Commun. Math. Phys. (2024) 405:93
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-024-04986-9

Communications in Mathematical Physics

Features of a Spin Glass in the Random Field Ising Model

Sourav Chatterjee

Department of Statistics, Stanford University, 390 Jane Stanford Way, Stanford, CA 94305, USA.

Received: 4 August 2023 / Accepted: 7 March 2024
Published online: 9 April 2024 –

Abstract: A longstanding open question in the theory of disordered systems is whether short-range models, such as the random field Ising model or the Edwards–Anderson model, can indeed have the famous properties that characterize mean-field spin glasses at nonzero temperature. This article shows that this is at least partially possible in the case of the random field Ising model. Consider the Ising model on a discrete d-dimensional cube under free boundary condition, subjected to a very weak i.i.d. random external field, where the field strength is inversely proportional to the square-root of the number of sites. It turns out that in $d \geq 2$ and at subcritical temperatures, this model has some of the key features of a mean-field spin glass. Namely, (a) the site overlap exhibits one step of replica symmetry breaking, (b) the quenched distribution of the overlap is non-self-averaging, and (c) the overlap has the Parisi ultrametric property. Furthermore, it is shown that for Gaussian disorder, replica symmetry does not break if the field strength is taken to be stronger than the one prescribed above, and non-self-averaging fails if it is weaker, showing that the above order of field strength is the only one that allows all three properties to hold. However, the model does not have two other features of mean-field models. Namely, (a) it does not satisfy the Ghirlanda–Guerra identities, and (b) it has only two pure states instead of many.

1. Introduction

The random field Ising model (RFIM) was introduced as a simple model of a disordered system by [38] in 1975. The model is defined as follows. Take any $d \geq 1$ and $\Lambda \subseteq \mathbb{Z}^d$. Let E denote the set of edges connecting neighboring points in Λ. Given a field strength $h \in \mathbb{R}$, define the (random) Hamiltonian $H : \{-1, 1\}^\Lambda \to \mathbb{R}$ as

$$H(\sigma) := -\sum_{\{i,j\}\in E} \sigma_i\sigma_j - h\sum_{i\in\Lambda} J_i\sigma_i, \tag{1.1}$$

where $J = (J_i)_{i\in\Lambda}$ is a fixed realization of i.i.d. random variables from some distribution. At inverse temperature $\beta > 0$, the RFIM prescribes a random Gibbs measure on $\{-1, 1\}^\Lambda$ with probability mass function proportional to $e^{-\beta H(\sigma)}$.

A large body of deep mathematics has grown around this model, such as the early works of [36,37] on the multiplicity of ground states in the 3D RFIM, the proof of phase transition in $d \geq 3$ by [13,14], the absence of phase transition in $d \leq 2$ proved by [3,4], and the more recent works on quantifying the Aizenman–Wehr theorem [2,5,16,18], culminating in the proof of exponential decay of correlations in the 2D RFIM by [24]. The recent developments have led to a resurgence of interest in this model in the mathematical community, yielding a number of new and important results [8,11,22,23,25,26].

In spite of all this progress, one major question that has not yet been settled is whether the RFIM has a spin glass phase. A disordered system is said to exhibit spin glass behavior if it has the properties that characterize mean-field spin glasses. In the formulation laid out by Giorgio Parisi [51], the main features of mean-field spin glasses are replica symmetry breaking (RSB), non-self-averaging (NSA), ultrametricity, and the presence of many pure states. These properties are defined as follows. Consider a system of N particles, with spins $\sigma = (\sigma_1, \ldots, \sigma_N) \in \{-1, 1\}^N$. In a disordered system, the probability law μ of σ is random. Let $\sigma^1, \sigma^2, \ldots$ be i.i.d. spin configurations drawn from a fixed realization of the random probability measure μ. The overlap between the configurations σ^i and σ^j is defined as

$$R_{i,j} := \frac{1}{N}\sum_{k=1}^{N} \sigma_k^i \sigma_k^j.$$

Let $\langle R_{1,2}\rangle$ denote the expected value of $R_{1,2}$ with respect to μ. Roughly speaking, we say that the system exhibits replica symmetry if $R_{1,2} \approx \langle R_{1,2}\rangle$ with high probability (i.e., probability $\to 1$ as the system size $\to \infty$). Otherwise, we say that replica symmetry breaks. The breaking of replica symmetry is usually quite difficult to prove rigorously. RSB has been established rigorously only in mean-field systems, where every particle interacts with every other particle. The primary example of this is the Sherrington–Kirkpatrick (SK) model [54], where the discovery of RSB led to the development of Parisi's broken replica method [44]. Rigorous proofs of RSB in the SK and other mean-field models came much later (see [57,59] and references therein).

For short-range models such as the RFIM and the Edwards–Anderson (EA) model [30], there is no proof of RSB as of now. Settling a longstanding debate [41], it was shown in [17, Lemma 2.6] that replica symmetry does not break in the RFIM at any fixed temperature and nonzero field strength. The question of RSB in the EA model is still open, although some aspects of spin glass behavior have been established at zero temperature in the recent preprint [19], confirming some old conjectures from physics [12,31].

The second basic property of spin glass models in Parisi's formulation is non-self-averaging (NSA). NSA is the property that the quenched law of $R_{1,2}$ (i.e., its law conditional on a realization of μ) does not converge to a deterministic limit in probability as the system size goes to infinity. Rigorous proofs of NSA are now known for mean-field systems [57,59], but there is no short-range model that has been rigorously proved to have the NSA property. In fact, there are mathematical arguments based on ergodic theory that seem to rule out NSA in translation-invariant short-range models in infinite volume [45], but there is a counter-argument that infinite volume systems do not truly represent finite volume behavior [50]. The main result of [17] implies that the RFIM does not have the NSA property at non-critical field strengths, but that leaves open the possibility that NSA may hold at critical temperatures in the RFIM. Nothing is known about NSA in the EA model.

The third basic property of spin glasses is ultrametricity. This means, roughly speaking, that for any given $\varepsilon > 0$, the probability of the event $R_{1,3} \geq \min\{R_{1,2}, R_{2,3}\} - \varepsilon$ tends to 1 as the system size goes to infinity. Ultrametricity implies that the Gibbs measure "organizes the states like a tree", a notion that has recently been made mathematically precise in [20]. Ultrametricity also has the important consequence that it allows one to write down the joint distribution of arbitrarily many overlaps from the distribution of a single overlap, and thereby understand almost everything about the system. Ultrametricity has been rigorously proved in mean-field systems, most notably by [49] for a variant of the SK model, followed by extensions to other mean-field systems [7,21,39,55,56]. As of today, there are no rigorous results about ultrametricity for systems with purely local interactions.

The fourth property—the existence of many pure states—means, very roughly, that the Gibbs measure behaves like a mixture of a large number of ergodic measures. Again, this is known rigorously only for mean-field models [48] and certain special models on lattices [46]. Incidentally, it is rather unclear how to define a pure state outside the setting of Markov random fields on finite-dimensional lattices [32]. For certain kinds of mean-field spin glasses, a rigorous definition was given by [48]. In the next section, we will give a general definition of the number of pure states that encompasses both mean-field and lattice models.

Proving that short-range models of disordered systems can have the above features of mean-field spin glasses has long been one of the main unsolved questions in this area, first posed in the seminal monograph of [44]. As mentioned above, there is a negative result from [17], where it was established that the RFIM does not have a phase where replica symmetry breaks. In this article, we show that in spite of this negative result, the first three features of a spin glass listed above—RSB, NSA, and ultrametricity—can in fact arise in the RFIM, if instead of keeping the field strength h in (1.1) fixed, we take it to zero like $|\Lambda|^{-\frac{1}{2}}$ as $|\Lambda| \to \infty$, and take β bigger than the critical inverse temperature of the Ising model. Moreover, if the J_i's are Gaussian, then we show that this is the only scaling of h where this happens. However, the fourth property does not hold, because the system appears to be a mixture of two pure states instead of many. Another common (but perhaps not essential) feature of mean-field spin glasses, called the Ghirlanda–Guerra identities, also does not hold for this system.

2. Results

Take any $d \geq 2$. For each n, let $B_n := \{-n, \dots, n\}^d$, and let E_n be the set of undirected nearest neighbor edges of B_n. Let $\Sigma_n := \{-1, 1\}^{B_n}$ be the set of ± 1-valued spin configurations on B_n. Let $(J_i)_{i \in B_n}$ be a collection of i.i.d. random variables with mean zero, variance one, and finite moment generating function in an open neighborhood of the origin. Let $h \in \mathbb{R}$ be a parameter. Define the Hamiltonian $H_n : \Sigma_n \to \mathbb{R}$ as

$$H_n(\sigma) := -\sum_{\{i,j\} \in E_n} \sigma_i \sigma_j - \frac{h}{\sqrt{|B_n|}} \sum_{i \in B_n} J_i \sigma_i \tag{2.1}$$

This is the Hamiltonian for the Ising model on B_n subjected to a random external field of strength $h J_i |B_n|^{-\frac{1}{2}}$ at site i for each $i \in B_n$. That is, we have replaced the parameter h in (1.1) by $h|B_n|^{-\frac{1}{2}}$. The Gibbs measure for this model at inverse temperature β is the random probability measure on Σ_n with probability mass function proportional to $e^{-\beta H_n(\sigma)}$ at each $\sigma \in \Sigma_n$. For a function $f : \Sigma_n \to \mathbb{R}$, let $\langle f \rangle$ denote its expected value with respect to the Gibbs measure. The "quenched distribution" of f is the law of $f(\sigma)$ conditional on $(J_i)_{i \in B_n}$, where σ is drawn from the Gibbs measure.

2.1. Replica symmetry breaking and non-self-averaging. Let σ^1 and σ^2 be drawn independently from the Gibbs measure defined by a single realization of the disorder $(J_i)_{i \in B_n}$. Recall from the previous section that the site overlap (or spin overlap) between σ^1 and σ^2 is defined as

$$R_{1,2} := \frac{1}{|B_n|} \sum_{i \in B_n} \sigma_i^1 \sigma_i^2.$$

If we have a sequence of configurations $\sigma^1, \sigma^2, \ldots$ drawn independently from the Gibbs measure, then $R_{i,j}$ denotes the overlap between σ^i and σ^j. The following theorem is the first main result of this paper.

Theorem 2.1 (Replica symmetry breaking and non-self-averaging). *Take any $d \geq 2$ and $n \geq 1$ and consider the model defined above on $B_n = \{-n, \ldots, n\}^d$ at inverse temperature $\beta > \beta_0$, where β_0 is the critical inverse temperature for the ordinary Ising model on $\mathbb{Z}^d$. Then there is a deterministic value $q > 0$ depending only on β and d, such that $\mathbb{E}\langle (R_{1,2}^2 - q^2)^2 \rangle \to 0$ as $n \to \infty$. Moreover, if we define*

$$X_n := \frac{\sqrt{q}\beta h}{\sqrt{|B_n|}} \sum_{i \in B_n} J_i, \tag{2.2}$$

then we have that

$$\lim_{n \to \infty} \mathbb{E}[(\langle R_{1,2} \rangle - q \tanh^2 X_n)^2] = 0. \tag{2.3}$$

Consequently, as $n \to \infty$, $\langle R_{1,2} \rangle$ converges in law to $q \tanh^2(\sqrt{q}\beta h Z)$, where Z is a standard Gaussian random variable.

For the reader's convenience, let us briefly explain the significances of the two assertions of the above theorem. The first assertion, that $\mathbb{E}\langle (R_{1,2}^2 - q^2)^2 \rangle \to 0$ as $n \to \infty$, shows that when n is large, the overlap $R_{1,2}$ is close to either q or $-q$ with high probability. The second assertion shows that the quenched expectation of $R_{1,2}$ is a random variable that converges to a non-degenerate limiting distribution as $n \to \infty$. Jointly, this proves two things. First, it shows that $R_{1,2}$ does indeed behave like a random variable that is close to one of two values, and not just one value (because otherwise, $\langle R_{1,2} \rangle$ would be close to q or $-q$). This is known as one step of replica symmetry breaking (1RSB). Second, it shows that the quenched distribution of the overlap is not self-averaging—that is, it does not converge to a deterministic limiting distribution as $n \to \infty$. Equation (2.3) shows that the mass near q is approximately

$$\frac{1}{2}(1 + \tanh^2 X_n), \tag{2.4}$$

and the mass near $-q$ is 1 minus the above. An important thing to note is that q depends only on β and d, and not on h. Thus, q is the limiting absolute value of the overlap in the ordinary Ising model—that is, the case $h = 0$. In particular, Theorem 2.1 implies that for the Ising model, the quenched law of $R_{1,2}$ converges in probability to the uniform distribution on $\{-q, q\}$ as $n \to \infty$. The presence of h only changes the masses near q and $-q$.

Theorem 2.1 shows that non-self-averaging can occur even in a system that only has local interactions. It is to be noted that the system under consideration here has no obvious representative in the infinite volume limit (because the field strength is tending to zero but with a non-trivial effect which cannot be captured by a model in infinite volume in any obvious way), thereby posing no contradiction to the results of [45] on the impossibility of NSA in translation-invariant infinite volume systems.

2.2. Ultrametricity. The next result says that the overlap satisfies the Parisi ultrametric property in the large n limit, meaning that $R_{1,3} \geq \min\{R_{1,2}, R_{2,3}\} - o(1)$ with probability $1 - o(1)$ as $n \to \infty$.

Theorem 2.2 (Ultrametricity). *Let d, n, β_0, β and q be as in Theorem 2.1. Then, as $n \to \infty$, the quenched distribution of $(R_{1,2}, R_{1,3}, R_{2,3})$ converges in law to a random limiting distribution with support*

$$\{(q, q, q), (-q, -q, q), (-q, q, -q), (q, -q, -q)\}. \tag{2.5}$$

Consequently, for any $\varepsilon > 0$, the quenched probability of the event $R_{1,3} \geq \min\{R_{1,2}, R_{2,3}\} - \varepsilon$ tends to 1 in probability as $n \to \infty$.

Combined with Theorem 2.1, it is easy to deduce the approximate masses assigned by the law of $(R_{1,2}, R_{1,3}, R_{2,3})$ near the four points displayed in (2.5). Let a be the approximate mass near (q, q, q), and let b be the approximate mass near each of the other three points (which must be equal, by symmetry). Then $a + 3b \approx 1$, and $a + b \approx$ the probability of the event $R_{1,2} \approx q$, which is given by the formula (2.4). Solving, we get

$$a \approx \frac{1}{4}(1 + 3\tanh^2 X_n), \quad b \approx \frac{1}{4}(1 - \tanh^2 X_n).$$

Just like Theorems 2.1, 2.2 is valid even if $h = 0$, that is, for the Ising model. It shows that at subcritical temperatures, the overlap in the Ising model has the ultrametricity property.

2.3. Behavior of the magnetization. The magnetization of a configuration σ is defined as

$$m = m(\sigma) := \frac{1}{|B_n|} \sum_{i \in B_n} \sigma_i.$$

The following theorem identifies the limiting behavior of the magnetization of a configuration drawn from the Gibbs measure when β is bigger than the critical inverse temperature of the Ising model. It also gives a relation between the magnetizations of two independently drawn configurations and their overlap.

Theorem 2.3 (Behavior of the magnetization). *Let d, n, β_0, β and q be as in Theorem 2.1. The magnetization m of a configuration σ drawn from the model satisfies $\mathbb{E}\langle (m^2 - q)^2\rangle \to 0$ as $n \to \infty$, and with X_n defined as in (2.2), we have*

$$\lim_{n\to\infty} \mathbb{E}[(\langle m\rangle - \sqrt{q}\tanh X_n)^2] = 0. \tag{2.6}$$

In particular, $\langle m\rangle$ converges in law to $\sqrt{q}\tanh(\sqrt{q}\beta h Z)$, where Z is a standard Gaussian random variable. Moreover, for most values of $j \in B_n$, $\langle \sigma_j\rangle \approx \langle m\rangle$ with high probability, in the sense that

$$\lim_{n\to\infty} \frac{1}{|B_n|}\sum_{j\in B_n} \mathbb{E}[(\langle \sigma_j\rangle - \langle m\rangle)^2] = 0. \tag{2.7}$$

Lastly, if $m(\sigma^1)$ and $m(\sigma^2)$ are the magnetizations in two configurations σ^1 and σ^2 chosen independently from the same Gibbs measure, then $\mathbb{E}\langle (R_{1,2} - m(\sigma^1)m(\sigma^2))^2\rangle \to 0$ as $n \to \infty$.

This theorem is the basis for proving the previously stated results about the overlap, because it says that the overlap between two configuration is approximately equal to the product of their magnetizations with high probability, and gives the approximate distribution of the magnetization, which is concentrated near q or $-q$ with high probability. In addition to the previously stated results, it also gives the asymptotic quenched distribution of any number of overlaps, because conditional on the disorder, $m(\sigma^1), m(\sigma^2), \ldots$ behave like i.i.d. random variables taking values in $\{-\sqrt{q}, \sqrt{q}\}$ with a certain distribution, and $R_{i,j} \approx m(\sigma^i)m(\sigma^j)$ for each $i \neq j$.

2.4. Two pure states. As mentioned in the introduction, it is unclear how to rigorously define pure states outside the setting of Markov random fields on a lattice, where it is well-understood [32]. We will now give a general definition of the number of pure states in a sequence of models, and show that according to this definition, our model has two pure states in the $n \to \infty$ limit.

Let $\{N_n\}_{n\geq 1}$ be a sequence of positive integers tending to infinity, and let $(X_{n,i})_{n\geq 1, 1\leq i\leq N_n}$ be a triangular array of real-valued random variables. For each n, let π_n be a uniform random permutation of $1, \ldots, N_n$, independent of the $X_{n,i}$'s. Let $Y_{n,i} := X_{n,\pi_n(i)}$. Let $Z = (Z_1, Z_2, \ldots)$ be a sequence of random variables such that for each k, $(Y_{n,1}, \ldots, Y_{n,k})$ converges to $(Z_1, \ldots, Z_k)$ in distribution as $n \to \infty$. Then note that Z is an infinite exchangeable sequence of random variables. By De Finetti's theorem [40, Theorem 1.1], the law of Z is a mixture of probability laws of i.i.d. sequences, with a unique mixing measure [40, Proposition 1.4].

Definition 2.4. In the above setting, let μ be the mixing measure of the law of Z. Let p be the size of the support of μ, which may be a positive integer or infinity. Then, we will say that the law of $(X_{n,i})_{1\leq i\leq N_n}$ has p pure states asymptotically as $n \to \infty$.

For example, if the $X_{n,i}$'s are i.i.d., then so are the Z_i's, and therefore $p = 1$. On the other hand, suppose that $N_n = n$ and $X_{n,i} = Y + W_i$, $i = 1, \ldots, n$, where Y and $W_1, W_2, \ldots$ are i.i.d. standard Gaussian random variables. If π_n is a uniform random permutation of $1, \ldots, n$, then for any n and k, the law of $(X_{n,\pi_n(1)}, \ldots, X_{n,\pi_n(k)})$ is the same as the law of $(Z_1, \ldots, Z_k)$, where $Z_i = Y + W_i$. Now, $Z_1, Z_2, \ldots$ is an infinite exchangeable sequence, which is conditionally i.i.d. given Y. Since the support of Y

contains infinitely many points, we deduce that the law of $(X_{n,i})_{1\leq i\leq n}$ has infinitely many pure states as $n \to \infty$.

In the setting of disordered systems, the law of $(X_{n,i})_{1\leq i\leq N_n}$ is itself random, and may not be converging to a deterministic limit in any reasonable sense as $n \to \infty$. Thus, we have to modify Definition 2.4 to accommodate this scenario. Let $Y_{n,i} = X_{n,\pi_n(i)}$ be defined as before. For each k, let $\nu_{n,k}$ be the law of $(Y_{n,1}, \ldots, Y_{n,k})$, which is now a random probability measure. Let ν be a random probability measure taking value in the set of laws of infinite exchangeable sequences. Let ν_k be the (random) law of the first k coordinates of a sequence with law ν.

Definition 2.5. Let ν be as above, and let μ be the (random) mixing measure of a random probability measure with law ν. Suppose that there is a deterministic $p \in \{1, 2, \ldots\}\cup\{\infty\}$ such that with probability one, the support of μ has p points. Also, suppose that for each k, the law of $\nu_{n,k}$ converges weakly to the law of ν_k. Then, we will say that the (random) law of $(X_{n,i})_{1\leq i\leq N_n}$ has p pure states asymptotically as $n \to \infty$.

The following result shows that under the above definition, our model has two pure states asymptotically as $n \to \infty$. This holds for any h, and in particular $h = 0$, which is the case of the ordinary Ising model.

Theorem 2.6. *Let d, n, β_0 and β be as in Theorem 2.1. Then the random probability measure on Σ_n defined by the model from Theorem 2.1 has two pure states asymptotically as $n \to \infty$, as defined in Definition 2.5.*

2.5. Failure of the Ghirlanda–Guerra identities. The Ghirlanda–Guerra (GG) identities are a set of identities that are satisfied in the infinite volume limits of many mean-field spin glass models [33]. A symmetric array of random variables $(S_{i,j})_{1\leq i,j<\infty}$ is said to satisfy the GG identities if for any k, any bounded measurable function f of $(S_{i,j})_{1\leq i,j\leq k}$, and any bounded measurable function $\psi : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}(f\psi(S_{1,k+1})) = \frac{1}{k}\mathbb{E}(f)\mathbb{E}(\psi(S_{1,2})) + \frac{1}{k}\sum_{i=2}^{k}\mathbb{E}(f\psi(S_{1,i})). \tag{2.8}$$

These identities have been proved for the limiting joint law of overlaps for a variety of mean-field models of spin glasses. (Here, the "joint law" refers to the unconditional distribution, averaged over the disorder.) They form the basis of Panchenko's proof of ultrametricity in [49], following a line of prior work connecting the GG identities with ultrametricity [1,6,47,58]. The following theorem shows that the GG identities are not valid for our model. This shows that while the GG identities are sufficient for ultrametricity of the overlap (as shown by Panchenko [49]), they are not necessary.

Theorem 2.7 (Failure of the Ghirlanda–Guerra identities). *Let d, n, β_0 and β be as in Theorem 2.1. Then the limiting joint distribution of the overlaps, as $n \to \infty$, does not satisfy the Ghirlanda–Guerra identities.*

2.6. Failure of spin glass behavior at other field strengths. One may wonder if taking the field strength to be proportional to $|B_n|^{-\frac{1}{2}}$ is the only way to get replica symmetry breaking and non-self-averaging in the large n limit. Our next result shows that this is indeed the case for Gaussian disorder (and it is reasonable to conjecture that the same holds for any i.i.d. disorder). Replica symmetry does not break if the parameter h is allowed to go to $\pm\infty$ as $n \to \infty$, and the non-self-averaging of the quenched law of the overlap breaks down if h is allowed to go to zero as $n \to \infty$.

Theorem 2.8 (Failure of spin glass behavior at other field strengths). *Suppose that the parameter h in the Hamiltonian H_n is allowed to vary with n. If $h \to 0$ as $n \to \infty$, then the distance between the quenched law of $R_{1,2}$ under our model and the law of $R_{1,2}$ under the Ising model on B_n at the same temperature and free boundary condition tends to zero in probability as $n \to \infty$, for any metric that metrizes weak convergence of probability measures. In particular, non-self-averaging fails. On the other hand, if $|h| \to \infty$ as $n \to \infty$, and if the J_i's are i.i.d. standard Gaussian random variables, then $\mathbb{E}\langle (R_{1,2} - \langle R_{1,2}\rangle)^2\rangle \to 0$, meaning that replica symmetry does not break. These conclusions hold at any temperature.*

The second assertion of the above theorem extends [17, Lemma 2.6] by showing that replica symmetry holds not only when the parameter h in the standard form (1.1) of the RFIM Hamiltonian is fixed and nonzero, but is even allowed to go to zero slower than $|\Lambda|^{-\frac{1}{2}}$ (for $\Lambda = B_n$).

2.7. The antiferromagnetic RFIM. For the sake of completeness, let us also consider the random field antiferromagnetic Ising model on B_n under free boundary condition. This is the model where the minus in front of the first term on the right side in (2.1) is replaced by a plus. That is, the Hamiltonian is

$$H_n(\sigma) := \sum_{\{i,j\}\in E_n} \sigma_i\sigma_j - \frac{h}{\sqrt{|B_n|}} \sum_{i\in B_n} J_i\sigma_i. \tag{2.9}$$

All of the results for the ferromagnetic model continue to hold for the antiferromagnetic version, except one—the magnetization tends to zero instead of converging in law to a non-degenerate distribution.

Theorem 2.9 (Results for the antiferromagnetic RFIM). *Theorems 2.1, 2.2 and 2.8 remain valid for the antiferromagnetic model, with J_i replaced by $(-1)^{|i|_1} J_i$ in the (2.2), where $|i|_1$ is the ℓ^1 norm of i. The magnetization, however, satisfies $\mathbb{E}\langle m^2\rangle \to 0$ as $n \to \infty$.*

2.8. Uniformity of correlations in the ordinary Ising model. In addition to the above results, our analysis also reveals the following "uniformity of correlations" for the ordinary Ising model on B_n under free boundary condition and subcritical temperatures. Namely, $\langle\sigma_i\sigma_j\rangle \approx q$ for most $i, j \in B_n$. More generally, for any even l and most $i_1, \ldots, i_l \in B_n$, $\langle\sigma_{i_1}\cdots\sigma_{i_l}\rangle \approx q^{\frac{l}{2}}$. This result is the foundation for most of the other results in this paper. Note that the correlation is zero if l is odd due to the invariance of the model under the transform $\sigma \to -\sigma$.

Theorem 2.10 (Uniformity of correlations in the Ising model). *Let d, n, β_0, β and q be as in Theorem 2.1. Consider the ferromagnetic Ising model on B_n at inverse temperature β and free boundary condition (i.e., the model with Hamiltonian given in (2.1) but with $h = 0$). Then for any even positive integer l,*

$$\lim_{n\to\infty} \frac{1}{|B_n|^l} \sum_{i_1,\ldots,i_l\in B_n} |\langle\sigma_{i_1}\cdots\sigma_{i_l}\rangle - q^{\frac{l}{2}}| = 0.$$

Uniformity of correlations in infinite volume is a simple consequence of a result of [10] (see also [53]), which says that the infinite volume Gibbs measure for the Ising model under free boundary condition is the average of the infinite volume measures under plus and minus boundary conditions.

The finite volume result stated above does not follow easily from the infinite volume result, even though we know that correlations decay exponentially under plus and minus boundary conditions [29]. This is because Bodineau's theorem does not imply that the finite volume Gibbs measure under free boundary is approximately the average of the finite volume measures under plus and minus boundary conditions. The proof presented in this draft is due to Hugo Duminil-Copin (private communication). It uses the random cluster representation of the Ising model and Pisztora's renormalization scheme [52]. A different proof was given in the original draft of this paper, which had the disadvantage of not covering the full supercritical regime and has therefore been omitted.

This completes the statements of the results. The rest of the paper is devoted to proofs.

3. Proof of Theorem 2.10

We now present the proof of Theorem 2.10 due to Hugo Duminil-Copin (private communication), which uses coupling with the FK-Ising (random cluster) model.

3.1. The FK-Ising model. Recall that the FK-Ising model on B_n is defined as follows [35]. Let E_n be the set of edges of B_n, as before, and let $\Omega_n := \{0, 1\}^{E_n}$. Each element $\omega \in \Omega_n$ defines a graph on B_n, with (open) edges corresponding to those $e \in E_n$ for which $\omega_e = 1$. Edges of B_n that are not in this graph are said to be "closed". Let $E(\omega)$ denote the number of open edges and $k(\omega)$ denote the number of connected components of this graph. The FK-Ising model with parameter p, under free boundary condition, assigns a probability proportional to

$$p^{E(\omega)}(1-p)^{|E_n|-E(\omega)}2^{k(\omega)} \tag{3.1}$$

at each $\omega \in \Omega_n$. A different kind of boundary condition, called the "wired boundary condition", has an identical form of the probability mass function but with a different definition of $k(\omega)$. Under the wired boundary condition, all the boundary vertices of B_n are assumed to be connected to each other, and so all connected components that touch the boundary are merged into a single component. Fixing p, we will denote probabilities computed under the free and wired boundary conditions by P_n^0 and P_n^1, respectively.

It is known that the infinite volume limits of these measures exist and are equal if p is not equal to its critical value (which corresponds to the critical β in the Ising model if we reparametrize $p = 1 - e^{-2\beta}$) (by [10, Theorem 2.1] and [34, Theorem 5.3(b)]); that is, for any event A determined by finitely many edges, the limits $\lim_{n\to\infty} P_n^0(A)$ and $\lim_{n\to\infty} P_n^1(A)$ exist and are equal. We will denote this limit by $P(A)$.

Following standard convention, we will denote by $x \leftrightarrow y$ the event that two vertices x and y are connected by a path of open edges. Similarly, $x \leftrightarrow \partial B_n$ will denote the event that x is connected by a path to the boundary of B_n, and $x \leftrightarrow \infty$ will denote the event that x belongs to an infinite open cluster. It is known that when p is greater than the critical value, the infinite volume FK-Ising model has a unique infinite open cluster with probability one [15, Theorem 2] (see also [27, Theorem 1.10]). In the following, we will assume throughout that p is greater than the critical value.

Lastly, define

$$q := \lim_{n\to\infty} (P(0 \leftrightarrow \partial B_n))^2, \tag{3.2}$$

where the existence of the limit follows from monotonicity of the probability as a function of n. We will hold this q fixed throughout the remaining discussion. Note that

$$\begin{aligned} P(0 \leftrightarrow \infty) &= P(0 \leftrightarrow \partial B_n \text{ for all } n) \\ &= \lim_{n\to\infty} P(0 \leftrightarrow \partial B_n) = \sqrt{q}. \end{aligned} \tag{3.3}$$

The numbers p and q will remain fixed throughout the remainder of this section, unless otherwise mentioned.

3.2. Uniformity of connectivities in infinite volume. The identity (3.3) leads to the following lemma, which shows that $P(0 \leftrightarrow x) \approx q$ whenever $|x|$ is large.

Lemma 3.1. *For any x, $P(0 \leftrightarrow x) \geq q$, and given any $\varepsilon > 0$, there exists C depending on ε such that whenever $|x|_\infty > C$ (where $|x|_\infty$ denotes the ℓ^∞ norm of x), we have $P(0 \leftrightarrow x) \leq q + \varepsilon$.*

Proof. Take any x. By the uniqueness of the infinite open cluster, the event $0 \leftrightarrow x$ is implied by the events that $0 \leftrightarrow \infty$ and $x \leftrightarrow \infty$ (with probability one). By the FKG property and the identity (3.3), this implies that

$$\begin{aligned} P(0 \leftrightarrow x) &\geq P(0 \leftrightarrow \infty,\ x \leftrightarrow \infty) \\ &\geq P(0 \leftrightarrow \infty) P(x \leftrightarrow \infty) = q. \end{aligned}$$

This completes the proof of the lower bound. Next, for each n, let $B_n(x)$ denote the cube B_n shifted by x, that is, the set $x + B_n$. Let $\partial B_n(x)$ denote the boundary of $B_n(x)$. Take any $x \neq 0$ and $k < \frac{1}{2}|x|_\infty - 1$. Then the cubes ∂B_k and $\partial B_k(x)$ are disjoint. Moreover, there is a finite set S of edges in $\mathbb{Z}^d$ that are not edges of B_k or $B_k(x)$, such that

- if the edges in S are all open, then all vertices of ∂B_k and $\partial B_k(x)$ are in the same connected component, and
- every edge that is incident to a vertex in $\partial B_k \cup \partial B_k(x)$ but is not an edge of B_k or $B_k(x)$, is a member of S.

Let F denote the event that all edges in S are open. Conditional on F, the configurations of open edges in B_k and $B_k(x)$ are independent, and follow the random cluster models on these cubes with wired boundary condition. Take any $l < k$, and let E be the event $\{0 \leftrightarrow \partial B_l\} \cap \{x \leftrightarrow \partial B_l(x)\}$. Since E and F are increasing events and $P(F) > 0$, the FKG property implies that $P(E|F) \geq P(E)$. Consequently,

$$\begin{aligned} P(0 \leftrightarrow x) &\leq P(E) \leq P(E|F) \\ &= P(0 \leftrightarrow \partial B_l,\ x \leftrightarrow \partial B_l(x)|F) \\ &= P_k^1(0 \leftrightarrow \partial B_l) P_k^1(x \leftrightarrow \partial B_l(x)) \\ &= (P_k^1(0 \leftrightarrow \partial B_l))^2. \end{aligned}$$

Take any $\varepsilon > 0$. For fixed l, if k is large enough, then

$$(P_k^1(0 \leftrightarrow \partial B_l))^2 \leq (P(0 \leftrightarrow \partial B_l))^2 + \frac{\varepsilon}{2}.$$

But if l is large enough, then $(P(0 \leftrightarrow \partial B_l))^2 \le q + \frac{\varepsilon}{2}$. Thus, if $|x|$ is large enough, then we can choose l and k so that both inequalities are satisfied. This proves the claimed upper bound. $\square$

3.3. Uniformity of two-point correlations in finite volume. Our next goal, roughly speaking, is to show that the conclusion of Lemma 3.1 holds even if we consider the model restricted to a cube, as long as 0 and x are not too close to the boundary of the cube. The following lemma provides the upper bound.

Lemma 3.2. *Given any $\varepsilon > 0$ and n, there is some $k > 0$ depending only on d and ε (and not on n), such that whenever $x, y \in B_n$ and $|x - y|_\infty > k$, we have $P_n^0(x \leftrightarrow y) \le q + \varepsilon$.*

Proof. It is a simple consequence of the FKG property that $P_n^0(x \leftrightarrow y)$ is an increasing function of n. As a result, we have

$$P_n^0(x \leftrightarrow y) \le P(x \leftrightarrow y)$$

for any $x, y \in B_n$. But by Lemma 3.1 and the translation-invariance of the infinite volume measure, there is some k depending only on d and ε such that $P(x \leftrightarrow y) \le q + \varepsilon$ whenever $|x - y|_\infty > k$. This completes the proof of the lemma. $\square$

The lower bound in a finite cube is more complicated. It requires a version of the so-called "Pisztora renormalization argument" [52], due to [29]. First, recall the definition of the FK-Ising model on B_n under arbitrary boundary condition. A general boundary condition ξ refers to a partition of the set of boundary vertices of B_n, where we think of all vertices within the same member of the partition as being connected, when defining $k(\omega)$ in (3.1). So, for example, ξ consists of only singletons for the free boundary condition, and ξ consists of only the full set ∂B_n for the wired boundary condition. Let P_k^ξ denote the model on B_k under boundary condition ξ. For a given realization of the model on B_n, we say that a "block" $B_k(x) \subseteq B_n$ is "good" if $x \in k\mathbb{Z}^d$, and the following hold:

- There is an open cluster in $B_k(x)$ that touches all faces of $B_k(x)$.
- Any open path in $B_k(x)$ of length k is contained in this cluster.

We will frequently refer to the above open cluster as the "giant open cluster" of $B_k(x)$. Two blocks $B_k(x)$ and $B_k(y)$ are said to be neighbors if x and y are neighbors in $k\mathbb{Z}^d$. Note that two neighboring blocks have a substantial overlap. In particular, if two neighboring blocks are both good (in a realization of the model), then the conditions imply that the large clusters in the two blocks also intersect. In this situation, we say that the two blocks are "connected".

The result of [29] that we are going to use is that for any k, any boundary condition ξ on B_{2k} and any p greater than the critical value,

$$P_{2k}^\xi(B_k \text{ is good}) \ge 1 - e^{-ck}, \tag{3.4}$$

where c depends only on p and d. This was proved for $d \ge 3$ in [29, Equation (3.1)] (see also [9, Theorem 2.1]). For $d = 2$, it follows from the RSW estimate in [28]. A consequence of this is the following lemma.

Lemma 3.3. *Given any $\varepsilon > 0$, the following is true for all large enough even k (with the threshold depending only on d and ε). Suppose that $B_k(x)$ and $B_k(y)$ are both contained in B_n, and are disjoint. Then, under P_n^0, the probability that any open path of size $\frac{k}{2}$ in $B_k(x)$ is connected to any open path of size $\frac{k}{2}$ in $B_k(y)$ is at least $1 - \varepsilon$.*

Proof. Let P be an open path of length $\frac{k}{2}$ in $B_k(x)$. Consider blocks of the form $B_l(z)$, $z \in l\mathbb{Z}^d$, where $l = \frac{k}{2}$. Let a be the starting point of P. By the nature of the blocks, there is at least one block D such that $a \in D$ and the ℓ^1 distance of a from ∂D is at least $\frac{ld}{2}$. (For example, if $a = (a_1, \dots, a_d)$, then we can choose $D = B_l(z)$ where z_i is the integer multiple of l that is closest to a_i, so that $|(z_i \pm l) - a_i| \geq \frac{l}{2}$.) Then the part of P starting from a and continuing until the first time P hits ∂D, has length at least $\frac{ld}{2} \geq l$ (because $d \geq 2$). Thus, if D is a good block, then this part of P lies within the giant open cluster of P as in the definition of good block above.

If k is chosen large enough (depending on d and ε), then (3.4) shows that with probability at least $1 - \frac{\varepsilon}{2}$, all blocks intersecting $B_k(x)$ or $B_k(y)$ are good. On the other hand, as argued in the proof of [29, Lemma 3.1] with the help of the main result of [43], the collection of good blocks forms a finitely dependent percolation process on $l\mathbb{Z}^d \cap B_n$, which dominates an i.i.d. percolation process with parameter q, where q can be made as close to 1 as we want by choosing k large enough. Consequently, by choosing k large enough we can guarantee that with probability at least $1 - \frac{\varepsilon}{2}$, the giant open cluster of any good block intersecting $B_k(x)$ is connected to the giant open cluster of any good block intersecting $B_k(y)$. Combining this with our previous deductions, we get that with probability at least $1 - \varepsilon$, any open path of size $\frac{k}{2}$ in $B_k(x)$ is connected to any open path of size $\frac{k}{2}$ in $B_k(y)$. □

We are now ready to prove the lower bound.

Lemma 3.4. *Given any n and $\varepsilon > 0$, there exist positive integers k, l depending only on d and ε (and not on n) such that whenever $x, y \in B_n$, $|x - y|_\infty \geq 2k$, and x, y are at an ℓ^∞ distance at least l from ∂B_n, we have $P_n^0(x \leftrightarrow y) \geq q - \varepsilon$.*

Proof. Take any n, k, l, x and y as in the statement of the theorem, where k and l will be chosen later. We will choose k to be even and $k < l < n$. Let E be the event that there is an open cluster C_x in $B_k(x) \setminus B_{\frac{k}{2}}(x)$ connecting $\partial B_k(x)$ and $\partial B_{\frac{k}{2}}(x)$, and an open cluster C_y in $B_k(y) \setminus B_{\frac{k}{2}}(y)$ connecting $\partial B_k(y)$ and $\partial B_{\frac{k}{2}}(y)$, such that $C_x \not\leftrightarrow C_y$ in B_n. Note that if $x \leftrightarrow \partial B_k(x)$, $y \leftrightarrow \partial B_k(y)$, and E fails to happen, then the open clusters connecting x to $\partial B_k(x)$ and y to $\partial B_k(y)$ must be connected in B_n, and therefore $x \leftrightarrow y$. Thus, by the FKG inequality,

$$\begin{aligned} P_n^0(x \leftrightarrow y) &\geq P_n^0(x \leftrightarrow \partial B_k(x),\ y \leftrightarrow \partial B_k(y)) - P_n^0(E) \\ &\geq P_n^0(x \leftrightarrow \partial B_k(x)) P_n^0(y \leftrightarrow \partial B_k(y)) - P_n^0(E). \end{aligned} \tag{3.5}$$

Let Q_x denote the FK-Ising model on $B_l(x)$ under free boundary condition. Since $k < l < n$ and $B_l(x) \subseteq B_n$, the FKG property implies that

$$P_n^0(x \leftrightarrow \partial B_k(x)) \geq Q_x(x \leftrightarrow \partial B_k(x)) = P_l^0(0 \leftrightarrow \partial B_k).$$

Similarly,

$$P_n^0(y \leftrightarrow \partial B_k(y)) \geq P_l^0(0 \leftrightarrow \partial B_k).$$

By the definition of q, we can choose k large enough (depending on d and ε) such that

$$P(0 \leftrightarrow \partial B_k) \geq \sqrt{q} - \frac{\varepsilon}{8}.$$

Having chosen k like this, we can then use the definition of the infinite volume measure to find l large enough (depending on d, ε and k) such that

$$P_l^0(0 \leftrightarrow \partial B_k) \geq P(0 \leftrightarrow \partial B_k) - \frac{\varepsilon}{8}.$$

Thus, with such choices of k and l, we get that $P_n^0(x \leftrightarrow \partial B_k(x))$ and $P_n^0(y \leftrightarrow \partial B_k(y))$ are both bounded below by $\sqrt{q} - \frac{\varepsilon}{4}$. Plugging this into (3.5), we get

$$P_n^0(x \leftrightarrow y) \geq q - \frac{\varepsilon}{2} - P_n^0(E).$$

Finally, with a large enough choice of k (depending on d and ε), Lemma 3.3 implies that $P_n^0(E) < \frac{\varepsilon}{2}$, which completes the proof. □

Using Lemmas 3.2 and 3.4, and a standard coupling of the Ising and FK-Ising models, we are now ready to state and prove the following theorem, which is the main result of this subsection.

Theorem 3.5. *Let β_0 be the critical temperature of the ordinary Ising model in dimension d. Take any $\beta > \beta_0$ and let $p := 1 - e^{-2\beta}$, so that p is a supercritical probability for the FK-Ising model. Let q be defined as in equation* (3.2)*. Take any $\varepsilon \in (0, 1)$. For each n, let*

$$\delta_n = \delta_n(\beta, d, \varepsilon) := \max\{|\langle\sigma_i\sigma_j\rangle - q| : i, j \in B_{\lfloor(1-\varepsilon)n\rfloor}, \ |i - j|_1 \geq \varepsilon n\},$$

where $\langle\cdot\rangle$ denotes averaging with respect to the Ising model on B_n at inverse temperature β and free boundary condition. Then $\delta_n \to 0$ as $n \to \infty$.

Proof. Take any n. Let $P_n^0(\cdot)$ denote probability computed under the FK-Ising model with parameter p on B_n under free boundary condition. It is a standard fact [35, Theorem 1.16] that for any $i, j \in B_n$,

$$\langle\sigma_i\sigma_j\rangle = P_n^0(i \leftrightarrow j). \tag{3.6}$$

Using this identity and Lemmas 3.2 and 3.4, it is now straightforward to prove Theorem 3.5. □

3.4. Uniformity of four-point correlations in finite volume. In this subsection, we show that for most quadruples of vertices $i, j, k, l \in B_n$, $\langle\sigma_i\sigma_j\sigma_k\sigma_l\rangle \approx q^2$ if n is large, where the expectation is with respect to the Ising model with free boundary condition on B_n at a supercritical inverse temperature β, and q is as in (3.2). First, we need the following analogue of Lemma 3.2 for four-point connectivities. Let p be as in Theorem 3.5.

Lemma 3.6. *Given any $\varepsilon > 0$ and n, there is some $k > 0$ depending only on d and ε (and not on n), such that if $x, y, w, z \in B_n$ are such that all interpoint ℓ^∞ distances are greater than $2k$, and all four points are at least at an ℓ^∞ distance k from the boundary, then we have*

$$P_n^0(x \leftrightarrow y \leftrightarrow w \leftrightarrow z) \leq q^2 + \varepsilon, \quad P_n^0(x \leftrightarrow y \not\leftrightarrow w \leftrightarrow z) \leq \varepsilon.$$

Proof. For the proof of the first inequality, we proceed as in the proof of Lemma 3.1. Since the interpoint distances are all greater than $2k$, the cubes $B_k(x)$, $B_k(y)$, $B_k(w)$ and $B_k(z)$ are disjoint. Let F be the event that all edges of B_n that do not belong to these cubes are open. Then by the FKG property, we have that for any $l < k$,

$$\begin{aligned} P_n^0(x \leftrightarrow y \leftrightarrow w \leftrightarrow z) &\le P_n^0(x \leftrightarrow \partial B_l(x),\ y \leftrightarrow \partial B_l(y),\ w \leftrightarrow \partial B_l(w),\ z \leftrightarrow \partial B_l(z)) \\ &\le P_n^0(x \leftrightarrow \partial B_l(x),\ y \leftrightarrow \partial B_l(y),\ w \leftrightarrow \partial B_l(w),\ z \leftrightarrow \partial B_l(z)|F). \end{aligned}$$

But, given F, the configurations inside the cubes $B_k(x)$, $B_k(y)$, $B_k(w)$ and $B_k(z)$ are independent, and follow the FK-Ising models in these cubes with wired boundary condition. Thus, we get

$$P_n^0(x \leftrightarrow y \leftrightarrow w \leftrightarrow z) \le (P_k^1(0 \leftrightarrow \partial B_l))^4.$$

By the equality of the infinite volume measures under free and wired boundary conditions, we have that for l fixed and k sufficiently large (depending on l, d and ε),

$$(P_k^1(0 \leftrightarrow \partial B_l))^4 \le (P(0 \leftrightarrow \partial B_l))^4 + \frac{\varepsilon}{2}.$$

By the definition of q, a large enough value of l ensures that

$$(P(0 \leftrightarrow \partial B_l))^4 \le q^2 + \frac{\varepsilon}{2}.$$

Combining the last three displays proves the first claim of the lemma.

For the second claim, note that the event $x \leftrightarrow y \not\leftrightarrow w \leftrightarrow z$ implies that $x \leftrightarrow \partial B_k(x)$, $y \leftrightarrow \partial B_k(y)$, $w \leftrightarrow \partial B_k(w)$, and $z \leftrightarrow \partial B_k(z)$, but the open cluster joining y to $\partial B_k(y)$ is not connected to the open cluster joining w to $\partial B_k(w)$. By Lemma 3.3, the probability of this event can be made as small as we like by choosing k large enough. □

The next lemma is the analogue of Lemma 3.4 for four-point connectivities.

Lemma 3.7. *Given any $\varepsilon > 0$ and n, there exist k and l depending only on d and ε (and not on n), such that if $x, y, w, z \in B_n$ are such that all interpoint ℓ^∞ distances are greater than $2k$, and all four points are at least at an ℓ^∞ distance l from the boundary, then we have*

$$P_n^0(x \leftrightarrow y \leftrightarrow w \leftrightarrow z) \ge q^2 - \varepsilon.$$

Proof. Let C_x and C_y be as in the proof of Lemma 3.4, and define C_w and C_z analogously. Let E be the event at least two of these clusters are not connected to each other. Then, as in the proof of Lemma 3.4, we can use Lemma 3.3 to conclude that $P_n^0(E) < \frac{\varepsilon}{2}$ if k is chosen large enough. Also, as in the proof of Lemma 3.4, we deduce that

$$\begin{aligned} &P_n^0(x \leftrightarrow y \leftrightarrow w \leftrightarrow z) \\ &\ge P_n^0(x \leftrightarrow \partial B_k(x),\ y \leftrightarrow \partial B_k(y),\ w \leftrightarrow \partial B_k(w),\ z \leftrightarrow \partial B_k(z)) - P_n^0(E). \end{aligned}$$

The proof is now completed by applying the FKG inequality to replace the first probability on the right by the product of the probabilities of the four events, and then proceeding as in the proof of Lemma 3.4 to show that these probabilities are all bounded below by $\sqrt{q} - \frac{\varepsilon}{100}$ if k and l are large enough. □

Finally, the following lemma gives the analogue of equation (3.6) for four-point correlatons.

Lemma 3.8. *Take any $\beta > 0$ and let $p := 1 - e^{-2\beta}$. Take any n. Let $\langle \cdot \rangle$ denote averaging with respect to the Ising model on B_n at inverse temperature β and free boundary condition, and let $P_n^0(\cdot)$ denote probability computed under the FK-Ising model with parameter p on B_n under free boundary condition. Then for any distinct $i, j, k, l \in B_n$,*

$$\begin{aligned}\langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle &= P_n^0(\text{Any open cluster contains even number of elements from } \{i, j, k, l\}) \\ &= P_n^0(i \leftrightarrow j \leftrightarrow k \leftrightarrow l) + P_n^0(i \leftrightarrow j \not\leftrightarrow k \leftrightarrow l) \\ &\quad + P_n^0(i \leftrightarrow k \not\leftrightarrow j \leftrightarrow l) + P_n^0(i \leftrightarrow l \not\leftrightarrow j \leftrightarrow k).\end{aligned}$$

Proof. It is a standard fact that a configuration from the Ising model on B_n at inverse temperature β and free boundary condition may be obtained as follows. First, generate a configuration from the FK-Ising model on B_n with parameter p, under free boundary condition. Then, take the connected components of vertices in this configuration, and independently for each component, assign the same spin to all vertices, where the spin is chosen to be 1 or -1 with equal probability. (For a proof, see [35, Theorem 1.16].)

Now take any distinct $i, j, k, l \in B_n$. To compute $\langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle$, we consider the above coupling and compute the conditional expectations given the FK-Ising configuration, which we denote by $\langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle'$. The following are easy to see:

- If i, j, k, l are all in the same cluster, them $\langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle' = 1$.
- If two of i, j, k, l are in one cluster and the other two are in a different cluster, then $\langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle' = 1$.
- In all other cases, $\langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle' = 0$.

Taking unconditional expectation gives the desired result. □

We now arrive at the main result of this subsection.

Theorem 3.9. *Let d, β_0, β and q be as in Theorem 3.5. Take any $\varepsilon \in (0, 1)$. For each n, let*

$$\gamma_n = \gamma_n(\beta, d, \varepsilon) := \max\{|\langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle - q^2| : i, j, k, l \in B_{\lfloor (1-\varepsilon)n \rfloor}, \\ \text{all pairwise } \ell^1 \textit{ distances between } i, j, k, l \textit{ are } \geq \varepsilon n\},$$

where $\langle \cdot \rangle$ denotes averaging with respect to the Ising model on B_n at inverse temperature β and free boundary condition. Then $\gamma_n \to 0$ as $n \to \infty$.

Proof. It is easy to see that Theorem 3.9 follows from the representation of the four-point correlation given in Lemma 3.8, together with the upper and lower bounds given in Lemmas 3.6 and 3.7. □

3.5. Concentration of the magnetization and the overlap in the Ising model. To generalize the results for two-point and four-point correlations to l-point correlations for all even l, as well as for other purposes, we need the following theorem.

Theorem 3.10. *Let d, β_0, β and q be as in Theorem 3.5. Let m be the magnetization and $R_{1,2}$ be the overlap between two independent replicas in the ferromagnetic Ising model on B_n under free boundary condition and inverse temperature β. Then $\langle (m^2 - q)^2 \rangle \to 0$ and $\langle (R_{1,2}^2 - q^2)^2 \rangle \to 0$ as $n \to \infty$.*

Proof. Throughout this proof, $C, C_1, C_2, \ldots$ will denote constants that depend only on d, whose values may change from line to line. Fix some n and some $\varepsilon \in (0, 1)$. Let $m := \lfloor (1-\varepsilon)n \rfloor$. Let δ_n be as in Theorem 3.5 and γ_n be as in Theorem 3.9. Let

$$S := \{(i, j) \in B_n \times B_n : i, j \in B_m, \ |i-j|_1 \geq \varepsilon n\}, \tag{3.7}$$

and let

$$T := \{(i, j, k, l) \in B_m^4 : \text{all pairwise } \ell^1 \text{ distances between } i, j, k, l \text{ are } \geq \varepsilon n\}.$$

Let $S^c := B_n^2 \setminus S$ and $T^c := B_n^4 \setminus T$. Now, if $(i, j) \in S^c$, then either at least one of i and j is in $B_n \setminus B_m$, or $|i-j|_1 < \varepsilon n$. From this observation, it follows that

$$|S^c| \leq C\varepsilon n^{2d} + C\varepsilon^d n^{2d}, \tag{3.8}$$

where C depends only on d. Since

$$|\langle m^2 \rangle - q| \leq \frac{1}{|B_n|^2} \sum_{i,j \in B_n} |\langle \sigma_i \sigma_j \rangle - q| \leq \frac{|S^c|}{|B_n|^2} + \frac{|S|\delta_n}{|B_n|^2},$$

this shows that

$$\limsup_{n\to\infty} |\langle m^2 \rangle - q| \leq C\varepsilon. \tag{3.9}$$

On the other hand,

$$\begin{aligned} |\langle m^4 \rangle - q^2| &\leq \frac{1}{|B_n|^4} \sum_{i,j,k,l \in B_n} |\langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle - q^2| \\ &\leq \frac{|T^c|}{|B_n|^4} + \frac{|T|\gamma_n}{|B_n|^4}, \end{aligned}$$

which shows that

$$\limsup_{n\to\infty} |\langle m^4 \rangle - q^2| \leq C\varepsilon. \tag{3.10}$$

Combining (3.9) and (3.10), we get

$$\begin{aligned} \limsup_{n\to\infty} \langle (m^2 - q)^2 \rangle &= \limsup_{n\to\infty} (\langle m^4 \rangle - 2\langle m^2 \rangle q + q^2) \\ &= \limsup_{n\to\infty} (\langle m^4 \rangle - q^2 - 2(\langle m^2 \rangle - q)q) \\ &\leq C\varepsilon. \end{aligned}$$

Since ε is arbitrary, this completes the proof of the first assertion of the theorem. Next, note that

$$\begin{aligned} \langle R_{1,2}^2 \rangle &= \left\langle \left(\frac{1}{|B_n|} \sum_{i \in B_n} \sigma_i^1 \sigma_i^2 \right)^2 \right\rangle \\ &= \frac{1}{|B_n|^2} \sum_{i,j \in B_n} \langle \sigma_i^1 \sigma_i^2 \sigma_j^1 \sigma_j^2 \rangle \\ &= \frac{1}{|B_n|^2} \sum_{i,j \in B_n} \langle \sigma_i \sigma_j \rangle^2. \end{aligned}$$

Proceeding as above, this shows that $\langle R_{1,2}^2\rangle \to q^2$ as $n \to \infty$. Similarly,

$$\langle R_{1,2}^4\rangle = \frac{1}{|B_n|^4}\sum_{i,j,k,l}\langle \sigma_i\sigma_j\sigma_k\sigma_l\rangle^2,$$

which can be used as above to show that $\langle R_{1,2}^4\rangle \to q^4$ as $n \to \infty$. Combining, we get that $\langle (R_{1,2}^2 - q^2)^2\rangle \to 0$. □

Corollary 3.11. *In the setting of Theorem 3.10, as $n \to \infty$, the law of m tends to the probability measure that puts equal mass on $\pm\sqrt{q}$, and the law of $R_{1,2}$ tends to the probability measure that puts equal mass on $\pm q$.*

Proof. Simply combine Theorem 3.10 with the observation that $\langle R_{1,2}\rangle = \langle m\rangle = 0$ by the invariance of the model under the transform $\sigma \to -\sigma$. □

3.6. Proof of Theorem 2.10. For $l = 2$, the proof is contained in the proof of Theorem 3.10. Take any even $l \geq 4$. Note that

$$\frac{1}{|B_n|^l}\sum_{i_1,\ldots,i_l\in B_n}\langle \sigma_{i_1}\cdots\sigma_{i_l}\rangle = \langle m^l\rangle, \quad \frac{1}{|B_n|^l}\sum_{i_1,\ldots,i_l\in B_n}\langle \sigma_{i_1}\cdots\sigma_{i_l}\rangle^2 = \langle R_{1,2}^l\rangle.$$

Thus, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
\frac{1}{|B_n|^l}\sum_{i_1,\ldots,i_l\in B_n}|\langle \sigma_{i_1}\cdots\sigma_{i_l}\rangle - q^{\frac{l}{2}}| &\leq \left[\frac{1}{|B_n|^l}\sum_{i_1,\ldots,i_l\in B_n}(\langle \sigma_{i_1}\cdots\sigma_{i_l}\rangle - q^{\frac{l}{2}})^2\right]^{\frac{1}{2}} \\
&= \left[\frac{1}{|B_n|^l}\sum_{i_1,\ldots,i_l\in B_n}(\langle \sigma_{i_1}\cdots\sigma_{i_l}\rangle^2 - 2q^{\frac{l}{2}}\langle \sigma_{i_1}\cdots\sigma_{i_l}\rangle + q^l)\right]^{\frac{1}{2}} \\
&= [\langle R_{1,2}^l\rangle - 2q^{\frac{l}{2}}\langle m^l\rangle + q^l]^{\frac{1}{2}}. \qquad (3.11)
\end{aligned}$$

Now, by the fact that $R_{1,2}$ and q are both in [0, 1], and the inequality

$$|x^{\frac{l}{2}} - y^{\frac{l}{2}}| \leq \frac{l}{2}|x - y|$$

that holds for all $x, y \in [-1, 1]$, we have

$$|\langle R_{1,2}^l\rangle - q^l| \leq \langle |R_{1,2}^l - q^l|\rangle \leq \frac{l}{2}\langle |R_{1,2}^2 - q^2|\rangle.$$

Thus, by Theorem 3.10, $\langle R_{1,2}^l\rangle \to q^l$ as $n \to \infty$. Similarly,

$$|\langle m^l\rangle - q^{\frac{l}{2}}| \leq \langle |m^l - q^{\frac{l}{2}}|\rangle \leq \frac{l}{2}\langle |m^2 - q|\rangle,$$

and so, $\langle m^l\rangle \to q^{\frac{l}{2}}$ as $n \to \infty$. Using these in (3.11) completes the proof.

4. Proofs of the Main Results

In this section, we will complete the proofs of the results from Sect. 2 (except Theorem 2.10, which has already been proved in the previous section). Throughout this section, we will let $\langle\cdot\rangle$ denote averaging with respect to the model on B_n with Hamiltonian H_n defined in (2.1), at inverse temperature β. Diverging from the notation used in the previous section, we will use $\langle\cdot\rangle_0$ to denote averaging with respect to the Ising model on B_n at inverse temperature β and free boundary condition (because this model corresponds to the case $h = 0$ of our model). The following lemma (which is just the central limit theorem for the moment generating function) will be used several times.

Lemma 4.1. *Take any $a_i \in \mathbb{R}$, $i \in B_n$. Let $\theta > 0$ be a constant such that $|a_i| \le \theta/2$ for all i, and $\mathbb{E}(e^{\theta|J_0|}) < \infty$. Then*

$$\left|\mathbb{E}\left[\exp\left(\frac{1}{\sqrt{|B_n|}}\sum_{i\in B_n} a_i J_i\right)\right] - \exp\left(\frac{1}{2|B_n|}\sum_{i\in B_n} a_i^2\right)\right| \le \frac{C}{|B_n|^{\frac{3}{2}}}\sum_{i\in B_n}|a_i|^3,$$

where C is a positive constant that depends only on the law of the J_i's and the choice of θ.

Proof. Take any θ as in the statement of the theorem. We will let $C, C_1, C_2, \ldots$ denote any positive constants whose values depend only on d, on the law of the J_i's and on the choice of θ, and whose values may change from line to line. First, note that for any k,

$$\mathbb{E}|J_0|^k \le \frac{k!}{\theta^k}\mathbb{E}(e^{\theta|J_0|}) \le \frac{Ck!}{\theta^k}. \tag{4.1}$$

By the above inequality and the facts that $\mathbb{E}(J_0) = 0$, $\mathbb{E}(J_0^2) = 1$, we get that for any i,

$$\mathbb{E}\left[\exp\left(\frac{a_i J_i}{\sqrt{|B_n|}}\right)\right] = 1 + \frac{a_i^2}{2|B_n|} + R_i, \tag{4.2}$$

where

$$R_i := \sum_{k=3}^{\infty}\mathbb{E}\left(\frac{a_i^k J_i^k}{k!|B_n|^{\frac{k}{2}}}\right).$$

Note that by (4.1) and the fact that $|a_i| \le \theta/2$ for all i,

$$|R_i| \le \sum_{k=3}^{\infty}\frac{|a_i|^k}{k!|B_n|^{\frac{k}{2}}}\mathbb{E}|J_i|^k \le \sum_{k=3}^{\infty}\frac{C_1|a_i|^3(\theta/2)^{k-3}}{|B_n|^{\frac{k}{2}}\theta^k} \le \frac{C_2|a_i|^3}{|B_n|^{\frac{3}{2}}}. \tag{4.3}$$

Similarly,

$$\left|\exp\left(\frac{a_i^2}{2|B_n|}\right) - 1 - \frac{a_i^2}{2|B_n|}\right| = \sum_{k=2}^{\infty}\frac{a_i^{2k}}{k!2^k|B_n|^k} \le \frac{Ca_i^4}{|B_n|^2}. \tag{4.4}$$

Now, for any N, and any $x_1, \ldots, x_N, y_1, \ldots, y_N \in \mathbb{R}$, if we let K be the maximum of $|x_i|$ and $|y_i|$ over all i, then

$$\left|\prod_{i=1}^{N} x_i - \prod_{i=1}^{N} y_i\right| \le \sum_{i=1}^{N} |x_1 \cdots x_{i-1} y_i \cdots y_N - x_1 \cdots x_i y_{i+1} \cdots y_N|$$
$$\le \sum_{i=1}^{N} K^{N-1} |x_i - y_i|. \tag{4.5}$$

By (4.2), (4.3), and the inequalities $1 + x \le e^x$ and $|a_i| \le \theta/2$,

$$0 \le \mathbb{E}\left[\exp\left(\frac{a_i J_i}{\sqrt{|B_n|}}\right)\right] \le e^{C/|B_n|}.$$

Similarly, by (4.4),

$$0 \le \exp\left(\frac{a_i^2}{2|B_n|}\right) \le e^{C/|B_n|}.$$

Thus, by (4.2), (4.3), (4.4) and (4.5),

$$\left|\mathbb{E}\left[\exp\left(\frac{1}{\sqrt{|B_n|}} \sum_{i \in B_n} a_i J_i\right)\right] - \exp\left(\frac{1}{2|B_n|} \sum_{i \in B_n} a_i^2\right)\right|$$
$$= \left|\prod_{i \in B_n} \mathbb{E}\left[\exp\left(\frac{a_i J_i}{\sqrt{|B_n|}}\right)\right] - \prod_{i \in B_n} \exp\left(\frac{a_i^2}{2|B_n|}\right)\right|$$
$$\le (e^{C/|B_n|})^{|B_n|} \sum_{i \in B_n} \left|\mathbb{E}\left[\exp\left(\frac{a_i J_i}{\sqrt{|B_n|}}\right)\right] - \exp\left(\frac{a_i^2}{2|B_n|}\right)\right|$$
$$\le \frac{C}{|B_n|^{\frac{3}{2}}} \sum_{i \in B_n} |a_i|^3.$$

This completes the proof of the lemma. □

4.1. Proof of Theorem 2.3. In this proof, $o(1)$ will denote any quantity, deterministic or random, whose absolute value can be bounded by a deterministic quantity depending only on n (and the law of the J_i's and our choices of β and d) that tends to zero as $n \to \infty$. We begin with the derivation of the approximate formula for the quenched expectation of the magnetization. Let X_n be defined as in (2.2), and define the random variable

$$L = L(\sigma) := \frac{\beta h}{\sqrt{|B_n|}} \sum_{i \in B_n} J_i \sigma_i, \tag{4.6}$$

where σ is drawn from the Ising model on B_n at inverse temperature β and free boundary condition. Let m and $R_{1,2}$ be the magnetization of σ and the overlap between two configurations drawn independently from the Gibbs measure of the Ising model, respectively. Let β and q be as in Theorem 3.5. The first step in the proof of Theorem 2.3 is the following lemma.

Lemma 4.2. *Let L be as above. Then*

$$\lim_{n\to\infty} \mathbb{E}[(\langle e^L\rangle_0 - e^{\frac{1}{2}\beta^2h^2(1-q)}\cosh X_n)^2] = 0.$$

Proof. Note that by Lemma 4.1,

$$\begin{aligned}\mathbb{E}\langle e^L\rangle_0^2 &= \mathbb{E}\langle e^{L(\sigma^1)+L(\sigma^2)}\rangle_0 \\ &= \mathbb{E}\biggl\langle \exp\biggl(\frac{\beta h}{\sqrt{|B_n|}}\sum_{i\in B_n} J_i(\sigma_i^1+\sigma_i^2)\biggr)\biggr\rangle_0 \\ &= \biggl\langle \exp\biggl(\frac{\beta^2h^2}{2|B_n|}\sum_{i\in B_n}(\sigma_i^1+\sigma_i^2)^2\biggr) + o(1)\biggr\rangle_0 \\ &= e^{\beta^2h^2}\langle e^{\beta^2h^2R_{1,2}}\rangle_0 + o(1). \end{aligned}\tag{4.7}$$

By Corollary 3.11, this shows that

$$\lim_{n\to\infty}\mathbb{E}\langle e^L\rangle_0^2 = e^{\beta^2h^2}\cosh(\beta^2h^2q). \tag{4.8}$$

Next, again by Lemma 4.1,

$$\begin{aligned} e^{\beta^2h^2(1-q)}\mathbb{E}\cosh^2 X_n &= \frac{1}{4}e^{\beta^2h^2(1-q)}\mathbb{E}(e^{2X_n}+e^{-2X_n}+2) \\ &= \frac{1}{2}e^{\beta^2h^2(1-q)}(e^{2\beta^2h^2q}+1)+o(1) \\ &= e^{\beta^2h^2}\cosh(\beta^2h^2q)+o(1). \end{aligned}\tag{4.9}$$

Finally, by another application of Lemma 4.1,

$$\begin{aligned} &e^{\frac{1}{2}\beta^2h^2(1-q)}\mathbb{E}[\langle e^L\rangle_0\cosh X_n] = \frac{1}{2}e^{\frac{1}{2}\beta^2h^2(1-q)}\mathbb{E}[\langle e^{L+X_n}\rangle_0 + \langle e^{L-X_n}\rangle_0] \\ &= \frac{1}{2}e^{\frac{1}{2}\beta^2h^2(1-q)}\mathbb{E}\biggl[\biggl\langle \exp\biggl(\frac{\beta h}{\sqrt{|B_n|}}\sum_{i\in B_n}J_i(\sigma_i+\sqrt{q})\biggr)\biggr\rangle_0 \\ &\qquad + \biggl\langle \exp\biggl(\frac{\beta h}{\sqrt{|B_n|}}\sum_{i\in B_n}J_i(\sigma_i-\sqrt{q})\biggr)\biggr\rangle_0\biggr] \\ &= \frac{1}{2}e^{\frac{1}{2}\beta^2h^2(1-q)}\biggl[\biggl\langle \exp\biggl(\frac{\beta^2h^2}{2|B_n|}\sum_{i\in B_n}(\sigma_i+\sqrt{q})^2\biggr)\biggr\rangle_0 \\ &\qquad + \biggl\langle \exp\biggl(\frac{\beta^2h^2}{2|B_n|}\sum_{i\in B_n}(\sigma_i-\sqrt{q})^2\biggr)\biggr\rangle_0 + o(1)\biggr] \\ &= \frac{1}{2}e^{\beta^2h^2}[\langle e^{\beta^2h^2\sqrt{q}m}\rangle_0 + \langle e^{-\beta^2h^2\sqrt{q}m}\rangle_0] + o(1). \end{aligned}\tag{4.10}$$

But, by Corollary 3.11,

$$\lim_{n\to\infty}\langle e^{\beta^2h^2\sqrt{q}m}\rangle_0 = \lim_{n\to\infty}\langle e^{-\beta^2h^2\sqrt{q}m}\rangle_0 = \cosh(\beta^2h^2q).$$

Thus,

$$\lim_{n\to\infty} e^{\frac{1}{2}\beta^2h^2(1-q)}\mathbb{E}[\langle e^L\rangle_0 \cosh X_n] = e^{\beta^2h^2}\cosh(\beta^2h^2q). \tag{4.11}$$

Combining (4.8), (4.9) and (4.11), we get

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{E}[(\langle e^L\rangle_0 - e^{\frac{1}{2}\beta^2h^2(1-q)}\cosh X_n)^2]\\ &= \lim_{n\to\infty} \mathbb{E}[\langle e^L\rangle_0^2 - 2e^{\frac{1}{2}\beta^2h^2(1-q)}\langle e^L\rangle_0 \cosh X_n + e^{\beta^2h^2(1-q)}\cosh^2 X_n] = 0.\end{aligned}$$

This completes the proof of the lemma. □

The next step in the proof of Theorem 2.3 is the following lemma.

Lemma 4.3. *Let L be as above. Then*

$$\lim_{n\to\infty} \mathbb{E}[(\langle me^L\rangle_0 - \sqrt{q}e^{\frac{1}{2}\beta^2h^2(1-q)}\sinh X_n)^2] = 0.$$

Proof. Take any $j \in B_n$. By a computation similar to the one that led to equation (4.7), we get

$$\mathbb{E}\langle\sigma_j e^L\rangle_0^2 = e^{\beta^2h^2}\langle\sigma_j^1\sigma_j^2 e^{\beta^2h^2R_{1,2}}\rangle_0 + o(1).$$

Averaging over j, we get

$$\frac{1}{|B_n|}\sum_{j\in B_n}\mathbb{E}\langle\sigma_j e^L\rangle_0^2 = e^{\beta^2h^2}\langle R_{1,2}e^{\beta^2h^2R_{1,2}}\rangle_0 + o(1).$$

By Corollary 3.11, this shows that

$$\lim_{n\to\infty}\frac{1}{|B_n|}\sum_{j\in B_n}\mathbb{E}\langle\sigma_j e^L\rangle_0^2 = qe^{\beta^2h^2}\sinh(\beta^2h^2q). \tag{4.12}$$

Next, by a computation similar to the one that led to equation (4.9), we get

$$qe^{\beta^2h^2(1-q)}\mathbb{E}\sinh^2 X_n = qe^{\beta^2h^2}\sinh(\beta^2h^2q) + o(1). \tag{4.13}$$

Finally, by a computation similar to the one that led to equation (4.10), we get

$$\begin{aligned}&\sqrt{q}e^{\frac{1}{2}\beta^2h^2(1-q)}\mathbb{E}[\langle\sigma_j e^L\rangle_0\sinh X_n]\\ &= \frac{\sqrt{q}}{2}e^{\beta^2h^2}[\langle\sigma_j e^{\beta^2h^2\sqrt{q}m}\rangle_0 - \langle\sigma_j e^{-\beta^2h^2\sqrt{q}m}\rangle_0] + o(1).\end{aligned}$$

Averaging this over j, we get

$$\frac{1}{|B_n|}\sum_{j\in B_n}\sqrt{q}e^{\frac{1}{2}\beta^2h^2(1-q)}\mathbb{E}[\langle\sigma_j e^L\rangle_0\sinh X_n] = \sqrt{q}e^{\beta^2h^2}\langle m\sinh(\beta^2h^2\sqrt{q}m)\rangle_0 + o(1).$$

By Corollary 3.11, this shows that

$$\lim_{n\to\infty}\frac{1}{|B_n|}\sum_{j\in B_n}\sqrt{q}e^{\frac{1}{2}\beta^2h^2(1-q)}\mathbb{E}[\langle\sigma_j e^L\rangle_0\sinh X_n] = qe^{\beta^2h^2}\sinh(\beta^2h^2q). \tag{4.14}$$

Combining (4.12), (4.13) and (4.14), we get

$$\lim_{n\to\infty} \frac{1}{|B_n|} \sum_{j\in B_n} \mathbb{E}[(\langle \sigma_j e^L\rangle_0 - \sqrt{q} e^{\frac{1}{2}\beta^2 h^2(1-q)} \sinh X_n)^2] = 0. \tag{4.15}$$

An application of the Cauchy–Schwarz inequality shows that the above quantity is an upper bound for the quantity that we want to show is converging to zero, thereby completing the proof. □

The final ingredient we need is the following.

Lemma 4.4. *Let β be as in Theorem 3.5. Then $\langle (R_{1,2} - m(\sigma^1)m(\sigma^2))^2\rangle_0 \to 0$ as $n \to \infty$, where σ^1 and σ^2 are drawn independently from the Ising model on B_n at inverse temperature β and free boundary condition.*

Proof. Note that

$$\langle (R_{1,2} - m(\sigma^1)m(\sigma^2))^2\rangle_0 = \langle R_{1,2}^2\rangle_0 - 2\langle R_{1,2} m(\sigma^1)m(\sigma^2)\rangle_0 + \langle m^2\rangle_0^2.$$

By Corollary 3.11, $\langle R_{1,2}^2\rangle_0 \to q^2$ and $\langle m^2\rangle_0 \to q$ as $n \to \infty$. Now, note that

$$\begin{aligned}\langle R_{1,2} m(\sigma^1)m(\sigma^2)\rangle_0 &= \left\langle \frac{1}{|B_n|^3} \sum_{i,j,k\in B_n} \sigma_i^1\sigma_i^2\sigma_j^1\sigma_k^2 \right\rangle_0 \\ &= \frac{1}{|B_n|^3} \sum_{i,j,k\in B_n} \langle \sigma_i\sigma_j\rangle_0 \langle \sigma_i\sigma_k\rangle_0.\end{aligned}$$

Using the same tactics as in the proof of Theorem 3.10, it is now easy to show that the above quantity tends to q^2 as $n \to \infty$. This completes the proof. □

We are now ready to complete the proof of Theorem 2.3.

Proof of Theorem 2.3. First, note that by Jensen's inequality,

$$\langle e^L\rangle_0 \geq e^{\langle L\rangle_0} = 1. \tag{4.16}$$

Thus, by (4.16) and the Cauchy–Schwarz inequality,

$$\begin{aligned}\langle |m^2 - q|\rangle &= \frac{\langle |m^2 - q| e^L\rangle_0}{\langle e^L\rangle_0} \\ &\leq \langle |m^2 - q| e^L\rangle_0 \leq \sqrt{\langle (m^2-q)^2\rangle_0 \langle e^{2L}\rangle_0}.\end{aligned}$$

Taking expectation on both sides and applying Jensen's inequality, we get

$$\mathbb{E}(\langle |m^2 - q|\rangle) \leq \sqrt{\langle (m^2-q)^2\rangle_0 \mathbb{E}(\langle e^{2L}\rangle_0)}.$$

By Theorem 3.10, the first term within the square-root tends to zero as $n \to \infty$. By Lemma 4.1, the second term is uniformly bounded in n. Thus, $\mathbb{E}(\langle |m^2 - q|\rangle) \to 0$ as $n \to \infty$. Since

$$\langle (m^2 - q)^2\rangle \leq 2\langle |m^2 - q|\rangle,$$

this proves the first claim of the theorem. Next, note that

$$\langle m\rangle = \frac{\langle me^L\rangle_0}{\langle e^L\rangle_0}.$$

Thus, if we let

$$a := \sqrt{q}e^{\frac{1}{2}\beta^2h^2(1-q)}\sinh X_n, \quad b := e^{\frac{1}{2}\beta^2h^2(1-q)}\cosh X_n, \quad c := \frac{a}{b} = \sqrt{q}\tanh X_n,$$

then by (4.16),

$$\begin{aligned}|\langle m\rangle - c| &= \left|\frac{\langle me^L\rangle_0}{\langle e^L\rangle_0} - \frac{a}{b}\right| = \frac{|b\langle me^L\rangle_0 - a\langle e^L\rangle_0|}{b\langle e^L\rangle_0}\\ &\le \frac{|b\langle me^L\rangle_0 - a\langle e^L\rangle_0|}{b} \le |\langle me^L\rangle_0 - a| + \frac{a}{b}|\langle e^L\rangle_0 - b|.\end{aligned}$$

Since $b \ge 1$, this shows that

$$\begin{aligned}\mathbb{E}|\langle m\rangle - c| &\le \mathbb{E}|\langle me^L\rangle_0 - a| + \mathbb{E}(a|\langle e^L\rangle_0 - b|)\\ &\le \sqrt{\mathbb{E}[(\langle me^L\rangle_0 - a)^2]} + \sqrt{\mathbb{E}(a^2)\mathbb{E}[(\langle e^L\rangle_0 - b)^2]}.\end{aligned} \tag{4.17}$$

By Lemmas 4.2, 4.3, and the fact that $\mathbb{E}(a^2)$ is uniformly bounded in n (by Lemma 4.1), we get that the above quantity tends to zero as $n \to \infty$. But, since $\langle m\rangle$ and c are both in $[-1, 1]$,

$$\mathbb{E}[(\langle m\rangle - c)^2] \le 2\mathbb{E}|\langle m\rangle - c|.$$

This proves (2.6). To prove (2.7), note that by Lemma 4.2 and the inequality (4.16), proceeding as in the derivation of (4.17), we get

$$\begin{aligned}\frac{1}{|B_n|}\sum_{j\in B_n}\mathbb{E}|\langle\sigma_j\rangle - \sqrt{q}\tanh X_n| &= \frac{1}{|B_n|}\sum_{j\in B_n}\mathbb{E}\left|\frac{\langle\sigma_j e^L\rangle_0}{\langle e^L\rangle_0} - \frac{a}{b}\right|\\ &= \frac{1}{|B_n|}\sum_{j\in B_n}\mathbb{E}\left(\frac{|b\langle\sigma_j e^L\rangle_0 - a\langle e^L\rangle_0|}{b\langle e^L\rangle_0}\right)\\ &\le \frac{1}{|B_n|}\sum_{j\in B_n}\mathbb{E}\left(\frac{|b\langle\sigma_j e^L\rangle_0 - a\langle e^L\rangle_0|}{b}\right)\\ &\le \sqrt{\mathbb{E}(a^2)\mathbb{E}[(\langle e^L\rangle_0 - b)^2]} + \frac{1}{|B_n|}\sum_{j\in B_n}\sqrt{\mathbb{E}[(\langle\sigma_j e^L\rangle_0 - a)^2]}.\end{aligned}$$

We have already seen that the first term tends to zero as $n \to \infty$. The second term is bounded above by

$$\left[\frac{1}{|B_n|}\sum_{j\in B_n}\mathbb{E}[(\langle\sigma_j e^L\rangle_0 - a)^2]\right]^{\frac{1}{2}}.$$

By (4.15), this also tends to zero as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \frac{1}{|B_n|} \sum_{j\in B_n} \mathbb{E}|\langle\sigma_j\rangle - \sqrt{q}\tanh X_n| = 0.$$

Thus, by (2.6) and the fact that

$$(\langle\sigma_j\rangle - \sqrt{q}\tanh X_n)^2 \le 2|\langle\sigma_j\rangle - \sqrt{q}\tanh X_n|,$$

we get (2.7). Finally, note that by (4.16) and the Cauchy–Schwarz inequality,

$$\begin{aligned}\mathbb{E}\langle|R_{1,2} - m(\sigma^1)m(\sigma^2)|\rangle &= \mathbb{E}\biggl(\frac{\langle|R_{1,2} - m(\sigma^1)m(\sigma^2)|e^{L(\sigma^1)+L(\sigma^2)}\rangle_0}{\langle e^L\rangle_0^2}\biggr)\\ &\le \mathbb{E}\langle|R_{1,2} - m(\sigma^1)m(\sigma^2)|e^{L(\sigma^1)+L(\sigma^2)}\rangle_0\\ &\le \sqrt{\langle(R_{1,2} - m(\sigma^1)m(\sigma^2))^2\rangle_0 \mathbb{E}\langle e^{2L(\sigma^1)+2L(\sigma^2)}\rangle_0}\end{aligned}$$

By Lemma 4.4, the first term inside the square-root tends to zero as $n \to \infty$. The second term is uniformly bounded in n, by Lemma 4.1. This shows that the expression on the left tends to zero. But note that

$$\mathbb{E}\langle(R_{1,2} - m(\sigma^1)m(\sigma^2))^2\rangle \le 2\mathbb{E}\langle|R_{1,2} - m(\sigma^1)m(\sigma^2)|\rangle.$$

This completes the proof of Theorem 2.3.

4.2. Proof of Theorem 2.1. All assertions of Theorem 2.1 are direct consequences of the properties of m from Theorem 2.3 and the result that $\mathbb{E}\langle(R_{1,2} - m(\sigma^1)m(\sigma^2))^2\rangle \to 0$ as $n \to \infty$.

4.3. Proof of Theorem 2.2. Let $A := \{-\sqrt{q}, \sqrt{q}\}^3 \subseteq \mathbb{R}^3$, and let B denote the set displayed in (2.5). Consider the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $f(x, y, z) := (xy, yz, zx)$. Then f is a continuous map, and an easy verification shows that $f(A) = B$. (For example, $f(\sqrt{q}, \sqrt{q}, \sqrt{q}) = (q, q, q)$, $f(\sqrt{q}, \sqrt{q}, -\sqrt{q}) = (q, -q, -q)$, $f(\sqrt{q}, -\sqrt{q}, -\sqrt{q}) = (-q, q, -q)$, etc.) Take any open set $V \supseteq B$, and let $U = f^{-1}(V)$. Then U is also open, and $U \supseteq A$. Let $\sigma^1, \sigma^2, \sigma^3$ be three configurations drawn independently from the Gibbs measure of our model, and define the overlaps as usual. By Theorem 2.3, the difference between the random vectors $(R_{1,2}, R_{2,3}, R_{3,1})$ and $f(m(\sigma^1), m(\sigma^2), m(\sigma^3))$ converges to the zero vector in L^2 (unconditionally, after integrating out the disorder). This shows, first of all, that the quenched law of $(R_{1,2}, R_{2,3}, R_{3,1})$ converges in distribution, because so does the quenched law of $(m(\sigma^1), m(\sigma^2), m(\sigma^3))$. Next, note that by Theorem 2.3,

$$\lim_{n\to\infty} \mathbb{P}((m(\sigma^1), m(\sigma^2), m(\sigma^3)) \in U) = 1,$$

where $\mathbb{P}$ denotes the unconditional probability, after integrating out the disorder. Thus,

$$\lim_{n\to\infty} \mathbb{P}(f(m(\sigma^1), m(\sigma^2), m(\sigma^3)) \in V) = 1.$$

Combining this with the previous observation, we see that for any open set $V \supseteq B$,

$$\lim_{n\to\infty} \mathbb{P}((R_{1,2}, R_{2,3}, R_{3,1}) \in V) = 1.$$

This shows that the quenched probability of the event $(R_{1,2}, R_{2,3}, R_{3,1}) \in V$ converges to 1 in probability. From this, it is easy to complete the proof of the theorem.

4.4. Proof of Theorem 2.7. Taking $k = 2$, $f = S_{1,2}$ and $\psi(x) = x$ in (2.8) gives the equation

$$\mathbb{E}(S_{1,2}S_{1,3}) = \frac{1}{2}(\mathbb{E}(S_{1,2}))^2 + \frac{1}{2}\mathbb{E}(S_{1,2}^2).$$

We will show that this equation fails for the overlap in the infinite volume limit of our model. Indeed, by Theorem 2.3,

$$\begin{aligned}
&\lim_{n\to\infty} (2\mathbb{E}\langle R_{1,2}R_{1,3}\rangle - (\mathbb{E}\langle R_{1,2}\rangle)^2 - \mathbb{E}\langle R_{1,2}^2\rangle) \\
&= \lim_{n\to\infty} (2\mathbb{E}\langle m(\sigma^1)^2 m(\sigma^2) m(\sigma^3)\rangle - (\mathbb{E}\langle m(\sigma^1) m(\sigma^2)\rangle)^2 - \mathbb{E}\langle m(\sigma^1)^2 m(\sigma^2)^2\rangle) \\
&= \lim_{n\to\infty} (2\mathbb{E}(\langle m^2\rangle\langle m\rangle^2) - (\mathbb{E}(\langle m\rangle^2))^2 - \mathbb{E}(\langle m^2\rangle^2)), \qquad (4.18)
\end{aligned}$$

provided that the limits exist (which we will prove shortly). Let X_n be defined as in (2.2), and let $Y_n := \tanh X_n$. Then by Theorem 2.3, the right side of (4.18) equals

$$\lim_{n\to\infty} (2q^2\mathbb{E}(Y_n^2) - q^2(\mathbb{E}(Y_n^2))^2 - q^2) = -q^2 \lim_{n\to\infty} (1 - \mathbb{E}(Y_n^2))^2.$$

But Y_n is a bounded random variable with converges in distribution to $\tanh(\sqrt{q}\beta h Z)$, where Z is a standard Gaussian random variable. Thus, for any finite h, the above limit is nonzero. This completes the proof.

4.5. Proof of Theorem 2.9. In this subsection, we will denote averaging with respect to the antiferromagnetic Ising model on B_n at inverse temperature β and free boundary condition by $\langle\cdot\rangle_{a,0}$, and averaging with respect to the model on B_n with Hamiltonian (2.9) at inverse temperature β by $\langle\cdot\rangle_a$.

Let σ be a configuration drawn from the ferromagnetic Ising model on B_n at inverse temperature β and free boundary condition. Define $\eta \in \Sigma_n$ as

$$\eta_i := (-1)^{|i|_1}\sigma_i \text{ for all } i \in B_n. \qquad (4.19)$$

Then, it is easy to see that η is drawn from antiferromagnetic Ising model on B_n at inverse temperature β and free boundary condition. Thus, we have

$$\langle m^2\rangle_{a,0} = \frac{1}{|B_n|^2}\sum_{i,j\in B_n} \langle\sigma_i\sigma_j\rangle_{a,0} = \frac{1}{|B_n|^2}\sum_{i,j\in B_n} (-1)^{|i|_1+|j|_1}\langle\sigma_i\sigma_j\rangle_0.$$

Take any $\varepsilon \in (0, 1)$. Let δ_n be as in Theorem 3.5 and let $m := \lfloor(1-\varepsilon)n\rfloor$. Let S be defined as in equation (3.7), and let $S^c := (B_n \times B_n)\setminus S$. Then

$$\frac{1}{|B_n|^2}\sum_{i,j\in B_n} (-1)^{|i|_1+|j|_1}(\langle\sigma_i\sigma_j\rangle_0 - q) \le \frac{|S^c|}{|B_n|^2} + \frac{|S|\delta_n}{|B_n|^2} \le \frac{|S^c|}{|B_n|^2} + \delta_n.$$

By Theorem 3.5, $\delta_n \to 0$ as $n \to \infty$. Combining these observations and the upper bound (3.8), we get

$$\limsup_{n\to\infty} \frac{1}{|B_n|^2} \sum_{i,j\in B_n} (-1)^{|i|_1+|j|_1} (\langle \sigma_i \sigma_j \rangle_0 - q) \le C\varepsilon.$$

It is easy to see that

$$\lim_{n\to\infty} \frac{1}{|B_n|^2} \sum_{i,j\in B_n} (-1)^{|i|_1+|j|_1} = 0.$$

Combining all of the above, we get that $\limsup_{n\to\infty} \langle m^2 \rangle_{a,0} \le C\varepsilon$. Since this holds for every $\varepsilon \in (0, 1)$, and $\langle m^2 \rangle_{a,0} \ge 0$, we conclude that $\langle m^2 \rangle_{a,0} \to 0$ as $n \to \infty$.

Now let L be defined as in (4.6). Then, as in (4.16), we have $\langle e^L \rangle_{a,0} \ge e^{\langle L \rangle_{a,0}} = 1$. Thus,

$$\mathbb{E}\langle |m| \rangle_a = \mathbb{E}\left(\frac{\langle |m| e^L \rangle_{a,0}}{\langle e^L \rangle_{a,0}} \right) \le \mathbb{E}\langle |m| e^L \rangle_{a,0} \le \sqrt{\mathbb{E}\langle m^2 \rangle_{a,0} \mathbb{E}\langle e^L \rangle_{a,0}}.$$

We have shown above that the first term inside the square-root tends to zero as $n \to \infty$. By Lemma 4.1 and the above relationship between η and σ, the second term is uniformly bounded in n. Thus, $\mathbb{E}\langle |m| \rangle_a \to 0$ as $n \to \infty$. Since $|m| \le 1$, this implies that $\mathbb{E}\langle m^2 \rangle_a \to 0$.

Lastly, let σ^1 and σ^2 are configurations drawn independently from the model on B_n with Hamiltonian (2.1) at inverse temperature β, but with J_i replaced by $(-1)^{|i|_1} J_i$. Define η^1 and η^2 via the relationship (4.19). Then, it is easy to see that η^1 and η^2 are drawn independently from the model on B_n with Hamiltonian (2.9) at inverse temperature β. Moreover, the overlap between η^1 and η^2 is exactly the same as the overlap between σ^1 and σ^2. Thus, all of the claims about the overlap that we have proved for the ferromagnetic model continue to hold for the antiferromagnetic model, after replacing J_i by $(-1)^{|i|_1} J_i$ in the theorem statements.

4.6. Proof of Theorem 2.8. Let F denote the free energy of our model. That is,

$$F = \log \sum_{\sigma\in\Sigma_n} e^{-\beta H_n(\sigma)},$$

with H_n defined as in (2.1). Then note that

$$\frac{\partial F}{\partial J_i} = \frac{\beta h \langle \sigma_i \rangle}{\sqrt{|B_n|}}.$$

This implies, by the Gaussian Poincaré inequality [42, p. 49], that

$$\mathrm{Var}(F) \le \sum_{i\in B_n} \mathbb{E}\left[\left(\frac{\partial F}{\partial J_i} \right)^2 \right] \le \beta^2 h^2. \tag{4.20}$$

On the other hand,

$$\frac{\partial^2 F}{\partial J_i \partial J_j} = \frac{\beta^2 h^2}{|B_n|} (\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle),$$

and therefore, by [18, Theorem 3.1],

$$\begin{aligned}\mathrm{Var}(F) &\geq \frac{1}{2}\sum_{i,j\in B_n}\left[\mathbb{E}\left(\frac{\partial^2 F}{\partial J_i \partial J_j}\right)\right]^2 \\ &= \frac{\beta^4 h^4}{2|B_n|^2}\sum_{i,j\in B_n}[\mathbb{E}(\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle)]^2. \end{aligned} \tag{4.21}$$

By the FKG inequality for the RFIM [17, Lemma 2.5], $\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle \geq 0$ for i, j. Thus,

$$\begin{aligned}\langle R_{1,2}^2\rangle - \langle R_{1,2}\rangle^2 &= \frac{1}{|B_n|^2}\sum_{i,j\in B_n}(\langle\sigma_i\sigma_j\rangle^2 - \langle\sigma_i\rangle^2\langle\sigma_j\rangle^2) \\ &= \frac{1}{|B_n|^2}\sum_{i,j\in B_n}(\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle)(\langle\sigma_i\sigma_j\rangle + \langle\sigma_i\rangle\langle\sigma_j\rangle) \\ &\leq \frac{2}{|B_n|^2}\sum_{i,j\in B_n}|\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle| \\ &= \frac{2}{|B_n|^2}\sum_{i,j\in B_n}(\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle).\end{aligned}$$

Combining this with (4.21), we get

$$\begin{aligned}\mathbb{E}\langle(R_{1,2} - \langle R_{1,2}\rangle)^2\rangle &= \mathbb{E}(\langle R_{1,2}^2\rangle - \langle R_{1,2}\rangle^2) \\ &\leq \frac{2}{|B_n|^2}\sum_{i,j\in B_n}\mathbb{E}(\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle) \\ &\leq 2\left[\frac{1}{|B_n|^2}\sum_{i,j\in B_n}(\mathbb{E}(\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle))^2\right]^{\frac{1}{2}} \\ &\leq 2\left[\frac{2\mathrm{Var}(F)}{\beta^4 h^4}\right]^{\frac{1}{2}}.\end{aligned}$$

Plugging in the upper bound on $\mathrm{Var}(F)$ from (4.20), we get

$$\mathbb{E}\langle(R_{1,2} - \langle R_{1,2}\rangle)^2\rangle \leq \frac{2^{\frac{3}{2}}}{\beta|h|}.$$

The upper bound tends to zero if $|h| \to \infty$ as $n \to \infty$. This proves the second claim of Theorem 2.8. For the first claim, note that for any k,

$$\langle R_{1,2}^k\rangle = \frac{\langle R_{1,2}^k e^L\rangle_0}{\langle e^L\rangle_0},$$

where L is the function defined in (4.6). By (4.16), this shows that

$$\begin{aligned}|\langle R_{1,2}^k\rangle - \langle R_{1,2}^k\rangle_0| &= \left|\frac{\langle R_{1,2}^k e^L\rangle_0}{\langle e^L\rangle_0} - \langle R_{1,2}^k\rangle_0\right| \\ &= \frac{|\langle R_{1,2}^k e^L\rangle_0 - \langle R_{1,2}^k\rangle_0\langle e^L\rangle_0|}{\langle e^L\rangle_0} \\ &\le |\langle R_{1,2}^k e^L\rangle_0 - \langle R_{1,2}^k\rangle_0\langle e^L\rangle_0| \\ &= |\langle R_{1,2}^k (e^L - \langle e^L\rangle_0)\rangle_0| \\ &\le \langle |e^L - \langle e^L\rangle_0|\rangle_0 \\ &= \langle |(e^L - 1) - \langle e^L - 1\rangle_0|\rangle_0 \le 2\langle |e^L - 1|\rangle_0.\end{aligned}$$

Now, note that for any given $\sigma \in \Sigma_n$, by Lemma 4.1,

$$\begin{aligned}\mathbb{E}|e^{L(\sigma)} - 1| &\le \sqrt{\mathbb{E}[(e^{L(\sigma)} - 1)^2]} \\ &= \sqrt{\mathbb{E}[e^{2L(\sigma)} - 2e^{L(\sigma)} + 1]} \\ &= \sqrt{e^{2\beta^2h^2} - 2e^{\frac{1}{2}\beta^2h^2} + 1 + o(1)} \\ &= \sqrt{(e^{2\beta^2h^2} - 1) - 2(e^{\frac{1}{2}\beta^2h^2} - 1) + o(1)}.\end{aligned}$$

By the inequality $e^x - 1 \le ex$ that holds for $0 \le x \le 1$, we get that the above quantity is bounded above by $C\beta|h| + o(1)$ when $|h| \le \frac{1}{\beta}$, where C is a universal constant. In particular, it tends to zero as $h \to 0$. Thus, if $h \to 0$ as $n \to \infty$, then for every k,

$$\lim_{n\to\infty} \mathbb{E}|\langle R_{1,2}^k\rangle - \langle R_{1,2}^k\rangle_0| = 0.$$

This shows that if n is large, then all quenched moments of $R_{1,2}$ under our model are, with high probability, close to the corresponding moments of $R_{1,2}$ under the Ising model. From this, it is not hard to prove the claim stated in the theorem (e.g., using Bernstein approximation).

4.7. Proof of Theorem 2.6. By Theorem 2.3 and Theorem 2.1, $\mathbb{E}\langle (R_{1,2}^2 - q^2)^2\rangle \to 0$ and $\mathbb{E}\langle (m^2 - q)^2\rangle \to 0$ for our model. This allows us to repeat the proof of Theorem 2.10 from Sect. 3.6 verbatim to deduce that the conclusion of Theorem 2.10 holds even if $h \neq 0$, with the same q. This shows, in particular, that if π_n is a uniform random permutation of the elements of B_n, then for any even l,

$$\langle \sigma_{\pi_n(1)}\sigma_{\pi_n(2)}\cdots\sigma_{\pi_n(l)}\rangle \to q^{\frac{l}{2}} \tag{4.22}$$

in probability as $n \to \infty$. Next, let us consider the case of odd l. For the Ising model, the above expectation is zero if l is odd. This is no longer true if $h \neq 0$. Recall the random variable X_n defined in equation (2.2). Take any odd positive integer l. We claim that

$$\langle \sigma_{\pi_n(1)}\sigma_{\pi_n(2)}\cdots\sigma_{\pi_n(l)}\rangle - q^{\frac{l}{2}}\tanh X_n \to 0 \tag{4.23}$$

in probability as $n \to \infty$. To prove this, let m be the magnetization. We claim that

$$\langle m^l\rangle - q^{\frac{l}{2}}\tanh X_n \to 0 \tag{4.24}$$

in probability as $n \to \infty$. To see this, note that by Theorem 2.3, $\mathbb{E}\langle (m^2 - q)^2 \rangle \to 0$. From this, it is easy to deduce that

$$\langle m^l \rangle - q^{\frac{1}{2}(l-1)} \langle m \rangle \to 0$$

in probability, since m^{l-1} can be replaced by $q^{\frac{1}{2}(l-1)}$ asymptotically. But again by Theorem 2.3, $\langle m \rangle - \sqrt{q} \tanh X_n \to 0$ in probability. Combining these two observations yields (4.24). Next, we claim that

$$\langle R_{1,2}^l \rangle - q^l \tanh^2 X_n \to 0 \tag{4.25}$$

in probability as $n \to \infty$. To see this, note that by Theorem 2.3, $\langle (R_{1,2} - m(\sigma^1) m(\sigma^2))^2 \rangle \to 0$ in probability. This implies that

$$\langle R_{1,2}^l \rangle - \langle m(\sigma^1)^l m(\sigma^2)^l \rangle \to 0$$

in probability. But $\langle m(\sigma^1)^l m(\sigma^2)^l \rangle = \langle m^l \rangle^2$. Thus, (4.25) follows from (4.24). Now, proceeding just as in the derivation of (3.11), we get

$$\begin{aligned}
&\frac{1}{|B_n|^l} \sum_{i_1,\ldots,i_l \in B_n} |\langle \sigma_{i_1} \cdots \sigma_{i_l} \rangle - q^{\frac{l}{2}} \tanh X_n| \\
&\le \left[\frac{1}{|B_n|^l} \sum_{i_1,\ldots,i_l \in B_n} (\langle \sigma_{i_1} \cdots \sigma_{i_l} \rangle - q^{\frac{l}{2}} \tanh X_n)^2 \right]^{\frac{1}{2}} \\
&= \left[\frac{1}{|B_n|^l} \sum_{i_1,\ldots,i_l \in B_n} (\langle \sigma_{i_1} \cdots \sigma_{i_l} \rangle^2 - 2q^{\frac{l}{2}} \langle \sigma_{i_1} \cdots \sigma_{i_l} \rangle \tanh X_n + q^l \tanh^2 X_n) \right]^{\frac{1}{2}} \\
&= [\langle R_{1,2}^l \rangle - 2q^{\frac{l}{2}} \langle m^l \rangle \tanh X_n + q^l \tanh^2 X_n]^{\frac{1}{2}}.
\end{aligned}$$

By (4.24) and (4.25), the last expression tends to zero in probability as $n \to \infty$. This proves the claim (4.23).

Now take any n, and let $\tau_{n,1}, \tau_{n,2}, \ldots$ be an infinite exchangeable sequence of random variables with the following random law. Given X_n, let Z_n be a random variable that takes value $\sqrt{q}$ with probability $\frac{1}{2}(1 + \tanh X_n)$ and $-\sqrt{q}$ with probability $\frac{1}{2}(1 - \tanh X_n)$. Having generated Z_n, let $\tau_{n,1}, \tau_{n,2}, \ldots$ be i.i.d. random variables taking value 1 with probability $\frac{1}{2}(1+Z_n)$ and -1 with probability $\frac{1}{2}(1-Z_n)$. Then note that $\mathbb{E}(\tau_{n,i} | Z_n, X_n) = Z_n$, and therefore, for any positive integer l,

$$\mathbb{E}(\tau_{n,1} \cdots \tau_{n,l} | Z_n, X_n) = Z_n^l.$$

This give us

$$\mathbb{E}(\tau_{n,1} \cdots \tau_{n,l} | X_n) = \mathbb{E}(Z_n^l | X_n) = \begin{cases} q^{\frac{l}{2}} \tanh X_n & \text{if } l \text{ is odd,} \\ q^{\frac{l}{2}} & \text{if } l \text{ is even.} \end{cases}$$

Comparing this with (4.22) and (4.23), it is now easy to show that for any l, the Lévy–Prokhorov distance between the (random) laws of $(\sigma_{\pi_n(1)}, \ldots, \sigma_{\pi_n(l)})$ and $(\tau_{n,1}, \ldots, \tau_{n,l})$ converges to zero in probability as $n \to \infty$. But, the random law of $(\tau_{n,1}, \ldots, \tau_{n,l})$ converges in distribution to the random law of $(\tau_1, \ldots, \tau_l)$, where $\tau_1, \tau_2, \ldots$ are defined just like the $\tau_{n,i}$'s, but with X_n replaced by $X = \sqrt{q} \beta h W$, where W is a standard Gaussian random variable. This suffices to complete the proof.

Acknowledgement I thank Louis-Pierre Arguin, Andrew Chen, Persi Diaconis, Hugo Duminil-Copin, Zhihan Li and Gourab Ray for many helpful comments and references. In particular, I thank Hugo for sketching the alternative proof of Theorem 2.10 and Gourab for helping expand the sketch to a complete argument, and the referee for explaining why the proof works for all subcritical temperatures. This work was partially supported by NSF grants DMS-2113242 and DMS-2153654.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Aizenman, M., Contucci, P.: On the stability of the quenched state in mean-field spin-glass models. J. Stat. Phys. **92**, 765–783 (1998)
2. Aizenman, M., Peled, R.: A power-law upper bound on the correlations in the 2D random field Ising model. Commun. Math. Phys. **372**(3), 865–892 (2019)
3. Aizenman, M., Wehr, J.: Rounding of first-order phase transitions in systems with quenched disorder. Phys. Rev. Lett. **62**(21), 2503 (1989)
4. Aizenman, M., Wehr, J.: Rounding effects of quenched randomness on first-order phase transitions. Commun. Math. Phys. **130**(3), 489–528 (1990)
5. Aizenman, M., Harel, M., Peled, R.: Exponential decay of correlations in the 2D random field Ising model. J. Stat. Phys. **180**, 304–331 (2020)
6. Arguin, L.-P., Aizenman, M.: On the structure of quasi-stationary competing particle systems. Ann. Probab. **37**(3), 1080–1113 (2009)
7. Auffinger, A., Chen, W.-K.: Universality of chaos and ultrametricity in mixed p-spin models. Commun. Pure Appl. Math. **69**(11), 2107–2130 (2016)
8. Bar-Nir, Y.: Upper and lower bounds for the correlation length of the two-dimensional random-field Ising model (2022). arXiv preprint arXiv:2205.01522
9. Bodineau, T.: Slab percolation for the Ising model. Probab. Theory Relat. Fields **132**, 83–118 (2005)
10. Bodineau, T.: Translation invariant Gibbs states for the Ising model. Probab. Theory Relat. Fields **135**, 153–168 (2006)
11. Bowditch, A., Sun, R.: The two-dimensional continuum random field Ising model. Ann. Probab. **50**(2), 419–454 (2022)
12. Bray, A.J., Moore, M.A.: Chaotic nature of the spin-glass phase. Phys. Rev. Lett. **58**(1), 57 (1987)
13. Bricmont, J., Kupiainen, A.: Lower critical dimension for the random-field Ising model. Phys. Rev. Lett. **59**(16), 1987 (1829)
14. Bricmont, J., Kupiainen, A.: Phase transition in the 3d random field Ising model. Commun. Math. Phys. **116**, 539–572 (1988)
15. Burton, R.M., Keane, M.: Density and uniqueness in percolation. Commun. Math. Phys. **121**, 501–505 (1989)
16. Camia, F., Jiang, J., Newman, C.M.: A note on exponential decay in the random field Ising model. J. Stat. Phys. **173**(2), 268–284 (2018)
17. Chatterjee, S.: Absence of replica symmetry breaking in the random field Ising model. Commun. Math. Phys. **337**(1), 93–102 (2015)
18. Chatterjee, S.: On the decay of correlations in the random field Ising model. Commun. Math. Phys. **362**(1), 253–267 (2018)
19. Chatterjee, S.: Spin glass phase at zero temperature in the Edwards–Anderson model (2023). arXiv preprint arXiv:2301.04112
20. Chatterjee, S., Sloman, L.: Average Gromov hyperbolicity and the Parisi ansatz. Adv. Math. **376**, 107417 (2021)
21. Contucci, P., Mingione, E., Starr, S.: Factorization properties in d-dimensional spin glasses. Rigorous results and some perspectives. J. Stat. Phys. **151**, 809–829 (2013)
22. Dario, P., Harel, M., Peled, R.: Quantitative disorder effects in low-dimensional spin systems (2021). arXiv preprint arXiv:2101.01711

23. Ding, J., Wirth, M.: Correlation length of the two-dimensional random field Ising model via greedy lattice animal. Duke Math. J. **1**(1), 1–31 (2023)
24. Ding, J., Xia, J.: Exponential decay of correlations in the two-dimensional random field Ising model. Invent. Math. **224**(3), 999–1045 (2021)
25. Ding, J., Zhuang, Z.: Long range order for random field Ising and Potts models (2021). arXiv preprint arXiv:2110.04531
26. Ding, J., Liu, Y., Xia, A.: Long range order for three-dimensional random field Ising model throughout the entire low temperature regime (2022). arXiv preprint arXiv:2209.13998
27. Duminil-Copin, H.: Lectures on the Ising and Potts models on the hypercubic lattice. In: PIMS-CRM Summer School in Probability, pp. 35–161. Springer (2017)
28. Duminil-Copin, H., Hongler, C., Nolin, P.: Connection probabilities and RSW-type bounds for the two-dimensional FK Ising model. Commun. Pure Appl. Math. **64**(9), 1165–1198 (2011)
29. Duminil-Copin, H., Goswami, S., Raoufi, A.: Exponential decay of truncated correlations for the Ising model in any dimension for all but the critical temperature. Commun. Math. Phys. **374**(2), 891–921 (2020)
30. Edwards, S.F., Anderson, P.W.: Theory of spin glasses. J. Phys. F Met. Phys. **5**(5), 965 (1975)
31. Fisher, D.S., Huse, D.A.: Ordered phase of short-range Ising spin-glasses. Phys. Rev. Lett. **56**(15), 1601 (1986)
32. Georgii, H.-O.: Gibbs Measures and Phase Transitions. de Gruyter (2011)
33. Ghirlanda, S., Guerra, F.: General properties of overlap probability distributions in disordered spin systems. Towards Parisi ultrametricity. J. Phys. A Math. Gen. **31**(46), 9149 (1998)
34. Grimmett, G.: The stochastic random-cluster process and the uniqueness of random-cluster measures. Ann. Probab. **8**, 1461–1510 (1995)
35. Grimmett, G.: The Random Cluster Model. Springer (2006)
36. Imbrie, J.Z.: Lower critical dimension of the random-field Ising model. Phys. Rev. Lett. **53**(18), 1747 (1984)
37. Imbrie, J.Z.: The ground state of the three-dimensional random-field Ising model. Commun. Math. Phys. **98**(2), 145–176 (1985)
38. Imry, Y., Ma, S.-K.: Random-field instability of the ordered state of continuous symmetry. Phys. Rev. Lett. **35**(21), 1399 (1975)
39. Jagannath, A.: Approximate ultrametricity for random measures and applications to spin glasses. Commun. Pure Appl. Math. **70**(4), 611–664 (2017)
40. Kallenberg, O.: Probabilistic Symmetries and Invariance Principles. Springer (2005)
41. Krzakala, F., Ricci-Tersenghi, F., Zdeborová, L.: Elusive spin-glass phase in the random field Ising model. Phys. Rev. Lett. **104**(20), 207208 (2010)
42. Ledoux, M.: The Concentration of Measure Phenomenon. American Mathematical Society, Providence (2001)
43. Liggett, T.M., Schonmann, R.H., Stacey, A.M.: Domination by product measures. Ann. Probab. **25**(1), 71–95 (1997)
44. Mézard, M., Parisi, G., Virasoro, M.A.: Spin Glass Theory and Beyond: An Introduction to the Replica Method and Its Applications, vol. 9. World Scientific Publishing Company (1987)
45. Newman, C.M., Stein, D.L.: Non-mean-field behavior of realistic spin glasses. Phys. Rev. Lett. **76**(3), 515 (1996)
46. Newman, C.M., Stein, D.L.: Ground-state structure in a highly disordered spin-glass model. J. Stat. Phys. **82**, 1113–1132 (1996)
47. Panchenko, D.: A connection between the Ghirlanda–Guerra identities and ultrametricity. Ann. Probab. **38**(1), 327–347 (2010)
48. Panchenko, D.: The Sherrington–Kirkpatrick Model. Springer (2013a)
49. Panchenko, D.: The Parisi ultrametricity conjecture. Ann. Math. **8**, 383–393 (2013)
50. Parisi, G.: Recent rigorous results support the predictions of spontaneously broken replica symmetry for realistic spin glasses (1996). arXiv preprint arxiv:cond-mat/9603101
51. Parisi, G.: Mean field theory of spin glasses: statics and dynamics. In: Complex Systems, Volume 85 of Les Houches, pp. 131–178. Elsevier (2007)
52. Pisztora, A.: Surface order large deviations for Ising, Potts and percolation models. Probab. Theory Relat. Fields **104**, 427–466 (1996)
53. Raoufi, A.: Translation-invariant Gibbs states of the Ising model: general setting. Ann. Probab. **48**(2), 760–777 (2020)
54. Sherrington, D., Kirkpatrick, S.: Solvable model of a spin-glass. Phys. Rev. Lett. **35**(26), 1792 (1975)
55. Subag, E.: The geometry of the Gibbs measure of pure spherical spin glasses. Invent. Math. **210**(1), 135–209 (2017)
56. Subag, E.: Free energy landscapes in spherical spin glasses (2018). arXiv preprint arXiv:1804.10576
57. Talagrand, M.: Mean Field Models for Spin Glasses: Volume I: Basic Examples. Springer (2010a)

58. Talagrand, M.: Construction of pure states in mean field models for spin glasses. Probab. Theory Relat. Fields **148**(3–4), 601 (2010)
59. Talagrand, M.: Mean Field Models for Spin Glasses. Advanced Replica-Symmetry and Low Temperature, vol. II. Springer, London (2011)