r+n_{k'+m'+k+m})+\cdots+n_{k'+m'+k+m-1}}$.
(b) n_r corresponds to a reference particle azimuthal angle, and $n_{k'+m'+k+m}$ corresponds to a POI angle. In this case, we replace $\mathrm{Q}_{n_r,1}$ with $\mathrm{q}_{n_r+n_{k'+m'+k+m},2}$.
(c) n_r corresponds to a POI azimuthal angle, and $n_{k'+m'+k+m}$ corresponds to a reference particle angle. As before, we replace $\mathrm{p}_{n_r,1}$ with $\mathrm{q}_{n_r+n_{k'+m'+k+m},2}$.
(d) n_r and $n_{k'+m'+k+m}$ both correspond to POI angles. In this case, we replace $\mathrm{p}_{n_r,1}$ with $\mathrm{p}_{n_r+n_{k'+m'+k+m},2}$.

We summarize the recurrence relation below, while noting that the evaluation of $\mathrm{N}\langle k'+m'+k+m-1\rangle$ requires the use of the rules for each scenario detailed above:

$$N\langle k'+m'+k+m\rangle_{n_1,\cdots n_k,n'_{k+m+1},\ldots,n'_{k+m+k'}|n_{k+1}\cdots n_m,n_{k+m+k'+1}\cdots n_{k'+m'+k+m}}$$
$$=\prod_{i=1}^{k+m}\mathrm{Q}_{n_i,1}\prod_{j=1}^{k'+m'}\mathrm{p}_{n'_j,1}-\sum_{r=1}^{k'+m'+m+k-1}N\langle k'+m'+k+m-1\rangle_{n_1,\ldots,(n_r+n_{k'+m'+k+m})+\cdots+n_{k'+m'+k+m-1}},\tag{B10}$$

where we show that $N\langle k'+m'+k+m\rangle$ can be recursively evaluated by considering a product of $k+m+k'+m$ for the Q_n vectors and then subtracting a sum of correlators with one fewer particle angle $N\langle k'+m'+k+m-1\rangle$, where for the rth subtracted correlator, the $k'+m'+k+m$th subscript $n_{k'+m'+k+m}$ is added to the rth harmonic of the correlator. A very simple example is shown below, where the two-particle correlation between particles of interest and reference particles is written using Q_n vectors:

$$\langle 1'+1\rangle_{n,-n'}=\langle e^{in(\phi_1-\psi_2)}\rangle=\frac{\mathrm{Q}_{n,1}\mathrm{p}_{n,1}-\mathrm{q}_{2n,2}}{\mathrm{Q}_{0,1}\mathrm{p}_{0,1}-\mathrm{q}_{0,2}},\tag{B11}$$

where this result is consistent with the same definition given in Ref. [78]. There the weighting for each particle was uniform, and the denominator was rewritten in terms of N, N', and N_q.

APPENDIX C: CONTRIBUTION OF v_n, v'_n TO MULTI-POI CUMULANTS

We calculate the contribution of v_n and v'_n to the evaluation of $h^{k',m'}_{n/p}\{k'+m'+k+m\}$. We start with the definition of the generating function:

$$\mathcal{H}^{k',m'}_{n/p}(z)=\frac{\langle e^{ip(\psi_1+\ldots+\psi_{k'}-\psi_{k'+1}-\ldots-\psi_{k'+m'})}G_n(z)\rangle}{\langle G_n(z)\rangle},\tag{C1}$$

which was previously discussed in Eq. (48). Using the result from Refs. [6,94], we can determine the value of $\langle G_n(z)\rangle$ in terms of v_n, which approximates a Bessel function:

$$\langle G_n(z)\rangle=\sum_{j=0,1,2,\ldots}^{M/2}\frac{1}{M^{2j}}\frac{M!}{(M-2j)!(j!)^2}v_n^{2j}|z|^{2j}\approx I_0(2|z|v_n).\tag{C2}$$

We now calculate $\langle e^{ip(\psi_1+\ldots+\psi_{k'}-\psi_{k'+1}-\ldots-\psi_{k'+m'})}G_n(z)\rangle$ using the same method we used to evaluate $\mathcal{F}_{p/n}(z)$, by expressing the average as an integral around a fixed angle:

$$\langle e^{ip(\psi_1+\ldots+\psi_{k'}-\psi_{k'+1}-\ldots-\psi_{k'+m'})}G_n(z)\rangle$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\langle e^{ip(\psi_1+\ldots+\psi_{k'}-\psi_{k'+1}-\ldots-\psi_{k'+m'})}G_n(z)\rangle|_{\Phi_n}d\Phi_n.\tag{C3}$$

Then, for every $e^{ip\psi_i}$, we substitute in $v'_p e^{ip\Phi_n}$, again following Ref. [6], because the event averaged ψ angles at Φ_n are not correlated with the event averaged ϕ angles at Φ_n:

$$\langle e^{ip(\psi_1+\ldots+\psi_{k'}-\psi_{k'+1}-\ldots-\psi_{k'+m'})}G_n(z)\rangle$$
$$=\frac{v'^{\,k'+m'}_{p/n}}{2\pi}\int_0^{2\pi}e^{ip(k'-m')\Phi_n}\langle G_n(z)\rangle|_{\Phi_n}d\Phi_n.\tag{C4}$$

We then solve the equation on the right, which only has nonzero solutions if $p(k'-m')=qn$ for some integer q:

$$\int_0^{2\pi}e^{ip(k'-m')\Phi_n}\langle G_n(z)\rangle|_{\Phi_n}d\Phi_n$$
$$=\sum_{l=0}^{[(M+q)/2]}\frac{M!}{(M-q-2l)!l!(2l+q)!}\left(\frac{v_n}{M}\right)^{2l+q}z^{*l}z^{l+q}.\tag{C5}$$

TABLE II. We present the first few central moments, with different combinations of variables.

Central moments	1 stochastic variable (univariate)	2 stochastic variables	3 stochastic variables	4 stochastic variables
$\nu = \sum_i \nu_i = 2$	$\langle X^2\rangle - \langle X\rangle^2$ Variance: the fundamental measure of fluctuations.	$\langle XY\rangle - \langle X\rangle\langle Y\rangle$ Covariance: a measure of correlation between two variables, of which SC is a special case.		
$\nu = 3$	$\langle X^3\rangle - 3\langle X^2\rangle\langle X\rangle + 2\langle X\rangle^3$ Skewness: a measure of a distribution's asymmetry.	$\langle X^2Y\rangle - 2\langle XY\rangle\langle X\rangle$ $-\langle X^2\rangle\langle Y\rangle + 2\langle X\rangle^2\langle Y\rangle$ Coskewness: correlates linear deviations from the mean in Y with nonlinear (squared) deviations from the mean in X.	$\langle XYZ\rangle - \langle XY\rangle\langle Z\rangle$ $-\langle XZ\rangle\langle Y\rangle - \langle YZ\rangle\langle X\rangle$ $+ 2\langle X\rangle\langle Y\rangle\langle Z\rangle$ Coskewness: a correlation between deviations from the mean in X Y and Z.	
$\nu = 4$	$\langle X^4\rangle - 4\langle X^3\rangle\langle X\rangle$ $+6\langle X^2\rangle\langle X\rangle^2 - 3\langle X\rangle^4$ Kurtosis: a measure of a distribution's tail "heaviness."	Cokurtosis in two variables can correlate a deviation from the mean in Y to a deviation from the mean *cubed* in X. $\langle X^3Y\rangle - 3\langle X^2Y\rangle\langle X\rangle$ $-\langle X^3\rangle\langle Y\rangle + 3\langle XY\rangle\langle X\rangle^2$ $+3\langle X^2\rangle\langle X\rangle\langle Y\rangle - 3\langle X\rangle^3\langle Y\rangle$ Cokurtosis can also correlate a squared deviation from the mean in X to a squared deviation from the mean in Y. $\langle X^2Y^2\rangle - 2\langle X^2Y\rangle\langle Y\rangle$ $- 2\langle XY^2\rangle\langle X\rangle + \langle X^2\rangle\langle Y\rangle^2$ $+ 4\langle XY\rangle\langle X\rangle\langle Y\rangle$ $+\langle Y^2\rangle\langle X\rangle^2 - 3\langle X\rangle^2\langle Y\rangle^2$	$\langle X^2YZ\rangle - \langle X^2Y\rangle\langle Z\rangle$ $-\langle X^2Z\rangle\langle Y\rangle - 2\langle XYZ\rangle\langle X\rangle$ $+2\langle XY\rangle\langle X\rangle\langle Z\rangle$ $+ 2\langle XZ\rangle\langle X\rangle\langle Y\rangle$ $+\langle X^2\rangle\langle Y\rangle\langle Z\rangle + \langle YZ\rangle\langle X\rangle^2$ cokurtosis: correlating a squared deviation from the mean in X to a deviation from the mean in Y, and a deviation from the mean in Z.	$\langle XYZW\rangle - \langle XYZ\rangle\langle W\rangle$ $- \langle XYW\rangle\langle Z\rangle$ $-\langle XZW\rangle\langle Y\rangle - \langle YZW\rangle\langle X\rangle$ $+\langle XY\rangle\langle Z\rangle\langle W\rangle$ $+ \langle XZ\rangle\langle Y\rangle\langle W\rangle$ $+\langle XW\rangle\langle Y\rangle\langle Z\rangle$ $+ \langle YZ\rangle\langle X\rangle\langle W\rangle$ $+\langle YW\rangle\langle X\rangle\langle Z\rangle$ $+ \langle ZW\rangle\langle X\rangle\langle Y\rangle$ $-3\langle X\rangle\langle Y\rangle\langle Z\rangle\langle W\rangle$ cokurtosis between 4 variables: the correlation between deviations from the mean in X, Y, Z, and W.

Using the result from Ref. [6], we obtain the full expression for $\langle e^{ip(\psi_1+\ldots+\psi_{k'}-\psi_{k'+1}-\ldots-\psi_{k'+m'})}G_n(z)\rangle$:

$$\begin{aligned}
&\langle e^{ip(\psi_1+\ldots+\psi_{k'}-\psi_{k'+1}-\ldots-\psi_{k'+m'})}G_n(z)\rangle \\
&\quad = {v'_{p/n}}^{k'+m'} \sum_{l=0}^{\left[\left(M+\frac{p(k'+m)}{n}\right)/2\right]} \\
&\qquad \times \frac{M!}{\left(M-\frac{p(k'+m)}{n}-2l\right)!l!\left(2l+\frac{p(k'+m)}{n}\right)!} \\
&\qquad \times \left(\frac{v_n}{M}\right)^{2l+\frac{p(k'+m)}{n}} z^{*l} z^{l+\frac{p(k'+m)}{n}} \\
&\quad \approx {v'_{p/n}}^{k'+m} I_{\frac{p(k'-m')}{n}}(2|z|v_n)\left(\frac{z}{|z|}\right)^{\frac{p(k'-m')}{n}}. \qquad \text{(C6)}
\end{aligned}$$

Then, we obtain $\mathcal{H}^{k',m'}_{p/n}(z)$:

$$\mathcal{H}^{k',m'}_{p/n}(z) \approx \frac{I_{\frac{p(k'-m')}{n}}(2|z|v_n)}{I_0(2|z|v_n)}\left(\frac{z}{|z|}\right)^{\frac{p(k'-m')}{n}} {v'_{p/n}}^{k'+m'}. \qquad \text{(C7)}$$

Note that this conclusion is consistent with $\mathcal{F}(z)$ as defined in Sec. III B, because letting $k' = m' = 1$ returns $\mathcal{F}_{n/n}(z) = {v'_n}^2$. Now that we have an expression for $\mathcal{H}^{k',m'}_{p/n}(z)$, we can also produce an expression $h_{p/n}\{k'+m'+k+m\}$, the power series expansion coefficients for $\mathcal{H}^{k',m'}_{p/n}(z)$, and express an estimate for ${v'_n}^{k'+m}$.

APPENDIX D: SOME PROPERTIES OF CENTRAL MOMENTS

Since central moments are not typically used to evaluate fluctuations in v_n, we demonstrate here some mathematical properties of the central moments that are similar to those of cumulants.

(a) Central moments display *reduction*: if a stochastic variable X_i is duplicated j times, the resulting quantity is still a central moment, now of order j in X:

$$\begin{aligned}
&\tilde{\mu}_{\nu_1,\ldots,\nu_n}\left(\underbrace{X_1,\ldots,X_1}_{j},\ldots,X_n\right) \\
&\quad = \left\langle \prod_{i=1}^{j}(X_1-\langle X_1\rangle)^{\nu_i}\prod_{i=j+1}^{n}(X_i-\langle X_i\rangle)^{\nu_i}\right\rangle \\
&\quad = \left\langle (X_1-\langle X_1\rangle)^{\sum_{i=1}^{j}\nu_i}\prod_{i=j+1}^{n}(X_i-\langle X_i\rangle)^{\nu_i}\right\rangle \\
&\quad = \tilde{\mu}_{(\nu_1+\ldots+\nu_j),\ldots,\nu_n}(X_1,\ldots,X_n). \qquad \text{(D1)}
\end{aligned}$$

(b) Central moments display *translation invariance*: shifting the stochastic variable by some constant factor does not alter the value of $\tilde{\mu}$. We use the fact that $\langle X_i + c\rangle = \langle X_i\rangle + c$:

$$\begin{aligned}
&\tilde{\mu}_{\nu_1,..,\nu_g..,\nu_n}(X_1..(X_g+c)..X_n)\\
&= \left\langle \left(\prod_{i=1,i\neq g}^{n} (X_i - \langle X_i\rangle)^{\nu_i} \right) (X_g + c - \langle X_g + c\rangle)^{\nu_g} \right\rangle\\
&= \left\langle \left(\prod_{i=1,i\neq g}^{n} (X_i - \langle X_i\rangle)^{\nu_i} \right) (X_i - \langle X_g\rangle)^{\nu_g} \right\rangle\\
&= \tilde{\mu}_{\nu_1,..,\nu_g...,\nu_n}(X_1..X_g..X_n). \qquad \text{(D2)}
\end{aligned}$$

(c) Central moments display *homogeneity*: they scale with their variables to their order. Using the fact that $\langle cX\rangle = c\langle X\rangle$:

$$\begin{aligned}
&\tilde{\mu}_{\nu_1,..,\nu_g..,\nu_n}(X_1..(cX_g)..X_n)\\
&= \left\langle \left(\prod_{i=1,i\neq g}^{n} (X_i - \langle X_i\rangle)^{\nu_i} \right) (cX_g - \langle cX_g\rangle)^{\nu_g} \right\rangle\\
&= \left\langle \left(\prod_{i=1,i\neq g}^{n} (X_i - \langle X_i\rangle)^{\nu_i} \right) c^{\nu_g}(X_g - \langle X_g\rangle)^{\nu_g} \right\rangle\\
&= c^{\nu_g}\tilde{\mu}_{\nu_1,..,\nu_g..,\nu_n}(X_1..X_g..X_n). \qquad \text{(D3)}
\end{aligned}$$

Additionally, central moments of order $\sum_i \nu_i \leqslant 3$ are identical to multivariate cumulants of the same order in each stochastic variable, meaning they obey some useful properties of cumulants, specifically additivity, and multilinearity: $\kappa_{n,...}(X+Y,\ldots) = \kappa_{n,...}(X,\ldots) + \kappa_{n,j}(Y,\ldots)$.

APPENDIX E: EXPLICIT FORMS OF CENTRAL MOMENTS

In Table II, we introduce the first few central moments at orders $\nu = 2$, 3 and 4, for up to four unique stochastic variables X, Y, Z, and W, as defined in Sec. II C. Additionally, we briefly explain the various correlations and fluctuations measured by these central moments, and how they can be interpreted. The central moments with order $\nu = 2$ and $\nu = 3$ are also considered cumulants, as described in Sec. III C.

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