J. Evol. Equ. (2024)24:44

https://doi.org/10.1007/s00028-024-00969-4

Journal of Evolution Equations

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Fujita exponent for the global-in-time solutions to a semilinear heat equation with non-homogeneous weights

TATSUKI KAWAKAMI, YANNICK SIRE AND JIAYI NIKKI WANG

Abstract. We consider a non-homogeneous parabolic equation with degenerate coefficients of the form $u_t - L_\omega u = u^p$, where $L_\omega = \omega^{-1}\mathrm{div}(\omega\nabla)$. This paper establishes the existence/non-existence of global-in-time mild solutions based on a critical exponent, known as the Fujita exponent. Similar topics for a semilinear heat equation with degenerate coefficients are treated in Fujishima (Calc Var Partial Differ Equ 58:25, 2019). They considered an equation $u_t - \mathrm{div}(\omega\nabla u) = u^p$, which is not self-adjoint, with two types of homogeneous weights: $\omega(x) = |x_1|^a$ and $\omega(x) = |x|^b$ where $a, b > 0$. In this paper we consider the case of a self-adjoint operator, and extend to more general weights that meet certain restrictions such as being in the Muckenhoupt class A_2, non-decreasing, and where the limits $\alpha := \lim_{|x'|\to\infty}(\log\omega(x))/(\log|x'|)$ and $\beta := \lim_{|x'|\to 0}(\log\omega(x))/(\log|x'|)$ exist, where $x' = (x_1, \dots, x_n)$ and $1 \le n \le N$. The main result establishes that the Fujita exponent is given by $p_F = 1 + 2/(N + \alpha)$. This means that the asymptotic behavior of the weight at infinity affects global existence of solutions and the one at the origin does not.

Contents

1. Introduction

We consider the problem

$$\begin{cases} \partial_t u - L_\omega u = u^p, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x) \ge 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\partial_t := \partial/\partial t$, $L_\omega := \omega^{-1}\mathrm{div}(\omega\nabla)$, $N \ge 1$, and $p > 1$. Here the weight function $\omega(x) := \omega(x')$ satisfies the following conditions (A1)–(A3) for $x' \in \mathbb{R}^n$ with $n \in [1, N]$:

Published online: 11 May 2024

(A1) $\omega(x)$ belongs to the class A_2 of Muckenhoupt functions, i.e.

$$c_0 := \sup_{B\subset\mathbb{R}^N} \left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{-1}(x)\,dx\right) < \infty$$

holds for any ball $B \subset \mathbb{R}^N$ (see [22]).

(A2) The limits of $\frac{\log\omega(x)}{\log|x'|}$ exist $|x'| \to \infty$ and $|x'| \to 0$, respectively. Denote the limit by

$$\alpha := \lim_{|x'|\to\infty} \frac{\log\omega(x)}{\log|x'|}, \qquad \beta := \lim_{|x'|\to 0} \frac{\log\omega(x)}{\log|x'|}.$$

(A3) $\omega(x)$ is monotonically non-decreasing with respect to the distance from origin to $x^* := (x', 0, \dots, 0)$, i.e. if $|x'| \geq |y'|$, then $\omega(x) \geq \omega(y)$ for any $x, y \in \mathbb{R}^N$.

It follows from (A2) that, for any $\ell > 0$, there exist $M > 0$ and $0 < m < 1$ such that

$$|x'|^{\alpha-\ell} < \omega(x) < |x'|^{\alpha+\ell} \tag{1.2}$$

for all $|x'| > M > 1$, and

$$|x'|^{\beta+\ell} < \omega(x) < |x'|^{\beta-\ell} \tag{1.3}$$

for all $|x'| < m < 1$. Furthermore, (A3) implies that

$$\alpha, \beta \geq 0.$$

Notice that

$$\alpha - \beta = \lim_{|x'|\to\infty}\left(\frac{\log\omega(x)}{\log|x'|} - \frac{\log\omega(1/x)}{\log(1/|x'|)}\right) = \lim_{|x'|\to\infty} \frac{\log(\omega(x)\omega(1/x))}{\log|x'|} \geq 0$$

if and only if $\lim_{|x'|\to\infty}\omega(x)\omega(1/x) \geq 1$. So the relationship between α and β is undetermined. Here we provide some examples of weight function in our mind that satisfy the above set-up (see [19]*Sect. 2). A rather large class of inhomogeneities ω is covered by our assumptions:

1. $\omega(x) = |x_1|^a$ with $a \in [0, 1)$ and $\omega(x) = |x|^b$ with $b \in [0, N)$ are homogeneous examples, with $n = 1$ and $n = N$ respectively. Any small enough non-homogeneous perturbation of those are also satisfying our assumptions.
2. $\omega(x)$ is a polynomial of $|x'|$ with the highest degree within the interval $[0, n)$, and $\omega(x) \geq 0$ a.e. in $\mathbb{R}^N$, like, e.g., $\omega(x) = 2|x'|^2 + |x'|$.

More interestingly, it is important to note that no continuity assumption is required for the weight function $\omega(x)$. We can allow randomness by defining a weight function based on a measurable function θ defined on the unit sphere $\mathbb{S}^N$. Let $0 < c_0 \leq \theta < c_0^{-1}$. The weight function on $\mathbb{R}^N \setminus \{0\}$ is given by $\omega(x) = |x|^a \cdot \theta(x/|x|)$.

The main objective of the present paper is to establish a theory for the existence of global-in-time mild solutions to problem (1.1). In particular, we seek to determine a critical exponent, known as the Fujita exponent, which plays a pivotal role in determining the existence/non-existence of global-in-time solutions. The concept of the Fujita exponent was initially introduced by Fujita [9] in 1966. Beginning with this classical paper by Fujita, critical exponents for the existence of global-in-time solutions (not only positive ones but also sign-changing ones) were established for many classes of evolution problems, which include degenerate parabolic equations, fractional diffusion equations and so on. It seems almost impossible to make complete list of this topics. So we only refer a part of them for instance [8,10,11,16–18,23–25,31] and references therein. (See also [26], which includes a nice survey for the semilinear parabolic equation.) Among others, in [10] Fujishima and the first two authors consider the problem

$$\begin{cases} \partial_t u - \mathrm{div}(\omega(x)\nabla u) = u^p, \ x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, \qquad x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

where the coefficient ω is either $\omega(x) = |x_1|^a$ with $a \in [0, \min\{1, 2/N\})$ or $\omega(x) = |x|^b$ with $b \in [0, 1)$. They showed that the Fujita exponent of problem (1.4) is $p_F := 1 + \frac{2-\alpha}{N}$ with $\alpha = \{a, b\}$, namely, if $p \leq p_F$, then problem (1.4) has no nontrivial global-in-time solutions, and if $p > p_F$, then there exists a global-in-time (mild) solutions to (1.4).

One of the distinctive points of (1.4) is that the operator $\mathrm{div}(\omega\nabla)$ is not self-adjoint on $L^2(\mathbb{R}^N, \omega dx)$. At the end of [10, Section 1], the authors suggested one can run similar estimates and exhibit Fujita exponent for (1.1), which includes a self-adjoint operator L_ω. The aim of the paper is to consider the case of a self-adjoint operator with more general weights, which satisfies conditions (A1)–(A3), and to determine the Fujita exponent of problem (1.1). Here we recall that, for the case of $\omega(x) = |x_1|^a$, problem (1.1) and (1.4) are related to the fractional parabolic equations $(\partial_t - \Delta)^s u = u^p$ (see, i.e., [1]) and the nonlocal problem with fractional Laplacian $\partial_t u + (-\Delta)^s u = u^p$ with $s \in (0, 1)$ by Caffarelli–Silvestre extension [3], respectively.

We introduce the definition of a solution to (1.1). Let $\Gamma = \Gamma(x, y, t)$ be the fundamental solution of

$$\partial_t v - L_\omega v = 0, \ x \in \mathbb{R}^N, \ t > 0,$$

with a pole at $(y, 0)$. For any measurable function f, put

$$[S(t)f](x) := \int_{\mathbb{R}^N} \Gamma(x, y, t) f(y)\, \omega(y)\, dy, \qquad x \in \mathbb{R}^N, \quad t > 0. \tag{1.5}$$

Then, we define the mild solution of equation (1.1) as follows.

Definition 1.1. Let u_0 be a nonnegative measurable function in $\mathbb{R}^N$. Let $T \in (0, \infty]$ and u be a nonnegative measurable function in $\mathbb{R}^N \times (0, T)$ such that $u \in L^\infty(0, T; L^\infty(\mathbb{R}^N))$. Then u is a **mild solution** of (1.1) in $\mathbb{R}^N \times (0, T)$ if u satisfies

$$u(x,t) = [S(t)u_0](x) + \int_0^t [S(t-s)u(s)^p](x)\, ds < \infty \tag{1.6}$$

for almost all $x \in \mathbb{R}^N$ and $t \in (0, T)$. In particular, u is called a global-in-time mild solution of (1.1) if u is a solution of (1.1) in $\mathbb{R}^N \times (0, \infty)$.

When considering parabolic problems, a natural notion of solution is given by *mild solutions*, which incorporate the fundamental solution of the operator. A fundamental instrument in our argument relies on estimating the fundamental solution of the operator $\partial_t - \omega^{-1}\mathrm{div}(\omega\nabla)$. However, the fundamental solution of this parabolic PDE does not have explicit formula as the operator is not translation-invariant. On the other hand, by the assumption (A1), the coefficient $\omega(x)$ belongs to the class A_2, and it is easy to check that L_ω is a self-adjoint operator in $L^2(\mathbb{R}^N, \omega dx)$. Therefore, even though Γ does not have explicit formula, Proposition 2.1 provides us with some useful properties of the fundamental solution. In this context, the assumption of A_2 weights is particularly significant.

Remark 1.1. (i) In studying equations with coefficients, especially for degenerate coefficients, it is useful to look at the problem from a geometric perspective. By treating the equation as a diffusion operator on a weighted manifold, we can invoke the spectral theory and functional calculus of self-adjoint operator, and hence construct the associated heat semigroup (see, e.g., [13]) and investigate their properties. We refer the reader to the lecture notes by F. Baudoin [2] for a very nice survey on such an approach.
(ii) In the realm of degenerate equations, numerous studies have focused on the regularity of both nonlinear and linear equations. We refer, e.g., to the references [21,27–30] where several variations of the model under consideration here are considered.

Before stating our main results we introduce some notations. For any $x \in \mathbb{R}^N$ and $R > 0$, we put $B_R(x) := \{y \in \mathbb{R}^N : |x - y| < R\}$. For any $1 \le r \le \infty$, we define the weighted Lebesgue space $L^r_\omega := L^r(\mathbb{R}^N, \omega(x)dx)$ by

$$L^r_\omega := \{f : \ f \text{ is measurable in } \mathbb{R}^N, \quad \|f\|_{L^r_\omega} < \infty\},$$

where

$$\|f\|_{L^r_\omega} := \begin{cases} \left(\displaystyle\int_{\mathbb{R}^N} |f(x)|^r \,\omega(x)\,dx\right)^{\frac{1}{r}} & \text{if} \quad 1 \le r < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^N} |f(x)| & \text{if} \quad r = \infty, \end{cases} \tag{1.7}$$

namely, $\|f\|_{L^\infty_\omega} = \|f\|_{L^\infty}$. Then the weighted Lebesgue space L^r_ω is a Banach space. Let $\mathbb{1}\{\Omega\}$ be the characteristic function of the set Ω. For any measurable function f in $\mathbb{R}^N$,

$$\mu_f(\lambda) := \int_{\mathbb{R}^N} \mathbb{1}\,\{x : |f(x)| > \lambda\}\ \omega(x)\,dx, \qquad \lambda \ge 0 \tag{1.8}$$

is the distribution function of f, and we define the non-increasing rearrangement of f by

$$f^*(s) := \inf\{\lambda > 0 : \mu_f(\lambda) \le s\}, \qquad s \ge 0.$$

Then, for any $1 \le r \le \infty$ and $1 \le \sigma \le \infty$, we define the weighted Lorentz space $L^{r,\sigma}_\omega := L^{r,\sigma}(\mathbb{R}^N, \omega(x)dx)$ by

$$L^{r,\sigma}_\omega := \{f : f \text{ is measurable in } \mathbb{R}^N, \ \|f\|_{L^{r,\sigma}_\omega} < \infty\},$$

where

$$\|f\|_{L^{r,\sigma}_\omega} := \begin{cases} \left(\int_0^\infty \left[s^{\frac{1}{r}} f^*(s)\right]^\sigma \frac{ds}{s}\right)^{\frac{1}{\sigma}} & \text{if } \quad 1 \le \sigma < \infty, \\ \sup_{s>0} s^{\frac{1}{r}} f^*(s) & \text{if } \quad \sigma = \infty. \end{cases}$$

The Lorentz $L^{r,\sigma}_\omega$ is a Banach space and the following holds (see, e.g., [12,14,32]):

(P1) $L^{r,r}_\omega = L^r_\omega$ if $1 < r \le \infty$ and $L^{1,\infty}_\omega = L^1_\omega$:

(P2) $L^{r,\sigma_1}_\omega \subset L^{r,\sigma_2}_\omega$ if $1 \le r \le \infty$ and $1 \le \sigma_1 \le \sigma_2 \le \infty$;

(P3) (The interpolation) Let $1 \le r_0 \le r \le r_1 \le \infty$ be such that

$$\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} \qquad \text{for} \quad \theta \in [0, 1].$$

Then it holds that

$$\|f\|_{L^{r,\infty}_\omega} \le \|f\|^{1-\theta}_{L^{r_0,\infty}_\omega} \|f\|^{\theta}_{L^{r_1,\infty}_\omega}, \qquad f \in L^{r_0,\infty}_\omega \cap L^{r_1 \infty}_\omega; \tag{1.9}$$

(P4) (Hölder inequality) Let $1 \le r_1 \le \infty$ and r_2 be the Hölder conjugate number of r_1, namely $1/r_1 + 1/r_2 = 1$. Then it holds that

$$\|fg\|_{L^1_\omega} \le \|f\|_{L^{r_1,1}_\omega} \|g\|_{L^{r_2,\infty}_\omega}, \qquad f \in L^{r_1,1}_\omega, \quad g \in L^{r_2,\infty}_\omega. \tag{1.10}$$

Now we state the main results of this paper. Put

$$p_F := 1 + \frac{2}{N+\alpha}. \tag{1.11}$$

Then we have the following two theorems.

Theorem 1.1. *Assume* (A1)–(A3). *Let $1 < p < p_F$. Then (1.1) has no nontrivial global-in-time solutions.*

Theorem 1.2. *Assume* (A1)–(A3). *Let $p > p_F$. Choose ϵ small enough such that*

$$r_* := \frac{N+\alpha-\epsilon}{2}(p-1) > 1, \tag{1.12}$$

then the following holds:

(i) *There exists a positive constant δ such that, for any $u_0 \in L^\infty \cap L^{r_*,\infty}_\omega$ with*

$$\max(\|u_0\|_{L^\infty}, \|u_0\|_{L^{r_*,\infty}_\omega}) < \delta, \tag{1.13}$$

a unique global-in-time solution u of (1.1) exists and it satisfies

$$\sup_{t>0} (1+t)^{\frac{N+\alpha-\epsilon}{2}(\frac{1}{r_*}-\frac{1}{q})} \|u(t)\|_{L^{q,\infty}_\omega} < \infty, \qquad r_* \le q \le \infty. \tag{1.14}$$

(ii) *Let* $1 \le r \le r_*$. *There exists a positive constant* δ *such that, for any* $u_0 \in L^\infty \cap L^r_\omega$ *with*

$$\max\{\|u_0\|_{L^\infty}, \|u_0\|_{L^r_\omega}\} < \delta, \tag{1.15}$$

a unique global-in-time solution u of (1.1) exists and it satisfies

$$\sup_{t>0} (1+t)^{\frac{N+\alpha-\epsilon}{2}(\frac{1}{r}-\frac{1}{q})} \|u(t)\|_{L^q_\omega} < \infty, \qquad r \le q \le \infty. \tag{1.16}$$

By Theorems 1.1 and 1.2, there does not exist nontrivial global solution if $p \in (1, p_F)$, while there exists unique nontrivial global solution under certain condition of initial data u_0 and $p > p_F$. This implies that the exponent p_F given by (1.11) is the Fujita exponent of problem (1.1).

We would like to emphasize that the critical case $p = p_F$ is left open. Our conjecture is that no nontrivial global solution exists in this critical scenario. Specifically, for $p = p_F$, the strategy is to argue by contradiction to Lemma 3.1 utilizing the already provided global-in-time solution u, achieved by selecting an appropriate initial data $u(x, T)$ for sufficiently large T. In the presence of homogeneous weights such as $\omega(x) = |x_1|^a$ and $\omega(x) = |x|^b$, we observe that $\int_{|x|\le\sqrt{t}} u(x, t+1)w(x)dx$ blows up over time. Consequently, for any given constant C^*, there exists a $T > 0$ such that $U(x) = u(x, T)$ satisfies the contradicting condition $t^{\frac{1}{p-1}} \|S(t)U\|_\infty > C^*$. However, given the current weight assumptions (A1)–(A3), we are unable to achieve the desired blow-up phenomenon. This suggests the need for alternative methods and approaches in this context.

Corollary 1.1. *Assume* (A1)–(A3). *Let* $p > p_F$. *Choose* ϵ *small enough such that (1.12) holds. Then there exists a positive constant* δ *such that, if*

$$0 \le u_0(x) \le \frac{\delta}{1 + |x|^{\frac{2}{p-1}\frac{N+\alpha+\epsilon}{N+\alpha-\epsilon}}}, \qquad x \in \mathbb{R}^N, \tag{1.17}$$

then a unique global-in-time solution u of (1.1) exists and it satisfies (1.14).

Remark 1.2. (i) For $\omega \equiv 1$, it is well-known that the decay rate for initial data given by

$$0 \le u_0(x) \le \frac{\delta}{1 + |x|^{\frac{2}{p-1}}}, \qquad x \in \mathbb{R}^N,$$

at spatial infinity is optimal to obtain the global existence of solutions to (1.1) (see, e.g., [20]). For a more general weight $\omega(x)$ as we consider, if $u_0(x)$ satisfies (1.17), then it follows that $u_0 \in L^\infty \cap L^{r_*,\infty}_\omega$. As a direct consequence of Theorem 1.2 (i), a unique global-in-time solution u of (1.1) exists. However if $u_0(x) = O(|x|^{-2/(p-1)})$ as $|x| \to \infty$, then $u_0 \notin L^{r_*}_\omega$. This is a clear advantage in using Lorentz spaces in place of the classical L^p spaces.

(ii) Besides the existence/non-existence theory, A_p weights have significant applications in examining the regularity of degenerate parabolic equations and are ubiquitous in harmonic analysis. Chiarenza and Serapioni [4,5] emphasize the necessity of assuming that the weight belongs to the A_2 class in order to achieve L^2 continuity of the weak solutions. This is one of the reason for choosing $\omega(x)$ to be A_2 in (1.1). In order to upgrade the mild solution to a weak solution and further to a C^α solution, we need a spatial gradient bound, and then the time derivative bound for the fundamental solution. To be able to derive Hölder regularity invoking the results of Chiarenza and Serapioni, one could try to produce directly a weak solution. We leave this aspect to possible future work.

2. Preliminaries

We introduce certain properties that are very useful in checking the bounds of the fundamental solution. Then, we proceed to estimate the heat semigroup, which plays a key role in the definition of the mild solutions as expressed in (1.6). Finally, we apply upper estimates of the heat semigroup in the weighted L^p space and the weighted Lorentz space. In what follows, by the letter C we denote generic positive constants (independent of x and t) and they may have different values also within the same line.

2.1. Estimates for the fundamental solution Γ

In order to apply geometric inequalities from the heat semigroup technique, one has to assume the coefficient $\omega(x)$ is an A_2 weight in the sense of Muckenhoupt class (our assumption (A1)). Consequently, it follows that $\omega(x)$ satisfies the volume doubling and reverse doubling properties, as established in the work of [15]. Here we say $\omega(x)$ satisfies the doubling and reverse doubling conditions if there exist positive constants c_1 and c_2 such that

$$c_1 w_t(x) \leq w_{2t}(x) \leq c_2 w_t(x) \tag{2.1}$$

for all $x \in \mathbb{R}^N$ and $t > 0$, where $w_t(x) := \int_{B_{\sqrt{t}}(x)} \omega(y)\, dy$. Under the assumptions that ω is an A_2 weight and L_ω is self-adjoint, we can establish the following properties (see [2,6,7]).

Proposition 2.1. *The fundamental solution $\Gamma(x, y, t)$ of (1.1) has the following properties:*

(1) *Symmetry of heat kernel and stochastic completeness of heat semigroup:*

$$\int_{\mathbb{R}^N} \Gamma(x, y, t)\omega(x)\, dx = \int_{\mathbb{R}^N} \Gamma(x, y, t)\omega(y)\, dy = 1 \tag{2.2}$$

for all $x, y \in \mathbb{R}^N$ and $t > 0$.

(2) *Chapman-Kolmogorov relation:*

$$\Gamma(x, y, t) = \int_{\mathbb{R}^N} \Gamma(x, \xi, t-s)\Gamma(\xi, y, s)\,\omega(\xi)\,d\xi \tag{2.3}$$

for all $x, y \in \mathbb{R}^N$ *and* $t > s > 0$.

(3) *Li-Yau inequality: There exist positive constants* c_* *and* C_* *depending only on* N *and* c_0 *such that*

$$\begin{aligned} \frac{c_*^{-1}}{\sqrt{w_t(x)}\sqrt{w_t(y)}} \exp\left(-c_*\frac{|x-y|^2}{t}\right) &\le \Gamma(x, y, t) \\ \le \frac{C_*^{-1}}{\sqrt{w_t(x)}\sqrt{w_t(y)}} \exp\left(-C_*\frac{|x-y|^2}{t}\right) & \end{aligned} \tag{2.4}$$

for all $x, y \in \mathbb{R}^N$ *and* $t > 0$.

The properties in Proposition 2.1 allow us to estimate the fundamental solution if we are able to estimate $w_t(x)$. We state the following lemma on upper and lower estimates of $w_t(x)$ based on our choice of $\omega(x)$ and the doubling and reverse doubling condition.

Lemma 2.1. *Let* $T > 0$*, and let* $\epsilon > 0$ *be an arbitrary sufficiently small positive constant, Then, there exist constants* C_1 *and* C_2 *depending only on* N *and* α *such that*

$$C_1 t^{\frac{N+\alpha-\epsilon}{2}} < w_t(x) < C_2 \begin{cases} |x|^{\alpha+\epsilon}\, t^{\frac{N}{2}} & \text{if} \quad 0 < t \le |x|^2, \\ t^{\frac{N+\alpha+\epsilon}{2}} & \text{if} \quad |x|^2 \le t < \infty, \end{cases} \tag{2.5}$$

for all $x \in \mathbb{R}^N$ *and* $t \ge T$*. Furthermore, there exists a constant* C_3 *depending only on* N *and* β *such that*

$$w_t(x) > C_3 t^{\frac{N+\beta+\epsilon}{2}} \tag{2.6}$$

for all $x \in \mathbb{R}^N$ *and* $0 < t < T$.

Proof. Let $\epsilon > 0$ be an arbitrary sufficiently small positive constant. Then, by (1.2) and (1.3), we can fix constants $M > 1$ and $m < 1$. It is sufficient to show (2.5) for $t > \lambda M^2$ and (2.6) for $0 < t < m^2$, where λ is a fixed large number to be chosen later. Once these are proved, we can apply the doubling property (2.1) to scale the domain. Indeed, if $T < \lambda M^2$, for $T \le t \le \lambda M^2$, we scale $w_t(x)$ using

$$\frac{1}{c_2^k} w_{2^k t}(x) \le w_t(x) \le \frac{1}{c_1^k} w_{2^k t}(x)$$

where k is chosen to be large enough such that $2^k t > \lambda M^2$. Similarly, if $T > m^2$, for $m^2 \le t \le T$, we scale $w_t(x)$ using

$$w_t(x) \ge c_1^k w_{\frac{t}{2^k}}(x)$$

where k is chosen to be large enough such that $t < 2^k m^2$.

We first prove (2.5) for $t > \lambda M^2$. Since $\omega(x)$ is monotonically non-decreasing functions with respect to the distance from the origin to $x^* = (x', 0, \dots, 0)$, for any $x \in \mathbb{R}^N$ and $t > 0$, we see that

$$w_t(x) \geq w_t(0). \tag{2.7}$$

In fact, since $\omega(x) = \omega(x')$ for $x' \in \mathbb{R}^n$ with $n \in [1, N]$, it holds that $\omega_t(x) = \omega_t(x_*)$ for $x_* = (x', 0) \in \mathbb{R}^N$. Furthermore, we put $B_1 := B_{\sqrt{t}}(x_*) \setminus B_{\sqrt{t}}(0)$ and $B_2 := B_{\sqrt{t}}(0) \setminus B_{\sqrt{t}}(x_*)$. Then, it follows from the monotonicity of $\omega(y)$ that $\sup_{y \in B_2} \omega(y) \leq \inf_{y \in B_1} \omega(y)$. This implies (2.7). Combining with (1.2) and (2.7), we have

$$\begin{aligned}
w_t(x) \geq w_t(0) &= \int_{B_{\sqrt{t}}(0)} \omega(y)\, dy \geq \int_{B_{\sqrt{t}}(0) \setminus B_M(0)} |y'|^{\alpha - \epsilon}\, dy \\
&= \int_{\tilde{B}_{\sqrt{t}}(0)} \left(\int_{|y''|_{N-n} < \sqrt{t - |y'|^2}} dy'' \right) |y'|^{\alpha-\epsilon}\, dy' \\
&\quad - \int_{\tilde{B}_M(0)} \left(\int_{|y''|_{N-n} < \sqrt{M^2 - |y'|^2}} dy'' \right) |y'|^{\alpha-\epsilon}\, dy' \\
&= d_{N-n} \left(\int_{\tilde{B}_{\sqrt{t}}(0)} \left(t - |y'|^2\right)^{\frac{N-n}{2}} |y'|^{\alpha-\epsilon}\, dy' - \int_{\tilde{B}_M(0)} \left(M^2 - |y'|^2\right)^{\frac{N-n}{2}} |y'|^{\alpha-\epsilon}\, dy' \right) \\
&= d_{N-n} |\partial(\tilde{B}_1(0))| \left(\int_0^{\sqrt{t}} \left(t - r^2\right)^{\frac{N-n}{2}} r^{n+\alpha-\epsilon-1}\, dr \right. \\
&\quad \left. - \int_0^M \left(M^2 - r^2\right)^{\frac{N-n}{2}} r^{n+\alpha-\epsilon-1}\, dr \right) \\
&= \frac{d_{N-n} |\partial(\tilde{B}_1(0))|}{2} \int_0^1 (1-r)^{\frac{N-n}{2}} r^{\frac{n+\alpha-\epsilon}{2} - 1}\, dr \left(t^{\frac{N+\alpha-\epsilon}{2}} - M^{N+\alpha-\epsilon} \right) \\
&= \frac{d_{N-n} |\partial(\tilde{B}_1(0))|}{2} B\left(\frac{n+\alpha-\epsilon}{2}, \frac{N-n+2}{2} \right) \left(t^{\frac{N+\alpha-\epsilon}{2}} - M^{N+\alpha-\epsilon} \right) \geq C_1 t^{\frac{N+\alpha-\epsilon}{2}}
\end{aligned} \tag{2.8}$$

for all x sin $\mathbb{R}^N$ and $t > 2M^2$, where $y = (y', y'') \in \mathbb{R}^N$, $|\cdot|_{N-n}$ denotes the usual Euclidian norm in $\mathbb{R}^{N-n}$, $\tilde{B}_R(0)$ denotes the ball in $\mathbb{R}^n$ with radius R centered at the origin, $\partial(\tilde{B}_1(0))$ denotes the boundary of ball in $\mathbb{R}^n$ with radius 1 centered at the origin, d_m denotes the volume of the unit ball in $\mathbb{R}^m$, and $B(\cdot, \cdot)$ denotes the Beta function.

On the other hand, since $|y'| = |y^*| \leq |y| \leq |x| + \sqrt{t}$ for all $y \in B_{\sqrt{t}}(x)$, it follows from (1.2) that

$$\begin{aligned}
w_t(x) &= \int_{B_{\sqrt{t}}(x) \setminus B_M(0)} \omega(y)\, dy + \int_{B_{\sqrt{t}}(x) \cap B_M(0)} \omega(y)\, dy \\
&< \int_{B_{\sqrt{t}}(x) \setminus B_M(0)} |y'|^{\alpha+\epsilon}\, dy + \int_{B_M(0)} \omega(y)\, dy \\
&\leq \int_{B_{\sqrt{t}}(x)} (|x| + \sqrt{t})^{\alpha+\epsilon}\, dy + d_N M^N \max_{x \in B_M(0)} \omega(x) \\
&= d_N \left(t^{\frac{N}{2}} (|x| + \sqrt{t})^{\alpha+\epsilon} + M^N \max_{x \in B_M(0)} \omega(x) \right)
\end{aligned}$$

$$\le C_2 \begin{cases} |x|^{\alpha+\epsilon} t^{\frac{N}{2}} & \text{if} \quad 0 < \sqrt{t} \le |x|, \\ t^{\frac{N+\alpha+\epsilon}{2}} & \text{if} \quad |x| \le \sqrt{t} < \infty, \end{cases}$$

for all $x \in \mathbb{R}^N$ and $t > \lambda M^2$, where λ is large enough such that $\lambda^{\frac{N}{2}} > \max_{x \in B_M(0)} \omega(x)$. Thus (2.5) holds when $t \ge T$.

Next we prove (2.5) for $0 < t < m^2$. Similar to (2.8), we have

$$\begin{aligned} w_t(x) \ge w_t(0) &= \int_{B_{\sqrt{t}}(0)} \omega(y)\,dy > \int_{B_{\sqrt{t}}(0)} |y'|^{\beta+\epsilon}\,dy \\ &= \int_{B_{\sqrt{t}}(0)} \left(\int_{|y''|_{N-n} < \sqrt{t-|y'|^2}} dy'' \right) |y'|^{\beta+\epsilon}\,dy' \\ &= d_{N-n} \int_{B_{\sqrt{t}}(0)} \left(t - |y'|^2 \right)^{\frac{N-n}{2}} |y'|^{\beta+\epsilon}\,dy' \\ &= C t^{\frac{N+\beta+\epsilon}{2}} \int_0^1 (1-r)^{\frac{N-n}{2}} r^{\frac{n+\beta+\epsilon}{2}-1}\,dr \ge C_3 t^{\frac{N+\beta+\epsilon}{2}} \end{aligned}$$

for all $x \in \mathbb{R}^N$ and $0 < t < m^2$. Thus (2.5) holds when $0 < t < T$, and Lemma 2.1 follows. □

Applying Lemma 2.1 with the Li-Yau inequality (2.4), we see that

$$\begin{aligned} & D_1^{-1} \min\left\{ |x|^{-\frac{\alpha+\epsilon}{2}} t^{-\frac{N}{4}}, t^{-\frac{N+\alpha+\epsilon}{4}} \right\} \min\left\{ |y|^{-\frac{\alpha+\epsilon}{2}} t^{-\frac{N}{4}}, t^{-\frac{N+\alpha+\epsilon}{4}} \right\} \\ & \exp\left(-D_1 \frac{|x-y|^2}{t} \right) \\ & < \Gamma(x,y,t) < D_2^{-1} t^{-\frac{N+\alpha-\epsilon}{2}} \exp\left(-D_2 \frac{|x-y|^2}{t} \right) \end{aligned} \tag{2.9}$$

and

$$\Gamma(x,y,t) < D_3^{-1} t^{-\frac{N+\beta+\epsilon}{2}} \exp\left(-D_3 \frac{|x-y|^2}{t} \right) \tag{2.10}$$

for all $x, y \in \mathbb{R}^N$ and $T \le t < \infty$, where D_1 and D_2 are positive constants depending only on N, c_0, and α, and D_3 is a positive constant depending only on N, c_0, and β. By (2.9) and (2.10), we obtain

$$\Gamma(x,y,t) < \begin{cases} D_2^{-1} t^{-\frac{N+\alpha-\epsilon}{2}} & \text{if} \quad T \le t < \infty, \\ D_3^{-1} t^{-\frac{N+\beta+\epsilon}{2}} & \text{if} \quad 0 < t < T, \end{cases}$$

for all $x, y \in \mathbb{R}^N$. This together with (1.7) and (2.2) implies that

$$\begin{aligned} \|\Gamma(\cdot,y,t)\|_{L^r_\omega} &< \begin{cases} C_\alpha t^{-\frac{N+\alpha-\epsilon}{2}\left(1-\frac{1}{r}\right)} & \text{if} \quad T \le t < \infty, \\ C_\beta t^{-\frac{N+\beta+\epsilon}{2}\left(1-\frac{1}{r}\right)} & \text{if} \quad 0 < t < T, \end{cases} \\ \|\Gamma(x,\cdot,t)\|_{L^r_\omega} &< \begin{cases} C_\alpha t^{-\frac{N+\alpha-\epsilon}{2}\left(1-\frac{1}{r}\right)} & \text{if} \quad T \le t < \infty, \\ C_\beta t^{-\frac{N+\beta+\epsilon}{2}\left(1-\frac{1}{r}\right)} & \text{if} \quad 0 < t < T, \end{cases} \end{aligned} \tag{2.11}$$

for any $1 \le r \le \infty$ and all $x, y \in \mathbb{R}^N$, where C_α and C_β are constants depending only on N, c_0, and $\{\alpha, \beta\}$, respectively.

To establish the $L^{q,\infty}_\omega$–$L^{r,\infty}_\omega$ estimate for the heat semigroup, we rely on the utilization of (1.9) and (1.10). So, we introduce a lemma that estimate the fundamental solution in the weighted $L^{r,1}$ norm.

Lemma 2.2. *Let $T > 0$, and let $\epsilon > 0$ be an arbitrary sufficiently small positive constant. Assume* (A1)–(A3). *Then, for any $1 < r < \infty$, there exist positive constants $\tilde{C}_\alpha$ and $\tilde{C}_\beta$ depending only on r, N, c_0, and $\{\alpha, \beta\}$ respectively such that*

$$\|\Gamma(x, \cdot, t)\|_{L^{r,1}_\omega} < \begin{cases} \tilde{C}_\alpha t^{-\frac{N+\alpha-\epsilon}{2}\left(1-\frac{1}{r}\right)} & \text{if} \quad T \le t < \infty, \\ \tilde{C}_\beta t^{-\frac{N+\beta+\epsilon}{2}\left(1-\frac{1}{r}\right)} & \text{if} \quad 0 < t < T, \end{cases} \tag{2.12}$$

for $x \in \mathbb{R}^N$.

Proof. Since $L^r_\omega = L^{r,r}_\omega$ for $1 < r \le \infty$ (see (P1)), we have

$$\begin{aligned} \|\Gamma(x, \cdot, t)\|_{L^{r,1}_\omega} &= \left(\int_0^A + \int_A^\infty \right) s^{\frac{1}{r}-1} \Gamma^*(x, s, t)\, ds \\ &\le \sup_{0<s<A} \Gamma^*(x, s, t) \int_0^A s^{\frac{1}{r}-1}\, ds + \left(\int_A^\infty s^{\frac{q}{r}-q}\, ds \right)^{\frac{1}{q}} \left(\int_A^\infty (\Gamma^*(x, s, t))^{q'}\, ds \right)^{\frac{1}{q'}} \\ &\le C A^{\frac{1}{r}} \|\Gamma(x, \cdot, t)\|_{L^\infty} + C A^{\frac{1}{r}-1+\frac{1}{q}} \|\Gamma(x, \cdot, t)\|_{L^{q'}_\omega} \end{aligned}$$

for any $1 < r < \infty$, where q is a constant satisfying $q > r/(r-1)$ with $1/q + 1/q' = 1$ and C depends only on r and q.

For the case $T \le t < \infty$, we put $A = t^{\frac{N+\alpha-\epsilon}{2}}$. Combining with (2.11), we have

$$\begin{aligned} \|\Gamma(x, \cdot, t)\|_{L^{r,1}_\omega} &< C t^{\frac{N+\alpha-\epsilon}{2r}} t^{-\frac{N+\alpha-\epsilon}{2}} + C t^{\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{r}-1+\frac{1}{q}\right)} t^{-\frac{N+\alpha-\epsilon}{2}\left(1-\frac{1}{q'}\right)} \\ &\le C t^{-\frac{N+\alpha-\epsilon}{2}\left(1-\frac{1}{r}\right)} \end{aligned}$$

The proof for $0 < t < T$ follows the same by letting $A = t^{\frac{N+\beta+\epsilon}{2}}$. □

2.2. Estimates for $S(t)\varphi$

In this section, we give several estimates for $S(t)\varphi$ as a basis for the proof of the main theorems.

We first prove the following estimate, which is the lower bound for $S(t)\varphi$.

Lemma 2.3. *Assume the same conditions as in Lemma 2.2. Let $\varphi \in L^\infty$ be a nontrivial measurable function such that $\varphi \ge 0$ in $\mathbb{R}^N$. Then there exists a positive constant C depending only on N, c_0, and α such that*

$$[S(t)\varphi](x) > C^{-1} t^{-\frac{N+\alpha+\epsilon}{2}} \int_{|y| \le \sqrt{t}} \varphi(y)\, \omega(y)\, dy$$

for all $|x| \le \sqrt{t}$ and $t \ge T$.

Proof. Since $|x-y|^2 \le (|x|+|y|)^2 \le 4t$ for $|x|, |y| \le \sqrt{t}$, it follows from (2.9) that

$$\Gamma(x,y,t) > D_1^{-1} t^{-\frac{N+\alpha+\epsilon}{2}} \exp\left(-D_1 \frac{|x-y|^2}{t}\right) \ge D_1^{-1} e^{-4D_1} t^{-\frac{N+\alpha+\epsilon}{2}}$$

for all $t \ge T$, where D_1 is given in (2.9). Then, combining with (1.5), if $t \ge T$, then we have

$$[S(t)\varphi](x) \ge \int_{|y|\le\sqrt{t}} \Gamma(x,y,t)\varphi(y)\,\omega(y)\,dy > C^{-1} t^{-\frac{N+\alpha+\epsilon}{2}} \int_{|y|\le\sqrt{t}} \varphi(y)\,\omega(y)\,dy$$

for all $|x| \le \sqrt{t}$. □

Next we give L^q_ω–L^r_ω estimate and $L^{q,\infty}_\omega$–$L^{r,\infty}_\omega$ estimate for $S(t)\varphi$.

Lemma 2.4. *Assume the same conditions as in Lemma 2.2.*

(i) *For any $\varphi \in L^q_\omega$ and $1 \le q \le r \le \infty$, it holds that*

$$\|S(t)\varphi\|_{L^r_\omega} < \begin{cases} c_\alpha t^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|\varphi\|_{L^q_\omega} & \text{if} \quad T \le t < \infty, \\ c_\beta t^{-\frac{N+\beta+\epsilon}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|\varphi\|_{L^q_\omega} & \text{if} \quad 0 < t < T. \end{cases} \tag{2.13}$$

Here c_α and c_β depend only on N, c_0, and $\{\alpha, \beta\}$, respectively. In particular, it holds that

$$\|S(t)\varphi\|_{L^r_\omega} \le \|\varphi\|_{L^r_\omega}, \qquad t > 0. \tag{2.14}$$

(ii) *For any $\varphi \in L^{q,\infty}_\omega$ and $1 < q < \infty$, it holds that*

$$\|S(t)\varphi\|_{L^{r,\infty}_\omega} < \begin{cases} d_\alpha t^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|\varphi\|_{L^{q,\infty}_\omega} & \text{if} \quad T \le t < \infty, \\ d_\beta t^{-\frac{N+\beta+\epsilon}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|\varphi\|_{L^{q,\infty}_\omega} & \text{if} \quad 0 < t < T, \end{cases} \tag{2.15}$$

for any $q \le r \le \infty$. Here d_α and d_β depends only on q, N, c_0, and $\{\alpha, \beta\}$, respectively. In particular, In particular, it holds that

$$\|S(t)\varphi\|_{L^{q,\infty}_\omega} \le C_q \|\varphi\|_{L^{q,\infty}_\omega}, \qquad t > 0. \tag{2.16}$$

Here d_α, d_β, and C_q are bounded in $q \in (1+\varepsilon, \infty)$ for any fixed $\varepsilon > 0$ and $d_\alpha, d_\beta, C_q \to \infty$ as $q \to 1$.

Proof. It suffices to prove for $T \le t < \infty$ case, then $0 < t < T$ case follows the same argument.

We first prove the assertion (i). Let $T > 0$. Then, applying the Hölder inequality with (2.11), we have

$$\|S(t)\varphi\|_{L^\infty} \le \sup_{x\in\mathbb{R}^N} \|\Gamma(x,\cdot,t)\|_{L_\omega^{\frac{q}{q-1}}} \|\varphi\|_{L^q_\omega} < C(N,c_0,\alpha) t^{-\frac{N+\alpha-\epsilon}{2q}} \|\varphi\|_{L^q_\omega}, \qquad t \ge T,$$

for any $1 \le q \le \infty$. Furthermore, by (2.2) we apply the Jensen inequality and the Fubini theorem to obtain

$$\begin{aligned}\|S(t)\varphi\|_{L^q_\omega}^q &\le \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\Gamma(x,y,t)|\varphi(y)|\,\omega(y)\,dy\right)^q\omega(x)\,dx\\ &\le \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\Gamma(x,y,t)|\varphi(y)|^q\,\omega(y)\,dy\right)\omega(x)\,dx\\ &\le \int_{\mathbb{R}^N}|\varphi(y)|^q\left(\int_{\mathbb{R}^N}\Gamma(x,y,t)\,\omega(x)\,dx\right)\omega(y)\,dy=\|\varphi\|_{L^q_\omega}^q,\qquad t\ge T,\end{aligned}$$

for any $1 \le q \le \infty$. This implies (2.14). Furthermore, combining the two inequalities with interpolation, we show

$$\begin{aligned}\|S(t)\varphi\|_{L^r_\omega} &\le \|S(t)\varphi\|_{L^\infty}^{\frac{r-q}{r}}\|S(t)\varphi\|_{L^q_\omega}^{\frac{q}{r}}\\ &< C(N,c_0,\alpha)^{\frac{r-q}{r}}t^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|\varphi\|_{L^q_\omega},\qquad t\ge T,\end{aligned}$$

for any $1 \le q \le r \le \infty$.

Next we prove the assertion (ii). For the case $q = r$, the estimate (2.16) holds by [10]*Lemma 2.3. On the other hand, by (1.7), (1.10) with $r_2 = q$ and (2.12), for any $1 < q < \infty$, we have

$$\begin{aligned}\|S(t)\varphi\|_{L^\infty} &\le \operatorname{ess\,sup}_{x\in\mathbb{R}^N}\|\Gamma(x,\cdot,t)\varphi\|_{L^1_\omega}\\ &\le \operatorname{ess\,sup}_{x\in\mathbb{R}^N}\|\Gamma(x,\cdot,t)\|_{L^{\frac{q}{q-1},1}_\omega}\|\varphi\|_{L^{q,\infty}_\omega}\\ &< C(N,c_0,q,\alpha)t^{-\frac{N+\alpha-\epsilon}{2q}}\|\varphi\|_{L^{q,\infty}_\omega}\end{aligned}\tag{2.17}$$

for $x \in \mathbb{R}^N$ and $t \ge T$. Therefore, by (1.9) with $(r_0, r_1) = (q, \infty)$, (2.15) with $q = r$ and (2.17) we have

$$\begin{aligned}\|S(t)\varphi\|_{L^{r,\infty}_\omega} &\le \|S(t)\varphi\|_{L^{q,\infty}_\omega}^{1-\theta}\|S(t)\varphi\|_{L^\infty_\omega}^{\theta}\\ &< C\|\varphi\|_{L^{q,\infty}_\omega}^{1-\theta}C(N,c_0,q,\alpha)t^{-\frac{N+\alpha-\epsilon}{2q}\theta}\|\varphi\|_{L^{q,\infty}_\omega}^{\theta}\\ &= Ct^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|\varphi\|_{L^{q,\infty}_\omega}\end{aligned}$$

for some C depending on α, c_0, q, and N. □

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1, which means that problem (1.1) has no nonnegative nontrivial global-in-time solutions in the case $1 < p < p_F$.

To begin with, we present a crucial lemma that plays a key role in determining the Fujita exponent.

Lemma 3.1. *Let u be a solution of (1.1) in $\mathbb{R}^N \times (0, T)$ with $0 < T \le \infty$. Then there exists a constant C^* depending only on p such that, for any $t \in [0, T)$, we have*

$$t^{\frac{1}{p-1}} \|S(t)u_0\|_{L^\infty} \le C^*.$$

Proof. Since the fundamental solution Γ satisfies (2.2) and (2.3), the proof of this lemma is almost the same as in the proof of [10, Lemma 3.1] (see also [31, Theorem 5]). So we omit the details here. □

We prove Theorem 1.1 by using Lemma 3.1.

Proof of Theorem 1.1. Suppose by contradiction, u is a global-in-time solution to (1.1). Since $u(\cdot, 1)$ is a positive measurable function in $\mathbb{R}^N$, we can find a non-trivial measurable function $U_1 \in L^\infty$ such that supp $U_1 \subset B_1(0)$ and

$$0 \le U_1(x) \le u(x, 1) \tag{3.1}$$

for almost all $x \in \mathbb{R}^N$. If $t \ge 1$, then $B_1(0) \subset \{x : |x| \le \sqrt{t}\}$, and hence it follows from Lemma 2.3 that

$$\begin{aligned}[S(t)U_1](x) &> C^{-1} t^{-\frac{N+\alpha+\epsilon}{2}} \int_{|y|\le\sqrt{t}} U_1(y)\,\omega(y)\,dy \\ &\ge C^{-1} t^{-\frac{N+\alpha+\epsilon}{2}} \int_{B_1(0)} U_1(y)\,\omega(y)\,dy = C^{-1} \tilde{M} t^{-\frac{N+\alpha+\epsilon}{2}}\end{aligned}$$

for any $|x| \le \sqrt{t}$ and $t \ge 1$, where $\tilde{M} := \int_{B_1(0)} U_1(y)\,\omega(y)\,dy$ is bounded. This together with (3.1) yields

$$[S(t)u(1)](x) \ge [S(t)U_1](x) > C^{-1} \tilde{M} t^{-\frac{N+\alpha+\epsilon}{2}} \tag{3.2}$$

for any $|x| \le \sqrt{t}$ and $t \ge 1$.

On the other hand, by (1.5), (1.6) and (2.3) we can see that

$$u(x, t+1) = [S(t)u(1)](x) + \int_1^{t+1} [S(t+1-s)u(s)^p](x)\,ds$$

for almost all $x \in \mathbb{R}^N$ and all $t > 0$. This means that if u is a global-in-time solution to (1.1), then Lemma 3.1 holds for $u_0(x)$ replaced by $u(x, 1)$. Let $1 < p < p_F$. Then we can choose ϵ to be small enough such that

$$\epsilon < \frac{2}{p-1} - (N + \alpha),$$

and by (3.2) we have

$$t^{\frac{1}{p-1}} \|S(t)u(1)\|_{L^\infty} > t^{\frac{1}{p-1}} C^{-1} \tilde{M} t^{-\frac{N+\alpha+\epsilon}{2}} \to \infty \text{ as } t \to \infty.$$

This contradicts Lemma 3.1, and we see that problem (1.1) does not possess any nonnegative nontrivial global-in-time solutions. Hence, we have completed the proof of Theorem 1.1. □

4. Proof of Theorem 1.2 and Corollary 1.1

In this section, we prove Theorem 1.2 and its Corollary 1.1, which assert the existence of a unique global-in-time mild solution to problem (1.1) in the case $p > p_F$. Throughout this section we fixed a $\epsilon > 0$ satisfying (1.12).

We first give the uniqueness property of the solution to problem (1.1).

Lemma 4.1. *Assume* (A1)–(A3). *Let $\tau > 0$ and u_1, u_2 be solutions to (1.1) in $\mathbb{R}^N \times (0, \tau)$ with $u_{0,1}, u_{0,2} \in L^\infty$. Then, for any $\eta \in (0, \tau)$, there exists a constant C such that*

$$\sup_{0<t\le\eta} \|u_1(t) - u_2(t)\|_{L^\infty} \le C\|u_{0,1} - u_{0,2}\|_{L^\infty}.$$

Here the constant C depends on $\|u_1\|_{L^\infty(0,\eta:L^\infty)}$ and $\|u_2\|_{L^\infty(0,\eta:L^\infty)}$.

Proof. The proof of this lemma is almost same as in [10]*Lemma 4.1. So we omit the details here. □

Remark 4.1. Let $\tau > 0$ and u be a solution of (1.1) in $\mathbb{R}^N \times (0, \tau)$. If $\|u\|_{L^\infty(0,\tau:L^\infty)}$ is bounded, then we can take a constant C independent from η. Therefore, we can extend the interval $(0, \tau)$ where the uniqueness property holds. If $\tau = \infty$ and u is a global-in-time bounded solution to (1.1), then we see that u is a unique solution to (1.1).

Next we construct local-in-time mild solutions to (1.1). For any nonnegative function $u_0 \in L^\infty$, define $\{u_n\}$ inductively by

$$\begin{aligned} u_1(x,t) &:= [S(t)u_0](x) = \int_{\mathbb{R}^N} \Gamma(x,y,t)u_0(y)\,\omega(y)\,dy, \\ u_{n+1}(x,t) &:= u_1(x,t) + \int_0^t [S(t-s)u_n(s)^p](x)\,ds, \quad n = 1, 2, \ldots, \end{aligned} \tag{4.1}$$

for almost all $x \in \mathbb{R}^N$ and all $t > 0$. Then we can easily prove that

$$0 \le u_n(x,t) \le u_{n+1}(x,t) \tag{4.2}$$

for almost all $x \in \mathbb{R}^N$ and all $t > 0$, $n \in \mathbb{N}$. In fact, it is clear to obtain $u_2 \ge u_1$ since Γ and u_1 are nonnegative functions. If there exists a number $k \in \mathbb{N}$ such that $u_k(x,t) \le u_{k+1}(x,t)$ for almost all $x \in \mathbb{R}^N$ and all $t > 0$, then

$$\begin{aligned} u_{k+2}(x,t) &= u_1(x,t) + \int_0^t \int_{\mathbb{R}^N} \Gamma(x,y,t-s)u_{k+1}(y,s)^p\,\omega(y)\,dy\,ds \\ &\ge u_1(x,t) + \int_0^t \int_{\mathbb{R}^N} \Gamma(x,y,t-s)u_k(y,s)^p\,\omega(y)\,dy\,ds = u_{k+1}(x,t) \end{aligned}$$

for almost all $x \in \mathbb{R}^N$ and all $t > 0$. This means that (4.2) holds true for all $n \in \mathbb{N}$. Therefore, the limit function

$$u_*(x,t) := \lim_{n\to\infty} u_n(x,t) \in [0,\infty] \tag{4.3}$$

can be defined for almost all $x \in \mathbb{R}^N$ and all $t > 0$. Furthermore, by (2.13) and (2.15) we can put a constant

$$c_* = \max\{1, c_\alpha, c_\beta, d_\alpha, d_\beta, C_{r_*}\}$$

such that

$$\begin{aligned} &\sup_{0<t<\infty} \|u_1(t)\|_{L^\infty} \le c_*\|u_0\|_{L^\infty}, \\ &\sup_{T\le t<\infty} t^{\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{r_*}-\frac{1}{q}\right)} \|u_1(t)\|_{L^{q,\infty}_\omega} < c_*\|u_0\|_{L^{r_*,\infty}_\omega}, \end{aligned} \tag{4.4}$$

for a fixed $T > 0$ and any $q \in [r_*, \infty]$ if $u_0 \in L^\infty \cap L^{r_*,\infty}_\omega$, where r_* is given in (1.12) and $c_\alpha, c_\beta, d_\alpha, d_\beta, C_{r_*}$ are given in (2.13), (2.15), and (2.16), respectively. Then we have the following lemma, which implies the local existence of solutions to (1.1).

Lemma 4.2. *Assume* (A1)–(A3). *Let $u_0 \in L^\infty$. Then there exists a positive constant T such that the problem (1.1) possesses a unique solution u of (1.1) in $\mathbb{R}^N \times (0,T)$ satisfying*

$$\sup_{0<t<T} \|u(t)\|_{L^\infty} \le 2c_*\|u_0\|_{L^\infty}.$$

Proof. This proof follows from [10]*Lemma 4.2. Let T be a sufficiently small positive constant to be chosen later. By induction we prove

$$\sup_{0<t<T} \|u_n(t)\|_{L^\infty} \le 2c_*\|u_0\|_{L^\infty}, \quad n = 1, 2, \ldots. \tag{4.5}$$

By (4.4), we have (4.5) for $n = 1$. Assume that (4.5) holds true for $n = n_* \in \{1, 2, \ldots\}$. Then, by (2.13), (4.1) and (4.4) we have

$$\begin{aligned} \|u_{n_*+1}(t)\|_{L^\infty} &\le \|u_1(t)\|_{L^\infty} + \int_0^t \|S(t-s)u_{n_*}(s)^p\|_{L^\infty}\, ds \\ &\le c_*\|u_0\|_{L^\infty} + \int_0^t \|u_{n_*}(s)\|^p_{L^\infty}\, ds \\ &\le c_*\|u_0\|_{L^\infty} + T(2c_*\|u_0\|_{L^\infty})^p \end{aligned} \tag{4.6}$$

for all $t \in (0,T)$. Let T be a sufficiently small constant such that

$$T2^p(c_*\|u_0\|_{L^\infty})^{p-1} \le 1. \tag{4.7}$$

Then, by (4.6) we have (4.5) for $n = n_* + 1$. Therefore (4.5) holds true for all $n = 1, 2, \ldots$. By (4.2), (4.3) and (4.5) we see that the limit function u_* satisfies (1.6) and

$$\sup_{0<t<T} \|u_*(t)\|_{L^\infty} \le 2c_*\|u_0\|_{L^\infty}.$$

This together with Lemma 4.1 implies that u_* is a mild solution of (1.1) in $\mathbb{R}^N \times (0, T)$. □

Now we are ready to prove Theorem 1.2.

Proof of the assertion (i) of Theorem 1.2. Assume (1.12). Let δ be a sufficiently small positive constant. Assume and (1.13). Fix $T < 1$ that satisfies (4.7), so u_* is a mild solution to (1.1) in $\mathbb{R}^N \times (0, T)$ and all u_n satisfy (4.5) by Lemma 4.2. By induction we prove

$$\|u_n(t)\|_{L^{r_*,\infty}_\omega} \le 2c_*\delta, \qquad \|u_n(t)\|_{L^\infty} \le 2c_*\delta t^{-\frac{N+\alpha-\epsilon}{2r_*}}, \tag{4.8}$$

for all $n \in \mathbb{N}$ and $t > 0$. By (2.14), (2.16), and (4.4) we have (4.8) for $n = 1$. Assume that (4.8) holds for some $n = n_* \in \{1, 2, \ldots\}$, that is,

$$\|u_{n_*}(t)\|_{L^{r_*,\infty}_\omega} \le 2c_*\delta, \qquad \|u_{n_*}(t)\|_{L^\infty} \le 2c_*\delta t^{-\frac{N+\alpha-\epsilon}{2r_*}},$$

for all $t > 0$. Then by (1.9) we have

$$\|u_{n_*}(t)\|_{L^{q,\infty}_\omega} \le \|u_{n_*}(t)\|^{\frac{r_*}{q}}_{L^{r_*,\infty}_\omega} \|u_{n_*}(t)\|^{1-\frac{r_*}{q}}_{L^\infty} \le 2c_*\delta t^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{r_*}-\frac{1}{q}\right)} \tag{4.9}$$

for any $r_* \le q \le \infty$ and all $t > 0$. So, for any $\zeta > 1$ with $\zeta \le r_* < \zeta p$, by (1.12) and (4.9) we obtain

$$\|u_{n_*}(t)^p\|_{L^{\zeta,\infty}_\omega} = \|u_{n_*}(t)\|^p_{L^{\zeta p,\infty}_\omega} \le (2c_*\delta)^p t^{-\frac{N+\alpha-\epsilon}{2r_*}+\frac{N+\alpha-\epsilon}{2\zeta}-1} \tag{4.10}$$

and

$$\|u_{n_*}(t)^p\|_{L^\infty} = \|u_{n_*}(t)\|^p_{L^\infty} \le (2c_*\delta t^{-\frac{N+\alpha-\epsilon}{2r_*}})^p = (2c_*\delta)^p t^{-\frac{N+\alpha-\epsilon}{2r_*}-1} \tag{4.11}$$

for all $t > 0$. Combining with (2.14), (2.16), (4.10), and (4.11), we have

$$\begin{aligned}
\left\| \int_{t/2}^t S(t-s)u_{n_*}(s)^p\,ds \right\|_{L^\infty} &\le \int_{t/2}^t \|S(t-s)u_{n_*}(s)^p\|_{L^\infty}\,ds \\
&\le \int_{t/2}^t \|u_{n_*}(s)^p\|_{L^\infty}\,ds \\
&\le C\delta^p \int_{t/2}^t s^{-\frac{N+\alpha-\epsilon}{2r_*}-1}\,ds \le C\delta^p t^{-\frac{N+\alpha-\epsilon}{2r_*}},
\end{aligned}$$

and

$$\left\| \int_{t/2}^{t} S(t-s)u_{n_*}(s)^p\,ds \right\|_{L^{r_*,\infty}_\omega} \le \int_{t/2}^{t} \|S(t-s)u_{n_*}(s)^p\|_{L^{r_*,\infty}_\omega}\,ds$$

$$\le C\int_{t/2}^{t} \|u_{n_*}(s)^p\|_{L^{r_*,\infty}_\omega}\,ds \le C\delta^p \int_{t/2}^{t} s^{-1}\,ds \le C\delta^p,$$

for all $t > 0$. On the other hand, to estimate the part of $0 < s < t/2$ for the Duhamel term, that is,

$$\int_0^{t/2} S(t-s)u_{n_*}(s)^p\,ds,$$

we need to separate into two cases $t \ge 2T$ and $0 < t < 2T < 2$ in order to avoid having β in our estimation. For the case $0 < t < 2T < 2$, we have (4.5) because of our choice of T. Combining with (1.9), (2.14), (2.16), and (4.8), we have

$$\left\| \int_0^{t/2} S(t-s)u_{n_*}(s)^p\,ds \right\|_{L^\infty} \le \int_0^{t/2} \|u_{n_*}(s)^p\|_{L^\infty}\,ds$$

$$\le \int_0^{t/2} (2c_*\delta)^p\,ds \le T(2c_*\delta)^p \le C\delta^p t^{-\frac{N+\alpha-\epsilon}{2r_*}},$$

and

$$\left\| \int_0^{t/2} S(t-s)u_{n_*}(s)^p\,ds \right\|_{L^{r_*,\infty}_\omega} \le C\int_0^{t/2} \|u_{n_*}(s)^p\|_{L^{r_*,\infty}_\omega}\,ds$$

$$\le C\int_0^{t/2} \|u_{n_*}(s)\|_{L^{r_*,\infty}_\omega} \|u_{n_*}(s)\|_{L^\infty}^{p-1}\,ds$$

$$\le C\int_0^{t/2} (2c_*\delta)^p\,ds \le C\delta^p,$$

for all $t < 2T < 2$. Furthermore, for the case $t \ge 2T$, combining with (2.13), (2.15), (4.10) and (4.11) with $\zeta < r_* < \zeta p$, we have

$$\left\| \int_0^{t/2} S(t-s)u_{n_*}(s)^p\,ds \right\|_{L^\infty} \le \int_0^{t/2} \|S(t-s)u_{n_*}(s)^p\|_{L^\infty}\,ds$$

$$\le C\int_0^{t/2} (t-s)^{-\frac{N+\alpha-\epsilon}{2\zeta}} \|u_{n_*}(s)^p\|_{L^{\zeta,\infty}_\omega}\,ds$$

$$\le C\delta^p t^{-\frac{N+\alpha-\epsilon}{2\zeta}} \int_0^{t/2} s^{-\frac{N+\alpha-\epsilon}{2r_*}+\frac{N+\alpha-\epsilon}{2\zeta}-1}\,ds$$

$$\le C\delta^p t^{-\frac{N+\alpha-\epsilon}{2r_*}},$$

and

$$\left\| \int_0^{t/2} S(t-s)u_{n_*}(s)^p\,ds \right\|_{L^{r_*,\infty}_\omega} \le \int_0^{t/2} \|S(t-s)u_{n_*}(s)^p\|_{L^{r_*,\infty}_\omega}\,ds$$

$$
\begin{aligned}
&\le C\int_0^{t/2}(t-s)^{-\frac{N+\alpha-\epsilon}{2}(\frac{1}{\zeta}-\frac{1}{r_*})}\|u_{n_*}(s)^p\|_{L^{\zeta,\infty}_\omega}\,ds\\
&\le C\delta^p t^{-\frac{N+\alpha-\epsilon}{2}(\frac{1}{\zeta}-\frac{1}{r_*})}\int_0^{t/2}s^{-\frac{N+\alpha-\epsilon}{2r_*}+\frac{N+\alpha-\epsilon}{2\zeta}-1}\,ds\\
&\le C\delta^p,
\end{aligned}
$$

for any $t\ge 2T$. Then, taking a sufficiently small δ if necessary, combining the above estimates for the Duhamel term with (4.4), we see that

$$
\left.\begin{array}{l} t^{\frac{N+\alpha-\epsilon}{2r_*}}\|u_{n_*+1}(t)\|_{L^\infty}\\ \|u_{n_*+1}(t)\|_{L^{r_*,\infty}_\omega}\end{array}\right\}\le c_*\|u_0\|_{L^{r_*,\infty}_\omega}+C\delta^p\le c_*\delta+C\delta^p\le 2c_*\delta
$$

for all $t>0$. Thus we obtain (4.8) for $n=n_*+1$, and (4.8) holds true for $n=1,2,\dots$. Therefore, applying an argument similarly to the proof of Lemma 4.2, by (4.8) we see that there exists a unique global-in-time solution u to (1.1) such that

$$
\|u(t)\|_{L^{r_*,\infty}_\omega}\le 2c_*\delta,\qquad \|u(t)\|_{L^\infty}\le 2c_*\delta t^{-\frac{N+\alpha-\epsilon}{2r_*}}
$$

This together with Lemma 4.2 implies that

$$
\|u(t)\|_{L^\infty}\le C(1+t)^{-\frac{N+\alpha-\epsilon}{2r_*}}
$$

for all $t>0$. Furthermore, by (1.9) we have

$$
\|u(t)\|_{L^{q,\infty}_\omega}\le\|u(t)\|^{\frac{r_*}{q}}_{L^{r_*,\infty}_\omega}\|u(t)\|^{1-\frac{r_*}{q}}_{L^\infty}\le C(1+t)^{-\frac{N+\alpha-\epsilon}{2}(\frac{1}{r_*}-\frac{1}{q})},\quad r_*\le q\le\infty,
$$

for all $t>0$. Thus we have (1.14), and the proof of assertion of Theorem 1.2 is completed. □

Proof of the assertion (ii) of Theorem 1.2. Assume (1.12). Let δ be a sufficiently small constant and assume (1.15). Then, by the assertion (i) of Theorem 1.2 we see that there exists a unique global-in-time solution u to (1.1) satisfying (1.14).

We prove the existence of a global-in-time solution of (1.1) satisfying (1.16). For $r=r_*$, it follows from a similar argument as in the proof of the assertion (i) of Theorem 1.2. So we assume $1\le r<r_*$. By (4.1) we see that u_n satisfies

$$
u_n(x,t)=[S(t-\tau)u_n(\tau)](x)+\int_\tau^t[S(t-s)u_{n-1}(s)^p](x)\,ds, \tag{4.12}
$$

for all $x\in\mathbb{R}^N$ and $t>\tau\ge 0$. On the other hand, by (1.15) and (2.13) we can find a constant C_{**} independent of δ, q and r such that

$$
\|S(t)u_0\|_{L^q_\omega}\le C_{**}\delta(1+t)^{-\frac{N+\alpha-\epsilon}{2}(\frac{1}{r}-\frac{1}{q})},\qquad t>0, \tag{4.13}
$$

for any $q\in[r,\infty]$.

By induction we first prove that

$$\|u_n(t)\|_{L^q_\omega} \le 2C_{**}\delta, \quad 0 < t \le 2, \tag{4.14}$$

for any $q \in [r, \infty]$ and $n = 1, 2, \ldots$. By (4.13) we have (4.14) for $n = 1$. Assume that (4.14) holds for some $n = n_*$, that is,

$$\|u_{n_*}(t)\|_{L^q_\omega} \le 2C_{**}\delta, \quad 0 < t \le 2, \tag{4.15}$$

for any $q \in [r, \infty]$. Then, by (4.15), for any $q \in [r, \infty]$, we have

$$\|u_{n_*}(t)^p\|_{L^q_\omega} = \|u_{n_*}(t)\|^p_{L^{pq}_\omega} \le (2C_{**}\delta)^p \tag{4.16}$$

for all $0 < t \le 2$. Taking a sufficiently small δ if necessary, by (2.13), (4.12), (4.13) and (4.16) we obtain

$$\begin{aligned}\|u_{n_*+1}(t)\|_{L^q_\omega} &\le \|S(t)u_0\|_{L^q_\omega} + \int_0^t \|S(t-s)u_{n_*}(s)^p\|_{L^q_\omega}\,ds\\ &\le C_{**}\delta + C_1\int_0^t \|u_{n_*}(s)^p\|_{L^q_\omega}\,ds\\ &\le C_{**}\delta + C_2\delta^p \le 2C_{**}\delta, \qquad 0 < t \le 2,\end{aligned} \tag{4.17}$$

for any $q \in [r, \infty]$, where C_1 and C_2 are constants independent of n_* and δ. Thus we have (4.14) for $n = n_* + 1$, and (4.14) holds for all $n = 1, 2, \ldots$.

Let C'_* be a constant to be chosen later such that $C'_* \ge 2C_{**}$. Next, by induction we prove that

$$\|u_n(t)\|_{L^q_\omega} \le C'_*\delta t^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}, \quad t > 1/2, \tag{4.18}$$

for any $q \in [r, \infty]$ and $n = 1, 2, \ldots$. By (4.13) we have (4.18) for $n = 1$. Assume that (4.18) holds for some $n = n_*$. Then, similarly to (4.17), since $r_* = \frac{N+\alpha-\epsilon}{2}(p-1) > r$, taking a sufficiently small δ if necessary, by (2.13), (4.12) and (4.14) we have

$$\begin{aligned}\|u_{n_*+1}(t)\|_{L^q_\omega} &\le C_3(t-1/2)^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\|u_{n_*+1}(1/2)\|_{L^r_\omega}\\ &\quad + C_3\int_{1/2}^{t/2}(t-s)^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\|u_{n_*}(s)^p\|_{L^r_\omega}\,ds\\ &\quad + C_3\int_{t/2}^{t}\|u_{n_*}(s)^p\|_{L^q_\omega}\,ds\\ &\le C_4C_{**}\delta t^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{r}-\frac{1}{q}\right)} + C_4(C'_*\delta)^p t^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\int_{1/2}^{t/2} s^{-\frac{r_*}{r}}\,ds\\ &\quad + C_4(C'_*\delta)^p\int_{t/2}^{t} s^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{p}{r}-\frac{1}{q}\right)}\,ds\\ &\le C_5C_{**}\delta t^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{r}-\frac{1}{q}\right)} + C_5(C'_*\delta)^p t^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{r_*}{r}+1}\end{aligned}$$

for all $t > 1$, where C_3, C_4 and C_5 are constants independent of n_* and δ. Let $C_*' \geq 2C_5C_{**}$. Then, taking a sufficiently small δ if necessary, we have

$$\|u_{n_*+1}(t)\|_{L^q_\omega} \leq C_*'\delta t^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}, \quad t > 1.$$

This together with (4.15) implies (4.18) with $n = n_* + 1$. Thus (4.18) holds for all $n = 1, 2, \ldots$.

By (4.15) and (4.18) we can find a constant C such that

$$\|u_n(t)\|_{L^q_\omega} \leq C\delta(1+t)^{-\frac{N+\alpha-\epsilon}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}, \qquad t > 0,$$

for all $q \in [r, \infty]$ and $n = 1, 2, \ldots$. Then, by the same argument as in the proof of the assertion (i) of Theorem 1.2, we see that there exists a solution u to (1.1) satisfying (1.16). Thus the assertion (ii) of Theorem 1.2 follows, and the proof of Theorem 1.2 is complete. □

Proof of Corollary 1.1. Let

$$f(x) := \frac{\delta}{1 + |x|^{\frac{2}{p-1}\frac{N+\alpha+\epsilon}{N+\alpha-\epsilon}}}$$

for all $x \in \mathbb{R}^N$. Then, for $\lambda < \delta$, by (1.8) and (2.5) we see that

$$\begin{aligned}
\mu_f(\lambda) &= \int_{\mathbb{R}^N} \mathbb{1}\,\{x : |f(x)| > \lambda\}\, \omega(x)\, dx \\
&= \int_{\mathbb{R}^N} \mathbb{1}\left\{x : |x| < \left(\frac{\delta}{\lambda} - 1\right)^{\frac{p-1}{2}\frac{N+\alpha-\epsilon}{N+\alpha+\epsilon}}\right\} \omega(x)\, dx \\
&= w_{(\frac{\delta}{\lambda}-1)^{(p-1)\frac{N+\alpha-\epsilon}{N+\alpha+\epsilon}}}(0) \leq C\left(\frac{\delta}{\lambda} - 1\right)^{\frac{N+\alpha-\epsilon}{2}(p-1)} = C\left(\frac{\delta}{\lambda} - 1\right)^{r_*}.
\end{aligned}$$

Notice that if $\mu_f(\lambda) \leq F(\lambda)$ for some function F, then the non-increasing rearrangement of f would satisfies

$$f^*(s) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq s\} \leq \inf\{\lambda > 0 : F(\lambda) \leq s\}, \qquad s \geq 0.$$

So we have

$$f^*(s) \leq \frac{\delta}{1 + Cs^{\frac{1}{r_*}}}, \qquad s \geq 0$$

This implies that

$$\|f\|_{L^{r_*,\infty}_\omega} = \sup_{s>0} s^{\frac{1}{r_*}} f^*(s) \leq \sup_{s>0} s^{\frac{1}{r_*}} \frac{\delta}{1 + Cs^{\frac{1}{r_*}}} < C\delta.$$

Furthermore, it is obvious that $\|f\|_{L^\infty} \leq \delta$. Therefore, there exists a constant C independent of δ such that

$$\max(\|u_0\|_{L^{r_*,\infty}_\omega}, \|u_0\|_{L^\infty}) < C\delta,$$

and applying the assertion (i) of Theorem 1.2, we see that if δ is sufficiently small, then a global-in-time solution of (1.1) exists and it satisfies (1.14) □

Acknowledgements

The authors are grateful to the referees for their helpful comments on the paper. Y.S. is supported by NSF DMS grant 2154219, " Regularity *vs* singularity formation in elliptic and parabolic equations". T.K. is supported in part by JSPS KAKENHI Grant Number JP 20K03689 and 22KK0035.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Data availability. Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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Tatsuki Kawakami
Faculty of Advanced Science and Technology
Ryukoku University
1-5 Yokotani
Seta Oe-cho, Otsu Shiga 520-2194
Japan
E-mail: kawakami@math.ryukoku.ac.jp

Yannick Sire and Jiayi Nikki Wang
Department of Mathematics
Johns Hopkins University
3400 N. Charles Street
Baltimore MD 21218
USA
E-mail: ysire1@jhu.edu

Jiayi Nikki Wang
E-mail: jwang344@jhu.edu

Accepted: 19 March 2024