A NEW GRADIENT ESTIMATE FOR THE COMPLEX MONGE-AMPÈRE EQUATION [1]

Bin Guo, Duong H. Phong, and Freid Tong

Abstract

A gradient estimate for complex Monge-Ampère equations which improves in some respects on known estimates is proved using the ABP maximum principle.

1 Introduction

Gradient estimates occupy a special position in the theory of complex Monge-Ampère equations. In Yau's original proof of the Calabi conjecture for compact manifolds [12], they can be bypassed, as C^2 estimates can be obtained directly once C^0 estimates are known. But this is no longer the case for subsequent extensions of the theory. The first gradient bounds appear to be due to Hanani [7], but this paper did not seem to be widely known. More recent approaches are due to Blocki [1], P. Guan [3], B. Guan-Q. Li [4], and Phong-Sturm [10]. The sharpest result to date may be [10], which builds on the approach of [1], gives a pointwise estimate, and requires only a lower bound for the solution φ of the equation, and not an upper bound. These features are essential for applications to boundary value problems or the case of degenerating background metrics [10, 9].

In [5], the authors developed a new method for establishing the classical L^∞ estimates for the complex Monge-Ampère equation without recourse to pluripotential theory. This method builds on works of Wang, Wang, Zhou [11] and particular of Chen and Cheng [2], who introduced the idea of using an auxiliary Monge-Ampère equation. The methods of [5] turn out not just to recover the classical L^∞ estimates, but to improve and widen them in many significant ways. Thus it is natural to examine their possibilities for other estimates. In this paper, we examine the case of gradient estimates. We shall show below that the methods of [5] can recapture the sharp gradient estimate of [10, 9], in fact with a weaker assumption on the right hand side which may be of geometric significance.

Let (X, ω_0) be a compact Kähler manifold with or without boundary and φ be a ω-plurisubharmonic function solving the following complex Monge-Ampère equation

$$(\omega_0 + i\partial\bar\partial\varphi)^n = e^F \omega_0^n, \tag{1.1}$$

where $F \in C^\infty$. When X has no boundary, we assume that F satisfies the compatibility condition $\int_X e^F \omega_0^n = \int_X \omega_0^n$. When X has a smooth non-empty boundary ∂X, we impose the boundary condition $\varphi = \phi$ on ∂X for some $\phi \in C^2(\overline{X})$ with $\omega_\phi = \omega_0 + i\partial\bar\partial\phi$ a smooth Kähler metric on $\overline{X}$.

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Theorem 1 *Under the above conditions, we have the gradient estimate*

$$|\nabla\varphi|^2_{\omega_0} \le Ce^{\lambda(\varphi-\inf_X\varphi)},$$

where $\lambda > 0$ and C are positive constants, depending respectively only on a lower bound for the bisectional curvature of ω_0, and on $n, \omega_0, \sup_X F$, $\|\nabla F\|_{L^{2n}(e^{2F}\omega_0^n)}$, $\|\nabla_{\omega_0}\varphi\|_{L^\infty(\partial X)}$ and $\|\phi\|_{C^2(\overline{X})}$.

We observe that previous results had required control of the full L^∞ norm of the gradient of the right-hand side. With our method, we can relax this to an L^{2n}-control. We illustrate later an application of this improvement.

2 Proof of the Theorem

By replacing φ and ϕ respectively by $\varphi-\inf_X\varphi$ and $\phi-\inf_X\phi$, we may assume $\inf_X\varphi = 0$. Let $\omega = \omega_0 + i\partial\bar\partial\varphi$ be the Kähler metric associated with the complex Monge-Ampère equation (1.1).

Lemma 1 *The following equation holds*

$$\Delta_\omega|\nabla\varphi|^2_{\omega_0} = 2Re\langle\nabla F, \bar\nabla\varphi\rangle_{\omega_0} + g^{i\bar j}g_0^{k\bar l}(\varphi_{ki}\varphi_{\bar j\bar l} + \varphi_{k\bar j}\varphi_{i\bar l}) + g^{i\bar j}R(g_0)_{i\bar jk\bar l}\varphi_p\varphi_{\bar q}g_0^{k\bar q}g_0^{p\bar l} \qquad (2.1)$$

where $\omega = (g_{i\bar j})$, $\omega_0 = ((g_0)_{i\bar j})$, $\varphi_{ki} = (\nabla_{\omega_0}\nabla_{\omega_0}\varphi)_{ki}$ are the second covariant derivatives with respect to ω_0, and $R(g_0)_{i\bar jk\bar l}$ is the bisectional curvature of ω_0.

The proof of Lemma 1 is a standard calculation, so we omit the details. Let $-K$ be a lower bound of the bisectional curvature $R(g_0)$. From the equation (2.1) we have

$$\Delta_\omega|\nabla\varphi|^2_{\omega_0} \ge 2Re\langle\nabla F, \bar\nabla\varphi\rangle_{\omega_0} + g^{i\bar j}g_0^{k\bar l}(\varphi_{ki}\varphi_{\bar j\bar l} + \varphi_{k\bar j}\varphi_{i\bar l}) - 2K\mathrm{tr}_\omega\omega_0|\nabla\varphi|^2_{\omega_0}$$

Denote $H = e^{-\lambda\varphi}|\nabla\varphi|^2_{\omega_0}$ for $\lambda = 2K + 10$. We calculate at an arbitrary point $x \in X$, and choose a normal coordinates system for ω_0 such that ω is diagonal at x.

$$\begin{aligned}
\Delta_\omega H &= \Delta(e^{-\lambda\varphi}|\nabla\varphi|^2_{\omega_0}) \\
&= e^{-\lambda\varphi}\Delta|\nabla\varphi|^2_{\omega_0} + |\nabla\varphi|^2_{\omega_0}\Delta(e^{-\lambda\varphi}) - 2\lambda e^{-\lambda\varphi}Re\langle\nabla\varphi, \bar\nabla|\nabla\varphi|^2_{\omega_0}\rangle_\omega \\
&\ge e^{-\lambda\varphi}\Big(2Re\langle\nabla F, \bar\nabla\varphi\rangle_{\omega_0} + g^{i\bar j}g_0^{k\bar l}(\varphi_{ki}\varphi_{\bar j\bar l} + \varphi_{k\bar j}\varphi_{i\bar l}) - 2K\mathrm{tr}_\omega\omega_0|\nabla\varphi|^2_{\omega_0}\Big) \\
&\quad + |\nabla\varphi|^2_{\omega_0}e^{-\lambda\varphi}\Big(-\lambda n + \lambda\mathrm{tr}_\omega\omega_0 + \lambda^2|\nabla\varphi|^2_\omega\Big) - 2\lambda e^{-\lambda\varphi}Re\langle\nabla\varphi, \bar\nabla|\nabla\varphi|^2_{\omega_0}\rangle_\omega
\end{aligned}$$

The last term on the right hand side is

$$\begin{aligned}
& -2\lambda e^{-\lambda\varphi}Re\langle\nabla\varphi, \bar\nabla|\nabla\varphi|^2_{\omega_0}\rangle_\omega \\
=\ & -2\lambda e^{-\lambda\varphi}Re(g^{i\bar i}\varphi_i(\varphi_k\varphi_{\bar k})_{\bar i}) \\
=\ & -2\lambda e^{-\lambda\varphi}Re(g^{i\bar i}\varphi_i\varphi_{k\bar i}\varphi_{\bar k} + g^{i\bar i}\varphi_i\varphi_k\varphi_{\bar k\bar i}) \\
\ge\ & -2\lambda e^{-\lambda\varphi}g^{i\bar i}\varphi_i\varphi_{k\bar i}\varphi_{\bar k} - \lambda^2e^{-\lambda\varphi}g^{i\bar i}\varphi_i\varphi_{\bar i}|\nabla\varphi|^2_{\omega_0} - e^{-\lambda\varphi}g^{i\bar i}\varphi_{ki}\varphi_{\bar k\bar i} \\
=\ & -2\lambda e^{-\lambda\varphi}g^{i\bar i}\varphi_i\varphi_{\bar i}(g_{i\bar i} - 1) - \lambda^2e^{-\lambda\varphi}g^{i\bar i}\varphi_i\varphi_{\bar i}|\nabla\varphi|^2_{\omega_0} - e^{-\lambda\varphi}g^{i\bar i}\varphi_{ki}\varphi_{\bar k\bar i} \\
=\ & -2\lambda e^{-\lambda\varphi}|\nabla\varphi|^2_{\omega_0} + 2\lambda e^{-\lambda\varphi}|\nabla\varphi|^2_\omega - \lambda^2e^{-\lambda\varphi}|\nabla\varphi|^2_\omega|\nabla\varphi|^2_{\omega_0} - e^{-\lambda\varphi}g^{i\bar i}\varphi_{ki}\varphi_{\bar k\bar i}.
\end{aligned}$$

where we applied the Cauchy-Schwarz inequality. We thus obtain

$$\Delta_\omega H \quad \geq 2e^{-\lambda\varphi} Re\langle \nabla F, \bar{\nabla}\varphi\rangle_{\omega_0} + (\lambda - 2K)H\mathrm{tr}_\omega\omega_0 - \lambda(n+2)H.$$

Note that this inequality holds at any point of X, since it is independent of the choice of normal coordinates. Let $\alpha > 1$ be a positive constant. We calculate

$$\begin{aligned} \Delta_\omega H^\alpha &= \alpha H^{\alpha-1}\Delta H + \alpha(\alpha-1)H^{\alpha-2}|\nabla H|^2_\omega \\ &\geq \alpha H^{\alpha-1}\Big(e^{-\lambda\varphi} Re\langle\nabla F, \bar{\nabla}\varphi\rangle_{\omega_0} + (\lambda-2K)H\mathrm{tr}_\omega\omega_0 - \lambda(n+2)H\Big) \\ &\quad +\alpha(\alpha-1)H^{\alpha-2}|\nabla H|^2_\omega. \end{aligned} \tag{2.2}$$

Since X is compact, we can assume H attains its maximum at a point x_0, with $H(x_0) =: M > 0$. We may suppose x_0 lies in the interior of X, otherwise we are done. Since ω_0 is smooth up to ∂X, we may assume $(\overline{X}, \omega_0)$ isometrically embeds to another Kähler manifold $(\hat{X}, \hat{\omega}_0)$ as a compact subset[1].Let $r > 0$ be the injectivity radius of the Riemannian manifold $(\hat{X}, \hat{\omega}_0)$. Without loss of generality we may identify the metric ball $B_{g_0}(x_0, r)$ with an open domain in the Euclidean space $\mathbf{C^n}$, where we denote $B_{g_0}(x_0, r) = \{x \in X | d_{g_0}(x, x_0) < r\}$. We will apply a trick of Chen-Cheng [2]. Let $\theta = \min\{\frac{1}{10nC_0}, \frac{r^2}{10nC_0}\}$ be a given constant (where $C_0 > 1$ depends only on ω_0) and choose an auxiliary function η such that $\eta = 1$ on $B_{g_0}(x_0, r/2)$ and $\eta = 1 - \theta$ on $B_{g_0}(x_0, r)\backslash B_{g_0}(x_0, 3r/4)$, and $\eta \in [1-\theta, 1]$ in the annulus between. We also have (this η may be chosen as $\hat{\eta}(\frac{d_0(x)^2}{r^2})$ where d_0 is a smoothing of the g_0-distance to x_0 and $\hat{\eta}$ is some appropriate function on $\mathbf{R}$)

$$|\nabla\eta|^2_{g_0} \leq \frac{C_0\theta^2}{r^2}, \quad |\nabla^2\eta|_{g_0} \leq \frac{C_0\theta}{r^2}$$

We calculate as follows

$$\Delta_\omega(\eta H^\alpha) = \eta\Delta H^\alpha + 2\alpha H^{\alpha-1} Re\langle\nabla\eta, \bar{\nabla}H\rangle_\omega + H^\alpha\Delta_\omega\eta.$$

Note that the last term satisfies

$$H^\alpha\Delta_\omega\eta \geq -C_0\frac{\theta}{r^2}H^\alpha\mathrm{tr}_\omega\omega_0,$$

and the middle term is

$$\begin{aligned} 2\alpha H^{\alpha-1} Re\langle\nabla\eta, \bar{\nabla}H\rangle_\omega &\geq -2\alpha H^{\alpha-1}|\nabla H|_\omega|\nabla\eta|_\omega \\ &\geq -\frac{\alpha(\alpha-1)}{2}H^{\alpha-2}|\nabla H|^2_\omega - \frac{2\alpha}{\alpha-1}H^\alpha|\nabla\eta|^2_\omega \\ &\geq \underbrace{-\frac{\alpha(\alpha-1)}{2}H^{\alpha-2}|\nabla H|^2_\omega}_{\text{controlled by the last term in } \eta\Delta H^\alpha} - \frac{2\alpha}{\alpha-1}H^\alpha\frac{C_0\theta^2}{r^2}\mathrm{tr}_\omega\omega_0 \end{aligned}$$

[1] We can alternatively cover ∂X by finitely many Euclidean half balls, and apply similar calculations.

Combining the above inequalities we get

$$\begin{aligned}\Delta(\eta H^\alpha) &\geq \alpha\eta H^{\alpha-1}e^{-\lambda\varphi}\langle\nabla F,\bar{\nabla}\varphi\rangle_{\omega_0}+\Big(\alpha\eta(\lambda-2K)-\frac{C_0\theta}{r^2}-\frac{2\alpha}{\alpha-1}\frac{C_0\theta^2}{r^2}\Big)H^\alpha\mathrm{tr}_\omega\omega_0\\ &\quad -\lambda\alpha(n+2)H^\alpha. \end{aligned}\tag{2.3}$$

Note that $\eta\geq 9/10$. We can choose $\alpha=2$. Together with the choice of θ and λ, the middle term of the right hand side of the above inequality is nonnegative, so

$$\Delta_\omega(\eta H^\alpha)\geq -\alpha\eta H^{\alpha-\frac{1}{2}}e^{-\lambda\varphi/2}|\nabla F|_{\omega_0}-\lambda\alpha(n+2)H^\alpha \tag{2.4}$$

We may assume $(1-\theta)M^\alpha\geq \sup_{\partial X}|\nabla\varphi|^\alpha_{\omega_0}$, otherwise we are done. Applying the ABP maximum principle to the function ηH^α on the ball $B_{g_0}(x_0,r)$, we obtain (with $B_0=B_{g_0}(x_0,r)$)

$$\begin{aligned}M^\alpha &= \sup_{B_0}(\eta H^\alpha)\\ &\leq \sup_{\partial B_0}(\eta H^\alpha)+C(n,\omega_0)r\Big(\int_{B_0}\frac{[\alpha\eta H^{\alpha-\frac{1}{2}}e^{-\lambda\varphi/2}|\nabla F|_{\omega_0}+\lambda\alpha(n+2)H^\alpha]^{2n}}{e^{-2F}}\omega_0^n\Big)^{1/2n}\\ &\leq \sup_{\partial B_0}\eta H^\alpha+C(n,\omega_0)r\Big[\Big(\int_{B_0}H^{2n\alpha}\omega_0^n\Big)^{1/2n}+M^{\alpha-1/2}\big(\int_{B_0}e^{2F}|\nabla F|^{2n}_{\omega_0}\omega_0^n\big)^{1/2n}\Big]\\ &\leq \sup_{\partial B_0}\eta H^\alpha+C(n,\omega_0)r\Big[M^{\alpha(1-\frac{1}{2n})}\Big(\int_{B_0}H\omega_0^n\Big)^{1/2n}+M^{\alpha-1/2}\big(\int_{B_0}e^{2F}|\nabla F|^{2n}_{\omega_0}\omega_0^n\big)^{1/2n}\Big]\\ &\leq (1-\theta)M^\alpha+C(n,\omega_0,F)r\Big[M^{\alpha(1-\frac{1}{2n})}+M^{\alpha-1/2}\Big],\end{aligned}$$

where the last constant $C(n,\omega_0,F)$ depends on $\||\nabla F|_{\omega_0}\|_{L^{2n}(X,e^{2F}\omega_0^n)}$ and $\sup_X F$. Note that $\theta>c_0>0$ for some constant c_0 depending only on ω_0. We conclude that

$$c_0M^\alpha\leq\theta M^\alpha\leq C(n,\omega_0,F)r(M^{\alpha-1/2}+M^{\alpha(1-\frac{1}{2n})}),$$

from which we derive $M\leq C(n,\omega_0,F)$, since the RHS are powers of M with degree smaller than α. Finally in the estimate above we implicitly use the uniform bound on $\int_X H\omega_0^n$, which follows from the lemma below. It is because of this lemma that we need the C^2-bound of the boundary value ϕ.

Lemma 2 *We have* $\int_X H\omega_0^n=\int_X e^{-\lambda\varphi}|\nabla\varphi|^2_{\omega_0}\omega_0^n\leq C(n,\omega_0,\omega_\phi)$.

Proof. From the equation $\omega^n=e^F\omega_0^n$, we obtain

$$e^F\omega_0^n-\omega_\phi^n=\omega^n-\omega_\phi^n=i\partial\bar{\partial}(\varphi-\phi)\wedge(\omega^{n-1}+\cdots+\omega_\phi^{n-1}),$$

Multiplying both sides by $e^{-\lambda\varphi+\lambda\phi}-1$ and applying integration by parts, we can write the right hand side as

$$\int_X(e^{-\lambda\varphi+\lambda\phi}-1)i\partial\bar{\partial}(\varphi-\phi)\wedge(\omega^{n-1}+\cdots+\omega_0^{n-1})$$

$$
\begin{aligned}
&= \int_X \lambda e^{-\lambda(\varphi-\phi)}\partial(\varphi-\phi)\wedge\bar\partial(\varphi-\phi)\wedge(\omega^{n-1}+\cdots+\omega_\phi^{n-1}) \\
&\geq \int_X \lambda e^{-\lambda(\varphi-\phi)}\partial(\varphi-\phi)\wedge\bar\partial(\varphi-\phi)\wedge\omega_\phi^{n-1} \\
&\geq C\int_X e^{-\lambda\varphi}|\nabla\varphi|^2_{\omega_0}\omega_0^n - C(\phi,\omega_0),
\end{aligned}
$$

since ω_ϕ is equivalent to ω_0 by assumption. On the other hand the left hand side can be bounded as follows,

$$
\int_X (e^{-\lambda(\varphi-\phi)}-1)(e^F\omega_0^n-\omega_\phi^n) \leq \int_X e^{\lambda\phi}(e^F\omega_0^n+\omega_\phi^n) \leq C(n,\omega_0,\phi,\sup_X F).
$$

This proves the lemma, and the proof of the theorem is complete.

3 Application

We can give now an application of the improved gradient estimates:

Corollary 1 *Let X be a compact Kähler manifold, $f \geq 0$ be a smooth function with $\int_X f\omega_0^n = \int_X \omega_0^n$, and assume that f satisfies*

$$
\int_X \frac{|\nabla f|^{2n}}{f^{2n-2}}\omega_0^n < \infty \tag{3.1}
$$

Then the solution of the complex Monge-Ampère equation

$$
(\omega_0 + i\partial\bar\partial\varphi)^n = f\omega_0^n \tag{3.2}
$$

is Lipschitz continuous.

Proof. Let f_k be a regularization of f chosen with the following properties: $\int_X f_k\omega_0^n = \int_X f\omega_0^n$, $f_k > 0$ for $k > 0$, f_k converges to f smoothly as $k \to \infty$, and moreover in a small neighborhood of the vanishing locus of f, we require that $f_k = f + \frac{1}{k}$. It's not hard to see that such a regularization can be arranged.

Then by our choice of f_k, we observe that in a neighborhood of the vanishing locus of f, we have

$$
\frac{|\nabla f_k|^{2n}}{f_k^{2n-2}} = \frac{|\nabla f|^{2n}}{f_k^{2n-2}} \leq \frac{|\nabla f|^{2n}}{f^{2n-2}} \tag{3.3}
$$

and hence $\int_X \frac{|\nabla f_k|^{2n}}{f_k^{2n-2}}\omega_0^n$ is uniformly bounded. By Theorem 1, we know that the sequence of solutions of the Monge-Ampère equations

$$
(\omega_0 + i\partial\bar\partial\varphi_k)^n = f_k\omega^n \tag{3.4}
$$

has uniform C^1 bounds independent of k. By the stability of complex Monge-Ampère equations [8, 6], the solutions φ_k converge uniformly to φ, hence φ must be Lipschitz.

Our theorem applies for example, when f has isolated zeroes, near which f is asymptotically $f(z) \sim \frac{1}{|\log |z|^2|}$ or $f(z) \sim |z|^{\varepsilon_0}$ for $\varepsilon_0 > 0$. Observe that $\|\nabla f^{1/n}\|_{L^\infty}$ is not finite, so the usual gradient estimate in [1, 10] do not apply. However we still have $\int_X |f|^2 |\nabla \log f|^{2n} \omega_0^n < \infty$, hence our result applies.

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Department of Mathematics & Computer Science, Rutgers University, Newark, NJ 07102 USA
bguo@rutgers.edu

Department of Mathematics, Columbia University, New York, NY 10027 USA
phong@math.columbia.edu, tong@math.columbia.edu