frontiers | Frontiers in Physics

TYPE Brief Research Report
PUBLISHED 08 December 2023
DOI 10.3389/fphy.2023.1232691

OPEN ACCESS

EDITED BY
Alberto Martinez Torres,
University of São Paulo, Brazil

REVIEWED BY
Gaotsiwe Joel Rampho,
University of South Africa, South Africa
Francesco Giovanni Celiberto,
Bruno Kessler Foundation (FBK), Italy

*CORRESPONDENCE
M. R. Hadizadeh,
mhadizadeh@centralstate.edu

RECEIVED 01 June 2023
ACCEPTED 06 November 2023
PUBLISHED 08 December 2023

CITATION
Mohammadzadeh M, Radin M, Mohseni K and Hadizadeh MR (2023), Four-body bound states in momentum space: the Yakubovsky approach without two-body t – matrices.
Front. Phys. 11:1232691.
doi: 10.3389/fphy.2023.1232691

Four-body bound states in momentum space: the Yakubovsky approach without two-body t – matrices

M. Mohammadzadeh[1], M. Radin[1], K. Mohseni[2] and M. R. Hadizadeh[3,4]*

[1]Department of Physics, K. N. Toosi University of Technology, Tehran, Iran, [2]Departamento de Física, Instituto Tecnológico de Aeronáutica, Departamento de Ciência e Tecnologia Aeroespacial (DCTA), São José dos Campos, Brazil, [3]College of Engineering Science, Technology and Agriculture, Central State University, Wilberforce, OH, United States, [4]Department of Physics and Astronomy, Ohio University, Athens, OH, United States

This study presents a solution to the Yakubovsky equations for four-body bound states in momentum space, bypassing the common use of two-body t – matrices. Typically, such solutions are dependent on the fully-off-shell two-body t – matrices, which are obtained from the Lippmann-Schwinger integral equation for two-body subsystem energies controlled by the second and third Jacobi momenta. Instead, we use a version of the Yakubovsky equations that does not require t – matrices, facilitating the direct use of two-body interactions. This approach streamlines the programming and reduces computational time. Numerically, we found that this direct approach to the Yakubovsky equations, using 2B interactions, produces four-body binding energy results consistent with those obtained from the conventional t – matrix dependent Yakubovsky equations, for both separable (Yamaguchi and Gaussian) and non-separable (Malfliet-Tjon) interactions.

KEYWORDS

Yakubovsky equations, four-body bound state, momentum space, Malfliet-Tjon potential, Yamaguchi potential, Gaussian potential

1 Introduction

The Yakubovsky equations provide a non-perturbative framework for investigating few-body bound and scattering systems in different sectors of physics, including atomic, nuclear, and particle physics. These equations have been utilized extensively in both momentum [1–24] and configuration [25–45] spaces. The Yakubovsky equations fundamentally rely on two-body (2B) transition matrices, denoted as $t(\epsilon)$, which are derived from the solution of the Lippmann-Schwinger (LS) equation, considering either positive (scattering states) or negative (bound states) 2B subsystem energies ϵ. Notably, solving the LS equation for positive energies can prove to be a numerically demanding task due to the presence of singularities. Contrarily, for negative energies, the LS equation must be computed for the 2B subsystem energies, which are determined by the Jacobi momenta of the third and fourth particles.

In this study, we utilize a version of the Yakubovsky equations for four-body (4B) bound states that directly incorporates 2B interactions, eliminating the need for 2B t – matrices. This t – matrix-free approach to the Yakubovsky equations has been previously solved in configuration space by Lazauskas *et al.* [29, 39, 40, 44]. In our

TABLE 1 The parameters for the Malfliet-Tjon (MT-V), Yamaguchi (Y-IV), and Gaussian potentials utilized in this study.

MT-V	λ_1	μ_1 (fm^{-1})	λ_2	μ_2 (fm^{-1})
	−2.93	1.55	7.39	3.11
Yamaguchi-IV		**λ (MeV·fm^{-1})**		**β (fm^{-1})**
		−7.42313		1.15
Gaussian		**λ (MeV·fm^{3})**		**Λ (fm^{-1})**
		−15.825		0.70

work, we present the t − matrix-free coupled Yakubovsky equations in momentum space. Here, the Yakubovsky components linked to the 3 + 1 and 2 + 2 chains are derived as a function of the Jacobi momentum vectors, directly including 2B interactions. To determine the 4B binding energies, we solve the coupled Yakubovsky integral equations using separable potentials with Yamaguchi and Gaussian form factors, as well as the non-separable Malfliet-Tjon potential, with all potentials projected into the s − wave channel. Our numerical findings highlight that the t − matrix-free version of the Yakubovsky integral equations, when utilizing 2B interactions, aligns perfectly with results derived from the conventional Yakubovsky integral equations that employ 2B t − matrices. In a related research, this t − matrix-free approach is successfully employed for relativistic three-body (3B) bound states [46]. This led to a version of the relativistic Faddeev equation that directly employs 2B boosted interactions [47, 48], eliminating the need for 2B boosted t − matrices [19, 20, 24].

This paper has been structured into several sections. Section 2. provides a brief overview of the Yakubovsky equations for 4B bound states, bypassing the use of 2B t − matrices. Herein, we formulate the coupled Yakubovsky equations in momentum space, considering Jacobi momentum vectors and presenting a simplified form for s − wave interactions. Section 3. Outlines our numerical results for 4B binding energies as calculated from the t − matrix-free Yakubovsky equations, alongside a comparison with results obtained from conventional t − matrix-dependent Yakubovsky equations. Lastly, Section 4. presents our conclusion and discusses prospects for succeeding studies.

2 The t − matrix-free coupled Yakubovsky equations for 4B bound states

The conventional form of Yakubovsky equations that describe the bound state of four identical particles interacting through pairwise interactions reads as follows [4].

$$\begin{aligned}\psi_1 &= G_0 t P\left[(1+P_{34})\psi_1+\psi_2\right],\\ \psi_2 &= G_0 t \tilde{P}\left[(1+P_{34})\psi_1+\psi_2\right],\end{aligned} \tag{1}$$

where ψ_1 and ψ_2 denote Yakubovsky components of the 4B wave function, corresponding to 3 + 1 and 2 + 2 type chains, respectively. $G_0 = (E - H_0)^{-1}$ represents 4B free propagator, while $P = P_{12}P_{23} + P_{13}P_{23}$, $\tilde{P} = P_{13}P_{24}$ and P_{34} serve as the permutation operators. The 2B t − matrix is determined by the LS equation

$$t = V + VG_0 t. \tag{2}$$

The coupled Yakubovsky Eq. 1 can be restructured to yield another form of Yakubovsky equations as follows

$$\begin{aligned}\psi_1 &= G_0 V P\left[(1+P_{34})\psi_1+\psi_2\right] + G_0 V \psi_1,\\ \psi_2 &= G_0 V \tilde{P}\left[(1+P_{34})\psi_1+\psi_2\right] + G_0 V \psi_2,\end{aligned} \tag{3}$$

where the 2B interactions V are being directly utilized as input to the Yakubovsky equations, consequently eliminating the need for the 2B t − matrices. The representation of the t − matrix-free form of the coupled Yakubovsky Eq. 3 in momentum space leads to the following coupled 3D integral equations

TABLE 2 Convergence of the 4B binding energy obtained from the t −matrix-free (E_v), Eq. 6, and the conventional t −matrix-dependent ($E_{t\text{-matrix}}$), Eq. (A4), versions of the coupled Yakubovsky equations. The convergence is shown as a function of the number of mesh points for the magnitude of Jacobi momenta u_i and v_i, denoted as $N_{u_i} = N_{v_i}$. Results are presented for MT-V, Yamaguchi-IV, and Gaussian potentials. The number of mesh points for angle variables is 40. All calculations were performed with $\hbar^2/m$ =41.47 MeV · fm^2.

$N_{u_i} = N_{v_i}$	E_v (MeV)	$E_{t\text{-matrix}}$ (MeV)	E_v (MeV)	$E_{t\text{-matrix}}$ (MeV)	E_v (MeV)	$E_{t\text{-matrix}}$ (MeV)
	s − wave MT-V		Yamaguchi-IV		Gaussian	
30	−30.323	−30.668	−36.156	−36.151	−31.281	−31.276
40	−30.206	−30.367	−36.236	−36.233	−31.272	−31.272
50	−30.313	−30.220	−36.258	−36.256	−31.272	−31.271
60	−30.182	−30.145	−36.266	−36.264	−31.271	−31.271
70	−30.135	−30.116	−36.270	−36.268	−31.271	−31.271
80	−30.103	−30.098	−36.272	−36.270	−31.271	−31.270
90	−30.092	−30.089	−36.273	−36.271	−31.270	−31.270
100	−30.087	−30.084	−36.273	−36.271	−31.270	−31.270
150	−30.078	−30.077	−36.274	−36.272	−31.270	−31.270
	–	−30.07 [58]	–	−36.3 [59]	–	–

$$\psi_1(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3) = G_0(u_1,u_2,u_3)\Big(\int d^3u_2' V_s\Big(\mathbf{u}_1,\frac{1}{2}\mathbf{u}_2+\mathbf{u}_2'\Big) \times\Big\{\psi_1\Big(\mathbf{u}_2+\frac{1}{2}\mathbf{u}_2',\mathbf{u}_2',\mathbf{u}_3\Big)+\psi_1\Big(\mathbf{u}_2+\frac{1}{2}\mathbf{u}_2',\frac{1}{3}\mathbf{u}_2'+\frac{8}{9}\mathbf{u}_3,\mathbf{u}_2'-\frac{1}{3}\mathbf{u}_3\Big) +\psi_2\Big(\mathbf{u}_2+\frac{1}{2}\mathbf{u}_2',-\mathbf{u}_2'-\frac{2}{3}\mathbf{u}_3,\frac{1}{2}\mathbf{u}_2'-\frac{2}{3}\mathbf{u}_3\Big)\Big\} +\frac{1}{2}\int d^3u_1' V_s(\mathbf{u}_1,\mathbf{u}_1')\psi_1(\mathbf{u}_1',\mathbf{u}_2,\mathbf{u}_3)\Big),$$

$$\langle\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3|\psi_2\rangle = \frac{1}{2}G_0(v_1,v_2,v_3)\Big(\int d^3v_3' V_s(\mathbf{v}_1,\mathbf{v}_3') \times\Big\{2\psi_1\Big(\mathbf{v}_3,\frac{2}{3}\mathbf{v}_2+\frac{2}{3}\mathbf{v}_3',\frac{1}{2}\mathbf{v}_2-\mathbf{v}_3'\Big)+\psi_2(\mathbf{v}_3,-\mathbf{v}_2,\mathbf{v}_3')\Big\} +\int d^3v_1' V_s(\mathbf{v}_1,\mathbf{v}_1')\psi_1(\mathbf{v}_1',\mathbf{v}_2,\mathbf{v}_3)\Big), \quad (4)$$

with the symmetrized 2B interaction defined as $V_s(\mathbf{a},\mathbf{b}) = V(\mathbf{a},\mathbf{b}) + V(\mathbf{a},-\mathbf{b})$ and 4B free propagators characterized through the following expressions

$$G_0(u_1,u_2,u_3) = \Big(E-\frac{u_1^2}{m}-\frac{3u_2^2}{4m}-\frac{2u_3^2}{3m}\Big)^{-1},$$
$$G_0(v_1,v_2,v_3) = \Big(E-\frac{v_1^2}{m}-\frac{v_2^2}{2m}-\frac{v_3^2}{m}\Big)^{-1}. \quad (5)$$

The Jacobi momenta $\mathbf{u}_i$ and $\mathbf{v}_i$ $(i = 1, 2, 3)$ correspond to 3 + 1 and 2 + 2 chains, respectively [11]. The coupled Yakubovsky Eq. 4 can be simplified for the s – wave interactions as

$$\psi_1(u_1,u_2,u_3) = 4\pi G_0(u_1,u_2,u_3) \times\Big(\int_0^\infty du_2' u_2'^2\int_{-1}^1 dx'\, V(u_1,\Pi_1(u_2,u_2',x')) \times\Big[\psi_1(\Pi_1(u_2',u_2,x'),u_2',u_3) +\frac{1}{2}\int_{-1}^1 dx\Big\{\psi_1\big(\Pi_1(u_2',u_2,x'),\Pi_2(u_2',u_3,x),\Pi_3(u_2',u_3,x)\big) +\psi_2\big(\Pi_1(u_2',u_2,x'),\Pi_4(u_2',u_3,x),\Pi_5(u_2',u_3,x)\big)\Big\}\Big] +\int_0^\infty du_1' u_1'^2 V(u_1,u_1')\psi_1(u_1',u_2,u_3)\Big),$$

$$\psi_2(v_1,v_2,v_3) = 4\pi G_0(v_1,v_2,v_3) \times\Big(\int_0^\infty dv_3' v_3'^2 V(v_1,v_3') \times\Big[\int_{-1}^1 dx_3'\,\psi_1\Big(v_3,\Pi_6(v_2,v_3',x_3'),\Pi_7(v_2,v_3',x)\Big)+\psi_2(v_3,v_2,v_3')\Big] +\int_0^\infty dv_1' v_1'^2 V(v_1,v_1')\psi_2(v_1',v_2,v_3)\Big), \quad (6)$$

where the shifted momentum arguments are defined as [49].

$$\tilde{\Pi}_1 = \Big(\frac{1}{4}u_2^2+u_2'^2+u_2u_2'x\Big)^{\frac{1}{2}},\quad \Pi_1 = \Big(u_2^2+\frac{1}{4}u_2'^2+u_2u_2'x\Big)^{\frac{1}{2}},$$
$$\Pi_2 = \frac{1}{3}\Big(u_2^2+\frac{64}{9}u_3^2+\frac{16}{3}u_2u_3x'\Big)^{\frac{1}{2}},\quad \Pi_3 = \Big(u_2^2+\frac{1}{9}u_3^2-\frac{2}{3}u_2u_3x'\Big)^{\frac{1}{2}},$$
$$\Pi_4 = \Big(\frac{1}{4}u_2^2+\frac{4}{9}u_3^2+\frac{2}{3}u_2u_3x'\Big)^{\frac{1}{2}},\quad \Pi_5 = \Big(u_2^2+\frac{4}{9}u_3^2-\frac{4}{3}u_2u_3x'\Big)^{\frac{1}{2}},$$
$$\Pi_6 = \frac{2}{3}(v_2^2+v_3^2+2v_2v_3x)^{\frac{1}{2}},\quad \Pi_7 = \Big(v_3^2+\frac{1}{4}v_2^2-v_2v_3x\Big)^{\frac{1}{2}}. \quad (7)$$

For comparison purposes, the representation of the conventional t – matrix-dependent form of the coupled Yakubovsky Eq. 1 in momentum space can be found in Appendix A. Upon comparing the t – matrix-free and conventional t – matrix-dependent forms of the Yakubovsky equations - specifically, Eqs. 4 and (A1) - it is evident that the t – matrix-free form incorporates an extra term. This term involves integration over the 2B interaction and the Yakubovsky components without interpolations on momenta or angles. Despite this, its numerical solution proves to be more straightforward and cost-effective than that of the conventional t – matrix-dependent that necessitates solving the LS equation to compute the 2B t – matrices for all required 2B subsystem energies, which depend on the magnitude of the second and third Jacobi momenta.

3 Numerical results

The numerical solution of the coupled Yakubovsky integral Eq. 6 for the calculation of 4B binding energy demands solving an eigenvalue equation, where the physical binding energy corresponds to an eigenvalue equal to one. The Lanczos iterative method is implemented for solving such eigenvalue equation [24, 49–52]. We employ Gauss-Legendre quadratures to discretize the continuous momentum and angular variables with a hyperbolic mapping for Jacobi momenta and a linear mapping for angle variables [53]. This allows us to properly capture the behavior of the Yakubovsky components of the 4B wave function at both small and large momenta. In each iteration step of solving the coupled Yakubovsky integral equations, to accurately perform multi-dimensional interpolations on the shifted momentum arguments given in Eq. 7, we employ the Cubic-Hermite spline method due to its combination of high accuracy and computational speed [54].

Our numerical analysis presents a comparison between 4B binding energy obtained from the t – matrix-free approach and the conventional t – matrix-dependent formulation of the coupled Yakubovsky equations, namely, Eqs. 6 and A4. For our numerical analysis, we employ two models of one-term separable potential with the following general form

$$V(p,p') = \lambda g(p)g(p'), \quad (8)$$

where λ represents the potential strength. The potential form factor $g(p)$ for the Yamaguchi-type potential is defined as $g(p) = 1/(p^2+\beta^2)$ [55], while for the Gaussian potential, it takes the form $g(p) = \exp(-p^2/\Lambda^2)$ [56]. Furthermore, to provide a comprehensive validation of our formalism and code, we also incorporate an s-wave non-separable Malfliet-Tjon (MT) potential, comprises two attractive and repulsive terms [57].

$$V(p,p') = \sum_{i=1}^{2}\frac{\lambda_i}{2\pi pp'}\ln\Big(\frac{\mu_i^2+p^2+p'^2+2pp'}{\mu_i^2+p^2+p'^2-2pp'}\Big). \quad (9)$$

In Table 1, we provide the parameters for the potentials employed in our calculations, which include MT model V (MT-V), Yamaguchi potential model IV (Y-IV), and a Gaussian potential. The strength of the Gaussian potential was adjusted to reproduce the deuteron binding energy of −2.225 MeV, with a form factor parameter $\Lambda = 0.7$ fm^{-1}.

Table 2 presents the convergence of the 4B binding energy as a function of the number of mesh points for the Jacobi momenta magnitudes u_i and v_i. The table provides a side-by-side comparison of results obtained using both the t-matrix-free and conventional t-matrix-dependent forms of the coupled Yakubovsky equations.

Specifically, our results show that the t − matrix-free Yakubovsky equations yield a 4B binding energy of −30.08 MeV for the MT-V potential and −36.27 MeV for the Yamaguchi-IV potential. These values closely align with the −30.07 MeV [58] and −36.3 MeV [59] obtained by other groups using the t − matrix-dependent Yakubovsky calculations. The comparison validates the potential of the t − matrix-free adoption of Yakubovsky formulation to solve 4B bound state problems efficiently, matching the precision of the conventional method but potentially offering more straightforward computational requirements. Moreover, the convergence behavior remains consistent across different numbers of mesh points for the magnitude of Jacobi momenta, emphasizing the computational robustness of the t − matrix-free formulation.

4 Summary and outlook

The Yakubovsky approach is a powerful method to study few-body bound and scattering systems. However, the solution of these equations can be computationally demanding due to inherent singularities in 2B t − matrices when dealing with scattering problems and the need to calculate them for 2B subsystem energies dictated by second and third Jacobi momenta when 4B bound state problems are considered. This study utilizes a version of the coupled Yakubovsky equations for 4B bound states that directly incorporates 2B interactions in momentum space, avoiding the use of the 2B t − matrices. The efficacy of this approach is validated through the calculation of 4B binding energies in momentum space using, both the separable potentials with Yamaguchi and Gaussian form factors, and the non-separable Malfliet-Tjon potential. Our findings align well with results from the conventional form of the coupled Yakubovsky integral equations incorporating 2B t − matrices. The extension of calculations to include more general interactions, beyond just the s − wave, is currently in progress.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

MR took the lead in project design, while MR and MRH collaborated on developing the theoretical formalism and computer codes. MM and KM calculated the four-body binding energies. MRH, MR, and KM engaged in the result discussion and contributed to the final manuscript. All authors contributed to the article and approved the submitted version.

Funding

The work of MRH was supported by the National Science Foundation under Grant No. NSF-PHY-2000029 with Central State University. KM was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) Grant No. 400789/2019-0.

Acknowledgments

KM acknowledges a Ph.D. scholarship from the Brazilian agency CNPq.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

References

1. Kamada H, Glöckle W. Solutions of the four-body Yakubovsky equations for the α-particle using realistic 2N interactions. *Phys Lett B* (1992) 292:1–4. doi:10.1016/0370-2693(92)90598-x

2. Glöckle W, Kamada H. On the inclusion of 3*N*-forces into the 4*N*-Yakubovsky equations. *Nucl Phys A* (1993) 560:541–7. doi:10.1016/0375-9474(93)90113-c

3. Nogga A, Kamada H, Glöckle W. Modern nuclear force predictions for the α particle. *Phys Rev Lett* (2000) 85:944–7. doi:10.1103/physrevlett.85.944

4. Nogga A, Kamada H, Glöckle W, Barrett BR. The α particle based on modern nuclear forces. *Phys Rev C* (2002) 65:054003. doi:10.1103/physrevc.65.054003

5. Uzu E, Kamada H, Koike Y. Complex energy method in four-body Faddeev-Yakubovsky equations. *Phys Rev C* (2003) 68:061001. doi:10.1103/physrevc.68.061001

6. Platter L, Hammer H-W, Meißner U-G. Four-boson system with short-range interactions. *Phys Rev A* (2004) 70:052101. doi:10.1103/physreva.70.052101

7. Nogga A. Faddeev-Yakubovsky calculations for $A = 4$ hypernuclear systems. *Nucl Phys A* (2005) 754:36–42. doi:10.1016/j.nuclphysa.2005.01.013

8. Nogga A. Application of chiral nuclear forces to light nuclei. In: *Chiral dynamics 2006*. Singapore: World Scientific (2007). p. 182–93.

9. Bayegan S, Hadizadeh M, Harzchi M. A realistic three-dimensional calculation of ^{3}H binding energy. *Few-body Syst* (2008) 44:65–7. doi:10.1007/s00601-008-0258-y

10. Deltuva A, Fonseca AC. Four-nucleon scattering: *ab initio* calculations in momentum space. *Phys Rev C* (2007) 75:014005. doi:10.1103/physrevc.75.014005

11. Hadizadeh M, Bayegan S. Four-body bound-state calculations in three-dimensional approach. *Few-Body Syst* (2007) 40:171–91. doi:10.1007/s00601-006-0169-8

12. Bayegan S, Hadizadeh MR, Harzchi M. Three-nucleon bound state in a spin-isospin dependent three dimensional approach. *Phys Rev C* (2008) 77:064005. doi:10.1103/physrevc.77.064005

13. Bayegan S, Hadizadeh MR, Glöckle W. A realistic formalism for 4*N* bound state in a three-dimensional Yakubovsky scheme. *Prog Theor Phys* (2008) 120:887–916. doi:10.1143/ptp.120.887

14. Hadizadeh M, Bayegan S. Towards a three-dimensional solution for 3*N* bound states with 3NFs. *Mod Phys Lett A* (2009) 24:816–22. doi:10.1142/s0217732309000073

15. Hadizadeh MR, Tomio L, Bayegan S. Solutions of the bound-state Faddeev-Yakubovsky equations in three dimensions by using *NN* and 3*N* potential models. *Phys Rev C* (2011) 83:054004. doi:10.1103/physrevc.83.054004

16. Frederico T, Delfino A, Hadizadeh M, Tomio L, Yamashita MT. Universality in four-boson systems. *Few-Body Syst* (2013) 54:559–68. doi:10.1007/s00601-012-0463-6

17. Tomio L, Hadizadeh M, Yamashita MT, Delfino A, Frederico T. Four-Boson systems close to a universal regime. *Few-Body Syst* (2013) 54:1543–6. doi:10.1007/s00601-012-0588-7

18. Kamada H. Four-body scattering equations including a three-body force in the Faddeev-Yakubovsky theory. *Few-Body Syst* (2019) 60:33. doi:10.1007/s00601-019-1501-4

19. Hadizadeh MR, Elster C, Polyzou WN. Relativistic three-body bound state in a 3D formulation. *Phys Rev C* (2014) 90:054002. doi:10.1103/physrevc.90.054002

20. Hadizadeh M, Elster C, Polyzou W. The relativistic three-body bound state in three-dimensions. *EPJ Web of Conferences* (2016) 113:03011. doi:10.1051/epjconf/201611303011

21. Radin M, Mohseni H, Nazari F, Hadizadeh M. Four-body bound state calculations using three-dimensional low-momentum effective interaction $V_{\mathrm{low-k}}$. *Int J Mod Phys E* (2017) 26:1750083. doi:10.1142/s0218301317500835

22. Bazak B, Petrov DS. Energy of N two-dimensional bosons with zero-range interactions. *New J Phys* (2018) 20:023045. doi:10.1088/1367-2630/aaa64f

23. Bazak B, Kirscher J, König S, Valderrama MP, Barnea N, van Kolck U. Four-body scale in universal few-boson systems. *Phys Rev Lett* (2019) 122:143001. doi:10.1103/physrevlett.122.143001

24. Hadizadeh M, Radin M, Mohseni K. A three-dimensional momentum-space calculation of three-body bound state in a relativistic Faddeev scheme. *Scientific Rep* (2020) 10:1949. doi:10.1038/s41598-020-58577-4

25. Barnea N, Viviani M. Projected Faddeev-Yakubovsky equations for the **N**-body problem. *Phys Rev C* (2000) 61:034003. doi:10.1103/physrevc.61.034003

26. Filikhin I, Yakovlev S. ^{16}O nucleus in the 4α cluster model. *Phys At Nuclei* (2000) 63:343–52. doi:10.1134/1.855641

27. Kezerashvili RY. Elastic N-body to N-body scattering in the hyperspherical representation. In: *150 Years of quantum many-body theory*. Singapore: World Scientific (2001). p. 89–95.

28. Filikhin I, Yakovlev S, Roudnev V, Vlahovic B. The ^{4}He tetramer ground state in the Faddeev-Yakubovsky differential equations formalism. *J Phys B: At Mol Opt Phys* (2002) 35:501–8. doi:10.1088/0953-4075/35/3/305

29. Lazauskas R, Carbonell J. Ab-initio calculations of four-nucleon elastic scattering. *Few-Body Syst* (2004) 34:105. doi:10.1007/s00601-004-0049-z

30. Uzu E, Koike Y. Four-body Faddeev-Yakubovsky calculation using the finite range expansion method. *Prog Theor Phys* (2006) 115:309–23. doi:10.1143/ptp.115.309

31. Lekala ML, Masita DF (2010). Four-body reactions of light nuclei. *AIP Conf. Proc.* 1265 (1):88–91. doi:10.1063/1.3480268

32. Schellingerhout NW. Numerical methods in configuration-space a = 3, 4 bound-state and scattering calculations. In: Bakker BLG, van Dantzig R, editors. *Few-body problems in physics '93*. Vienna: Springer (1994). p. 361–70.

33. Filikhin I, Suslov V, Vlahovic B. Modeling of 6ΛHe hypernucleus within configuration space Faddeev approach (2014). Available at: https://arxiv.org/abs/1401.2189.

34. Lazauskas R. Modern nuclear force predictions for n –^{3}H scattering above the three- and four-nucleon breakup thresholds. *Phys Rev C* (2015) 91:041001. doi:10.1103/physrevc.91.041001

35. Hiyama E, Lazauskas R, Carbonell J, Kamimura M. Possibility of generating a 4-neutron resonance with a $T = 3/2$ isospin 3-neutron force. *Phys Rev C* (2016) 93:044004. doi:10.1103/physrevc.93.044004

36. Kezerashvili RY. Few-body neutron and kaonic clusters. *Phys Sci Tech* (2017) 3:63–72. doi:10.26577/phst-2016-1-95

37. Kezerashvili RY. Strange dibaryonic and tribaryonic clusters, Physics, properties and dynamics (2017) 227.

38. Ferrari Ruffino F, Lonardoni D, Barnea N, Deflorian S, Leidemann W, Orlandini G, et al. Benchmark results for few-body hypernuclei. *Few-Body Syst* (2017) 58:113. doi:10.1007/s00601-017-1273-7

39. Lazauskas R. Solution of the n –^{4}He elastic scattering problem using the Faddeev-Yakubovsky equations. *Phys Rev C* (2018) 97:044002. doi:10.1103/physrevc.97.044002

40. Lazauskas R, Carbonell J. The faddeev-yakubovsky symphony. *Few-Body Syst* (2019) 60:62. doi:10.1007/s00601-019-1529-5

41. Lazauskas R, Song Y-H. Parity-violating neutron spin rotation in ^{4}He. *Phys Rev C* (2019) 99:054002. doi:10.1103/physrevc.99.054002

42. Kezerashvili RY. Few-body systems in condensed matter physics. *Few-Body Syst* (2019) 60:52. doi:10.1007/s00601-019-1520-1

43. Filikhin I, Kezerashvili RY, Suslov VM, Tsiklauri SM, Vlahovic B. Three-body model for $K(1460)$ resonance. *Phys Rev D* (2020) 102:094027. doi:10.1103/physrevd.102.094027

44. Lazauskas R, Carbonell J. Description of four- and five-nucleon systems by solving Faddeev-Yakubovsky equations in configuration space. *Front Phys* (2020) 7:251. doi:10.3389/fphy.2019.00251

45. Kezerashvili RY, Tsiklauri SM, Takibayev NZ. Search and research of K$^-$NNN and K-K-NN antikaonic clusters. *Prog Part Nucl Phys* (2021) 121:103909. doi:10.1016/j.ppnp.2021.103909

46. Mohammadzadeh M, Radin M, Hadizadeh M. *Relativistic Faddeev equations for three-body bound states without 2B T – matrices, submitted for publication* (2023).

47. Hadizadeh M, Radin M. Calculation of relativistic nucleon-nucleon potentials in three dimensions. *The Eur Phys J A* (2017) 53:18. doi:10.1140/epja/i2017-12209-6

48. Hadizadeh M, Radin M, Nazari F. Relativistic nucleon-nucleon potentials in a spin-dependent three-dimensional approach. *Scientific Rep* (2021) 11:17550. doi:10.1038/s41598-021-96924-1

49. Hadizadeh MR, Yamashita MT, Tomio L, Delfino A, Frederico T. Binding and structure of tetramers in the scaling limit. *Phys Rev A* (2012) 85:023610. doi:10.1103/physreva.85.023610

50. Ahmadi M, Hadizadeh MR, Radin M, Bayegan S. Novel regularization scheme for nucleon-nucleon lattice simulations with effective field theory. *Phys Rev C* (2020) 102:044001. doi:10.1103/physrevc.102.044001

51. Mohseni K, Chaves A, da Costa D, Frederico T, Hadizadeh M. Three-boson stability for boosted interactions towards the zero-range limit. *Phys Lett B* (2021) 823:136773. doi:10.1016/j.physletb.2021.136773

52. Hadizadeh MR, Yamashita MT, Tomio L, Delfino A, Frederico T. Scaling properties of universal tetramers. *Phys Rev Lett* (2011) 107:135304. doi:10.1103/physrevlett.107.135304

53. Mohseni K, Hadizadeh MR, Frederico T, da Costa DR, Chaves AJ. Trion clustering structure and binding energy in two-dimensional semiconductor materials: Faddeev equations approach. *Phys Rev B* (2023) 107:165427. doi:10.1103/physrevb.107.165427

54. Hüber D, Witała H, Nogga A, Glöckle W, Kamada H. A new look into the partial-wave decomposition of three-nucleon forces. *Few-Body Syst* (1997) 22:107–35. doi:10.1007/s006010050057

55. Yamaguchi Y. Two-Nucleon problem when the potential is nonlocal but separable. I. *Phys Rev* (1954) 95:1628–34. doi:10.1103/physrev.95.1628

56. Deltuva A, Lazauskas R, Platter L. Universality in four-body scattering. *Few-Body Syst* (2011) 51:235–47. doi:10.1007/s00601-011-0227-8

57. Kessler B, Payne GL, Polyzou W. Scattering calculations with wavelets. *Few-body Syst* (2003) 33:1–26. doi:10.1007/s00601-003-0008-0

58. Kamada H, Glöckle W. Solutions of the yakubovsky equations for four-body model systems. *Nucl Phys A* (1992) 548:205–26. doi:10.1016/0375-9474(92)90009-9

59. Gibson BF, Lehman DR. Erratum: four-body bound states from the Schrödinger equation with separable potentials. *Phys Rev C* (1977) 15:2257. doi:10.1103/physrevc.15.2257.2

Appendix A: Representation of conventional t – matrix-dependent coupled Yakubovsky equations in momentum space

The conventional form of the coupled Yakubovsky Eq. 1 in momentum space is represented as follows [11].

$$\begin{aligned}\psi_1(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3) &= G_0(u_1,u_2,u_3)\int d^3u_2'\; t_s\Big(\mathbf{u}_1,\frac{1}{2}\mathbf{u}_2+\mathbf{u}_2';\epsilon\Big)\\ &\times\Big\{\psi_1\Big(\mathbf{u}_2+\frac{1}{2}\mathbf{u}_2',\mathbf{u}_2',\mathbf{u}_3\Big)+\psi_1\Big(\mathbf{u}_2+\frac{1}{2}\mathbf{u}_2',\frac{1}{3}\mathbf{u}_2'+\frac{8}{9}\mathbf{u}_3,\mathbf{u}_2'-\frac{1}{3}\mathbf{u}_3\Big)\\ &+\psi_2\Big(\mathbf{u}_2+\frac{1}{2}\mathbf{u}_2',-\mathbf{u}_2'-\frac{2}{3}\mathbf{u}_3,\frac{1}{2}\mathbf{u}_2'-\frac{2}{3}\mathbf{u}_3\Big)\Big\},\\ \psi_2(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) &= \frac{1}{2}G_0(v_1,v_2,v_3)\int d^3v_3'\; t_s(\mathbf{v}_1,\mathbf{v}_3';\epsilon^*)\\ &\times\Big\{2\psi_1\Big(\mathbf{v}_3,\frac{2}{3}\mathbf{v}_2+\frac{2}{3}\mathbf{v}_3',\frac{1}{2}\mathbf{v}_2-\mathbf{v}_3'\Big)+\psi_2(\mathbf{v}_3,-\mathbf{v}_2,\mathbf{v}_3')\Big\},\end{aligned} \tag{A1}$$

with the symmetrized 2B t – matrices defined as

$$t_s(\mathbf{a},\mathbf{b};\epsilon) = t(\mathbf{a},\mathbf{b};\epsilon)+t(\mathbf{a},-\mathbf{b};\epsilon). \tag{A2}$$

The matrix elements of 2B t – matrices $t(\mathbf{a},\ \mathbf{b};\ \epsilon)$ needs to be calculated from the solution of the LS Eq. 2 for the 2B subsystem energies associated with the 3 + 1 and 2 + 2 chains, given by

$$\epsilon = E-\frac{3u_2^2}{4m}-\frac{2u_3^2}{3m},\qquad \epsilon^\star = E-\frac{v_2^2}{2m}-\frac{v_3^2}{m}. \tag{A3}$$

The coupled Yakubovsky equations (A1) can be simplified for the s – wave interactions as

$$\begin{aligned}&\psi_1(u_1,u_2,u_3)\\ &\quad= 4\pi G_0(u_1,u_2,u_3)\\ &\qquad\times\int_0^\infty du_2'u_2'^2\int_{-1}^1 dx'\; t(u_1,\Pi_1(u_2,u_2',x'),\epsilon)\\ &\qquad\times\Big(\psi_1(\Pi_1(u_2',u_2,x'),u_2',u_3)\\ &\qquad+\frac{1}{2}\int_{-1}^1 dx\,\{\psi_1(\Pi_1(u_2',u_2,x'),\Pi_2(u_2',u_3,x),\Pi_3(u_2',u_3,x))\\ &\qquad+\psi_2(\Pi_1(u_2',u_2,x'),\Pi_4(u_2',u_3,x),\Pi_5(u_2',u_3,x))\}\Big),\\ &\psi_2(v_1,v_2,v_3)\\ &\quad= 4\pi G_0(v_1,v_2,v_3)\\ &\qquad\times\int_0^\infty dv_3'v_3'^2\,t(v_1,v_3',\epsilon^*)\\ &\qquad\times\Big(\int_{-1}^1 dx_3'\,\psi_1(v_3,\Pi_6(v_2,v_3',x_3'),\Pi_7(v_2,v_3',x))\\ &\qquad+\psi_2(v_3,v_2,v_3')\Big).\end{aligned} \tag{A4}$$