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(Quasi-)conformal methods in two-dimensional free boundary problems

Abstract. In this paper we relate the theory of quasi-conformal maps to the regularity of the solutions to nonlinear thin-obstacle problems; we prove that the contact set is locally a finite union of intervals and we apply this result to the solutions of one-phase Bernoulli free boundary problems with geometric constraint. We also introduce a new conformal hodograph transform, which allows to obtain the precise expansion at branch points of both the solutions to the one-phase problem with geometric constraint and a class of symmetric solutions to the two-phase problem, as well as to construct examples of free boundaries with cusp-like singularities.

1. Introduction

This note is dedicated to the analysis of the branch singularities arising in two different types of free boundary problems in dimension two: non-linear thin-obstacle problems and one-phase Bernoulli problems with geometric constraint. In the last part of the paper we will present some results about branch points of the two-phase problem.

Our main motivation is the description of the structure of branch points arising in free boundary problems of Bernoulli type. Our main model example is the following one-phase problem with geometric constraint, which for simplicity we state for nonnegative functions u defined on the unit ball B_1 in $\mathbb{R}^d$:

$$
\begin{aligned}
\Delta u &= 0 && \text{in} && \Omega_u \subset B_1 \cap \{x_d > 0\} \\
u &= 0 && \text{on} && B_1 \cap \{x_d = 0\} \\
|\nabla u| &= 1 && \text{on} && \partial\Omega_u \cap \{x_d > 0\} \\
|\nabla u| &\geq 1 && \text{on} && \partial\Omega_u \cap \{x_d = 0\},
\end{aligned}
$$

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in which

$$\Omega_u := \{u > 0\}$$

and the geometric constraint is the inclusion $\Omega_u \subset B_1 \cap \{x_d > 0\}$. The (optimal) $C^{1,1/2}$ regularity of the free boundary $\partial\Omega_u \cap B_1$ for this specific problem was proved by Chang-Lara and Savin in [5]. On the other hand, as in the case of other Bernoulli free boundary problems as the two-phase problem [10] and the vectorial problem [22], the $C^{1,\alpha}$ regularity of the free boundary $\partial\Omega_u \cap B_1$ by itself does not give any information on the contact set

$$\partial\Omega_u \cap \{x_d = 0\} \cap B_1,$$

nor the structure of its relative boundary in $B_1 \cap \{x_d = 0\}$, which is the set of points at which $\partial\Omega_u$ branches away from $\{x_d = 0\}$. In dimension two, it is natural to expect that this set is discrete and that around each branch point the set $\{u = 0\} \cap \{x_d > 0\}$ forms a cusp. This is precisely the content of one of our main results, Theorem 1.1.

We will study these singularities in two different ways. First we will prove that branch singularities for minimizers of a general non-linear thin-obstacle problem are isolated, using the theory of quasiconformal maps, and then we will deduce the same result for solutions of the problem above via an hodograph transform. Secondly, we will introduce a *conformal hodograph transform* and use it to deduce the result directly. This second method has two advantages: it allows us to give a precise description of the cuspidal behavior of the free boundary at branch singularities and moreover, being reversible, it allows to show that solutions of the 2-dimensional one phase problem with obstacle are in a 1 to 1 correspondence with solution to the thin-obstacle problem, thus producing many examples of cuspidal singularities. Finally we will describe a special symmetric situation in which our techniques apply to the branch points of solutions to the two-phase problem. Extending our results to the general two-phase situation seems to require an entirely new idea, which is similar in spirit to an analogue of the Almgren's center manifold for this problem (see [4]).

The quasi-conformal technique is needed to prove Theorem 1.1 and it is the only one available there. On the other hand the conformal hodograph transform is the only technique which allows to give the precise analytic expansion in Theorem 1.3 (b) and Theorem 1.6 (b), and to construct the corresponding examples in Theorem 1.4 and Theorem 1.8. Both techniques can be used to prove Theorem 1.3 (a) and Theorem 1.6 (a).

We wish to remark that such precise results at branch points, that is singular points at which the tangent to the free boundary is a plane, usually with multiplicity, are quite rare. To our knowledge, the only such examples are the results of Chang on 2-dimensional area minimizing currents ([4, 7–9]), of Sakai on the 2-dimensional obstacle problem ([20, 21]), and of Lewy on the 2-dimensional thin-obstacle problem ([17], and also [16] for a less precise result); like in the present paper, all these results are 2-dimensional.

Our approach is similar in spirit to the results of Sakai and Lewy, and makes use of (quasi)-conformal techniques to prove both the local finiteness of the branch set and to give a precise description of the cuspidal behavior at such points. A possible alternative

approach, which could also be applicable in higher dimensions, would be to look for a monotone quantity, such as the Almgren's frequency function as done for instance in the Chang's paper [4]; in fact, for some thin-obstacle problems, as for instance the one involving the classical Laplace operartor, the monotonicity of the Almgren's frequency function is known (see [1, 16]) and can still be used to get information on the dimension of the branch set (see [14]). However, the operators we study are not regular enough to guarantee the monotonicity of the frequency function, and so we were naturally led to consider (quasi-)conformal techniques. Furthermore, our techniques have the additional benefit of yielding a very precise local description of the free-boundary at branch points (see Items (b) of Theorems 1.1, 1.3 and 1.6) in a straightforward way, much simpler than the induction procedure that would be needed using the frequency function as in [4].

Concerning the possible extensions of Theorem 1.1, Theorem 1.3 and Theorem 1.6 to higher dimensions we point out that, since the monotonicity of the frequency function doesn't seem to hold in none of these cases (and it certainly does not hold in the full generality of Theorem 1.1), a dimension reduction argument seems completely out of reach.

1.1. Non-linear thin-obstacle problem

Let B_1 be the unit ball in $\mathbb{R}^2$ and let

$$B_1^+ := \{(x, y) \in B_1 \ : \ y > 0\} \qquad \text{and} \qquad B_1' = \{(x, y) \in B_1 \ : \ y = 0\}.$$

Let $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}$ be a C^2-regular function, and let $\mathcal{F}_j$, $j = 1, 2$ and $\mathcal{F}_{ij}$, $1 \leq i, j \leq 2$, be the first and second order partial derivatives of $\mathcal{F}$. Moreover, we identify $\mathbb{R}^2$ with the field of complex numbers $\mathbb{C}$, so we will often think of the functions on $\mathbb{R}^2 = \mathbb{C}$ as functions of two real variables $(x, y) \in \mathbb{R}^2$ and at the same time as a function of one complex variable $z = x + iy \in \mathbb{C}$.

We consider solutions $U \in C^1(B_1^+ \cup B_1')$ of the following nonlinear thin-obstacle problem

$$\text{div}(\nabla \mathcal{F}(\nabla U)) = 0 \quad \text{in} \quad B_1^+ \,, \tag{1.1}$$

$$U \geq 0 \quad \text{on} \quad B_1' \,, \tag{1.2}$$

$$\nabla \mathcal{F}(\nabla U) \cdot e_2 = 0 \quad \text{on} \quad \{U > 0\} \cap B_1' \,, \tag{1.3}$$

$$\nabla \mathcal{F}(\nabla U) \cdot e_2 \leq 0 \quad \text{on} \quad \{U = 0\} \cap B_1' \,, \tag{1.4}$$

where $e_2 = (0, 1)$. Our first main result is the following.

Theorem 1.1 (Non-linear thin-obstacle). *Suppose that $U \in C^1(B_1^+ \cup B_1')$ is a solution to* (1.1)-(1.2)-(1.3)-(1.4) *and that $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}$ is C^2-regular function satisfying*

$$\nabla \mathcal{F}(0) = 0 \qquad \text{and} \qquad \nabla^2 \mathcal{F}(0) = \text{Id}. \tag{1.5}$$

Then, the following holds:

(a) *The set of branch points*

$$\mathcal{S}(U) := \big\{ z \in B_1' \ : \ U(z) = 0, \ \nabla U(z) = 0 \big\}, \tag{1.6}$$

is a discrete (locally finite) subset of B_1'.

(b) *For every point* $z_0 \in \mathcal{S}(U)$ *(without loss of generality* $z_0 = 0$*), there are:*

- *a radius* $r > 0$ *and a quasi-conformal homeomorphism* $\Psi : B_r \to \Omega$, *between* B_r *and an open set* $\Omega \subset B_1$, *such that:*

$$\Psi \in W^{1,2}_{loc}(B_r; \mathbb{R}^2), \tag{1.7}$$

$$Im(\Psi(z)) \equiv 0 \quad on \quad Im(z) \equiv 0, \tag{1.8}$$

$$|\Psi(z) - z| = o(|z|); \tag{1.9}$$

- *a holomorphic function* $\Phi : B_1 \to \mathbb{C}$ *of the form*

$$\Phi(z) = az^k + O(z^{k+1}) \quad where \quad k \geq 3 \quad and \quad a \in \mathbb{C}; \tag{1.10}$$

such that we can write the solution U *as*

$$U(z) = Re\Big(\Phi\big(\Psi(z)\big)^{1/2} \Big) \quad for\ every \quad z \in B_r(z_0). \tag{1.11}$$

Remark 1.2 (Optimal regularity). *We notice that one particular consequence of the previous theorem, is the optimal regularity for solutions of the non-linear thin-obstacle problem* (1.1)-(1.2)-(1.3)-(1.4). *In fact, if* $U \in C^1(B_1^+ \cup B_1')$ *is as in Theorem 1.1, then from* (1.11), (1.10) *and* (1.9) *it follows that* $U \in C^{1,1/2}(B_1^+ \cup B_1')$.

In the case of the classical thin-obstacle problem in which the operator is the Laplacian, that is $\mathcal{F}(x, y) = x^2 + y^2$, the results (a) and (b) of Theorem 1.1 were obtained by Lewy in [17]; moreover, in this case, the claim (a) can also be obtained by means of the Almgren's monotonicity formula (see [1] and [16]); we also notice that for the classical thin-obstacle problem, the map Ψ from Theorem 1.1 is the identity.

However, in order to apply this result to the one-phase problem described in the next subsection, we will be interested in solutions u of the thin-obstacle problem with

$$\mathcal{F}(x, y) := \frac{x^2 + y^2}{1 + y}$$

and for which $\nabla u \in C^{0,1/2}$ and no better. In particular, it is easy to check that U is a solution of an equation of the form

$$\mathrm{div}(A(x)\nabla U) = 0$$

where $A(x)$ is no better than $C^{0,1/2}$. For these type of equations the results in [15] can not be applied (and actually are known to fail) so in order to obtain our result we need to exploit the "quasi-linear" structure of the problem and our approach, based on the use of quasi-conformal maps, seems to be more suitable, although limited to dimension 2.

1.2. One-phase problem with geometric constraint

Next, we consider the following one-phase problem constrained above an hyperplane, that is let $u : B_1 \cap \{x_d \geq 0\} \to \mathbb{R}$ be a continuous non-negative function solution of the problem

$$\Delta u = 0 \quad \text{in} \quad \Omega_u := \{u > 0\} \subset B_1, \tag{1.12}$$

$$u = 0 \quad \text{on} \quad B_1 \cap \{x_d = 0\}, \tag{1.13}$$

$$|\nabla u| = 1 \quad \text{on} \quad \partial \Omega_u \cap \{x_d > 0\}, \tag{1.14}$$

$$|\nabla u| \geq 1 \quad \text{on} \quad \partial \Omega_u \cap \{x_d = 0\}. \tag{1.15}$$

In the recent paper by Chang-Lara and Savin [5] it was shown that if u is a viscosity solution of this problem (that is, if the boundary conditions (1.14) and (1.15) are intended in viscosity sense), then in a neighborhood of any contact point $x = (x', 0) \in \partial \Omega_u \cap \{x_d = 0\}$ the boundary $\partial \Omega_u$ is a $C^{1,\alpha}$-regular graph over the hyperplane $\{x_d = 0\}$. More precisely in a neighborhood of a point $z_0 \in \partial \Omega_u \cap \{x_d = 0\}$, the boundary $\partial \Omega$ is a $C^{1,1/2}$-regular surfaces, that is, there are a radius $\rho > 0$ and a $C^{1,1/2}$-regular function

$$f : B'_\rho(z_0) \to [0, +\infty),$$

such that, up to a rotation and translation of the coordinate system, we have

$$\begin{cases} u(x) > 0 & \text{for} \quad x \in (x', x_d) \in B_\rho(z_0) \quad \text{such that} \quad x_d > f(x'); \\ u(x) = 0 & \text{for} \quad x \in (x', x_d) \in B_\rho(z_0) \quad \text{such that} \quad x_d \leq f(x'). \end{cases} \tag{1.16}$$

We denote by $C_1(u)$ the contact set of the free boundary $\partial \Omega_u$ with the hyperplane $\{x_d = 0\}$

$$C_1(u) := \{x_d = 0\} \cap \partial \Omega_u,$$

and by $\mathcal{B}_1(u)$ the set of points at which the free boundary separates from $\{x_d = 0\}$:

$$\mathcal{B}_1(u) := \left\{ x \in C_1(u) \ : \ B_r(x) \cap \left(\partial \Omega_u \setminus \{x_d = 0\}\right) \neq \emptyset \ \text{for every} \ r > 0 \right\}.$$

By $\mathcal{S}_1(u)$ we denote the set of points in $C_1(u)$ at which u has gradient precisely equal to 1

$$\mathcal{S}_1(u) := \left\{ z \in C_1(u) \ : \ |\nabla u|(z) = 1 \right\}. \tag{1.17}$$

We notice that a priori the set $C_1(u)$ is no more than a closed subset of $\{x_d = 0\}$. Moreover, if at a point $x = (x', 0)$ we have that $|\nabla u|(x', 0) > 1$, then this point is necessarily in the interior of $C_1(u)$ in the hyperplane $\{x_d = 0\}$. Thus,

$$\mathcal{S}_1(u) \text{ contains all branch points}, \mathcal{B}_1(u) \subset \mathcal{S}_1(u).$$

Theorem 1.3 (Analyticity at the branch points in the one phase problem with obstacle). *Let u be a solution of* (1.12)–(1.15) *in dimension $d = 2$. Then, the following holds:*
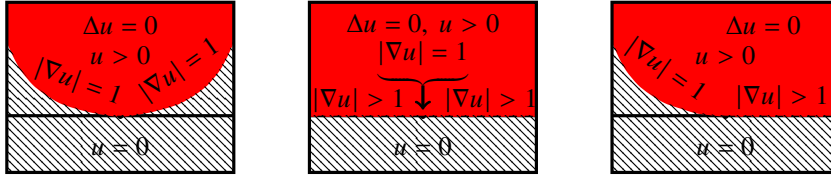
(a) $\mathcal{S}_1(u)$ *is locally finite and $C_1(u)$ is a locally finite union of disjoint closed intervals of the axis $\{x_2 = 0\}$;*

(b) *For every point $z_0 \in \mathcal{S}_1(u)$, one of the following holds:*

    (b.1) *z_0 is an isolated point of $C_1(u)$ and, in a neighborhood of z_0, the free boundary $\partial \Omega_u$ is the graph of an analytic function that vanishes only at z_0;*

    (b.2) *z_0 lies in the interior of $C_1(u)$ and there is $r > 0$ such that u is harmonic in $B_r(z_0)$ and $|\nabla u| > 1$ at all points of $\{x_2 = 0\} \cap B_r(z_0)$ except z_0;*

    (b.3) *z_0 is an endpoint of a non-trivial interval in the contact set $C_1(u)$; moreover, there is an interval $\mathcal{I}_\rho = (-\rho, \rho)$ and analytic function $\phi : \mathcal{I}_\rho \to \mathbb{R}$ such that $\phi(0) > 0$ and, up to setting $z_0 = 0$ and rotating the coordinate axis,*

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ x^{k/2}\,\phi(x) & \text{if } x < 0\,, \end{cases} \tag{1.18}$$

    where f is the function from (1.16).

As we mentioned above we will give two proofs of this result. The first will be obtained combining Theorem 1.1 with the standard hodograph transform. The second proof instead, more geometric in spirit, will be achieved via a conformal hodograph transform. This proof has the advantage of being reversible, thus allowing us to construct examples of solutions and free boundaries with any prescribed cuspidal behavior (without invoking any fixed point argument, as usual in the literature).

Theorem 1.4 (Cuspidal points for one-phase problem). *For any positive integer $n \in \mathbb{N}$, there exists a solution of* (1.12)–(1.15) *in dimension $d = 2$ such that* (1.18) *in Theorem 1.3 holds with $k = 4n - 1$.*

    We point out that Theorem 1.3 and Theorem 1.4 can be deduced respectively from Theorem 1.6 and Theorem 1.8 below by performing an odd reflection.

1.3. Symmetric two-phase problem

Finally, we consider solutions to the two-phase free boundary problem in viscosity sense, that is we let $u : B_1 \to \mathbb{R}$ be a continuous function and we denote by u_+ and u_- the functions

$$u_+ = \max\{u, 0\} \qquad \text{and} \qquad u_- := \min\{u, 0\}.$$

and by Ω_u^+ and Ω_u^- the sets

$$\Omega_u^\pm := \{\pm u > 0\}.$$

Notice that with this notation u_- is negative. Then u is a viscosity solution of the problem

$$\Delta u = 0 \quad \text{in} \quad \Omega_u^+ \cup \Omega_u^-, \tag{1.19}$$

$$|\nabla u_+| = 1 \quad \text{on} \quad \partial\Omega_u^+ \setminus \partial\Omega_u^- \cap B_1, \tag{1.20}$$

$$|\nabla u_-| = 1 \quad \text{on} \quad \partial\Omega_u^- \setminus \partial\Omega_u^+ \cap B_1, \tag{1.21}$$

$$|\nabla u_+| = |\nabla u_-| \geq 1 \quad \text{on} \quad \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1. \tag{1.22}$$

In [10], we proved that if u is a viscosity solution of this problem in any dimension $d \geq 2$, then in a neighborhood of any two-phase point

$$x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1,$$

both free boundaries $\partial\Omega_u^+ \cap B_1$ and $\partial\Omega_u^- \cap B_1$ are $C^{1,\alpha}$ regular. Thus, by the classical elliptic regularity theory, also the functions $u_\pm$ are $C^{1,\alpha}$ regular respectively on $\overline{\Omega_u^+} \cap B_1$ and $\overline{\Omega_u^-} \cap B_1$ and the equations (1.19)-(1.22) hold in the classical sense.

We will denote with $C_2(u_+, u_-)$ the two-phase free boundary, which is the contact set between the free boundaries $\partial\Omega_u^+$ and $\partial\Omega_u^-$, and with $O_\pm$ the remaining one-phase parts:

$$C_2(u_+, u_-) := \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1 \qquad \text{and} \qquad O_\pm := \left(\partial\Omega_u^\pm \cap B_1\right) \setminus C_2(u_+, u_-).$$

We notice that the set $C_2(u_+, u_-)$ is closed, while O_+ and O_- are relatively open subsets respectively of $\partial\Omega_u^\pm \cap B_1$. We define the set of branch points $\mathcal{B}_2(u_+, u_-)$ as the set of points at which the two free boundaries $\partial\Omega_u^\pm$ separate, that is

$$\mathcal{B}_2(u_+, u_-) = \left\{x \in C_2(u_+, u_-) \ : \ B_r(x) \cap O_\pm \neq \emptyset \text{ for every } r > 0\right\}. \tag{1.23}$$

By C^1-regularity of $u_\pm$, if $x \in \left(\partial\Omega_u^+ \cup \partial\Omega_u^-\right) \cap B_1$ is such that

$$|\nabla u_+|(x) > 1 \quad \text{or} \quad |\nabla u_-|(x) > 1,$$

then it is necessarily a two-phase non-branch point: $x \in C_2(u_+, u_-) \setminus \mathcal{B}_2(u_+, u_-)$. In particular, this implies that the set

$$S_2(u_+, u_-) := \left\{x \in C_2(u_+, u_-) \ : \ |\nabla u_+|(x) = |\nabla u_-|(x) = 1\right\}, \tag{1.24}$$

contains the set of branch points $\mathcal{B}_2(u_+, u_-)$.

In dimension $d = 2$, $\partial\Omega_u^\pm$ are locally parametrized by two $C^{1,\alpha}$ curves. Precisely, suppose that $z_0 = (x_0, y_0) \in C_2(u_+, u_-)$, without loss of generality we may assume that $z_0 = (0, 0)$, and that there is an interval $\mathcal{I}_\rho := (-\rho, \rho)$ and two $C^{1,\alpha}$-regular functions

$$f_\pm : \mathcal{I}_\rho \to \mathbb{R},$$

such that

$$f_+ \geq f_- \quad \text{on} \quad \mathcal{I}_\rho \qquad \text{and} \qquad f_+(0) = f_-(0) = \partial_x f_+(0) = \partial_x f_-(0) = 0,$$

and, up to rotations and translations,

$$\begin{cases} u(x, y) > 0 & \text{for} \quad (x, y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho \quad \text{such that} \quad y > f_+(x); \\ u(x, y) = 0 & \text{for} \quad (x, y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho \quad \text{such that} \quad f_-(x) \leq y \leq f_+(x); \\ u(x, y) < 0 & \text{for} \quad (x, y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho \quad \text{such that} \quad y < f_-(x). \end{cases} \tag{1.25}$$

Thus, in the square $\mathcal{I}_\rho \times \mathcal{I}_\rho$, the one-phase parts O_+ and O_- of the free boundary are the union of $C^{1,\alpha}$ (actually analytic) graphs over a countable family of disjoint open intervals:

$$O_\pm := \bigcup_{i \in \mathbb{N}} \Gamma^i_\pm,$$

where, for every $i \in \mathbb{N}$, there is an open interval $\mathcal{I}_i \subset \mathcal{I}_\rho$ such that

$$\Gamma^i_\pm = \big\{ (x, f_\pm(x)) \, : \, x \in \mathcal{I}_i \big\}. \tag{1.26}$$

Definition 1.5 (Symmetric solutions of the two-phase problem). *In dimension $d = 2$, we will say that a continuous function $u : B_1 \to \mathbb{R}$ is a* symmetric *solution to the two-phase problem if u satisfies (1.19)-(1.22) and moreover*

$$\mathcal{H}^1(\Gamma^i_+) = \mathcal{H}^1(\Gamma^i_-) \quad \text{for every} \quad i \in \mathbb{N} \quad \text{such that} \quad \overline{\mathcal{I}_i} \subset \mathcal{I}_\rho. \tag{1.27}$$

The main result of this section is the following.

Theorem 1.6 (Cuspidal points for the symmetric solutions of the two-phase problem). *Let $u : B_1 \to \mathbb{R}$ be a viscosity solution of the two-phase problem (1.19)-(1.22). Then the following holds.*
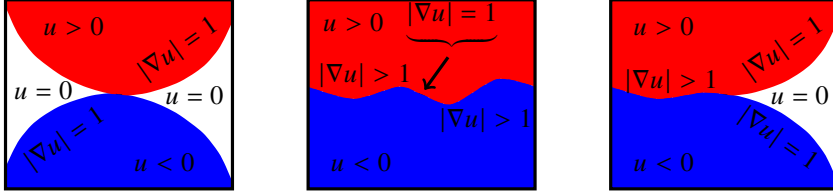
(a) *If u is symmetric in the sense of Definition 1.5, then the singular set $\mathcal{S}_2(u_+, u_-)$ defined in (1.24) is locally finite, so in particular the two-phase free boundary $\mathcal{C}_2(u_+, u_-) = \big(\partial\Omega^+_u \cup \partial\Omega^-_u\big) \cap B_1$ is a locally finite union of disjoint $C^{1,\alpha}$-arcs;*

(b) *If $z_0 \in \mathcal{S}_2(u_+, u_-)$ is an isolated point of $\mathcal{S}_2(u_+, u_-)$, then we have one of the following possibilities:*

    (b.1) *z_0 is an isolated point of $\mathcal{C}_2(u_+, u_-)$ and, in a neighborhood of z_0, the free boundaries $\partial\Omega^+_u$ and $\partial\Omega^-_u$ are analytic graphs meeting only in z_0;*

    (b.2) *z_0 lies in the interior of $\mathcal{C}_2(u_+, u_-)$ and moreover there is $r > 0$ such that: $\Delta u = 0$ in $B_r(z_0)$ and $|\nabla u| > 1$ at all points of $\{u = 0\} \cap B_r(z_0)$ except z_0;*

    (b.3) *z_0 is an endpoint of a non-trivial arc in $\mathcal{C}_2(u_+, u_-)$, and there are an interval $\mathcal{I}_\rho = (-\rho, \rho)$ a constant $k \in \mathbb{N}$, $k \geq 3$, and an analytic function $\phi : \mathcal{I}_\rho \to \mathbb{R}$ such that $\phi(0) \neq 0$ and, up to setting $z_0 = 0$ and changing the coordinates,*

$$f_+(x) - f_-(x) = \begin{cases} x^{k/2} \phi(|x|^{1/2}) & \text{if } x \leq 0 \\ 0 & \text{if } x \geq 0. \end{cases} \tag{1.28}$$

Precisely, there are analytic functions Φ, $\beta_\pm$ *and* Θ *such that for every* $x \le 0$

$$f_\pm(x) = \Phi\left(x + |x|^{5/2}\beta_\pm\left(|x|^{1/2}\right)\right) \pm \Psi\left(x + |x|^{5/2}\beta_\pm\left(|x|^{1/2}\right)\right), \qquad (1.29)$$

where Ψ *is of the form* $\Psi(x) = |x|^{3/2}\Theta(x)$.

Notice that (a) of the previous theorem requires that the function u is symmetric in the generalized sense of Definition 1.5, while (b.3) is always true at isolated branch points. The question of whether the statement of Theorem 1.6 (a) is true without the generalized symmetry assumption is extremely interesting and would probably require the introduction of new techniques.

We also have the following result, which simply follows from the fact that if z_0 is an isolated point of $\mathcal{B}_2(u_+, u_-)$, then it is also an isolated point of $\mathcal{S}_2(u_+, u_-)$ for which Theorem 1.6 (b.2) does not hold.

Corollary 1.7 (Isolated cuspidal points of two-phase problem). *Let u be a solution of the two-phase problem as in Definition 1.5. If $z_0 \in \mathcal{B}_2(u_+, u_-)$ is an isolated point of the set $\mathcal{B}_2(u_+, u_-)$ defined in (1.23), then at least one of the points (b.1) and (b.3) is true at z_0.*

We will prove Theorem 1.6 in Section 5, where we will also discuss the obstructions in applying the conformal hodograph transform to the study of the branch points of the two-phase problem in the absence of symmetries or in the presence of weights $\lambda_\pm$ on the volume of the positivity and the negativity sets.

Finally, as in Theorem 1.4, by reversing the argument from the proof of Theorem 1.6, we can construct two-phase cusps with prescribed behavior.

Theorem 1.8 (Cuspidal points for two-phase problem). *For any positive integer $n \in \mathbb{N}$, there exists a solution of (1.19)–(1.22) in dimension $d = 2$ such that (1.28) holds with $k = 4n - 1$ and (1.29) with $\Phi(x) = x^m + o(x)$, with $m \ge 2$.*

The particular case $\Phi \equiv 0$ is an immediate consequence from Theorem 1.4 as a solution of the one-phase problem, together with its reflection, gives a solution of the two-phase one. However, the same method provides also non-symmetric examples in which the asymmetry is given by the function Φ.

We notice that the examples constructed in Theorem 1.8 are minimizers of the two-phase functional. Indeed, any flat monotone solution to (1.19)–(1.22) is unique and so it minimizes the two-phase functonal; we prove this in Appendix A as a direct consequence

of the maximum principle. We refer to the recent work [6] for some interesting examples of almost-minimizing free boundaries.

2. Non-linear thin-obstacle problem

In this section we prove Theorem 1.1 using the theory of quasi-conformal map.

2.1. Notation and known results

Let $U \in C^1(B_1^+ \cup B_1')$ be a solution of the thin-obstacle problem (1.1)-(1.2)-(1.3)-(1.4), where the function $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}$ is C^2 regular.

2.1.1. Variational inequality formulation.
The system (1.1)–(1.4) can be equivalently written in the form of a variational inequality. Precisely, the following are equivalent:

(1) $U \in C^1(B_1^+ \cup B_1')$ and satisfies (1.1), (1.2), (1.3) and (1.4);

(2) $U \in H^1_{loc}(B_1^+ \cup B_1')$ (that is $u \in H^1(B_r^+)$ for every $r < 1$) and

$$\int_{B_1^+} \nabla \mathcal{F}(\nabla U) \cdot \nabla(U - v) \, dx \leq 0 \quad \text{for every} \quad v \in \mathcal{K}_U, \tag{2.1}$$

where $\mathcal{K}_U$ is the convex set

$$\mathcal{K}_U := \left\{ v \in H^1_{loc}(B_1^+ \cup B_1') : v \geq 0 \text{ on } B_1' , v = U \text{ in a neighborhood of } \partial B_1 \cap \{x_d > 0\} \right\}.$$

Indeed, the implication (1) $\Rightarrow$ (2) follows simply by an integration by parts, while (2) $\Rightarrow$ (1) was proved by Frehse [13]. In particular, if $U \in H^1(B_1^+)$ minimizes the integral functional

$$\mathcal{I}(v) := \int_{B_1^+} \mathcal{F}(\nabla v) \, dx , \tag{2.2}$$

among all functions in $\mathcal{K}_U$, then U satisfies the variational inequality (2.1).

2.1.2. Higher regularity of the solutions.
It was proved by Frehse in [13, Lemma 2.2] that if $U \in H^1(B_1^+)$ is a solution of the variational inequality (2.1), then U is in $H^2(B_r^+)$ for every $r < 1$. Moreover, in [11, Theorem 4.1] it was shown that the solution U is actually in $C^{1,\alpha}(B_1^+ \cup B_1')$ for some $\alpha > 0$.

2.2. Local finiteness of the set of branch points

In this subsection we prove Theorem 1.1 (a). We introduce a special function Q that we prove to be quasi-regular in the half-ball, then we obtain Theorem 1.1 (a) by applying the Stoïlow's factorization theorem for quasi-conformal and quasi-regular maps (see [2, Chapter 5]).

Given a solution $U : B_1 \cap \{y \geq 0\} \to \mathbb{R}$ of (1.1)-(1.2)-(1.3)-(1.4), we consider the function

$$Q : B_1^+ \cap \{y \geq 0\} \to \mathbb{C}, \qquad Q(x + iy) = \partial_x U - i\mathcal{F}_2(\nabla U(x, y)) \qquad (2.3)$$

We gather the fundamental properties of this function in the next lemma.

Lemma 2.1. *The function Q defined in* (2.3) *satisfies the following properties:*

(1) $Q^2 \in W^{1,2}(B_r^+; \mathbb{C})$, *for every $r < 1$;*

(2) *there is $r_0 > 0$ such that, for every $r < r_0$, Q satisfies the Beltrami equation*

$$\partial_{\bar{z}} Q = \mu(\nabla U, \nabla^2 U)\, \partial_z Q \qquad in \qquad B_r^+,$$

and if for some $\delta \in (0, 1]$

$$\|Id - \nabla^2 \mathcal{F}(\nabla U(z))\|_2 \leq \delta \qquad for\ every \qquad z = (x, y) \in B_r^+,$$

then

$$|\mu(\nabla U(z), \nabla^2 U(z))| \leq \frac{\delta}{\sqrt{4 - 4\delta - \delta^2}} \qquad for\ every \qquad z = (x, y) \in B_r^+,$$

where for any real matrix $A = (a_{ij})_{ij}$, $\|A\|_2 := \left(\sum_{i,j} a_{ij}^2 \right)^{1/2}$.

In particular, properties (1) and (2) imply that Q is a quasi-conformal map.

Proof. We first prove (1). By [13], we know that $U \in H^2(B_r^+)$ and that $|\nabla U| \in L^\infty(B_r^+)$. Thus, (1) follows directly by the definition of Q. Let us now prove (2).

For simplicity, we set

$$A := \partial_x U \qquad and \qquad B := \mathcal{F}_2(\nabla U).$$

Thus, $Q = A - iB$ and

$$\begin{cases} \partial_{\bar{z}} Q = \frac{1}{2}(\partial_x + i\partial_y)(A - iB) = \frac{1}{2}(\partial_x A + \partial_y B) + \frac{i}{2}(\partial_y A - \partial_x B), \\ \partial_z Q = \frac{1}{2}(\partial_x - i\partial_y)(A - iB) = \frac{1}{2}(\partial_x A - \partial_y B) - \frac{i}{2}(\partial_y A + \partial_x B), \end{cases}$$

which implies

$$\begin{cases} 4\,|\partial_{\bar{z}} Q|^2 &= (\partial_x A + \partial_y B)^2 + (\partial_y A - \partial_x B)^2, \\ 4\,|\partial_z Q|^2 &= (\partial_x A - \partial_y B)^2 + (\partial_y A + \partial_x B)^2. \end{cases} \qquad (2.4)$$

We first compute

$$\begin{cases} \partial_x A &= \partial_{xx} U \\ \partial_y A &= \partial_{xy} U \\ \partial_x B &= \mathcal{F}_{12}(\nabla U)\partial_{xx} U + \mathcal{F}_{22}(\nabla U)\partial_{xy} U \\ \partial_y B &= \mathcal{F}_{12}(\nabla U)\partial_{xy} U + \mathcal{F}_{22}(\nabla U)\partial_{yy} U, \end{cases} \qquad (2.5)$$

and, using the equation for U, we obtain

$$\begin{cases} \partial_x A + \partial_y B & = & \big(1 - \mathcal{F}_{11}(\nabla U)\big)\partial_{xx}U - \mathcal{F}_{12}(\nabla U)\partial_{xy}U \\ \partial_y A - \partial_x B & = & -\mathcal{F}_{12}(\nabla U)\partial_{xx}U + \big(1 - \mathcal{F}_{22}(\nabla U)\big)\partial_{xy}U. \end{cases} \tag{2.6}$$

For simplicity, we use the following notation

$$m_{ij} := \delta_{ij} - \mathcal{F}_{ij}(\nabla U) \quad \text{for every} \qquad 1 \le i, j \le 2,$$

and

$$\mathcal{M} := \mathrm{Id} - \nabla^2 \mathcal{F}(\nabla U) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

We also set

$$\|\mathcal{M}\|_2^2 := m_{11}^2 + 2m_{12}^2 + m_{22}^2.$$

Then, by (2.6) and the Cauchy-Schwartz inequality, we immediately obtain

$$(\partial_x A + \partial_y B)^2 + (\partial_y A - \partial_x B)^2 \le \|\mathcal{M}\|_2^2 |\nabla A|^2. \tag{2.7}$$

In order to estimate $|\partial_z Q|^2$ in (2.4), we write

$$\begin{aligned} (\partial_x A - \partial_y B)^2 + (\partial_y A + \partial_x B)^2 &= \Big(2\partial_x A - (\partial_x A + \partial_y B)\Big)^2 + \Big(2\partial_y A - (\partial_y A - \partial_x B)\Big)^2 \\ &= 4|\nabla A|^2 - 4\nabla A \cdot \mathcal{M}(\nabla A) \\ &\qquad + (\partial_x A + \partial_y B)^2 + (\partial_y A - \partial_x B)^2 \\ &=: 4|\nabla A|^2 + \mathcal{R}, \end{aligned}$$

where by (2.6) and (2.7), we have the estimate

$$|\mathcal{R}| \le \Big(4\|\mathcal{M}\|_2 + \|\mathcal{M}\|_2^2\Big)|\nabla A|^2.$$

Now, if at some point $\nabla A = 0$, then $\partial_z Q = \partial_{\bar{z}} Q = 0$. Thus, we can define μ as follows:

$$\mu = 0, \quad \text{if} \quad \nabla A = 0 \,; \qquad \mu = \frac{\partial_{\bar{z}} Q}{\partial_z Q}, \quad \text{if} \quad \nabla A \ne 0 \,.$$

Since A, $\partial_{\bar{z}} Q$ and $\partial_z Q$ are all functions of ∇U and $\nabla^2 U$, also μ can be written in terms of the same variables, that is: $\mu = \mu(\nabla U, \nabla^2 U)$. We notice that with this definition, μ remains bounded. Indeed,

$$|\mu|^2 = \left| \frac{\partial_{\bar{z}} Q}{\partial_z Q} \right|^2 \le \frac{\|\mathcal{M}\|_2^2}{4 - 4\|\mathcal{M}\|_2 + \|\mathcal{M}\|_2^2},$$

so that for r sufficiently small the conclusion follows. ∎

Proof of Theorem 1.1 (a). Let Q be the function defined in (2.3) and let

$$S(z) := \begin{cases} Q(z)^2 & \text{if } \operatorname{Im}(z) \geq 0 \\ \overline{S(\overline{z})} & \text{if } \operatorname{Im}(z) \leq 0 \end{cases}$$

We notice that

$$\operatorname{Im}(Q^2(z)) = \partial_x U \cdot \mathcal{F}_2(\nabla U) = 0 \qquad \text{on} \qquad \{\operatorname{Im}(z) = 0\},$$

so that the function S is in $W^{1,2}(B_r)$ and satisfies the Beltrami equation

$$\partial_{\overline{z}} S = \psi(z) \, \partial_z S \qquad \text{in} \qquad B_r^+,$$

where

$$\psi(z) = \psi(x + iy) := \begin{cases} \mu\big(\nabla U(x, y), \nabla^2 U(x, y)\big) & \text{if } \operatorname{Im}(z) \geq 0, \\ \overline{\psi(\overline{z})} & \text{if } \operatorname{Im}(z) \leq 0. \end{cases}$$

Thus, by [2, Theorem 5.5.2], we conclude that the zeros of the function S are isolated, which is the claim. ∎

2.3. Local behavior of the solutions at branch points

In this subsection we prove Theorem 1.1 (b). Given a branch point $z_0 \in \mathcal{S}$, we construct a quasi-regular mapping whose real part is precisely the solution U. Assume that $z_0 = 0$. We consider the case where it exists a radius $r > 0$ such that

$$\{U = 0\} \cap B_r' = \{x \leq 0\} \cap B_r' \qquad \text{and} \qquad \{U > 0\} \cap B_r' = \{x > 0\} \cap B_r', \qquad (2.8)$$

which is the case of a branch point, the other two cases being analogous.

We notice that the differential form

$$\alpha = -\mathcal{F}_2(\nabla U) \, dx + \mathcal{F}_1(\nabla U) \, dy$$

is closed in B_r^+ and so the potential

$$V : B_r^+ \cup B_r' \to \mathbb{R}, \qquad V(x, y) := \int_0^1 \Big(-\mathcal{F}_2\big(\nabla U(tx, ty)\big) x + \mathcal{F}_1\big(\nabla U(tx, ty)\big) y \Big) \, dt$$

is Lipschitz continuous in $B_r^+ \cup B_r'$, C^2 in B_r^+ and satisfies

$$\begin{cases} \partial_x V = -\mathcal{F}_2(\nabla U) & \text{in} \quad B_r^+, \\ \partial_y V = \mathcal{F}_1(\nabla U) & \text{in} \quad B_r^+, \\ \qquad UV = 0 & \text{on} \quad B_r', \end{cases}$$

where the last equality follows from (2.8), (1.3) and the very definition of V. We next define the complex function

$$P : B_r^+ \cap \{y \geq 0\} \to \mathbb{C}, \qquad P(x + iy) = U(x, y) + iV(x, y). \qquad (2.9)$$

Remark 2.2. *Notice that, by the definition of V, we have $\partial_x P = Q$ in B_r^+.*

We now prove the following lemma.

Lemma 2.3. *The function P defined in* (2.9) *satisfied the following properties.*

(1) $P^2 \in W_{loc}^{1,\infty}(B_1^+ \cup B_1')$,

(2) *P satisfies the Beltrami equation*

$$\partial_{\bar{z}} P = \eta(\nabla U)\, \partial_z P \qquad in \qquad B_r^+\,, \tag{2.10}$$

where $\eta(\nabla U) = o(|\nabla U|)$.

Proof. The first claim follows from the Lipschitz continuity of U and V. In order to prove the second claim, we compute

$$\begin{cases} 2\partial_{\bar{z}} P = (\partial_x + i\partial_y)(U + iV) = (\partial_x U - \mathcal{F}_1(\nabla U)) + i(\partial_y U - \mathcal{F}_2(\nabla U)), \\ 2\partial_z P = (\partial_x - i\partial_y)(U + iV) = (\partial_x U + \mathcal{F}_1(\nabla U)) - i(\partial_y U + \mathcal{F}_2(\nabla U)), \end{cases}$$

Now, by the differentiability of $\mathcal{F}_1$ and $\mathcal{F}_2$ in zero and (1.5), we can write

$$\mathcal{F}_1(X) - X_1 = \varepsilon_1(X)|X| \qquad \text{and} \qquad \mathcal{F}_2(X) - X_2 = \varepsilon_2(X)|X|,$$

for every $X = (X_1, X_2) \in \mathbb{R}^2$, where the functions ε_1 and ε_2 are such that

$$\lim_{|X|\to 0} \varepsilon_1(X) = \lim_{|X|\to 0} \varepsilon_2(X) = 0\,,$$

from which the first part of the claim follows.

∎

Proof of Theorem 1.1 (b). Let P be the function defined in (2.9) and let

$$T(z) := \begin{cases} P(z)^2 & \text{if } \operatorname{Im}(z) \geq 0 \\ \overline{T}(\bar{z}) & \text{if } \operatorname{Im}(z) \leq 0 \end{cases}$$

Then

$$\operatorname{Im}(P^2(z)) = U(z)V(z) = 0 \qquad \text{on} \qquad \{\operatorname{Im}(z) = 0\}\,,$$

so T is Lipschitz continuous on B_r, and satisfies the Beltrami equation

$$\partial_{\bar{z}} T = \phi(z)\, \partial_z T \quad \text{in} \quad B_r\,, \tag{2.11}$$

where ϕ is the extension over the whole B_r of the Beltrami coefficient $\eta(\nabla U)$ from (2.10) :

$$\phi(z) = \phi(x + iy) := \begin{cases} \eta(\nabla U(x, y)) & \text{if } \operatorname{Im}(z) \geq 0\,, \\ \overline{\phi}(\bar{z}) & \text{if } \operatorname{Im}(z) \leq 0\,. \end{cases}$$

By [2, Theorem 5.5.1 and Corollary 5.5.3], there exists an homeomorphism $\Psi \in W^{1,2}(B_r; B_1)$, solution of (2.11) and such that $\Psi(0) = 0$ and $\Psi(\rho) = \rho$, for some $\rho < r$, and an holomorphic function $\Phi \colon \Omega \to \mathbb{C}$ such that

$$T(z) = \Phi(\Psi(z)) \qquad \forall z \in B_r. \tag{2.12}$$

Next we prove (1.8). Observe that if Ψ is a solution to (2.11), then also $\overline{\Psi}(\overline{z})$ is a solution to (2.11), and moreover $\overline{\Psi}(0) = \Psi(0) = 0$ and $\overline{\Psi}(\rho) = \Psi(\rho) = 1$. It follows, by uniqueness of normalized solutions, that $\overline{\Psi}(z) = \Psi(z)$, which implies (1.8).

Finally we come to (1.9). Suppose by contradiction that (1.9) is false. Then, there is a sequence of radii $\rho_k \to 0$ such that the sequence of homeomorphisms $\Psi_k \in W^{1,2}(B_r, B_1)$, solutions of

$$\partial_{\overline{z}} \Psi_k = \phi(z)\, \partial_z \Psi \text{ in } B_r, \quad \Psi_k(0) = 0, \quad \Psi_k(\rho_k) = \rho_k,$$

doesn't converge uniformly to the function z. Consider the sequence of functions $\tilde{\Psi}_k(z) := \rho_k^{-1}\, \Psi_k(\rho_k\, z)$, then they are solutions of

$$\partial_{\overline{z}} \tilde{\Psi}_k = \phi\,(\rho_k\, z)\; \partial_z \tilde{\Psi} \text{ in } B_{r/\rho_k} \quad \tilde{\Psi}_k(0) = 0, \quad \tilde{\Psi}_k(1) = 1.$$

Reasoning as in the proof of Lemma 2.3 and using the fact that $\nabla U(\rho_k z) \to 0$ as $k \to \infty$, since $U \in C^1$ and $\nabla U(0) = 0$, we have

$$\lim_{k \to 0} \phi\,(\rho_k\, z) = 0 \qquad \text{a.e. } z \in B_{r/\rho_k}.$$

Using [2, Lemma 5.3.5], we have that $\tilde{\Phi}_k$ converges locally uniformly to a homeomorphism $\tilde{\Psi} \colon \mathbb{C} \to \mathbb{C}$, which is a solution of

$$\partial_{\overline{z}} \tilde{\Psi} = 0 \text{ in } \mathbb{C}, \quad \tilde{\Psi}(0) = 0, \quad \tilde{\Psi}(1) = 1.$$

But this implies that $\tilde{\Psi}(z) = z$, which is a contradiction for k sufficiently large.

In particular notice that, if $\Phi(z) = z^k + O(z^{k+1})$, then the C^1 regularity of solutions to the non-linear thin-obstacle problem (see for instance [12]) implies that $k \geq 3$. ∎

3. Theorem 1.3: proof via quasiconformal maps

In this section, we will prove Theorem 1.3 as a consequence of Theorem 1.1 combined with an application of the hodograph transform.

3.1. The hodograph transform

In this section we write the hodograph transformation of a solution u of (1.12)–(1.15). We do this in every dimension $d \geq 2$.

3.1.1. Notation. We adopt the following notation. We write every point $x \in \mathbb{R}^d$ in coordinates as $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$. For every $\rho > 0$, we denote by B_ρ and B'_ρ the balls centered in zero of radius ρ in $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$, respectively. We will identify $\mathbb{R}^{d-1}$ with the hyperplane $\mathbb{R}^{d-1} \times \{0\} \subset \mathbb{R}^d$, thus

$$B'_\rho = B_\rho \cap \{x_d = 0\} \qquad \text{and} \qquad B^+_\rho = B_\rho \cap \{x_d > 0\}.$$

We denote by $\nabla_{x'}$ the gradient with respect to the first $d - 1$ coordinates $x' = (x_1, \ldots, x_{d-1})$. Thus, for every function $u : \mathbb{R}^d \to \mathbb{R}$, we can write the full gradient ∇u as

$$\nabla u = (\nabla_{x'} u, \partial_d u) \quad \text{and} \quad |\nabla u|^2 = |\nabla_{x'} u|^2 + |\partial_d u|^2.$$

Let us assume that $0 \in \mathcal{S}_1(u)$, that is 0 is a branch point, and let $f \in C^{1,\alpha}$ be the function that locally describes the free boundary $\partial \Omega_u$ as in (1.16), so that

$$f(0) = 0 \quad \text{and} \quad \nabla_{x'} f(0) = 0.$$

Now since $u(x', f(x'))$ vanishes for every $x' \in B'_\rho$, we have that $\nabla_{x'} u(0) = 0$. Thus

$$\nabla u(0) = \partial_d u(0)\, e_d \qquad \text{and} \qquad \partial_d u(0) \geq 1\;.$$

3.1.2. The hodograph transform. Let $0 \in \partial \Omega_u \cap \{x_d = 0\}$ and $f : B'_\rho \to [0, +\infty)$ be as above. We consider the change of coordinates

$$y' = x'\,, \quad y_d = u(x', x_d).$$

Since $u \in C^{1,\alpha}(\overline{\Omega}_u \cap B_1)$, and since $\partial_d u(0) \geq 1 > 0$, we have that the function

$$T : B_\rho \cap \overline{\Omega}_u \to \mathbb{R}^d \cap \{y_d \geq 0\}\,, \qquad T(x', x_d) = (y', y_d),$$

is invertible for ρ small enough. In particular, the set $T\big(B_\rho \cap \overline{\Omega}_u\big)$ is an open neighborhood of 0 in the upper half-space $\mathbb{R}_d \cap \{y_d \geq 0\}$. Let

$$S : T\big(B_\rho \cap \overline{\Omega}_u\big) \to B_\rho \cap \overline{\Omega}_u\,, \qquad S(y', y_d) = (x', x_d),$$

be the inverse of T. Since the map T does not change the first $d - 1$ coordinates, there is a $C^{1,\alpha}$ regular function v, defined on the set $T\big(B_\rho \cap \overline{\Omega}_u\big)$, such that

$$S(y', y_d) = \big(y', v(y', y_d)\big).$$

We will write this in coordinates as

$$x' = y'\,, \quad x_d = v(y', y_d).$$

Remark 3.1. *The function v contains all the information of the free boundary $\partial \Omega_u$. Precisely, for every x' in a neighborhood of $0 \in \mathbb{R}^{d-1}$, we have*

$$v(x', 0) = f(x'). \tag{3.1}$$

Indeed, it is immediate to check that for any point (x', x_d) in a neighborhood of zero,

$$x_d = f(x') \iff (x', x_d) \in \partial\Omega_u \iff x_d = v(x', u(x', x_d)) = v(x', 0).$$

As a consequence of (3.1)*, we get that*

$$v(x', 0) \geq 0 \quad \text{for every } x' \text{ in a neighborhood of zero in } \mathbb{R}^{d-1}. \tag{3.2}$$

Lemma 3.2 (Hodograph transform). *Let u, T, B_ρ and v be as above. Then, there is $r > 0$ such that*

$$B_r \cap \{x_d \geq 0\} \subset T(B_\rho \cap \overline{\Omega}_u),$$

and such that the function

$$v : B_r \cap \{x_d \geq 0\} \to \mathbb{R},$$

exists, is $C^{1,\alpha}$ in $B_r \cap \{x_d \geq 0\}$ and C^∞ in $B_r \cap \{x_d > 0\}$. Moreover, the function

$$w : B_r \cap \{x_d \geq 0\} \to \mathbb{R}, \qquad w(x', x_d) = v(x', x_d) - x_d$$

solves the nonlinear thin-obstacle problem

$$\mathrm{div}(\nabla\mathcal{F}(\nabla w)) = 0 \quad \text{in} \quad B_r^+, \tag{3.3}$$

$$w \geq 0 \quad \text{on} \quad B_r', \tag{3.4}$$

$$\mathcal{F}_d(\nabla w) = 0 \quad \text{on} \quad \{w > 0\} \cap B_r', \tag{3.5}$$

$$\mathcal{F}_d(\nabla w) \leq 0 \quad \text{on} \quad \{w = 0\} \cap B_r', \tag{3.6}$$

for the nonlinearity $\mathcal{F}(x', x_d) := \dfrac{|x'|^2 + x_d^2}{1 + x_d}$.

Remark 3.3. *We notice that* (3.1) *implies that the contact sets of the solution of the one-phase problem u and the solution of the nonlinear thin-obstacle problem w are mapped one into the other:*

$$C_1(u) = \partial\Omega_u \cap B_r' = S(\{w = 0\} \cap B_r')$$

as well as the singular sets defined in (1.6) *and* (1.17)

$$\mathcal{S}_1(u) = B_r' \cap \{u = 0\} \cap \{|\nabla u| = 1\} = S(B_r' \cap \{w = 0\} \cap \{|\nabla w| = 0\}).$$

Proof of Lemma 3.2. We first notice that

$$w(x', 0) = v(x', 0) = f(x') \quad \text{for every} \quad x' \in B_r'.$$

This proves (3.4) and the first part of (3.6). Next, we notice that since

$$v(x', u(x', x_d)) = x_d \quad \text{for every} \quad (x', x_d) \in B_\rho \cap \overline{\Omega}_u,$$

we have that

$$\partial_i v_+(x', u_+(x', x_d)) + \partial_d v_+(x', u(x', x_d))\partial_i u_+(x', x_d) = 0 \quad \text{for} \quad i = 1, \ldots, d-1, \tag{3.7}$$

and

$$\partial_d v\big(x', u(x', x_d)\big)\partial_d u(x', x_d) \equiv 1. \tag{3.8}$$

Thus, we can compute

$$\Big(1 + \partial_d w(x', 0)\Big)\partial_d u(x', f(x')) \equiv 1, \tag{3.9}$$

and since $\partial_d u(x', 0) \geq 1$, we obtain also the second part of (3.6).

Next, in order to prove that the boundary condition (3.5) holds, we notice that it is equivalent to

$$\big(\partial_d v(x', 0)\big)^2 = 1 + |\nabla_{x'} f(x')|^2 \quad \text{for} \quad x' \in B_r' \cap \{f > 0\},$$

and, in view of (3.9), also to

$$\big(\partial_d u(x', f(x'))\big)^2\Big(1 + |\nabla_{x'} f(x')|^2\Big) = 1 \quad \text{for} \quad x' \in B_r' \cap \{f > 0\},$$

which is a consequence of the identity

$$\partial_i u(x', f(x')) + \partial_d u(x', f(x'))\partial_i f(x') \equiv 0 \quad \text{for every} \quad i = 1, \ldots, d-1,$$

and the boundary condition

$$(-\nabla_{x'} f(x'), 1) \cdot \nabla u(x', f(x')) = -\big(|\nabla_{x'} f(x')|^2 + 1\big)^{1/2} \quad \text{on} \quad \{f > 0\}.$$

In order to prove (3.3) we notice that, in Ω_u, u is a local minimizer of the Dirichlet integral

$$J(u) = \int |\nabla u|^2 \, dx,$$

which can be expressed in terms of w by applying (3.7) and (3.8):

$$|\nabla u|^2(x', x_d) = \frac{|\nabla_{x'} v|^2\big(x', u(x', x_d)\big) + 1}{|\partial_d v|^2\big(x', u(x', x_d)\big)} \quad \text{and} \quad \det(\nabla T)(x', x_d) = \partial_d u(x', x_d).$$

Now, by the change of coordinates $y' = x'$, $y_d = u(x', x_d)$, we get

$$\int_{B_\rho \cap \overline{\Omega}_u} |\nabla u|^2 \, dx = \int \frac{|\nabla_{y'} v|^2(y', y_d) + 1}{|\partial_d v|^2(y', y_d)} \frac{1}{|\partial_d u(x', x_d)|} \, dy = \int \frac{|\nabla_{y'} v|^2(y', y_d) + 1}{\partial_d v(y', y_d)} \, dy$$

where all the integrals in dy are over $T(B_\rho \cap \overline{\Omega}_u)$. Now, by the definition of w, we get

$$\int_{B_\rho \cap \Omega_u} |\nabla u|^2 \, dx = \int_{T(B_\rho \cap \overline{\Omega}_u)} \left(\frac{|\nabla w|^2(y', y_d)}{1 + \partial_d w(y', y_d)} + 2 \right) \, dy.$$

Thus, w minimimizes the functional

$$J(w) = \int \frac{|\nabla w|^2(y', y_d)}{1 + \partial_d w(y', y_d)} \, dy$$

in the open set $T(B_\rho \cap \Omega_u)$ with respect to perturbations of the form $w + \varepsilon\varphi$ for small ε and smooth φ. This concludes the proof of (3.3). ∎

Proof of Theorem 1.3. Theorem 1.3 follows by combining Lemma 3.2 with Theorem 1.1.

∎

4. Theorems 1.3 and 1.4: proof via conformal hodograph transform

In this section we prove Theorem 1.3 by introducing a new, conformal version, of the hodograph transform, which not only provides another proof of the fact that the one-phase branch points are isolated, but also provides the full expansion of the solution, and a way to construct examples of solutions with prescribed vanishing order (see Theorem 1.4).

4.1. The harmonic conjugate

Let u be a solution of the one-phase problem (1.12)–(1.15), let $\mathcal{S}_1(u)$ be the singular set defined in (1.17) and let $0 \in \mathcal{S}_1(u)$. Let $\mathcal{I}_\rho = (-\rho, \rho)$ and let $f : \mathcal{I}_\rho \to \mathbb{R}$ be the $C^{1,\alpha}$ function from (1.16) that describes locally the free boundary $\partial\Omega_u \cap B_\rho$; we recall that f is non-negative and $f(0) = f'(0) = 0$. Now, since the function

$$\mathcal{I}_\rho \ni x \mapsto u(x, f(x)),$$

vanishes for every $x \in \mathcal{I}_\rho$, we have that $\partial_x u(0,0) = 0$. Thus

$$\nabla u(0,0) = \partial_y u(0,0)\, e_2 \qquad \text{and} \qquad \partial_y u(0,0) \geq 1\,,$$

where $e_2 = (0,1)$. We next define the open set

$$\Omega_\rho = \Big\{(x,y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho \ : \ f(x) > y\Big\},$$

and the boundary

$$\Gamma_\rho := \Big\{(x,y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho \ : \ f(x) = y\Big\}.$$

Since Ω_ρ is simply connected, and u is harmonic in Ω_ρ, there is a function

$$U : \Omega_\rho \cup \Gamma_\rho \to \mathbb{R}$$

which solves the problem

$$U(0,0) = 0, \quad \partial_x U = \partial_y u \quad \text{and} \quad \partial_y U = -\partial_x u \quad \text{in} \quad \Omega_\rho.$$

We recall that, for any $(x,y) \in \Omega_\rho \cup \Gamma_\rho$, $U(x,y)$ is the line integral $\displaystyle\int_\sigma \alpha$ of the 1-form

$$\alpha := \partial_y u(x,y)\, dx - \partial_x u(x,y)\, dy$$

over any curve

$$\sigma : [0,1] \to \Omega_\rho \cup \Gamma_\rho$$

connecting the origin $(0,0)$ to (x,y). In particular, U is as regular as u:

$$U \in C^{1,\alpha}(\Omega_\rho \cup \Gamma_\rho).$$

If we choose σ to be the curve parametrizing the free boundary Γ_ρ,

$$\sigma : [0,x] \to \mathbb{R}^2, \quad \sigma(t) = (t, f(t)),$$

then, by integrating α over σ and using that

$$\partial_x u(t, f(t)) + f'(t)\partial_y u(t, f(t)) = 0 \qquad \text{for every} \qquad t \in \mathcal{I}_\rho \,,$$

we obtain the formula

$$
\begin{aligned}
U(x, f(x)) :&= \int_0^x \Big(\partial_y u(t, f(t)) - \partial_x u(t, f(t)) f'(t) \Big) \, dt \\
&= \int_0^x |\nabla u|(t, f(t))\sqrt{1 + f'(t)^2} \, dt = \int_\sigma |\nabla u|.
\end{aligned}
$$

In what follows, we will use the notation

$$\eta(x) := U(x, f(x)) = \int_\sigma |\nabla u|.$$

4.2. The conformal hodograph transform

With the notation from Section 4.1, we consider the change of coordinates

$$x' = U(x, y)\,, \quad y' = u(x, y)\,,$$

given by the $C^{1,\alpha}$-regular map

$$T : \Omega_\rho \cup \Gamma_\rho \to \mathbb{R}^2 \cap \{y' \geq 0\}\,, \qquad T(x, y) = (x', y')\,.$$

Now, by the definition of U and the fact that $\partial_y u(0, 0) \geq 1$, we have that the map T is invertible for ρ small enough. In particular, the set $T(\Omega_\rho \cup \Gamma_\rho)$ is an open neighborhood of $(0, 0)$ in the upper half-plane $\mathbb{R}^2 \cap \{y' \geq 0\}$. Let

$$S : T(\Omega_\rho \cup \Gamma_\rho) \to \Omega_\rho \cup \Gamma_\rho\,, \qquad S(x', y') = (x, y)\,,$$

be the inverse of T. We can write S as

$$S(x', y') = \big(V(x', y'), v(x', y') \big),$$

which in coordinates reads as

$$x = V(x', y')\,, \quad y = v(x', y')\,.$$

As in the case of the classical hodograph transform, the function v contains all the information of the free boundary Γ_ρ. Precisely, for every $x \in \mathcal{I}_\rho$, we have

$$y = f(x) \iff (x, y) \in \Gamma_\rho \iff y = v\big(U(x, y), u(x, y)\big) = v(x', 0).$$

As a consequence, we obtain the equation

$$f(x) = v(\eta(x), 0) \quad \text{for every} \quad x \in \mathcal{I}_\rho\,.$$

In particular, for $x' \in \mathbb{R}$ in a neighborhood of zero, $v(x', 0) \geq 0$ and

$$v(x', 0) > 0 \quad \iff \quad f(\eta^{-1}(x')) > 0. \tag{4.1}$$

Remark 4.1. *We notice that, in terms of the contact sets*

$$C_1(u) = \{y = 0\} \cap \partial\Omega_u \qquad and \qquad C(v) = \{y' = 0\} \cap \{v(x', 0) = 0\},$$

the map η is locally a C^1 diffeomorphism, which is sending $C_1(u)$ into $C(v)$.

Lemma 4.2 (Equations for v). *Let $T = (U, u)$ and $S = (V, v)$ be as above. Then, there is $r > 0$ such that*

$$B_r \cap \{y' \geq 0\} \subset T(\Omega_\rho \cup \Gamma_\rho),$$

and such that the function

$$v : B_r \cap \{y' \geq 0\} \to \mathbb{R},$$

is $C^{1,\alpha}$-regular in $B_r \cap \{y' \geq 0\}$ and C^∞ in $B_r \cap \{y' > 0\}$. Moreover, if we denote by C_v the contact set

$$C_v := \Big\{ (x', 0) \ : \ x' = \eta(x), \ x \in \mathcal{I}_\rho, \ f(x) = 0 \Big\}, \tag{4.2}$$

then v solves the problem

$$\Delta v = 0 \quad in \quad B_r \cap \{y' > 0\}, \tag{4.3}$$

$$v \geq 0 \quad on \quad B_r \cap \{y' = 0\}, \tag{4.4}$$

$$|\nabla v| = 1 \quad on \quad B_r \cap \{y' = 0\} \setminus C_v, \tag{4.5}$$

$$v = 0 \quad and \quad |\nabla v| \leq 1 \quad on \quad B_r \cap \{y' = 0\} \cap C_v. \tag{4.6}$$

Moreover, for every $x \in \Gamma_\rho$, we have the identities

$$f'(x) = \frac{\partial_{x'} v(\eta(x), 0)}{\partial_{y'} v(\eta(x), 0)} \qquad and \qquad \eta'(x) = \frac{1}{\partial_{y'} v(\eta(x), 0)}. \tag{4.7}$$

Proof. We start by proving that v satisfies the equations (4.3)–(4.6). First notice that v is harmonic since it is the second component of a conformal map. Moreover, since

$$v\big(U(x, y), u(x, y)\big) = y \quad \text{for every} \quad (x, y) \in \Omega_\rho,$$

taking the derivatives with respect to x and y, we obtain that

$$\partial_{x'} v\big(U(x, y), u(x, y)\big)\partial_x U(x, y) + \partial_{y'} v\big(U(x, y), u(x, y)\big)\partial_x u(x, y) = 0,$$
$$\partial_{x'} v\big(U(x, y), u(x, y)\big)\partial_y U(x, y) + \partial_{y'} v\big(U(x, y), u(x, y)\big)\partial_y u(x, y) = 1.$$

By exploiting that $\partial_x U = \partial_y u$ and $\partial_y U = -\partial_x u$, we get

$$\partial_{x'} v(x', y') \, \partial_y u(x, y) + \partial_{y'} v(x', y') \, \partial_x u(x, y) = 0, \tag{4.8}$$

$$-\partial_{x'} v(x', y') \, \partial_x u(x, y) + \partial_{y'} v(x', y') \, \partial_y u(x, y) = 1. \tag{4.9}$$

Solving the system (4.8)-(4.9) leads to

$$\partial_{y'} v(x', y') = \frac{\partial_y u(x, y)}{|\nabla u|^2(x, y)} \qquad and \qquad \partial_{x'} v(x', y') = -\frac{\partial_x u(x, y)}{|\nabla u|^2(x, y)}. \tag{4.10}$$

Thus, we obtain

$$|\nabla u|(x, y)\, |\nabla v|(x', y') = 1\,, \tag{4.11}$$

which gives both (4.6) and (4.5). We next prove (4.7). Using that $u(x, f(x)) \equiv 0$, we get

$$f'(x) = -\frac{\partial_x u(x, f(x))}{\partial_y u(x, f(x))}\,,$$

which together with (4.10) gives the first part of (4.7). For the second part, we notice that the identity $v(\eta(x), 0) = f(x)$ gives that

$$f'(x) = \eta'(x)\partial_{x'} v(\eta(x), 0)\,,$$

which, combined with the first identity in (4.7), concludes the proof. ∎

4.3. Proof of Theorem 1.3

Let v be as in the previous section and let

$$Q := \partial_{z'} v = \partial_{x'} v - i\partial_{y'} v,$$

where $z' = x' + iy'$. Since v satisfies (4.3)-(4.6), we get that

$$\begin{cases} \partial_{\bar{z}'} Q = 0 & \text{in} \quad B_r \cap \{y' > 0\}, \\ |Q| = 1 & \text{on} \quad B_r \cap \{y' = 0\} \setminus C_v, \\ \text{Re}\, Q = 0 & \text{on} \quad B_r \cap \{y' = 0\} \cap C_v, \end{cases}$$

where the set C_v was defined in (4.2). Consider now the function

$$P = -i\frac{Q+i}{Q-i} = -i\frac{(Q+i)(\bar{Q}+i)}{|Q-i|^2} = \frac{2\,\text{Re}\, Q}{|Q-i|^2} - i\frac{|Q|^2-1}{|Q-i|^2}\,.$$

Then, we have that $P(0) = 0$ and

$$\begin{cases} \partial_{\bar{z}'} P = 0 & \text{in} \quad B_r \cap \{y' > 0\}, \\ \text{Re}\, P = 0 & \text{on} \quad B_r \cap \{y' = 0\} \cap C_v, \\ \text{Im}\, P = 0 & \text{on} \quad B_r \cap \{y' = 0\} \setminus C_v, \end{cases}$$

which implies that $P^2(0) = 0$ and

$$\begin{cases} \partial_{\bar{z}'}(P^2) = 0 & \text{in} \quad B_r \cap \{y' > 0\}, \\ \text{Im}\,(P^2) = 0 & \text{on} \quad B_r \cap \{y' = 0\}. \end{cases}$$

As a consequence, the zero set

$$\mathcal{Z}(P) = \left\{ z' \in B_r \ : \ P(z') = 0,\ \text{Im}\, z' = 0 \right\},$$

is discrete or coincides with $B_r \cap \{y' = 0\}$. Now, Theorem 1.3 (a) follows since

$$P(z') = 0 \qquad \Leftrightarrow \qquad \begin{cases} \partial_x u(x, y) = 0, \\ \partial_y u(x, y) = 1, \end{cases}$$

that is, every branch point $(x, y) \in \mathcal{S}_1(u)$ corresponds to a zero z' of P.

We next prove Theorem 1.3 (b). Let $z_0 = 0$ be an isolated point of $\mathcal{S}_1(u)$ and $z'_0 = 0$ be the corresponding point in $\mathcal{Z}(P)$. Since zero is an isolated point of $\mathcal{Z}(P)$ and since

$$\operatorname{Re} P(z') \cdot \operatorname{Im} P(z') = 0 \quad \text{on} \quad \{\operatorname{Im} z' = 0\},$$

we have the following three possibilities in a neighborhood of zero:

(1) $\operatorname{Re} P(z') \equiv 0$ on $\{y' = 0\}$, and $\operatorname{Im} P(z') \neq 0$ on $\{y' = 0\} \setminus \{x' = 0\}$;

(2) $\operatorname{Im} P(z') \equiv 0$ on $\{y' = 0\}$, and $\operatorname{Re} P(z') \neq 0$ on $\{y' = 0\} \setminus \{x' = 0\}$;

(3) up to changing the direction of the real axis $\{y' = 0\}$ we have

$$\begin{cases} \operatorname{Re} P(z') \equiv 0 \quad \text{and} \quad \operatorname{Im} P(z') \neq 0 \quad \text{on} \quad \{y' = 0\} \cap \{x' > 0\}; \\ \operatorname{Re} P(z') \neq 0 \quad \text{and} \quad \operatorname{Im} P(z') \equiv 0 \quad \text{on} \quad \{y' = 0\} \cap \{x' < 0\}. \end{cases}$$

We will show that each of these cases corresponds to one of the points (b.1), (b.2) and (b.3) of Theorem 1.3. We first suppose that (3) holds. Then P solves the problem

$$\begin{cases} \partial_{\bar{z}'} P = 0 \quad \text{in} \quad B_r \cap \{y' > 0\}, \\ \operatorname{Re} P = 0 \quad \text{on} \quad B'_r \cap \{x' \geq 0\}, \\ \operatorname{Im} P = 0 \quad \text{on} \quad B'_r \cap \{x' < 0\}. \end{cases}$$

We next notice that

$$\partial_{x'} v - i \partial_{y'} v = Q = \frac{1 + iP}{P + i} = \frac{2\operatorname{Re}(P)}{|P + i|^2} - i \frac{1 - |P|^2}{|P + i|^2}.$$

so that

$$\partial_{x'} v = \frac{2\operatorname{Re}(P)}{|P + i|^2} \qquad \text{and} \qquad \partial_{y'} v = \frac{1 - |P|^2}{|P + i|^2}.$$

In particular, since the function η is increasing and $\eta(0) = 0$, we get

$$\partial_{x'} v(\eta(x), 0) \equiv 0 \quad \text{for} \quad x \geq 0.$$

Integrating this identity and taking into account that $v(\eta(0), 0) = v(0, 0) = 0$, we obtain

$$f(x) = v(\eta(x), 0) = \int_0^x \partial_{x'} v(\eta(t), 0)\, \eta'(t)\, dt = 0 \quad \text{for} \quad x \geq 0.$$

Conversely, assume that $x < 0$ and let $x' = \eta(x) < 0$. Then, $\operatorname{Im}(P(x')) = 0$ and

$$\partial_{x'} v(x', 0) = \frac{2P(x')}{1 + P^2(x')} \qquad \text{and} \qquad \partial_{y'} v(x', 0) = \frac{1 - P^2(x')}{1 + P^2(x')} \qquad \text{for} \qquad z' = x' < 0.$$

In particular, from (4.7) it follows that

$$
\begin{cases}
\eta'(x) = \dfrac{1 + P^2(\eta(x))}{1 - P^2(\eta(x))} & \text{if } x < 0 \\
\eta(0) = 0,
\end{cases}
$$

which implies, by Cauchy-Kovalevskaya theorem, that $\eta \colon (-\rho, 0] \to \mathbb{R}$ is an analytic function, with $\eta'(0) = 1$, since $P(0) = 0$. Since for $x < 0$ we have

$$
\eta'(x) = \sqrt{1 + f'(x)^2} \qquad \Rightarrow \qquad f'(x) = \sqrt{\eta'(x)^2 - 1}, \tag{4.12}
$$

we get that $f' \colon (-\rho, 0] \to \mathbb{R}$ is of the form

$$
f'(x) = x^{k/2} \psi(x),
$$

for some $k \geq 1$ and some analytic function $\psi \colon (-\rho, 0] \to \mathbb{R}$ with $\psi(0) > 0$. It follows that there is an analytic function ϕ, such that $\phi(0) > 0$ and

$$
f(x) = 0 \quad \text{if} \quad x \geq 0 \qquad \text{and} \qquad f(x) = x^{\frac{k+2}{2}} \phi(x) \quad \text{if} \quad x < 0.
$$

Suppose now that (2) holds. Then $\operatorname{Im} P \equiv 0$ on the real axis $\{y' = 0\}$ and so, P (not only P^2) is an holomorphic function. As a consequence, also Q is holomorphic. Thus, $\partial_{y'} v(x', 0)$ is analytic. Since, $\eta \colon (-\rho, \rho) \to \mathbb{R}$ solves the equation

$$
\eta'(x) = \frac{1}{\partial_{y'} v(\eta(x), 0)}, \qquad \eta(0) = 0,
$$

we get that η is analytic and, by (4.12), so is f. This gives (b.2).

Finally, we suppose that (1) holds. Since $\operatorname{Im} P \neq 0$ on $\{y' = 0\} \setminus \{0\}$, we get that the contact set C_v contains a neighborhood of zero. As a consequence also the contact set $C_1(u)$ contains a neighborhood of zero (see Remark 4.1), from which we obtain (b.1). ∎

4.4. Proof of Theorem 1.4

Finally we come to the proof of Theorem 1.4, which is obtained by reversing the construction from the previous subsection.

Proof of Theorem 1.4. For any k of the form $k = 2n - \frac{3}{2}$ with $n \in \mathbb{N}_{\geq 1}$, we define

$$
P(z) = (iz)^k = \rho^k \Big(-\sin(k\theta) + i\cos(k\theta) \Big).
$$

In particular, setting $C_P := \{(x, 0) \in \mathbb{R}^2 \,:\, x \geq 0\}$ we have

$$
\begin{cases}
\partial_{\bar{z}} P = 0 & \text{in} \quad \{y > 0\}, \\
\operatorname{Re} P = 0 \quad \text{and} \quad \operatorname{Im} P > 0 & \text{on} \quad \{x > 0\} \\
\operatorname{Re} P < 0 \quad \text{and} \quad \operatorname{Im} P = 0 & \text{on} \quad \{x < 0\}.
\end{cases}
$$

Then we consider a radius $r \in (0, 1)$ and the function $Q \colon B_r \cap \{y \geq 0\} \to \mathbb{C}$

$$Q = \frac{1 + iP}{P + i} = \frac{2\,\mathrm{Re}(P)}{|P + i|^2} - i\frac{1 - |P|^2}{|P + i|^2}\,.$$

Notice that Q is still conformal in $B_r \cap \{y > 0\}$ and that we have

$$\begin{cases} \partial_{\bar{z}} Q = 0 \quad \text{in} \quad \{y > 0\}, \\ \mathrm{Re}\,Q = 0\,, \quad \mathrm{Im}\,Q \in (-1, 0) \quad \text{and} \quad |Q| < 1 \quad \text{on} \quad \{x > 0\}\,, \\ \mathrm{Re}\,Q < 0, \quad \mathrm{Im}\,Q \in (-1, 0) \quad \text{and} \quad |Q| = 1 \quad \text{on} \quad \{x < 0\}\,. \end{cases}$$

Since $B_r \cap \{y > 0\}$ is simply connected, there is a function $v \colon B_r \cap \{y \geq 0\} \to \mathbb{R}$ such that

$$\partial_z v = \partial_x v - i\partial_y v = Q \quad \text{in} \quad B_r \cap \{y > 0\}\,.$$

Precisely, for every $z = x + iy$ in $B_r \cap \{y \geq 0\}$, v is given by the formula

$$v(z) = v(x, y) = \int_0^1 \Big(x\,\mathrm{Re}\,Q(tz) - y\,\mathrm{Im}\,Q(tz) \Big)\,dt.$$

Thus, v is a solution to the problem

$$\begin{cases} \Delta v = 0 \quad \text{in} \quad B_r \cap \{y > 0\}, \\ v = 0 \quad \text{and} \quad |\nabla v| < 1 \quad \text{on} \quad B_r \cap \{x > 0\}\,, \\ v > 0 \quad \text{and} \quad |\nabla v| = 1 \quad \text{on} \quad B_r \cap \{x < 0\}\,. \end{cases}$$

Moreover, we notice that $v(0, 0) = 0$ and $\partial_y v(0, 0) = 1$. Thus, by choosing $r > 0$ small enough, we may suppose that $v > 0$ in $B_r \cap \{y > 0\}$. We next consider the harmonic conjugate $V \colon B_r \cap \{y > 0\} \to \mathbb{R}$ of v and the inverse hodograph transform

$$S \colon B_r \cap \{y \geq 0\} \to \mathbb{R}^2\,, \quad S(x, y) := \big(V(x, y), v(x, y)\big)\,.$$

Tracing backwards the argument from Section 4.2, we have that when r is small enough, S is a diffeomorphism; we can then consider its inverse

$$T \colon S\big(B_r \cap \{y \geq 0\}\big) \to B_r \cap \{y \geq 0\}\,, \qquad T(x', y') = \big(U(x', y'), u(x', y')\big),$$

where we notice that the positivity set $\Omega_u = \{u > 0\}$ of the second component u of T is precisely $S\big(B_r \cap \{y > 0\}\big)$ and that, since $v \geq 0$, $\Omega_u = S\big(B_r \cap \{y > 0\}\big)$ is contained in the upper half-plane $\{y' > 0\}$. Now, reasoning as in Lemma 4.2 (see (4.11)), we get that

$$|\nabla u(x', y')|\,|\nabla v(x, y)| = 1,$$

and that, in a small ball B_ρ, u is a solution to the problem

$$\Delta u = 0 \quad \text{in} \quad \Omega_u \cap B_\rho, \tag{4.13}$$

$$u = 0 \quad \text{on} \quad B_\rho \cap \{y' = 0\}, \tag{4.14}$$

$$|\nabla u| = 1 \quad \text{on} \quad \partial\Omega_u \cap \{y' > 0\}, \tag{4.15}$$

$$|\nabla u| \geq 1 \quad \text{on} \quad \partial\Omega_u \cap \{y' = 0\}, \tag{4.16}$$

where $\partial\Omega_u \cap \{y' = 0\} = \{x' \geq 0\} \cap \{y' = 0\}$ and $|\nabla u| \geq 1$ on $\{x' \geq 0\} \cap \{y' = 0\}$. We now define the function f describing the boundary $\partial\Omega_u$ (see (1.16)) and the function $\eta(x) = U(x, f(x))$ to be as in the proof of Theorem 1.3. Then, η is a solution to

$$\begin{cases} \eta'(x) = \dfrac{1 + P^2(\eta(x))}{1 - P^2(\eta(x))} & \text{if } x < 0 \\ \eta(0) = 0 \, , \end{cases}$$

and so, it is analytic since $P^2(z) = iz^{4n-3}$ with $n \in \mathbb{N}$. Finally, since $\eta(x) = x + o(x)$, we can write the function η as

$$|\eta(x)|^{1/2} = |x|^{1/2}\psi(x) \quad \text{for } x \leq 0,$$

where ψ is analytic and $\psi(0) = 1$. Thus, we get the precise form of f by the formula

$$f(x) = v(\eta(x), 0) = \begin{cases} \displaystyle\int_0^x \dfrac{-|\eta(t)|^{2n-1/2}}{|\eta(t)|^{4n-3} + 1} dt & \text{if } x < 0, \\ 0 & \text{if } x \geq 0 \, , \end{cases}$$

and we notice that $f(x) = |x|^{2n-1/2}(1 + o(1))$ for $x < 0$. This concludes the proof. ∎

5. The symmetric two-phase problem and some remarks

Let $0 = z_0 \in \mathcal{S}$ and let $f_\pm$ be as in (1.25). We define

$$\Omega_\rho^\pm = \Big\{(x, y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho \ : \ f_\pm(x) > y\Big\},$$

and

$$\Gamma_\rho^\pm := \Big\{(x, y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho \ : \ f_\pm(x) = y\Big\}.$$

In what follows, we perform the hodograph transform of u_+ in Ω_ρ^+ and in u_- in Ω_ρ^-. In order to simplify the notation, we set

$$i := + \text{ or } - \, .$$

Let $\eta_\pm, T_\pm = (U_\pm, u_\pm), S_\pm = (V_\pm, v_\pm)$ be the functions constructed in Section 4.1 and Section 4.2 separately for u_+ and u_-. Recall that the functions v_i, $i = \pm$, contain all the information of the free boundaries Γ_ρ^i. Precisely, for every $x \in \mathcal{I}_\rho$, we have

$$y = f_i(x) \iff (x, y) \in \Gamma_\rho^i \iff y = v_i\big(U_i(x, y), u_i(x, y)\big) = v_i(x', 0).$$

As a consequence, we get the equation

$$f_i(x) = v_i(\eta_i(x), 0) \quad \text{for every} \quad x \in \mathcal{I}_\rho.$$

In particular, we have

$$v_+(\eta_+(x), 0) \geq v_-(\eta_-(x), 0) \quad \text{for every} \quad x \in \mathcal{I}_\rho. \tag{5.1}$$

Lemma 5.1. *There is $r > 0$ such that*

$$B_r \cap \{y' \geq 0\} \subset T_+\big(\Omega_\rho^+ \cup \Gamma_\rho^+\big) \qquad \text{and} \qquad B_r \cap \{y' \leq 0\} \subset T_-\big(\Omega_\rho^- \cup \Gamma_\rho^-\big).$$

The functions

$$v_\pm : B_r \cap \{\pm y' \geq 0\} \to \mathbb{R},$$

are both $C^{1,\alpha}$-regular respectively in the half-disks $B_r \cap \{\pm y' \geq 0\}$ up to the hyperplane $\{y' = 0\}$, and are C^∞ respectively in $B_r \cap \{\pm y' > 0\}$. Furthermore they solve the following thin two-membrane problem

$$
\begin{aligned}
\Delta v_+ = 0 \quad &\text{in} \quad B_r \cap \{y' > 0\}, \\
\Delta v_- = 0 \quad &\text{in} \quad B_r \cap \{y' < 0\}, \\
v_+\big(\eta_+(x), 0\big) \geq v_-\big(\eta_-(x), 0\big) \quad &\text{for} \quad x \in \mathcal{I}_\rho, \\
|\nabla v_\pm|(\eta_\pm(x), 0) = 1 \quad &\text{when} \quad v_+(\eta_+(x), 0) > v_-(\eta_-(x), 0), \\
\eta_+'(x)\, \partial_{y'} v_+(\eta_+(x), 0) = \eta_-'(x)\, \partial_{y'} v_-(\eta_-(x), 0) \leq 1 \quad &\text{when} \quad v_+(\eta_+(x), 0) = v_-(\eta_-(x), 0),
\end{aligned}
$$

Moreover, for every $x \in \Gamma_\rho$ we have the identities

$$f_\pm'(x) = \pm \frac{\partial_{x'} v_\pm(\eta_\pm(x), 0)}{\partial_{y'} v_\pm(\eta_\pm(x), 0)} \qquad \text{and} \qquad \eta_\pm'(x) = \frac{1}{\partial_{y'} v_\pm(\eta_\pm(x), 0)}. \tag{5.2}$$

Proof. We reason precisely as in [Lemma 4.2](#). Since

$$v_i\big(U_i(x, y), u_i(x, y)\big) = y \quad \text{for every} \quad (x, y) \in \Omega_\rho^i,$$

taking the derivatives with respect to x and y, we obtain that

$$
\begin{cases}
\partial_{x'} v_i\big(U_i(x, y), u_i(x, y)\big)\partial_x U_i(x, y) + \partial_{y'} v_i\big(U_i(x, y), u_i(x, y)\big)\partial_x u_i(x, y) = 0, \\
\partial_{x'} v_i\big(U_i(x, y), u_i(x, y)\big)\partial_y U_i(x, y) + \partial_{y'} v_i\big(U_i(x, y), u_i(x, y)\big)\partial_y u_i(x, y) = 1.
\end{cases}
$$

Since, $\partial_x U_i = \partial_y u_i$ and $\partial_y U_i = -\partial_x u_i$, we get

$$
\begin{cases}
-\partial_{x'} v_i(x', y')\partial_y u_i(x, y) + \partial_{y'} v_i(x', y')\partial_x u_i(x, y) = 0, \\
\partial_{x'} v_i(x', y')\partial_x u_i(x, y) + \partial_{y'} v_i(x', y')\partial_y u_i(x, y) = 1.
\end{cases}
$$

When $y' = 0$, we can write

$$x' = \eta_i(x) \qquad \text{and} \qquad y = f_i(x).$$

Thus, we have

$$
\begin{cases}
-\partial_{x'} v_i(\eta_i(x), 0)\partial_y u_i(x, f_i(x)) + \partial_{y'} v_i(\eta_i(x), 0)\partial_x u_i(x, f_i(x)) = 0, \\
\partial_{x'} v_i(\eta_i(x), 0)\partial_x u_i(x, f_i(x)) + \partial_{y'} v_i(\eta_i(x), 0)\partial_y u_i(x, f_i(x)) = 1,
\end{cases}
$$

which we will simply write as

$$
\begin{cases}
-\partial_{x'} v_i\, \partial_y u_i + \partial_{y'} v_i\, \partial_x u_i = 0, \\
\partial_{x'} v_i\, \partial_x u_i + \partial_{y'} v_i\, \partial_y u_i = 1,
\end{cases}
\tag{5.3}
$$

and we remember that all the derivatives of v are computed in $(\eta_i(x), 0)$, while all the derivatives of u are calculated in $(x, f_i(x))$. We next consider two cases:

Case 1. $v_+(\eta_+(x), 0) = v_-(\eta_-(x), 0)$. We set

$$f(x) := f_+(x) = f_-(x) \qquad \text{and} \qquad f'(x) := f'_+(x) = f'_-(x),$$

and we notice that we have the system

$$\partial_x u_+ + f'(x)\partial_y u_+ = 0 = \partial_x u_- + f'(x)\partial_y u_- \tag{5.4}$$

$$-f'(x)\partial_x u_+ + \partial_y u_+ = -f'(x)\partial_x u_- + \partial_y u_- \tag{5.5}$$

$$-f'(x)\partial_x u_\pm + \partial_y u_\pm \geq \left(1 + (f'(x))^2\right)^{1/2}. \tag{5.6}$$

where again all the partial derivatives of u_+ and u_- are computed in $(x, f(x))$. Now, using (5.4) in (5.5) and (5.6), we get

$$\partial_y u_+ = \partial_y u_- \tag{5.7}$$

$$\sqrt{1 + (f'(x))^2}\,\partial_y u_\pm \geq 1. \tag{5.8}$$

On the other hand, using (5.4) in the system (5.3), it becomes

$$\begin{cases} \left(\partial_{x'} v_i + \partial_{y'} v_i\, f'(x)\right)\partial_y u_i = 0, \\ \left(-f'(x)\,\partial_{x'} v_i + \partial_{y'} v_i\right)\partial_y u_i = 1, \end{cases} \tag{5.9}$$

so we get

$$\left(1 + f'(x)^2\right)\partial_{y'} v_\pm\, \partial_y u_\pm = 1,$$

which gives that

$$\partial_{y'} v_+ = \partial_{y'} v_-\,, \qquad \partial_{x'} v_+ = \partial_{x'} v_- \qquad \text{and} \qquad \sqrt{1 + (f'(x))^2}\,\partial_{y'} v_\pm \leq 1\,,$$

all the derivatives of $v_\pm$ being calculated in $(\eta_\pm(x), 0)$.

Case 2. $v_+(\eta_+(x), 0) > v_-(\eta_-(x), 0)$. In this case the two free boundaries separate, that is $f_+ > f_-$ in a neighborhood of x. Then, for each $i = \pm$, we can proceed as in the proof of (4.5) in Lemma 4.2.

Finally, we notice that (5.2) follows by taking the reflection $\bar{u}(x, y) := -u_-(x, -y)$ and applying the identities from (4.7) to u_+ and $\bar{u}$. ∎

When u is a symmetric solution to the two-phase problem, we have the following

Corollary 5.2. *Let u be a symmetric solution to the two-phase problem, then, up to taking a smaller radius $r > 0$, the functions $v_\pm$ constructed in Lemma 5.1 satisfy*

$$\Delta v_+ = 0 \quad in \quad B_r \cap \{y' > 0\},$$

$$\Delta v_- = 0 \quad in \quad B_r \cap \{y' < 0\},$$

$$|\nabla v_\pm|(x', 0) = 1 \quad when \quad x' \in B'_r \setminus C_v$$

$$|\nabla v_+|(x', 0) = |\nabla v_-|(x', 0) \leq 1 \quad when \quad B'_r \cap C_v,$$

where we denote by C_v the contact set

$$C_v := \left\{ (x', 0) \ : \ x' = \eta(x), \ x \in \mathcal{I}_\rho, \ f_+(x) = f_-(x) \right\}. \tag{5.10}$$

Proof. By definition

$$\eta_\pm(x) = \int_0^x |\nabla u_\pm|(t, f_\pm(t)) \sqrt{1 + |f'_\pm(t)|^2} \, dt \,.$$

Let $\mathcal{I}_i$ be the intervals defined in (1.26), then notice that

- if $t \in \mathcal{I}_i$, then $|\nabla u_\pm|(t, f_\pm(t)) = 1$;

- if $t \in (-\rho, \rho) \setminus (\bigcup_i \mathcal{I}_i)$, then $f_+(t) = f_-(t)$ and $|\nabla u_+|(t, f(t)) = |\nabla u_-|(t, f(t))$.

In particular the first bullet implies that

$$\eta_+(\mathcal{I}_i) = \eta_-(\mathcal{I}_i) \qquad \forall i \,,$$

which combined with the second bullet implies that

$$\eta_+\left(\left\{ x \in (-\rho, \rho) \ : \ f_+(x) > f_-(x) \right\}\right) = \eta_-\left(\left\{ x \in (-\rho, \rho) \ : \ f_+(x) > f_-(x) \right\}\right),$$

from which the conclusion follows from the previous lemma. ∎

Remark 5.3. *Notice that, in the above proof, we are not claiming that $\eta_+ \equiv \eta_-$, but only that branch points are sent in branch points.*

5.1. Proof of Theorem 1.6 (a)

Let $v_\pm$ be the functions from Corollary 5.2 and let

$$Q_\pm := \partial_{x'} v_\pm - i \partial_{y'} v_\pm \tag{5.11}$$

As in the proof of Theorem 1.3, we have that Q is a solution to

$$\begin{cases} \partial_{\bar{z}} Q_\pm = 0 & \text{in} \quad B_r \cap \{\pm y' > 0\} \,, \\ |Q_\pm| = 1 & \text{on} \quad B_r \cap \{y' = 0\} \setminus C_v \,, \\ Q_+ = Q_- & \text{on} \quad B_r \cap \{y' = 0\} \cap C_v \,. \end{cases} \tag{5.12}$$

We then define

$$P_\pm = -i \frac{Q_\pm + i}{Q_\pm - i} = -i \frac{(Q_\pm + i)(\bar{Q}_\pm + i)}{|Q_\pm - i|^2} = \frac{2 \operatorname{Re} Q_\pm}{|Q_\pm - i|^2} - i \frac{|Q_\pm|^2 - 1}{|Q_\pm + i|^2} \,, \tag{5.13}$$

and we notice that

$$\begin{cases} \partial_{\bar{z}} P_\pm = 0 & \text{in} \quad B_r \cap \{\pm y' > 0\} \,, \\ P_+ = P_- & \text{on} \quad B_r \cap \{y' = 0\} \cap C_v \,, \\ \operatorname{Im} P_\pm = 0 & \text{on} \quad B_r \cap \{y' = 0\} \setminus C_v \,. \end{cases}$$

We now consider the reflection

$$P' : B_r \cap \{y' \geq 0\} \to \mathbb{C}, \qquad P'(z) := \overline{P}_-(\bar{z}),$$

so that the functions P_+ and P' are both defined on the same domain and we can take

$$M(z) := \frac{P_+(z) + P'(z)}{2} \qquad \text{and} \qquad D(z) := \frac{P_+(z) - P'(z)}{2}, \qquad (5.14)$$

which satisfy the equations

$$\begin{cases} \partial_{\bar{z}} M = 0 & \text{in} \quad B_r \cap \{y' > 0\}, \\ \operatorname{Im} M = 0 & \text{on} \quad B_r \cap \{y' = 0\}. \end{cases} \qquad (5.15)$$

and

$$\begin{cases} \partial_{\bar{z}} D = 0 & \text{in} \quad B_r \cap \{y' > 0\}, \\ \operatorname{Re} D = 0 & \text{on} \quad B_r \cap \{y' = 0\} \cap C_v, \\ \operatorname{Im} D = 0 & \text{on} \quad B_r \cap \{y' = 0\} \setminus C_v, \end{cases}$$

Reasoning as in the proof of Theorem 1.3, D^2 we get that $\operatorname{Im}(D^2) = 2\operatorname{Re} D \operatorname{Im} D = 0$ on $\{y' = 0\}$ so that D^2 can be extended to a conformal map on to the whole of B_r, so the set

$$\{D = 0\} \cap B_r \cap \{y' = 0\},$$

is either discrete or coincides with $B_r \cap \{y' = 0\}$. This proves Theorem 1.6 (a) since at every z' on the real line $\{y' = 0\}$ we have

$$D(z') = 0 \quad \Leftrightarrow \quad \begin{cases} P^+ = P^- \\ \operatorname{Im} P_\pm = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} Q^+ = Q^- \\ |Q_\pm| = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \nabla u_+ = \nabla u_- \\ |\nabla u_\pm| = 1, \end{cases}$$

that is every branch point of u corresponds to a zero of D. ∎

5.2. Proof of Theorem 1.6 (b) and Corollary 5.2

Remark 5.4. *We notice that in this part of Theorem 1.6 we do not assume any symmetry of the solutions, but only that the branch points are isolated.*

Let $z_0 \in \mathcal{S}_2(u_+, u_-)$ be an isolated point of $\mathcal{S}_2(u_+, u_-)$. If z_0 is in the interior of the contact set $C_2(u_+, u_-)$, then (b.2) is immediate as the function $u = u_+ - u_-$ is harmonic in a neighborhood of z_0. Suppose then that z_0 is a branch point: $z_0 \in \mathcal{B}_2(u_+, u_-)$; moreover, since $\mathcal{B}_2 \subset \mathcal{S}_2$, we have that z_0 is isolated in the set of branch points $\mathcal{B}_2(u_+, u_-)$. This means that in order to complete the proof of Theorem 1.6 (b) we only need to prove Corollary 5.2. We set $z_0 = 0$ and we consider the following two cases:

Case 1. 0 is isolated also as point of the contact set $C_2(u_+, u_-)$, that is $B_r \cap C_2(u_+, u_-) = \{0\}$ for some radius $r > 0$. In this case, on the free boundaries $\partial \Omega_u^\pm$ we have that $|\nabla u_\pm| = 1$ and so, Corollary 5.2 (b.1) follows as in the proof of Theorem 1.3 (b.1).

Case 2. 0 is not isolated in the set $C_2(u_+, u_-)$. Then, since there are no other branch points in a neighborhood of 0, we can assume that:

$$f_+(x) = f_-(x) \quad \text{when} \quad x \geq 0 \qquad \text{and} \qquad f_+(x) > f_-(x) \quad \text{when} \quad x < 0.$$

As above, we define $\eta_\pm$ as

$$\eta_\pm(x) = \int_0^x |\nabla u_\pm|(t, f_\pm(t))| \sqrt{1 + (f_\pm'(t))^2} \, dt, \tag{5.16}$$

while $v_\pm$ are the hodograph transforms of $u_\pm$, for which we recall the identities

$$f_\pm(x) = v_\pm(\eta_\pm(x), 0) \quad \text{and} \quad |\nabla v_\pm|(\eta_\pm(x), 0) = \frac{1}{|\nabla u|(x, f_\pm(x))}.$$

for every x in a neighborhood of zero. Then, since $\eta_+(x) = \eta_-(x)$ for $x \geq 0$, we get that:

$$\begin{cases} v_+(x', 0) = v_-(x', 0) \quad \text{and} \quad \nabla v_+(x', 0) = \nabla v_-(x', 0) \qquad \text{when} \quad x' \geq 0, \\ |\nabla v_+|(x', 0) = |\nabla v_-|(x', 0) \qquad\qquad\qquad\qquad\qquad \text{when} \quad x' < 0. \end{cases}$$

Remark 5.5. *Notice that when $x < 0$ we cannot say if $\eta_+(x) = \eta_-(x)$. In particular, we cannot say if $v_+(x', 0) \geq v_-(x', 0)$ when $x' < 0$ and so, we don't know if $\{x' \geq 0\}$ is the contact set $\{x' \, : \, v_+(x', 0) = v_-(x', 0)\}$.*

We next consider the functions $Q_\pm$ and $P_\pm$ given by (5.11) and (5.13), and the functions D and M defined in (5.14). Then, in a neighborhood $(-r, r) \times [0, r)$ of zero, the difference D satisfies

$$\begin{cases} \partial_{\bar{z}} D = 0 \quad \text{in} \quad (-r, r) \times (0, r), \\ \text{Re} D = 0 \quad \text{on} \quad (0, r) \times \{0\} \\ \text{Im} D = 0 \quad \text{on} \quad (-r, 0) \times \{0\}. \end{cases} \tag{5.17}$$

Recall that by the definitions of M, D and P', we have

$$P_+(z) = M(z) + D(z) \qquad \text{and} \qquad P_-(z) = \overline{M(\bar{z})} - \overline{D(\bar{z})}$$

and moreover

$$\partial_{x'} v_\pm = \text{Re}(Q_\pm) = \frac{2 \, \text{Re}(P_\pm)}{|P_\pm + i|^2} \qquad \text{and} \qquad \partial_{y'} v_\pm = -\text{Im}(Q_\pm) = \frac{1 - |P_\pm|^2}{|P_\pm + i|^2}.$$

We set $g_\pm(x') := \eta_\pm^{-1}(x')$ and $\tilde{f}_\pm(x') := f_\pm(g_\pm(x'))$. Since,

$$f_\pm(x) = v_\pm(\eta_\pm(x), 0) \qquad \text{and} \qquad \eta_\pm'(x) = \frac{1}{\partial_{y'} v_\pm(\eta_\pm(x), 0)},$$

we get that

$$\tilde{f}_\pm(x') = v_\pm(x', 0) \qquad \text{and} \qquad g_\pm'(x') = \partial_{y'} v_\pm(x', 0).$$

In particular,

$$\tilde{f}_\pm(x') = \int_0^{x'} \partial_{x'} v_\pm(t, 0) \, dt = \int_0^{x'} \frac{2 \, \text{Re}(P_\pm(t))}{|P_\pm(t) + i|^2} \, dt$$

and

$$g_\pm(x') = \int_0^{x'} \partial_{y'} v_\pm(t,0)\, dt = \int_0^{x'} \frac{1 - |P_\pm(t)|^2}{|P_\pm(t) + i|^2}\, dt\,.$$

Now, by (5.17) and (5.15), we have that

$$M = \operatorname{Re} M \quad \text{and} \quad D = i \operatorname{Im} D \quad \text{on} \quad [0,r) \times \{0\}\,,$$

which gives that on $[0,r) \times \{0\}$, $P_+ = P_-$, precisely:

$$\operatorname{Re}(P_+) = \operatorname{Re}(P_-) = M \quad \text{and} \quad \operatorname{Im}(P_+) = \operatorname{Im}(P_-) = \operatorname{Im} D = -iD\,.$$

This implies that

$$\tilde{f}_\pm(x') = \int_0^{x'} \frac{2\,M(t)}{M^2(t) + \big(1 + \operatorname{Im} D(t)\big)^2}\, dt\,,$$

so that $\tilde{f}_+ \equiv \tilde{f}_-$ on $\{x' \geq 0\}$. Similarly,

$$g_\pm(x') = \int_0^{x'} \frac{1 - M^2(t) - \big(\operatorname{Im} D(t)\big)^2}{M^2(t) + \big(1 + \operatorname{Im} D(t)\big)^2}\, dt\,,$$

which again implies that $g_+ \equiv g_-$. Combining these two identities, we get that

$$f_+ \equiv f_- \quad \text{on} \quad \{x' \geq 0\}\,.$$

Using again (5.17) and (5.15), this time for $x' \leq 0$, we get that

$$M = \operatorname{Re} M \quad \text{and} \quad D = \operatorname{Re} D \quad \text{on} \quad (-r,0) \times \{0\}\,,$$

which implies that $P_\pm$ are both real and

$$P_+ = M + D \quad \text{and} \quad P_- = M - D \quad \text{on} \quad (-r,0) \times \{0\}\,.$$

As above, we compute

$$\tilde{f}_\pm(x') = 2 \int_0^{x'} \frac{M(t) \pm D(t)}{1 + (M(t) \pm D(t))^2}\, dt$$

and

$$g_\pm(x') = \int_0^{x'} \frac{1 - (M(t) \pm D(t))^2}{1 + (M(t) \pm D(t))^2}\, dt\,.$$

We now define

$$\Psi(x') := \frac{\tilde{f}_+(x') - \tilde{f}_-(x')}{2} = 2 \int_0^{x'} D(t) \frac{1 + D^2 - M^2}{(1 + M^2 + D^2)^2 - 4D^2 M^2}\, dt$$

and

$$\Phi(x') := \frac{\tilde{f}_+(x') + \tilde{f}_-(x')}{2} = 2 \int_0^{x'} M(t) \frac{1 + M^2 - D^2}{(1 + M^2 + D^2)^2 - 4D^2 M^2}\, dt$$

and we notice that:

- Φ is an analytic function of the form $\Phi(x') = O(x'^2)$;
- Ψ is of the form $\Psi(x') = (x')^{3/2}\Theta(x')$, where Θ is an analytic function.

Also, let

$$\psi := \frac{g_+(x') - g_-(x')}{2} = \int_0^{x'} \frac{-4D(t)M(t)}{(M^2 + D^2 + 1)^2 - 4M^2D^2} \, dt \, ,$$

and

$$\phi := \frac{g_+(x') + g_-(x')}{2} = \int_0^{x'} \frac{1 - (M^2 - D^2)^2}{(M^2 + D^2 + 1)^2 - 4M^2D^2} \, dt \, ,$$

where, as above,

- ϕ is an analytic function of the form $\phi(x') = x' + o(x')$;
- ψ is of the form $\psi(x') = (x')^{5/2}\theta(x')$, where θ is an analytic function.

Therefore we have

$$\begin{cases} f_+\big(\phi(x') + \psi(x')\big) - f_-\big(\phi(x') - \psi(x')\big) = 2\Psi(x'), \\ f_+\big(\phi(x') + \psi(x')\big) + f_-\big(\phi(x') - \psi(x')\big) = 2\Phi(x'), \end{cases}$$

and

$$f_+\big(\phi(x') + \psi(x')\big) = \Phi(x') + \Psi(x') \qquad \text{and} \qquad f_-\big(\phi(x') - \psi(x')\big) = \Phi(x') - \Psi(x').$$

Since $\eta_\pm$ is the inverse of $\phi \pm \psi$, we get that $\eta_\pm$ of the form

$$\eta_\pm(x) = x + x^{5/2}\beta_\pm(x^{1/2}),$$

where $\beta_\pm$ are analytic functions. Thus,

$$f_\pm(x) = \Phi\Big(x + x^{5/2}\beta_\pm\big(x^{1/2}\big)\Big) \pm \Psi\Big(x + x^{5/2}\beta_\pm\big(x^{1/2}\big)\Big),$$

which concludes the proof of Corollary 5.2 and Theorem 1.6 (b.3). ∎

5.3. Remarks on the non-symmetric case

For non-symmetric solutions, or more generally when different weights are put on the gradients of $u_\pm$ (as in the more general Alt-Caffarelli-Friedman energy, see for instance [10]), we cannot guarantee the validity of Corollary 5.2, and so branch points of the original problem might not be sent into branch points of the thin two-membrane problem. In fact, suppose that $(x_0, f_\pm(x_0))$ and $(x_1, f_\pm(x_1))$ are two consecutive points in $\mathcal{B}_2(u_+, u_-)$ such that $x_0 < x_1$ and

$$\begin{cases} f_+(x) = f_-(x) & \text{when} \quad x \le x_0, \\ f_+(x) > f_-(x) & \text{when} \quad x_0 < x < x_1, \\ f_+(x) = f_-(x) & \text{when} \quad x \ge x_1. \end{cases}$$

Suppose that $x_0 = 0$ and define $\eta_\pm$ as in (5.16). Now, we might have that

$$\eta_+(x_1) = \int_0^{x_1} \sqrt{1 + (f'_+(t))^2}\, dt > \int_0^{x_1} \sqrt{1 + (f'_-(t))^2}\, dt = \eta_-(x_1). \tag{5.18}$$

But then, for a generic point x' between $\eta_-(x_1)$ and $\eta_+(x_1)$, we get that $|\nabla v_+|(x', 0) = 1$, while $|\nabla v_-|(x', 0) < 1$, so that the equations (5.12) for $Q_\pm$ are not satisfied.

We notice that the symmetry assumption in point (a) of Theorem 1.6 is precisely what prevents (5.18) from happening. In particular, this assumption is fulfilled when

$$f_+(x) + f_-(x) \equiv 0 \quad \text{on} \quad B'_1. \tag{5.19}$$

We also notice that (5.19) is equivalent to assuming that $\eta_+ \equiv \eta_-$.

Lemma 5.6. *Suppose that $\eta_+ \equiv \eta_-$ on $(-1, 1)$, then $u_\pm : B_1^\pm \cup B'_1 \to \mathbb{R}$ and moreover*

$$u_-(x, y) = -u_+(x, -y) \quad \text{and} \quad f_+(x) + f_-(x) = 0 \quad \text{for every} \quad x \in (-1, 1).$$

Proof. Since $\eta'_+ \equiv \eta'_-$, (5.2) implies that $\partial_{y'} v_+(\eta_+(x), 0) = \partial_{y'} v_-(\eta_-(x), 0)$. In particular,

- if $f_+(x) > f_-(x)$, then $|\nabla v_\pm(\eta(x), 0)| = 1$ and so $\partial_x v_+(\eta_+(x), 0) = \partial_x v_-(\eta_-(x), 0)$;
- if $f_+(x) = f_-(x)$, then $\partial_x v_+(\eta_+(x), 0) = \partial_x v_-(\eta_-(x), 0)$.

In conclusion we have that

$$\nabla v_+(\eta_+(x), 0) = \nabla v_-(\eta_-(x), 0),$$

which using again (5.2) implies that $f'_+(x) \equiv -f'_-(x)$. Since $f_\pm(0) = 0$, integrating we get

$$f_+(x) + f_-(x) = \int_0^x (f'_+(t) + f'_-(t))\, dt = 0.$$

Finally, $u_-(x, y) + u_+(x, -y)$ is a harmonic function in Ω_u^- which vanishes together with its gradient on $\partial \Omega_u^-$. This implies that $u_-(x, y) + u_+(x, -y) = 0$ for every $(x, y) \in \Omega_u^-$. $\blacksquare$

Appendix A. The flat monotone solutions are minimizers

In this section we show that the solutions constructed in Theorem 1.4 and Theorem 1.8 are minimizers. We prove this fact for monotone solutions to the two-phase problem, the one-phase case being analogous.

Theorem A.1. *There is a constant $\varepsilon > 0$ depending only on the dimenson d such that the following holds. Let B'_r be the ball of radius r in $\mathbb{R}^{d-1}$; let $\eta_\pm : B'_2 \to \mathbb{R}$ be two C^1 functions with $\eta_\pm(0) = |\nabla_{x'} \eta_\pm(0)| = 0$ and*

$$|\eta_\pm| + |\nabla \eta_\pm| \le \varepsilon \quad \text{on} \quad B'_2.$$

Let

$$\Gamma_\pm := \left\{ (x', \eta_\pm(x')) \, : \, x' \in B'_2 \right\} \quad \text{and} \quad \Omega_\pm := \left\{ (x', x_d) \in B'_2 \times (-2, 2) \, : \, \pm x_d > \eta_\pm(x') \right\}.$$

Let $u_\pm : \Omega_\pm \cup \Gamma_\pm \to \mathbb{R}$ be two C^1 functions on $\Omega_\pm \cup \Gamma_\pm$ that solve

$$\Delta u_\pm = 0 \quad in \quad \Omega_+ \cup \Omega_-, \tag{A.1}$$

$$u_+ = 0 \quad and \quad |\nabla u_+| = 1 \quad on \quad \Gamma_+ \setminus \Gamma_-, \tag{A.2}$$

$$u_- = 0 \quad and \quad |\nabla u_-| = 1 \quad on \quad \Gamma_- \setminus \Gamma_+, \tag{A.3}$$

$$|\nabla u_+| = |\nabla u_-| \geq 1 \quad on \quad \Gamma_+ \cap \Gamma_-, \tag{A.4}$$

and are such that

$$1 - \varepsilon \leq \partial_{x_d} u_\pm \leq 1 + \varepsilon \quad on \quad \Omega_\pm.$$

Then, the function $u = u_+ - u_-$ is the unique minimizer of the two-phase functional in $\Omega := B'_1 \times (-1, 1)$, with $u = u_+ - u_-$ as boundary datum on $\partial\Omega$.

Proof. We first notice that

$$x_d - 2\varepsilon \leq u(x', x_d) \leq x_d + 2\varepsilon \quad \text{for} \quad (x', x_d) \in B'_1 \times (-2, 2).$$

Let $v = v_+ - v_-$ be a minimum of the two-phase functonal in $\Omega = B'_1 \times (-1, 1)$, with boundary datum u on $\partial\Omega$. Then,

$$x_d - 2\varepsilon \leq v(x', x_d) \leq x_d + 2\varepsilon \quad \text{for} \quad (x', x_d) \in \Omega.$$

Now, consider the family of functions (which are all defined on Ω when $|t| < 1$)

$$u_t(x', x_d) := u(x', x_d - t).$$

Then u_t are solutions to (A.1)–(A.4) in $B'_1 \times (-1, 1)$ and are monotone, that is,

$$u_t \leq u_s \quad \text{whenever} \quad t \leq s.$$

Now, for t small enough, we have that $u_t \leq x_d - 2\varepsilon \leq v(x)$. Let $t \leq 0$ be the largest parameter for which $u_t \leq v$. In particular,

$$\{u_t > 0\} \subset \{v > 0\} \qquad \text{and} \qquad \{u_t < 0\} \supset \{v < 0\}.$$

Suppose that $t < 0$. By the monotonicity of $u_\pm$, we have that $u_t < u = v$ on $\partial\Omega$. Thus, u_t touches v from below at a point $(x', x_d) \in \Omega$ and we have the following three possibilities:

(1) $u_t(x', x_d) = v(x', x_d) > 0$;

(2) $u_t(x', x_d) = v(x', x_d) = 0$ and $(x', x_d) \in \partial\{u_t > 0\} \cap \partial\{v > 0\}$;

(3) $u_t(x', x_d) = v(x', x_d) = 0$ and $(x', x_d) \in \partial\{u_t < 0\} \cap \partial\{v < 0\}$.

Now, (1) cannot happen by the strict maximum principle. Suppose that (2) holds. Then, both $\partial\{v > 0\}$ and v_+ are C^1 in a neighborhood of (x', x_d). Since u_t touches v from below, we have that $|\nabla u_t^+|(x', x_d) \leq |\nabla v_+|(x', x_d)$. Now, if both gradients are strictly greater than one, then both u_t and v are harmonic in a neighborhood of (x', x_d), so by the strong maximum principle and the unique continuation property they should coincide. Then, since at least one of the gradients should be smaller than 1, so necessarily $|\nabla u_t^+|(x', x_d) = 1$. In order to

rule out this possibility, we consider two further cases. Suppose first that $|\nabla v_+|(x', x_d) > 1$. Then, $(x', x_d) \in \partial\{v > 0\} \cap \partial\{v < 0\}$. But this means that $(x', x_d) \in \partial\{u_t > 0\} \cap \partial\{u_t < 0\}$ and $|\nabla u_t^-|(x', x_d) = 1$. But this is impossible since $-u_t^-$ should remain smaller than $-v_-$. Finally, the last possibility is that $|\nabla v_+|(x', x_d) = |\nabla u_t^+|(x', x_d) = 1$. But this is impossible since it violates the Hopf maximum principle. Thus, we have showed that (2) cannot happen. By the same argument, (3) cannot happen neither. Then, the only possibility is that $t = 0$, so $v \geq u$ in Ω. Analogously, $v \leq u$ in Ω, so we have that $u = v$ in Ω. ∎

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