Research Article

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On the heat kernel of the Rumin complex and Calderón reproducing formula

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Abstract: We derive several properties of the heat equation with the Hodge operator associated with the Rumin's complex on Heisenberg groups and prove several properties of the fundamental solution. As an application, we use the heat kernel for Rumin's differential forms to construct a Calderón reproducing formula on Rumin's forms.

Keywords: Heisenberg groups; differential forms; currents; Laplace operators; heat kernel

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To Ermanno, in friendship and admiration of his mathematics.

1 Introduction and basic objects

Heisenberg groups $\mathbb{H}^n$, $n \geq 1$, are connected, simply connected Lie groups whose Lie algebra is a one-dimensional central extension of $\mathfrak{h}_1 = \mathbb{R}^{2n}$,

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2, \quad \text{with} \ \ \mathfrak{h}_2 = \mathbb{R} = Z(\mathfrak{h}), \tag{1.1}$$

with bracket $\mathfrak{h}_1 \otimes \mathfrak{h}_1 \to \mathfrak{h}_2 = \mathbb{R}$ being a nondegenerate skew-symmetric 2-form. Due to its stratification (1.1), the Heisenberg Lie algebra admits a one-parameter group of automorphisms δ_t,

$$\delta_t = t \quad \text{on } \mathfrak{h}_1, \quad \delta_t = t^2 \quad \text{on } \mathfrak{h}_2,$$

which are counterparts of the usual Euclidean dilations in $\mathbb{R}^N$. The stratification of the Lie algebra $\mathfrak{h}$ yields a lack of homogeneity of de Rham's exterior differential with respect to group dilations δ_λ. The so-called Rumin complex is meant precisely to bypass the lack of homogeneity of de Rham complex through a new complex that is still homotopic to de Rham complex. In Appendix A, we shall provide a more exhaustive description of Rumin complex.

In this article, we investigate several properties of the heat kernel associated with the Rumin complex on Heisenberg groups, i.e., of the distributional kernel of the "heat operator"

$$\mathcal{L} \coloneqq \partial_s + \Delta_{\mathbb{H},h} \quad \text{in } \mathbb{R}_+ \times \mathbb{H}^n,$$

where $\Delta_{\mathbb{H},h}$ is the homogeneous Hodge Laplacian associated with Rumin complex (1.2), and derive a natural reproducing formula in the spirit of Calderón reproducing formula. Beside this application, which has its own interest, we collect several basic results on the heat kernel, which seem to be not all available in the literature

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(in this spirit, we quote the studies by Albin and Quan [1] and Rumin [46] for the heat kernel in contact manifolds, as well as the study by Dave and Haller [19] in filtered manifolds).

This project grew out of understanding compensation–compactness phenomena for differential forms on nilpotent groups. Several div-curl lemma have been proved in the setting of Heisenberg groups by the second author jointly with different co-authors in previous studies [8,10,11,29]. The present article stems from the following observation: in a very interesting article, Lou and McIntosh [40] introduced Hardy spaces of *exact* differential forms for the De Rham complex on Euclidean spaces and generalized the foundational work of Coifman et al. [17]. The work of Lou and McIntosh contains several ideas around the use of differential forms with coefficients in a suitable Hardy spaces (and their atomic decompositions) but also their analysis via a reproducing formula *à la Calderón*. Thanks to the works of the second author with Baldi et al. [4–7], several important functional inequalities are now available for the Rumin complex. However, as mentioned, the full generalization to the Rumin complex of div-curl lemma of Lou and McInstosh requires the introduction of Hardy spaces and their atomic decomposition. At this point, the theory of such spaces for the Rumin complex departs from the Euclidean setting, even if every Heisenberg group is a space of homogeneous type, because of the structural properties inherent to the Rumin complex. In a subsequent article, we will address the construction of such spaces and the applications to compensated compactness on the Rumin complex. This application to div-curl lemmas is also the motivation behind our choice to present the Calderón reproducing formula in the space L^1. However, we must stress that, unlike in [2], our reproducing formula is not associated with a semigroup with finite speed of propagation, and therefore, following the study by Lou and McIntosh [40], we are lead to work with a decomposition in molecules, replacing the usual decomposition in atoms of the functions in real Hardy spaces.

Classically, approximation on groups or manifolds can be done through the heat operator. The scalar case, i.e., the heat operator associated with a subelliptic Laplacian on stratified nilpotent Lie groups is nowadays well understood. We refer to previous studies [25,33,49] and to the historical introduction of the study by Bramanti et al. [15]. On the contrary, much less is known for the heat kernel on differential forms in both the Riemannian and the non-Riemannian setting. We refer to the previous study Coulhon et al. [18] and to the reference therein. In particular, the literature is rather poor on the properties of the heat kernel for differential forms in Heisenberg groups for the Rumin Laplacian. The primary goal of the present work is to fill in this gap and provide several ready-to-use properties of the heat equation on the Rumin complex. As an application, we use this heat kernel to prove a general Calderón reproducing formula on Rumin forms. Our contribution can then be seen as a further expansion of the noncommutative harmonic analysis of differential complexes on the Heisenberg group.

To state our main results, we first recall some basic notations related to the Heisenberg group and the Rumin complex of differential forms. The subsequent sections introduce all the necessary tools and the appendices expand on more details on the geometry and analysis on Heisenberg together with the Rumin complex. We refer the reader to those for a more detailed account.

In this section, we present some basic notations and introduce both the structure of Heisenberg groups together with the formulation of the Rumin complex. We denote by $\mathbb{H}^n$ the $(2n+1)$-dimensional Heisenberg group, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted by $p = (x, y, t)$, with both $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. If p and $p' \in \mathbb{H}^n$, the group operation is defined by

$$p \cdot p' = (x + x', y + y', t + t' + \frac{1}{2}\sum_{j=1}^{n}(x_j y_j' - y_j x_j')).$$

Notice that $\mathbb{H}^n$ can be equivalently identified with $\mathbb{C}^n \times \mathbb{R}$ endowed with the group operation

$$(z, t)\cdot(\zeta, \tau) \coloneqq \left(z + \zeta, t + \tau - \frac{1}{2}\mathrm{Im}(z\bar{\zeta})\right).$$

The unit element of $\mathbb{H}^n$ is the origin, which will be denoted by e. For any $q \in \mathbb{H}^n$, the *(left) translation* $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ is defined as follows:

$$p \mapsto \tau_q p \coloneqq q \cdot p.$$

We denote by $\mathfrak{h}$ the Lie algebra of the left invariant vector fields of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}$ is given, for $i = 1, \dots, n$, by

$$X_i := \partial_{x_i} - \frac{1}{2} y_i \partial_t, \quad Y_i := \partial_{y_i} + \frac{1}{2} x_i \partial_t, \quad T := \partial_t.$$

The only nontrivial commutation relations are $[X_j, Y_j] = T$, for $j = 1, \dots, n$. The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$: $\mathfrak{h}_1 := \operatorname{span}_{\mathbb{R}} \{X_1, \dots, X_n, Y_1, \dots, Y_n\}$.

Denoting by $\mathfrak{h}_2$ the linear span of T, the two-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

The stratification of the Lie algebra $\mathfrak{h}$ induces a family of nonisotropic dilations $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$, $\lambda > 0$ as follows: if $p = (x, y, t) \in \mathbb{H}^n$, then

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t).$$

Throughout this article, we also write

$$W_i := X_i, \quad W_{i+n} := Y_i \quad \text{and} \quad W_{2n+1} := T, \quad \text{for } i = 1, \dots, n.$$

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1 \mathfrak{h}$. The basis of $\bigwedge^1 \mathfrak{h}$, dual to the basis $\{X_1, \dots, Y_n, T\}$, is the family of covectors $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n, \theta\}$, where

$$\theta := dt - \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j)$$

is called the *contact form* in $\mathbb{H}^n$. We also denote by $\langle \cdot, \cdot \rangle$ the inner product in $\bigwedge^1 \mathfrak{h}$ that makes $(dx_1, \dots, dy_n, \theta)$ an orthonormal basis, and we set

$$\omega_i := dx_i, \quad \omega_{i+n} := dy_i \quad \text{and} \quad \omega_{2n+1} := \theta, \quad \text{for } i = 1, \dots, n.$$

We put $\bigwedge_0 \mathfrak{h} := \bigwedge^0 \mathfrak{h} = \mathbb{R}$ and, for $1 \le h \le 2n + 1$,

$$\textstyle\bigwedge^h \mathfrak{h} := \operatorname{span}_{\mathbb{R}} \{\omega_{i_1} \wedge \cdots \wedge \omega_{i_h} : 1 \le i_1 < \cdots < i_h \le 2n + 1\}.$$

We shall denote by Θ^h the basis of $\bigwedge^h \mathfrak{h}$ defined by

$$\Theta^h := \{\omega_{i_1} \wedge \cdots \wedge \omega_{i_h} : 1 \le i_1 < \cdots < i_h \le 2n + 1\}.$$

The inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge^1 \mathfrak{h}$ yields an inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge^h \mathfrak{h}$ making Θ^h an orthonormal basis. The elements of $\bigwedge^h \mathfrak{h}$ are identified with *left invariant* differential forms of degree h on $\mathbb{H}^n$.

The same construction can be performed starting from the vector subspace $\mathfrak{h}_1 \subset \mathfrak{h}$, obtaining the *horizontal h-covectors*

$$\textstyle\bigwedge^h \mathfrak{h}_1 := \operatorname{span} \{\omega_{i_1} \wedge \cdots \wedge \omega_{i_h} : 1 \le i_1 < \cdots < i_h \le 2n\}.$$

It is easy to see that

$$\Theta_0^h := \Theta^h \cap \textstyle\bigwedge^h \mathfrak{h}_1$$

provides an orthonormal basis of $\bigwedge^h \mathfrak{h}_1$.

Keeping in mind that the Lie algebra $\mathfrak{h}$ can be identified with the tangent space to $\mathbb{H}^n$ at $x = e$, the neutral element (see [31], Proposition 1.72), starting from $\bigwedge^h \mathfrak{h}$, we can define by left translation a fiber bundle over $\mathbb{H}^n$ that we can still denote by $\bigwedge^h \mathfrak{h}$. We can think of h-forms as sections of $\bigwedge^h \mathfrak{h}$. We denote by Ω^h the vector space of all smooth h-forms on $\mathbb{H}^n$.

As we stressed earlier, the stratification of the Lie algebra $\mathfrak{h}$ yields a lack of homogeneity of de Rham's exterior differential with respect to group dilations δ_λ. Thus, to keep into account the different degrees of homogeneity of the covectors when they vanish on different layers of the stratification, we introduce the notion of *weight* of a covector as follows. This is at the core of Rumin construction of the differential complex.

Definition 1.1. If $\eta \neq 0$, $\eta \in \bigwedge^1 \mathfrak{h}_1$, we say that η has *weight 1*, and we write $w(\eta) = 1$. If $\eta = \theta$, we say $w(\eta) = 2$. More generally, if $\eta \in \bigwedge^h \mathfrak{h}$, $\eta \neq 0$, we say that η has *pure weight p* if η is a linear combination of covectors $\omega_{i_1} \wedge \cdots \wedge \omega_{i_h}$ with $w(\omega_{i_1}) + \cdots + w(\omega_{i_h}) = p$.

The following result holds (see [8], formula (16)):

$$\bigwedge^h \mathfrak{h} = \bigwedge^{h,h} \mathfrak{h} \oplus \bigwedge^{h,h+1} \mathfrak{h} = \bigwedge^h \mathfrak{h}_1 \oplus (\bigwedge^{h-1} \mathfrak{h}_1) \wedge \theta,$$

where $\bigwedge^{h,p}\mathfrak{h}$ denotes the linear span of the h-covectors of weight p and a basis of $\bigwedge^{h,p}\mathfrak{h}$ is given by $\Theta^{h,p} := \Theta^h \cap \bigwedge^{h,p}\mathfrak{h}$ (such a basis is usually called an adapted basis). Consequently, the weight of a h-form is either h or $h+1$, and there are no h-forms of weight $h+2$, since there is only one 1-form of weight 2. Starting from $\bigwedge^{h,p}\mathfrak{h}$, we can define by left translation a fiber bundle over $\mathbb{H}^n$ that we can still denote by $\bigwedge^{h,p}\mathfrak{h}$. Thus, if we denote by $\Omega^{h,p}$ the vector space of all smooth h-forms in $\mathbb{H}^n$ of weight p, i.e., the space of all smooth sections of $\bigwedge^{h,p}\mathfrak{h}$, we have

$$\Omega^h = \Omega^{h,h} \oplus \Omega^{h,h+1}.$$

Starting from the notion of weight of a differential form, it is possible to define a new complex of differential forms $(E_0^\bullet, d_c)$ that is homotopic to the de Rham complex and respects the homogeneities of the group. This is the Rumin complex. A crucial feature of $(E_0^\bullet, d_c)$ is that the "exterior differential" d_c is an operator of order 1 with respect to group dilations when acting on forms of degree $h \neq n$, but of order 2 on n-forms.

Following [44], we define the operator $\Delta_{\mathbb{H},h}$ on E_0^h by setting

$$\Delta_{\mathbb{H},h} = \begin{cases} d_c d_c^* + d_c^* d_c & \text{if } h \neq n, n+1; \\ (d_c d_c^*)^2 + d_c^* d_c & \text{if } h = n; \\ d_c d_c^* + (d_c^* d_c)^2 & \text{if } h = n+1. \end{cases} \tag{1.2}$$

We point out that Rumin Laplacian $\Delta_{\mathbb{H},h}$ is an operator of order 2 with respect to group dilations when acting on forms of degree $h \neq n$, but of order 4 on n-forms.

We stress also that Rumin Laplacian differs from the "Riemannian" Hodge Laplacian in $\mathbb{H}^n$ [42] associated with de Rham complex, which fails to be homogeneous.

1.1 Main results

Consider now the heat operator associated with the Rumin Laplacian $\Delta_{\mathbb{H},h}$ associated with the complex $(E_0^\bullet, d_c)$, i.e.,

$$\mathcal{L} := \partial_s + \Delta_{\mathbb{H},h} \quad \text{in } \mathbb{R}_+ \times \mathbb{H}^n,$$

where ∂_s stands for $\partial_s I_d$, I_d being the identity $N_h \times N_h$ matrix, where $N_h = \dim E_0^h$. Our first result is

Theorem 1.2. *The operator $\mathcal{L}$ is hypoelliptic on $\mathbb{R}_+ \times \mathbb{H}^n$.*

Building on the latter, we also prove the following basic properties of the heat kernel:

Theorem 1.3. *If $0 \leq h \leq 2n+1$, the operators $-\Delta_{\mathbb{H},h} : \mathcal{D}(\mathbb{H}^n, E_0^h) \subset L^2(\mathbb{H}^n, E_0^h) \to L^2(\mathbb{H}^n, E_0^h)$ are densely defined, self-adjoint and dissipative, and therefore generate strongly continuous analytic semigroup $(\exp(-s\Delta_{\mathbb{H},h}))_{s\geq 0}$ in $L^2(\mathbb{H}^n, E_0^h)$.*

Furthermore, there exists a matrix-valued kernel

$$h = h(s,p) = (h_{ij}(s,p))_{i,j=1,\dots,N_h} \in (\mathcal{D}'(\mathbb{H}^n))^{N_h \times N_h}$$

such that

$$\exp(-s\Delta_{\mathbb{H},h})\alpha = \alpha * h(s,\cdot) \quad \textit{for } \alpha \in \mathcal{D}(\mathbb{H}^n, E_0^h).$$

The kernel constructed in the previous statement has the following crucial properties.

Theorem 1.4. *We have*:

(*i*) $\mathcal{L}h = 0$ *in* $(\mathcal{D}'(\mathbb{R}_+ \times \mathbb{H}^n))^{N_h\times N_h}$, *i.e.*

$$\langle \mathcal{L}h | A(s,y)\rangle = 0 \quad \textit{for all } A \in \mathcal{D}((0,\infty)\times \mathbb{H}^n),$$

where the action of the heat operator $\mathcal{L}$ *on* h *must be understood as the formal matrix product*

$$\mathcal{L}h \coloneqq (\mathcal{L}^{i,j})_{i,j=1,\dots,N_h} \cdot (h_{i,j})_{i,j=1,\dots,N_h},$$

defined in the sense of distributions.

(*ii*) *the matrix-valued distribution* h *is smooth on* $(0,\infty)\times\mathbb{H}^n$. *In particular, if* $\phi = \sum_j \phi_j \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{H}^n, E_0^h)$, *we can write*

$$\langle h|\phi\rangle = \sum_{i,j}\left(\int_{\mathbb{R}_+\times\mathbb{H}^n} h_{i,j}(s,p)\phi_j(s,p)\,\mathrm{d}s\mathrm{d}p\right)\xi_i;$$

(*iii*) *if* $r > 0$

$$h(r^a s, y) = r^{-Q} h(s, \delta_{1/r} y) \quad \textit{for } s > 0 \textit{ and } y \in \mathbb{H}^n.$$

We now finally state how we use the heat kernel to build a reproducing formula. Denote $\alpha \in L^1(\mathbb{H}^n, E_0^h)$, such that $d_c\alpha = 0$ and define the map

$$F(s,x) \coloneqq d_c^*\left(h\left(\frac{s}{2},\cdot\right) * \alpha\right)(x) \quad s > 0.$$

We then have

Theorem 1.5. *If* $\alpha \in L^1(\mathbb{H}^n, E_0^h)$ *is a* d_c*-closed form, we have*:

$$\alpha = -\int_0^\infty d_c\left(h\left(\frac{s}{2},\cdot\right) * F(s,\cdot)\right)\mathrm{d}s.$$

1.2 Notations

We refer the reader to the Appendices for notations that are used in the article.

This article is organized as follows: in Section 2, we introduce currents on Heisenberg groups. Section 3 is the core of the article and is devoted to a thorough investigation of the heat kernel associated with the Rumin complex and its application to the reproducing formula. In the subsequent appendices, we recall the necessary tools from the construction of Rumin and the Heisenberg groups (Appendix A) and from the analysis on groups as developed by Folland and Stein (Appendix B).

2 Currents on Heisenberg groups

Let $U \subset \mathbb{H}^n$ be an open set. We shall use the following classical notations: $\mathcal{E}(U)$ is the space of all smooth function on U, and $\mathcal{D}(U)$ is the space of all compactly supported smooth functions on U, endowed with the standard topologies [53]. The spaces $\mathcal{E}'(U)$ and $\mathcal{D}'(U)$ are their dual spaces of distributions.

Definition 2.1. If $\Omega \subset \mathbb{H}^n$ is an open set, we say that T is a h-current on Ω if T is a continuous linear functional on $\mathcal{D}(\Omega, E_0^h)$ endowed with the usual topology. We write $T \in \mathcal{D}'(\Omega, E_0^h)$. The definition of $\mathcal{E}'(\Omega, E_0^h)$ is given analogously.

If $T \in \mathcal{D}'(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$, we shall denote the action of T on ϕ by $\langle T|\phi\rangle$. An analogous notation will be used for currents versus differential forms.

Proposition 2.2. *If $\Omega \subset \mathbb{H}^n$ is an open set, and $T \in \mathcal{D}'(\Omega)$ is a (usual) distribution, then T can be identified canonically with a $2n+1$-current $\tilde{T} \in \mathcal{D}'(\Omega, E_0^{2n+1})$ through the formula*

$$\langle \tilde{T}|\alpha\rangle \coloneqq \langle T| * \alpha\rangle \tag{2.1}$$

for any $\alpha \in \mathcal{D}(\Omega, E_0^{2n+1})$. Reciprocally, by (2.1), any n-current $\tilde{T}$ can be identified with an usual distribution $T \in D'(\Omega)$.

Proof. See [21], Section 17.5, and [9], Proposition 4. □

Following [22], 4.1.7, we give the following definition.

Definition 2.3. If $T \in \mathcal{D}'(\Omega, E_0^{2n+1})$, and $\phi \in \mathcal{E}(\Omega, E_0^k)$, with $0 \le k \le 2n+1$, we define $T \llcorner \phi \in \mathcal{D}'(\Omega, E_0^{2n+1-k})$ by the identity

$$\langle T \llcorner \phi|\alpha\rangle \coloneqq \langle T|\alpha \wedge \phi\rangle$$

for any $\alpha \in \mathcal{D}(\Omega, E_0^{2n+1-k})$.

The following result is taken from the study by Baldi et al. [9], Propositions 5 and 6, and Definition 10, but we refer also to the study by Dieudonné et al. [21], Sections 17.3, 17.4, and 17.5.

Proposition 2.4. *Let $\Omega \subset \mathbb{H}^n$ be an open set. If $1 \le h \le 2n+1$, $N_h = \dim E_0^h$ and $\Xi_0^h = \{\xi_1^h, \dots \xi_{N_h}^h\}$ is a left invariant basis of E_0^h and $T \in \mathcal{D}'(\Omega, E_0^h)$, then*

(i) *There exist (uniquely determined) $T_1, \dots, T_{N_h} \in \mathcal{D}'(\Omega)$ such that we can write*

$$T = \sum_j \tilde{T}_j \llcorner (*\xi_j^h),$$

with $\tilde{T}_j \in \mathcal{D}'(\Omega, E_0^{2n+1})$ constructed from T_j as in Proposition 2.2.

(ii) *If $\alpha \in \mathcal{E}(\Omega, E_0^h)$, then α can be identified canonically with a h-current T_α through the formula*

$$\langle T_\alpha|\beta\rangle \coloneqq \int_\Omega * \alpha \wedge \beta \tag{2.2}$$

for any $\beta \in \mathcal{D}(\Omega, E_0^h)$. Moreover, if $\alpha = \sum_j \alpha_j \xi_j^h$, then

$$T_\alpha = \sum_j \tilde{\alpha}_j \llcorner (*\xi_j^h),$$

where $\tilde{\alpha}_j$ is the $2n+1$-current associated with $\alpha_j \in \mathcal{D}(\Omega)$.

(iii) *We say that T is smooth in Ω when $T_1, \dots, T_{N_h}$ are (identified with) smooth functions. This is clearly equivalent to saying that there exists $\beta \in \mathcal{E}(\Omega, E_0^h)$ such that*

$$\langle T|\alpha\rangle = \int_\Omega \langle \beta, \alpha\rangle dV$$

for any $\alpha \in \mathcal{D}(\Omega, E_0^{2n+1})$ (in fact, we choose $\beta = \sum_j T_j \xi_j^h$).

Remark 2.5. If $1 \le h \le 2n+1$, let

$$\Xi_0^h = \{\xi_1^h, \dots \xi_{N_h}^h\}$$

be a left invariant basis of E_0^h. Then the linear maps on E_0^h

$$\alpha \to (\xi_j^h)^*(\alpha) := *(\alpha \wedge * \xi_j^h)$$

belong to $(E_0^h)^*$ (the dual of E_0^h) and

$$(\xi_j^h)^*(\xi_i^h) = *(\xi_i^h \wedge * \xi_j^h) = \delta_{i,j} * \mathrm{d}V = \delta_{i,j},$$

i.e., $(\Xi_0^h)^* = \{(\xi_1^h)^*, \dots, (\xi_{N_h}^h)^*\}$ is a left invariant dual basis of $(E_0^h)^*$.

Remark 2.6. Let us remind the notion of distribution section of a finite-dimensional vector bundle $\mathcal{F}$: a distribution section is a continuous linear map on the space of compactly supported sections of the dual vector bundle $\mathcal{F}^*$ [53, p. 77].

Let T be a current on E_0^h,

$$T = \sum_j \tilde{T}_j \llcorner (*\xi_j^h),$$

where $T_1, \dots, T_{N_h} \in \mathcal{D}'(\Omega)$. Then T can be seen as a section of $(E_0^h)^*$. Indeed, if $\alpha = \sum_i \alpha_i \xi_i^h \in \mathcal{D}(\Omega, E_0^h)$

$$\begin{aligned}\langle T|\alpha\rangle &= \sum_j \langle \tilde{T}_j \llcorner (*\xi_j^h)|\alpha\rangle \\ &= \sum_j \langle \tilde{T}_j|\alpha \wedge (*\xi_j^h)\rangle \\ &= \sum_j \langle T_j|\alpha_j\rangle = \sum_{i,j} \langle T_j|(\xi_j^h)^*(\alpha_i\xi_i^h)\rangle \\ &= \sum_j \langle T_j|(\xi_j^h)^*(\alpha)\rangle,\end{aligned}$$

where the dualities in the first line are meant as dualities between currents and test forms, while the dualities in the second line are meant as dualities between distributions and test functions. Thus, we can write formally

$$T = \sum_j T_j(\xi_j^h)^*, \tag{2.3}$$

and we can identify T with a vector-valued distribution $(T_1, \dots, T_{N_h})$.

We notice also that, if $\alpha = \sum_j \alpha_j \xi_j^h \in \mathcal{E}(\Omega, E_0^h)$, then

$$T_\alpha = \sum_j \alpha_j (\xi_j^h)^*.$$

Definition 2.7. If $T_{i,j} \in \mathcal{D}'(\mathbb{H}^n)$ for $i, j = 1, \dots, N_h$, we shall refer to the matrix $T := (T_{i,j})_{i,j=1,\dots,N_h}$ as to matrix-valued distribution

$$(T_{i,j})_{i,j=1,\dots,N_h} : \mathcal{D}(\mathbb{H}^n, E_0^h) \to E_0^h$$

defined through the identity

$$\langle T|\alpha\rangle := \sum_i \sum_j \langle T_{i,j}|\alpha_j\rangle \xi_i^h \tag{2.4}$$

if $\alpha = \sum_j \alpha_j \xi_j^h \in \mathcal{D}(\Omega, E_0^h)$.

A matrix-valued distribution $T := (T_{i,j})_{i,j=1,\dots,N_h}$ can also be seen as a distribution section of the fiber bundle $(\mathbb{H}^n, E_0^h \otimes (E_0^h)^*)$ ([52], p. 76) through the action

$$\langle T|A\rangle = \sum_{i,j} \langle T_{i,j}|A_{i,j}\rangle$$

for $A := \sum_{i,j} A_{i,j} (\xi_i^h)^* \otimes \xi_j^h \in \mathcal{D}(\mathbb{H}^n, (E_0^h)^* \otimes E_0^h)$.

As we did in Remark 2.6, we can write

$$T = \sum_{i,j} T_{i,j}\xi_i^h \otimes (\xi_j^h)^*.$$

3 Rumin Laplacian and heat operator in $E_0^\bullet$

This section is our main contribution. After a brief introduction on the Rumin Laplacian, we derive several basic properties of the associated heat operator. The last section is then devoted to an application to the construction of a Calderón formula in this setting.

3.1 Rumin Laplacian and its fundamental solution

Definition 3.1. In $\mathbb{H}^n$, following the study by Rumin [44], we define the operator $\Delta_{\mathbb{H},h}$ on E_0^h by setting

$$\Delta_{\mathbb{H},h} = \begin{cases} d_c d_c^* + d_c^* d_c & \text{if } h \neq n, n+1; \\ (d_c d_c^*)^2 + d_c^* d_c & \text{if } h = n; \\ d_c d_c^* + (d_c^* d_c)^2 & \text{if } h = n+1. \end{cases}$$

Notice that $-\Delta_{\mathbb{H},0}$ is the usual positive sub-Laplacian of $\mathbb{H}^n$.

Definition 3.2. (Laplacian of a current) In the sequel, when T is a h-current identified with its components $(T_1, \dots, T_{N_h})$ with respect to a fixed basis $(\xi_1)^*, \dots, (\xi_{N_h})^*$ of $(E_0^h)^*$ as in Remark 2.6, it will be useful to think of $\Delta_{\mathbb{H},h}$ as a matrix-valued differential operator $(\Delta_{\mathbb{H},h}^{i,j})_{i,j=1,\dots,N_h}$ acting as follows (again with the notations of Remark 2.6):

$$\Delta_{\mathbb{H},h} T = \Delta_{\mathbb{H},h}\Big(\sum_j T_j(\xi_j)^*\Big) = \sum_{i,j} (\Delta_{\mathbb{H},h}^{i,j} T_j)(\xi_i)^*. \tag{3.1}$$

It is easy to see that

Lemma 3.3. *If the basis $\xi_1, \dots, \xi_{N_h}$ of E_0^h is orthonormal with respect to the scalar product used to define d_c^*, then*

$$(\Delta_{\mathbb{H},h}^{i,j})^* = \Delta_{\mathbb{H},h}^{j,i}, \tag{3.2}$$

where $(\Delta_{\mathbb{H},h}^{i,j})^*$ is the formal adjoint of $\Delta_{\mathbb{H},h}^{i,j}$ on $\mathcal{D}(\mathbb{H}^n, E_0^h)$.

Definition 3.4. (The Laplacian of a matrix-valued distribution) If $T = (T_{i,j})_{i,j=1,\dots,N_h}$ is a matrix-valued distribution, we shall denote by $\Delta_{\mathbb{H},h} T$ the matrix-valued distribution defined by

$$\langle \Delta_{\mathbb{H},h} T | A \rangle \coloneqq \sum_{i,j,\ell} \langle T_{i,j} | \Delta_{\mathbb{H},h}^{i,\ell} A_{\ell,j} \rangle = \sum_{i,j,\ell} \langle \Delta_{\mathbb{H},h}^{\ell,i} T_{i,j} | A_{\ell,j} \rangle \tag{3.3}$$

for all test matrices $A = (A_{\ell,j})$.

Remark 3.5. We stress that the notation $\langle \Delta_{\mathbb{H},h} T | A \rangle$ may conflict with the notation $\langle T | \alpha \rangle$ of (2.6) if $\alpha = \sum_j \alpha_j \xi_j^h$ is a test form. If there is no way to misunderstanding, we shall use this ambiguous notation, using Greek lower case characters for forms and capital Latin characters for matrices.

In addition, if $T = (T_{i,j})_{i,j=1,\dots,N_h}$ and $S = (S_{i,j})_{i,j=1,\dots,N_h}$ are matrix-valued distributions, then the convolution $T * S$ is defined by

$$(T * S)_{i,j=1,\dots,N_h} := \sum_{\ell} T_{i,\ell} * S_{\ell,j}, \tag{3.4}$$

provided all convolutions in (3.4) are well defined.

Theorem 3.6. ([12], Theorem 3.1) *If $0 \le h \le 2n+1$, then the differential operator $\Delta_{\mathbb{H},h}$ is homogeneous of degree a with respect to the group dilations, where $a = 2$ if $h \ne n, n+1$ and $a = 4$ if $h = n, n+1$. We have:*

(*i*) *By* [44], [43], *$\Delta_{\mathbb{H},h}$ is a Rockland operator and hence is maximal hypoelliptic (in particular hypoelliptic), in the sense of* [35], *i.e., if $\Omega \subset \mathbb{H}^n$ is a bounded open set, then there exists $C = C_\Omega$ such that for any $p \in (1,\infty)$ and for any multi-index I with $|I| = a$, we have*

$$\|W^I \alpha\|_{L^p(\mathbb{H}^n, E_0^h)} \le C(\|\Delta_{\mathbb{H},h}\alpha\|_{L^p(\mathbb{H}^n, E_0^h)} + \|\alpha\|_{L^p(\mathbb{H}^n, E_0^h)}) \tag{3.5}$$

for any $\alpha \in \mathcal{D}(\Omega, E_0^h)$ and where W^I are defined in (A6).

(*ii*) *For $j = 1, \dots, N_h$ there exists*

$$K_j = (K_{1j}, \dots, K_{N_h j}), \quad j = 1, \dots N_h \tag{3.6}$$

with $K_{ij} \in \mathcal{D}'(\mathbb{H}^n) \cap \mathcal{E}(\mathbb{H}^n \backslash \{e\})$, $i, j = 1, \dots, N$ such that $\sum_\ell \Delta_{\mathbb{H},h}^{i,\ell} K_{\ell,j} = 0$ if $i \ne j$ and $\sum_\ell \Delta_{\mathbb{H},h}^{i,\ell} K_{\ell,i} = \delta_e$ (where δ_e denotes the Dirac mass at $p = e$);

(*iii*) *If $a < Q$, then the K_{ij}'s are kernels of type a in the sense of Definition* B.7 (*and hence belong to $\mathbf{K}^{a-Q}$ in the sense of Definition* B.9) *for $i, j = 1, \dots, N_h$. In particular, $K_{i,j}$ are tempered distributions. If $a = Q$, then the K_{ij}'s satisfy the logarithmic estimate $|K_{ij}(p)| \le C(1 + |\ln \rho(p)|)$ and hence belong to $L^1_{\mathrm{loc}}(\mathbb{H}^n)$. Moreover, their horizontal derivatives $W_\ell K_{ij}$, $\ell = 1, \dots, 2n$, are kernels of type $Q-1$. In particular, the K_{ij}'s belong to $\mathcal{S}'(\mathbb{H}^n)$ for $a \le Q$ for $i, j = 1, \dots, N_h$;*

(*iv*) *When $\alpha = \sum_j \alpha_j \xi_j \in \mathcal{D}(\mathbb{H}^n, E_0^h)$, if we set*

$$\Delta_{\mathbb{H},h}^{-1}\alpha := \sum_{i,j} (\alpha_j * K_{i,j})\xi_i = \alpha * (K_{i,j})_{i,j} \quad \text{(Definition 5.2)}, \tag{3.7}$$

then

$$\Delta_{\mathbb{H},h}\Delta_{\mathbb{H},h}^{-1}\alpha = \alpha. \tag{3.8}$$

Moreover, if $a < Q$, also $\Delta_{\mathbb{H},h}^{-1}\Delta_{\mathbb{H},h}\alpha = \alpha$. Thus, if we identify the operator $\Delta_{\mathbb{H},h}^{-1}$ with its distributional kernel, we can write

$$(K_{i,j})_{i,j} =: \Delta_{\mathbb{H},h}^{-1}.$$

With the notation of (3.3), (3.8) *can be written as follows*:

$$\Delta_{\mathbb{H},h}\Delta_{\mathbb{H},h}^{-1} = \delta_{e,h}, \tag{3.9}$$

where $\delta_{e,h}$ is the matrix-valued distribution $(a_{i,j})_{i,j=1,\dots,N_h}$ where $a_{i,j} = 0$ if $i \ne j$, and $a_{i,i} = \delta_e$ for $i = 1, \dots, N_h$, so that

$$\delta_{e,h} u = u(e) \quad \text{for all } u \in \mathcal{D}(\mathbb{H}^n, E_0^h);$$

(*v*) *If $a = Q$, then for any $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$, there exists $\beta_\alpha := (\beta_1, \dots, \beta_{N_h}) \in \mathbb{R}^{N_h}$, such that*

$$\Delta_{\mathbb{H},h}^{-1}\Delta_{\mathbb{H},h}\alpha - \alpha = \beta_\alpha. \tag{3.10}$$

This situation arises only when $n = 1$ and $h = 1, 2$.

The following vector-valued Liouville type theorem has been proved in [12], Proposition 3.2.

Proposition 3.7. *Suppose $\mathcal{L}$ is a left invariant hypoelliptic differential operator which is formally self-adjoint. Suppose also that $\mathcal{L}$ is homogeneous of degree $a \le Q$. If $T = (T_1, \dots, T_N) \in \mathcal{S}'(\mathbb{H}^n)^N$ satisfies $\mathcal{L}T = 0$, then T is a (vector-valued) polynomial.*

In particular, by Theorem 3.6, (*i*), *the proposition applies to $\mathcal{L} = \Delta_{\mathbb{H},h}$.*

As a consequence, the following results can be proved as in the study by Bonfiglioli et al. [13], Propositions 5.3.10 and 5.3.11.

Theorem 3.8. *Suppose $Q > a$. We have*:

(*i*) *if $\tilde{K} := (\tilde{K}_{i,j})_{i,j}$ with $\tilde{K}_{i,j} \in \mathcal{S}'(\mathbb{H}^n) \cap \mathcal{E}(\mathbb{H}^n \backslash \{e\})$, $i, j = 1, \dots, N_h$, vanishes at infinity and satisfies* (3.9), *then $\tilde{K} = \Delta_{\mathbb{H},h}^{-1}$;*

(*ii*) $\Delta_{\mathbb{H},h}^{-1} = {}^{\mathrm{v}}\Delta_{\mathbb{H},h}^{-1}$ (*identity among convolution kernels*).

Proof. Let us prove (i). Set $\Gamma := \tilde{K} - \Delta_{\mathbb{H},h}^{-1}$. By Theorem 3.6, (iii) Γ belongs to L^1_{loc}. In addition, $\Delta_{\mathbb{H},h}\Gamma = 0$, so that, by Proposition 3.7, Γ is a vector-valued polynomial. But, by Theorem 3.6, (iii), Γ has at most a logarithmic behavior at infinity and hence vanishes.

Let us prove (ii). Take $\phi = \sum_j \phi_j \xi_j \in \mathcal{D}(\mathbb{H}^n, E_0^h)$, and set

$$u(p) := \Delta_{\mathbb{H},h}\phi * \Delta_{\mathbb{H},h}^{-1}(p) = \sum_k \sum_{j,\ell} \left(\int \Delta^{\ell,j}\phi_j(q) K_{\ell,k}(q^{-1}p)\mathrm{d}q \right) \xi_k.$$

Arguing on the entries, it turns out that the matrix-valued distribution u is well defined and smooth. In addition, if $h \neq n, n+1$, by Lemma B.8,

$$u(p) = O(|p|^{2-Q}) \quad \text{as } p \to \infty. \tag{3.11}$$

Analogously, if $h = n, n+1$ and $n > 1$,

$$u(p) = O(|p|^{4-Q}) \quad \text{as } p \to \infty, \tag{3.12}$$

and, eventually,

$$u(p) = O(\ln|p|) \quad \text{as } p \to \infty \tag{3.13}$$

when $n = 1$ and $h = 1, 2$. Take now $\psi = \sum_j \psi_j \xi_j \in \mathcal{D}(\mathbb{H}^n, E_0^h)$. We have

$$\begin{aligned} \int \langle u, \Delta_{\mathbb{H},h}\psi \rangle \mathrm{d}p &= \sum_{k,i} \sum_{j,\ell} \int \left(\int \Delta_{\mathbb{H},h}^{\ell,j}\phi_j(q) K_{\ell,k}(q^{-1}p)\mathrm{d}q \right) \Delta_{\mathbb{H},h}^{k,i}\psi_i(p)\mathrm{d}p \\ &= \sum_{j,\ell} \int \mathrm{d}q \Delta_{\mathbb{H},h}^{\ell,j}\phi_j(q) \int \mathrm{d}p \sum_{k,i} K_{\ell,k}(q^{-1}p)\Delta_{\mathbb{H},h}^{k,i}\psi_i(p) \\ &= \sum_{j,\ell} \int \mathrm{d}q \Delta_{\mathbb{H},h}^{\ell,j}\phi_j(q) \int \mathrm{d}p \sum_k K_{\ell,k}(q^{-1}p)(\Delta_{\mathbb{H},h}\psi)_k(p). \end{aligned} \tag{3.14}$$

Now, putting $q^{-1}p =: \eta$, and keeping in mind that $\Delta_{\mathbb{H},h}$ is left invariant,

$$\begin{aligned} \int \mathrm{d}p \sum_k K_{\ell,k}(q^{-1}p)(\Delta_{\mathbb{H},h}\psi)_k(p) &= \int \mathrm{d}\eta \sum_k K_{\ell,k}(\eta)(\Delta_{\mathbb{H},h}\psi)_k(\tau_q\eta) \\ &= \int \mathrm{d}\eta \sum_k K_{\ell,k}(\eta)(\Delta_{\mathbb{H},h}(\psi \circ \tau_q))_k(\eta) \\ &= (\Delta_{\mathbb{H},h}^{-1}\Delta_{\mathbb{H},h}(\psi \circ \tau_q))_\ell(e) = \psi_\ell(q), \end{aligned} \tag{3.15}$$

by Theorem 3.6, (iv), provided $h \neq 1, 2$ if $n = 1$. If $n = 1$ and $h = 1, 2$, the last line must be replaced by

$$\psi_\ell(q) + \beta_\ell,$$

where $\sum_k \beta_k \xi_k$ is a constant coefficients form (depending on ψ).

Plugging (3.15) in (3.14), we obtain

$$\int \langle u, \Delta_{\mathbb{H},h}\psi\rangle dp = \sum_{j,\ell}\int dq \Delta_{\mathbb{H},h}^{\ell,j}\phi_j(q)\psi_\ell(q) = \langle \Delta_{\mathbb{H},h}\phi, \psi\rangle,$$

i.e., $\Delta_{\mathbb{H},h}\phi = \Delta_{\mathbb{H},h}u$ in the sense of distributions. On the other hand,

$$\langle \Delta_{\mathbb{H},h}\phi, \beta\rangle = 0,$$

and the conclusion still holds when $n = 1$ and $h = 1, 2$.

We can conclude that $\Delta_{\mathbb{H},h}(\phi - u) = 0$, so that, by Proposition 3.7. $\phi - u$ is a polynomial form. On the other hand, by (3.13), $\phi - u$ has at most a logarithmic behavior at infinity, so that $\phi - u = 0$. In particular,

$$\phi(e) = u(e) = \sum_k \sum_{j,\ell} (\Delta^{\ell,j}\phi_j(q) K_{\ell,k}(q^{-1}))\xi_k = \phi * {}^{\mathrm{v}}\Delta_{\mathbb{H},h}^{-1}.$$

Since ${}^{\mathrm{v}}\Delta_{\mathbb{H},h}^{-1}$ satisfies the assumptions of (i), we obtain ${}^{\mathrm{v}}\Delta_{\mathbb{H},h}^{-1} = \Delta_{\mathbb{H},h}^{-1}$. □

The aim of the following result is the characterization of some integer order Sobolev spaces of forms in $E_0^\bullet$ in terms of integer powers of Rumin Laplacian. More precisely, we prove that

Proposition 3.9. *If $k \in \mathbb{N}$, $\alpha \in L^2(\mathbb{H}^n, E_0^h) \cap \mathcal{D}(\Delta_{\mathbb{H}^n,h}^k)$, then*

$$\|\alpha\|_{L^2(\mathbb{H}^n,E_0^h)} + \|\Delta_{\mathbb{H},h}^k\alpha\|_{L^2(\mathbb{H}^n,E_0^h)}$$

is equivalent to the norm of α in $W^{ak,2}(\mathbb{H}^n, E_0^h)$ (*we remind that $a = 2$ if $h \neq n, n+1$ and $a = 4$ if $h = n, n+1$*).

Proof. For the sake of simplicity, we take $n > 1$. The case $n = 1$ can be handled in the same way.

Obviously, we have just to show that

$$\|\alpha\|_{L^2(\mathbb{H}^n,E_0^h)} + \|\Delta_{\mathbb{H},h}^k\alpha\|_{L^2(\mathbb{H}^n,E_0^h)} \geq c\ \|\alpha\|_{W^{ak,2}(\mathbb{H}^n,E_0^h)}.$$

Suppose first $\alpha \in \mathcal{S}_0(\mathbb{H}^n, E_0^h)$. By Proposition B.10, $\Delta_{\mathbb{H},h}^{-1} : \mathcal{S}_0(\mathbb{H}^n, E_0^h) \to \mathcal{S}_0(\mathbb{H}^n, E_0^h)$, so that we can write

$$\alpha = (\Delta_{\mathbb{H},h}^{-1})^k \circ \Delta_{\mathbb{H},h}^k\alpha.$$

Notice now that, by Proposition B.15 and Theorem 3.6, (i)

$$(\Delta_{\mathbb{H},h}^{-1})^k = O_0(K_k), \quad \text{with } K_k \in \mathbf{K}^{ak-Q}.$$

Moreover, by Lemma B.11, if $d(I) = ak$

$$X^I(\Delta_{\mathbb{H},h}^{-1})^k = O_0(X^I K_k), \quad \text{where } X^I K_k \in \mathbf{K}^{-Q}.$$

Thus, keeping in mind Theorem B.12, taking $d(I) = ak$,

$$\begin{aligned}\|\alpha\|_{W^{ak,2}(\mathbb{H}^n,E_0^h)} &\leq C(\|X^I\alpha\|_{L^2(\mathbb{H}^n,E_0^h)} + \|\alpha\|_{L^2(\mathbb{H}^n,E_0^h)})\\ &= C(\|O_0(X^I K_k)\Delta_{\mathbb{H},h}^k\alpha\|_{L^2(\mathbb{H}^n,E_0^h)} + \|\alpha\|_{L^2(\mathbb{H}^n,E_0^h)})\\ &\leq C(\|\Delta_{\mathbb{H},h}^k\alpha\|_{L^2(\mathbb{H}^n,E_0^h)} + \|\alpha\|_{L^2(\mathbb{H}^n,E_0^h)}).\end{aligned}$$

Then the assertion follows by density, thanks to Lemma B.18. □

3.2 Heat equation on $E_0^\bullet$

We consider now the heat operator associated with the Rumin Laplacian $\Delta_{\mathbb{H},h}$, i.e.,

$$\mathcal{L} := \partial_s + \Delta_{\mathbb{H},h} \quad \text{in } \mathbb{R}_+ \times \mathbb{H}^n,$$

where ∂_s stands for $\partial_s I_d$, I_d being the identity $N_h \times N_h$ matrix. Arguing as in (7), $\mathcal{L}$ can be written as a matrix-valued operator of the form

$$(\delta_{i,j}\partial_s + \Delta^{i,j}_{\mathbb{H},h})_{i,j=1,\dots N_h} =: (\mathcal{L}^{i,j})_{i,j=1,\dots N_h}, \tag{3.16}$$

where $\delta_{i,j}$ is the Kronecker symbol, so that, arguing as in (3.1), if $T_1, \dots, T_{N_h} \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{H}^n)$, with the convention of Remark 2.6,

$$\mathcal{L}\left(\sum_j T_j(\xi_j)^*\right) = \sum_i \left(\sum_j \mathcal{L}^{i,j} T_j\right)(\xi_i)^* = \sum_i (\partial_s T_i + (\Delta_{\mathbb{H},h}T)_i)(\xi_i)^*, \tag{3.17}$$

where $\xi_i \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{H}^n)$.

The following results are basically contained in [25], Chapter 4.B and [43] (in particular Lemma 5.4.9). However, we point out that the arguments of Folland [24,25] rely on the fact that the heat kernel is nonnegative (Hunt's theorem). Clearly this is not the case in the present situation, since h is a vector-valued kernel.

Arguing as in [25], Chapter 4.B and keeping in mind that $\Delta_{\mathbb{H},h}$ is a Rockland operator (Theorem 3.6, (i) above), we have:

Theorem 3.10. *The operator $\mathcal{L}$ is hypoelliptic on $\mathbb{R}_+ \times \mathbb{H}^n$.*

Proposition 3.11. *If $0 \le h \le 2n+1$, the operators $-\Delta_{\mathbb{H},h} : \mathcal{D}(\mathbb{H}^n, E_0^h) \subset L^2(\mathbb{H}^n, E_0^h) \to L^2(\mathbb{H}^n, E_0^h)$ are densely defined, self-adjoint, and dissipative, and therefore generate strongly continuous analytic semigroup $(\exp(-s\Delta_{\mathbb{H},h}))_{s\ge 0}$ in $L^2(\mathbb{H}^n, E_0^h)$.*

In addition,

(*i*) *for any $s > 0$, the operator $\exp(-s\Delta_{\mathbb{H},h})$ is left invariant;*

(*ii*) *if $I \subset [0,\infty)$ is a compact interval, then*

$$\sup_{s\in I} \|\exp(-s\Delta_{\mathbb{H},h})\|_{\mathcal{L}(L^2(\mathbb{H}^n,E_0^h),L^2(\mathbb{H}^n,E_0^h))} = C_I < \infty; \tag{3.18}$$

(*iii*) *for any $s > 0$, if $\alpha \in \mathcal{D}(\mathbb{H}^n, E_0^h)$, then $\exp(-s\Delta_{\mathbb{H},h})\alpha \in \mathcal{E}(\mathbb{H}^n, E_0^h)$;*

(*iv*) *for any $s > 0$,*

$$\exp(-s\Delta_{\mathbb{H},h}) : \mathcal{D}(\mathbb{H}^n, E_0^h) \to \mathcal{D}'(\mathbb{H}^n, E_0^h). \tag{3.19}$$

Proof. By a density argument, $\Delta_{\mathbb{H},h}$ is symmetric since is formally self-adjoint in $\mathcal{D}(\mathbb{H}^n, E_0^h)$. In addition, arguing as in the study by Franchi and Tesi [30], Proposition 6.18, $\Delta_{\mathbb{H},h}$ is self-adjoint and dissipative, so that generates an analytic semigroup $(\exp(-s\Delta_{\mathbb{H},h}))_{s\ge 0}$ ([36], Example 1.25). Thus, by Lunardi [41], Proposition 2.1.4, $\exp(-s\Delta_{\mathbb{H},h})$ is strongly continuous on $[0,\infty)$.

Assertion (i) follows straightforwardly by the left invariance of $\Delta_{\mathbb{H},h}$, whereas assertion (ii) follows by Banach-Steinhaus' Theorem. As for (iii), take now $k > Q/2a$ and $\beta \in (0, \min\{1, ak - Q/2\})$, so that, by [24], Theorem 5.15 and Proposition 5.10,

$$W^{ak,2}(\mathbb{H}^n) \hookrightarrow \Gamma_\beta(\mathbb{H}^n),$$

where the Folland-Stein Hölder spaces $\Gamma_\beta(\mathbb{H}^n)$ will be defined in Section B.2. If $s \in I$,

$$\begin{aligned}\|\exp(-s\Delta_{\mathbb{H},h})\alpha\|_{\Gamma_\beta(\mathbb{H}^n,E_0^h)} &\le C\|\exp(-s\Delta_{\mathbb{H},h})\alpha\|_{W^{ak,2}(\mathbb{H}^n,E_0^h)}\\ &\le C\{\|\Delta^k_{\mathbb{H},h}\exp(-s\Delta_{\mathbb{H},h})\alpha\|_{L^2(\mathbb{H}^n,E_0^h)} + \|\exp(-s\Delta_{\mathbb{H},h})\alpha\|_{L^2(\mathbb{H}^n,E_0^h)}\},\end{aligned} \tag{3.20}$$

by Proposition 3.9. Let us consider the first term, the second one can be handled in the same way. By Lunardi [41], Proposition 2.1.1, and (3.18)

$$
\begin{aligned}
\|\Delta^k_{\mathbb{H},h}\exp(-s\Delta_{\mathbb{H},h})\alpha\|_{L^2(\mathbb{H}^n,E_0^h)} &= \|\exp(-s\Delta_{\mathbb{H},h})\Delta^k_{\mathbb{H},h}\alpha\|_{L^2(\mathbb{H}^n,E_0^h)}\\
&\le C_I\|\Delta^k_{\mathbb{H},h}\alpha\|_{L^2(\mathbb{H}^n,E_0^h)} \le C_I\|\alpha\|_{W^{ak,2}(\mathbb{H}^n,E_0^h)} < \infty.
\end{aligned}
$$

Thus, (iii) is proved. Finally, (iv) follows trivially from (iii). □

Remark 3.12. Suppose $\alpha \in \mathcal{D}(\mathbb{H}^n, E_0^h)$. The arguments of the proof of (iii) in Proposition 3.11 (with the same notations) yield also that, if $s, s' \ge 0$, then

$$
\begin{aligned}
|\langle h(s,\cdot)|\alpha\rangle - \langle h(s',\cdot)|\alpha\rangle| &\le C\|\exp(-s\Delta_{\mathbb{H},h})^{\mathrm{v}}\alpha - \exp(-s'\Delta_{\mathbb{H},h})^{\mathrm{v}}\alpha\|_{\Gamma_\beta(\mathbb{H}^n,E_0^h)}\\
&\le C\{\|\exp(-s\Delta_{\mathbb{H},h})\Delta^k_{\mathbb{H},h}{}^{\mathrm{v}}\alpha - \exp(-s'\Delta_{\mathbb{H},h})\Delta^k_{\mathbb{H},h}{}^{\mathrm{v}}\alpha\|_{L^2(\mathbb{H}^n,E_0^h)}\\
&\quad + \|\exp(-s\Delta_{\mathbb{H},h})^{\mathrm{v}}\alpha - \exp(-s'\Delta_{\mathbb{H},h})^{\mathrm{v}}\alpha\|_{L^2(\mathbb{H}^n,E_0^h)}\} \to 0 \quad \text{as } s' \to s,
\end{aligned}
$$

since the semigroup is strongly continuous. This proves that

$$
\text{the map } s \to \langle h(s,\cdot)|\alpha\rangle \text{ is continuous.} \tag{3.21}
$$

In particular, if $I \subset [0,\infty)$ is a compact interval, then

$$
\sup_{s\in I}|\langle h(s,\cdot)|\alpha\rangle| < \infty. \tag{3.22}
$$

In addition, if $k > Q/2a$,

$$
\sup_{s\in I}|\langle h(s,\cdot)|\alpha\rangle| \le C\|{}^{\mathrm{v}}\alpha\|_{W^{ak,2}(\mathbb{H}^n,E_0^h)}. \tag{3.23}
$$

Proposition 3.13. *For any $s > 0$, by Proposition 3.11, (i), and (iv), there exists a matrix-valued kernel*

$$
h = h(s,p) = (h_{ij}(s,p))_{i,j=1,\dots,N_h} \in (\mathcal{D}'(\mathbb{H}^n))^{N_h\times N_h}, \tag{3.24}
$$

such that

$$
\exp(-s\Delta_{\mathbb{H},h})\alpha = \alpha * h(s,\cdot) \quad \textit{for } \alpha \in \mathcal{D}(\mathbb{H}^n, E_0^h).
$$

Here, if $\alpha = \sum_j \alpha_j\xi_j$, then $\alpha * h := \sum_i(\sum_j \alpha_j * h_{ij})\xi_i$.

In addition,

(*i*) *If $s > 0$,*

$$
\langle h(s,\cdot)|\alpha\rangle = (\exp(-s\Delta_{\mathbb{H},h})({}^{\mathrm{v}}\alpha))(e), \tag{3.25}
$$

(*ii*) *If $s > 0$,*

$$
\langle h(s,\cdot)|\,{}^{\mathrm{v}}\Delta_{\mathbb{H},h}\alpha\rangle = -\frac{\partial}{\partial s}\langle h(s,\cdot)|{}^{\mathrm{v}}\alpha\rangle. \tag{3.26}
$$

Proof. Since the convolution maps $\mathcal{D}\times\mathcal{D}'$ into $\mathcal{E}$ (see [53], Theorem 27.3), keeping in mind Proposition 3.11 and (B6), for all $\alpha \in \mathcal{D}(\mathbb{H}^n, E_0^h)$, we have

$$
\begin{aligned}
\langle h(s,\cdot)|\alpha\rangle &:= \sum_{i,j}\langle h_{ij}(s,\cdot)|\alpha_j\rangle\xi_i = \sum_{i,j}\langle{}^{\mathrm{v}}h_{ij}(s,\cdot)|{}^{\mathrm{v}}\alpha_j\rangle\xi_i = \lim_{p\to e}{}^{\mathrm{v}}\alpha * h(s,\cdot)(p)\\
&= \lim_{p\to e}(\exp(-s\Delta_{\mathbb{H},h})({}^{\mathrm{v}}\alpha))(p) = (\exp(-s\Delta_{\mathbb{H},h})({}^{\mathrm{v}}\alpha))(e).
\end{aligned} \tag{3.27}
$$

This proves (i). On the other hand, since both $\exp(-s\Delta_{\mathbb{H},h})^{\mathrm{v}}\alpha$ and $\exp(-s\Delta_{\mathbb{H},h})\,{}^{\mathrm{v}}\Delta_{\mathbb{H},h}\alpha$ are smooth functions, it follows from the identity

$$
-\frac{\partial}{\partial s}\exp(-s\Delta_{\mathbb{H},h}) = \exp(-s\Delta_{\mathbb{H},h})\Delta_{\mathbb{H},h}
$$

that the same identity holds at e. Then (ii) follows. □

Proposition 3.14. *For $i, j = 1, \dots, N_h$, we have ${}^{\mathrm{v}}h_{i,j} = h_{j,i}$, i.e.,*

$$ {}^{\mathrm{v}}h = {}^{\mathrm{t}}h. $$

Proof. By the spectral theorem, $\exp(-s\Delta_{\mathbb{H},h})$ is self-adjoint in $L^2(\mathbb{H}^n, E_0^h)$ for $s > 0$. Thus, if ϕ and ψ are arbitrary test functions, then, by (B5),

$$ \begin{aligned} \int (\psi * {}^{\mathrm{v}}h_{i,j})\phi \mathrm{d}p &= \int (\phi * h_{i,j})\psi \mathrm{d}p = \int \langle \exp(-s\Delta_{\mathbb{H},h})(\phi\xi_j), (\psi\xi_i)\rangle \mathrm{d}p \\ &= \int \langle \exp(-s\Delta_{\mathbb{H},h})(\psi\xi_i), (\phi\xi_j)\rangle \mathrm{d}p \\ &= \int (\psi * h_{j,i})\phi \mathrm{d}p. \end{aligned} $$

Thus,

$$ \psi * {}^{\mathrm{v}}h_{i,j} = \psi * h_{j,i}. $$

Take now $\psi = \psi_k$, where $(\psi_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{D}(\mathbb{H}^n)$ supported in a fixed neighborhood of e and convergent in $\mathcal{E}'$ to the Dirac δ concentrated at $p = e$ (see [53], Theorem 28.2). Taking the limit as $k \to \infty$, by [50], Théorème V, p. 157, ${}^{\mathrm{v}}h_{i,j} = \delta * {}^{\mathrm{v}}h_{i,j} = \delta * h_{j,i} = h_{j,i}$, and the assertion follows. □

Definition 3.15. The kernel $h = h(s, p)$ can be identified with a matrix-valued distribution

$$ h \in (\mathcal{D}'((0, \infty) \times \mathbb{H}^n))^{N_h \times N_h} $$

as follows: first, we notice that by [53], Theorem 39.2, a distribution in $(0, \infty) \times \mathbb{H}^n$ can be defined by its action on $\mathcal{D}(\mathbb{R}) \otimes \mathcal{D}(\mathbb{H}^n)$. Thus, arguing on the entries of h, if $v \in \mathcal{D}((0, \infty))$ and $\alpha \in \mathcal{D}(\mathbb{H}^n, E_0^h)$, we can set

$$ \langle h | v \otimes \alpha \rangle = \int_I v(s)\langle h(s, \cdot)|\alpha\rangle \mathrm{d}s \coloneqq \sum_{i,j} \Big(\int_I v(s)\langle h_{ij}(s, \cdot)|\alpha_j\rangle \mathrm{d}s\Big)\xi_i. \tag{3.28} $$

Keeping in mind (3.22), (3.28) defines a distribution.

Proposition 3.16. *We have:*

(*i*) *$\mathcal{L}h = 0$ in $(\mathcal{D}'(\mathbb{R}_+ \times \mathbb{H}^n))^{N_h \times N_h}$, i.e.,*

$$ \langle \mathcal{L}h | A(s, y)\rangle = 0 \quad \text{for all } A \in \mathcal{D}((0, \infty) \times \mathbb{H}^n), \tag{3.29} $$

where the action of the heat operator $\mathcal{L}$ on h must be understood as the formal matrix product

$$ \mathcal{L}h \coloneqq (\mathcal{L}^{i,j})_{i,j=1,\dots,N_h} \cdot (h_{i,j})_{i,j=1,\dots,N_h}, \tag{3.30} $$

defined in the sense of distributions.

(*ii*) *the matrix-valued distribution h is smooth on $(0, \infty) \times \mathbb{H}^n$. In particular, if $\phi = \sum_j \phi_j \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{H}^n, E_0^h)$, we can write*

$$ \langle h | \phi \rangle = \sum_{i,j} \Big(\int_{\mathbb{R}_+ \times \mathbb{H}^n} h_{i,j}(s, p)\phi_j(s, p) \mathrm{d}s \mathrm{d}p\Big)\xi_i; \tag{3.31} $$

(*iii*) *if $r > 0$,*

$$ h(r^a s, y) = r^{-Q} h(s, \delta_{1/r} y) \quad \text{for } s > 0 \text{ and } y \in \mathbb{H}^n; \tag{3.32} $$

(*iv*) *combining (ii) and (i), it follows that*

$$ \partial_s h_{i,j} - \sum_{\ell} (\Delta_{\mathbb{H},h})^{i,\ell} h_{\ell,j} = 0, \quad i, j = 1, \dots, N_h. \tag{3.33} $$

Proof. To prove assertion (i), by [53], Theorem 39.2, we check the identity on forms

$$\psi \otimes A \in \mathcal{D}(\mathbb{R}) \otimes \mathcal{D}(\mathbb{H}^n, (E_0^h)^* \otimes E_0^h).$$

We have:

$$\begin{aligned}\langle(\partial_s + \Delta_{\mathbb{H},h})h|\psi \otimes A\rangle &:= \langle h|(-\partial_s + \Delta_{\mathbb{H},h})(\psi \otimes A)\rangle\\ &= \langle h|-\psi'(s) \otimes A + \psi(s) \otimes \Delta_{\mathbb{H},h}A\rangle\\ &= -\sum_{i,j}\int ds\psi'(s)\langle h_{ij}|A_{i,j}\rangle + \sum_{i,j}\int ds\psi(s)\langle h_{i,j}|(\Delta_{\mathbb{H},h}A)_{i,j}\rangle.\end{aligned}$$

Put

$$\Phi_j = \sum_\ell A_{\ell,j}\xi_\ell, \quad \text{so that } A_{\ell,j} = (\Phi_j)_\ell.$$

Now we have

$$\begin{aligned}\sum_{i,j}\langle h_{i,j}|(\Delta_{\mathbb{H},h}A)_{i,j}\rangle &= \sum_{i,j}\sum_\ell\langle h_{i,j}|\Delta_{\mathbb{H},h}^{i,\ell}A_{\ell,j}\rangle\\ &= \sum_{i,j}\sum_\ell\langle h_{i,j}|\Delta_{\mathbb{H},h}^{i,\ell}(\Phi_j)_\ell\rangle = \sum_{i,j}\langle h_{i,j}|(\Delta_{\mathbb{H},h}\Phi_j)_i\rangle\\ &= \sum_{i,j}\langle h_{j,i}|\,{}^{\mathrm{v}}(\Delta_{\mathbb{H},h}\Phi_j)_i\rangle = \sum_j(\exp(-s\Delta_{\mathbb{H},h})\Delta_{\mathbb{H},h}\Phi_j(e))_j\\ &= -\frac{\partial}{\partial s}\sum_j(\exp(-s\Delta_{\mathbb{H},h})\Phi_j(e))_j.\end{aligned}$$

Integrating by parts,

$$\sum_{i,j}\int ds\psi(s)\langle h_{i,j}|(\Delta_{\mathbb{H},h}A)_{i,j}\rangle = \sum_j\int ds\psi'(s)(\exp(-s\Delta_{\mathbb{H},h})\Phi_j(e))_j.$$

On the other hand,

$$\sum_{i,j}\langle h_{i,j}|A_{i,j}\rangle = \sum_{i,j}(\Phi_j)_i * h_{j,i}(e) = \sum_j(\exp(-s\Delta_{\mathbb{H},h})\Phi_j(e))_j,$$

and the assertion is proved.

To prove (ii), let us consider the currents $H_\lambda = \sum_\ell h_{\ell,\lambda}\xi_\ell^*$, for $\lambda = 1,\dots,N_h$. We want to show that

$$(\partial_s + \Delta_{\mathbb{H},h})H_\lambda = 0.$$

If $\alpha = \sum\alpha_\ell\xi_\ell$ is a test form and $\psi \in \mathcal{D}(\mathbb{R})$, this means that

$$\langle(\partial_s + \Delta_{\mathbb{H},h})H_\lambda|\psi \otimes \alpha\rangle = 0.$$

Now

$$\begin{aligned}\langle(\partial_s + \Delta_{\mathbb{H},h})H_\lambda|\sum_\ell(\psi \otimes \alpha_\ell)\xi_\ell\rangle &= \langle H_\lambda|\sum_{j,\ell}(-\partial_s + \Delta_{\mathbb{H},h}^{j\ell})(\psi \otimes \alpha_\ell)\xi_j\rangle = \sum_{i,\ell}\langle h_{i,\lambda}|(-\partial_s + \Delta_{\mathbb{H},h}^{i\ell})(\psi \otimes \alpha_\ell)\rangle\\ &= \sum_{i,j,\ell}\langle h_{i,j}|(-\partial_s + \Delta_{\mathbb{H},h}^{i\ell})(\psi \otimes \delta_{\lambda,j}\alpha_\ell)\rangle = 0\end{aligned}$$

by (i), if we choose $A_{\ell,j}^{(\lambda)} := \delta_{\lambda,j}\alpha_\ell$.

Thus, by Theorem 3.10, H_λ is smooth for $\lambda = 1,\dots,N_h$, and hence, the $h_{i,j}$'s are smooth for $i,j = 1,\dots,N_h$.

Finally, to prove (iii), let us consider the case $a = 2$. The case $a = 4$ can be handled in the same way. Keeping in mind (3.31) and [53], Theorem 39.2, by density, it will be enough to prove that, if $u = \sum_j u_j\xi_j \in \mathcal{D}(\mathbb{H}^n, E_0^h)$ and $v \in \mathcal{D}(0,\infty)$, then

$$
\begin{aligned}
\langle h(r^2 s, y)|v \otimes u\rangle &= \sum_{i,j}\left\{\int_0^\infty\left(\int_{\mathbb{H}^n} h_{i,j}(r^2 s, y)u_j(y)\mathrm{d}y\right)v(s)\mathrm{d}s\right\}\xi_i \\
&= r^{-Q}\sum_{i,j}\left\{\int_0^\infty\left(\int_{\mathbb{H}^n} h_{i,j}(s, \delta_{1/r}y)u_j(y)\mathrm{d}y\right)v(s)\mathrm{d}s\right\}\xi_i \\
&= r^{-Q}\langle h(s, \delta_{1/r}y)|v \otimes u\rangle.
\end{aligned}
\tag{3.34}
$$

Now, if we put $v_r(s) \coloneqq v(r^{-2}s)$,

$$
\begin{aligned}
&\sum_{i,j}\left\{\int_0^\infty\left(\int_{\mathbb{H}^n} h_{i,j}(r^2 s, y)u_j(y)\mathrm{d}y\right)v(s)\mathrm{d}s\right\}\xi_i \\
&= r^{-2}\sum_{i,j}\left\{\int_0^\infty\left(\int_{\mathbb{H}^n} h_{i,j}(s, y)u(y)\mathrm{d}y\right)v(r^{-2}s)\mathrm{d}s\right\}\xi_i \\
&\eqqcolon r^{-2}\langle h|v_r \otimes u\rangle \\
&= r^{-2}\int_0^\infty v_r(s)(\exp(-s\Delta_{\mathbb{H},h})\ {}^{\mathrm{v}}u)(e)\mathrm{d}s \\
&= \int_0^\infty v(s)(\exp(-r^2 s\Delta_{\mathbb{H},h})\ {}^{\mathrm{v}}u)(e)\mathrm{d}s \\
&= \int_0^\infty v(s)(\exp(-s\Delta_{\mathbb{H},h})({}^{\mathrm{v}}u \circ \delta_r))(e)\mathrm{d}s \\
&\quad (\text{since} \quad \exp(-s\Delta_{\mathbb{H},h}) \quad \text{commutes with group dilations}) \\
&= \langle h|v \otimes u \circ \delta_r\rangle \\
&= \sum_{i,j}\left\{\int_0^\infty\left(\int_{\mathbb{H}^n} h(s, y)u_j(\delta_r y)\mathrm{d}y\right)v(s)\mathrm{d}s\right\}\xi_i \\
&= r^{-Q}\sum_{i,j}\left\{\int_0^\infty\left(\int_{\mathbb{H}^n} h(s, \delta_{1/r}y)u_j(y)\mathrm{d}y\right)v(s)\mathrm{d}s\right\}\xi_i,
\end{aligned}
$$

and (3.34) follows. □

Theorem 3.17. *Denote by $\tilde{h}$ the matrix-valued function on $\mathbb{R} \times \mathbb{H}^n$ defined continuing h by zero for $s \le 0$. Then*

(i) *keeping in mind* (3.31), *$\tilde{h}$ defines a matrix-valued distribution*

$$\tilde{h} \in (\mathcal{D}'(\mathbb{R} \times \mathbb{H}^n))^{N_h \times N_h}$$

by the identity

$$\langle \tilde{h}|v \otimes u\rangle \coloneqq \int_0^\infty v(s)\langle h(s, \cdot)|u\rangle \mathrm{d}s \tag{3.35}$$

when $v \in \mathcal{D}(-\infty, \infty)$ and $u \in \mathcal{D}(\mathbb{H}^n, E_0^h)$;

(ii) *$\mathcal{L}\tilde{h} = \delta_0 \otimes \delta_{e,h}$, where $\delta_{e,h}$ is the matrix-valued distribution $(a_{i,j})_{i,j=1,\dots,N_h}$, where $a_{i,j} = 0$ if $i \ne j$, and $a_{i,i} = \delta_e$ for $i = 1, \dots, N_h$, so that*

$$\delta_{e,h}u = u(e) \quad \textit{for all } u \in \mathcal{D}(\mathbb{H}^n, E_0^h);$$

(iii) *$\tilde{h} \in \mathbb{C}^\infty((\mathbb{R} \times \mathbb{H}^n)\setminus(0, e))$.*

Proof. Again by [53], Theorem 39.2, identity (3.35) defines a distribution. This proves (i).

Let us prove (ii). Arguing as in (3.29),

$$\langle \mathcal{L}\tilde{h}|v\otimes u\rangle = \langle \tilde{h}| -\partial_s v\otimes u + v\otimes \Delta_{\mathbb{H},h}u\rangle = -\int_0^\infty \partial_s v(s)\langle h(s,\cdot)|u\rangle + \int_0^\infty v(s)\langle h(s,\cdot)|\Delta_{\mathbb{H},h}u\rangle ds.$$

By (3.23), the integrals

$$\int_{\mathbb{H}^n} h(s,y)u(y)dy \quad \text{and} \quad \int_{\mathbb{H}^n} h(s,y)\Delta_{\mathbb{H},h}u(y)dy$$

are both bounded for $s\in \operatorname{supp} v\cap[0,\infty)$, and hence,

$$\partial_s v(s)\int_{\mathbb{H}^n} h(s,y)(y)u dy \quad \text{and} \quad v(s)\int_{\mathbb{H}^n} h(s,y)\Delta_{\mathbb{H},h}u(y)dy$$

belong to $L^1([0,\infty))$, and we can write

$$\begin{aligned}
&-\int_0^\infty \partial_s v(s)\left(\int_{\mathbb{H}^n} h(s,y)u(y)dy\right) + \int_0^\infty v(s)\left(\int_{\mathbb{H}^n} h(s,y)\Delta_{\mathbb{H},h}u(y)dy\right)ds\\
&= \lim_{\varepsilon\to 0}\left\{-\int_\varepsilon^\infty \partial_s v(s)\left(\int_{\mathbb{H}^n} h(s,y)u(y)dy\right) + \int_\varepsilon^\infty v(s)\left(\int_{\mathbb{H}^n} h(s,y)\Delta_{\mathbb{H},h}u(y)dy\right)ds\right\} := \lim_{\varepsilon\to 0}\{I_\varepsilon + J_\varepsilon\}.
\end{aligned}$$

Since

$$(s,y)\to \partial_s v(s)u(y) \quad \text{belongs to } L^1([\varepsilon,\infty)\times\mathbb{H}^n),$$

we have

$$\begin{aligned}
I_\varepsilon &= -\int_\varepsilon^\infty \partial_s v(s)\left(\int_{\mathbb{H}^n} h(s,y)u(y)dy\right)\\
&= -\int_{\mathbb{H}^n} u(y)\left(\int_\varepsilon^\infty h(s,y)\partial_s v(s)ds\right)dy\\
&= v(\varepsilon)\int_{\mathbb{H}^n} u(y)h(\varepsilon,y)dy + \int_{\mathbb{H}^n} u(y)\left(\int_\varepsilon^\infty \partial_s h(s,y)v(s)ds\right)dy.
\end{aligned}$$

Thus, keeping in mind Proposition 24 (iii)(2),

$$I_\varepsilon + J_\varepsilon = v(\varepsilon)\int_{\mathbb{H}^n} u(y)h(\varepsilon,y)dy = v(\varepsilon)(\exp(-\varepsilon\Delta_{\mathbb{H},h})\,{}^{\mathrm{v}}u)(e).$$

Arguing as in Remark 3.12, if we take $k > Q/2a$

$$|(\exp(-\varepsilon\Delta_{\mathbb{H},h})\,{}^{\mathrm{v}}u)(e) - u(e)| = |(\exp(-\varepsilon\Delta_{\mathbb{H},h})\,{}^{\mathrm{v}}u)(e) - {}^{\mathrm{v}}u(e)| \le C\|(\exp(-\varepsilon\Delta_{\mathbb{H},h}) - I)\Delta_{\mathbb{H},h}^k({}^{\mathrm{v}}u)\|_{L^2(\mathbb{H}^n,E_0^h)} \to 0$$

as $\varepsilon\to 0$, since $s\to\exp(-s\Delta_{\mathbb{H},h})$ is a strongly continuous semigroup. This proves (ii). Finally, (iii) follows straighforwardly from (ii) and Theorem 3.10. □

Theorem 3.18. *For any $s,\sigma>0$, we have*
(*i*) $h_{i,j}(s,\cdot)\in\mathcal{S}(\mathbb{H}^n)$, $i,j=1,\dots,N_h$;
(*ii*) $h(s,\cdot)*h(\sigma,\cdot) = h(\sigma,\cdot)*h(s,\cdot) = h(s+\sigma,\cdot)$.

Proof. By the very definition of $\tilde{h}$, $k,\ell\in\mathbb{N}\cup\{0\}$, then the map $(s,y)\mapsto s^{-k}\,\partial_s^\ell\tilde{h}_{i,j}(s,y)$ is continuous away from $(0,e)$.

Thus, if $K\subset\mathbb{H}^n$ is a compact set, $e\notin K$,

$$\sup_K |s^{-k}\partial_s^\ell h_{i,j}(s,\cdot)| \to 0 \quad \text{as } s \to 0\,.$$

Then the proof of (i) can be carried out as in [25], Proposition 1.74. In addition, (i) implies that the convolutions in (ii) well defined and so (ii) follows from the semigroup property of $s \to \exp(-s\Delta_{\mathbb{H},h})$. □

Thanks to the density of $\mathcal{D}(\mathbb{H}^n)$ in $\mathcal{D}'(\mathbb{H}^n)$, the following lemma holds ([53], p. 272).

Lemma 3.19. *If $T \in \mathcal{D}'(\mathbb{H}^n)$, and there exist $C > 0$ and $N \in \mathbb{N}$ such that*

$$|\langle T|\phi\rangle| \le C \sup_{m+|\alpha|\le N} \sup_{p\in\mathbb{H}^n} (1+|p|)^m |D^\alpha \phi(p)|$$

for all $\phi \in \mathcal{D}(\mathbb{H}^n)$, then $T \in \mathcal{S}'(\mathbb{H}^n)$.

Combining the previous lemma with Theorem XIII p. 74 of [50], we have

Proposition 3.20. *Let $(T_j)_{j\in\mathbb{N}}$ a sequence in $\mathcal{S}'(\mathbb{H}^n) \subset \mathcal{D}'(\mathbb{H}^n)$ such that*

(i) *$(T_j)_{j\in\mathbb{N}}$ is bounded in $\mathcal{S}'(\mathbb{H}^n)$, i.e., there exist $C > 0$ and $N \in \mathbb{N}$ such that*

$$|\langle T_j|\phi\rangle| \le C \sup_{m+|\alpha|\le N} \sup_{p\in\mathbb{H}^n} (1+|p|)^m |D^\alpha \phi(p)| \tag{3.36}$$

for all $\phi \in \mathcal{D}(\mathbb{H}^n)$ and $j \in \mathbb{N}$;

(ii) *the sequence $(\langle T_j|\phi\rangle)_{j\in\mathbb{N}}$ has a limit $\langle T|\phi\rangle$ as $j \to \infty$ for all $\phi \in \mathcal{D}(\mathbb{H}^n)$,*

then $T \in \mathcal{S}'(\mathbb{H}^n)$ and $T_j \to T$ in $\mathcal{D}'(\mathbb{H}^n)$ as $j \to \infty$.

Proof. By Theorem XIII p. 74 of [50], $T \in \mathcal{D}'(\mathbb{H}^n)$. On the other hand, (3.36) still holds for T, and the assertion follows from Lemma 3.19. □

Proposition 3.21. *If $\phi \in \mathcal{D}(\mathbb{H}^n)$ and $i,j = 1,\dots,N_h$, we have*

(i) *by (3.21), the function $s \to \langle h_{i,j}(s,\cdot)|\phi\rangle$ is continuous for $s \ge 0$, and the identity*

$$\left\langle \int_0^M h_{i,j}(s,\cdot)\,\mathrm{d}s \,\middle|\, \phi \right\rangle := \int_0^M \langle h_{i,j}(s,\cdot)|\phi\rangle\,\mathrm{d}s \tag{3.37}$$

defines a tempered distribution for all $M > 1$;

(ii) *the function $s \to \langle h_{i,j}(s,\cdot)|\phi\rangle$ belongs to $L^1([0,\infty))$*

(iii) *the identity*

$$\left\langle \int_0^\infty h_{i,j}(s,\cdot)\,\mathrm{d}s \,\middle|\, \phi \right\rangle := \lim_{M\to\infty} \int_0^M \langle h_{i,j}(s,\cdot)|\phi\rangle\,\mathrm{d}s \tag{3.38}$$

defines a tempered distribution. Thus,

$$\int_0^\infty h(s,\cdot)\,\mathrm{d}s := \left(\int_0^\infty h_{i,j}(s,\cdot)\,\mathrm{d}s\right)_{i,j=1,\dots,N_h}$$

is a matrix-valued tempered distribution. Notice that, by Proposition 3.20, we can write also

$$\int_0^\infty h_{i,j}(s,\cdot)\,\mathrm{d}s = \lim_{M\to\infty} \int_0^M h_{i,j}(s,\cdot)\,\mathrm{d}s \quad \text{in } \mathcal{D}'(\mathbb{H}^n). \tag{3.39}$$

Proof. Let us prove the following statement: there exist $C > 0$ and $N > 0$ independent of $s > 0$ and ϕ such that

$$|\langle h_{i,j}(s,\cdot)|\phi\rangle| \le C\min\{1, s^{-Q/a}\} \sup_{m+|\alpha|\le N} \sup_{p\in\mathbb{H}^n}(1+|p|)^m|D^\alpha\phi(p)|. \tag{3.40}$$

Then (i), (ii), and (iii) will follow by Proposition 3.20.

First of all, we prove that, if $M > 1$, there exist $C_M > 0$ and $N \in \mathbb{N}$ such that, if $0 < s \le M$,

$$|\langle h_{i,j}(s,\cdot)|\phi\rangle| \le C_M \sup_{m+|\alpha|\le N} \sup_{p\in\mathbb{H}^n}(1+|p|)^m|D^\alpha\phi(p)| \tag{3.41}$$

for all $\phi \in \mathcal{D}(\mathbb{H}^n)$. Indeed, by (3.23), if $k > Q/2a$ and $I \subset [0,\infty)$ is a compact interval, then

$$\sup_{s\in I}|\langle h(s,\cdot)|\alpha\rangle| \le C_I\ \|{}^{\mathrm{v}}\alpha\|_{W^{ak,2}(\mathbb{H}^n)}. \tag{3.42}$$

On the other hand, if J is a multi-index with $d(J) \le ak$ there exists a family of polynomials P_σ, $|\sigma| \le ak$, such that for any function $u \in \mathcal{S}(\mathbb{H}^n)$

$$\begin{aligned}
\|W^J u\|^2_{L^2(\mathbb{H}^n)} &\le \sum_{|\sigma|\le ak}\int_{\mathbb{H}^n}|P_\sigma(p)D^\sigma u|^2 dp\\
&\le C\int_{\mathbb{H}^n}(1+|p|)^{-2m}dp\cdot\sum_{|\sigma|\le ak}\sup_{p\in\mathbb{H}^n}(1+|p|)^{2m}|D^\sigma(p)|^2\\
&= C\sum_{|\sigma|\le ak}\sup_{p\in\mathbb{H}^n}(1+|p|)^{2m}|D^\sigma\phi(p)|^2
\end{aligned}$$

for m large enough. This proves (3.36).

On the other hand, keeping in mind (3.32) and Theorem 3.18, if $s > 1$, then

$$\begin{aligned}
|\langle h_{i,j}(s,\cdot)|\phi\rangle| &\le \int_{\mathbb{H}^n}|h_{i,j}(s,p)\phi(p)|dp\\
&= s^{-Q/a}\int_{\mathbb{H}^n}|h_{i,j}(1,\delta_{s^{-1/a}}p)\phi(p)|dp\\
&\le s^{-Q/a}\sup_{p\in\mathbb{H}^n}|h_{i,j}(1,p)|\|\phi\|_{L^1(\mathbb{H}^n,E_0^h)}\\
&\le Cs^{-Q/a}\sup_{p\in\mathbb{H}^n}|h_{i,j}(1,p)|\sup_{p\in\mathbb{H}^n}(1+|p|)^{2n+2}|\phi(p)|.
\end{aligned} \tag{3.43}$$

Then, combining (3.41) and (3.43), (3.40) follows. □

Theorem 3.22. *We have*

$$\int_0^\infty h(s,\cdot)ds = \Delta^{-1}_{\mathbb{H},h} \tag{3.44}$$

(*identity between convolution kernels*).

Proof. First, let us prove that

$$\Delta_{\mathbb{H},h}\int_0^\infty h(s,\cdot)ds = \delta_{e,h}. \tag{3.45}$$

To this end, let $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^h)$ be a test form. Suppose that supp $\phi \subset K$, where $K \subset \mathbb{H}^n$ is a compact set. We notice first that

$$\int_0^1|\langle h(s,\cdot)|\Delta_{\mathbb{H},h}\phi\rangle|ds < \infty \tag{3.46}$$

(the integral is well defined by Remark 3.12). Then by (3.39), Proposition 3.21, (ii), (3.33), and (3.46), we have

$$\begin{aligned}\langle \Delta_{\mathbb{H},h}\int_0^\infty h(s,\cdot)\mathrm{d}s|\phi\rangle &= \lim_{M\to\infty}\int_0^M \langle h(s,\cdot)|\Delta_{\mathbb{H},h}\phi\rangle \mathrm{d}s\\ &= \lim_{M\to\infty}\int_1^M \langle h(s,\cdot)|\Delta_{\mathbb{H},h}\phi\rangle \mathrm{d}s + \lim_{\varepsilon\to 0}\int_\varepsilon^1 \langle h(s,\cdot)|\Delta_{\mathbb{H},h}\phi\rangle \mathrm{d}s\\ &= -\lim_{M\to\infty}\int_1^M \langle \partial_s h(s,\cdot)|\phi\rangle \mathrm{d}s - \lim_{\varepsilon\to 0}\int_\varepsilon^1 \langle \partial_s h(s,\cdot)|\phi\rangle \mathrm{d}s\\ &= -\lim_{M\to\infty}\langle h(M,\cdot)|\phi\rangle + \lim_{\varepsilon\to 0}\langle h(\varepsilon,\cdot)|\phi\rangle\\ &= \lim_{\varepsilon\to 0}(\exp(-\varepsilon\Delta_{\mathbb{H},h})\phi)(e) = \phi(e).\end{aligned}$$

Thus, (3.45) holds, and then

$$\int_0^\infty h(s,\cdot)\mathrm{d}s \in (\mathcal{E}(\mathbb{H}^n\backslash\{e\}))^{N_h\times N_h}$$

since $\Delta_{\mathbb{H},h}$ is hypoelliptic, by Theorem 3.6, (i). Thus, by keeping into account Proposition 3.21, (ii), we have

$$\int_0^\infty h(s,\cdot)\mathrm{d}s \in (\mathcal{S}'(\mathbb{H}^n,E_0^\bullet)\cap\mathcal{E}(\mathbb{H}^n\backslash\{e\}))^{N_h\times N_h}.$$

and (3.44) follows from Proposition 3.8, provided we prove that

$$\lim_{p\to\infty}\int_0^\infty h(s,p)\mathrm{d}s = 0. \tag{3.47}$$

But, thanks to (3.32), it follows easily that

$$\int_0^\infty h(s,\cdot)\mathrm{d}s \quad \text{is a (vector-valued) kernel of type } \alpha$$

(Definition B.7), which vanishes at infinity since $Q > \alpha$. This completes the proof of the theorem. □

Corollary 3.23. *By Theorems* 3.22 *and* 3.8,

$$\Delta_{\mathbb{H},h}\int_0^\infty {}^{\mathrm{v}}h(s,\cdot)\mathrm{d}s = \Delta_{\mathbb{H},h}\ {}^{\mathrm{v}}\Delta_{\mathbb{H},h}^{-1} = \Delta_{\mathbb{H},h}\Delta_{\mathbb{H},h}^{-1} = \delta_{e,h}.$$

Lemma 3.24. *If* $\alpha\in L^1(\mathbb{H}^n,E_0^\bullet)\subset\mathcal{S}'(\mathbb{H}^n,E_0^\bullet)$ *and* $s>0$, *then*

$$F(s,\cdot) \coloneqq h\left(\frac{s}{2},\cdot\right) * d_c^*\alpha = d_c^*\left(h\left(\frac{s}{2},\cdot\right)*\alpha\right) \in \mathcal{O}_M(\mathbb{H}^n,E_0^\bullet)\subset\mathcal{E}(\mathbb{H}^n,E_0^\bullet)\cap\mathcal{S}'(\mathbb{H}^n,E_0^\bullet) \tag{3.48}$$

(*we recall that* $\mathcal{O}_M$ *denotes the space of the smooth functions slowly increasing at infinity: see the study by Treves* [53], *Theorem 25.5, p. 275 or* [50], *p.* 243).

Proof. Since $L^1(\mathbb{H}^n)\subset\mathcal{S}'(\mathbb{H}^n)$ then both $h\left(\frac{s}{2},\cdot\right)*d_c^*\alpha$ and $d_c^*\left(h\left(\frac{s}{2},\cdot\right)*\alpha\right)$ belong to $\mathcal{O}_M$ (see [50], p. 248). On the other hand, given $\phi\in\mathcal{D}(\mathbb{H}^n,E_0^\bullet)$, we have

$$\begin{aligned}\left\langle d_c^*\left(h\left(\frac{s}{2},\cdot\right)*\alpha\right)\,\middle|\,\phi\right\rangle &= \left\langle h\left(\frac{s}{2},\cdot\right)*\alpha\,\middle|\,d_c\phi\right\rangle \\ &= \left\langle \alpha|{}^{\mathrm{v}}h\left(\frac{s}{2},\cdot\right)*d_c\phi\right\rangle = \left\langle \alpha|d_c\left({}^{\mathrm{v}}h\left(\frac{s}{2},\cdot\right)*\phi\right)\right\rangle \\ &= \left\langle d_c^*\alpha|{}^{\mathrm{v}}h\left(\frac{s}{2},\cdot\right)*\phi\right\rangle = \left\langle h\left(\frac{s}{2},\cdot\right)*d_c^*\alpha\,\middle|\,\phi\right\rangle.\end{aligned}$$

□

Remark 3.25. Again by [50], p. 248, for any $s > 0$,

$$h\left(\frac{s}{2},\cdot\right)*F(s,\cdot)\in O_M\subset\mathcal{E}(\mathbb{H}^n,E_0^\bullet)\cap\mathcal{S}'(\mathbb{H}^n,E_0^\bullet).$$

Lemma 3.26. *The function*

$$s\to\left\langle d_c\left(h\left(\frac{s}{2},\cdot\right)*F(s,\cdot)\right)\,\middle|\,\phi\right\rangle$$

belongs to $L^1([0,\infty))$ *for all* $\phi\in\mathcal{D}(\mathbb{H}^n,E_0^\bullet)$. *In particular, for all* $\phi\in\mathcal{D}(\mathbb{H}^n,E_0^\bullet)$, *there exists*

$$\lim_{M\to\infty}\int_0^M\left\langle d_c\left(h\left(\frac{s}{2},\cdot\right)*F(s,\cdot)\right)\,\middle|\,\phi\right\rangle ds.$$

Proof. If $s > 0$, keeping in mind Theorem 3.18, we have

$$\begin{aligned}\left|\left\langle d_c\left(h\left(\frac{s}{2},\cdot\right)*F(s,\cdot)\right)\,\middle|\,\phi\right\rangle\right| &= \left|\left\langle h\left(\frac{s}{2},\cdot\right)*F(s,\cdot)\,\middle|\,d_c^*\phi\right\rangle\right| \\ &= \left|\left\langle F(s,\cdot)|{}^{\mathrm{v}}h\left(\frac{s}{2},\cdot\right)*d_c^*\phi\right\rangle\right| = \left|\left\langle h\left(\frac{s}{2},\cdot\right)*d_c^*\alpha\,\middle|\,{}^{\mathrm{v}}h\left(\frac{s}{2},\cdot\right)*d_c^*\phi\right\rangle\right| \\ &= |\langle d_c^*\alpha|{}^{\mathrm{v}}h(s,\cdot)*d_c^*\phi\rangle| \quad \text{(by Theorem 3.8 and (91))} \\ &= |\langle\alpha|{}^{\mathrm{v}}h(s,\cdot)*d_cd_c^*\phi\rangle| \\ &\le \int|\langle\alpha,{}^{\mathrm{v}}h(s,\cdot)*d_cd_c^*\phi\rangle|dp \\ &\le \|\alpha\|_{L^1(\mathbb{H}^n,E_0^\bullet)}\cdot\|{}^{\mathrm{v}}d_cd_c^*\phi*h(s,\cdot)\|_{L^\infty(\mathbb{H}^n,E_0^\bullet)}.\end{aligned}$$

On the other hand, by (3.23),

$$\sup_{[0,1]}\|{}^{\mathrm{v}}d_cd_c^*\phi*h(s,\cdot)\|_{L^\infty(\mathbb{H}^n,E_0^\bullet)}\le C_\phi,$$

whereas, if $s > 1$, by (3.32),

$$\begin{aligned}\|{}^{\mathrm{v}}d_cd_c^*\phi*h(s,\cdot)\|_{L^\infty(\mathbb{H}^n,E_0^\bullet)} &\le C_\phi\|h(s,\cdot)\|_{L^\infty(\mathbb{H}^n,E_0^\bullet)} \\ &\le C_\phi s^{-Q/a}\|h(1,\cdot)\|_{L^\infty(\mathbb{H}^n,E_0^\bullet)}.\end{aligned}$$

Since $a < Q$, the assertion is proved. □

Thanks to Lemma 3.26 and Theorem XIII p. 74 of [50], we can define the following distribution:

Definition 3.27. We set

$$\int_0^\infty d_c\left(h\left(\frac{s}{2},\cdot\right)*F(s,\cdot)\right)ds := \lim_{M\to\infty}\int_0^M d_c\left(h\left(\frac{s}{2},\cdot\right)*F(s,\cdot)\right)ds$$

in $\mathcal{D}'(\mathbb{H}^n, E_0^\bullet)$, where

$$\left\langle \int_0^M d_c\left(h\left(\frac{s}{2}, \cdot\right) * F(s, \cdot)\right) \,\middle|\, \phi \right\rangle ds := \int_0^M \left\langle d_c\left(h\left(\frac{s}{2}, \cdot\right) * F(s, \cdot)\right) \,\middle|\, \phi \right\rangle ds$$

for all $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^\bullet)$.

3.3 The Calderón reproducing formula

If $\alpha \in L^1(\mathbb{H}^n, E_0^h)$, $d_c\alpha = 0$, let us set

$$F(s, x) := d_c^*\left(h\left(\frac{s}{2}, \cdot\right) * \alpha\right)(x) \quad s > 0. \tag{3.49}$$

By (3.48), for any $s > 0$ $F(s, \cdot) \in \mathcal{O}_M \subset \mathcal{E}(\mathbb{H}^n, E_0^{2n-h}) \cap \mathcal{S}'(\mathbb{H}^n, E_0^{2n-h})$. In particular, $F(s, \cdot)$ is smooth for any $s > 0$. In addition, again by (3.48), we can write

$$F(s, \cdot) = h\left(\frac{s}{2}, \cdot\right) * d_c^*\alpha.$$

If $\alpha = \sum_j \alpha_j \xi_j^h \in L^1(\mathbb{H}^n, E_0^h)$, there exist homogeneous differential operators in the horizontal derivative $P_{j,\ell}$, say of order 1 or 2 according to the degree of the forms, such that (with the formal notation of (5))

$$d_c^*\alpha = \sum_{j,\ell} (P_{j,\ell}\alpha_j)(\xi_\ell^{h-1})^*.$$

Theorem 3.28. *If $\alpha \in L^1(\mathbb{H}^n, E_0^h)$ is a d_c-closed form, we have*

$$\alpha = -\int_0^\infty d_c\left(h\left(\frac{s}{2}, \cdot\right) * F(s, \cdot)\right) ds. \tag{3.50}$$

Proof. Since both α and $\int_0^\infty d_c\left(h\left(\frac{s}{2}, \cdot\right) * F(s, \cdot)\right) ds$ belong to $\mathcal{D}'(\mathbb{H}^n, E_0^h)$ (see Definition 3.27), it will be enough to show that, if $\phi \in \mathcal{D}(\mathbb{H}^n, E_0^h)$, then

$$\begin{aligned} \langle \alpha | \phi \rangle &= -\left\langle \int_0^\infty d_c\left(h\left(\frac{s}{2}, \cdot\right) * F(s, \cdot)\right) ds \,\middle|\, \phi \right\rangle \\ &:= -\lim_{M\to\infty} \int_0^M \left\langle d_c\left(h\left(\frac{s}{2}, \cdot\right) * F(s, \cdot)\right) \,\middle|\, \phi \right\rangle ds. \end{aligned} \tag{3.51}$$

Suppose first that $\alpha \in \mathcal{D}(\mathbb{H}^n, E_0^h)$. If $h \neq n + 1$, we have

$$
\begin{aligned}
\int_0^M \left\langle d_c\left(h\left(\frac{s}{2},\cdot\right) * F(s,\cdot)\right) \,\middle|\, \phi \right\rangle \mathrm{d}s &= \int_0^M \left\langle \left(h\left(\frac{s}{2},\cdot\right) * F(s,\cdot)\right) \,\middle|\, d_c^*\phi \right\rangle \mathrm{d}s \\
&= \int_0^M \left\langle F(s,\cdot) | {}^{\mathrm{v}}h\left(\frac{s}{2},\cdot\right) * d_c^*\phi \right\rangle \mathrm{d}s \quad \text{(by (89))} \\
&:= \int_0^M \left\langle h\left(\frac{s}{2},\cdot\right) * \alpha \,\middle|\, {}^{\mathrm{v}}h\left(\frac{s}{2},\cdot\right) * d_c d_c^*\phi \right\rangle \mathrm{d}s \quad \text{(by (55))} \\
&= \int_0^M \left\langle \alpha | {}^{\mathrm{v}}h\left(\frac{s}{2},\cdot\right) * {}^{\mathrm{v}}h\left(\frac{s}{2},\cdot\right) * d_c d_c^*\phi \right\rangle \mathrm{d}s \quad \text{(again by (89))} \\
&= \int_0^M \langle \alpha | {}^{\mathrm{v}}h(s,\cdot) * (d_c d_c^* + d_c^* d_c)\phi \rangle \mathrm{d}s \quad (\text{since } d_c\alpha = 0) \\
&:= \int_0^M \langle \alpha | {}^{\mathrm{v}}h(s,\cdot) * \Delta_{\mathbb{H},h}\phi \rangle \mathrm{d}s \quad (\text{since } d_c\alpha = 0) \\
&:= \int_0^M \int_{\mathbb{H}^n} \langle \alpha, {}^{\mathrm{v}}h(s,\cdot) * \Delta_{\mathbb{H},h}\phi \rangle \mathrm{d}p\mathrm{d}s,
\end{aligned}
\tag{3.52}
$$

since $\alpha \in L^1(\mathbb{H}^n, E_0^h)$ and ${}^{\mathrm{v}}h(s,\cdot)*\Delta_{\mathbb{H},h}\phi \in \mathcal{S}(\mathbb{H}^n, E_0^h)$ (if $h = n+1$, we must replace $d_c^* d_c$ with $(d_c^* d_c)^2$ to obtain the homogeneous Laplacian).

We notice now that, arguing as in the proof of Lemma 3.26,

$$
\langle \alpha, {}^{\mathrm{v}}h(s,\cdot) * \Delta_{\mathbb{H},h}\phi \rangle \in L^1([0,\infty) \times \mathbb{H}^n), \tag{3.53}
$$

since

$$
\int_0^\infty \int_{\mathbb{H}^n} |\alpha| \cdot |{}^{\mathrm{v}}h(s,\cdot)*\Delta_{\mathbb{H},h}\phi| \mathrm{d}p\mathrm{d}s < \infty. \tag{3.54}
$$

Thus, by Fubini's theorem,

$$
\begin{aligned}
\int_0^M \int_{\mathbb{H}^n} \langle \alpha, {}^{\mathrm{v}}h(s,\cdot) * \Delta_{\mathbb{H},h}\phi \rangle \mathrm{d}p\mathrm{d}s &= \int_{\mathbb{H}^n} \left\langle \alpha, \int_0^M {}^{\mathrm{v}}h(s,\cdot) * \Delta_{\mathbb{H},h}\phi \mathrm{d}s \right\rangle \mathrm{d}p \\
&= \left\langle \int_0^M {}^{\mathrm{v}}h(s,\cdot) * \Delta_{\mathbb{H},h}\phi \mathrm{d}s \,\middle|\, \alpha \right\rangle.
\end{aligned}
\tag{3.55}
$$

Let us write (61) in terms of components. We obtain

$$
\sum_{i,j,\ell} \int_{\mathbb{H}^n} \alpha_i \int_0^M {}^{\mathrm{v}}h_{i,j}(s,\cdot) * \Delta_{\mathbb{H},h}^{j,\ell}\phi_\ell \mathrm{d}s\mathrm{d}p. \tag{3.56}
$$

We want to prove that

$$
\int_{\mathbb{H}^n} \alpha_i \int_0^M {}^{\mathrm{v}}h_{i,j}(s,\cdot) * \Delta_{\mathbb{H},h}^{j,\ell}\phi_\ell \mathrm{d}s\mathrm{d}p = \int_{\mathbb{H}^n} \alpha_i \left(\int_0^M {}^{\mathrm{v}}h_{i,j}(s,\cdot)\mathrm{d}s \right) * \Delta_{\mathbb{H},h}^{j,\ell}\phi_\ell \mathrm{d}p, \tag{3.57}
$$

all integrals in (3.57) being well defined.

For any $s > 0$, since $h_{i,j}(s,\cdot) \in \mathcal{S}(\mathbb{H}^n)$, we can write

$$\begin{aligned}({}^{\mathrm{v}}h_{i,j}(s,\cdot)*\Delta^{j,\ell}_{\mathbb{H},h}\phi_\ell)(p) &= \int_{\mathbb{H}^n} {}^{\mathrm{v}}h_{i,j}(s,q)(\Delta^{j,\ell}_{\mathbb{H},h}\phi_\ell)(q^{-1}p)\mathrm{d}q \\ &= \int_{\mathbb{H}^n} {}^{\mathrm{v}}h_{i,j}(s,q)\ {}^{\mathrm{v}}(\Delta^{j,\ell}_{\mathbb{H},h}\phi_\ell)(p^{-1}q)\mathrm{d}q \\ &= \int_{\mathbb{H}^n} {}^{\mathrm{v}}h_{i,j}(s,q)\ {}^{\mathrm{v}}(\Delta^{j,\ell}_{\mathbb{H},h}(\phi_\ell\circ\tau_{p^{-1}}))(q)\mathrm{d}q.\end{aligned} \tag{3.58}$$

Now, by (3.42), if $k > Q/2a$,

$$\begin{aligned}&\int_0^M \mathrm{d}s \int |{}^{\mathrm{v}}h_{i,j}(s,q)|\cdot|\ {}^{\mathrm{v}}(\Delta^{j,\ell}_{\mathbb{H},h}(\phi_\ell\circ\tau_{p^{-1}}))(q)|\mathrm{d}q \\ &\quad\le CM\|(\Delta^{j,\ell}_{\mathbb{H},h}(\phi_\ell\circ\tau_{p^{-1}}))\|_{W^{2,k}(\mathbb{H}^n)} \\ &\quad\le CM\sum_{|I|\le k+a}\|\phi_\ell\circ\tau_{p^{-1}}\|_{L^2(\mathbb{H}^n)} \\ &\quad= CM\sum_{|I|\le k+a}\|\phi_\ell\|_{L^2(\mathbb{H}^n)} < \infty.\end{aligned} \tag{3.59}$$

Thus, combining (3.58) and (3.59), the map

$$(s,p)\to({}^{\mathrm{v}}h_{i,j}(s,\cdot)*\Delta^{j,\ell}_{\mathbb{H},h}\phi_\ell)(p)$$

belongs to $L^1([0,M]\times\operatorname{supp}\alpha_i)$ and, by Fubini theorem,

$$\begin{aligned}&\int_{\mathbb{H}^n}\alpha_i(p)\int_0^M({}^{\mathrm{v}}h_{i,j}(s,\cdot)*\Delta^{j,\ell}_{\mathbb{H},h}\phi_\ell)(p)\mathrm{d}s\mathrm{d}p \\ &\quad= \int_{\mathbb{H}^n}\alpha_i(p)\int_0^M\int_{\mathbb{H}^n}{}^{\mathrm{v}}h_{i,j}(s,q)\ {}^{\mathrm{v}}(\Delta^{j,\ell}_{\mathbb{H},h}(\phi_\ell\circ\tau_{p^{-1}}))(q)\mathrm{d}q\mathrm{d}s\mathrm{d}p \\ &\quad= \int_{\mathbb{H}^n}\alpha_i(p)\int_{\mathbb{H}^n}\left\{\int_0^M{}^{\mathrm{v}}h_{i,j}(s,q)\mathrm{d}s\right\}\ {}^{\mathrm{v}}(\Delta^{j,\ell}_{\mathbb{H},h}(\phi_\ell\circ\tau_{p^{-1}}))(q)\mathrm{d}q\mathrm{d}p \\ &\quad= \int_{\mathbb{H}^n}\alpha_i(p)\left(\left\{\int_0^M{}^{\mathrm{v}}h_{i,j}(s,\cdot)\mathrm{d}s\right\}*\Delta^{j,\ell}_{\mathbb{H},h}\phi_\ell\right)(p)\mathrm{d}p \\ &\quad= \left\langle\left\{\int_0^M{}^{\mathrm{v}}h_{i,j}(s,\cdot)\mathrm{d}s\right\}*\Delta^{j,\ell}_{\mathbb{H},h}\phi_\ell\ \middle|\ \alpha_i\right\rangle.\end{aligned} \tag{3.60}$$

Thus, (3.55) becomes

$$\int_0^M\int_{\mathbb{H}^n}\langle\alpha,{}^{\mathrm{v}}h(s,\cdot)*\Delta_{\mathbb{H},h}\phi\rangle\mathrm{d}p\mathrm{d}s = \left\langle\left\{\int_0^M{}^{\mathrm{v}}h(s,\cdot)\mathrm{d}s\right\}*\Delta_{\mathbb{H},h}\phi\ \middle|\ \alpha\right\rangle. \tag{3.61}$$

On the other hand, by Proposition 3.21, we know that

$$\int_0^M{}^{\mathrm{v}}h(s,\cdot)\mathrm{d}s\to\int_0^\infty{}^{\mathrm{v}}h(s,\cdot)\mathrm{d}s \quad \text{in } \mathcal{D}'(\mathbb{H}^n,E_0^h).$$

But the map $T\to T*\psi$ is continuous from $\mathcal{D}'(\mathbb{H}^n)$ to $\mathcal{D}'(\mathbb{H}^n)$ for fixed $\psi\in\mathcal{D}(\mathbb{H}^n)$ (this is a special instance of Théorème V, p. 157 of [50]), so that

$$\left\langle \left\{ \int_0^M {}^{\mathrm{v}}h(s,\cdot)\mathrm{d}s \right\} * \Delta_{\mathbb{H},h}\phi \,\middle|\, \alpha \right\rangle \to \left\langle \left\{ \int_0^\infty {}^{\mathrm{v}}h(s,\cdot)\mathrm{d}s \right\} * \Delta_{\mathbb{H},h}\phi \,\middle|\, \alpha \right\rangle \tag{3.62}$$

as $M \to \infty$. Thus, combining (3.52), (3.61), and (3.62), the reproducing formula (3.50) is proved when $\alpha \in \mathcal{D}(\mathbb{H}^n, E_0^h)$.

Eventually, let us consider the case $\alpha \in L^1(\mathbb{H}^n, E_0^h)$. Suppose for a while we are able to prove the reproducing formula (3.50) when α is replaced by a form $\tilde{\alpha} \in L^2(\mathbb{H}^n, E_0^h) \cap \mathcal{E}'(\mathbb{H}^n, E_0^h)$. We argue as follows: if $\alpha = \sum_j \alpha_j \xi j$ and $N \in \mathbb{N}$, we set

$$\alpha_{N,j} \coloneqq \min\{|\alpha_j|\chi_{B(e,N)}, N\}\, \frac{\alpha_j}{|\alpha_j|}, \quad \text{where } \alpha \neq 0$$

and

$$\alpha_N = \sum_j \alpha_{N,j}\xi j.$$

Since $\alpha_{N,j}$ is compactly supported and bounded, then $\alpha_N \in L^1(\mathbb{H}^n, E_0^h)$. In addition, a.e. $\alpha_{N,j} \to \alpha_j$ as $N \to \infty$, and $|\alpha_{N,j}| \le |\alpha_j|$, $j = 1, \dots, N_h$.

In addition, set

$$F_N(s,x) \coloneqq d_c^* \left(h\left(\frac{s}{2},\cdot\right) * \alpha_N \right)(x) \quad s > 0. \tag{3.63}$$

By our temporary assumption, if ϕ is a test form, arguing as in (58), we obtain

$$\begin{aligned} \langle \alpha_N, \phi \rangle &= -\left\langle \int_0^\infty d_c \left(h\left(\frac{s}{2},\cdot\right) * F_N(s,\cdot) \right) \mathrm{d}s, \phi \right\rangle \\ &= \int_0^\infty \int_{\mathbb{H}^n} \langle \alpha_N, {}^{\mathrm{v}}h(s,\cdot) * \Delta_{\mathbb{H},h}\phi \rangle \mathrm{d}p\mathrm{d}s. \end{aligned} \tag{3.64}$$

Since $|\alpha_N| \le |\alpha|$, we can take the limit in (3.64) as $N \to \infty$, and we obtain (3.50).

Thus, we are left with the case

$$\alpha = \sum_j \alpha_j \xi_j \quad \text{with } \alpha_j \in L^2(\mathbb{H}^n) \cap \mathcal{E}'(\mathbb{H}^n).$$

If $(\omega_\varepsilon)_{\varepsilon>0}$ are the (usual) Friedrichs' mollifiers, we set

$$\alpha_{j,\varepsilon} \coloneqq \alpha_j * \omega_\varepsilon$$

and

$$\alpha_\varepsilon = \sum_j \alpha_{j,\varepsilon}\xi_j.$$

Denote now by $\gamma \in E_0^h$ the Rumin form

$$\sum_j (M\alpha_{j,\varepsilon})\xi_j,$$

where M is the Hardy-Littlewood maximal function. It is well known that

$$|\alpha_{j,\varepsilon}| \le \gamma_j \quad \text{a.e. in } \mathbb{H}^n \quad \text{for } j = 1, \dots, N_h. \tag{3.65}$$

Moreover, since $\alpha \in L^2(\mathbb{H}^n, E_0^h)$, then

$$\gamma \in L^2(\mathbb{H}^n, E_0^h).$$

Let us prove now that

$$|y|\cdot|^{\mathrm{v}}h(s,\cdot) * \Delta_{\mathbb{H},h}\phi| \in L^1([0,\infty)\times\mathbb{H}^n). \tag{3.66}$$

Indeed, we have:

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{H}^n} |y|\cdot|^{\mathrm{v}}h(s,\cdot)*\Delta_{\mathbb{H},h}\phi|\mathrm{d}p\mathrm{d}s \\
&\le \|y\|_{L^2(\mathbb{H}^n,E_0^h)}\cdot\int_0^\infty \|^{\mathrm{v}}h(s,\cdot) * \Delta_{\mathbb{H},h}\phi\|_{L^2(\mathbb{H}^n,E_0^h)}\mathrm{d}s \\
&= \|y\|_{L^2(\mathbb{H}^n,E_0^h)}\left(\int_0^1 \cdots\,\mathrm{d}s + \int_1^\infty \cdots\,\mathrm{d}s\right).
\end{aligned}$$

Now

$$\int_0^1 \|^{\mathrm{v}}h(s,\cdot) * \Delta_{\mathbb{H},h}\phi\|_{L^2(\mathbb{H}^n,E_0^h)}\mathrm{d}s = \int_0^1 \|\exp(-s\Delta_{\mathbb{H},h})\ {}^{\mathrm{v}}\Delta_{\mathbb{H},h}\phi\|_{L^2(\mathbb{H}^n,E_0^h)}\mathrm{d}s \le C_\phi \quad \text{(by (24))},$$

whereas, keeping in mind that $h(1,\cdot) \in \mathcal{S}$,

$$\int_1^\infty \|^{\mathrm{v}}h(s,\cdot) * \Delta_{\mathbb{H},h}\phi\|_{L^2(\mathbb{H}^n,E_0^h)}\mathrm{d}s = \sup_{\mathbb{H}^n}|h(1,\cdot)|\cdot\int_1^\infty s^{-Q/a}\|\Delta_{\mathbb{H},h}\phi\|_{L^2(\mathbb{H}^n,E_0^h)}\mathrm{d}s < \infty.$$

Then we can write (3.50) for $\alpha_\varepsilon \in \mathcal{D}(\mathbb{H}^n, E_0^h)$ and (by dominate convergence theorem) take the limit as $\varepsilon \to 0$. This completes the proof of the theorem. □

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Appendixes

A Rumin complex on Heisenberg groups

In this appendix, we present some basic notations and introduce both the structure of Heisenberg groups together with the formulation of the Rumin complex. We denote by $\mathbb{H}^n$ the $(2n+1)$-dimensional Heisenberg group, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted by $p = (x, y, t)$, with both $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. If p and $p' \in \mathbb{H}^n$, the group operation is defined by

$$p \cdot p' = \left(x + x', y + y', t + t' + \frac{1}{2}\sum_{j=1}^{n}(x_j y_j' - y_j x_j')\right).$$

Notice that $\mathbb{H}^n$ can be equivalently identified with $\mathbb{C}^n \times \mathbb{R}$ endowed with the group operation

$$(z, t)\cdot(\zeta, \tau) \coloneqq (z + \zeta, t + \tau - \frac{1}{2}\operatorname{Im}(z\bar{\zeta})).$$

For any $q \in \mathbb{H}^n$, the *(left) translation* $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ is defined as follows:

$$p \mapsto \tau_q p \coloneqq q \cdot p.$$

For a general review on Heisenberg groups and their properties, we refer to [34,51,54]. See also [27] for notations.

The Heisenberg group $\mathbb{H}^n$ can be endowed with a homogeneous norm (Cygan-Korányi norm): if $p = (x, y, t) \in \mathbb{H}^n$, then we set

$$\varrho(p) = ((x^2 + y^2)^2 + 16t^2)^{1/4}, \tag{A1}$$

and we define the gauge distance (a true distance, see [51], p. 638, with a different normalization in the group law), that is left invariant, i.e., $d(\tau_q p, \tau_q p') = d(p, p')$ for all $p, p' \in \mathbb{H}^n$) as follows:

$$d(p, q) \coloneqq \varrho(p^{-1} \cdot q). \tag{A2}$$

Notice that d is equivalent to the Carnot-Carathéodory distance on $\mathbb{H}^n$ ([13], Corollary 5.1.5). Finally, the balls for the metric d are the so-called Cygan-Korányi balls

$$B(p, r) \coloneqq \{q \in \mathbb{H}^n;\ d(p, q) < r\}. \tag{A3}$$

Notice that Cygan-Korányi balls are convex smooth sets. A straightforward computation shows that, if $\rho(p) < 1$, then

$$|p| \le \rho(p) \le |p|^{1/2}. \tag{A4}$$

It is well known that the topological dimension of $\mathbb{H}^n$ is $2n + 1$, since as a smooth manifold it coincides with $\mathbb{R}^{2n+1}$, whereas the Hausdorff dimension of $(\mathbb{H}^n, d)$ is $Q \coloneqq 2n + 2$ (the so-called *homogeneous dimension* of $\mathbb{H}^n$).

We denote by $\mathfrak{h}$ the Lie algebra of the left invariant vector fields of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}$ is given, for $i = 1, \dots, n$, by

$$X_i \coloneqq \partial_{x_i} - \frac{1}{2}y_i\partial_t, \quad Y_i \coloneqq \partial_{y_i} + \frac{1}{2}x_i\partial_t, \quad T \coloneqq \partial_t.$$

The only nontrivial commutation relations are $[X_j, Y_j] = T$, for $j = 1, \dots, n$. The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$: $\mathfrak{h}_1 \coloneqq \operatorname{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$.

Coherently, from now on, we refer to $X_1, \dots, X_n, Y_1, \dots, Y_n$ (identified with first order differential operators) as the *horizontal derivatives*. Denoting by $\mathfrak{h}_2$ the linear span of T, the two-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

The stratification of the Lie algebra $\mathfrak{h}$ induces a family of nonisotropic dilations $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$, $\lambda > 0$ as follows: if $p = (x, y, t) \in \mathbb{H}^n$, then

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t). \tag{A5}$$

Remark A.1. Heisenberg groups are special instance of the so-called *Carnot groups*. A *graded group* of step κ is a connected, simply connected Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ is the direct sum of κ subspaces $\mathfrak{g}_i$, $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\kappa$, satisfying

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \quad \text{for } 1 \le i, j \le \kappa,$$

where $\mathfrak{g}_i = 0$ for $i > \kappa$. The group is called stratified if it is generated by the first layer $\mathfrak{g}_1$. We denote as n the dimension of $\mathfrak{g}$ and as n_j the dimension of $\mathfrak{g}_j$, for $1 \le j \le \kappa$.

A *Carnot group* $\mathbb{G}$ of step κ is a graded group of step $\mathfrak{g}$, where $\mathfrak{g}_1$ generates all of $\mathfrak{g}$. That is, $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$, for $i = 1, \dots, \kappa$. We refrain from dealing with such generality.

Going back to Heisenberg groups, the vector space $\mathfrak{h}$ can be endowed with an inner product, denoted by $\langle \cdot, \cdot \rangle$, making $X_1, \dots, X_n, Y_1, \dots, Y_n$ and T orthonormal.

Throughout this article, we also write

$$W_i := X_i, \quad W_{i+n} := Y_i \quad \text{and} \quad W_{2n+1} := T, \quad \text{for } i = 1, \dots, n. \tag{A6}$$

Following [25], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_n)$ is a multi-index, we set

$$W^I = W_1^{i_1} \cdots W_n^{i_n}. \tag{A7}$$

Remark A.2. By the Poincaré-Birkhoff-Witt theorem ([14], I.2.7), the differential operators W^I form a basis for the algebra of left invariant differential operators on $\mathbb{G}$. Furthermore, we denote by $|I| := i_1 + \cdots + i_n$ the order of the differential operator W^I, and by $d(I) := d_1 i_1 + \cdots + d_n i_n$ its degree of homogeneity with respect to group dilations. From the Poincaré-Birkhoff-Witt theorem, it follows, in particular, that any homogeneous linear differential operator in the horizontal derivatives can be expressed as a linear combination of the operators W^I of the special form above. Thus, often we can restrict ourselves to consider only operators of the special form W^I.

The dual space of $\mathfrak{h}$ is denoted by $\textstyle\bigwedge^1 \mathfrak{h}$. The basis of $\textstyle\bigwedge^1 \mathfrak{h}$, dual to the basis $\{X_1, \dots, Y_n, T\}$, is the family of covectors $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n, \theta\}$, where

$$\theta := dt - \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j) \tag{A8}$$

is called the *contact form* in $\mathbb{H}^n$. We also denote by $\langle \cdot, \cdot \rangle$ the inner product in $\textstyle\bigwedge^1 \mathfrak{h}$ that makes $(dx_1, \dots, dy_n, \theta)$ an orthonormal basis.

Coherently with the previous notation (A6), we set

$$\omega_i := dx_i, \quad \omega_{i+n} := dy_i \quad \text{and} \quad \omega_{2n+1} := \theta, \quad \text{for } i = 1, \dots, n.$$

We put $\textstyle\bigwedge_0 \mathfrak{h} := \textstyle\bigwedge^0 \mathfrak{h} = \mathbb{R}$ and, for $1 \le h \le 2n + 1$,

$$\textstyle\bigwedge^h \mathfrak{h} := \operatorname{span}\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_h} : 1 \le i_1 < \cdots < i_h \le 2n + 1\}.$$

In the sequel, we shall denote by Θ^h the basis of $\textstyle\bigwedge^h \mathfrak{h}$ defined by

$$\Theta^h := \{\omega_{i_1} \wedge \cdots \wedge \omega_{i_h} : 1 \le i_1 < \cdots < i_h \le 2n + 1\}.$$

To avoid cumbersome notations, if $I := (i_1, \dots, i_h)$, we write

$$\omega_I := \omega_{i_1} \wedge \cdots \wedge \omega_{i_h}.$$

The inner product $\langle \cdot, \cdot \rangle$ on $\textstyle\bigwedge^1 \mathfrak{h}$ naturally yields an inner product $\langle \cdot, \cdot \rangle$ on $\textstyle\bigwedge^h \mathfrak{h}$ making Θ^h an orthonormal basis.

The volume $(2n+1)$-form $\omega_1 \wedge \cdots \wedge \omega_{2n+1}$ will be also written as dV.

Throughout this article, the elements of $\bigwedge^h \mathfrak{h}$ are identified with *left invariant* differential forms of degree h on $\mathbb{H}^n$.

Definition A.3. A h-form α on $\mathbb{H}^n$ is said left invariant if

$$\tau_q^\# \alpha = \alpha \quad \text{for any } \; q \in \mathbb{H}^n.$$

The pull-back of differential forms is well defined as follows ([31], Proposition 1.106);

Definition A.4. If $\mathcal{U}, \mathcal{V}$ are open subsets of $\mathbb{H}^n$, and $f : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism, then for any differential form α of degree h, we denote by $f^\sharp \alpha$ the pull-back form on $\mathcal{U}$ defined by

$$\langle f^\sharp \alpha(p)|v_1, \ldots, v_h\rangle := \langle \alpha(f(p))|df(p)v_1, \ldots, df(p)v_h\rangle$$

for any h-tuple $(v_1, \ldots, v_h)$ of tangent vectors at p.

The same construction can be performed starting from the vector subspace $\mathfrak{h}_1 \subset \mathfrak{h}$, obtaining the *horizontal h-covectors*

$$\bigwedge^h \mathfrak{h}_1 := \operatorname{span}\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_h} : 1 \le i_1 < \cdots < i_h \le 2n\}.$$

It is easy to see that

$$\Theta_0^h := \Theta^h \cap \bigwedge^h \mathfrak{h}_1$$

provides an orthonormal basis of $\bigwedge^h \mathfrak{h}_1$.

Keeping in mind that the Lie algebra $\mathfrak{h}$ can be identified with the tangent space to $\mathbb{H}^n$ at $x = e$ ([31], Proposition 1.72), starting from $\bigwedge^h \mathfrak{h}$ we can define by left translation a fiber bundle over $\mathbb{H}^n$ that we still denote by $\bigwedge^h \mathfrak{h}$. We can think of h-forms as sections of $\bigwedge^h \mathfrak{h}$. We denote by Ω^h the vector space of all smooth h-forms.

The stratification of the Lie algebra $\mathfrak{h}$ yields a lack of homogeneity of de Rham's exterior differential with respect to group dilations δ_λ. Thus, to keep into account the different degrees of homogeneity of the covectors when they vanish on different layers of the stratification, we introduce the notion of *weight* of a covector as follows.

Definition A.5. If $\eta \neq 0$, $\eta \in \bigwedge^1 \mathfrak{h}_1$, we say that η has *weight 1*, and we write $w(\eta) = 1$. If $\eta = \theta$, we say $w(\eta) = 2$. More generally, if $\eta \in \bigwedge^h \mathfrak{h}$, $\eta \neq 0$, we say that η has *pure weight p* if η is a linear combination of covectors $\omega_{i_1} \wedge \cdots \wedge \omega_{i_h}$ with $w(\omega_{i_1}) + \cdots + w(\omega_{i_h}) = p$.

Notice that, if $\eta, \zeta \in \bigwedge^h \mathfrak{h}$ and $w(\eta) \neq w(\zeta)$, then $\langle \eta, \zeta\rangle = 0$ ([8], Remark 2.4). Also, we point out that $w(d\theta) = w(\theta)$, since, if α is a left invariant h-form of weight p and $d\alpha \neq 0$, then $w(d\alpha) = w(\alpha)$ ([47], Section 2.1).

We stress that generic covectors may fail to have pure weight: It is enough to consider $\mathbb{H}^1$ and the covector $dx_1 + \theta \in \bigwedge^1 \mathfrak{h}$. However, the following result holds ([8], formula (16)):

$$\bigwedge^h \mathfrak{h} = \bigwedge^{h,h} \mathfrak{h} \oplus \bigwedge^{h,h+1} \mathfrak{h} = \bigwedge^h \mathfrak{h}_1 \oplus (\bigwedge^{h-1} \mathfrak{h}_1) \wedge \theta, \tag{A9}$$

where $\bigwedge^{h,p} \mathfrak{h}$ denotes the linear span of the h-covectors of weight p. By our previous remark, the decomposition (A9) is orthogonal. In addition, since the elements of the basis Θ^h have pure weights, a basis of $\bigwedge^{h,p} \mathfrak{h}$ is given by $\Theta^{h,p} := \Theta^h \cap \bigwedge^{h,p} \mathfrak{h}$ (such a basis is usually called an adapted basis).

We notice that, according to (A9), the weight of a h-form is either h or $h+1$, and there are no h-forms of weight $h+2$, since there is only one 1-form of weight 2. Something analogous happens in $\mathbb{H}^n \times \mathbb{R}$, but it fails already in the case of general step 2 groups with higher dimensional center.

As mentioned earlier, starting from $\bigwedge^{h,p}\mathfrak{h}$, we can define by left translation a fiber bundle over $\mathbb{H}^n$ that we can still denote by $\bigwedge^{h,p}\mathfrak{h}$. Thus, if we denote by $\Omega^{h,p}$ the vector space of all smooth h-forms in $\mathbb{H}^n$ of weight p, i.e., the space of all smooth sections of $\bigwedge^{h,p}\mathfrak{h}$, we have

$$\Omega^h = \Omega^{h,h} \oplus \Omega^{h,h+1}. \tag{A10}$$

Starting from the notion of weight of a differential form, it is possible to define a new complex of differential forms $(E_0^\bullet, d_c)$ that is homotopic to the de Rham complex and respects the homogeneities of the group.

We sketch here the construction of the Rumin complex. For a more detailed presentation, we refer to Rumin's papers [47]. Here, we follow the presentation of the study by Baldi et al. [8]. The exterior differential d does not preserve weights. It splits into

$$d = d_0 + d_1 + d_2,$$

where d_0 preserves weight, d_1 increases weight by 1 unit, and d_2 increases weight by 2 units.

More explicitly, let $\alpha \in \Omega^{h,h}$ be a smooth h-form of pure weight h. We can write

$$\alpha = \sum_{\omega_I \in \Theta_0^h} \alpha_I \omega_I, \quad \text{with } \alpha_I \in C^\infty(\mathbb{H}^n).$$

Then

$$\mathrm{d}\alpha = \sum_{\omega_I \in \Theta_0^h} \sum_{j=1}^{2n} (W_j \alpha_I) \omega_j \wedge \omega_I + \sum_{\omega_I \in \Theta_0^h} (W_{2n+1} \alpha_I) \theta \wedge \omega_I = d_1 \alpha + d_2 \alpha,$$

and $d_0 \alpha = 0$. On the other hand, if $\alpha \in \Omega^{h,h+1}$ has pure weight $h + 1$, then

$$\alpha = \sum_{\omega_J \in \Theta_0^{h-1}} \alpha_J \theta \wedge \omega_J,$$

and

$$\mathrm{d}\alpha = \sum_{\omega_J \in \Theta_0^h} \alpha_J \mathrm{d}\theta \wedge \omega_J + \sum_{\omega_J \in \Theta_0^h} \sum_{j=1}^{2n} (W_j \alpha_J) \omega_j \wedge \theta \wedge \omega_I = d_0 \alpha + d_1 \alpha,$$

and $d_2 \alpha = 0$.

It is crucial to notice that d_0 is an algebraic operator, in the sense that for any real-valued $f \in C^\infty(\mathbb{H}^n)$, we have

$$d_0(f\alpha) = f d_0 \alpha,$$

so that its action can be identified at any point with the action of a linear operator from $\bigwedge^h\mathfrak{h}$ to $\bigwedge^{h+1}\mathfrak{h}$ (that we denote again by d_0).

Following Rumin [45,47], we give the following definition:

Definition A.6. If $0 \le h \le 2n + 1$, keeping in mind that $\bigwedge^h\mathfrak{h}$ is endowed with a canonical inner product, we set

$$E_0^h \coloneqq \ker d_0 \cap (\mathrm{Im}\ d_0)^\perp.$$

Straightforwardly, E_0^h inherits from $\bigwedge^h\mathfrak{h}$ the inner product.

As mentioned earlier, $E_0^\bullet$ defines by left translation a fiber bundle over $\mathbb{H}^n$, that we still denote by $E_0^\bullet$. To avoid cumbersome notations, we denote also by $E_0^\bullet$ the space of sections of this fiber bundle.

Let $L : \bigwedge^h\mathfrak{h} \to \bigwedge^{h+2}\mathfrak{h}$, the Lefschetz operator defined by

$$L\xi = \mathrm{d}\theta \wedge \xi. \tag{A11}$$

Then the spaces $E_0^\bullet$ can be defined explicitly as follows:

Theorem A.7. (See [44,46]) *We have*:

(*i*) $E_0^1 = \bigwedge^1\mathfrak{h}_1$;

(*ii*) *if* $2 \le h \le n$, *then* $E_0^h = \bigwedge^h\mathfrak{h}_1 \cap (\bigwedge^{h-2}\mathfrak{h}_1 \wedge \mathrm{d}\theta)^\perp$ (*i.e.*, E_0^h *is the space of the so-called primitive covectors of* $\bigwedge^h\mathfrak{h}_1$);

(*iii*) *if* $n < h \le 2n+1$, *then* $E_0^h = \{\alpha = \beta \wedge \theta,\ \beta \in \bigwedge^{h-1}\mathfrak{h}_1,\ \beta \wedge \mathrm{d}\theta = 0\} = \theta \wedge \ker L$;

(*iv*) *if* $1 < h \le n$, *then* $N_h := \dim E_0^h = \binom{2n}{h} - \binom{2n}{h-2}$;

(*v*) *if* $*$ *denotes the Hodge duality associated with the inner product in* $\bigwedge^\bullet\mathfrak{h}$ *and the volume form* $\mathrm{d}V$, *then* $*E_0^h = E_0^{2n+1-h}$.

Notice that all forms in E_0^h *have weight* h *if* $1 \le h \le n$ *and weight* $h+1$ *if* $n < h \le 2n+1$.

A further geometric interpretation (in terms of decomposition of $\mathfrak{h}$ and of graphs within $\mathbb{H}^n$) can be found in [28].

Notice that there exists a left invariant basis

$$\Xi_0^h = \{\xi_1^h, \dots, \xi_{N_h}^h\} \tag{A12}$$

of E_0^h that is adapted to the filtration (A9). Such a basis is explicitly constructed by induction in [3,55]. To avoid cumbersome notations, if there is no risk of misunderstandings and the degree h of the forms is evident or uninfluential, we write ξ_j for ξ_j^h.

The core of Rumin's theory consists in the construction of a suitable "exterior differential" $d_c : E_0^h \to E_0^{h+1}$ making $\mathcal{E}_0 := (E_0^\bullet, d_c)$ a complex homotopic to the de Rham complex.

Let us sketch Rumin's construction: first the next result ([8], Lemma 2.11 for a proof) allows us to define a (pseudo) inverse of d_0:

Lemma A.8. *If* $1 \le h \le n$, *then* $\ker d_0 = \bigwedge^h\mathfrak{h}_1$. *Moreover, if* $\beta \in \bigwedge^{h+1}\mathfrak{h}$, *then there exists a unique* $\gamma \in \bigwedge^h\mathfrak{h} \cap (\ker d_0)^\perp$ *such that*

$$d_0\gamma - \beta \in \mathcal{R}(d_0)^\perp.$$

With the notations of the previous lemma, we set

$$\gamma := d_0^{-1}\beta.$$

We notice that d_0^{-1} preserves the weights.

The following theorem summarizes the construction of the intrinsic differential d_c (for details, see [47] and [8], Section 2).

Theorem A.9. *The de Rham complex* $(\Omega^\bullet, d)$ *splits into the direct sum of two sub-complexes* $(E^\bullet, d)$ *and* $(F^\bullet, d)$, *with*

$$E := \ker d_0^{-1} \cap \ker(d_0^{-1}d) \quad \text{and} \quad F := \mathcal{R}(d_0^{-1}) + \mathcal{R}(dd_0^{-1}).$$

Let Π_E *be the projection on* E *along* F (*that is not an orthogonal projection*). *We have*

(*i*) *If* $\gamma \in E_0^h$, *then*
- $\Pi_E\gamma = \gamma - d_0^{-1}d_1\gamma$ *if* $1 \le h \le n$;
- $\Pi_E\gamma = \gamma$ *if* $h > n$.

(*ii*) Π_E *is a chain map, i.e.*,

$$\mathrm{d}\Pi_E = \Pi_E d.$$

(*iii*) *Let* Π_{E_0} *be the orthogonal projection from* $\bigwedge^\bullet\mathfrak{h}$ *on* $E_0^\bullet$, *then*

$$\Pi_{E_0} = I - d_0^{-1}d_0 - d_0d_0^{-1}, \quad \Pi_{E_0^\perp} = d_0^{-1}d_0 + d_0d_0^{-1}. \tag{A13}$$

(*iv*) $\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0}$ *and* $\Pi_E\Pi_{E_0}\Pi_E = \Pi_E$.
Set now

$$d_c = \Pi_{E_0}\mathrm{d}\,\Pi_E : E_0^h \to E_0^{h+1}, \quad h = 0, \ldots, 2n.$$

We have:

(*v*) $d_c^2 = 0$;

(*vi*) *the complex* $\mathcal{E}_0 := (E_0^\bullet, d_c)$ *is homotopic to the de Rham complex*;

(*vii*) $d_c : E_0^h \to E_0^{h+1}$ *is a homogeneous differential operator in the horizontal derivatives of order 1 if* $h \neq n$, *whereas* $d_c : E_0^n \to E_0^{n+1}$ *is a homogeneous differential operator in the horizontal derivatives of order 2.*

Remark A.10. The construction of Rumin complex can be carried out on general Carnot groups; we refer for instance to [8, 47,48]. The starting point is a notion of *weight* of a covector in term of homogeneity with respect to group dilations. For an alternative presentation, we refer to the previous studies [20,23,28,39].

Since the exterior differential d_c on E_0^h can be written in coordinates as a left invariant homogeneous differential operator in the horizontal variables, of order 1 if $h \neq n$ and of order 2 if $h = n$, the proof of the following Leibniz' formula is easy.

Lemma A.11. *If* ζ *is a smooth real function, then*

- *if* $h \neq n$, *then on* E_0^h *we have*

$$[d_c, \zeta] = P_0^h,$$

where $P_0^h : E_0^h \to E_0^{h+1}$ *is a linear homogeneous differential operator of degree zero, with coefficients depending only on the horizontal derivatives of* ζ;

- *if* $h = n$, *then on* E_0^n *we have*

$$[d_c, \zeta] = P_1^n + P_0^n,$$

where $P_1^n : E_0^n \to E_0^{n+1}$ *is a linear homogeneous differential operator of degree 1, with coefficients depending only on the horizontal derivatives of* ζ, *and where* $P_0^h : E_0^n \to E_0^{n+1}$ *is a linear homogeneous differential operator in the horizontal derivatives of degree 0 with coefficients depending only on second-order horizontal derivatives of* ζ.

B Kernels in Carnot groups and Folland-Stein spaces

B.1 Convolution in $\mathbb{H}^n$

If $f : \mathbb{H}^n \to \mathbb{R}$, we set ${}^{\mathrm{v}}f(p) = f(p^{-1})$, and, if $T \in \mathcal{D}'(\mathbb{H}^n)$, then $\langle {}^{\mathrm{v}}T|\phi\rangle := \langle T|{}^{\mathrm{v}}\phi\rangle$ for all $\phi \in \mathcal{D}(\mathbb{H}^n)$. Obviously, the map $T \to {}^{\mathrm{v}}T$ is continuous from $\mathcal{D}'(\mathbb{H}^n)$ to $\mathcal{D}'(\mathbb{H}^n)$.

Following [25], p. 15, we can define a group convolution in $\mathbb{H}^n$: if, for instance, $f \in \mathcal{D}(\mathbb{H}^n)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$, we set

$$f * g(p) := \int f(q)g(q^{-1} \cdot p)\mathrm{d}q \quad \text{for} \quad q \in \mathbb{H}^n. \tag{B1}$$

We recall that, if, say, g is a smooth function and P is a left invariant differential operator, then

$$P(f * g) = f * Pg.$$

We also recall that the convolution is well defined when $f, g \in \mathcal{D}'(\mathbb{H}^n)$, provided at least one of them has compact support.

In this case, the following identities hold

(i)
$$\langle f * g|\phi\rangle = \langle g|\, {}^{\mathrm{v}}f * \phi\rangle \quad \text{and} \quad \langle f * g|\phi\rangle = \langle f|\phi * {}^{\mathrm{v}}g\rangle \tag{B2}$$
for any test function ϕ. Analogously, for any function $\phi \in \mathcal{D}(\mathbb{H}^n)$,
$$\langle f * g|\phi\rangle = \langle g|\, {}^{\mathrm{v}}f * \phi\rangle \quad \text{if } f \in \mathcal{S}'(\mathbb{H}^n) \text{ and } g \in \mathcal{S}(\mathbb{H}^n), \tag{B3}$$
([50], p. 248) $\mathcal{S}'(\mathbb{H}^n)*\mathcal{S}(\mathbb{H}^n) \subset \mathcal{O}_M \subset \mathcal{E}(\mathbb{H}^n) \cap \mathcal{S}'(\mathbb{H}^n)$ and $\mathcal{S}(\mathbb{H}^n)*\mathcal{D}(\mathbb{H}^n) \subset \mathcal{S}(\mathbb{H}^n)$, where $\mathcal{O}_M$ denotes the space of the smooth functions slowly increasing at infinity ([53], Theorem 25.5, [50], p. 243). Analogously,
$$\langle f * g|\phi\rangle = \langle g|\, {}^{\mathrm{v}}f * \phi\rangle \quad \text{if } f \in \mathcal{S}(\mathbb{H}^n) \text{ and } g \in \mathcal{S}'(\mathbb{H}^n) \tag{B4}$$
(notice that $\mathcal{S}(\mathbb{H}^n) * \mathcal{D}(\mathbb{H}^n) \subset \mathcal{S}(\mathbb{H}^n)*\mathcal{S}(\mathbb{H}^n) \subset \mathcal{S}(\mathbb{H}^n)$). Indeed, by [53], Remark 28.3, there exists a sequence $(g_k)_{k\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{H}^n)$ such that $g_k \to g$ in $\mathcal{S}'(\mathbb{H}^n)$, so that $f * g_k \to f * g$ in $\mathcal{D}'(\mathbb{H}^n)$ as $k \to \infty$. Since ${}^{\mathrm{v}}f * \phi \in \mathcal{S}(\mathbb{H}^n)$, the assertion follows from (B2);

(ii) If $\psi \in \mathcal{D}(\mathbb{H}^n) \subset \mathcal{E}'(\mathbb{H}^n)$ and $h \in \mathcal{E}(\mathbb{H}^n) \subset \mathcal{D}'(\mathbb{H}^n)$, then $\langle \psi|h\rangle = \langle h|\psi\rangle$, so that, if $\phi, \psi \in \mathcal{D}(\mathbb{H}^n)$ and $g \in \mathcal{D}'(\mathbb{H}^n)$, (B2) yields
$$\langle \psi * {}^{\mathrm{v}}g|\phi\rangle = \langle \phi|\psi * {}^{\mathrm{v}}g\rangle = \langle \phi * {}^{\mathrm{v}}g|\psi\rangle. \tag{B5}$$

(iii) if the convolution $g * f$ is well defined, then
$${}^{\mathrm{v}}f * {}^{\mathrm{v}}g = {}^{\mathrm{v}}(g * f) \tag{B6}$$

The notion of convolution can be extended by duality to currents.

Definition B.1. Let $\phi \in \mathcal{D}(\mathbb{H}^n)$ and $T \in \mathcal{E}'(\mathbb{H}^n, E_0^h)$ be given, and denote by ${}^{\mathrm{v}}\phi$ the function defined by ${}^{\mathrm{v}}\phi(p) := \phi(p^{-1})$ (if S is a distribution, then ${}^{\mathrm{v}}S$ is defined by duality). Then we set
$$\langle \phi * T|\alpha\rangle := \langle T|\, {}^{\mathrm{v}}\phi * \alpha\rangle$$
for any $\alpha \in \mathcal{D}(\mathbb{H}^n, E_0^h)$.

Definition B.2. Let $h = 1, \dots, 2n+1$ be fixed, and let $\xi_1^h, \dots, \xi_{N_h}^h$ be an orthonormal basis of E_0^h. If
$$\phi := (\phi_{ij})_{i,j=1,\dots,N_m}$$
is a matrix-valued distribution, and $\alpha = \sum_j \alpha_j \xi_j \in \mathcal{D}(\mathbb{H}^n, E_0^h)$, we set
$$\alpha * \phi = \sum_{i,j} (\alpha_j * \phi_{ij})\xi_i.$$
Obviously, this notion still makes sense whenever all convolutions $\phi_{ij} * \alpha_j$ are well defined.

B.2 Folland-Stein-Sobolev spaces and homogeneous kernels

The following sections deal with Sobolev spaces (the so-called Folland-Stein-Sobolev spaces: see [24,25]), and with the calculus for homogeneous kernels [16] in the more general setting of Carnot groups. Heisenberg groups will provide a special instance. We refer to the previous studies [24,25] for the standard definitions of Sobolev spaces and their Hölder counterpart $\Gamma_\beta(\mathbb{H}^n)$. Recall that we adopt the following multi-index notation for higher-order derivatives: if $I = (i_1, \dots, i_n)$ is a multi-index, we define

$$W^I = W_1^{i_1} \cdots W_n^{i_n}.$$

Definition B.3. We denote by $\Delta_{\mathbb{H}}$ the positive sub-Laplacian

$$\Delta_{\mathbb{H}} \coloneqq -\sum_{i=1}^{2n} W_i^2.$$

Definition B.4. Let $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, $W^{m,p}_{\mathrm{Euc}}(U)$ denotes the usual Sobolev space.

Definition B.5. If $U \subset \mathbb{H}^n$ is an open set, $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, then the space $W^{m,p}(U)$ is the space of all $u \in L^p(U)$ such that, with the notation of (A7),

$$W^I u \in L^p(U) \quad \text{for all multi-indices } I \text{ with } d(I) \leq m,$$

endowed with the natural norm

$$\| u\|_{W^{k,p}(U)} \coloneqq \sum_{d(I)\leq m} \|W^I u\|_{L^p(U)}.$$

Folland-Stein Sobolev spaces enjoy the following properties akin to those of the usual Euclidean Sobolev spaces [24,26].

Theorem B.6. *If $U \subset \mathbb{H}^n$, $1 \leq p \leq \infty$, and $k \in \mathbb{N}$, then*
(*i*) *$W^{k,p}(U)$ is a Banach space.*
In addition, if $p < \infty$,
(*ii*) *$W^{k,p}(U) \cap C^\infty(U)$ is dense in $W^{k,p}(U)$;*
(*iii*) *if $U = \mathbb{H}^n$, then $\mathcal{D}(\mathbb{H}^n)$ is dense in $W^{k,p}(U)$;*
(*iv*) *if $1 < p < \infty$, then $W^{k,p}(U)$ is reflexive.*

Definition B.7. Following [24,25], a kernel of type α is a homogeneous distribution of degree $\alpha - Q$ (with respect to the group dilations δ_r), that is smooth outside of the origin.

The following estimate has been proved in [9], Lemma 3.7. It will turn useful in the sequel.

Lemma B.8. *Let g be a kernel of type $\mu > 0$. Then, if $f \in \mathcal{D}(\mathbb{H}^n)$ and R is a homogeneous polynomial of degree $\ell \geq 0$ in the horizontal derivatives, we have*

$$R(f * g)(p) = O(|p|^{\mu-Q-\ell}) \quad \text{as } p \to \infty.$$

In addition, let g be a smooth function in $\mathbb{H}^n \setminus \{e\}$ satisfying the logarithmic estimate

$$|g(p)| \leq C(1 + |\ln|p||),$$

and suppose its first-order horizontal derivatives are kernels of type $Q - 1$ with respect to group dilations. Then, if $f \in \mathcal{D}(\mathbb{H}^n)$ and R is a homogeneous polynomial of degree $\ell \geq 0$ in the horizontal derivatives, we have

$$R(f * g)(p) = O(|p|^{-\ell}) \quad \text{as } p \to \infty \quad \text{if } \ell > 0;$$
$$R(f * g)(p) = O(\ln|p|) \quad \text{as } p \to \infty \quad \text{if } \ell = 0.$$

We set now

$$\mathcal{S}_0(\mathbb{H}^n) \coloneqq \{u \in \mathcal{S}(\mathbb{H}^n) \ : \ \int_{\mathbb{H}^n} x^\alpha u(x)\, \mathrm{d}x = 0\}$$

for all monomials x^α.

Definition B.9. If $\alpha \in \mathbb{R}$ and $\alpha \notin \mathbb{Z}^+ := \mathbb{N} \cup \{0\}$, then we denote by $\mathbf{K}^\alpha$ the set of the distributions in $\mathbb{H}^n$ that are smooth away from the origin and homogeneous of degree α, whereas, if $\alpha \in \mathbb{Z}^+$, we say that $K \in \mathcal{D}'(\mathbb{H}^n)$ belongs to $\mathbf{K}^\alpha$ if has the form

$$K = \tilde{K} + p(x)\ln|x|,$$

where $\tilde{K}$ is smooth away from the origin and homogeneous of degree α, and p is a homogeneous polynomial of degree α.

In particular, kernels of type α according to Definition B.7 belong to $\mathbf{K}^{\alpha-Q}$.
If $K \in \mathbf{K}^\alpha$, we denote by $O_0(K)$ the operator defined on $\mathcal{S}_0(\mathbb{H}^n)$ by $O_0(K)u := u * K$.

Proposition B.10. ([16], *Proposition* 2.2) $O_0(K) : \mathcal{S}_0(\mathbb{H}^n) \to \mathcal{S}_0(\mathbb{H}^n)$.

A straightforward computation shows that

Lemma B.11. *If $K \in \mathbf{K}^\alpha$, and X^I is a left invariant homogeneous differential operator, then*

$$X^I O_0(K) = O_0(X^I K), \quad \text{and } X^I K \in \mathbf{K}^{\alpha-d(I)}.$$

Theorem B.12. [37,38] *If $K \in \mathbf{K}^{-Q}$, then $O_0(K) : L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)$.*

Remark B.13. We stress that we also have

$$\mathcal{S}_0(\mathbb{H}^n) \subset \operatorname{Dom}(\Delta_{\mathbb{H},h}^{-\alpha/2}) \quad \text{with } \alpha > 0.$$

Indeed, take $M \in \mathbb{N}$, $M > \alpha/2$. If $u \in \mathcal{S}_0(\mathbb{H}^n)$, we can write $u = \Delta_{\mathbb{H},h}^M v$, where

$$v := (O_0(R_2) \circ O_0(R_2) \circ \cdots \circ O_0(R_2))u \in \mathcal{S}_0(\mathbb{H}^n)$$

(M times). Since $v \in \operatorname{Dom}(\Delta_{\mathbb{H},h}^M) \cap \operatorname{Dom}(\Delta_{\mathbb{H},h}^{M-\alpha/2})$ by density, then $u = \Delta_{\mathbb{H},h}^M v \in \operatorname{Dom}(\Delta_{\mathbb{H},h}^{-\alpha/2})$, and $\Delta_{\mathbb{H},h}^{M-\alpha/2} v = \Delta_{\mathbb{H},h}^{-\alpha/2}\Delta_{\mathbb{H},h}^M v$, by [24], Proposition 3.15, (iii).

Theorem B.14. (see [16,32], Theorem 5.11) *Take $K \in \mathbf{K}^{-Q}$ and let the following Rockland condition hold: for every nontrivial irreducible unitary representation π of $\mathbb{H}^n$, the operator $\overline{\pi_K}$ is injective on $\mathbf{C}^\infty(\pi)$, the space of smooth vectors of the representation π. Then the operator $O_0(K) : L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)$ is left invertible.*

Obviously, if $O_0(K)$ is formally self-adjoint, i.e., if $K = {}^{\mathrm{v}}K$, then $O_0(K)$ is also right invertible.

Proposition B.15. ([16], *Proposition* 2.3) *If $K_i \in \mathbf{K}^{\alpha_i}$, $i = 1, 2$, then there exists at least one $K \in \mathbf{K}^{\alpha_1+\alpha_2+Q}$ such that*

$$O_0(K_2) \circ O_0(K_1) = O_0(K).$$

It is possible to provide a standard procedure yielding such a K (see [16], *p.* 42). *Following* [16], *we write $K = K_2 \,\underline{*}\, K_1$.*

Definition B.16. Throughout this article, if $\mathcal{L}$ is an operator acting on functions, then we still denote by $\mathcal{L}$ the diagonal operator $(\delta_{ij}\mathcal{L})_{i,j=1,\dots,M_h}$.

Lemma B.17. *If $m > 0$ and $u \in \mathcal{S}_0(\mathbb{H}^n)$, then $(1 - \Delta_{\mathbb{H}})^{-m/2}u \in \mathcal{S}_0(\mathbb{H}^n)$.*

Proof. By [24], p. 185 (3), $(1 - \Delta_{\mathbb{H}^n})^{-m/2}u = u * J_m$, where J_m is the Bessel potential defined therein. For our purpose, it is important to stress that
(i) $J_m \in L^1(\mathbb{H}^n)$;
(ii) $J_m(p) = O(|p|^{-N})$ for all $N \in \mathbb{N}$ (see again [24] p. 185 (2)).

It is easy to see that $u * J_m$ is smooth. To prove that $u * J_m \in \mathcal{S}(\mathbb{H}^n)$, we can follow basically the same arguments we shall use later to prove that all moments of $u * J_m$ vanish. Thus, we shall not repeat twice the same computations (that, by the way, are elementary though cumbersome).

Thus, we have to show that all moments of $u * J_m$ vanish. In the sequel, we denote by $\tilde{J}$ a smooth function in $\mathbb{H}^n \backslash \{e\}$ satisfying (i) and (ii) earlier.

To start with, we can write

$$\int_{\mathbb{H}^n}\left(\int_{\mathbb{H}^n} u(y) J_m(y^{-1}x)\mathrm{d}y\right)\mathrm{d}x = \int_{\mathbb{H}^n} u(y)\left(\int_{\mathbb{H}^n} J_m(y^{-1}x)\ \mathrm{d}x\right)\mathrm{d}y = \int_{\mathbb{H}^n} J_m(\xi)\mathrm{d}\xi \cdot \int_{\mathbb{H}^n} u(y)\mathrm{d}y = 0.$$

Denote now by $(x_1, \ldots, x_{2n}, x_{2n+1})$ a generic point in $\mathbb{H}^n$.

Take, for instance,

$$\int_{\mathbb{H}^n} x_j\left(\int_{\mathbb{H}^n} u(y)\tilde{J}(y^{-1}x)\mathrm{d}y\right)\mathrm{d}x \quad \text{with } j = 1, \ldots, 2n.$$

We write

$$\begin{aligned}\int_{\mathbb{H}^n} x_j\left(\int_{\mathbb{H}^n} u(y)\tilde{J}(y^{-1}x)\mathrm{d}y\right) &= \int_{\mathbb{H}^n} u(y)\left(\int_{\mathbb{H}^n} x_j\,\tilde{J}(y^{-1}x)\ \mathrm{d}x\right)\mathrm{d}y\\ &= \int_{\mathbb{H}^n} u(y)\left(\int_{\mathbb{H}^n}(x_j - y_j)\,\tilde{J}(y^{-1}x)\ \mathrm{d}x\right)\mathrm{d}y + \int_{\mathbb{H}^n} y_j\ u(y)\left(\int_{\mathbb{H}^n}\tilde{J}(y^{-1}x)\ \mathrm{d}x\right)\mathrm{d}y.\end{aligned}$$

As mentioned earlier,

$$\int_{\mathbb{H}^n} y_j\ u(y)\left(\int_{\mathbb{H}^n}\tilde{J}(y^{-1}x)\ \mathrm{d}x\right)\mathrm{d}y = 0,$$

since $u \in \mathcal{S}_0(\mathbb{H}^n)$. On the other hand,

$$\begin{aligned}\int_{\mathbb{H}^n} u(y)\left(\int_{\mathbb{H}^n}(x_j - y_j)\,\tilde{J}(y^{-1}x)\ \mathrm{d}x\right)\mathrm{d}y &= \int_{\mathbb{H}^n} u(y)\left(\int_{\mathbb{H}^n}(y^{-1}x)_j\,\tilde{J}(y^{-1}x)\ \mathrm{d}x\right)\mathrm{d}y\\ &= \int_{\mathbb{H}^n}\xi_j\tilde{J}(\xi)\mathrm{d}\xi \cdot \int_{\mathbb{H}^n} u(y)\mathrm{d}y = 0.\end{aligned}$$

If $j = 2n + 1$, the argument is similar, but requires some further tricks. We write:

$$\begin{aligned}x_{2n+1} &= (y^{-1}x)_{2n+1} + y_{2n+1} + \frac{1}{2}\sum_{j=1}^{n}(y_j x_{n+j} - x_j y_{n+j})\\ &= (y^{-1}x)_{2n+1} + y_{2n+1} + \frac{1}{2}\sum_{j=1}^{n}(y_j(x-y)_{n+j} - (x-y)_j y_{n+j}).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{H}^n} x_{2n+1}\left(\int_{\mathbb{H}^n} u(y)\tilde{J}(y^{-1}x)\mathrm{d}y\right)\mathrm{d}x &= \int_{\mathbb{H}^n}\left(\int_{\mathbb{H}^n} u(y)(y^{-1}x)_{2n+1}\tilde{J}(y^{-1}x)\mathrm{d}y\right)\mathrm{d}x + \int_{\mathbb{H}^n}\left(\int_{\mathbb{H}^n} u(y)y_{2n+1}\tilde{J}(y^{-1}x)\mathrm{d}y\right)\mathrm{d}x\\ &+ \frac{1}{2}\sum_{j=1}^{n}\left\{\int_{\mathbb{H}^n}\left(\int_{\mathbb{H}^n} u(y)y_j(x-y)_{n+j}\tilde{J}(y^{-1}x)\mathrm{d}y\right)\mathrm{d}x + \int_{\mathbb{H}^n}\left(\int_{\mathbb{H}^n} u(y)y_{n+j}(x-y)_j\tilde{J}(y^{-1}x)\mathrm{d}y\right)\mathrm{d}x\right.\\ &= 0\end{aligned}$$

arguing as earlier. Thus, by iteration, $(1 - \Delta_{\mathbb{H}^n})^{-m/2}u \in \mathcal{S}_0(\mathbb{H}^n)$. □

Lemma B.18. *If $m \geq 0$, then $\mathcal{S}_0(\mathbb{H}^n)$ is dense in $W^{m,2}(\mathbb{H}^n)$.*

Proof. If $m = 0$, then the assertion follows straighforwardly via Fourier transform. Suppose now $m > 0$ and let $v \in W^{m,2}(\mathbb{H}^n)$ be normal to $\mathcal{S}_0(\mathbb{H}^n)$, i.e.,

$$\langle (1 - \Delta_{\mathbb{H}})^m v, u \rangle_{L^2(\mathbb{H}^n)} = 0 \quad \text{for all} \;\; u \in \mathcal{S}_0(\mathbb{H}^n). \tag{B7}$$

Let now $\phi \in \mathcal{S}_0(\mathbb{H}^n)$ arbitrary. By Lemma B.17, we can take in (B7) $u := (1 - \Delta_{\mathbb{H}})^{-m}\phi \in \mathcal{S}_0(\mathbb{H}^n)$. Therefore,

$$\langle v, \phi \rangle_{L^2(\mathbb{H}^n)} = \langle (1 - \Delta_{\mathbb{H}})^m v, u \rangle_{L^2(\mathbb{H}^n)} = 0,$$

and the assertion follows since $\mathcal{S}_0(\mathbb{H}^n)$ is dense in $L^2(\mathbb{H}^n)$. □

Definition B.19. Once a basis of $E_0^\bullet$ is fixed, and $1 \leq p \leq \infty$, we denote by $L^p(\mathbb{H}^n, E_0^\bullet)$ the space of all sections of $E_0^\bullet$ such that their components with respect to the given basis belong to $L^p(\mathbb{H}^n)$, endowed with its natural norm. Clearly, this definition is independent of the choice of the basis itself. The notations, $\mathcal{D}(\mathbb{H}^n, E_0^\bullet)$, $\mathcal{S}(\mathbb{H}^n, E_0^\bullet)$, $\mathcal{S}_0(\mathbb{H}^n, E_0^\bullet)$, as well as $W^{m,p}(\mathbb{H}^n, E_0^\bullet)$ have the same meaning.

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