Journal of Functional Analysis 287 (2024) 110472

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Non-compactness results for the spinorial Yamabe-type problems with non-smooth geometric data

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ARTICLE INFO

Article history:
Received 2 June 2023
Accepted 15 April 2024
Available online 24 April 2024
Communicated by J. Wei

MSC:
primary 53C27
secondary 35R01

Keywords:
Dirac operator
Spinorial Yamabe problem
Blow-up phenomenon

ABSTRACT

Let (M, g, σ) be an m-dimensional closed spin manifold, with a fixed Riemannian metric g and a fixed spin structure σ; let $\mathbb{S}(M)$ be the spinor bundle over M. The spinorial Yamabe-type problems address the solvability of the following equation

$$D_g\psi = f(x)|\psi|_g^{\frac{2}{m-1}}\psi, \quad \psi : M \to \mathbb{S}(M),\ x \in M$$

where D_g is the associated Dirac operator and $f : M \to \mathbb{R}$ is a given function. The study of such nonlinear equation is motivated by its important applications in Spin Geometry: when $m = 2$, a solution corresponds to a conformal isometric immersion of the universal covering $\widetilde{M}$ into $\mathbb{R}^3$ with prescribed mean curvature f; meanwhile, for general dimensions and $f \equiv$ *constant* $\neq 0$, a solution provides an upper bound estimate for the Bär-Hijazi-Lott invariant.

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[1] Y.S. is partially supported by NSF DMS Grant 2154219, "Regularity *vs* singularity formation in elliptic and parabolic equations".
[2] T.X. is partially supported by the National Science Foundation of China (NSFC 12371117) and the Alexander von Humboldt Foundation of Germany.

https://doi.org/10.1016/j.jfa.2024.110472
0022-1236/

The aim of this paper is to establish non-compactness results related to the spinorial Yamabe-type problems. Precisely, concrete analysis is made for two specific models on the manifold (S^m, g) where the solution set of the spinorial Yamabe-type problem is not compact: 1). the geometric potential f is constant (say $f \equiv 1$) with the background metric g being a C^k perturbation of the canonical round metric g_{S^m}, which is not conformally flat somewhere on S^m; 2). f is a perturbation from constant and is of class C^2, while the background metric $g \equiv g_{S^m}$.

Contents

1. Introduction

On a closed Riemannian m-manifold (M, g) with $m \geq 3$, the scalar curvature problem (or simply known as the Yamabe-type problem) is given by the differential equation

$$-\frac{4(m-1)}{m-2}\Delta_g u + R_g u = f(x)u^{\frac{m+2}{m-2}}, \quad u > 0, \tag{1.1}$$

where Δ_g is the Laplace operator with respect to g and R_g stands for the scalar curvature of g. Here, the problem is to decide which function f on M can be the scalar curvature of a conformal metric $\tilde{g} = u^{4/(m-2)}g \in [g]$. In case $f \equiv$ *constant*, this problem is referred to as the classical Yamabe problem, and is completely solved by a series of works of Yamabe [56], Trudinger [54], Aubin [13] and Schoen [47]. See also the survey paper [42] by Lee & Parker.

In the setting of spin geometry there exists a conformally covariant operator, the Dirac operator, which enjoys analogous properties to the conformal Laplacian. This operator was formally introduced by M.F. Atiyah in 1962 in connection with his elaboration of the index theory of elliptic operators.

Let (M, g, σ) be an m-dimensional closed spin manifold, $m \geq 2$, with a fixed Riemannian metric g and a fixed spin structure $\sigma : P_{\mathrm{Spin}}(M) \to P_{\mathrm{SO}}(M)$. The Dirac operator D_g is defined in terms of a representation $\rho : \mathrm{Spin}(m) \to \mathrm{Aut}(\mathbb{S}_m)$ of the spin group which is compatible with Clifford multiplication. Let $\mathbb{S}(M) := P_{\mathrm{Spin}}(M) \times_\rho \mathbb{S}_m$ be the associated bundle, which we call the spinor bundle over M, with $\dim_{\mathbb{C}} \mathbb{S}(M) = 2^{[\frac{m}{2}]}$. Then the Dirac operator D_g is a first order differential operator acting on smooth sections of $\mathbb{S}(M)$, i.e. $D_g : C^\infty(M, \mathbb{S}(M)) \to C^\infty(M, \mathbb{S}(M))$. We are concerned with the spinorial Yamabe-type problem

$$D_g\psi = f(x)|\psi|_g^{\frac{2}{m-1}}\psi, \quad \psi : M \to \mathbb{S}(M) \tag{1.2}$$

where $|\cdot|_g$ is the hermitian metric on $\mathbb{S}(M)$ induced from g. This equation appears in the study of different problems from conformal geometry, and has attracted much attention recently, see for instance [6–8,10,12,15,17,18,25,33,36,46,49,50] and references therein.

We point out here that there are at least two motivations for studying Eq. (1.2). One of them is that, when $f \equiv$ *constant* $\neq 0$ (say $f \equiv 1$), Eq. (1.2) is closely related to the study of a conformal spectral invariant, i.e., the *Bär-Hijazi-Lott* invariant (see [28, Section 8.5] for an overview)

$$\lambda_{min}^+(M, g, \sigma) := \inf_{\tilde{g} \in [g]} \lambda_1^+(\tilde{g}) \,\mathrm{Vol}(M, \tilde{g})^{\frac{1}{m}},$$

where $\lambda_1^+(\tilde{g})$ stands for the smallest (i.e. first) positive eigenvalue of $D_{\tilde{g}}$ with respect to $\tilde{g} \in [g]$. In fact, as was pointed out in [6,8,10], the value of the Bär-Hijazi-Lott invariant for an arbitrary closed spin m-manifold can not be larger than that for the round sphere (with the same dimension), that is

$$\lambda_{min}^+(M, g, \sigma) \leq \lambda_{min}^+(S^m, g_{S^m}, \sigma_{S^m}) = \frac{m}{2}\omega_m^{\frac{1}{m}} \tag{1.3}$$

where g_{S^m} is the standard round metric, σ_{S^m} stands for the unique spin structure on S^m and ω_m denotes the volume of (S^m, g_{S^m}). In this regard, the next stage would consist in showing that (1.3) is a strict inequality when (M, g) is not conformally equivalent to (S^m, g_{S^m}). And it is important to notice that, if there exists a nontrivial solution to Eq. (1.2) (with $f \equiv 1$) such that $\int_M |\psi|_g^{2m/(m-1)} d\,\mathrm{vol}_g < (\frac{m}{2})^m \omega_m$, then the strict inequality in (1.3) holds true (see [49,50]). This can be viewed as the spinorial analogue of the Yamabe problem in geometric analysis. However, the strict inequality in (1.3) is only verified for some special cases (for instance, if M is locally conformally flat, if D_g is invertible and if the so-called *Mass endomorphism* is not identically zero [12], and all rectangular tori [49], and non-locally conformally flat manifolds [50]), but a general result is still lacking (cf. [9,28,30]). The methods that can be used are sometimes similar to the ones of the Yamabe problem, but since we work with Dirac operator and spinors, the reasoning is more involved as the eigenvalues of the Dirac operator tend to both $-\infty$ and $+\infty$ and there is no adequate replacement for the maximum principle.

Another reason that makes Eq. (1.2) interesting is that, in dimension $m = 2$, its solution provides a strong tool for showing the existence of prescribed mean curvature surfaces in $\mathbb{R}^3$ (here the function f plays the role of the mean curvature). Special cases of such surfaces are constant mean curvature (CMC) surfaces (that is $f \equiv constant$) which have been studied before by completely different techniques, see for instance [29]. The correspondence between a solution of Eq. (1.2) on a Riemannian surface M and a periodic conformal immersion (possibly with branching points) of the universal covering $\widetilde{M}$ into $\mathbb{R}^3$ with mean curvature f is known as the spinorial Weierstraß representation. For details in this direction, we refer to [6,8,26,37,39,40,44,51–53] and references therein.

Although the existence problem for Eq. (1.2) is not settled in full generality, there are several partial existence results in the literature, see for instance [34,35,55], and it is often true that many solutions exist for Eq. (1.2). As a first step towards multiple existence results, consider the problem on the Torus $S^1(L) \times S^1(1)$ with product metric, there are many non-minimizing solutions if L is large, see [49] (and also see [36] for more examples in the non-locally conformally flat setting). In this paper, we address a very fundamental question

Question 1. *Let M be a closed oriented spin m-manifold, equipped with the data (g, f) on M (a metric and a real function), so that either (M, g) is not conformally equivalent to (S^m, g_{S^m}) or $f : M \to \mathbb{R}$ is not a constant. Whether or not the set of all solutions to the spinorial Yamabe-type PDE (1.2) is compact (in the C^1-topology, say)?*

The case of the round sphere (S^m, g_{S^m}) and $f \equiv constant \neq 0$ is exceptional since (1.2) is invariant under the action of the conformal group on S^m, which is not compact. Let us also mention here that, in the context of the spinorial Weierstraß representation, the above question may lead us to think

Question 2. *Given a connected closed oriented surface Σ and arbitrary data (g, f) on Σ, is it possible to characterize a non-compact family of immersions $\Pi_i : \Sigma \to \mathbb{R}^3$ conformally realizing (g, f), that is*

$$\Pi_i^*(g_{\mathbb{R}^3}) \in [g] \quad \textit{and} \quad H_{\Pi_i} = f, \quad \textit{for all } i = 1, 2, \dots$$

where H_Π stands for the mean curvature of an immersion Π?

Remark 1.1. Usually, a generic immersion is uniquely determined up to a rigid motion by its first fundamental form and its mean curvature function, but there are some exceptions, for instance most constant mean curvature immersions. A classical result by Bonnet states that if there exists a diffeomorphism $\Psi : \Sigma_1 \to \Sigma_2$ between two closed immersed surfaces Σ_1, Σ_2 of genus zero in $\mathbb{R}^3$ such that Ψ preserves both the metric and the mean curvature function of the surfaces, then Σ_1 and Σ_2 are congruent in $\mathbb{R}^3$ (i.e., they differ by a rigid motion). Note that in Question 2 we are not assuming that the immersed surfaces are isometric, which is a critical hypothesis of Bonnet's result.

Generally speaking, the answer to Question 2 is definitely no. On the one hand, for any immersion Π of a compact surface in $\mathbb{R}^3$, one must have $H_\Pi > 0$ somewhere. This means that the function f cannot be arbitrarily taken. And in fact there are further obstructions, at least on the sphere $\Sigma = S^2$. Indeed, the mean curvature H_Π of a conformal immersion $\Pi : S^2 \to \mathbb{R}^3$ must satisfy

$$\int_{S^2} V(H_\Pi) d\,\mathrm{vol}_{\Pi^*(g_{\mathbb{R}^3})} = 0$$

for any conformal vector field V on S^2, see [11]. In particular, if $x_3 : S^2 \to \mathbb{R}$ stands for the third component of the standard inclusion of S^2 in $\mathbb{R}^3$, then for any $\varepsilon \neq 0$ the function $f(x) = 1 + \varepsilon x_3$ cannot be realized as the mean curvature of a conformal immersion $S^2 \to \mathbb{R}^3$. On the other hand, it is well-known that the round sphere is the only possible shape of an immersed closed CMC surface in $\mathbb{R}^3$ having genus 0 (see Hopf [32]). Therefore, the questions which concern us here are only interesting in the case where the solution set of Eq. (1.2) is non-empty and having rich characterization to reflect the geometric interpretations.

Let us point out that a similar non-compactness question has been raised for the classical Yamabe problem, which is well-known as the Compactness Conjecture, see [48]. And such conjecture has been verified up to dimension 24 and disproved for dimensions $m \geq 25$, see [20,21,38]. So far, to the best of our knowledge, there is no result characterizing the compactness or non-compactness of the solution set for Eq. (1.2). One of the reasons is that, since the validity of the strict inequality in (1.3) is still open, the solvability of Eq. (1.2) is far from complete. Moreover, it is also not even clear if the positive mass theorem (or its variants) can be employed to the study of Eq. (1.2) as the Schoen-Yau positive energy theorem does for the classical Yamabe problem.

In this paper, we intend to construct specific geometric data on a Riemannian spin manifold such that the set of solutions to Eq. (1.2) fails to be compact. To be more precise, we will focus on the case $M = S^m$ and attack the problem from two perspectives. For starters, let us take $f \equiv 1$ in (1.2) and consider the effects of the background metric g. In this case, we are facing with the equation

$$D_g\psi = |\psi|_g^{\frac{2}{m-1}}\psi \quad \text{on } (S^m, g) \tag{1.4}$$

where g is not conformally related to the round metric. It is of particular interest since the integral $\int_{S^m} |\psi|_g^{\frac{2m}{m-1}} d\,\mathrm{vol}_g$ of a solution gives an upper bound of the Bär-Hijazi-Lott invariant. Hence, it would be interesting if one can derive a conformal spectral estimate for the Dirac operator D_g so that (1.3) is a strict inequality. Another perspective is to fix $g = g_{S^m}$ (that is the canonical round metric) and to consider the problem with a non-constant function $f : S^m \to \mathbb{R}$, so that the effects of the external potential function can be detected. This leads us to consider the equation

$$D_{g_{S^m}}\psi = f(x)|\psi|_{g_{S^m}}^{\frac{2}{m-1}}\psi \quad \text{on } (S^m, g_{S^m}) \tag{1.5}$$

where $f \not\equiv constant$. In this setting, when $m = 2$, it is of geometric interest to show the existence of a non-compact collection of immersed spheres in $\mathbb{R}^3$ with a prescribed mean curvature function f.

Our first main result reads as

Theorem 1.2. *For $k \geq 1$ and $m \geq 4k + 2$. There exists a Riemannian metric g on S^m of class C^k and a sequence of spinors $\{\psi_i\}_{i=1}^\infty \subset C^1(S^m, \mathbb{S}(S^m))$ with the following properties:*

(1) *g is not locally conformally flat;*
(2) *ψ_i is a nontrivial solution of the equation (1.4) for all $i \in \mathbb{N}$;*
(3) $\displaystyle\int_{S^m} |\psi_i|_g^{\frac{2m}{m-1}} d\,\mathrm{vol}_g < \big(\frac{m}{2}\big)^m \omega_m$ *for all $i \in \mathbb{N}$, and*

$$\lim_{i\to\infty} \int_{S^m} |\psi_i|_g^{\frac{2m}{m-1}} d\,\mathrm{vol}_g = \big(\frac{m}{2}\big)^m \omega_m;$$

(4) $\sup_{S^m} |\psi_i|_g \to +\infty$ *as $i \to \infty$.*

Moreover, the strict inequality in (1.3) holds true for g, i.e.

$$\lambda^+_{min}(S^m, g, \sigma_{S^m}) < \frac{m}{2}\omega_m^{\frac{1}{m}}.$$

Remark 1.3.

(1) Let us point out here that, following from our construction, the metric g in Theorem 1.2 cannot be smooth. But the above result indicates that higher the dimension is better the regularity of the metric g will be.
(2) Theorem 1.2 provides an example of non-smooth metric g on a spin manifold such that the strict inequality in (1.3) holds. This is the first result of this kind in the study of Bär-Hijazi-Lott invariant.
(3) Theorem 1.2 can be considered as a counterpart of Berti-Malchiodi's result for the Yamabe problem, see [16, Theorem 1.2]. It is worth noting that, by Hijazi's inequality [31], we have

$$\lambda_{min}^{+}(M,g,\sigma)^2 \geq \frac{m}{4(m-1)} Y(M,g)$$

where $Y(M,g)$ stands for the Yamabe constant of (M,g). Hence, Theorem 1.2 above implies

$$Y(S^m,g) < m(m-1)\omega_m^{\frac{2}{m}},$$

which played a crucial role in the solvability of the Yamabe problem. From this point of view, Theorem 1.2 provides an alternative construction of constant scalar curvature metrics in the conformal class of g. This extends the result of Berti and Malchiodi by relaxing the starting dimension from 6 instead of 11 as in [16]. Comparing Theorem 1.2 with the results of the Yamabe problem, it would be natural to expect a compactness result for Eq. (1.4) in low dimensions. And it is also interesting to see if the non-compactness results hold for some C^∞ smooth background metric.

Our next result is concerned with the external potential function f.

Theorem 1.4. *For every $m \geq 2$, there exists a non-constant function $f \in C^2(S^m)$, $f > 0$, and a sequence of spinors $\{\psi_i\}_{i=1}^{\infty} \subset C^1(S^m, \mathbb{S}(S^m))$ with the following properties:*

(1) *ψ_i is a nontrivial solution of the equation (1.5) for all $i \in \mathbb{N}$;*
(2) *$|\psi_i|_{g_{S^m}} > 0$ on S^m and there holds*

$$\lim_{i\to\infty} \int_{S^m} f(x)|\psi_i|_{g_{S^m}}^{\frac{2m}{m-1}} \, d\,\mathrm{vol}_{g_{S^m}} = \left(\frac{m}{2}\right)^m \omega_m$$

and

$$\lim_{i\to\infty} \int_{S^m} f(x)^2|\psi_i|_{g_{S^m}}^{\frac{2m}{m-1}} \, d\,\mathrm{vol}_{g_{S^m}} = \left(\frac{m}{2}\right)^m \omega_m;$$

(3) *$\sup_{S^m} |\psi_i|_{g_{S^m}} \to +\infty$ as $i \to \infty$.*

Remark 1.5.

(1) Theorem 1.4 (2) has its own geometric meaning. In fact, in dimension $m = 2$, we can introduce a conformal metric $g_i = |\psi_i|_{g_{S^2}}^4 g_{S^2}$ on S^2, for each i. Then, due to the conformal covariance of the Dirac operator (cf. [27,28]), we see that there is a spinor field φ_i on (S^2, g_i) such that

$$D_{g_i}\varphi_i = f(x)\varphi_i \quad \text{and} \quad |\varphi_i|_{g_i} \equiv 1.$$

Hence, by the spinorial Weierstraß representation, there is an isometric immersion $\Pi_i : (S^2, g_i) \to (\mathbb{R}^3, g_{\mathbb{R}^3})$ with mean curvature $H_{\Pi_i} = f$. Furthermore, since the pull-back of the Euclidean volume form under this immersion is $\Pi_i^*(d\,\mathrm{vol}_{g_{\mathbb{R}^3}}) = |\psi_i|^4_{g_{S^2}} d\,\mathrm{vol}_{g_{\mathbb{R}^2}}$, the associated Willmore energy $W(\Pi_i)$ for this immersion satisfies

$$W(\Pi_i) = \int_{S^2} f(x)^2 |\psi_i|^4_{g_{S^2}} d\,\mathrm{vol}_{g_{\mathbb{R}^2}} < 8\pi$$

for all i (large enough). Due to Li-Yau's inequality [43, Theorem 6], the immersion Π_i covers points in $\mathbb{R}^3$ at most once. Hence Π_i is actually an embedding.

(2) Theorem 1.4 provides a positive answer to Question 2. Indeed, let us consider the family of immersions

$$\mathcal{I} = \big\{\Pi : S^2 \to \mathbb{R}^3 : \Pi \text{ conformally realizes } (g_{S^2}, f)\big\}$$

and discuss its compactness (say, whether the images of S^2 via elements of $\mathcal{I}$ form a compact collection of surfaces in $\mathbb{R}^3$), we find that $\{\Pi_i\} \subset \mathcal{I}$ and $W(\Pi_i) \to 4\pi$ as $i \to \infty$. Notice that an immersion $\Pi : \Sigma \to \mathbb{R}^3$ of a Riemann surface Σ satisfies $W(\Pi) = 4\pi$ if and only if $\Pi(\Sigma)$ is the round sphere. Hence, we see that $\mathcal{I}$ cannot be compact since the limit of $\Pi_i(S^2)$ (even if it exists) will not realize the non-constant function f as the mean curvature.

Now as an immediate consequence of the above remark, we have

Corollary 1.6. *There exists a non-constant function $f \in C^2(S^2)$ such that the family*

$$\mathcal{E} = \big\{\Pi : S^2 \to \mathbb{R}^3 \textit{ is an embedding} : \Pi \textit{ conformally realizes } (g_{S^2}, f)\big\}$$

fails to be compact in the sense that $\{\Pi(S^2) : \Pi \in \mathcal{E}\}$ is not compact in $\mathbb{R}^3$.

Let us sketch the main steps involved in the proofs of the Theorems 1.2 and 1.4. In Section 2, after introducing some basic concepts and notations from the spin geometry, we will reformulate our problems and work on $\mathbb{R}^m$ instead of S^m via stereographic projection. Our goal is to construct solutions to the spinorial Yamabe-type PDEs (1.4) and (1.5) on $(\mathbb{R}^m, \tilde{g})$ and $(\mathbb{R}^m, g_{\mathbb{R}^m})$ respectively, where either $\tilde{g} = g_{\mathbb{R}^m} + \varepsilon\tilde{h}$ is a perturbation of the Euclidean metric or $f(x) = 1+\varepsilon\tilde{H}(x)$ is a perturbation from constant. In Section 3, we set up a perturbative variational framework so that we can reduce our problems to a kind of finite dimensional bifurcation problem. This idea has been employed for the study of classical Yamabe problem (see, e.g., [4,5,16,24]). Here, unlike the scalar cases, the finite dimensional problem associated to the spinorial Yamabe-type PDEs is degenerate, that is, any critical point of the main term of the reduced functional is not isolated, and the collection of these critical points appear as critical manifolds of positive dimension.

Thus, it is not clear whether critical points of the reduced functional create true solutions of the original problems. For this reason, the abstract framework in [5,16] can not be implemented in a straightforward manner, and somehow a delicate handling is required. In the subsequent sections, i.e., Sections 4 and 5, we check the hypothesis of the abstract framework in the two cases of our main problems mentioned above, and complete the proofs of the main results. The Appendix contains some technical computations.

2. Preliminaries

2.1. Projecting the problems to $\mathbb{R}^m$

Let us first consider Eq. (1.4) and rewrite it in a more precise manner as

$$D_{g_\varepsilon}\psi = |\psi|_{g_\varepsilon}^{\frac{2}{m-1}}\psi \quad \text{on } S^m, \tag{2.1}$$

in which the metric g_ε is a perturbation from the canonical one on S^m. Using the stereographic projection $\pi_P : S^m \setminus \{P\} \to \mathbb{R}^m$ (for an arbitrarily fixed $P \in S^m$), we obtain the following one-to-one correspondence between g_ε on $S^m \setminus \{P\}$ and a metric $\tilde{g}_\varepsilon$ on $\mathbb{R}^m$:

$$\tilde{g}_\varepsilon = \mu^{-2} \cdot (\pi_P^{-1})^* g_\varepsilon, \quad \mu(x) = \frac{2}{1+|x|^2},\ x \in \mathbb{R}^m. \tag{2.2}$$

Clearly, if $\tilde{g}_\varepsilon = g_{\mathbb{R}^m}$ is the canonical Euclidean metric, then the metric g_ε on $S^m \setminus \{P\}$ can be extended globally to the standard round metric. In what follows, we assume that $\tilde{g}_\varepsilon$ takes the form $\tilde{g}_\varepsilon = g_{\mathbb{R}^m} + \varepsilon\tilde{h}$ where $\tilde{h}$ is a smooth symmetric bilinear form on $\mathbb{R}^m$. In particular, let us consider a specific situation

$$\tilde{g}_\varepsilon(x) = \operatorname{diag}\big(\tilde{g}_{11}(x), \dots, \tilde{g}_{mm}(x)\big) \quad \text{with} \quad \tilde{g}_{ii}(x) = 1 + \varepsilon\tilde{h}_{ii}(x), \tag{2.3}$$

where $\tilde{h}_{ii} : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, m$, are smooth functions. Let us point out here that, for a general choice of $\tilde{h}$, the pull-back metric g_ε on $S^m \setminus \{P\}$ may be discontinuous at the point P. Hence, in order to extend g_ε globally on S^m, it is natural to require the entries $\tilde{h}_{ii}$, $i = 1, \dots, m$, and their derivatives behave "nicely" at infinity.

We also mention that the Eq. (2.1) on S^m is equivalent to an equation on $\mathbb{R}^m$ by conformal equivalence. More precisely, the equation $D_g\psi = |\psi|_g^{\frac{2}{m-1}}\psi$ on a spin manifold (M, g) is invariant under conformal changes of the metric. In fact, let $\bar{g} = e^{2u}g$ for some function u on M, there is an isomorphism of vector bundles $F : \mathbb{S}(M, g) \to \mathbb{S}(M, \bar{g})$ (here $\mathbb{S}(M, g)$ and $\mathbb{S}(M, \bar{g})$ are spinor bundles on M with respect to the metrics g and $\bar{g}$, respectively) which is a fiberwise isometry such that

$$D_{\bar{g}}\big(F(\varphi)\big) = F\big(e^{-\frac{m+1}{2}u} D_g(e^{\frac{m-1}{2}u}\varphi)\big)$$

for $\varphi \in C^\infty((M,g),\mathbb{S}(M,g))$ (for more detailed definitions and facts about Clifford algebras, spin structures on manifolds and Dirac operators, please consult [27,41]). Thus, when ψ is a solution to the equation $D_g\psi = |\psi|_g^{\frac{2}{m-1}}\psi$ on (M,g), then $\varphi := F(e^{-\frac{m-1}{2}u}\psi)$ satisfies the same equation on $(M,\bar g)$: $D_{\bar g}\varphi = |\varphi|_{\bar g}^{\frac{2}{m-1}}\varphi$ on $(M,\bar g)$.

Applying the above observation to Eq. (2.1) with $M = S^m$ and using (2.2)–(2.3), we find that if $\psi \in C^1(S^m,\mathbb{S}(S^m))$ is a solution then $\varphi = \mu^{\frac{m-1}{2}}F(\psi\circ\pi_P^{-1})$ satisfies the equation

$$D_{\tilde g_\varepsilon}\varphi = |\varphi|_{\tilde g_\varepsilon}^{\frac{2}{m-1}}\varphi \quad \text{on } \mathbb{R}^m. \tag{2.4}$$

Conversely, by the regularity theorem and the removal of singularities theorem for Dirac equations on spin manifolds (see [33, Appendix] and [8, Theorem 5.1]), if φ is a solution to Eq. (2.4) and $\varphi \in L^{\frac{2m}{m-1}}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$ then it corresponds to a global C^1-solution ψ to Eq. (1.4) on S^m. Therefore, the study of Eq. (2.1) is equivalent to the study of Eq. (2.4).

Now, to characterize the metric g_ε, let us set

$$\tilde h(x) = \sum_{i=1}^{+\infty} a_i h(x-x_i) \tag{2.5}$$

where $a_i \in \mathbb{R}$, $|x_i| \to +\infty$ as $i \to +\infty$, and h is a smooth symmetric matrix function with compact support. Roughly speaking, with this choice of $\tilde h$ in the definition of metric $\tilde g_\varepsilon$, the Dirac operator in (2.4) becomes

$$D_{\tilde g_\varepsilon} = D_{g_{\mathbb{R}^m}} + R(\varepsilon, x, \tilde h, \nabla) \tag{2.6}$$

where $R(\varepsilon,x,\tilde h,\nabla)$ is a suitable perturbation term. In this way, we expect that Eq. (2.4) shall be handled by means of a perturbation method in nonlinear analysis. When $\tilde h$ consists of only a finite number of terms, the existence problem of Eq. (2.4) has been firstly treated in [36]. In particular, a very specific construction of the matrix function h has been introduced in [36] so that the effect of the perturbation term in (2.6) can be explicitly computed from a variational point of view. Here, for the sake of completeness, we present the very formulation for the matrix h in (2.5) as follows.

Definition 2.1. Given a smooth $m\times m$ diagonal matrix function

$$h(x) = \operatorname{diag}\big(h_{11}(x),\dots,h_{mm}(x)\big) \quad \text{for } x \in \mathbb{R}^m,\ m \ge 2,$$

and a point $\xi = (\xi_1,\dots,\xi_m) \in \mathbb{R}^m$. For $k \in \{1,\dots,m\}$ and $p \in [1,\infty)$, we say that h is (k,p)-elementary at ξ, if $\xi \notin \operatorname{supp} h_{kk}$ and, for $x = (x_1,\dots,x_m) \in \mathbb{R}^m$ close to ξ and $i \neq k$,

$$h_{ii}(x) = h_{ii}(\xi) + c_i(x_i - \xi_i) + c_k(x_k - \xi_k) + o(|x - \xi|^p)$$

where $c_i \in \mathbb{R}$, $i = 1, \dots, m$, are constants with particularly $c_k \neq 0$. Moreover, if the $o(|x - \xi|^p)$ term vanishes identically in the above local expansion of h_{ii}'s, then we say h is (k, ∞)-elementary at ξ. In this way, we call $p \in [1, \infty) \cup \{\infty\}$ the remainder exponent of h at ξ.

Remark 2.2. Let us present here a simple example of $(1, p)$-elementary matrix at the origin, in dimension 3:

$$h(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a + c_1 x_1 + c_2 x_2 & 0 \\ 0 & 0 & b + c_1 x_1 + c_3 x_3 \end{pmatrix} + o(|x|^p)$$

for $|x| < r$, where a, b, c_1, c_2, $c_3 \in \mathbb{R}$ are real constants with particularly $c_1 \neq 0$. This very specific definition is first introduced in [36] for the study of multiple solutions for the spinorial Yamabe-type problems. More examples and a brief explanation of such (k, p)-elementary matrices have been given in [36, Appendix]. We mention here that the main reason we introduce those (k, p)-elementary matrices lies in Proposition 4.3, where we find such matrices are surprisingly compatible with the perturbed Dirac operator (2.6) and they guarantee the implementation of our abstract result in Section 3.

It can be seen from Definition 2.1 that "*elementary*" matrix is a local concept. In the sequel, if it is clear from the context to which dimension we refer, we will simply use the name "elementary matrix" to designate a member h (without specifying its tag numbers k and the location point ξ). In order to classify the perturbation term in (2.5), let us set

$$\mathcal{H}(p) = \left\{ \tilde{h}(\cdot) = \sum_{i=1}^{+\infty} a_i h(\cdot - x_i) \,\middle|\, \begin{array}{l} h \text{ is a compactly supported elementary matrix} \\ \text{with remainder exponent } p, \\ \{a_i\} \subset \mathbb{R} \text{ and } \displaystyle\sum_{i=1}^{\infty} |a_i|^\tau < +\infty, \text{ for some } \tau > 1, \\ \{x_i\} \subset \mathbb{R}^m \text{ and } |x_j - x_i| > 4\,\mathrm{diam}(\mathrm{supp}\, h) \text{ for } i \neq j \end{array} \right\}.$$

Then Theorem 1.2 is nothing but a direct consequence of the following result.

Theorem 2.3. *Let $p \in [2, \infty) \cup \{\infty\}$, $k \geq 1$ and $m \geq 4k + 2$. There exist $\tilde{h} \in \mathcal{H}(p)$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ the metric g_ε in (2.2)–(2.5) is of class C^k on S^m, and the following properties hold:*

(1) $\|g_\varepsilon - g_{S^m}\|_{C^k} \to 0$ *as* $\varepsilon \to 0$,
(2) *Eq.* (2.1) *possesses a sequence of solutions* $\{\psi_\varepsilon^{(i)}\}_{i=1}^\infty$ *satisfying* $\|\psi_\varepsilon^{(i)}\|_{L^\infty(S^m)} \to +\infty$ *as* $i \to +\infty$,

(3) *there holds*

$$\int_{S^m} |\psi_\varepsilon^{(i)}|_{g_\varepsilon}^{\frac{2m}{m-1}} d\,\mathrm{vol}_{g_\varepsilon} = \Big(\frac{m}{2}\Big)^m \omega_m + C_{i,m} a_i^2 \varepsilon^2 + o(a_i^2\varepsilon^2)$$

where $C_{i,m} < 0$ *is a negative constant depending only on* $\tilde{h}$*, the dimension* m *and* $i \in \mathbb{N}$*.*

Next, in order to study Eq. (1.5), let us focus on the case where the function f takes the form $f(x) = 1 + \varepsilon \tilde{H}(x)$, i.e.

$$D_{g_{S^m}}\psi = (1 + \varepsilon \tilde{H}(x))|\psi|_{g_{S^m}}^{\frac{2}{m-1}}\psi \quad \text{on } (S^m, g_{S^m}) \tag{2.7}$$

with $\varepsilon \neq 0$ and some $\tilde{H} : S^m \to \mathbb{R}$ at least being Hölder continuous. As before, denote by $\pi_{p_0} : S^m \setminus \{p_0\} \to \mathbb{R}^m$ the stereographic projection from p_0 (this point will be fixed later according to our choice of $\tilde{H}$), we have $(\pi_{p_0}^{-1})^* g_{S^m} = \mu^2 g_{\mathbb{R}^m}$. And then, via the conformal transformation, Eq. (1.5) can be converted to

$$D_{g_{\mathbb{R}^m}}\varphi = (1 + \varepsilon \tilde{K}(x))|\varphi|_{g_{\mathbb{R}^m}}^{\frac{2}{m-1}}\varphi \quad \text{on } \mathbb{R}^m \tag{2.8}$$

where $\tilde{K}(x) = \tilde{H}(\pi_{p_0}^{-1}(x))$. We remark here that we shall write it simply x for the argument of a function when no confusion can arise.

Similarly to the way we handle Eq. (2.4), let us consider a situation where the function $\tilde{H}$ can be decomposed into a series of components such that each component generates a solution to Eq. (2.8). In order to do so, for a continuously differentiable function H on S^m (which plays the role of an individual component of $\tilde{H}$), let us simply denote $Crit[H]$ the critical set of H. For later use, we assume the following two standing conditions on H:

(H-1) $H \in C^2(S^m)$ is a Morse function such that $\Delta_{g_{S^m}} H(p) \neq 0$ for $p \in Crit[H]$.
(H-2) H satisfies that

$$\sum_{p \in Crit[H],\ \Delta_{g_{S^m}} H(p) < 0} (-1)^{\mathfrak{M}(H,p)} \neq (-1)^m$$

where $\mathfrak{M}(H,p)$ is the Morse index of H at $p \in Crit[H]$.

Here we mention that condition (H-2) is the well-known index counting condition which was first introduced in the scalar curvature problem in [14,23].

Then we collect the following family of continuous functions on S^m

$$\mathscr{H} = \left\{ \tilde{H} = \sum_{i=1}^{\infty} a_i H\big(\pi_{p_0}^{-1}(\pi_{p_0}(\cdot) - z_i)\big) \;\middle|\; \begin{array}{l} H \text{ satisfies the conditions (H-1) and (H-2),} \\ p_0 \in Crit[H] \text{ and } \Delta_{g_{S^m}} H(p_0) > 0, \\ \{a_i\} \subset \mathbb{R} \text{ and } \displaystyle\sum_{i=1}^{\infty} |a_i| < +\infty, \\ \{z_i\} \subset \mathbb{R}^m \text{ and } |z_i - z_j| > 1 \text{ for } i \neq j \end{array} \right\}.$$

It is clear that, when $H \in C^2(S^m)$, $H \circ \pi_{p_0}^{-1}$ defines a C^2-function on $\mathbb{R}^m$ and $\lim_{|y|\to\infty} H \circ \pi_{p_0}(y) = H(p_0)$. The function $H\big(\pi_{p_0}^{-1}(\pi_{p_0}(\cdot) - z_i)\big) : S^m \setminus \{p_0\} \to \mathbb{R}$ can be viewed as a translation of H on S^m with $H(p_0)$ being fixed. Hence the above family describes a function that is (approximately) concentrated on the points $\pi_{p_0}^{-1}(z_i) \in S^m \setminus \{p_0\}$, $i = 1, 2, \ldots$, and is well-defined on S^m. We remark that the elements in $\mathscr{H}$ are not necessarily differentiable at p_0 and, as was indicated in its geometric background, the function $1+\varepsilon\tilde{H}$ plays a role of mean curvature. Hence, one may expect $\tilde{H}$ to have certain regularity at the point p_0. With all these in mind, let us present the following result that explains Theorem 1.4.

Theorem 2.4. *For every $m \geq 2$, there exist $\tilde{H} \in \mathscr{H} \cap C^2(S^m)$ and $\varepsilon_0 > 0$ such that for $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ the following properties hold:*

(1) *Eq. (2.7) possesses a sequence of solutions $\{\psi_\varepsilon^{(i)}\}_{i=1}^{\infty}$ satisfy $\|\psi_\varepsilon^{(i)}\|_{L^\infty(S^m)} \to +\infty$ as $i \to +\infty$,*
(2) *$|\psi_\varepsilon^{(i)}|_{g_{S^m}} > 0$ on S^m provided that $|\varepsilon|$ is small, moreover,*

$$\lim_{i\to\infty} \int_{S^m} (1 + \varepsilon\tilde{H}(x))|\psi_\varepsilon^{(i)}|_{g_{S^m}}^{\frac{2m}{m-1}} d\,\mathrm{vol}_{g_{S^m}} = \Big(\frac{m}{2}\Big)^m \omega_m$$

and

$$\lim_{i\to\infty} \int_{S^m} (1 + \varepsilon\tilde{H}(x))^2|\psi_\varepsilon^{(i)}|_{g_{S^m}}^{\frac{2m}{m-1}} d\,\mathrm{vol}_{g_{S^m}} = \Big(\frac{m}{2}\Big)^m \omega_m.$$

We end this subsection by pointing out that the following equation

$$D_{g_{\mathbb{R}^m}}\psi = |\psi|_{g_{\mathbb{R}^m}}^{\frac{2}{m-1}}\psi \quad \text{on } \mathbb{R}^m \tag{2.9}$$

can be viewed as the unperturbed equation of both Eq. (2.4) and Eq. (2.8). Hence, in the sequel, our framework will be build upon the study of Eq. (2.9) and its Euler-Lagrange functional

$$\mathcal{J}_0(\psi) = \frac{1}{2}\int_{\mathbb{R}^m} (\psi, D_{g_{\mathbb{R}^m}}\psi)_{g_{\mathbb{R}^m}}\, d\,\mathrm{vol}_{g_{\mathbb{R}^m}} - \frac{m-1}{2m}\int_{\mathbb{R}^m} |\psi|_{g_{\mathbb{R}^m}}^{\frac{2m}{m-1}}\, d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \tag{2.10}$$

where $(\cdot,\cdot)_{g_{\mathbb{R}^m}}$ and $|\cdot|_{g_{\mathbb{R}^m}}$ are the canonical hermitian product and its induced metric on the spinor bundle $\mathbb{S}(\mathbb{R}^m)$.

2.2. Configuration spaces

To treat Eq. (2.4) and (2.8) from a variational point of view, it is necessary to set up a functional framework. Suitable function spaces are $H^{\frac{1}{2}}(M, \mathbb{S}(M))$ and $\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ of spinor fields which are introduced in [33,34]. For completeness, we give the definitions as follows.

Recall that the Dirac operator D_g on a compact spin manifold (M, g) is self-adjoint on $L^2(M, \mathbb{S}(M))$ and has compact resolvents (see [27,41]). Particularly, there exists a complete orthonormal basis $\psi_1, \psi_2, \ldots$ of the Hilbert space $L^2(M, \mathbb{S}(M))$ consisting of the eigenspinors of D_g: $D_g\psi_k = \lambda_k\psi_k$. Moreover, $|\lambda_k| \to \infty$ as $k \to \infty$.

Now, we define the operator $|D_g|^{1/2} : L^2(M, \mathbb{S}(M)) \to L^2(M, \mathbb{S}(M))$ by

$$|D_g|^{1/2}\psi = \sum_{k=1}^{\infty} |\lambda_k|^{1/2}\alpha_k\psi_k,$$

for $\psi = \sum_{k=1}^{\infty}\alpha_k\psi_k \in L^2(M, \mathbb{S}(M))$ and consider its domain

$$H^{1/2}(M, \mathbb{S}(M)) := \Big\{\psi = \sum_{k=1}^{\infty}\alpha_k\psi_k \in L^2(M, \mathbb{S}(M)) : \sum_{k=1}^{\infty}|\lambda_k||\alpha_k|^2 < \infty\Big\}.$$

We can equip $H^{1/2}(M, \mathbb{S}(M))$ with the inner product

$$\langle\psi, \varphi\rangle_{1/2,2} := \mathrm{Re}(|D_g|^{1/2}\psi, |D_g|^{1/2}\varphi)_2 + \mathrm{Re}(\psi, \varphi)_2$$

and the induced norm $\|\cdot\|_{1/2,2}$, where $(\cdot,\cdot)_2$ is the L^2-inner product on spinors. It follows that $H^{1/2}(M, \mathbb{S}(M))$ coincides with the usual Sobolev space $W^{1/2,2}(M, \mathbb{S}(M))$ (cf. [1,6]). In the sequel, we are mainly concerned with the space $H^{\frac{1}{2}}(M, \mathbb{S}(M))$ for $M = S^m$. Notice that the spectrum of $D_{g_{S^m}}$ on S^m is bounded away from 0 and one checks easily that $\|\psi\|_{1/2} = \big||D_{g_{S^m}}|^{\frac{1}{2}}\psi\big|_2$ defines an equivalent norm on $H^{\frac{1}{2}}(S^m, \mathbb{S}(S^m))$.

On $\mathbb{R}^m$, a similar function space will also be useful in our argument. For simplicity of notation, we denote $L^q := L^q(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ with the norm $|\psi|_q^q = \int_{\mathbb{R}^m}|\psi|^q d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$ for $q \geq 1$ and denote $2^* = \frac{2m}{m-1}$ the critical Sobolev exponent of the embedding $H^{1/2}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m)) \hookrightarrow L^q(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ for $1 \leq q \leq 2^*$. Then, we recall the space $\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ of spinor fields ψ on $\mathbb{R}^m$ such that $\big||D_{g_{\mathbb{R}^m}}|^{1/2}\psi\big|_2^2 < \infty$ with norm $\|\psi\| := \big||D_{g_{\mathbb{R}^m}}|^{1/2}\psi\big|_2$. Here, $|D_{g_{\mathbb{R}^m}}|^{1/2}$ is defined via the Fourier transformation: $\mathscr{F}(|D_{g_{\mathbb{R}^m}}|^{1/2}\psi)(\xi) = |\xi|^{1/2}\mathscr{F}(\psi)(\xi)$ and $\big||D_{g_{\mathbb{R}^m}}|^{1/2}\psi\big|_2 := \big||\cdot|^{1/2}\mathscr{F}(\psi)\big|_2$. Notice that

$\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ is isomorphic to $H^{\frac{1}{2}}(S^m, \mathbb{S}(S^m))$ via the stereographic projection. The dual space of $\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ will be denoted by $\mathscr{D}^{-\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$.

2.3. Geometric preliminaries and expansion of the perturbed functional

In this part, we shall collect some basic results that will enable us to expand the energy functional $\mathcal{J}_\varepsilon$ for (2.4) with respect to the small parameter ε in (2.2)–(2.3). This requires comparing spinor fields in spinor bundles associated with different metrics. In order to carry this out, we recall a construction by Bourguignon and Gauduchon [19] and some formulas given in [36, Section 4.1] which will be useful for our computations.

To begin with, for the metrics $g_{\mathbb{R}^m}$ and $\tilde{g}$ on $\mathbb{R}^m$, let us consider the unique endomorphism A_x at each point $x \in \mathbb{R}^m$ such that

$$\tilde{g}(v, w) = g_{\mathbb{R}^m}(A_x v, w)$$

for $v, w \in T_x\mathbb{R}^m$. Notice that A_x is nothing but a positive definite symmetric matrix, it has a well-defined square root B_x. Let $b_{ij}(x)$, $i, j = 1, \dots, m$, be the entries of B_x, we have

$$\begin{aligned} B_x : (T_x\mathbb{R}^m \cong \mathbb{R}^m, g_{\mathbb{R}^m}) &\to (T_x\mathbb{R}^m, \tilde{g}_x) \\ v = \sum_k v_k \partial_k &\mapsto B_x(v) := \sum_j \Big(\sum_k b_{jk}(x) v_k \Big) \partial_j \end{aligned}$$

defines an isometry for each $x \in \mathbb{R}^m$. Then we obtain an isomorphism of $SO(m)$-principal bundles: $\eta\{v_1, \dots, v_m\} = \{B(v_1), \dots, B(v_m)\}$ for an oriented frame $\{v_1, \dots, v_m\}$ on $(\mathbb{R}^m, g_{\mathbb{R}^m})$. Note that the map η commutes with the right action of $SO(m)$, it can be lifted to spin structures:

$$\begin{array}{ccc} P_{Spin}(\mathbb{R}^m, g_{\mathbb{R}^m}) & \xrightarrow{\tilde{\eta}} & P_{Spin}(\mathbb{R}^m, \tilde{g}) \\ \downarrow & & \downarrow \\ \mathbb{R}^m & \xrightarrow{\mathrm{Id}} & \mathbb{R}^m \end{array}$$

which induces an isomorphism between the spinor bundles $\mathbb{S}(\mathbb{R}^m, g_{\mathbb{R}^m})$ and $\mathbb{S}(\mathbb{R}^m, \tilde{g})$:

$$\begin{aligned} \mathbb{S}(\mathbb{R}^m, g_{\mathbb{R}^m}) := P_{Spin}(\mathbb{R}^m, g_{\mathbb{R}^m}) \times_\rho \mathbb{S}_m &\longrightarrow \mathbb{S}(\mathbb{R}^m, \tilde{g}) := P_{Spin}(\mathbb{R}^m, g) \times_\rho \mathbb{S}_m \\ \psi = [s, \varphi] &\longmapsto \tilde{\psi} = [\tilde{\eta}(s), \varphi] \end{aligned} \tag{2.11}$$

where $[s, \varphi]$ stands for the equivalence class of (s, φ) under the action of $Spin(m)$. This identifies the spinor fields.

For the Dirac operators, as was shown by [10, Proposition 3.2], the identification can be expressed in the following formula

$$D_{\tilde g}\tilde\psi = \widetilde{D_{g_{\mathbb{R}^m}}\psi} + W\cdot_{\tilde g}\tilde\psi + X\cdot_{\tilde g}\tilde\psi + \sum_{i,j}(b_{ij}-\delta_{ij})\tilde\partial_i\cdot_{\tilde g}\widetilde{\nabla_{\partial_j}\psi} \tag{2.12}$$

where $\cdot_{\tilde g}$ denotes the Clifford multiplication with respect to the metric $\tilde g$,

$$W = \frac{1}{4}\sum_{\substack{i,j,k\\ i\neq j\neq k\neq i}}\sum_{\alpha,\beta} b_{i\alpha}(\partial_\alpha b_{j\beta})b^{-1}_{\beta k}\,\tilde\partial_i\cdot_{\tilde g}\tilde\partial_j\cdot_{\tilde g}\tilde\partial_k,$$

with b^{-1}_{ij} being the entries of the inverse matrix of B, $\tilde\partial_i = B(\partial_i)$ and

$$X = \frac{1}{2}\sum_{i,k}\tilde\Gamma^i_{ik}\tilde\partial_k,$$

with $\tilde\Gamma^k_{ij} = \tilde g(\tilde\nabla_{\tilde\partial_i}\tilde\partial_j,\tilde\partial_k)$ being the Christoffel symbols of the second kind.

Remark 2.5. On spin manifolds, since the tangent bundle is embedded in the bundle of Clifford algebra, vector fields have two different actions on spinors, i.e. the Clifford multiplications and the covariant derivatives. Here, to distinguish the two actions on a spinor ψ, we denote $\partial_i\cdot_{g_{\mathbb{R}^m}}\psi$ the Clifford multiplication of ∂_i and $\nabla_{\partial_i}\psi$ the covariant derivative with respect to the metric $g_{\mathbb{R}^m}$ (respectively, $\tilde\partial_i\cdot_{\tilde g}\tilde\psi$ the Clifford multiplication of $\tilde\partial_i$ and $\tilde\nabla_{\tilde\partial_i}\tilde\psi$ the covariant derivative with respect to the metric $\tilde g$). For functions, we shall simply denote $\partial_i u$ for its partial derivative.

Now we collect some formulas given in [36] which are direct consequences of some elementary computations and will be useful in our framework. It will always be understood that the metric $\tilde g_\varepsilon$ is given by (2.2)–(2.5) and $\tilde G_\varepsilon$ stands for the matrix of the coefficients in $\tilde g_\varepsilon$, expressed in the basis ∂_i, $i=1,\dots,m$. Then we have

$$\sqrt{\det\tilde G_\varepsilon} = 1 + \frac{\varepsilon}{2}\operatorname{tr}\tilde h + \varepsilon^2\Big(\frac{1}{8}(\operatorname{tr}\tilde h)^2 - \frac{1}{4}\operatorname{tr}(\tilde h^2)\Big) + o(\varepsilon^2), \tag{2.13}$$

$$B_\varepsilon = I - \frac{\varepsilon}{2}\tilde h + \frac{3\,\varepsilon^2}{8}\tilde h^2 + o(\varepsilon^2) \tag{2.14}$$

and

$$B_\varepsilon^{-1} = I + \frac{\varepsilon}{2}\tilde h - \frac{\varepsilon^2}{8}\tilde h^2 + o(\varepsilon^2). \tag{2.15}$$

The energy functional associated to (2.4), which is defined for a spinor $\tilde\psi$ in $\mathbb{S}(\mathbb{R}^m,\tilde g_\varepsilon)$, is given by

$$\mathcal{J}_\varepsilon(\tilde\psi) = \frac{1}{2}\int_{\mathbb{R}^m}(\tilde\psi, D_{\tilde g_\varepsilon}\tilde\psi)_{\tilde g_\varepsilon}\,d\operatorname{vol}_{\tilde g_\varepsilon} - \frac{1}{2^*}\int_{\mathbb{R}^m}|\tilde\psi|^{2^*}_{\tilde g_\varepsilon}\,d\operatorname{vol}_{\tilde g_\varepsilon}, \tag{2.16}$$

where ε is implicitly involved in the formulation. The main point of this section is to obtain an alternative expression of $\mathcal{J}_\varepsilon(\tilde{\psi})$ by using the aforementioned Bourguignon-Gauduchon identification $\mathbb{S}(\mathbb{R}^m, g_{\mathbb{R}^m}) \ni \psi \leftrightarrow \tilde{\psi} \in \mathbb{S}(\mathbb{R}^m, \tilde{g}_\varepsilon)$ so that ε is explicitly separated out. This can be summarized by the following lemma, which has been shown in [36, Lemma 4.3].

Lemma 2.6. *Let $\tilde{g}$ be given by (2.2)–(2.5), then*

$$\mathcal{J}_\varepsilon(\tilde{\psi}) = \mathcal{J}_0(\psi) + \varepsilon\Gamma(\psi) + \varepsilon^2\Phi(\psi) + o(\varepsilon^2), \tag{2.17}$$

where

$$\mathcal{J}_0(\psi) = \frac{1}{2}\int_{\mathbb{R}^m} (\psi, D_{g_{\mathbb{R}^m}}\psi)_{g_{\mathbb{R}^m}}\, d\,\mathrm{vol}_{g_{\mathbb{R}^m}} - \frac{1}{2^*}\int_{\mathbb{R}^m} |\psi|_{g_{\mathbb{R}^m}}^{2^*}\, d\,\mathrm{vol}_{g_{\mathbb{R}^m}},$$

$$\Gamma(\psi) = \int_{\mathbb{R}^m} \frac{\mathrm{tr}\,\tilde{h}}{2}\Big[\frac{1}{2}\big(\psi, D_{g_{\mathbb{R}^m}}\psi\big)_{g_{\mathbb{R}^m}} - \frac{1}{2^*}|\psi|_{g_{\mathbb{R}^m}}^{2^*}\Big] - \frac{1}{4}\sum_i \tilde{h}_{ii}\,\mathrm{Re}\,\big(\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\psi, \psi\big)_{g_{\mathbb{R}^m}}\, d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

and

$$\begin{aligned}\Phi(\psi) = \int_{\mathbb{R}^m} &\Big(\frac{1}{8}(\mathrm{tr}\,\tilde{h})^2 - \frac{1}{4}\,\mathrm{tr}(\tilde{h}^2)\Big)\Big[\frac{1}{2}\big(\psi, D_{g_{\mathbb{R}^m}}\psi\big)_{g_{\mathbb{R}^m}} - \frac{1}{2^*}|\psi|_{g_{\mathbb{R}^m}}^{2^*}\Big] \\ &+ \frac{1}{16}\sum_i \Big(3\tilde{h}_{ii}^2 - 2(\mathrm{tr}\,\tilde{h})\tilde{h}_{ii}\Big)\,\mathrm{Re}(\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\psi, \psi)_{g_{\mathbb{R}^m}}\, d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\end{aligned}$$

for $\psi \in \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$.

3. Abstract settings

The aim of this section is to present a general approach, which is based on a well-adapted well-known technique in nonlinear analysis: the Lyapunov-Schmidt reduction. The emphasis here is that the nature of the spinorial Yamabe-type problems prevent applying known reductions. Here, the general approach has been recently carried out by Isobe and Xu in [36]. For the sake of completeness, let us sketch the results as follows.

3.1. Lyapunov–Schmidt reduction of the functional

In a general setting, a well adopted Lyapunov-Schmidt reduction technique provides a powerful tool to study perturbed variational problems, see for instance [45, Chapter 10] and [22, II, 6] where the reduced problem is compact and [2,3,5] for the case that the reduced problem is non-compact. Following the monograph [5], we outline the idea as follows.

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space with the associated norm $\|\cdot\| := \langle\cdot,\cdot\rangle^{1/2}$. Suppose that $L_0 \in C^2(\mathcal{H}, \mathbb{R})$ and $\Gamma \in C^2(\mathcal{H}, \mathbb{R})$ are given. For $\varepsilon > 0$ small, we consider the perturbed functional

$$L_\varepsilon(z) = L_0(z) + \varepsilon\Gamma(z) + o(\varepsilon). \tag{3.1}$$

Assume that L_0 has a non-degenerate critical manifold $\mathcal{M} \subset \mathcal{H}$, that is,

$(A1)$ $\mathcal{M}$ is a d-dimensional C^2-submanifold of $\mathcal{H}$ such that $\nabla L_0(z) = 0$ for all $z \in \mathcal{M}$,
$(A2)$ $\mathcal{M}$ is non-degenerate in the sense that for all $z \in \mathcal{M}$, we have $T_z\mathcal{M} = \ker \nabla^2 L_0(z)$,
$(A3)$ $\nabla^2 L_0(z) : \mathcal{H} \to \mathcal{H}$ is a Fredholm operator with index zero for all $z \in \mathcal{M}$.

Set $\mathcal{W}_z := T_z\mathcal{M}^\perp$, where the orthogonal complement is taking with respect to $\langle\cdot,\cdot\rangle$ in $\mathcal{H}$. We look for critical points of L_ε in the form $u = z + w$ where $z \in \mathcal{M}$ and $w \in \mathcal{W}_z$. Let $P_z : \mathcal{H} \to \mathcal{W}_z$ be the orthogonal projection onto $\mathcal{W}_z$, the Euler-Lagrange equation $\nabla L_\varepsilon(z + w) = 0$ is equivalent to

$$\begin{cases} P_z \nabla L_\varepsilon(z + w) = 0 & (\textit{auxiliary equation}) \\ (I - P_z)\nabla L_\varepsilon(z + w) = 0 & (\textit{bifurcation equation}). \end{cases} \tag{3.2}$$

Then, under the conditions $(A2)$ and $(A3)$, the auxiliary equation in (3.2) can be solved firstly for w by applying the implicit function theorem: for arbitrary $z \in \mathcal{M}$ there is a unique small solution $w = w_\varepsilon(z) \in \mathcal{W}_z$ for small values of ε. Furthermore, on any compact subset $\mathcal{M}_c \subset \mathcal{M}$, one can have the uniform estimate (see [5, Chapter 2]):

$$\mathcal{M}_c \ni z \mapsto w_\varepsilon(z) \in \mathcal{W}_z \text{ is } C^1 \text{ and } \|w_\varepsilon(z)\|,\ \|w_\varepsilon'(z)\| = O(\varepsilon) \text{ uniformly for } z \in \mathcal{M}_c. \tag{3.3}$$

The next step is to consider the bifurcation equation in (3.2). To this end, we introduce the reduced functional $L_\varepsilon^{red} : \mathcal{M} \to \mathbb{R}$ by

$$L_\varepsilon^{red}(z) = L_\varepsilon(z + w_\varepsilon(z))$$

Then we have the following theorem

Theorem 3.1 *(Theorem 2.12 in [5]). Suppose* $(A1)$–$(A3)$ *are satisfied. Assume that for a compact subset* $\mathcal{M}_c \subset \mathcal{M}$ *and* $\varepsilon > 0$ *small,* L_ε^{red} *has a critical point* $z_\varepsilon \in \mathcal{M}_c$. *Then* $u_\varepsilon = z_\varepsilon + w_\varepsilon(z_\varepsilon)$ *is a critical point of* L_ε *on* $\mathcal{H}$.

Thanks to the uniform estimate (3.3), the reduced functional L_ε^{red} is well approximated in the sense that

$$L_\varepsilon^{red}(z) = L_0(z) + \varepsilon\Gamma(z) + o(\varepsilon), \quad \nabla L_\varepsilon^{red}(z) = \varepsilon\nabla\Gamma(z) + o(\varepsilon) \tag{3.4}$$

and $L_0(z)$ is constant on any connected component of $\mathcal{M}$. Thus if $z \in \mathcal{M}$ is a non-degenerate critical point of Γ in some certain sense (for example, the local degree of $\nabla\Gamma$ at z is non-zero), then z generates a critical point of L_ε on $\mathcal{H}$ (see [5,22,45] for details).

Remark 3.2.

(1) Turning back to the problems (2.4) and (2.8), a very natural idea is to apply the above abstract framework to the functionals given by $L_0 = \mathscr{J}_0$ and $L_\varepsilon = \mathscr{J}_\varepsilon$ on $\mathcal{H} = \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ (see Lemma 2.6 for the functionals associated to Eq. (2.4), while the functionals associated to Eq. (2.8) are much easier to obtain). As was already shown in [34, Section 5, 6] that $\mathscr{J}_0$ satisfies $(A1)$-$(A3)$ for a critical manifold $\mathcal{M}$ defined as

$$\mathcal{M} := \big\{\psi_{\lambda,\xi,\gamma} : \lambda > 0,\ \xi \in \mathbb{R}^m,\ \gamma \in \mathbb{S}_m,\ |\gamma| = 1\big\}, \tag{3.5}$$

where

$$\psi_{\lambda,\xi,\gamma}(x) = \frac{m^{\frac{m-1}{2}}\lambda^{\frac{m-1}{2}}}{\big(\lambda^2 + |x-\xi|^2\big)^{\frac{m}{2}}}(\lambda - (x-\xi)) \cdot_{g_{\mathbb{R}^m}} \gamma \tag{3.6}$$

for $\lambda > 0$, $\xi \in \mathbb{R}^m$, $\gamma \in \mathbb{S}_m$ with $|\gamma| = 1$ ($\mathbb{S}_m$ is the spinor module, see [27,41]) and $\cdot_{g_{\mathbb{R}^m}}$ denotes the Clifford multiplication with respect to the Euclidean metric. Note that $\mathcal{M}$ is diffeomorphic to $(0,\infty) \times \mathbb{R}^m \times S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$ via the canonical map $(\lambda, \xi, \gamma) \mapsto \psi_{\lambda,\xi,\gamma}$, where $S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$ stands for the $(2^{[\frac{m}{2}]+1}-1)$-dimensional unit sphere in $\mathbb{S}_m$. And hence $\mathcal{M}$ is a non-compact manifold and the dimension of $\mathcal{M}$ is $m + 2^{[\frac{m}{2}]+1}$.

(2) Unfortunately, in the spinorial setting, the reduced functional L_ε^{red} happens to have much worse analytic properties than the usual cases, and one of these "bad" behaviors is the degeneracy on $\mathcal{M}$. This, for instance, can be seen from the explicit formulations of those perturbation terms in (2.17) where Γ and Φ do not depend on all variables of $\mathcal{M}$ (in fact, if we substitute (3.5)–(3.6) into (2.17), we find that Γ and Φ do not depend on the variable γ in $\mathcal{M}$). Hence, critical points of the functional $\mathscr{J}_\varepsilon$ can not be obtained via non-degenerate arguments, in particular, standard methods as in [4,5,22,45] do not apply.

3.2. Perturbation method with degenerate conditions

Here we recall a recent framework developed in [36, Section 2], which can be employed to handle spinor field equations like (2.4) and (2.8). To see this, besides the assumptions $(A1)$-$(A3)$, we will need the following additional conditions for the critical manifold $\mathcal{M}$:

$(A4)$ $\mathcal{M}$ admits a (globally) trivializable fiber bundle structure over a compact base space $\mathcal{N}$ with projection $\vartheta : \mathcal{M} \to \mathcal{N}$ and fiber $\mathcal{G}$. Precisely, there is a fiber preserving diffeomorphism $\iota : \mathcal{G} \times \mathcal{N} \to \mathcal{M}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{G} \times \mathcal{N} \arrow[r, "\iota"] \arrow[d, "Proj"'] & \mathcal{M} \arrow[d, "\vartheta"] \\ \mathcal{N} \arrow[r, "\mathrm{Id}"] & \mathcal{N} \end{tikzcd}$$

$(A5)$ $T_\gamma \mathcal{N} \subset \ker \nabla(\Gamma \circ \iota)(g, \gamma)$ for any $(g, \gamma) \in \mathcal{G} \times \mathcal{N}$, where we have identified $T_\gamma \mathcal{N}$ as a subspace of the total tangent space $T_{(g,\gamma)}(\mathcal{G} \times \mathcal{N})$.

Remark 3.3.

(1) In our application $\mathcal{N} = S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$, $\mathcal{G} = (0, +\infty) \times \mathbb{R}^m$ and $\iota(g, \gamma) := \psi_{\lambda,\xi,\gamma}$ for $g = (\lambda, \xi) \in \mathcal{G}$ and $\gamma \in \mathcal{N}$, hence we have a very natural bundle structure on $\mathcal{M}$. Particularly, we note that there is a continuous action $\mathcal{G} \times \mathcal{M} \to \mathcal{M}$ such that $\mathcal{G}$ preserves the fibers of $\mathcal{M}$ (i.e. if $(\mu, y) \in \mathcal{G}$ and $\psi_{\lambda,\xi,\gamma} \in \mathcal{M}_\gamma$ then $\psi_{\lambda,\xi,\gamma} * (\mu, y) = \psi_{\lambda\mu,\, \xi+y,\gamma} \in \mathcal{M}_\gamma$). Hence the critical manifold in (3.5) is essentially a principal $\mathcal{G}$-bundle. And since it admits a global section, we easily see that $\mathcal{M}$ is trivializable. This is the reason we introduce condition $(A4)$.

(2) Note that if $\mathcal{M}$ is parameterized via the map ι, condition $(A4)$ makes the variational problem even clearer: it is equivalent to consider the functional $L_\varepsilon^{red} \circ \iota : \mathcal{G} \times \mathcal{N} \to \mathbb{R}$. Comparing with the standard theory in [5,22,45], the distinct new feature $(A5)$ describes a certain degenerate situation and, particularly, it implies that $\Gamma \circ \iota(g, \gamma)$ depends only on the variables in the fiber space $\mathcal{G}$. Thus we shall turn to study $\tilde{\Gamma}(g) = \Gamma \circ \iota(g, \gamma)$. For later use, we distinguish $(A5)$ into the following two cases:

$$\begin{cases} \ker \nabla(\Gamma \circ \iota)(g, \gamma) \equiv T_{(g,\gamma)}(\mathcal{G} \times \mathcal{N}) & \text{for all } (g, \gamma) \in \mathcal{G} \times \mathcal{N}, \\ \ker \nabla(\Gamma \circ \iota)(g, \gamma) \neq T_{(g,\gamma)}(\mathcal{G} \times \mathcal{N}) & \text{for some } (g, \gamma) \in \mathcal{G} \times \mathcal{N}, \end{cases}$$

and we will collect two abstract results which are useful in the spinorial Yamabe-type problems.

Case 1: $\ker \nabla(\Gamma \circ \iota)(g, \gamma) \equiv T_{(g,\gamma)}(\mathcal{G} \times \mathcal{N})$ **for all** $(g, \gamma) \in \mathcal{G} \times \mathcal{N}$

In this setting, we have $\Gamma \circ \iota(g, \gamma) \equiv$ *constant* on $\mathcal{G} \times \mathcal{N}$ and we need to evaluate further terms in the expansion of L_ε^{red}. For this purpose, let us develop the expansion (3.1) in powers of ε as

$$L_\varepsilon(z) = L_0(z) + \varepsilon\Gamma(z) + \varepsilon^2\Phi(z) + o(\varepsilon^2) \tag{3.7}$$

Note that $\Gamma \circ \iota(g, \gamma) \equiv$ *constant* on $\mathcal{G} \times \mathcal{N}$ is equivalent to $\Gamma(z) \equiv$ *constant* on $\mathcal{M}$. It follows that $\nabla\Gamma(z) \in \mathcal{W}_z := T_z\mathcal{M}^\perp$. Recall that $w_\varepsilon(z)$ is the solution to the auxiliary equation $P_z \nabla L_\varepsilon(z+w) = 0$, hence we have $\nabla L_\varepsilon(z + w_\varepsilon(z)) \in T_z\mathcal{M}$. For a fixed $z \in \mathcal{M}$, using Taylor expansion, one sees

$$\begin{aligned}\nabla L_\varepsilon(z+w_\varepsilon(z)) &= \nabla L_0(z+w_\varepsilon(z)) + \varepsilon\nabla\Gamma(z+w_\varepsilon(z)) + o(\varepsilon)\\ &= \nabla^2 L_0(z)[w_\varepsilon(z)] + \varepsilon\nabla\Gamma(z) + \varepsilon\nabla^2\Gamma(z)[w_\varepsilon(z)] + o(\|w_\varepsilon(z)\|) + o(\varepsilon).\end{aligned}$$

Then, form (3.3) and the fact $\nabla L_\varepsilon(z+w_\varepsilon(z)) \in T_z\mathcal{M}$, it follows that

$$\nabla^2 L_0(z)[w_\varepsilon(z)] + \varepsilon\nabla\Gamma(z) + o(\varepsilon) \in T_z\mathcal{M}.$$

And hence, by projecting the above equation into $\mathcal{W}_z$, we deduce

$$w_\varepsilon(z) = -\varepsilon K_z(\nabla\Gamma(z)) + o(\varepsilon), \tag{3.8}$$

where K_z stands for the inverse of $\nabla^2 L_0(z)$ restricted to $\mathcal{W}_z$. Now, we can expand $L_\varepsilon^{red}(z) := L_\varepsilon(z+w_\varepsilon(z))$ as

$$\begin{aligned}L_\varepsilon^{red}(z) &= L_0(z) + \frac{1}{2}\nabla^2 L_0(z)[w_\varepsilon(z), w_\varepsilon(z)]\\ &\quad + \varepsilon\Gamma(z) + \varepsilon\nabla\Gamma(z)[w_\varepsilon(z)] + \varepsilon^2\Phi(z) + o(\varepsilon^2)\\ &= L_0(z) + \varepsilon\Gamma(z) + \varepsilon^2\Big(\Phi(z) - \frac{1}{2}\langle K_z(\nabla\Gamma(z)), \nabla\Gamma(z)\rangle\Big) + o(\varepsilon^2).\end{aligned} \tag{3.9}$$

Here, we emphasize that both $L_0(z)$ and $\Gamma(z)$ are constants on $\mathcal{M}$. The following result is due to [36, Theorem 2.6].

Theorem 3.4. *Let $L_0, \Gamma, \Phi \in C^2(\mathcal{H}, \mathbb{R})$ as in (3.7) and suppose that (A1)-(A5) are satisfied. If there is an open bounded subset $U \subset \mathcal{G}$ such that*

$$\inf_{\gamma\in\mathcal{N}}\Big(\min_{\partial U}\hat\Phi\big|_{\vartheta^{-1}(\gamma)} - \min_{\overline{U}}\hat\Phi\big|_{\vartheta^{-1}(\gamma)}\Big) > 0 \quad or \quad \sup_{\gamma\in\mathcal{N}}\Big(\max_{\partial U}\hat\Phi\big|_{\vartheta^{-1}(\gamma)} - \max_{\overline{U}}\hat\Phi\big|_{\vartheta^{-1}(\gamma)}\Big) < 0,$$

where $\hat\Phi\big|_{\vartheta^{-1}(\gamma)} = \hat\Phi\circ\iota(\cdot,\gamma)$ and

$$\hat\Phi(z) := \Phi(z) - \frac{1}{2}\langle K_z(\nabla\Gamma(z)), \nabla\Gamma(z)\rangle \quad for\ z \in \mathcal{M}.$$

Then, for $|\varepsilon|$ small, the functional L_ε has a critical point on $\mathcal{M}$.

Case 2: $\ker\nabla(\Gamma\circ\iota)(g,\gamma) \neq T_{(g,\gamma)}(\mathcal{G}\times\mathcal{N})$ **for some** $(g,\gamma) \in \mathcal{G}\times\mathcal{N}$

Clearly, in this case, $\tilde\Gamma(g) = \Gamma\circ\iota(g,\gamma) \neq$ *constant* on $\mathcal{G}\times\mathcal{N}$ (evidently, $\Gamma(z) \neq$ *constant* on $\mathcal{M}$). And the existence result is as follows, we refer the reads to [36, Theorem 2.4 and Remark 2.5].

Theorem 3.5. *Let $L_0, \Gamma \in C^2(\mathcal{H}, \mathbb{R})$ as in (3.1) and suppose that $(A1)$-$(A5)$ are satisfied. If $\tilde{\Gamma}$ is a Morse function on $\mathcal{G}$ and there is an open bounded subset $\Omega \subset \mathcal{G}$ such that the topological degree $\deg(\nabla\tilde{\Gamma}, \Omega, 0) \neq 0$. Then, for $|\varepsilon|$ small, the functional L_ε^{red} has at least $\mathrm{Cat}(\mathcal{N})$ critical points on $\mathcal{M}$. In particular, for each critical point $\bar{g}$ of $\tilde{\Gamma}$, there exist at least $\mathrm{Cat}(\mathcal{N})$ critical points $(g(\gamma), \gamma) \in \mathcal{G} \times \mathcal{N}$ of L_ε^{red}, each of which satisfies $g(\gamma) = \bar{g} + o(1)$ as $\varepsilon \to 0$.*

Here $\mathrm{Cat}(\mathcal{N})$ denotes the Lusternik-Schnierelman category of $\mathcal{N}$, namely the smallest integer k such that $\mathcal{N} \subset \cup_{i=1}^k A_k$, where the sets A_k are closed and contractible in $\mathcal{N}$.

4. The non-compactness caused by the background metric

In this section, let us consider the Eq. (2.4), where the background metric is given by (2.2) and (2.3). In this setting, our main purpose is to use our abstract result to prove Theorem 2.3. Here we emphasis that the functional $\mathscr{J}_0$ in Lemma 2.6 plays the role of L_0 in our abstract settings.

4.1. Some basic facts

We first report some important properties of the functionals Γ and Φ in Lemma 2.6, which have been shown in [36]. Recall that since the critical manifold $\mathcal{M} \subset \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ for $\mathscr{J}_0$ is given by (3.5)–(3.6), we have

Lemma 4.1. *Assume that we are in the hypotheses of Lemma 2.6, for $\psi_{\lambda,\xi,\gamma} \in \mathcal{M}$ with $\lambda > 0$, $\xi \in \mathbb{R}^m$ and $\gamma \in S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$, there hold*

$$\Gamma(\psi_{\lambda,\xi,\gamma}) \equiv 0$$

and

$$\Phi(\psi_{\lambda,\xi,\gamma}) = \frac{m^{m-1}\lambda^m}{16} \int_{\mathbb{R}^m} \frac{\mathrm{tr}(\tilde{h}^2) - (\mathrm{tr}\,\tilde{h})^2}{\big(\lambda^2 + |x-\xi|^2\big)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} .$$

Moreover

(1) $\lim\limits_{\lambda \to 0} \Phi(\psi_{\lambda,\xi,\gamma}) = C_0\big(\mathrm{tr}(\tilde{h}^2) - (\mathrm{tr}\,\tilde{h})^2\big)(\xi)$ *for any $\xi \in \mathbb{R}^m$, where*

$$C_0 = \frac{m^{m-1}}{16} \int_{\mathbb{R}^m} \frac{1}{\big(1+|x|^2\big)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}};$$

(2) *for all* $v \in \mathcal{W}_z := T_z\mathcal{M}^\perp$,

$$\begin{aligned}\langle \nabla\Gamma(z), v\rangle = \frac{1}{4}\operatorname{Re}\int\limits_{\mathbb{R}^m} (\nabla(\operatorname{tr}\tilde{h}) \cdot_{g_{\mathbb{R}^m}} z, v)_{g_{\mathbb{R}^m}} d\operatorname{vol}_{g_{\mathbb{R}^m}} \\ -\frac{1}{2}\sum_i \operatorname{Re}\int\limits_{\mathbb{R}^m} \tilde{h}_{ii}(\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i} z, v)_{g_{\mathbb{R}^m}} d\operatorname{vol}_{g_{\mathbb{R}^m}} \\ -\frac{1}{4}\sum_i \operatorname{Re}\int\limits_{\mathbb{R}^m} \partial_i\tilde{h}_{ii}(\partial_i \cdot_{g_{\mathbb{R}^m}} z, v)_{g_{\mathbb{R}^m}} d\operatorname{vol}_{g_{\mathbb{R}^m}};\end{aligned} \tag{4.1}$$

(3) $\lim_{\lambda\to 0} \langle K_{\psi_{\lambda,\xi,\gamma}}(\nabla\Gamma(\psi_{\lambda,\xi,\gamma})), \nabla\Gamma(\psi_{\lambda,\xi,\gamma})\rangle = C_1\big(\operatorname{tr}(\tilde{h}^2) - (\operatorname{tr}\tilde{h})^2\big)(\xi)$ *for any* $\xi \in \mathbb{R}^m$, *where*

$$C_1 = \frac{m^{m-1}}{4}\int\limits_{\mathbb{R}^m} \frac{|x|^2}{\big(1+|x|^2\big)^{m+1}} d\operatorname{vol}_{g_{\mathbb{R}^m}}$$

and K_z *stands for the inverse of* $\nabla^2\mathcal{J}_0(z)$ *restricted to* $\mathcal{W}_z := T_z\mathcal{M}^\perp \subset \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$.

Remark 4.2. Let us point out that the aforementioned two constants C_0 and C_1 can be computed explicitly as

$$C_0 = \frac{m^{m-1}\omega_{m-1}}{16}\int_0^\infty \frac{r^{m-1}}{(1+r^2)^m}dr = \frac{m^{m-1}\omega_{m-1}}{32}B\Big(\frac{m}{2}, \frac{m}{2}\Big)$$

and

$$C_1 = \frac{m^{m-1}\omega_{m-1}}{4}\int_0^\infty \frac{r^{m+1}}{(1+r^2)^{m+1}}dr = \frac{m^{m-1}\omega_{m-1}}{8}B\Big(\frac{m}{2}, \frac{m}{2}+1\Big)$$

where $B(x, y)$, defined for $x, y > 0$, is the beta function classified by the first kind of Euler's integral. Using the property

$$B(x, x+1) = \frac{1}{2}B(x, x), \quad \text{for } x > 0$$

we find $C_0 = \frac{1}{2}C_1$.

Next, let $\mathcal{J}_\varepsilon^h$ be the Euler functional corresponding to the metric $\tilde{g}^h = g_{\mathbb{R}^m} + \varepsilon h$, where h is a fixed elementary matrix (see Definition 2.1). Then Lemma 4.1 can be performed also for $\mathcal{J}_\varepsilon^h$. Let Γ^h and Φ^h be the corresponding functionals appearing in the expansion of $\mathcal{J}_\varepsilon^h$. A more detailed characterization for the reorganized functional

$$\hat{\Phi}^h(z) := \Phi^h(z) - \frac{1}{2}\left\langle K_z(\nabla\Gamma^h(z)), \nabla\Gamma^h(z)\right\rangle \quad \text{for } z \in \mathcal{M} \tag{4.2}$$

can be summarized in the following proposition. We emphasize that, by Lemma 4.1 and Remark 4.2, there holds

$$\lim_{\lambda\to 0} \hat{\Phi}^h(\psi_{\lambda,\xi,\gamma}) = 0$$

for any $\xi \in \mathbb{R}^m$ and $\gamma \in S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$.

Proposition 4.3. *For $m \geq 4$, assume that we are in the hypotheses of Lemma 2.6. Let $k \in \{1, \dots, m\}$, $p \in [2, \infty]$ and $h = \operatorname{diag}(h_{11}, \dots, h_{mm})$ be (k,p)-elementary at a point $\xi \in \mathbb{R}^m$ with $\partial_k h_{ii}(\xi) \equiv c_k \neq 0$, for $i \neq k$. If*

$$\begin{cases} p = \infty & m = 4, \\ p > 2 & m = 5, \\ p \geq 2 & m \geq 6, \end{cases}$$

then

$$\hat{\Phi}^h(\psi_{\lambda,\xi,\gamma}) = -\frac{3m^{m-2}(m-1)(m-2)c_k^2}{128}\lambda^2 \int_{\mathbb{R}^m} \frac{|x|^2}{(1+|x|^2)^m} d\operatorname{vol}_{g_{\mathbb{R}^m}} + o(\lambda^2) \quad \textit{as } \lambda \to 0.$$

In particular, $\hat{\Phi}^h(\psi_{\lambda,\xi,\gamma}) < 0$ for small values of λ. Furthermore,

$$\hat{\Phi}^h(\psi_{\lambda,\xi,\gamma}) \to 0 \quad \textit{as } \lambda + |\xi| \to \infty.$$

Remark 4.4. The proof of Proposition 4.3 is very technical, we refer to [36, Section 4.2] for more details. We only point out here that the main ingredient lies in characterizing the mapping $w_\varepsilon^h : \mathcal{M} \to T\mathcal{M}^\perp$, $w_\varepsilon^h(z) = K_z(\nabla\Gamma^h(z))$, or equivalently solving the equation $\nabla^2\mathcal{J}_0(z)[w_\varepsilon^h(z)] = \nabla\Gamma(z)$ for $z \in \mathcal{M}$. And the (k,p)-elementary matrix makes the computation more accessible than using general choices of h.

Now, through the perturbation framework introduced in Section 3, we intend to reduce the problem (2.1) to a finite-dimensional one. Notice that, when $\tilde{h} \in \mathcal{H}(p)$ (see the definition above Theorem 2.3), we find the functionals Γ and Φ (in the expansion of $\mathcal{J}_\varepsilon$, see Lemma 2.6) are actually in the form of summing up infinitely many distinguished terms. For this reason more careful analysis is required.

Lemma 4.5. *For $m \geq 2$, assume that we are in the hypotheses of Lemma 2.6 with $\tilde{h} \in \mathcal{H}(p)$, some $p \in (2, \infty) \cup \{\infty\}$. Let $\psi, \varphi \in \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ and $z \in \mathcal{M}$. Then, via the Bourguignon-Gauduchon identification (2.11), there exists a constant $C > 0$ such that the following estimates hold for all $|\varepsilon|$ small:*

$$\mathcal{J}(\tilde{\psi}) - \mathcal{J}_0(\psi) - \varepsilon\Gamma(\psi) - \varepsilon^2\Phi(\psi) = o(\varepsilon^2)\big(\|\psi\|^2 + \|\psi\|^{\frac{2m}{m-1}}\big); \tag{4.3}$$

$$\big\|\nabla\mathcal{J}_\varepsilon(\tilde{\psi}) - \nabla\mathcal{J}_0(\psi) - \varepsilon\nabla\Gamma(\psi)\big\| = O(\varepsilon^2)\big(\|\psi\| + \|\psi\|^{\frac{m+1}{m-1}}\big); \tag{4.4}$$

$$\big\|\nabla\mathcal{J}_\varepsilon(\tilde{z})\big\| = O(|\varepsilon|); \tag{4.5}$$

$$\big\|\nabla^2\mathcal{J}_\varepsilon(\tilde{\psi}) - \nabla^2\mathcal{J}_0(\psi)\big\| = O(|\varepsilon|)\big(1 + \|\psi\|^{\frac{2}{m-1}}\big); \tag{4.6}$$

$$|\mathcal{J}_\varepsilon(\tilde{\psi} + \tilde{\varphi}) - \mathcal{J}_\varepsilon(\tilde{\psi})| \le C\|\varphi\|\big(\|\psi\| + \|\varphi\| + \|\psi\|^{\frac{m+1}{m-1}} + \|\varphi\|^{\frac{m+1}{m-1}}\big); \tag{4.7}$$

$$\big\|\nabla\mathcal{J}_\varepsilon(\tilde{\psi} + \tilde{\varphi}) - \nabla\mathcal{J}_\varepsilon(\tilde{\psi})\big\| \le C\|\varphi\|\big(1 + \|\psi\|^{\frac{2}{m-1}} + \|\varphi\|^{\frac{2}{m-1}}\big); \tag{4.8}$$

$$\big\|\nabla\Gamma(\tilde{\psi} + \tilde{\varphi}) - \nabla\Gamma(\tilde{\psi})\big\| \le C\|\varphi\|\big(1 + \|\psi\|^{\frac{2}{m-1}} + \|\varphi\|^{\frac{2}{m-1}}\big); \tag{4.9}$$

$$\big\|\nabla^2\mathcal{J}_\varepsilon(\tilde{\psi} + \tilde{\varphi}) - \nabla^2\mathcal{J}_\varepsilon(\tilde{\psi})\big\| \le \begin{cases} C\|\varphi\|(\|\psi\| + \|\varphi\|) & m = 2, \\ C\big(\|\varphi\|^{\frac{2}{m-1}} + \|\varphi\|\big) & m \ge 3, \end{cases} \tag{4.10}$$

uniformly in ψ, φ and z.

Without breaking the reading, the proof of Lemma 4.5 will be given in Appendix A.1. Now, as an important consequence, we have

Proposition 4.6. *For $m \ge 2$, assume that we are in the hypotheses of Lemma 2.6 with $\tilde{h} \in \mathcal{H}(p)$, some $p \in (2, \infty) \cup \{\infty\}$. There exists a C^1 mapping*

$$(w, \chi) : (-\varepsilon_0, \varepsilon_0) \times \mathcal{M} \to \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m)) \times \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$$

for some $\varepsilon_0 > 0$, which satisfies

(1) *$w_\varepsilon(z) = \widetilde{w(\varepsilon, z)} \in T_z\mathcal{M}^\perp$;*
(2) *$\nabla\mathcal{J}_\varepsilon\big(\tilde{z} + \widetilde{w_\varepsilon(z)}\big) = \chi(\varepsilon, z) \in T_z\mathcal{M}$ for all $z \in \mathcal{M}$ (via the Bourguignon-Gauduchon identification);*
(3) *$w_\varepsilon(z) = -\varepsilon K_z(\nabla\Gamma(z)) + O(|\varepsilon|^\mu)$ with $\mu = 2$ for $m = 2$ and $\mu = \frac{m+1}{m-1}$ for $\mu \ge 3$;*
(4) *denoted by $\mathcal{J}_\varepsilon^{red}(z) = \mathcal{J}_\varepsilon\big(\tilde{z} + \widetilde{w_\varepsilon(z)}\big)$, then $\tilde{z} + \widetilde{w_\varepsilon(z)}$ is a critical point of $\mathcal{J}_\varepsilon$ provided that $z \in \mathcal{M}$ is a critical point of $\mathcal{J}_\varepsilon^{red}$.*

Proof. To obtain the existence of (w, χ), let us define a mapping $H : \mathcal{M} \times (-\varepsilon_1, \varepsilon_1) \times \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m)) \times T\mathcal{M} \to \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m)) \times T\mathcal{M}$

$$H(z, \varepsilon, w, \chi) = \big(\nabla\mathcal{J}_\varepsilon(\tilde{z} + \tilde{w}) - \chi, (I - P_z)w\big)$$

where $(-\varepsilon_1, \varepsilon_1)$ is an interval such that $\mathcal{J}_\varepsilon$ is well-defined and $P_z : \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m)) \to T_z\mathcal{M}^\perp$ is the orthogonal projection onto $T_z\mathcal{M}^\perp$.

Plainly, we have $H(z, 0, 0, 0) = (0, 0)$ for all $z \in \mathcal{M}$. And by elementary computation, we have

$$\nabla_{(w,\chi)}H(z,0,0,0)[\varphi,\phi] = \big(\nabla^2\mathcal{J}_0(z)[\varphi] - \phi, (I-P_z)\varphi\big)$$

for $\varphi \in \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ and $\phi \in T_z\mathcal{M}$. Hence, it follows from the invertibility of $\nabla^2\mathcal{J}_0(z)|_{T_z\mathcal{M}^\perp}$ that $\nabla_{(w,\chi)}H(z,0,0,0)$ is invertible and

$$\|\nabla_{(w,\chi)}H(z,0,0,0)^{-1}\| \le C_0, \quad \forall z \in \mathcal{M} \tag{4.11}$$

for some $C_0 > 0$. Then, by applying the Implicit Function Theorem, one soon obtains the existence of $(w(\varepsilon,z), \chi(\varepsilon,z))$ such that $H(z,\varepsilon,w(\varepsilon,z),\chi(\varepsilon,z)) = (0,0)$. This proves (1) and (2).

To see (3) we need more careful analysis of the mapping $w_\varepsilon(z) = w(\varepsilon,z)$. To start with, let us use the invertibility of $\nabla_{(w,\chi)}H(z,0,0,0)$ to define $F_{z,\varepsilon} : \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m)) \times T_z\mathcal{M} \to \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m)) \times T_z\mathcal{M}$

$$F_{z,\varepsilon}(\varphi,\phi) = -\nabla_{(w,\chi)}H(z,0,0,0)^{-1}\Big(H(z,\varepsilon,\varphi,\phi) - \nabla_{(w,\chi)}H(z,0,0,0)[\varphi,\phi]\Big).$$

Then we can see that $(w(\varepsilon,z), \chi(\varepsilon,z))$ is a fixed point of $F_{z,\varepsilon}$. We claim that

Claim. *There exist $L_0, \varepsilon_0 > 0$ such that, for any given $L > L_0$, $F_{z,\varepsilon}$ is a contraction mapping on $B_\varepsilon := \big\{(\varphi,\phi) \in \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m)) \times T_z\mathcal{M} : \|\varphi\|^2 + \|\phi\|^2 \le L^2\varepsilon^2\big\}$, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.*

We only need to show that $F_{z,\varepsilon}(\varphi,\phi) \in B_\varepsilon$ and

$$\|F_{z,\varepsilon}(\varphi_1,\phi_1) - F_{z,\varepsilon}(\varphi_2,\phi_2)\| \le \delta\|(\varphi_1,\phi_1) - (\varphi_1,\phi_1)\|$$

for all $(\varphi_1,\phi_1), (\varphi_2,\phi_2) \in B_\varepsilon$, where $\delta \in (0,1)$. And, by (4.11), it is enough to show that

$$\big\|\nabla\mathcal{J}_\varepsilon(\tilde z + \tilde\varphi) - \nabla^2\mathcal{J}_0(z)[\varphi]\big\| \le \frac{L|\varepsilon|}{C_0} \tag{4.12}$$

and

$$\begin{aligned} &\big\|\big(\nabla\mathcal{J}_\varepsilon(\tilde z + \tilde\varphi_1) - \nabla^2\mathcal{J}_0(z)[\varphi_1]\big) - \big(\nabla\mathcal{J}_\varepsilon(\tilde z + \tilde\varphi_2) - \nabla^2\mathcal{J}_0(z)[\varphi_2]\big)\big\| \\ &\quad \le \frac{\delta}{C_0}\|(\varphi_1,\phi_1) - (\varphi_1,\phi_1)\|. \end{aligned} \tag{4.13}$$

Here and in the sequel, we will use the expressions given in Lemma 2.6 so that the gradient map $\nabla\mathcal{J}_\varepsilon(\cdot)$ is appropriately identified in the space $\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$, and it will cause no confusion if we make the difference between $\nabla\mathcal{J}_\varepsilon$ and derivatives of $\mathcal{J}_0$. We shall also adopt such identification for higher order derivatives of $\mathcal{J}_\varepsilon$.

Using (4.5), (4.6) and (4.10), we find

$$
\begin{aligned}
&\big\|\nabla\mathcal{J}_\varepsilon(\tilde z+\tilde\varphi)-\nabla^2\mathcal{J}_0(z)[\varphi]\big\| \\
&\quad=\big\|\nabla\mathcal{J}_\varepsilon(\tilde z+\tilde\varphi)-\nabla\mathcal{J}_\varepsilon(\tilde z)-\nabla^2\mathcal{J}_\varepsilon(\tilde z)[\varphi]+\nabla\mathcal{J}_\varepsilon(\tilde z)+(\nabla^2\mathcal{J}_\varepsilon(\tilde z)-\nabla^2\mathcal{J}_0(z))[\varphi]\big\| \\
&\quad\le\int_0^1\big\|\nabla^2\mathcal{J}_\varepsilon(\tilde z+s\tilde\varphi)-\nabla^2\mathcal{J}_\varepsilon(\tilde z)\big\|\|\varphi\|ds+O(|\varepsilon|)+O(|\varepsilon|)\|\varphi\| \\
&\quad\le\begin{cases}O(1)\|\varphi\|^2+O(1)\|\varphi\|^3+O(|\varepsilon|)+O(|\varepsilon|)\|\varphi\| & \text{if } m=2\\ O(1)\|\varphi\|^{\frac{m+1}{m-1}}+O(1)\|\varphi\|^2+O(|\varepsilon|)+O(|\varepsilon|)\|\varphi\| & \text{if } m\ge 3\end{cases}
\end{aligned}
$$

since $\|z\|$ is uniformly bounded for $z\in\mathcal{M}$. This proves (4.12) when L is fixed reasonably large. To see (4.13), we point out that

$$
\begin{aligned}
&\nabla\mathcal{J}_\varepsilon(\tilde z+\tilde\varphi_1)-\nabla\mathcal{J}_\varepsilon(\tilde z+\tilde\varphi_2)-\nabla^2\mathcal{J}_0(z)[\varphi_1-\varphi_2] \\
&\qquad=\int_0^1\nabla^2\mathcal{J}_\varepsilon\big(\tilde z+\tilde\varphi_2+s(\tilde\varphi_1-\tilde\varphi_2)\big)[\tilde\varphi_1-\tilde\varphi_2]ds-\nabla^2\mathcal{J}_0(z)[\varphi_1-\varphi_2]
\end{aligned}
$$

and hence by (4.6) and (4.10) we get

$$
\begin{aligned}
&\big\|\big(\nabla\mathcal{J}_\varepsilon(\tilde z+\tilde\varphi_1)-\nabla^2\mathcal{J}_0(z)[\varphi_1]\big)-\big(\nabla\mathcal{J}_\varepsilon(\tilde z+\tilde\varphi_2)-\nabla^2\mathcal{J}_0(z)[\varphi_2]\big)\big\| \\
&\quad\le O(|\varepsilon|)\max_{s\in[0,1]}\big(1+\|z+\varphi_2+s(\varphi_1-\varphi_2)\|^{\frac{2}{m-1}}\big)\|\varphi_1-\varphi_2\| \\
&\quad+\begin{cases}O(1)\displaystyle\max_{s\in[0,1]}\|\varphi_2+s(\varphi_1-\varphi_2)\|\big(\|z\|+\|\varphi_2+s(\varphi_1-\varphi_2)\|\big)\|\varphi_1-\varphi_2\| & \text{if } m=2,\\ O(1)\displaystyle\max_{s\in[0,1]}\big(\|\varphi_2+s(\varphi_1-\varphi_2)\|^{\frac{2}{m-1}}+\|\varphi_2+s(\varphi_1-\varphi_2)\|\big)\|\varphi_1-\varphi_2\| & \text{if } m\ge 3.\end{cases}
\end{aligned}
$$

Therefore, when $|\varepsilon|$ is small enough, we obtain (4.13). And the claim is proved.

As an immediate consequence of the above claim, we find that $F_{z,\varepsilon}$ always has a fixed point in B_ε. Hence we conclude that $\|w_\varepsilon(z)\|=\|w(\varepsilon,z)\|\le L\varepsilon$, with $L>L_0$ being fixed. Now, in order to prove (3), we write

$$
\begin{aligned}
\nabla\mathcal{J}_\varepsilon\big(\tilde z+\widetilde{w_\varepsilon(z)}\big)=\nabla\mathcal{J}_\varepsilon\big(\tilde z+\widetilde{w_\varepsilon(z)}\big)&-\nabla\mathcal{J}_0(z+w_\varepsilon(z))-\varepsilon\nabla\Gamma(z+w_\varepsilon(z)) \\
&+\nabla\mathcal{J}_0(z+w_\varepsilon(z))-\nabla^2\mathcal{J}_0(z)[w_\varepsilon(z)] \\
&+\varepsilon\nabla\Gamma(z+w_\varepsilon(z))-\varepsilon\nabla\Gamma(z) \\
&+\nabla^2\mathcal{J}_0(z)[w_\varepsilon(z)]+\varepsilon\nabla\Gamma(z)
\end{aligned}
$$

in which we can use (4.4), (4.9) and (4.10) to get

$$
\big\|\nabla\mathcal{J}_\varepsilon\big(\tilde z+\widetilde{w_\varepsilon(z)}\big)-\nabla\mathcal{J}_0(z+w_\varepsilon(z))-\varepsilon\nabla\Gamma(z+w_\varepsilon(z))\big\|=O(\varepsilon^2),
$$

$$\begin{aligned}\left\|\nabla\mathcal{J}_0(z+w_\varepsilon(z))-\nabla^2\mathcal{J}_0(z)[w_\varepsilon(z)]\right\| &\le \int_0^1 \left\|\nabla^2\mathcal{J}_0(z+sw_\varepsilon(z))-\nabla^2\mathcal{J}_0(z)\right\|\|w_\varepsilon(z)\|ds\\ &\le \begin{cases} O(1)\|w_\varepsilon(z)\|^2=O(\varepsilon^2) & \text{if } m=2\\ O(1)\|w_\varepsilon(z)\|^{\frac{m+1}{m-1}}=O\big(|\varepsilon|^{\frac{m+1}{m-1}}\big) & \text{if } m\ge 3\end{cases}\end{aligned}$$

and

$$\left\|\nabla\Gamma(z+w_\varepsilon(z))-\nabla\Gamma(z)\right\|=O(1)\|w_\varepsilon(z)\|=O(|\varepsilon|).$$

Hence, we deduce from (2) that

$$\chi(\varepsilon,z)=\nabla^2\mathcal{J}_0(z)[w_\varepsilon(z)]+\varepsilon\nabla\Gamma(z)+\begin{cases} O(1)\|w_\varepsilon(z)\|^2=O(\varepsilon^2) & \text{if } m=2,\\ O(1)\|w_\varepsilon(z)\|^{\frac{m+1}{m-1}}=O\big(|\varepsilon|^{\frac{m+1}{m-1}}\big) & \text{if } m\ge 3.\end{cases}$$

Projecting this equation on $T_z\mathcal{M}^\perp$ and applying the operator $K_z=\big(\nabla^2\mathcal{J}_0(z)|_{T_z\mathcal{M}^\perp}\big)^{-1}$ on both sides, we obtain assertion (3).

Finally (4) is a direct consequence of (1) and (2). This completes the proof. □

Remark 4.7. Comparing with (3.3) and (3.8), though Proposition 4.6 is quite similar to the framework in Section 3, we carry out the details here mainly because the functional $\mathcal{J}_\varepsilon$ is involved with the infinite series $\tilde{h}\in\mathcal{H}(p)$. Clearly, Proposition 4.6 suggests that Theorem 3.4 can be applied to the functional $\mathcal{J}_\varepsilon$.

4.2. Proof of Theorem 2.3

In virtue of Lemma 4.1 and Proposition 4.3, let us denote $\mathcal{J}_\varepsilon^{(i)}$ the Euler functional corresponding to the metric $g_\varepsilon^{(i)}=g_{\mathbb{R}^m}+\varepsilon a_i h(x-x_i)$, where h is a given compactly supported elementary matrix satisfying the hypotheses of Proposition 4.3, $a_i\in\mathbb{R}$ and $x_i\in\mathbb{R}^m$ are fixed. Let $\Gamma^{(i)}$ and $\Phi^{(i)}$ be the corresponding functionals appearing in the expansion of $\mathcal{J}_\varepsilon^{(i)}$. According to the abstract setting in Section 3, we denote $w_\varepsilon(z)$ and $w_\varepsilon^{(i)}(z)$ the solutions to the auxiliary equations $P_z\nabla\mathcal{J}_\varepsilon(z+w)=0$ and $P_z\nabla\mathcal{J}_\varepsilon^{(i)}(z+w)=0$, respectively, where $P_z:\mathscr{D}^{\frac12}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))\to T_z\mathcal{M}^\perp$ is the orthogonal projection.

It can be seen from Proposition 4.3 that the reorganized functional $\hat{\Phi}^h$, which is given by (4.2), possesses some negative minimum and tends to zero at the boundary of $(0,+\infty)\times\mathbb{R}^m\times S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$. Hence, we can find an open bounded subset $U\subset\mathcal{G}=(0,+\infty)\times\mathbb{R}^m$ and $\delta>0$ such that $\overline{U}\subset\mathcal{G}$ and

$$\inf_{\gamma\in\mathcal{N}}\Big(\min_{\partial U}\hat{\Phi}^h(\psi_{\cdot,\cdot,\gamma})-\min_{\overline{U}}\hat{\Phi}^h(\psi_{\cdot,\cdot,\gamma})\Big)\ge\delta$$

where $\mathcal{N} = S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$. In what follows, we keep this precompact set U being fixed and denote

$$U_{x_i} = \big\{(\lambda,\xi) \in \mathcal{G} :\, (\lambda, \xi - x_i) \in U\big\}.$$

Lemma 4.8.

(1) *Let $\mathcal{M}_c$ be a compact subset of $\mathcal{M}$, then there exists $C > 0$ such that for $|\varepsilon|$ small there hold*

$$\big\|w_\varepsilon(z) - w_\varepsilon^{(i)}(z)\big\| \le C|\varepsilon|\big\|\nabla\Gamma(z) - \nabla\Gamma^{(i)}(z)\big\|$$

for all $z \in \mathcal{M}_c$.

(2) *For $z = \psi_{\lambda,\xi,\gamma} \in \mathcal{M}$ with $(\lambda,\xi) \in \overline{U}_{x_i}$, there exist $C, L > 0$ such that if $|x_j - x_i| \ge L$ for all $j \ne i$ then*

$$\big\|\nabla\Gamma(z) - \nabla\Gamma^{(i)}(z)\big\| \le C \sum_{\substack{j\ge 1\\ j\ne i}} \frac{|a_j|}{|x_j - x_i|^{m-1}}.$$

Proof. Since the linear operator $K_z = \nabla^2\mathcal{J}_0(z)^{-1} : T_z\mathcal{M}^\perp \to T_z\mathcal{M}^\perp$ is uniformly bounded for $z \in \mathcal{M}$ (see [36, Lemma 4.11]), we soon obtain from (3.8) and Proposition 4.6 that

$$\big\|w_\varepsilon(z) - w_\varepsilon^{(i)}(z)\big\| = |\varepsilon|\big\|K_z(\nabla\Gamma(z)) - K_z(\nabla\Gamma^{(i)}(z))\big\| + o(\varepsilon),$$

which proves the assertion (1).

To check (2), let us use Lemma 4.1 (2) (which can be also applied to compute $\nabla\Gamma^{(i)}$) to get the estimate: for any $v \in \mathscr{D}^{1/2}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$

$$\begin{aligned}
&\big|\, \langle \nabla\Gamma(\psi_{\lambda,\xi,\gamma}), v\rangle - \Big\langle \nabla\Gamma^{(i)}(\psi_{\lambda,\xi,\gamma}), v\Big\rangle \,\big| \\
&\quad \le C\Big(\sum_{j\ne i} |a_j| \int_{\Omega_j} |\nabla\psi_{\lambda,\xi,\gamma}| \cdot |v| + |\psi_{\lambda,\xi,\gamma}| \cdot |v| d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \Big) \\
&\quad \le C\Big(\sum_{j\ne i} |a_j| \int_{\Omega_j} \frac{\lambda^{\frac{m-1}{2}}}{\big(\lambda^2 + |x-\xi|^2\big)^{\frac{m}{2}}} \cdot |v| + \frac{\lambda^{\frac{m-1}{2}}}{\big(\lambda^2 + |x-\xi|^2\big)^{\frac{m-1}{2}}} \cdot |v| d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \Big),
\end{aligned}$$

where $\Omega_j = \operatorname{supp} h(\cdot - x_j)$ for $j \ge 1$, and in the last inequality we have used the facts

$$|\psi_{\lambda,\xi,\gamma}(x)|_{g_{\mathbb{R}^m}} \sim \frac{\lambda^{\frac{m-1}{2}}}{\big(\lambda^2 + |x-\xi|^2\big)^{\frac{m-1}{2}}} \quad \text{and} \quad |\nabla\psi_{\lambda,\xi,\gamma}(x)| \sim \frac{\lambda^{\frac{m-1}{2}}}{\big(\lambda^2 + |x-\xi|^2\big)^{\frac{m}{2}}}.$$

Now, using the Hölder and Sobolev inequalities, we know that for $(\lambda, \xi) \in \overline{U}_{x_i}$ there holds

$$\left| \langle \nabla\Gamma(\psi_{\lambda,\xi,\gamma}), v\rangle - \left\langle \nabla\Gamma^{(i)}(\psi_{\lambda,\xi,\gamma}), v \right\rangle \right| \le C\|v\| \sum_{j\neq i} \Big(\frac{|a_j|}{|x_j - x_i|^m} + \frac{|a_j|}{|x_j - x_i|^{m-1}} \Big)$$

provided $|x_j - x_i| \ge L$, $j \neq i$, with L large enough (say $L \ge \operatorname{diam} U + 1$). This completes the proof. □

The next result will be devoted to compare the values of $\mathcal{J}_\varepsilon^{red}(z) := \mathcal{J}_\varepsilon(z + w_\varepsilon(z))$ and $\mathcal{J}_\varepsilon^{(i),red}(z) := \mathcal{J}_\varepsilon^{(i)}(z + w_\varepsilon^i(z))$ for $z = \psi_{\lambda,\xi,\gamma} \in \mathcal{M}$ with $(\lambda, \xi) \in \overline{U}_{x_i}$.

Proposition 4.9. *There exists $C > 0$ such that for $|\varepsilon|$ small there holds*

$$\left|\mathcal{J}_\varepsilon^{red}(z) - \mathcal{J}_\varepsilon^{(i),red}(z)\right| \le C|\varepsilon| \Bigg(\sum_{\substack{j\ge 1\\ j\neq i}} \frac{1}{|x_j - x_i|^{\frac{(m-1)\tau}{\tau-1}}} \Bigg)^{\frac{\tau-1}{\tau}}$$

for all $z = \psi_{\lambda,\xi,\gamma} \in \mathcal{M}$ with $(\lambda, \xi) \in \overline{U}_{x_i}$, and $|x_j - x_i| \ge L$.

Proof. Following from Lemma 4.8 and the boundedness of $\nabla\mathcal{J}_\varepsilon(z + w_\varepsilon(z))$, we have

$$\begin{aligned}
&\left|\mathcal{J}_\varepsilon^{red}(z) - \mathcal{J}_\varepsilon^{(i),red}(z)\right| \\
&\quad \le \left|\mathcal{J}_\varepsilon(z + w_\varepsilon(z)) - \mathcal{J}_\varepsilon(z + w_\varepsilon^{(i)}(z))\right| + \left|\mathcal{J}_\varepsilon(z + w_\varepsilon^{(i)}(z)) - \mathcal{J}_\varepsilon^{(i)}(z + w_\varepsilon^{(i)}(z))\right| \\
&\quad \le C\left\|w_\varepsilon(z) - w_\varepsilon^{(i)}(z)\right\| + \left|\mathcal{J}_\varepsilon(z + w_\varepsilon^{(i)}(z)) - \mathcal{J}_\varepsilon^{(i)}(z + w_\varepsilon^{(i)}(z))\right| \\
&\quad \le |\varepsilon| C\left\|\nabla\Gamma(z) - \nabla\Gamma^{(i)}(z)\right\| + \left|\mathcal{J}_\varepsilon(z + w_\varepsilon^{(i)}(z)) - \mathcal{J}_\varepsilon^{(i)}(z + w_\varepsilon^{(i)}(z))\right| \\
&\quad \le |\varepsilon| C \sum_{\substack{j\ge 1\\ j\neq i}} \frac{|a_j|}{|x_j - x_i|^{m-1}} + \left|\mathcal{J}_\varepsilon(z + w_\varepsilon^{(i)}(z)) - \mathcal{J}_\varepsilon^{(i)}(z + w_\varepsilon^{(i)}(z))\right|.
\end{aligned}$$

By noting that

$$\mathcal{J}_\varepsilon(z + w_\varepsilon^{(i)}(z)) - \mathcal{J}_\varepsilon^{(i)}(z + w_\varepsilon^{(i)}(z)) = \varepsilon\Gamma(z + w_\varepsilon^{(i)}(z)) - \varepsilon\Gamma^{(i)}(z + w_\varepsilon^{(i)}(z)) + o(\varepsilon)$$

and $\Gamma|_{\mathcal{M}} = \Gamma^{(i)}|_{\mathcal{M}} \equiv 0$, by using (3.3) and Lemma 4.5, we have

$$\Gamma(z + w_\varepsilon^{(i)}(z)) = \nabla\Gamma(z)[w_\varepsilon^{(i)}(z)] + o(\varepsilon) = O(\varepsilon)$$

and

$$\Gamma^{(i)}(z + w_\varepsilon^{(i)}(z)) = \nabla\Gamma^{(i)}(z)[w_\varepsilon^{(i)}(z)] + o(\varepsilon) = O(\varepsilon)$$

for $z = \psi_{\lambda,\xi,\gamma} \in \mathcal{M}$ with $(\lambda, \xi) \in \overline{U}_{x_i}$. Then, as long as $|x_j - x_i| \geq L$ for all $j \neq i$, we deduce

$$\left|\mathcal{J}_\varepsilon^{red}(z) - \mathcal{J}_\varepsilon^{(i),red}(z)\right| \leq |\varepsilon| C \sum_{\substack{j\geq 1\\ j\neq i}} \frac{|a_j|}{|x_j - x_i|^{m-1}} \leq |\varepsilon| C \Bigg(\sum_{\substack{j\geq 1\\ j\neq i}} \frac{1}{|x_j - x_i|^{\frac{(m-1)\tau}{\tau-1}}} \Bigg)^{\frac{\tau-1}{\tau}}$$

provided that

$$\sum_{j\geq 1} |a_j|^\tau < +\infty. \quad \square$$

Now we are ready to prove our main results:

Complete proof of Theorem 2.3. For a given $k \geq 1$, let us fix arbitrarily $x_0 \in \mathbb{R}^m$ with $|x_0| = 1$ and take $\tilde{h}$ to be of the form (2.5) with h being a compactly supported elementary matrix and satisfying the hypotheses of Proposition 4.3, $a_j = j^{-\beta}$ and $x_j = j^\alpha r x_0$ for $j \in \mathbb{N}$, where

$$r = \frac{C_0}{|\varepsilon|^{1/m-1}}, \quad \alpha > 1, \quad 2\alpha k < \beta \tag{4.14}$$

and $C_0 > 0$ is a constant fixed large enough (see below). With the above choice of a_j, we have $\sum_j |a_j|^\tau < +\infty$ since $\beta > 1 > \frac{1}{\tau}$. Note also that $\alpha > 1$, we have $|x_j - x_i| > 4\,\mathrm{diam}(\mathrm{supp}\, h)$ for all $i \neq j$ when $|\varepsilon|$ is small enough.

From the expansions in Lemma 2.6 and (3.9), we have that, for a fixed $i \geq 1$,

$$\mathcal{J}_\varepsilon^{(i),red}(z) = \mathcal{J}_0(z) + \varepsilon^2 a_i^2 \hat{\Phi}^{h(\cdot - x_i)}(z) + o(\varepsilon^2 a_i^2)$$

and, by Proposition 4.3, $\mathcal{J}_\varepsilon^{(i),red}$ attains a local minimum $z_i = \psi_{\lambda_i,\xi_i,\gamma_i}$ with $(\lambda_i, \xi_i) \in U_{x_i}$ and $\gamma_i \in \mathcal{N}$. In particular,

$$\inf_{\gamma\in\mathcal{N}} \Big(\min_{\partial U_{x_i}} \hat{\Phi}^{h(\cdot - x_i)}(\psi_{\cdot,\cdot,\gamma}) - \min_{\overline{U}_{x_i}} \hat{\Phi}^{h(\cdot - x_i)}(\psi_{\cdot,\cdot,\gamma}) \Big) \geq \delta$$

for some $\delta > 0$ independent of ε and i.

If we choose C_0 in (4.14) so large that $\min_{j\neq i} |x_j - x_i| \geq L$, so Proposition 4.9 holds, then we have

$$\begin{aligned}
\left|\mathcal{J}_\varepsilon^{red}(z) - \mathcal{J}_\varepsilon^{(i),red}(z)\right| &\leq C|\varepsilon| \Bigg(\sum_{\substack{j\geq 1\\ j\neq i}} \frac{1}{|x_j - x_i|^{\frac{(m-1)\tau}{\tau-1}}} \Bigg)^{\frac{\tau-1}{\tau}} \\
&= \frac{C|\varepsilon|}{r^{m-1}} \Bigg(\sum_{\substack{j\geq 1\\ j\neq i}} \frac{1}{|j^\alpha - i^\alpha|^{\frac{(m-1)\tau}{\tau-1}}} \Bigg)^{\frac{\tau-1}{\tau}}
\end{aligned}$$

$$\le \frac{C|\varepsilon|}{r^{m-1}} \cdot \frac{1}{i^{(\alpha-1)(m-1)}}$$

when i is fixed large, where we have used the inequality (see for instance [16, Lemma 4.4])

$$\sum_{\substack{j\ge1\\ j\ne i}} \frac{1}{|j^\alpha - i^\alpha|^{\frac{(m-1)\tau}{\tau-1}}} \le \frac{C}{i^{\frac{(\alpha-1)(m-1)\tau}{\tau-1}}}$$

for $\alpha > 1$ and $m \ge 2$. By enlarging C_0 if necessary, we shall have

$$\frac{C|\varepsilon|}{r^{m-1}} \le \frac{\delta}{4}\varepsilon^2.$$

And hence, when

$$(\alpha-1)(m-1) \ge 2\beta, \tag{4.15}$$

we find $\mathcal{J}_\varepsilon^{red}$ has a strict local minimum $\tilde z_i = \psi_{\tilde\lambda_1,\tilde\xi_i,\tilde\gamma_i}$ "near" z_i in the sense that $(\tilde\lambda_i, \tilde\xi_i) \in U_{x_i}$.

Summing up, we have proved that if (4.15) holds then, for all i large and $|\varepsilon|$ small, the functional $\mathcal{J}_\varepsilon^{red}$ attains a strict local minimum in $U_{x_i} \times \mathcal{N}$. Hence there are infinitely many distinct solutions of Eq. (2.4), denoted by $\{\tilde\varphi_\varepsilon^{(i)}\}$.

To determine the C^k-regularity of the metrics $\tilde g_\varepsilon$ at infinity and the pull-back metrics g_ε on S^m using our choice (4.14), let us denote $\tilde g_\varepsilon^\sharp(x) = \tilde g_\varepsilon(x/|x|^2)$ and $\tilde g_\varepsilon^{(i),\sharp}(x) = \tilde g_\varepsilon^{(i)}(x/|x|^2)$, for $i \in \mathbb{N}$. Since g_ε is smooth on $S^m\setminus\{P\}$, we find that the regularity of g_ε at P is the same of $\tilde g_\varepsilon$ at infinity, and so it is the same of $\tilde g_\varepsilon^\sharp$ at 0. If we set $\Omega_i = \operatorname{supp} h(\cdot - x_i)$, it follows that $\tilde g_\varepsilon^{(i),\sharp}(x) - g_{\mathbb{R}^m}$ has support $\Omega_i^\sharp := \{x \in \mathbb{R}^m : x/|x|^2 \in \Omega_i\}$. In this setting, since $\operatorname{diam}(\Omega_i^\sharp) \sim |x_i|^{-2}$, we have the following basic estimate

$$\|\tilde g_\varepsilon^{(i),\sharp} - g_{\mathbb{R}^m}\|_{C^k} \le C|\varepsilon a_i||x_i|^{2k} \le C \cdot C_0^{2k}|\varepsilon|^{1-2k/(m-1)} i^{2k\alpha-\beta}$$

for $k \ge 1$. Let

$$\tilde g_{\varepsilon,j}^\sharp = g_{\mathbb{R}^m} + \sum_{i=1}^{j} \left(\tilde g_\varepsilon^{(i),\sharp}(x) - g_{\mathbb{R}^m}\right)$$

we find that if $2k\alpha - \beta < 0$ then

$$\|\tilde g_{\varepsilon,j}^\sharp - \tilde g_{\varepsilon,l}^\sharp\|_{C^k} \le \max_{i=j+1,\dots,l} \|\tilde g_\varepsilon^{(i),\sharp} - g_{\mathbb{R}^m}\|_{C^k} \le C \cdot C_0^{2k}|\varepsilon|^{1-2k/(m-1)}(j+1)^{2k\alpha-\beta}$$

for all $j < l$. And thus $\{\tilde g_{\varepsilon,j}^\sharp\}_{j=1}^\infty$ is a Cauchy sequence in $C^k(B_1)$, where B_1 stands for the open ball of radius 1 centered at the origin. Therefore, $\tilde g_\varepsilon^\sharp$ can be extended to $x = 0$ in the class of C^k. And if there holds additionally that $1 - 2k/(m-1) > 0$, then $\tilde g_\varepsilon^\sharp \to g_{\mathbb{R}^m}$ as $\varepsilon \to 0$.

There are three essential inequalities in the above arguments, namely (4.15),

$$\beta > 2k\alpha \quad \text{and} \quad 2k < m-1.$$

They are satisfied with $m \geq 4k+2$,

$$\alpha > 4k+1 \quad \text{and} \quad 2k\alpha < \beta < 2k\alpha + \frac{\alpha-(4k+1)}{2}.$$

Finally, since the solutions $\{\tilde{\varphi}_\varepsilon^{(i)}\}$ of Eq. (2.4) can be parameterized in the compact set $\overline{U}_{x_i} \times \mathcal{N}$, via the conformal transformation mentioned in Section 2.1, we find that the corresponding solutions $\{\psi_\varepsilon^{(i)}\}$ of Eq. (2.1) blow up at $P \in S^m$ in the following sense

$$\|\psi_\varepsilon^{(i)}\|_{L^\infty} \to +\infty \quad \text{as } i \to +\infty.$$

And standard regularity arguments, see [33, Appendix] and [6, Chapter 3], imply that the weak solutions $\psi_\varepsilon^{(i)}$ are indeed of class C^1 on S^m. This completes the proof. $\square$

5. The non-compactness caused by the geometric potential

In this section, we intend to prove Theorem 1.4, and our proof will be based upon the abstract result Theorem 3.5 in Section 3.

Let $H \in C^2(S^m)$ be a given Morse function, satisfying the conditions (H-1) and (H-2) mentioned in Subsection 2.1. For simplicity, let us assume $H \geq 0$ and takes its minimum at $p_0 \in S^m$ and $H(p_0) = 0$. Denote $\pi_{p_0} : S^m \setminus \{p_0\} \to \mathbb{R}^m$ the stereographic projection, we define $K(x) = H(\pi_{p_0}^{-1}(x))$ for $x \in \mathbb{R}^m$. Then $K \in L^\infty(\mathbb{R}^m) \cap C^2(\mathbb{R}^m)$ and satisfies the following (see [36, Lemma 3.1])

$$|\nabla K(x)| \leq C_0(1+|x|^2)^{-1} \quad \text{and} \quad |\nabla^2 K(x)| \leq C_0(1+|x|^2)^{-3/2} \tag{5.1}$$

for some constant $C_0 > 0$. Taking into account the additional condition $H(p_0) = 0$, we also have

Lemma 5.1. *By suitably enlarging the constant C_0 in (5.1) (if necessary), there holds*

$$|K(x)| \leq C_0(1+|x|^2)^{-1}$$

for $x \in \mathbb{R}^m$.

Proof. Since $H(p_0) = 0$ and $\nabla_{g_{S^m}} H(p_0) = 0$, we have

$$|K(x)| = |H(\pi_{p_0}^{-1}(x)) - H(p_0)| \leq C|\pi_{p_0}^{-1}(x) - p_0|^2$$

by the Taylor's formula, for some constant $C > 0$. By rotation, we may assume that $p_0 = (0, 0, \dots, 1)$ is the north pole and $\pi_{p_0}^{-1}(x) = \left(\frac{2x}{1+|x|^2}, \frac{|x|^2-1}{1+|x|^2}\right)$, for $x \in \mathbb{R}^m$. Then the assertion follows from a simple calculation. $\square$

Remark 5.2. Though the function K is non-negative in this context, we kept the absolute value symbol in Lemma 5.1 to emphasis that the inequality also holds true for sign-changing functions and the proof only needs the facts $H(p_0) = 0$ and $\nabla_{g_{S^m}} H(p_0) = 0$.

To proceed, let $\{z_i\}_{i=0}^{\infty} \subset \mathbb{R}^m$ and $\{a_i\}_{i=0}^{\infty} \subset \mathbb{R}$ be such that

(1) $|z_i - z_j| \gg 1$ for $i \neq j$. For reader's convenience, we may simply take $z_i = i^{\alpha} z_0$ with $z_0 \in \mathbb{R}^m$, $|z_0| = R \gg 1$ and $\alpha > 1$.
(2) $a_i = i^{-\beta}$ with $\beta > 1$.

From K, $\{z_i\}$ and $\{a_i\}$ as above, we define

$$\tilde{K}(x) = \sum_{i=1}^{\infty} a_i K(x - z_i).$$

Then it follows that the above summation converges uniformly in x so that $\tilde{K}$ is well-defined and $\tilde{K} \in L^{\infty}(\mathbb{R}^m) \cap C^2(\mathbb{R}^m)$. In the sequel, for $\psi_{\lambda,\xi,\gamma} \in \mathcal{M}$ (see (3.5)–(3.6)), let us set

$$\tilde{\Gamma}(\psi_{\lambda,\xi,\gamma}) = -\frac{m-1}{2m} \int_{\mathbb{R}^m} \tilde{K}(x) |\psi_{\lambda,\xi,\gamma}|_{g_{\mathbb{R}^m}}^{\frac{2m}{m-1}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

And it is clear that $\tilde{\Gamma}(\psi_{\lambda,\xi,\gamma})$ is independent of the factor $\gamma \in S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$. Hence, in order to study $\tilde{\Gamma}(\psi_{\lambda,\xi,\gamma})$, it is sufficient to consider (up to multiplication by a constant)

$$\tilde{\Psi}(\lambda, \xi) := \int_{\mathbb{R}^m} \tilde{K}(x) |\psi_{\lambda,\xi,\gamma}|_{g_{\mathbb{R}^m}}^{\frac{2m}{m-1}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}.$$

We will also denote $\Gamma^{(i)}$, $\Psi^{(i)}$ etc for functions corresponding to $K_i(x) = a_i K(x - z_i)$. Then we have

$$\tilde{\Psi}(\lambda, \xi) = \sum_{i=1}^{\infty} \Psi^{(i)}(\lambda, \xi)$$

and

$$\Psi^{(i)}(\lambda, \xi) = m^m a_i \int_{\mathbb{R}^m} \frac{K(\lambda x + \xi - z_i)}{(1 + |x|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

where the above formulation comes from a change of variables. The following result is a direct consequence of the computations in [36, Subsection 3.1], which characterizes the critical points of each $\Psi^{(i)}$ (hence $\Gamma^{(i)}$).

Proposition 5.3. *Let $H \in C^2(S^m)$ and $K = H \circ \pi_{p_0}^{-1}$ be as above. Then the critical points of the function $\Psi : \mathcal{G} = (0, +\infty) \times \mathbb{R}^m \to \mathbb{R}$,*

$$\Psi(\lambda, \xi) := m^m \int_{\mathbb{R}^m} \frac{K(\lambda x + \xi)}{(1 + |x|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}},$$

are isolated and there exists a bounded domain $\Omega_H \subset \mathcal{G}$ such that

$$\overline{\Omega}_H \subset \mathcal{G}, \quad Crit[\Psi] \subset \Omega_H \quad and \quad \deg(\nabla\Psi, \Omega_H, 0) \neq 0,$$

where the closure of Ω_H is taken with respect to the standard Euclidean norm and "deg" stands for the topological degree.

As a direct consequence of Proposition 5.3, we can find a bounded domain Ω_H such that all critical points of $\Psi^{(i)}$ are contained in $\Omega_H(z_i) = \{(\lambda, \xi) : (\lambda, \xi - z_i) \in \Omega_H\}$. Moreover, when z_i and z_j are located far apart, we have $\Omega_H(z_i) \cap \Omega_H(z_j) = \emptyset$ provided $i \neq j$.

Next, we intend to apply Theorem 3.5 to prove our second non-compactness result, i.e., Theorem 2.4. The main ingredient here is to show that $\tilde{\Psi}$ (or equivalently $\tilde{\Gamma}$) has at least one critical point in each $\Omega_H(z_i)$. By abuse of notation, we continue to use $\mathcal{J}_\varepsilon$ for the Euler functional associated to the perturbed problem (2.8), that is,

$$\mathcal{J}_\varepsilon(\psi) = \mathcal{J}_0(\psi) + \varepsilon\tilde{\Gamma}(\psi)$$

where $\mathcal{J}_0$ is as in (2.10) and

$$\tilde{\Gamma}(\psi) = -\frac{m-1}{2m} \int_{\mathbb{R}^m} \tilde{K}(x)|\psi|_{g_{\mathbb{R}^m}}^{\frac{2m}{m-1}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}.$$

We also denote $\mathcal{J}_\varepsilon^{(i)}$ the functional

$$\mathcal{J}_\varepsilon^{(i)}(\psi) = \mathcal{J}_0(\psi) + \varepsilon\Gamma^{(i)}(\psi)$$

with

$$\Gamma^{(i)}(\psi) = -\frac{m-1}{2m} \int_{\mathbb{R}^m} K_i(x)|\psi|_{g_{\mathbb{R}^m}}^{\frac{2m}{m-1}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}.$$

Following from the abstract settings in Section 3, we will introduce the notation $w_\varepsilon(z)$ and $w_\varepsilon^i(z)$ for the solutions to the auxiliary equations $P_z\nabla\mathcal{J}_\varepsilon(z+w) = 0$ and $P_z\nabla\mathcal{J}_\varepsilon^{(i)}(z+$

$w) = 0$, respectively, where $P_z : \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m)) \to T_z\mathcal{M}^\perp$ stands for the orthogonal projection. Then, analogous to Lemma 4.8, we have in the present case

Lemma 5.4.

(1) *Let $\mathcal{M}_c$ be a compact subset of $\mathcal{M}$, then there exists $C > 0$ such that for $|\varepsilon|$ small there hold*

$$\big\|w_\varepsilon(z) - w_\varepsilon^i(z)\big\| \le C|\varepsilon|\big\|\nabla\tilde{\Gamma}(z) - \nabla\Gamma^{(i)}(z)\big\|,$$

for all $z \in \mathcal{M}_c$.

(2) *For $z = \psi_{\lambda,\xi,\gamma} \in \mathcal{M}$ with $(\lambda, \xi) \in \Omega_H(z_i)$, there exists $C, L > 0$ such that if $|z_j - z_i| \ge L$ for all $j \ne i$ then*

$$\big\|\nabla\tilde{\Gamma}(z) - \nabla\Gamma^{(i)}(z)\big\| \le C \sum_{\substack{j\ge1 \\ j\ne i}} \frac{|a_j|}{|z_j - z_i|^2}.$$

Proof. Recall that $w_\varepsilon(z)$ satisfies $P_z\nabla\mathcal{J}_\varepsilon(z + w_\varepsilon(z)) = 0$, via Taylor expansion, we find

$$\begin{aligned}\nabla\mathcal{J}_\varepsilon(z + w_\varepsilon(z)) &= \nabla\mathcal{J}_0(z + w_\varepsilon(z)) + \varepsilon\nabla\tilde{\Gamma}(z + w_\varepsilon(z)) \\ &= \nabla\mathcal{J}_0(z) + \nabla^2\mathcal{J}_0(z)[w_\varepsilon(z)] + \varepsilon\nabla\tilde{\Gamma}(z) + \varepsilon\nabla^2\tilde{\Gamma}(z)[w_\varepsilon(z)] + o(\|w_\varepsilon(z)\|).\end{aligned}$$

Since $\nabla\mathcal{J}_0(z) = 0$ for all $z \in \mathcal{M}$, we get

$$\nabla\mathcal{J}_\varepsilon(z + w_\varepsilon(z)) = \nabla^2\mathcal{J}_0(z)[w_\varepsilon(z)] + \varepsilon\nabla\tilde{\Gamma}(z) + \varepsilon\nabla^2\tilde{\Gamma}(z)[w_\varepsilon(z)] + o(\|w_\varepsilon(z)\|),$$

and the equation $P_z\nabla\mathcal{J}_\varepsilon(z + w_\varepsilon(z)) = 0$ becomes

$$P_z\nabla^2\mathcal{J}_0(z)[w_\varepsilon(z)] + \varepsilon P_z\nabla\tilde{\Gamma}(z) + \varepsilon P_z\nabla^2\tilde{\Gamma}(z)[w_\varepsilon(z)] + o(\|w_\varepsilon(z)\|) = 0.$$

And a similar equation holds for $w_\varepsilon^i(z)$. Notice that $\mathcal{M}$ is a non-degenerate critical manifold of $\mathcal{J}_0$ and $\nabla^2\mathcal{J}_0(z)$ is invertible on $T_z\mathcal{M}^\perp$, we find (1) holds true.

To check (2), let us take arbitrarily $\varphi \in \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ with $\|\varphi\| \le 1$. Then we have

$$\begin{aligned}\Big|\big\langle\nabla\tilde{\Gamma}(\psi_{\lambda,\xi,\gamma}), \varphi\big\rangle - \Big\langle\nabla\Gamma^{(i)}(\psi_{\lambda,\xi,\gamma}), \varphi\Big\rangle\Big| &\le C\sum_{\substack{j\ge1 \\ j\ne i}} |a_j| \int_{\mathbb{R}^m} |\psi_{\lambda,\xi,\gamma}|_{g_{\mathbb{R}^m}}^{2^*-1}|K(x - z_j)||\varphi| d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \\ &\le C\sum_{\substack{j\ge1 \\ j\ne i}} |a_j| \int_{\mathbb{R}^m} \frac{\lambda^{\frac{m+1}{2}}}{\left(\lambda^2 + |x - \xi|^2\right)^{\frac{m+1}{2}}}|K(x - z_j)||\varphi| d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\end{aligned}$$

$$\le C \sum_{\substack{j\ge 1\\ j\ne i}} |a_j| \int_{\mathbb{R}^m} \frac{\lambda^{\frac{m+1}{2}}}{\big(\lambda^2+|x-(\xi-z_j)|^2\big)^{\frac{m+1}{2}}} |K(x)||\varphi(x+z_j)| d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

$$\le C \sum_{\substack{j\ge 1\\ j\ne i}} |a_j| \int_{\mathbb{R}^m} \frac{\lambda^{\frac{m+1}{2}}}{\big(\lambda^2+|x-(\xi-z_j)|^2\big)^{\frac{m+1}{2}}} \cdot \frac{1}{1+|x|^2} \cdot |\varphi(x+z_j)| d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

where the last inequality follows from Lemma 5.1. To proceed, let us define

$$I_j = \int_{\mathbb{R}^m} \frac{\lambda^{\frac{m+1}{2}}}{\big(\lambda^2+|x-(\xi-z_j)|^2\big)^{\frac{m+1}{2}}} \cdot \frac{1}{1+|x|^2} \cdot |\varphi(x+z_j)| d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

for $j \ge 1$. By the Hölder's inequality and $\|\varphi\| \le 1$, we have

$$I_j \le C\,\lambda^{\frac{m+1}{2}} \Big(\int_{\mathbb{R}^m} \frac{1}{(\lambda^2+|x-(\xi-z_j)|^2)^m} \cdot \frac{1}{(1+|x|^2)^{\frac{2m}{m+1}}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \Big)^{\frac{m+1}{2m}}.$$

Recall that we have assumed $(\lambda,\xi) \in \Omega_H(z_i)$, then we claim that

$$J_j := \int_{\mathbb{R}^m} \frac{1}{(\lambda^2+|x-(\xi-z_j)|^2)^m} \cdot \frac{1}{(1+|x|^2)^{\frac{2m}{m+1}}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \le \frac{C\lambda^{-m}}{|\xi-z_j|^{\frac{4m}{m+1}}} \tag{5.2}$$

for some constant $C > 0$ when $|\xi - z_j| \gg 1$.

Assuming (5.2) for the moment, we soon have $I_j \le C|\xi-z_j|^{-2}$, and hence

$$\big\|\nabla\tilde{\Gamma}(\psi_{\lambda,\xi,\gamma}) - \nabla\Gamma^{(i)}(\psi_{\lambda,\xi,\gamma})\big\| \le C \sum_{\substack{j\ge 1\\ j\ne i}} \frac{|a_j|}{|\xi-z_j|^2} \le CL^2 \sum_{\substack{j\ge 1\\ j\ne i}} \frac{|a_j|}{|z_j-z_i|^2}$$

provided that $|z_j - z_i| \ge L$ with L large enough (say $L \ge \operatorname{diam}\Omega_H + 1$). This proves (2).

Now it remains to prove (5.2). Let us decompose the integral into two parts $J_j = J_{j,1} + J_{j,2}$, where

$$J_{j,1} = \int_{|x|\le \frac{|\xi-z_j|}{2}} \frac{1}{(\lambda^2+|x-(\xi-z_j)|^2)^m} \cdot \frac{1}{(1+|x|^2)^{\frac{2m}{m+1}}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

and

$$J_{j,1} = \int_{|x|\ge \frac{|\xi-z_j|}{2}} \frac{1}{(\lambda^2+|x-(\xi-z_j)|^2)^m} \cdot \frac{1}{(1+|x|^2)^{\frac{2m}{m+1}}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}.$$

Then, via elementary computations, we find

$$J_{j,1} \le \frac{1}{\left(\lambda^2 + \left|\frac{\xi - z_j}{2}\right|^2\right)^m} \int\limits_{|x| \le \frac{|\xi - z_j|}{2}} \frac{1}{(1+|x|^2)^{\frac{2m}{m+1}}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

$$\le \frac{C}{|\xi - z_j|^{2m}} \int_0^{\frac{|\xi - z_j|}{2}} \frac{r^{m-1} dr}{(1+r^2)^{\frac{2m}{m+1}}} \le \begin{cases} C|\xi - z_j|^{-m-\frac{4m}{m+1}} & \text{if } m \ge 4 \\ C|\xi - z_j|^{-6} \ln|\xi - z_j| & \text{if } m = 3 \\ C|\xi - z_j|^{-4} & \text{if } m = 2 \end{cases}$$

and

$$J_{j,2} \le \frac{1}{\left(1 + \left|\frac{\xi - z_j}{2}\right|^2\right)^{\frac{2m}{m+1}}} \int\limits_{|x| \ge \frac{|\xi - z_j|}{2}} \frac{1}{(\lambda^2 + |x - (\xi - z_j)|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

$$\le \frac{C}{|\xi - z_j|^{\frac{4m}{m+1}}} \int\limits_{\mathbb{R}^m} \frac{1}{(\lambda^2 + |x|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \le \frac{C\lambda^{-m}}{|\xi - z_j|^{\frac{4m}{m+1}}}$$

which directly imply (5.2). And the proof is hereby complete. □

Our next result intends to estimate the difference of the derivatives of the reduced functionals $\mathcal{J}_\varepsilon^{red}(z) := \mathcal{J}_\varepsilon(z + w_\varepsilon(z))$ and $\mathcal{J}_\varepsilon^{(i),red}(z) := \mathcal{J}_\varepsilon^{(i)}(z + w_\varepsilon^i(z))$ for $z = \psi_{\lambda,\xi,\gamma} \in \mathcal{M}$ with $(\lambda, \xi) \in \Omega_H(z_i)$.

Proposition 5.5. *Let $\{z_i\} \subset \mathbb{R}^m$ and $\{a_i\} \subset \mathbb{R}$ be chosen as above, then there exists a constant $C > 0$ such that*

$$\|\nabla\tilde{\Gamma}(z) - \nabla\Gamma^{(i)}(z)\| \le CR^{-2}$$

for all $z = \psi_{\lambda,\xi,\gamma} \in \mathcal{M}$ with $(\lambda, \xi) \in \Omega_H(z_i)$, some $i \ge 1$, where $R \gg 1$ is given in the definition of the sequence $\{z_i\}$. Furthermore, there holds

$$\|\nabla\mathcal{J}_\varepsilon^{red}(z) - \nabla\mathcal{J}_\varepsilon^{(i),red}(z)\| \le CR^{-2}|\varepsilon| + o(\varepsilon).$$

Proof. Since

$$\nabla\mathcal{J}_\varepsilon^{red}(z) = \varepsilon\nabla\tilde{\Gamma}(z) + o(\varepsilon)$$

and

$$\nabla\mathcal{J}_\varepsilon^{(i),red}(z) = \varepsilon\nabla\Gamma^{(i)}(z) + o(\varepsilon),$$

it follows from Lemma 5.4 that

$$\|\nabla\mathcal{J}_\varepsilon^{red}(z)-\nabla\mathcal{J}_\varepsilon^{(i),red}(z)\|\le|\varepsilon|\|\nabla\tilde{\Gamma}(z)-\nabla\Gamma^{(i)}(z)\|+o(\varepsilon)$$
$$\le C|\varepsilon|\sum_{\substack{j\ge1\\ j\ne i}}\frac{|a_j|}{|z_j-z_i|^2}+o(\varepsilon). \tag{5.3}$$

Next, we estimate $\sum_{j\neq i}|a_j||z_j-z_i|^{-2}$. Let us recall $z_j=j^\alpha z_0$, $|z_0|=R$ and $a_j=j^{-\beta}$ for $\alpha,\beta>1$ and $R\gg1$. Then we have

$$\sum_{\substack{j\ge1\\ j\ne i}}\frac{|a_j|}{|z_j-z_i|^2}=R^{-2}\sum_{\substack{j\ge1\\ j\ne i}}\frac{1}{j^\beta}\cdot\frac{1}{|j^\alpha-i^\alpha|^2}.$$

Notice that, for $i\le j\le i-1$, we have

$$|j^\alpha-i^\alpha|=i^\alpha-j^\alpha\ge i^\alpha-(i-1)^\alpha\ge\alpha(i-1)^{\alpha-1}$$

and similarly for $j\ge i+1$, we have

$$|j^\alpha-i^\alpha|=j^\alpha-i^\alpha\ge(i+1)^\alpha-i^\alpha\ge\alpha i^{\alpha-1}.$$

We thus have for $i\ge2$

$$\sum_{\substack{j\ge1\\ j\ne i}}\frac{1}{j^\beta}\cdot\frac{1}{|j^\alpha-i^\alpha|^2}\le\sum_{1\le j<i}\frac{1}{j^\beta}\cdot\frac{1}{\alpha^2(i-1)^{2\alpha-2}}+\sum_{j>i}\frac{1}{j^\beta}\cdot\frac{1}{\alpha^2i^{2\alpha-2}}$$
$$\le\frac{C(\beta)}{\alpha^2(i-1)^{2\alpha-2}}\le C(\beta)$$

where $C(\beta)=\sum_{j\ge1}\frac{1}{j^\beta}<+\infty$. The case $i=1$ is even much easier since we have

$$\sum_{j=2}^{\infty}\frac{1}{j^\beta}\cdot\frac{1}{|j^\alpha-1|^2}<C(\beta).$$

Hence, by (5.3), we deduce

$$\|\nabla\mathcal{J}_\varepsilon^{red}(z)-\nabla\mathcal{J}_\varepsilon^{(i),red}(z)\|\le CC(\beta)R^{-2}|\varepsilon|+o(\varepsilon),$$

which completes the proof. □

Now we are ready to prove Theorem 2.4.

Proof of Theorem 2.4. Now, by Proposition 5.3, 5.5 and the homotopy invariance of the topological degree, we can conclude that for $R\gg1$ sufficiently large (this only depends on the size of Ω_H) there exists $\varepsilon_0>0$ (independent of i) such that

$$\begin{aligned}\deg(\nabla\tilde{\Psi},\Omega_H(z_i),0) &= \deg(\nabla\tilde{\Gamma},\Omega_H(z_i)\times\{\gamma\},0) = \deg(\nabla\Gamma^{(i)},\Omega_H(z_i)\times\{\gamma\},0)\\ &= \deg(\nabla\Psi^{(i)},\Omega_H(z_i),0)\neq 0\end{aligned}$$

for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Thus, there exists a critical point $(\lambda_i, \xi_i) \in \Omega_H(z_i)$ of the function $\tilde{\Psi} : \mathcal{G} \to \mathbb{R}$, $(\lambda, \xi) \mapsto \tilde{\Psi}(\lambda, \xi)$. As we will see later in Appendix A.2, the function $\tilde{K}$ comes from a C^2-function on S^m provided α, $\beta > 1$ satisfy $\beta > 4\alpha + 1$. Then it follows from [36, Proposition 3.6] that, by approximating $\tilde{K}$ if necessary, we may assume $\tilde{\Psi}$ is a Morse function. Therefore, by virtue of Theorem 3.5, we can choose $g_i(\gamma) = (\lambda_i(\gamma), \xi_i(\gamma)) \in \Omega_H(z_i)$ depending on $\gamma \in \mathcal{N} = S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$ and $\gamma_i \in \mathcal{N}$ such that $\zeta_i := (g_i(\gamma_i), \gamma_i) \in \mathcal{G}\times\mathcal{N}$ is a critical point of $\mathcal{J}_\varepsilon^{red}$. Then the critical point $\varphi_i = \zeta_i + w_\varepsilon(\zeta_i)$ of $\mathcal{J}_\varepsilon$ are positive and concentrates at infinity. And, similar to the very last step in proving Theorem 2.3, the corresponding pull-back spinors $\{\psi_i\}$ concentrate at p_0 and $\|\psi_i\|_{L^\infty} \to +\infty$ as $i \to \infty$. Thus $\{\psi_i\}$ is a non-compact family of solutions to Eq. (2.7). The estimates in Theorem 2.4 (2) simply come from some direct computations. $\square$

Data availability

No data was used for the research described in the article.

Appendix A

A.1. Proof of Lemma 4.5

Since (4.3) can be obtained by using the identification (2.12) and (2.13)–(2.15), we start with (4.4).

For $\psi, \varphi \in \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m)) \cap C^1(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$, there holds

$$\langle\nabla\mathcal{J}_\varepsilon(\tilde{\psi}),\tilde{\varphi}\rangle = \frac{1}{2}\operatorname{Re}\int_{\mathbb{R}^m}(\tilde{\psi},D_{\tilde{g}_\varepsilon}\tilde{\varphi})_{\tilde{g}_\varepsilon} + (\tilde{\varphi},D_{\tilde{g}_\varepsilon}\tilde{\psi})_{\tilde{g}_\varepsilon}\,d\,\mathrm{vol}_{\tilde{g}_\varepsilon} - \operatorname{Re}\int_{\mathbb{R}^m}|\tilde{\psi}|_{\tilde{g}_\varepsilon}^{2^*-2}(\tilde{\psi},\tilde{\varphi})_{g_\varepsilon}\,d\,\mathrm{vol}_{g_\varepsilon}.$$

Using (2.12), we have

$$\begin{aligned}&\operatorname{Re}(\tilde{\psi},D_{\tilde{g}_\varepsilon}\tilde{\varphi})_{\tilde{g}_\varepsilon} + \operatorname{Re}(\tilde{\varphi},D_{\tilde{g}_\varepsilon}\tilde{\psi})_{\tilde{g}_\varepsilon}\\ &\quad= \operatorname{Re}\big(\tilde{\varphi},\widetilde{D_{g_{\mathbb{R}^m}}\psi}\big)_{\tilde{g}_\varepsilon} + \operatorname{Re}\big(\tilde{\psi},\widetilde{D_{g_{\mathbb{R}^m}}\varphi}\big)_{\tilde{g}_\varepsilon} + \operatorname{Re}(\tilde{\varphi},W\cdot_{\tilde{g}_\varepsilon}\tilde{\psi})_{\tilde{g}_\varepsilon}\\ &\qquad+ \operatorname{Re}(\tilde{\psi},W\cdot_{\tilde{g}_\varepsilon}\tilde{\varphi})_{\tilde{g}_\varepsilon} + \operatorname{Re}(\tilde{\varphi},X\cdot_{\tilde{g}_\varepsilon}\tilde{\psi})_{\tilde{g}_\varepsilon} + \operatorname{Re}(\tilde{\psi},X\cdot_{\tilde{g}_\varepsilon}\tilde{\varphi})_{\tilde{g}_\varepsilon}\\ &\qquad+ \sum_{i,j}(b_{ij}-\delta_{ij})\operatorname{Re}\big(\tilde{\varphi},\tilde{\partial}_i\cdot_{\tilde{g}_\varepsilon}\widetilde{\nabla_{\partial_j}\psi}\big)_{\tilde{g}_\varepsilon}\\ &\qquad+ \sum_{i,j}(b_{ij}-\delta_{ij})\operatorname{Re}\big(\tilde{\psi},\tilde{\partial}_i\cdot_{\tilde{g}_\varepsilon}\widetilde{\nabla_{\partial_j}\varphi}\big)_{\tilde{g}_\varepsilon}.\end{aligned}\tag{A.1}$$

Notice that $X \in T\mathbb{R}^m$, we find

$$\mathrm{Re}(\tilde{\varphi}, X \cdot_{\tilde{g}_\varepsilon} \tilde{\psi})_{\tilde{g}_\varepsilon} + \mathrm{Re}(\tilde{\psi}, X \cdot_{\tilde{g}_\varepsilon} \tilde{\varphi})_{\tilde{g}_\varepsilon} = \mathrm{Re}(\tilde{\varphi}, X \cdot_{\tilde{g}_\varepsilon} \tilde{\psi})_{\tilde{g}_\varepsilon} - \mathrm{Re}(X \cdot_{\tilde{g}_\varepsilon} \tilde{\psi}, \tilde{\varphi})_{\tilde{g}_\varepsilon} = 0.$$

And using the explicit formula

$$W = \frac{1}{4} \sum_{\substack{i,j,k \\ i \neq j \neq k \neq i}} \sum_{\alpha,\beta} b_{i\alpha}(\partial_\alpha b_{j\beta}) b^{-1}_{\beta k} \, \tilde{\partial}_i \cdot_{\tilde{g}_\varepsilon} \tilde{\partial}_j \cdot_{\tilde{g}_\varepsilon} \tilde{\partial}_k,$$

and (2.14)–(2.15), we can see that $W \equiv 0$ in dimension 2 and

$$\begin{cases} b_{i\alpha} = \delta_{i\alpha} - \dfrac{\varepsilon}{2}\tilde{h}_{i\alpha} + \dfrac{3\,\varepsilon^2}{8} \displaystyle\sum_l \tilde{h}_{il}\tilde{h}_{l\alpha} + o(\varepsilon^2), \\ \partial_\alpha b_{j\beta} = -\dfrac{\varepsilon}{2}\partial_\alpha \tilde{h}_{j\beta} + \dfrac{3\,\varepsilon^2}{8} \displaystyle\sum_l \Big(\partial_\alpha \tilde{h}_{jl}\tilde{h}_{l\beta} + \tilde{h}_{jl}\partial_\alpha \tilde{h}_{l\beta} \Big) + o(\varepsilon^2), \\ b^{-1}_{\beta k} = \delta_{\beta k} + \dfrac{\varepsilon}{2}\tilde{h}_{\beta k} - \dfrac{\varepsilon^2}{8} \displaystyle\sum_l \tilde{h}_{\beta l}\tilde{h}_{lk} + o(\varepsilon^2), \end{cases}$$

for dimension $m \geq 3$. Hence

$$\begin{aligned} b_{i\alpha}(\partial_\alpha b_{j\beta}) b^{-1}_{\beta k} = &-\frac{\varepsilon}{2}\delta_{\beta k}\delta_{i\alpha}\partial_\alpha \tilde{h}_{j\beta} + \frac{\varepsilon^2}{4}\partial_\alpha \tilde{h}_{j\beta}\big(\delta_{\beta k}\tilde{h}_{i\alpha} - \delta_{i\alpha}\tilde{h}_{\beta k}\big) \\ &+ \frac{3\delta_{\beta k}\delta_{i\alpha}\varepsilon^2}{8} \sum_l \Big(\partial_\alpha \tilde{h}_{jl}\tilde{h}_{l\beta} + \tilde{h}_{jl}\partial_\alpha \tilde{h}_{l\beta} \Big) + o(\varepsilon^2). \end{aligned}$$

Note that we have assumed $\tilde{h}_{ij} = 0$ for $i \neq j$, we soon get

$$b_{i\alpha}(\partial_\alpha b_{j\beta}) b^{-1}_{\beta k} = o(\varepsilon^2).$$

Recalling that the map $\psi \mapsto \tilde{\psi}$ defined in (2.11) is fiberwisely isometric, we obtain

$$\mathrm{Re}\left(\tilde{\varphi}, \widetilde{D_{g_{\mathbb{R}^m}}\psi}\right)_{\tilde{g}_\varepsilon} = \mathrm{Re}\left(\varphi, D_{g_{\mathbb{R}^m}}\psi\right)_{g_{\mathbb{R}^m}}, \quad \mathrm{Re}\left(\tilde{\varphi}, \tilde{\partial}_i \cdot_{\tilde{g}_\varepsilon} \widetilde{\nabla_{\partial_j}\psi}\right)_{\tilde{g}_\varepsilon} = \mathrm{Re}\left(\varphi, \partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_j}\psi\right)_{g_{\mathbb{R}^m}}$$

and

$$\mathrm{Re}(\tilde{\varphi}, W \cdot_{\tilde{g}_\varepsilon} \tilde{\psi})_{\tilde{g}_\varepsilon} = \frac{1}{4} \sum_{\substack{i,j,k \\ i \neq j \neq k \neq i}} \Big(\sum_{\alpha,\beta} b_{i\alpha}(\partial_\alpha b_{j\beta}) b^{-1}_{\beta k} \Big) \mathrm{Re}(\partial_i \cdot_{g_{\mathbb{R}^m}} \partial_j \cdot_{g_{\mathbb{R}^m}} \partial_k \cdot_{g_{\mathbb{R}^m}} \psi, \varphi)_{g_{\mathbb{R}^m}}.$$

And thus (A.1) can be expanded as

$$\begin{aligned} &\mathrm{Re}(\tilde{\psi}, D_{\tilde{g}_\varepsilon}\tilde{\varphi})_{\tilde{g}_\varepsilon} + \mathrm{Re}(\tilde{\varphi}, D_{\tilde{g}_\varepsilon}\tilde{\psi})_{\tilde{g}_\varepsilon} \\ &\quad = \mathrm{Re}(\varphi, D_{g_{\mathbb{R}^m}}\psi)_{g_{\mathbb{R}^m}} + \mathrm{Re}(\psi, D_{g_{\mathbb{R}^m}}\varphi)_{g_{\mathbb{R}^m}} \\ &\qquad - \frac{\varepsilon}{2} \sum_i \tilde{h}_{ii} \big[\mathrm{Re}(\varphi, \partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\psi)_{g_{\mathbb{R}^m}} + \mathrm{Re}(\psi, \partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\varphi)_{g_{\mathbb{R}^m}} \big] \end{aligned}$$

$$+\frac{3\varepsilon^2}{8}\sum_i \tilde h_{ii}^2\big[\mathrm{Re}(\varphi,\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi)_{g_{\mathbb{R}^m}}+\mathrm{Re}(\psi,\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\varphi)_{g_{\mathbb{R}^m}}\big]$$
$$+\,o(\varepsilon^2)|\varphi|_{g_{\mathbb{R}^m}}|\psi|_{g_{\mathbb{R}^m}}$$

where the support of the last $o(\varepsilon^2)$ term is contained in $\operatorname{supp}\tilde h$. Then, by (2.13), the specific expression of $\mathcal{J}_0$ and Γ in Lemma 2.6 with $\tilde h\in\mathcal{H}(p)$ and the embedding $\mathscr{D}^{\frac12}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))\hookrightarrow L^{2^*}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$, we easily find

$$\Big|\big\langle\nabla\mathcal{J}_\varepsilon(\tilde\psi),\tilde\varphi\big\rangle-\langle\nabla\mathcal{J}_0(\psi),\varphi\rangle-\varepsilon\,\langle\nabla\Gamma(\psi),\varphi\rangle\Big|\le O(\varepsilon^2)\big(\|\psi\|\|\varphi\|+\|\psi\|^{2^*-1}\|\varphi\|\big)$$

and (4.4) is proved by using the fundamental fact that $\mathscr{D}^{\frac12}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))\cap C^1(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$ is dense in $\mathscr{D}^{\frac12}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$.

The estimate (4.5) can be obtained in a similar manner, in particular, we have $\|z\|$ is uniformly bounded for $z\in\mathcal{M}$.

To see (4.6), let us remark that,

$$\begin{aligned}\nabla^2\mathcal{J}_\varepsilon(\tilde\psi)[\tilde\phi,\tilde\varphi]&=\frac12\int_{\mathbb{R}^m}\mathrm{Re}(\tilde\phi,D_{\tilde g_\varepsilon}\tilde\varphi)_{\tilde g_\varepsilon}+\mathrm{Re}(\tilde\varphi,D_{\tilde g_\varepsilon}\tilde\phi)_{\tilde g_\varepsilon}\,d\,\mathrm{vol}_{\tilde g_\varepsilon}\\&\quad-\mathrm{Re}\int_{\mathbb{R}^m}|\tilde\psi|_{\tilde g_\varepsilon}^{2^*-2}(\tilde\phi,\tilde\varphi)_{\tilde g_\varepsilon}\,d\,\mathrm{vol}_{\tilde g_\varepsilon}\\&\quad-(2^*-2)\int_{\mathbb{R}^m}|\tilde\psi|_{\tilde g_\varepsilon}^{2^*-4}\,\mathrm{Re}(\tilde\psi,\tilde\phi)_{\tilde g_\varepsilon}\,\mathrm{Re}(\tilde\psi,\tilde\varphi)_{\tilde g_\varepsilon}\,d\,\mathrm{vol}_{\tilde g_\varepsilon}\end{aligned}$$

for two given spinors $\varphi,\phi\in\mathscr{D}^{\frac12}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))\cap C^1(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$. Then, by virtue of (A.1) and (2.13), we deduce

$$\big|\nabla^2\mathcal{J}_\varepsilon(\tilde\psi)[\tilde\phi,\tilde\varphi]-\nabla^2\mathcal{J}_0(\psi)[\phi,\varphi]\big|\le O(\varepsilon)\big(\|\phi\|\|\varphi\|+\|\psi\|^{2^*-2}\|\phi\|\|\varphi\|\big)$$

which proves (4.6) through the density of $\mathscr{D}^{\frac12}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))\cap C^1(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$ in $\mathscr{D}^{\frac12}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$.

Next, let us turn to (4.8). Notice that there holds

$$\begin{aligned}&\big\langle\nabla\mathcal{J}_\varepsilon(\tilde\psi+\tilde\varphi),\tilde\phi\big\rangle-\big\langle\nabla\mathcal{J}_\varepsilon(\tilde\psi),\tilde\phi\big\rangle\\&\quad=\frac12\int_{\mathbb{R}^m}\mathrm{Re}(\tilde\varphi,D_{\tilde g_\varepsilon}\tilde\phi)_{\tilde g_\varepsilon}+\mathrm{Re}(\tilde\phi,D_{\tilde g_\varepsilon}\tilde\varphi)_{\tilde g_\varepsilon}\,d\,\mathrm{vol}_{g_\varepsilon}\\&\qquad-\Big(\mathrm{Re}\int_{\mathbb{R}^m}|\tilde\psi+\tilde\varphi|_{\tilde g_\varepsilon}^{2^*-2}(\tilde\psi+\tilde\varphi,\tilde\phi)_{\tilde g_\varepsilon}-|\tilde\psi|_{\tilde g_\varepsilon}^{2^*-2}(\tilde\psi,\tilde\phi)_{\tilde g_\varepsilon}\,d\,\mathrm{vol}_{\tilde g_\varepsilon}\Big).\end{aligned}$$

This implies that

$$\begin{aligned}&\big\|\nabla\mathcal{J}_\varepsilon(\tilde\psi+\tilde\varphi)-\nabla\mathcal{J}_\varepsilon(\tilde\psi)\big\|\\&\quad\le O(1)\|\varphi\|+O(1)\Big(\int\limits_{\mathbb{R}^m}\Big||\psi+\varphi|_{g_{\mathbb{R}^m}}^{2^*-2}(\psi+\varphi)-|\psi|_{g_{\mathbb{R}^m}}^{2^*-2}\psi\Big|^{\frac{2m}{m+1}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\Big)^{\frac{m+1}{2m}}.\end{aligned}\tag{A.2}$$

Denoted by $f(s)=|\psi+s\varphi|_{g_{\mathbb{R}^m}}^{2^*-2}(\psi+s\varphi)$, we have

$$|\psi+\varphi|_{g_{\mathbb{R}^m}}^{2^*-2}(\psi+\varphi)-|\psi|_{g_{\mathbb{R}^m}}^{2^*-2}\psi=f(1)-f(0)=\int\limits_0^1 f'(s)ds$$

and

$$|f'(s)|\le(2^*-1)|\psi+\varphi|_{g_{\mathbb{R}^m}}^{2^*-2}|\varphi|_{g_{\mathbb{R}^m}}.$$

Using the Hölder inequality and Fubini Theorem, we have that

$$\begin{aligned}\int\limits_{\mathbb{R}^m}|f(1)-f(0)|^{\frac{2m}{m+1}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}&\le\int\limits_{\mathbb{R}^m}\int\limits_0^1|f'(s)|^{\frac{2m}{m+1}}ds\,d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\&=\int\limits_0^1\int\limits_{\mathbb{R}^m}|f'(s)|^{\frac{2m}{m+1}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\,ds\\&\le O(1)\int\limits_0^1\int\limits_{\mathbb{R}^m}|\psi+s\varphi|_{g_{\mathbb{R}^m}}^{\frac{2}{m-1}\cdot\frac{2m}{m+1}}|\varphi|_{g_{\mathbb{R}^m}}^{\frac{2m}{m+1}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\,ds\\&\le O(1)\int\limits_0^1|\psi+s\varphi|_{2^*}^{\frac{22^*}{m+1}}|\varphi|_{2^*}^{\frac{2m}{m+1}}ds\\&\le O(1)\|\varphi\|^{\frac{2m}{m+1}}\max_{s\in[0,1]}\|\psi+s\varphi\|^{\frac{22^*}{m+1}}\end{aligned}$$

So from (A.2) we deduce

$$\big\|\nabla\mathcal{J}_\varepsilon(\tilde\psi+\tilde\varphi)-\nabla\mathcal{J}_\varepsilon(\tilde\psi)\big\|\le O(1)\big(\|\varphi\|+\|\varphi\|\max_{s\in[0,1]}\|\psi+\varphi\|^{\frac{2}{m-1}}\big)$$

which suggests (4.8).

We point out that the estimates (4.7) and (4.9) can be obtained with similar procedures, and hence it remains to check (4.10).

Observe that, for two spinors $\phi_1,\phi_2\in\mathscr{D}^{\frac12}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$, we have

$$\nabla^2\mathcal{J}_\varepsilon(\tilde\psi+\tilde\varphi)[\tilde\phi_1,\tilde\phi_2]-\nabla^2\mathcal{J}_\varepsilon(\tilde\psi)[\tilde\phi_1,\tilde\phi_2]$$

$$
\begin{aligned}
&= -\operatorname{Re}\int_{\mathbb{R}^m} |\tilde\psi+\tilde\varphi|_{\tilde g_\varepsilon}^{2^*-2}(\tilde\phi_1,\tilde\phi_2)_{\tilde g_\varepsilon}\,d\operatorname{vol}_{g_\varepsilon} + \operatorname{Re}\int_{\mathbb{R}^m} |\tilde\psi|_{\tilde g_\varepsilon}^{2^*-2}(\tilde\phi_1,\tilde\phi_2)_{\tilde g_\varepsilon}\,d\operatorname{vol}_{g_\varepsilon} \\
&\quad -(2^*-2)\int_{\mathbb{R}^m} |\tilde\psi+\tilde\varphi|_{\tilde g_\varepsilon}^{2^*-4}\operatorname{Re}(\tilde\psi+\tilde\varphi,\tilde\phi_1)_{\tilde g_\varepsilon}\operatorname{Re}(\tilde\psi+\tilde\varphi,\tilde\phi_2)_{\tilde g_\varepsilon}\,d\operatorname{vol}_{g_\varepsilon} \qquad (A.3)\\
&\quad +(2^*-2)\int_{\mathbb{R}^m} |\tilde\psi|_{\tilde g_\varepsilon}^{2^*-4}\operatorname{Re}(\tilde\psi,\tilde\phi_1)_{\tilde g_\varepsilon}\operatorname{Re}(\tilde\psi,\tilde\phi_2)_{\tilde g_\varepsilon}\,d\operatorname{vol}_{g_\varepsilon},
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\operatorname{Re}\int_{\mathbb{R}^m} |\tilde\psi+\tilde\varphi|_{\tilde g_\varepsilon}^{2^*-2}(\tilde\phi_1,\tilde\phi_2)_{\tilde g_\varepsilon}\,d\operatorname{vol}_{g_\varepsilon} - \operatorname{Re}\int_{\mathbb{R}^m} |\tilde\psi|_{\tilde g_\varepsilon}^{2^*-2}(\tilde\phi_1,\tilde\phi_2)_{\tilde g_\varepsilon}\,d\operatorname{vol}_{g_\varepsilon}\Big| \\
&\quad \le O(1)\int_{\mathbb{R}^m} \Big||\tilde\psi+\tilde\varphi|_{\tilde g_\varepsilon}^{2^*-2}-|\tilde\psi|_{\tilde g_\varepsilon}^{2^*-2}\Big||\phi_1||\phi_2|\,d\operatorname{vol}_{g_{\mathbb{R}^m}} \qquad (A.4)\\
&\quad \le \begin{cases} O(1)\displaystyle\int_{\mathbb{R}^2} (|\psi|_{g_{\mathbb{R}^2}}|\varphi|_{g_{\mathbb{R}^2}}+|\varphi|_{g_{\mathbb{R}^2}}^2)|\phi_1|_{g_{\mathbb{R}^2}}|\phi_2|_{g_{\mathbb{R}^2}}\,d\operatorname{vol}_{g_{\mathbb{R}^2}} & \text{if } m=2\\ O(1)\displaystyle\int_{\mathbb{R}^m} |\varphi|_{g_{\mathbb{R}^m}}^{2^*-2}|\phi_1|_{g_{\mathbb{R}^m}}|\phi_2|_{g_{\mathbb{R}^m}}\,d\operatorname{vol}_{g_{\mathbb{R}^m}} & \text{if } m\ge 3\end{cases}
\end{aligned}
$$

where we have used the sub-additivity of the function $\psi \mapsto |\psi|^{2^*-2}$ for $2^*-2 \in (0,1]$ (that is $m \ge 3$). Thus, we only need to estimate the last two integrals in (A.3). For this purpose, let us set

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^m} |\tilde\psi+\tilde\varphi|_{\tilde g_\varepsilon}^{2^*-2}\frac{\operatorname{Re}(\tilde\psi+\tilde\varphi,\tilde\phi_1)_{\tilde g_\varepsilon}\operatorname{Re}(\tilde\psi+\tilde\varphi,\tilde\phi_2)_{\tilde g_\varepsilon}}{|\tilde\psi+\tilde\varphi|_{g_\varepsilon}^2}\,d\operatorname{vol}_{g_\varepsilon}\\
&\quad -\int_{\mathbb{R}^m} |\tilde\psi|_{\tilde g_\varepsilon}^{2^*-2}\frac{\operatorname{Re}(\tilde\psi+\tilde\varphi,\tilde\phi_1)_{\tilde g_\varepsilon}\operatorname{Re}(\tilde\psi+\tilde\varphi,\tilde\phi_2)_{\tilde g_\varepsilon}}{|\tilde\psi+\tilde\varphi|_{g_\varepsilon}^2}\,d\operatorname{vol}_{g_\varepsilon}
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^m} |\tilde\psi|_{\tilde g_\varepsilon}^{2^*-2}\frac{\operatorname{Re}(\tilde\psi+\tilde\varphi,\tilde\phi_1)_{\tilde g_\varepsilon}\operatorname{Re}(\tilde\psi+\tilde\varphi,\tilde\phi_2)_{\tilde g_\varepsilon}}{|\tilde\psi+\tilde\varphi|_{g_\varepsilon}^2}\,d\operatorname{vol}_{g_\varepsilon}\\
&\quad -\int_{\mathbb{R}^m} |\tilde\psi|_{\tilde g_\varepsilon}^{2^*-2}\frac{\operatorname{Re}(\tilde\psi,\tilde\phi_1)_{\tilde g_\varepsilon}\operatorname{Re}(\tilde\psi,\tilde\phi_2)_{\tilde g_\varepsilon}}{|\tilde\psi|_{g_\varepsilon}^2}\,d\operatorname{vol}_{g_\varepsilon}
\end{aligned}
$$

so that $I_1 + I_2$ is nothing but the last two integrals in (A.3). Clearly, I_1 can be estimated similar to (A.4). And for I_2, let us set $\Omega = \big\{x \in \mathbb{R}^m : |\tilde\psi|_{g_\varepsilon}/|\tilde\psi+\tilde\varphi|_{g_\varepsilon} < 2\big\}$, then we can have the decomposition $I_2 = I_2^{(1)} + I_2^{(2)}$ with $I_2^{(1)}$ and $I_2^{(2)}$ being the integration on Ω

and $\mathbb{R}^m \setminus \Omega$, respectively. Notice that, on $\mathbb{R}^m \setminus \Omega$, we have $|\tilde\psi|_{g_\varepsilon} \le 2|\tilde\varphi|_{g_\varepsilon}$. Hence, there holds

$$|I_2^{(2)}| \le O(1) \int\limits_{\mathbb{R}^m\setminus\Omega} |\tilde\psi|_{g_\varepsilon}^{2^*-2} |\tilde\phi_1|_{g_\varepsilon} |\tilde\phi_2|_{g_\varepsilon} d\,\mathrm{vol}_{g_\varepsilon} \le O(1) \int\limits_{\mathbb{R}^m} |\varphi|_{g_{\mathbb{R}^m}}^{2^*-2} |\phi_1|_{g_{\mathbb{R}^m}} |\phi_2|_{g_{\mathbb{R}^m}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} .$$

Meanwhile, by using the fact

$$\Big|\frac{\tilde\psi+\tilde\varphi}{|\tilde\psi+\tilde\varphi|_{g_\varepsilon}} - \frac{\tilde\psi}{|\tilde\psi|_{g_\varepsilon}}\Big|_{g_\varepsilon} = \Big|\frac{\tilde\psi|\tilde\psi|_{g_\varepsilon} + \tilde\varphi|\tilde\psi|_{g_\varepsilon} - \tilde\psi|\tilde\psi+\tilde\varphi|_{g_\varepsilon}}{|\tilde\psi+\tilde\varphi|_{g_\varepsilon}|\tilde\psi|_{g_\varepsilon}}\Big|_{g_\varepsilon} \le \frac{2|\tilde\varphi|_{g_\varepsilon}}{|\tilde\psi+\tilde\varphi|_{g_\varepsilon}}$$

and

$$\begin{aligned} I_2^{(1)} = &\int\limits_{\Omega} |\tilde\psi|_{\tilde g_\varepsilon}^{2^*-2} \frac{\mathrm{Re}(\tilde\psi+\tilde\varphi,\tilde\phi_1)_{\tilde g_\varepsilon}\,\mathrm{Re}(\tilde\psi+\tilde\varphi,\tilde\phi_2)_{\tilde g_\varepsilon}}{|\tilde\psi+\tilde\varphi|_{g_\varepsilon}^2} d\,\mathrm{vol}_{g_\varepsilon} \\ &- \int\limits_{\Omega} |\tilde\psi|_{\tilde g_\varepsilon}^{2^*-2} \frac{\mathrm{Re}(\tilde\psi,\tilde\phi_1)_{\tilde g_\varepsilon}\,\mathrm{Re}(\tilde\psi+\tilde\varphi,\tilde\phi_2)_{\tilde g_\varepsilon}}{|\tilde\psi|_{g_\varepsilon}|\tilde\psi+\tilde\varphi|_{g_\varepsilon}} d\,\mathrm{vol}_{g_\varepsilon} \\ &+ \int\limits_{\Omega} |\tilde\psi|_{\tilde g_\varepsilon}^{2^*-2} \frac{\mathrm{Re}(\tilde\psi,\tilde\phi_1)_{\tilde g_\varepsilon}\,\mathrm{Re}(\tilde\psi+\tilde\varphi,\tilde\phi_2)_{\tilde g_\varepsilon}}{|\tilde\psi|_{g_\varepsilon}|\tilde\psi+\tilde\varphi|_{g_\varepsilon}} d\,\mathrm{vol}_{g_\varepsilon} \\ &- \int\limits_{\Omega} |\tilde\psi|_{\tilde g_\varepsilon}^{2^*-2} \frac{\mathrm{Re}(\tilde\psi,\tilde\phi_1)_{\tilde g_\varepsilon}\,\mathrm{Re}(\tilde\psi,\tilde\phi_2)_{\tilde g_\varepsilon}}{|\tilde\psi|_{g_\varepsilon}^2} d\,\mathrm{vol}_{g_\varepsilon} \end{aligned}$$

we deduce

$$\begin{aligned} |I_2^{(1)}| &\le 2\int\limits_{\Omega} |\tilde\psi|_{g_\varepsilon}^{2^*-2} \Big|\frac{\tilde\psi+\tilde\varphi}{|\tilde\psi+\tilde\varphi|_{g_\varepsilon}} - \frac{\tilde\psi}{|\tilde\psi|_{g_\varepsilon}}\Big|_{g_\varepsilon} |\tilde\phi_1|_{g_\varepsilon}|\tilde\phi_2|_{g_\varepsilon} d\,\mathrm{vol}_{g_\varepsilon} \\ &\le O(1)\int\limits_{\Omega} \frac{|\tilde\psi|_{g_\varepsilon}^{2^*-2}|\tilde\varphi|_{g_\varepsilon}|\tilde\phi_1|_{g_\varepsilon}|\tilde\phi_2|_{g_\varepsilon}}{|\tilde\psi+\tilde\varphi|_{g_\varepsilon}} d\,\mathrm{vol}_{g_\varepsilon} \\ &\le O(1)\int\limits_{\mathbb{R}^m} |\psi|_{g_{\mathbb{R}^m}}^{2^*-3}|\varphi|_{g_{\mathbb{R}^m}}|\phi_1|_{g_{\mathbb{R}^m}}|\phi_2|_{g_{\mathbb{R}^m}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} . \end{aligned} \tag{A.5}$$

And thus, we obtain

$$|I_2| \le \begin{cases} O(1)\int\limits_{\mathbb{R}^2} (|\psi|_{g_{\mathbb{R}^2}}|\varphi|_{g_{\mathbb{R}^2}} + |\varphi|_{g_{\mathbb{R}^2}}^2)|\phi_1|_{g_{\mathbb{R}^2}}|\phi_2|_{g_{\mathbb{R}^2}} d\,\mathrm{vol}_{g_{\mathbb{R}^2}} & \text{if } m=2 \\ O(1)\int\limits_{\mathbb{R}^3} |\varphi|_{g_{\mathbb{R}^3}}|\phi_1|_{g_{\mathbb{R}^3}}|\phi_2|_{g_{\mathbb{R}^3}} d\,\mathrm{vol}_{g_{\mathbb{R}^2}} & \text{if } m=3 \end{cases} \tag{A.6}$$

Notice that $2^* = \frac{2m}{m-1} < 3$ for $m \geq 4$, we need to divide Ω into two parts, i.e. $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 := \{x \in \Omega : |\psi|_{g_{\mathbb{R}^m}} > |\varphi|_{g_{\mathbb{R}^m}}\}$ and $\Omega_2 := \{x \in \Omega : |\psi|_{g_{\mathbb{R}^m}} \leq |\varphi|_{g_{\mathbb{R}^m}}\}$. Then, from the first and second lines in (A.5), we obtain

$$|I_2^{(1)}| \leq O(1) \int_{\Omega_1} |\varphi|_{g_{\mathbb{R}^m}}^{2^*-2} |\phi_1|_{g_{\mathbb{R}^m}} |\phi_2|_{g_{\mathbb{R}^m}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} + O(1) \int_{\Omega_2} |\psi|_{g_{\mathbb{R}^m}} |\phi_1|_{g_{\mathbb{R}^m}} |\phi_2|_{g_{\mathbb{R}^m}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} .$$

Hence we have

$$|I_2| \leq O(1) \int_{\mathbb{R}^m} \big(|\varphi|_{g_{\mathbb{R}^m}}^{2^*-2} + |\varphi|\big) |\phi_1|_{g_{\mathbb{R}^m}} |\phi_2|_{g_{\mathbb{R}^m}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \quad \text{for } m \geq 4. \tag{A.7}$$

Now, combining (A.3)–(A.7), we find that

$$\big\|\nabla^2 \mathcal{J}_\varepsilon(\tilde\psi + \tilde\varphi) - \nabla^2 \mathcal{J}_\varepsilon(\tilde\psi)\big\| \leq \begin{cases} O(1)\big(\|\psi\|\|\varphi\| + \|\varphi\|^2\big) & \text{if } m = 2 \\ O(1)\big(\|\varphi\|^{2^*-2} + \|\varphi\|\big) & \text{if } m \geq 3 \end{cases}$$

which proves (4.10). And the proof is hereby completed.

A.2. The global C^2 smoothness of the pull-back function $\tilde K \circ \pi_{p_0}$ on S^m

Here we show that $\tilde K$ comes from a C^2-function on S^m when $\alpha, \beta > 0$ satisfy $\beta > 4\alpha + 1$. And this will complete the proof of Theorem 2.4.

Clearly, $\tilde K$ is C^2 on $\mathbb{R}^m$, because the series defining $\tilde K$ converges uniformly on $\mathbb{R}^m$ up to the second derivatives. To prove the differentiability at infinity (which correspond to the north pole of S^m), we need to show that $y \mapsto \tilde K(y/|y|^2)$ is twice continuously differentiable near $y = 0$. Without loss of generality, we assume $|y| < 1$ in the following context. And, by Lemma 5.4, we see that $\tilde K(y/|y|^2)$ converges uniformly in y. In particular, we have $\tilde K(y/|y|^2) \to 0$ as $y \to 0$.

To see the convergence of the derivatives, for the function K as before, we define $\hat K(y) = K\big(\frac{y}{|y|^2}\big)$. Then, an elementary computation shows that derivatives of $\hat K$ can be estimated as

$$|\nabla \hat K(y)| \leq C \Big|\nabla K\Big(\frac{y}{|y|^2}\Big)\Big| |y|^{-2}$$

and

$$|\nabla^2 \hat K(y)| \leq C\bigg(\Big|\nabla^2 K\Big(\frac{y}{|y|^2}\Big)\Big| |y|^{-4} + \Big|\nabla K\Big(\frac{y}{|y|^2}\Big)\Big| |y|^{-3}\bigg).$$

Recall (5.1), we notice that

$$\Big|\nabla K\Big(\frac{y}{|y|^2} - z_i\Big)\Big| \leq \frac{C_0}{1 + \big|\frac{y}{|y|^2} - z_i\big|^2} \leq \frac{C_0}{1 + \big|\frac{1}{|y|} - i^\alpha R\big|^2} \tag{A.8}$$

where in the last inequality we used $z_i = i^\alpha z_0$ with $|z_0| = R$ and the triangle inequality $\left|\frac{y}{|y|^2} - z_i\right| \geq \left|\frac{1}{|y|} - i^\alpha R\right|$. Then, by (A.8) and our choice $a_i = i^{-\beta}$, we have for $i \geq N$ ($N \in \mathbb{N}$ is arbitrarily large)

$$\sum_{i\geq N} |a_i| \left|\nabla\Big(K\Big(\frac{y}{|y|^2} - z_i\Big)\Big)\right| \leq C|y|^{-2} \sum_{i\geq N} i^{-\beta} \cdot \frac{1}{1 + \left|\frac{1}{|y|} - i^\alpha R\right|^2}. \tag{A.9}$$

Let us set

$$S(N, y) = \sum_{i\geq N} i^{-\beta} \cdot \frac{1}{1 + \left|\frac{1}{|y|} - i^\alpha R\right|^2}.$$

To obtain an uniform estimate of $S(N, y)$ for $|y| \leq 1$, we decompose the sum into two pieces: (i) $|y| \leq \frac{1}{2N^\alpha R}$ and (ii) $|y| > \frac{1}{2N^\alpha R}$. For (i), we have

$$\begin{aligned} S(N, y) &= \sum_{N\leq i\leq (2|y|R)^{-\frac{1}{\alpha}}} i^{-\beta} \cdot \frac{1}{1 + \left|\frac{1}{|y|} - i^\alpha R\right|^2} + \sum_{i>(2|y|R)^{-\frac{1}{\alpha}}} i^{-\beta} \cdot \frac{1}{1 + \left|\frac{1}{|y|} - i^\alpha R\right|^2} \\ &\leq C \sum_{i\geq N} \frac{|y|^2}{i^\beta} + \sum_{i>(2|y|R)^{-\frac{1}{\alpha}}} \frac{1}{i^\beta} \leq C|y|^2 N^{1-\beta} + C|y|^{\frac{\beta-1}{\alpha}}. \end{aligned} \tag{A.10}$$

And, for (ii), we have

$$S(N, y) \leq \sum_{i\geq N} \frac{1}{i^\beta} \leq CN^{1-\beta} \tag{A.11}$$

Therefore, by additionally requiring $\beta > 2\alpha + 1$, we can deduce for the case (i)

$$|y|^{-2} S(N, y) \leq CN^{1-\beta} + C|y|^{\frac{\beta-1}{\alpha}-2} \leq CN^{1-\beta} + CN^{1-\beta+2\alpha} \leq CN^{1-\beta+2\alpha},$$

and for the case (ii)

$$|y|^{-2} S(N, y) \leq C|y|^{-2} N^{1-\beta} \leq CN^{1-\beta+2\alpha},$$

where N is considered arbitrarily large. Thus, the estimates in (A.10) and (A.11) imply that

$$\sup_{|y|\leq 1} |y|^{-2} S(N, y) = O(N^{1-\beta+2\alpha}) \quad \text{as } N \to +\infty.$$

And hence, by (A.9), the series defining $\nabla\big(\tilde{K}(y/|y|^2)\big)$ converges uniformly on $|y| \leq 1$. This suggests that $\tilde{K}(y/|y|^2)$ can be extended to $y = 0$ in the class of C^1.

The second derivatives can be estimated in a similar manner. At this stage, instead of (5.1), we need the following improved estimates

$$|\nabla K(x)| \le C_0(1+|x|^2)^{-\frac{3}{2}} \quad \text{and} \quad |\nabla^2 K(x)| \le C_0(1+|x|^2)^{-2} \tag{A.12}$$

by the choice of p_0. In fact, we have

$$\nabla K(x) = \nabla H(\pi_{p_0}^{-1}(x))[\nabla \pi_{p_0}^{-1}(x)],$$
$$\nabla^2 K(x) = \nabla^2 H(\pi_{p_0}^{-1}(x))[\nabla \pi_{p_0}^{-1}(x), \nabla \pi_{p_0}^{-1}(x)] + \nabla H(\pi_{p_0}^{-1})[\nabla^2 \pi_{p_0}^{-1}(x)]$$

and, as in the proof of Lemma 5.1,

$$|\nabla H(\pi_{p_0}^{-1}(x))| = |\nabla H(\pi_{p_0}^{-1}(x)) - \nabla H(p_0)| \le C \max_{S^m} |\nabla^2 H| \cdot |\pi_{p_0}^{-1}(x) - p_0| \le C(1+|x|^2)^{-\frac{1}{2}}.$$

These, together with the facts

$$|\nabla \pi_{p_0}^{-1}(x)| \le C(1+|x|^2)^{-1} \quad \text{and} \quad |\nabla^2 \pi_{p_0}^{-1}(x)| \le C(1+|x|^2)^{-\frac{3}{2}},$$

we obtain (A.12).

Now, by using the estimate

$$\begin{aligned}&\Big|\nabla^2 K\Big(\frac{y}{|y|^2} - z_i\Big)\Big| |y|^{-4} + \Big|\nabla K\Big(\frac{y}{|y|^2} - z_i\Big)\Big| |y|^{-3} \\ &\qquad \le \frac{C}{|y|^4} \cdot \frac{1}{\big(1+\big|\frac{1}{|y|} - i^\alpha R\big|^2\big)^2} + \frac{C}{|y|^3} \cdot \frac{1}{\big(1+\big|\frac{1}{|y|} - i^\alpha R\big|^2\big)^{\frac{3}{2}}},\end{aligned}$$

we find

$$\begin{aligned}\sum_{i \ge N} |a_i| \Big|\nabla^2 \Big(K\Big(\frac{y}{|y|^2} - z_i\Big)\Big)\Big| &\le C|y|^{-4} \sum_{i \ge N} \frac{1}{i^\beta} \cdot \frac{1}{\big(1+\big|\frac{1}{|y|} - i^\alpha R\big|^2\big)^2} \\ &\quad + C|y|^{-3} \sum_{i \ge N} \frac{1}{i^\beta} \cdot \frac{1}{\big(1+\big|\frac{1}{|y|} - i^\alpha R\big|^2\big)^{\frac{3}{2}}},\end{aligned} \tag{A.13}$$

where N is arbitrarily large as before. Let us set

$$\tilde{S}_1(N,y) = \sum_{i \ge N} \frac{1}{i^\beta} \cdot \frac{1}{\big(1+\big|\frac{1}{|y|} - i^\alpha R\big|^2\big)^2} \quad \text{and} \quad \tilde{S}_2(N,y) = \sum_{i \ge N} \frac{1}{i^\beta} \cdot \frac{1}{\big(1+\big|\frac{1}{|y|} - i^\alpha R\big|^2\big)^{\frac{3}{2}}}$$

then, by performing the same arguments in (A.10) and (A.11), we soon get

$$|y|^{-4}\tilde{S}_1(N,y) = O(N^{1-\beta+4\alpha}) \quad \text{and} \quad |y|^{-3}\tilde{S}_2(N,y) = O(N^{1-\beta+4\alpha}) \quad \text{as } N \to +\infty$$

provided that $\beta > 4\alpha + 1$. Thus, in this case, the series defining $\nabla^2(\tilde{K}(y/|y|^2))$ converges uniformly on $|y| \le 1$. This proves that $\tilde{K}(y/|y|^2)$ can be extended to $y = 0$ in the class of C^2, when $\beta > 4\alpha + 1$.

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