Decomposition and the Gross-Taylor string theory

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It was recently argued by Nguyen-Tanizaki-Ünsal that two-dimensional pure Yang-Mills theory is equivalent to (decomposes into) a disjoint union of (invertible) quantum field theories, known as universes. In this paper we compare this decomposition to the Gross-Taylor expansion of two-dimensional pure $SU(N)$ Yang-Mills theory in the large N limit as the string field theory of a sigma model. Specifically, we study the Gross-Taylor expansion of individual Nguyen-Tanizaki-Ünsal universes. These differ from the Gross-Taylor expansion of the full Yang-Mills theory in two ways: a restriction to single instanton degrees, and some additional contributions not present in the expansion of the full Yang-Mills theory. We propose to interpret the restriction to single instanton degrees as implying a constraint, namely that the Gross-Taylor string has a global (higher-form) symmetry with Noether current related to the worldsheet instanton number. We compare two-dimensional pure Maxwell theory as a prototype obeying such a constraint, and also discuss in that case an analogue of the Witten effect arising under two-dimensional theta angle rotation. We also propose a geometric interpretation of the additional terms, in the special case of Yang-Mills theories on two-spheres. In addition, also for the case of theories on two-spheres, we propose a reinterpretation of the terms in the Gross-Taylor expansion of the Nguyen-Tanizaki-Ünsal universes, replacing sigma models on branched covers by counting disjoint unions of stacky copies of the target Riemann surface, that makes the Nguyen-Tanizaki-Ünsal decomposition into invertible field theories more nearly manifest. As the Gross-Taylor string is a sigma model coupled to worldsheet gravity, we also briefly outline the tangentially-related topic of decomposition in two-dimensional theories coupled to gravity.

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1 Introduction

The papers [1–6] proposed that pure two-dimensional Yang-Mills theory could be understood as the string field theory of a string theory. This was demonstrated by first expanding the $SU(N)$ pure Yang-Mills partition function in the large N limit as a formal sum of correlation functions in two-dimensional untwisted S_n Dijkgraaf-Witten theories, summed over all n, an expansion we will refer to as the Gross-Taylor expansion. Those correlation functions were then interpreted combinatorially in the form of a sum over maps [1–3], and then later, using an interpretation of the Ω-points, in terms of branched covers [4–7], suggesting the interpretation as a string field theory of some sigma model. Specific proposals have been made for the corresponding sigma model, at least at the level of proofs of principle, see [4–6] for a sigma model localizing on holomorphic maps, and [8,9] for a sigma model localizing on harmonic maps.

In this paper we will reexamine these arguments in the context of decomposition [10] of two-dimensional pure Yang-Mills theories. Decomposition is the observation that d-dimensional quantum field theories with global $(d-1)$-form symmetries are equivalent to ("decompose into") disjoint unions of other theories, known in this context as universes. It was first described in [10] as part of an effort to resolve questions concerning the consistency of string propagation on stacks. Decomposition has been checked in a wide variety of contexts (including not only orbifolds and gauge theories but also e.g. open string theory and K theory [10]) via techniques including for example mirror symmetry [10,11], supersymmetric localization [12], and numerical/lattice computations [13], in not only two dimensional theories but also in three (see e.g. [14–16]) and four dimensions (see e.g. [17,18]). Its applications have included phases of gauged linear sigma models (see e.g. [19–32]), predictions for Gromov-Witten invariants (see e.g. [33–38]), IR limits of pure supersymmetric gauge theories and elliptic genera [39], adjoint QCD_2 [40], and anomalies in orbifolds [41]. (See also [42] for a recent relation to quivers with multiple components.) See e.g. [43–46] for reviews.

In particular, decomposition has been applied to argue that two-dimensional pure Yang-Mills theories are equivalent to disjoint unions of invertible field theories (meaning, trivial field theories with only a vacuum state) [18,47,48], with universes in one-to-one correspondence with irreducible representations of the gauge group. We will refer to those universes of the decomposition of two-dimensional pure Yang-Mills as Nguyen-Tanizaki-Ünsal universes.

We begin the paper with a short review of decomposition in two-dimensional pure Yang-Mills in section 2. The rest of this paper is organized into three main sections:

1. First, in section 3, we discuss the combinatorics of the Gross-Taylor expansion, deriving expressions for the expansion of the individual Nguyen-Tanizaki-Ünsal universes.

   Viewed as a sum of two-dimensional untwisted Dijkgraaf-Witten theory correlation functions, the result is extremely natural: one restricts to a value of n defined by the representation of $SU(N)$ defining the universe, corresponding to the symmetric group S_n, and then in correlation functions, one inserts projectors onto a single S_n Dijkgraaf-Witten universe, using the known fact that two-dimensional Dijkgraaf-Witten theory also decomposes. In other words, a single Nguyen-Tanizaki-Ünsal universe receives contributions only from a single Dijkgraaf-Witten theory universe in the Gross-Taylor expansion – the two decompositions intertwine naturally.

Interpreted in terms of sigma models, the resulting expressions for the separate universes are more subtle. They appear naively to describe (1) restrictions of the Gross-Taylor theory to maps (worldsheet instantons) of fixed degrees, plus (2) some additional contributions. Both require further explanation, to which we turn in the next section.

2. In section 4, we suggest interpretations of the two points raised above. We propose that the restrictions to maps of fixed degree be interpreted physically as a new constraint on the Gross-Taylor sigma model, that it possess a higher-form symmetry with Noether current coupling to the pullback of the Kähler form (which integrates to worldsheet instanton number, the covering map degree). We discuss the prototypical example of two-dimensional pure Maxwell theory, which has a one-form symmetry coupling to $U(1)$ bundle curvature.

   In hindsight, existence of such a symmetry in the Gross-Taylor string is expected from the yoga relating target-space and worldsheet symmetries, as we discuss, though the coupling to worldsheet instanton degree is not predicted by yoga alone.

   We also discuss the interpretation of the additional contributions mentioned above. Such additional contributions are typical in a decomposition, and cancel out when one sums over universes, as we review, so their existence is not a surprise, but they do require interpretation. In the special case of Yang-Mills theories on S^2, we propose a geometric interpretation involving stacky worldsheets.

3. Finally, in section 5, in the special case of Yang-Mills theories on S^2, we propose an alternative geometric interpretation of the Gross-Taylor expansion of the Nguyen-Tanizaki-Ünsal universes, not in terms of sigma models on branched covers of Σ_T, but instead in terms of some sort of counting problem, counting stacky copies of Σ_T, in line with the interpretation as invertible field theories.

In subsection 4.1.2, we also discuss an analogue of the Witten effect in two-dimensional pure Maxwell theory, in which universes of the decomposition are interchanged under theta angle rotation.

In appendix A we include some relevant group algebra identities, of use in the series expansions in section 3. In appendix B, we collect some pertinent basics of stacks, to assist in understanding stacky worldsheets.

Appendix C discusses examples of decomposition in two-dimensional theories coupled to worldsheet gravity. (This is not necessarily our prediction for the Gross-Taylor string, but is certainly a possibility, as the Gross-Taylor sigma model is coupled to worldsheet gravity.) Briefly, unlike a typical field-theoretic decomposition in which the universes are completely decoupled, after coupling to gravity, the universes have gravitational interactions (but no non-gravitational interactions).

Appendix D discusses possible alternative interpretations of the restriction on maps in the Gross-Taylor expansion of the Nguyen-Tanizaki-Ünsal universes, as different quantum field theories constructed to localize onto particular map degree sectors. These alternatives are not completely satisfactory, for reasons discussed there, so we do not advocate them, but we do include them for completeness.

2 Short review of decomposition in two-dimensional pure Yang-Mills

Consider two-dimensional pure Yang-Mills with gauge group G and action

$$S \;=\; \frac{1}{g_{YM}^2} \int_\Sigma \mathrm{Tr}\, F^2. \tag{2.1}$$

On a Riemann surface Σ of genus g with b boundaries, along which are associated group elements $U_1, \cdots, U_b$ the partition function has the form[1] [6, section 3.7], [49–57], [58, section 2],

$$Z(U_1, \cdots, U_b) \;=\; \sum_R (\dim R)^{2-2g-b} \exp\left[-g_{YM}^2 A C_2(R) \right] \; \chi_R(U_1) \cdots \chi_R(U_b). \tag{2.2}$$

where the sum is over irreducible representations R of the gauge group G. One can glue the Riemann surfaces along boundaries using (see e.g. [6, equ'ns (3.17)-(3.19)])

$$\int dU \, \chi_R(U) \chi_S(U^{-1}) \;=\; \delta_{RS}, \tag{2.3}$$

$$\int dU \, \chi_R(VU) \chi_S(U^{-1}W) \;=\; \delta_{RS} \frac{\chi_R(VW)}{\dim R}, \tag{2.4}$$

$$\int dU \, \chi_R(UVU^{-1}W) \;=\; \frac{\chi_R(V)\chi_R(W)}{\dim R}. \tag{2.5}$$

Recently, it was observed in [18] (for abelian cases) and [47,48] (for nonabelian cases) that pure two-dimensional gauge theories are equivalent to disjoint unions of physical theories; in other words, they "decompose," in the sense of [10,46]. Specifically, the component physical

[1]For gauge groups with $U(1)$ factors, we shall discuss how the θ angle appears in the exact expression in section 4.1.2.

theories ("universes") are in one-to-one correspondence with irreducible representations R, which is naturally reflected in the form of the partition function expressions above. For example, the universe associated to irreducible representation R has partition function

$$(\dim R)^{2-2g} \exp\left[-g_{YM}^2 A C_2(R)\right] \tag{2.6}$$

on a closed Riemann surface Σ of genus g, so that summing over universes reproduces the Yang-Mills partition function (2.2).. To be clear, decomposition in this context is the statement that the basis of irreducible representations diagonalizes correlation functions; every correlation function (e.g. (2.2)) can be written in terms of a sum of contributions from the constitutent universes, indexed by irreducible representations. This implies, but is very much stronger than, the statement that the partition function can be written as a certain sum.

Decomposition is also visible at the level of Hilbert spaces. The Hilbert space of two-dimensional pure Yang-Mills consists of the class functions on G, meaning functions which are invariant under conjugation. Such functions can be expanded in a Fourier series in characters χ of G, as [59, chapter II]

$$f(g) \;=\; \sum_R c_R \chi_R(g), \tag{2.7}$$

where the c_R are constants determined by G, and the sum is over irreducible representations of G. Certainly decomposition into universes indexed by irreducible representations is consistent with the structure of the Fourier series above More to the point, the universe associated to irreducible representation R has a one-dimensional Hilbert space, generated[2] by the character χ_R. A quantum field theory with a one-dimensional Hilbert space is known as an invertible field theory, so we see that the universes of the decomposition are each examples of invertible field theories (as defined by that property).

More generally, unitary two-dimensional topological field theories (with semisimple local operator algebras) decompose into a disjoint union of invertible field theories [40,60–62]. The partition functions of such theories are determined by an Euler number counterterm and an area counterterm, so that on a worldsheet Σ the partition function is just the exponential of the integral of those counterterms, and has the form

$$Z(\Sigma) \;=\; (\phi_1)^{\chi(\Sigma)} \exp\left(-\phi_2 \mathrm{Area}\right), \tag{2.8}$$

universally, for some constants $\phi_{1,2}$. We can see this structure in the partition function in pure two-dimensional Yang-Mills, where the contribution from any fixed universe / irreducible representation R is

$$(\dim R)^{\chi(\Sigma)} \exp\left(-C_2(R) \mathrm{Area}\right). \tag{2.9}$$

[2]For example, the QFT admits a set of orthogonal projectors, which in bra-ket notation have the form $|R\rangle\langle R|$ for R an irreducible representation.

In the language above, for any irreducible representation R, $\phi_1 = \dim R$ and $\phi_2 = C_2(R)$.

In passing, there exist generalizations of two-dimensional pure Yang-Mills which are also area-preserving-diffeomorphism invariant and exactly soluble, see [55,63,64]. Their classical actions are of the form [63, equ'n (5)]

$$\int_{\Sigma} \mathrm{tr}\, \left(i\phi F \, - \Phi(\phi) \right), \tag{2.10}$$

where ϕ is an auxiliary Lie-algebra-valued scalar, and Φ a function of the form [63, equ'n (7)]

$$\Phi(\phi) \; = \; \sum_{\{k_i\}} a_{\{k_i\}} \prod_i \mathrm{tr}\, \left(\phi^i\right)^{k_i}. \tag{2.11}$$

This reduces to pure Yang-Mills in the special case that $\Phi(\phi) \propto \mathrm{tr}\, \phi^2$ (see e.g. [56, section 2.1]). Their partition functions are of the form [63, equ'n (10)]

$$Z \; = \; \sum_R (\dim R)^{\chi(\Sigma)} \exp\left(-A \Lambda(R)\right), \tag{2.12}$$

for

$$\Lambda(R) \; = \; \sum_{\{k_i\}} a_{\{k_i\}} C_{\{k_1 - 1 + k_2 - 2 + k_3 - 3 + \cdots\}}(R). \tag{2.13}$$

At least naively, the arguments of [47,48] appear to apply, hence it is natural to conjecture [65] that these theories decompose, into universes indexed by irreducible representations R.

3 Revisiting the Gross-Taylor argument

In this section, we review the Gross-Taylor asymptotic series expansion of the partition function of both pure two-dimensional $SU(N)$ Yang-Mills theory, as well as partition functions of Nguyen-Tanizaki-Ünsal universes therein, in the large N limit, on a genus p Riemann surface Σ_T. We also review the interpretation of the terms in that series expansion in terms of two-dimensional Dijkgraaf-Witten theory, and in terms of a sigma model (the 'Gross-Taylor sigma model') mapping a branched cover Σ_W to Σ_T.

Interpreted as an expansion in Dijkgraaf-Witten theories, the series expansion of the Nguyen-Tanizaki-Ünsal universes is extremely natural, as we explain. However, we encounter two puzzles when we interpret the expension of the Nguyen-Tanizaki-Ünsal universes in terms of maps from a branched cover:

- the expansion of a fixed universe involves maps of a fixed degree, (so that summing over universes recovers maps of all degrees,) and

- there are additional[3] contributions in the large N limit which cannot be ascribed to maps from smooth branched covering surfaces Σ_W.

We will propose resolutions of those puzzles in section 4.

Also, in order to help clarify the interpretation, we keep track of the leading area (A) dependence in the large N limit. (The careful reader will recall that in the full Yang-Mills partition function, summed over all universes, there is a phase transition as a function of area [66–70].)

Similar expansions have also been studied for gauge groups $SO(N)$ and $Sp(N)$, see e.g. [71–75]. We have not worked through them carefully, but we expect that their analyses should be similar to what we present for $SU(N)$ theories.

3.1 Series expansion of partition functions

In this section we will review the pertinent large N asymptotic series expansion of the partition functions of both the full two-dimensional pure $SU(N)$ Yang-Mills theory, as well as those of the Nguyen-Tanizaki-Ünsal universes. Along the way, we will develop some identities that will be used in both.

First, it will be useful to rescale certain normalizations and write the partition function (2.2) of two-dimensional pure $SU(N)$ Yang-Mills in the form [1, 49, 53], [6, equ'n (3.20)] [54, equ'n (2.51)]

$$Z \;=\; \sum_R (\dim R)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R) \right), \tag{3.1}$$

on a closed genus-p worldsheet Σ_T of area A.

The references [1–7] rewrite the two-dimensional pure $SU(N)$ Yang-Mills partition function in the large N limit in a form that looks like a string field theory, namely as a sum over other worldsheets Σ_W, suggesting the existence of a two-dimensional sigma model of maps $\Sigma_W \to \Sigma_T$, for which an expression was given in [4–6]. In this expansion, $1/N$ plays the role of string dilaton.

As part of that, to get the correct[4] $1/N$ asymptotics, it was argued in [2–7] that one should replace the sum over irreducible representations by a sum over 'coupled' representations. For

[3]Additional contributions to the series expansions of individual universes which cancel out when universes are summed over, as these do, are common in decomposition, as we review in detail in section 4.2. The puzzle here is the interpretation of the extra contributions, not their existence per se.

[4]See also [76] which discussed a reformulation of the non-chiral expansion in terms of purely holomorphic maps plus some line defects.

the moment, to try to make the analysis more clear, we will set aside the use of coupled representations, and formally derive an expansion using just the naive sum. We will rederive the expansion including coupled representations shortly.

Setting aside coupled representations for the moment, the first step is to use Schur-Weyl duality to show that for $SU(N)$ representations [6, equ'n (6.5)]

$$\dim R(Y) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{r(Y)}(\sigma)\, N^{\sum_i k_i(\sigma)}, \tag{3.2}$$

$$= \frac{N^n}{n!} \chi_{r(Y)}(\Omega_n), \tag{3.3}$$

where Y is the Young tableau associated with representation R of $SU(N)$, n is the number of boxes in Y, $r(Y)$ is the representation of the symmetric group S_n defined by the same Young tableau Y, and [6, equ'n (6.5)]

$$\Omega_n = \sum_{\sigma \in S_n} N^{K_\sigma - n} \sigma, \tag{3.4}$$

for K_σ the total number of cycles in the cycle decomposition of σ,

$$K_\sigma = \sum_i k_i(\sigma). \tag{3.5}$$

As an aside, later it may be helpful to note that[5] [4, equ'n (5.2)]

$$\Omega_n^{-1} = 1 + \sum_{k=1}^{\infty} \sum_{v_1, \cdots, v_k \in S_n}{}' \left(\frac{1}{N} \right)^{\sum_j (n - K_{v_j})} (v_1 \cdots v_k)\, (-)^k, \tag{3.6}$$

where the primed sum means that the $v_i \neq 1$.

More generally [6, equ'n (6.6)],

$$(\dim R(Y))^m = \left(\frac{N^n \dim r(Y)}{|S_n|} \right)^m \frac{\chi_{r(Y)}(\Omega_n^m)}{\dim r(Y)}. \tag{3.7}$$

In a moment, we will also need the identity

$$\sum_{s,t \in G} \chi_r(s t s^{-1} t^{-1}) = \frac{|G|}{\dim r} \sum_{s \in G} \chi_r(s) \chi_r(s^{-1}), \tag{3.8}$$

$$= \left(\frac{|G|}{\dim r} \right)^2 \chi_r(1) = \left(\frac{|G|}{\dim r} \right)^2 \dim r, \tag{3.9}$$

which follows from the orthogonality relations (A.1), (A.2), (A.3).

[5] From [6, section 6.1.2], Ω_n is invertible for $N > n$, which we will always assume.

With this in mind, we can now expand the R contribution to the pure $SU(N)$ Yang-Mills partition function (3.1), which we will write for any positive integer exponent m:

$$\begin{aligned}
&(\dim R(Y))^m \\
&= \left(\frac{N^n \dim r(Y)}{|S_n|} \right)^m \frac{\chi_{r(Y)}((\Omega_n)^m)}{\dim r(Y)} \text{ using (3.7)}, \qquad (3.10)\\
&= N^{nm} \left(\frac{\dim r(Y)}{n!} \right)^{m+2p} \left[\prod_{i=1}^{p} \sum_{s_i, t_i \in S_n} \left(\frac{\chi_{r(Y)}(s_i t_i s_i^{-1} t_i^{-1})}{\dim r(Y)} \right) \right] \frac{\chi_{r(Y)}((\Omega_n)^m)}{\dim r(Y)}, \\
&\quad \text{using (3.9)}, \\
&= N^{nm} \left(\frac{\dim r(Y)}{n!} \right)^{m+2p} \left[\sum_{s_1, t_1 \cdots \in S_n} \prod_{i=1}^{p} \frac{\chi_{r(Y)}(s_i t_i s_i^{-1} t_i^{-1})}{\dim r(Y)} \right] \frac{\chi_{r(Y)}((\Omega_n)^m)}{\dim r(Y)}, \\
&= N^{nm} \left(\frac{\dim r(Y)}{n!} \right)^{m+2p} \sum_{s_1, t_1 \cdots \in S_n} \frac{\chi_r \left((\Omega_n)^m \prod_{i=1}^{p} s_i t_i s_i^{-1} t_i^{-1} \right)}{\dim r(Y)} \qquad (3.11)
\end{aligned}$$

using (A.5) and the fact that Ω_n is central.

Next, we include the finite area corrections.

For $SU(N)$ representations [6, equ'n (6.11)],

$$C_2(R(Y)) \;=\; nN \;+\; 2\frac{\chi_{r(Y)}(T_2)}{\dim r(Y)} \;-\; \frac{n^2}{N}, \qquad (3.12)$$

so we can write

$$\begin{aligned}
&(\dim R(Y))^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R(Y)) \right) \\
&= N^{n(2-2p)} \left(\frac{\dim r(Y)}{n!} \right)^2 \sum_{s_1, t_1 \cdots \in S_n} \frac{\chi_r \left((\Omega_n)^{2-2p} \prod_{i=1}^{p} s_i t_i s_i^{-1} t_i^{-1} \right)}{\dim r(Y)} \\
&\qquad \cdot \exp\left(-g_{YM}^2 \frac{A}{2} n \;+\; g_{YM}^2 A \frac{1}{N} \frac{\chi_{r(Y)}(T_2)}{\dim r(Y)} \;-\; g_{YM}^2 \frac{A}{2} \frac{n^2}{N^2} \right). \qquad (3.13)
\end{aligned}$$

Let us take a moment to interpret the expression above. The reader will note that, to leading order in $1/N$, the second two terms in the exponential can[6] be dropped. The area dependence is precisely what one would expect from sigma model contributions if one replaces $g_{YM}^2/2$ with $1/\alpha'_{GT}$, which we will do henceforward, following e.g. [77, section 2]. (As a consistency test, note that α'_{GT} has units of area, so A/α'_{GT} is unitless, as needed for these expressions to be consistent.)

[6] In other treatments, they are retained in a power series expansion. For our purposes, it will suffice to only keep the leading order area dependence.

So, with that interpretation, we can write the partition function of the universe associated to R as

$$\begin{aligned}
& (\dim R(Y))^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R(Y)) \right) \\
= \;& N^{n(2-2p)} \left(\frac{\dim r(Y)}{n!} \right)^2 \sum_{s_i, t_i \cdots \in S_n} \frac{\chi_r\left((\Omega_n)^{2-2p} \prod_{i=1}^p s_i t_i s_i^{-1} t_i^{-1} \right)}{\dim r(Y)} \\
& \cdot \exp\left(-\frac{A}{\alpha'_{GT}} n \right) \; + \; \text{subleading.}
\end{aligned} \tag{3.14}$$

Using equation (A.15), we can write this as

$$\begin{aligned}
& (\dim R(Y))^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R(Y)) \right) \\
= \;& N^{n(2-2p)} \left(\frac{\dim r(Y)}{n!} \right) \sum_{s_i, t_i \cdots \in S_n} \frac{\delta\left((\Omega_n)^{2-2p} \prod_{i=1}^p s_i t_i s_i^{-1} t_i^{-1} P_r \right)}{\dim r(Y)} \\
& \cdot \exp\left(-\frac{A}{\alpha'_{GT}} n \right) \; + \; \text{subleading,}
\end{aligned} \tag{3.15}$$

where P_r is the projector defined in appendix A.2.

In this language, the zero-area limit is an $\alpha'_{GT} \to \infty$ limit. The result is reminiscent of results on high energy symmetries [78–80] which argue that in that limit, sums over Riemann surfaces are dominated by particular surfaces (points on the moduli space of curves).

More to the point, we see that the contributions to a single universe differ from contributions to the Gross-Taylor expansion of the full two-dimensional Yang-Mills theory in two ways:

1. First, n is fixed, meaning contributions are from maps of a fixed degree (equal to the number of boxes in the Young tableau). Restricting a sigma model to maps of a single degree is equivalent to restricting a gauge theory to instantons of a single degree, which ordinarily would not be consistent. We will return to this in section 4, where we will argue that this implies a new constraint on the Gross-Taylor sigma model.

2. Second, the δ function constraint on the group combinatorics now has an insertion of P_r. We shall see later that this implies that there exist additional contributions, not interpretable in terms of smooth branched covers. We will return to this also in section 4.

The next key observation, detailed in [2, section 2], [6, section 6.3], is that to get the correct $1/N$ asymptotics, one should replace the sum over irreducible representations R

by a sum over 'coupled representations' $S\overline{T}$. As explained in the references, this resolves technical difficulties in the summation involving contributions from representations R such that $C_2(R)/N$ is $\mathcal{O}(1)$ in the $1/N$ expansion, but whose contribution is expanded in the $1/N$ expansion across infinitely many terms. To that end, we will next repeat the analysis above using coupled representations. The results will have the same form as obtained above.

Such coupled representations are described in detail in [2, section 2], [6, section 6.3]. Briefly, following [2, section 2], if we let L denote the length of the first row of the Young diagram for T, then the Young diagram for $S\overline{T}$ is defined as follows:

1. start with an $N \times L$ rectangle,
2. subtract the Young diagram from T from the bottom of the rectangle,
3. add the Young diagram for S to the top right of the rectangle.

More formally, if c_i, $\tilde{c}_i$ denote the height of the ith column of S, T, respectively, then the height of the ith column of the coupled representation is [2, equ'n (2.6)]

$$\left\{ \begin{array}{cc} N - \tilde{c}_{L+1-i} & i \leq L, \\ c_{i-L} & i > L. \end{array} \right. \tag{3.16}$$

Some pertinent properties of coupled representations include [2, equ'n (2.7)]

$$C_2(R\overline{S}) \;=\; C_2(R) + C_2(S) \;+\; 2\frac{n_R n_S}{N}, \tag{3.17}$$

where n_R, n_S are the numbers of boxes in the Young diagrams for R, S, respectively, and [2, equ'n (2.8)]

$$\dim R\overline{S} \;=\; (\dim R)(\dim S)\left(1 \;+\; \mathcal{O}(1/N^2)\right). \tag{3.18}$$

Furthermore, as should be clear from the construction above, in the large N limit a single coupled representation $R\overline{S}$ uniquely determines both R and S separately. As a result, so long as we are working at large N, we can replace the original sum over irreducible representations with a sum over R and S separately.

One can then extract an expression for $\dim R\overline{S}$ in the same fashion as we did previously for $\dim R$. As details of e.g. projectors will be important for our later analysis, we repeat the key steps here.

From [6, equ'n (6.17)], we can write

$$\dim R\overline{S} \;=\; \frac{N^{n_+ + n_-}}{n_+!\, n_-!} \chi_{r \otimes s^*}\left(\Omega_{n_+ n_-}\right). \tag{3.19}$$

where n_+ is the number of boxes in the Young diagram for R, and n_- is the number of boxes in the Young diagram for S, and r, s are representations of S_{n_+}, S_{n_-} determined by R, S. More generally [81],

$$(\dim R\overline{S})^m \;=\; \left(\frac{N^{n_+ + n_-} (\dim r)(\dim s)}{n_+!\, n_-!} \right)^m \frac{\chi_{r \otimes s^*}\left(\Omega^m_{n_+,n_-}\right)}{(\dim r)(\dim s)}, \tag{3.20}$$

and then, using (3.9),

$$\begin{aligned}(\dim R\overline{S})^m \;=\;& N^{(n_+ + n_-)m} \left(\frac{\dim r}{n_+!}\right)^{m+2p} \left(\frac{\dim s}{n_-!}\right)^{m+2p} \\ & \cdot \left[\prod_{i=1}^p \sum_{s_i^+, t_i^+ \in S_{n_+}} \left(\frac{\chi_r([s_i^+, t_i^+])}{\dim r} \right) \right] \left[\prod_{i=1}^p \sum_{s_i^-, t_i^- \in S_{n_-}} \left(\frac{\chi_s([s_i^-, t_i^-])}{\dim s} \right) \right] \\ & \cdot \frac{\chi_{r \otimes s^*}\left(\Omega^m_{n_+,n_-}\right)}{(\dim r)(\dim s)},\end{aligned} \tag{3.21}$$

Then, since Ω_{n_+,n_-} is central in $S_{n_+} \otimes S_{n_-}$ [81], using (A.5),

$$\begin{aligned}(\dim R\overline{S})^m \;=\;& N^{(n_+ + n_-)m} \left(\frac{\dim r}{n_+!}\right)^{m+2p} \left(\frac{\dim s}{n_-!}\right)^{m+2p} \frac{1}{(\dim r)(\dim s)} \\ & \cdot \sum_{s_i^\pm, t_i^\pm \in S_{n_\pm}} \chi_{r \otimes s^*} \left((\Omega_{n_+,n_-})^m \prod_{i=1}^p [s_i^+, t_i^+] \otimes [s_i^-, t_i^-] \right).\end{aligned} \tag{3.22}$$

From its definition in [4, equ'n (10.20)],

$$\Omega_{n_+,n_-} \;=\; \Omega_{n_+} \otimes \Omega_{n_-} \;+\; \text{subleading in } 1/N, \tag{3.23}$$

so, to leading order in $1/N$, we have

$$\begin{aligned}(\dim R\overline{S})^m \;=\;& N^{(n_+ + n_-)m} \left(\frac{\dim r}{n_+!}\right)^{m+2p} \left(\frac{\dim s}{n_-!}\right)^{m+2p} \frac{1}{(\dim r)(\dim s)} \\ & \cdot \sum_{s_i^\pm, t_i^\pm \in S_{n_\pm}} \chi_r \left((\Omega_{n_+})^m \prod_{i=1}^p [s_i^+, t_i^+] \right) \chi_{s^*} \left((\Omega_{n_-})^m \prod_{i=1}^p [s_i^-, t_i^-] \right) \\ & + \text{subleading}.\end{aligned} \tag{3.24}$$

We are now ready to describe the series expansion of both the partition function of a single Nguyen-Tanizaki-Ünsal universe on a Riemann surface Σ_T of genus p, as well as the pure Yang-Mills partition function.

We begin with the series expansion of a Nguyen-Tanizaki-Ünsal universe. We define

$$\begin{aligned} Z_R^+(A,p,N) &= N^{n_+(2-2p)} \left(\frac{\dim r}{n_+!} \right)^2 \sum_{s_i^+, t_i^+ \in S_{n_+}} \frac{\chi_r\left((\Omega_{n_+})^{2-2p} \prod_{i=1}^p [s_i^+, t_i^+] \right)}{\dim r} \\ & \qquad \cdot \exp\left(-\frac{A}{\alpha'_{GT}} n_+ \right), \qquad (3.25) \\ Z_S^-(A,p,N) &= N^{n_-(2-2p)} \left(\frac{\dim s}{n_-!} \right)^2 \sum_{s_i^-, t_i^- \in S_{n_-}} \frac{\chi_{s^*}\left((\Omega_{n_-})^{2-2p} \prod_{i=1}^p [s_i^-, t_i^-] \right)}{\dim s} \\ & \qquad \cdot \exp\left(-\frac{A}{\alpha'_{GT}} n_- \right), \qquad (3.26) \end{aligned}$$

where n_+, n_- are the number of boxes in the Young diagrams for R, S, respectively, then we see that the partition function of a Nguyen-Tanizaki-Ünsal universe associated to the coupled representation $R\overline{S}$ is

$$(\dim R\overline{S})^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R\overline{S}) \right) = Z_R^+(A,p,N)\, Z_S^-(A,p,N) + \text{subleading}. \qquad (3.27)$$

(We are, again, using the fact that in the large N limit, a coupled representation $R\overline{S}$ uniquely determines both R and S separately. We are also, again, only keeping the leading area dependence in the large N limit, absorbing the rest into subleading terms.)

The full nonchiral partition function of the zero-area limit of pure Yang-Mills is then giving by summing over coupled representations:

$$\begin{aligned} Z_{\rm YM}(\Sigma_T) &= \sum_{R\overline{S}} (\dim R\overline{S})^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R\overline{S}) \right), \qquad (3.28) \\ &= \left(\sum_R Z_R^+(A,p,N) \right) \left(\sum_S Z_S^-(A,p,N) \right) \\ & \quad + \text{subleading}. \qquad (3.29) \end{aligned}$$

(The reader may compare e.g. [4, equ'n (10.18)]). We have implicitly used the fact that in the large N limit, a coupled representation $R\overline{S}$ uniquely determines R and S separately.

In passing, expressions for the subleading finite N contributions are given in [58, equ'n (2.4)], [82, section 3], [76, 77, 83, 84]. We will not study the subleading corrections in this paper.

As before, we simplify expression (3.25) for the chiral component of the partition function expansion associated to an irreducible representation R. First, using identity (A.15), we can replace the character with a delta function with a projector:

$$
\begin{aligned}
Z_R^+(A,p,N) \;&=\; N^{n_+(2-2p)} \left(\frac{\dim r}{n_+!}\right) \sum_{s_i,t_i\cdots\in S_{n_+}} \frac{\delta\left((\Omega_{n_+})^{2-2p}\left(\prod_{i=1}^p [s_i^+,t_i^+]\right) P_r\right)}{\dim r} \\
&\qquad \cdot \exp\left(-\frac{A}{\alpha'_{GT}} n_+\right), \qquad (3.30) \\
Z_S^-(A,p,N) \;&=\; N^{n_-(2-2p)} \left(\frac{\dim s}{n_-!}\right) \sum_{s_i^-,t_i^-\in S_{n_-}} \frac{\delta\left((\Omega_{n_-})^{2-2p}\prod_{i=1}^p [s_i^-,t_i^-] P_{s^*}\right)}{\dim s} \\
&\qquad \cdot \exp\left(-\frac{A}{\alpha'_{GT}} n_-\right), \qquad (3.31)
\end{aligned}
$$

where n_+ is the number of boxes in the Young diagram for R, n_- is the number of boxes in the Young diagram for S, and P_r, P_{s^*} are the projectors (A.8).

It will also be useful later to expand out the powers of Ω_n. Following [4, section 5.1] and using (3.4), we find

$$
\begin{aligned}
Z_R^+(A,p,N) \;&=\; N^{n(2-2p)} \sum_{s_i,t_i\in S_n} \sum_{L=0}^{\infty} \sum_{v_1,\cdots,v_L\in S_n}{}' N^{\sum_j (K_{v_j}-n)} \frac{d(2-2p,L)}{n!} \qquad (3.32) \\
&\qquad \cdot \delta\left(v_1\cdots v_L \left(\prod_{i=1}^p [s_i,t_i]\right) P_r\right) \exp\left(-\frac{A}{\alpha'_{GT}} n\right),
\end{aligned}
$$

where the primed sum means that the $v_i \neq 1$, and

$$
(1+x)^m \;=\; \sum_{L=0}^{\infty} d(m,L)\, x^L. \qquad (3.33)
$$

(As a consistency check, the reader may compare [4, equ'n (5.3)], which also sums over irreducible representations.) A similar expression, also involving an explicit projector, arises in [82, section 3]. We omit $Z_S^-(A,p,N)$ for simplicity, as the expression is very similar.

So far, we have given an explicit expression (3.32) for $Z_R^+(A,p,N)$, the chiral component of the partition function for a Nguyen-Tanizaki-Ünsal universe associated to a coupled representation $R\overline{S}$. It will also be useful to write down the chiral component of the full two-dimensional Yang-Mills partition function, which is given by summing over contributions from the various irreducible representations:

$$
\begin{aligned}
Z^{+}(A,p,N) &= \sum_R Z_R^{+}(A,p,N), && (3.34)\\
&= \sum_{n=0}^{\infty} N^{n(2-2p)} \sum_{s_i,t_i\in S_n} \sum_{L=0}^{\infty} {\sum_{v_1,\cdots,v_L\in S_n}}' N^{\sum_j (K_{v_j}-n)} \frac{d(2-2p,L)}{n!} \\
&\qquad \cdot \sum_r \delta\left(v_1\cdots v_L \left(\prod_{i=1}^{p} [s_i,t_i] \right) P_r \right) && (3.35)\\
&\qquad \cdot \exp\left(-\frac{A}{\alpha'_{GT}} n \right), \\
&= \sum_{n=0}^{\infty} N^{n(2-2p)} \sum_{s_i,t_i\in S_n} \sum_{L=0}^{\infty} {\sum_{v_1,\cdots,v_L\in S_n}}' N^{\sum_j (K_{v_j}-n)} \frac{d(2-2p,L)}{n!} \\
&\qquad \cdot \delta\left(v_1\cdots v_L \left(\prod_{i=1}^{p} [s_i,t_i] \right) \right) && (3.36)\\
&\qquad \cdot \exp\left(-\frac{A}{\alpha'_{GT}} n \right).
\end{aligned}
$$

Specifically, summing over the irreducible representations R generates a sum over degrees n, as well as a sum over the S_n representations r, removing the projector P_r from the delta function. (Both of these differences will be important in the subsequent Nguyen-Tanizaki-Ünsal analysis.) The result is, in effect, a weighted sum over two-dimensional Dijkgraaf-Witten theories for the symmetric group S_n for all values of n.

3.2 The Dijkgraaf-Witten interpretation

One interpretation of the terms above is in terms of the correlation functions in a series of two-dimensional Dijkgraaf-Witten theories[7] [85]. Briefly, this theory describes an orbifold of a point. On a genus p Riemann surface, the partition function of (untwisted[8]) Dijkgraaf-Witten theory with orbifold group G is

$$
Z_{DW,p} \propto \sum_{s_i,t_i\in G} \delta\left(\prod_{i=1}^{p} [s_i,t_i] \right), \qquad (3.37)
$$

[7]Two-dimensional Dijkgraaf-Witten theories have been described in many places, see for a few examples [40, appendix C.1], [86, appendix C], [87–92].

[8]The 'twist' of Dijkgraaf-Witten theory is a choice of discrete torsion in the orbifold. Only untwisted Dijkgraaf-Witten theories (without discrete torsion) will be relevant for this paper.

which the reader will already recognize is a component of the expressions for the Gross-Taylor series expansion. The operators of Dijkgraaf-Witten theory are twist fields, which can be expressed as commuting linear combinations[9] of group elements (technically, elements of the center of the group algebra). A correlation function of twist fields can therefore be expressed formally as a linear combination of a correlation function of products of group elements $v_1, \cdots, v_L \in G$, which on a Riemann surface of genus p is given by

$$\langle v_1 \cdots v_L \rangle_{DW,p} \propto \sum_{s_i, t_i \in G} \delta \left(v_1 \cdots v_L \left(\prod_{i=1}^{p} [s_i, t_i] \right) \right). \tag{3.38}$$

The chiral partition function $Z^+(A, p, N)$ given in equation (3.36) is clearly a linear combination of such correlation functions, for $G = S_n$ (the symmetric group), summed over n.

Now, any two-dimensional gauge theory in which a subgroup of the gauge group acts trivially, has a global 1-form symmetry and hence decomposes, into universes indexed by the irreducible representations of the trivially-acting subgroup. Indeed, this is at the heart of the Nguyen-Tanizaki-Ünsal decomposition of two-dimensional Yang-Mills theory. For two-dimensional orbifolds, decomposition has been discussed in for example [10, 41, 46, 93–97]. Two-dimensional Dijkgraaf-Witten theory, in which the entire orbifold group acts trivially, is simply a special case[10] of the orbifolds considered in the references just cited, and the picture is particularly simple. In terms of the state space, for example, the projectors onto states in each universe are the projectors P_r (associated to irreducible representations r) described in appendix A.2. In fact, when the entire group acts trivially, it is a standard mathematics result that there is a (noncanonical) one-to-one correspondence between the possible twist fields and the projectors P_r, as both form bases of the center of the group algebra $\mathbb{C}[G]$ for orbifold group G, see for example [98, section 6.3], [99–107]. (In fact, those references describe a more general case, that of twisted group algebras, which in an orbifold corresponds to adding discrete torsion. Here we only consider the untwisted case.)

A correlation function within a particular universe (corresponding to irreducible representation r) is obtained by inserting a projection operator P_r, the same projector appearing earlier:

$$\langle v_1 \cdots v_L \rangle_{DW,p,r} \propto \sum_{s_i, t_i \in G} \delta \left(v_1 \cdots v_L \left(\prod_{i=1}^{p} [s_i, t_i] \right) P_r \right). \tag{3.39}$$

Comparing equation (3.32), we see that the chiral component $Z_R^+(A, p, N)$ is precisely a linear combination of Dijkgraaf-Witten correlation functions restricted to universe r, where r is a representation of S_n associated to the same Young diagram as the representation R of $SU(N)$.

[9] In the group algebra $\mathbb{C}[G]$.

[10] We can also see this from another perspective. Unitary two-dimensional topological field theories also decompose, into a collection of invertible field theories, see e.g. [40, 60–62], and Dijkgraaf-Witten theory is also a special case in that sense. For our purposes, the orbifold perspective is more relevant.

In brief, we see that the Gross-Taylor expansion of a Nguyen-Tanizaki-Ünsal universe involves first restricting to n (equal to the number of boxes in the Young diagram for R), then restricting to one universe in the decomposition of Dijkgraaf-Witten theory for S_n. In other words, the Gross-Taylor expansion of a single Nguyen-Tanizaki-Ünsal universe involves a single universe of the decomposition of two-dimensional untwisted Dijkgraaf-Witten theory. The two decompositions are therefore closely and naturally linked.

In the next section, we shall see that the interpretation of the components in terms of branched covers is more subtle.

3.3 The branched cover interpretation

We computed the series expansions of a Nguyen-Tanizaki-Ünsal universe associated to coupled representation $R\overline{S}$ in equation (3.27), and reviewed the series expansion of the full two-dimensional Yang-Mills partition function (3.29), obtained by summing over Nguyen-Tanizaki-Ünsal universe partition functions. Results for a Nguyen-Tanizaki-Ünsal universe were written in terms of the chiral partition function $Z_R^+(A,p,N)$, and results for the full two-dimensional Yang-Mills partition function were written in terms of the full chiral partition function

$$Z^+(A,p,N) \;=\; \sum_R Z_R^+(A,p,N). \tag{3.40}$$

In this subsection we will review how to interpreted those chiral components of partition functions as a sum over branched covers $\Sigma_W \to \Sigma_T$, following [1–6]. (This description is standard, but has not appeared recently in the literature, and as we will be manipulating it extensively, we think it useful to review in detail.) Later, in section 5, we will reinterpret the terms above as a sum over disjoint unions $\hat{\Sigma}_W$ of stacky copies of Σ_T.

The construction of the branched n-cover Σ_W is described systematically in e.g. [108,109], [6, section 5], which we briefly review here. The idea is that the elements $v_1, \cdots, v_L \in S_n$ define monodromies about L branch points in the base curve Σ_T. The delta function is nonzero only when those monodromies are consistent with the existence of a smooth branched n-cover Σ_W of a genus p Riemann surface (specifically, Σ_T).

More systematically, let B be the branch locus on Σ_T (the locations of insertions $v_1, \cdots, v_L \in S_n$), and define $X = \Sigma_T - B$, then use the fact that there is a one-to-one correspondence between conjugacy classes of subgroups of $\pi_1(X)$ and equivalence classes of topological coverings of X (see for example [110, theorem V.6.6, theorem V.10.2]), which are glued according to the data of the homomorphism to build the branched n-cover Σ_W.

To make this more clear, let us consider an example. Specifically, consider $\Sigma_T = \mathbb{P}^1$ with two insertions at positions denoted A, B, and let $p \in \mathbb{P}^1$, as illustrated in the figure below.

Shown on the left is a schematic illustration of $\mathbb{P}^1$ with the two points A, B, and the basepoint p for paths. On the right is the same illustration with two nonintersecting paths from p marked.

Let the monodromies about the two points be denoted v_A, v_B. Suppose that $n = 3$, so that $v_A, v_B, v_C \in S_3$, and take

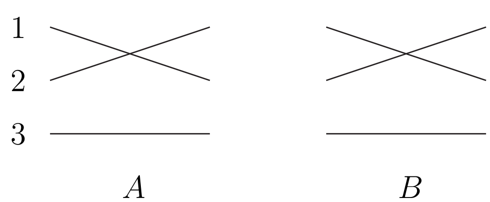
$$v_A \;=\; (12)(3) \;=\; v_B. \tag{3.41}$$

It is straightforward to check that the product

$$v_A v_B \;=\; 1, \tag{3.42}$$

so the delta function is nonzero.

In terms of branched 3-covers, the sheets of the cover in neighborhoods of the two points take the form



Locally near A and B, two of the sheets collide, but the third remains disjoint. The resulting branched 3-cover Σ_W is a disjoint union of one branched double cover of $\mathbb{P}^1$ (itself $\mathbb{P}^1$), formed from the first two sheets, and another $\mathbb{P}^1$ (the third sheet), so

$$\Sigma_W \;=\; \mathbb{P}^1 \coprod \mathbb{P}^1, \tag{3.43}$$

and $\chi(\Sigma_W) = 4$.

Next, we will consider Euler characteristics, and review how in general, the Euler characteristic of the branched cover Σ_W always matches that of the disjoint union $\hat{\Sigma}_W$, and both also match the power of N in the corresponding partition function term.

Consider the smooth branched cover Σ_W. The powers of N in the expansion above are

$$n(2 - 2p) \;+\; \sum_j (K_{v_j} - n), \tag{3.44}$$

and using the Riemann-Hurwitz formula:

$$\chi(\Sigma_W) \;=\; n\chi(\Sigma_T) \;-\; \sum_{i=1}^{L} (n - K_{v_i})\,, \tag{3.45}$$

where L is the number of branch points and K_{v_i} is the number of cycles in $v_i \in S_n$ corresponding to a branch point, we see that

$$n(2-2p) \;+\; \sum_j (K_{v_j} - n) \;=\; \chi(\Sigma_W), \tag{3.46}$$

and so the terms in $Z^+(0,p,N)$ are weighted by

$$N^{\chi(\Sigma_W)}, \tag{3.47}$$

as expected.

Rewriting in this language, we have that

$$\begin{aligned} Z^+(A,p,N) \;\;&=\;\; \sum_{n=0}^{\infty} \sum_{s_i,t_i \in S_n} \sum_{L=0}^{\infty} \sum_{v_1,\cdots,v_L \in S_n}{}' N^{\chi(\Sigma_W)} \, \frac{d(2-2p,L)}{n!} \\ &\qquad \cdot \delta\left(v_1 \cdots v_L \left(\prod_{i=1}^{p} [s_i,t_i] \right) \right) \exp\left(-\frac{A}{\alpha'_{GT}} n \right), \end{aligned} \tag{3.48}$$

where Σ_W is a smooth n-fold cover of the genus p base curve Σ_T, branched over L points.

In that language, the factor

$$\frac{d(2-2p,L)}{n!} \tag{3.49}$$

is interpreted in [4, 6] in terms of the orbifold Euler characteristic of the (Hurwitz) moduli space of maps $\Sigma_W \to \Sigma_T$. We refer the reader to thsoe references for further details, which are beyond the scope of this short overview.

Finally, the factor

$$\exp\left(-\frac{A}{\alpha'_{GT}} n \right) \tag{3.50}$$

is the weighting one expects in a sigma model describing maps of degree n. Its presence merely serves to confirm the interpretation.

To summarize, we have reviewed how the partition function of two-dimensional pure $SU(N)$ Yang-Mills on a Riemann surface Σ_T can be rewritten in the form of a sum (3.48) over smooth branched covers $\Sigma_W \to \Sigma_T$, which as noted in [1–3] is very suggestive of an interpretation as the string field theory of a sigma model with two-dimensional target space Σ_T.

Now, let us briefly compare to the corresponding chiral partition function of an Nguyen-Tanizaki-Ünsal universe (3.32), explicitly

$$Z_R^+(A,p,N) = N^{n(2-2p)} \sum_{s_i,t_i \in S_n} \sum_{L=0}^{\infty} \sum_{v_1,\cdots,v_L \in S_n}{}' N^{\sum_j (K_{v_j}-n)} \frac{d(2-2p,L)}{n!} \cdot \delta\left(v_1 \cdots v_L \left(\prod_{i=1}^{p} [s_i,t_i] \right) P_r \right) \exp\left(-\frac{A}{\alpha'_{GT}} n \right). \tag{3.51}$$

This is very similar to the expression above for the chiral contribution to the full Yang-Mills partition function, with two differences:

- The full chiral partition function $Z^+(A,p,N)$ contains a sum over values of n, whereas $Z_R^+(A,p,N)$ restricts to a single value of n (equal to the number of boxes in a Young diagram for the representation R).
- In $Z_R^+(A,p,N)$, the delta function contains a factor of a projector P_r (associated to a representation r of S_n, associated to the same Young diagram as R), which was not present in the full chiral partition function $Z^+(A,p,N)$.

These two differences have the following effects.

- The restriction to a single n means that $Z_R^+(A,p,N)$ only receives contributions from maps of a single degree. This is closely analogous in a gauge theory to restricting to instantons of a single degree.
- Because of the projector P_r, there can be nonzero contributions from monodromy insertions v_i that are not allowed for a smooth branched cover Σ_W. For example, the exponent of N can be odd, which is not possible for the Euler characteristic of a smooth oriented Riemann surface.

We will see these effects in concrete examples in subsection 3.4, and propose resolutions for the corresponding physics puzzles in section 4.

Before moving on, we note a few consequences of the Riemann-Hurwitz theorem that will be relevant later:

- If $\chi(\Sigma_W) > \chi(\Sigma_T)$, then no holomorphic maps exist. For example, there are no holomorphic maps $\mathbb{P}^1 \to T^2$, or more generally, from a lower-genus curve to a higher-genus curve. To have maps, Σ_W must have at least the same genus as Σ_T.

- For fixed Σ_W, Σ_T obeying the constraint above, maps can exist of (nearly) any degree. For example, in degree n, for branchings in which only two sheets collide,
  - maps $\mathbb{P}^1 \to \mathbb{P}^1$ exist and are branched over $2(n-1)$ branch points,
  - maps $T^2 \to \mathbb{P}^1$ exist and are branched over $2n$ branch points,
  - maps $\Sigma_W \to \mathbb{P}^1$ exist and are branched over $2n - \chi(\Sigma_W)$ branch points,
  - unbranched maps $T^2 \to T^2$ exist for any n,
  - maps $\Sigma_W \to T^2$ exist and are branched over $-\chi(\Sigma_W)$ points, for $\chi(\Sigma_W) < 0$.

  That said, the orbifold Euler characteristic of the Hurwitz moduli space will vanish in some cases[11], so not all such maps will be represented explicitly in the Gross-Taylor expansion.

3.4 Examples

In this section we will walk through a number of examples, to make our proposal more clear.

In each case, we will begin by reviewing the ordinary Gross-Taylor expansion, for terms of fixed degree n in the chiral partition functions $Z^+(A,p,N)$ of theories in the large N limit, of the form

$$N^{n(2-2p)} \sum_{s_i, t_i \in S_n} \frac{1}{n!} \delta \left((\Omega_n)^{2-2p} \prod_{i=1}^{p} s_i t_i s_i^{-1} t_i^{-1} \right) \exp\left(- \frac{A}{\alpha'_{GT}} n \right), \tag{3.52}$$

where

$$\Omega_n \; = \; \sum_{\sigma \in S_n} N^{K_\sigma - n} \sigma \; = \; 1 \, + \, \sum_{\sigma \neq 1} \left(\frac{1}{N} \right)^{n - K_\sigma} \sigma, \tag{3.53}$$

(due to the fact that the identity in S_n decomposes as $K_1 = n$ cycles).

Then, after discussing the interpretation of those terms as branched covers, we will describe the terms in the expansion of the partition function of the corresponding Nguyen-Tanizaki-Ünsal universe. The terms have a similar form (at fixed degree) but also with projectors P_r inserted, namely

$$Z_R^+(A,p,N) \; = \; N^{n(2-2p)} \sum_{s_i, t_i \in S_n} \frac{1}{n!} \delta \left((\Omega_n)^{2-2p} \left(\prod_{i=1}^{p} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right) \exp\left(- \frac{A}{\alpha'_{GT}} n \right). \tag{3.54}$$

[11]For example, for $p = 0$, $d(2, L) = 0$ for $L > 2$, hence the only branched covers of $\mathbb{P}^1$ that contribute have no more than two branch points.

For simplicity, and to clean up notation, in the rest of the analysis we will usually restrict to the zero-area limit. (One should be slightly careful, as the exact expression for the Yang-Mills partition function on surfaces of low genus diverges in the zero-area limit and must be regularized. We will simply examine the interpretation of individual terms, omitting areas merely for convenience, so the convergence of the sum in that limit is not a concern.)

3.4.1 $p = 0, n = 2$

First, consider the case that the Yang-Mills theory lives on $\Sigma_T = \mathbb{P}^1$ (so that $p = 0$), and restrict to maps of degree $n = 2$. Since $S_2 = \mathbb{Z}_2$, we can write

$$\Omega_{n=2} \;=\; 1 \;+\; \left(\frac{1}{N} \right) v, \tag{3.55}$$

for $v \in S_2$ the nontrivial element.

We begin by considering the pertinent part of the ordinary Gross-Taylor expansion in this case. Expanding (3.52) in the zero-area limit, we find

$$\begin{aligned} \frac{N^{2n}}{n!} \delta \left((\Omega_n)^2 \right) \;&=\; \frac{N^{2n}}{n!} \left(1 \;+\; \left(\frac{1}{N} \right)^2 v^2 \right), \qquad (3.56) \\ &=\; \frac{N^4}{2!} \;+\; \frac{N^2}{2!} \delta(v^2). \qquad (3.57) \end{aligned}$$

Following [1–6], we interpret the first term as describing maps $\mathbb{P}^1 \coprod \mathbb{P}^1 \to \mathbb{P}^1$. As a check, note that $\chi(\mathbb{P}^1 \coprod \mathbb{P}^1) = 4$. The first term is essentially describing a free $S_2 = \mathbb{Z}_2$ Dijkgraaf-Witten theory on $\Sigma_T = \mathbb{P}^1$, and as such, from decomposition, it descomposes into a disjoint union of two pieces, hence Σ_W is a disjoint union.

We interpret the second term as describing maps from a branched double cover of $\mathbb{P}^1$, branched over two points (the locations of each v). Such a double cover is precisely $\mathbb{P}^1$, and note that the exponent of N is correct for this case, as $\chi(\mathbb{P}^1) = 2$. The factor of $1/2$ represents the orbifold Euler characteristic of the Hurwitz moduli space of maps $\Sigma_W \to \Sigma_T$ [4].

In passing, note that the interpretation we have assigned to the two Dijkgraaf-Witten partition functions above is ambiguous, simply because $v^2 = 1$. We could have equivalently written

$$\frac{N^{2n}}{n!} \delta \left((\Omega_n)^2 \right) \;=\; \frac{N^4}{2!} \delta(v^2) \;+\; \frac{N^2}{2!} \delta(1), \tag{3.58}$$

and then tried to interpret the first term in terms of a branched double cover of $\mathbb{P}^1$, and the second term in terms of a disjoint union of two copies of $\mathbb{P}^1$, but this would not be consistent with the exponentials of the N's. Our point, however, is that merely giving the Dijkgraaf-Witten partition functions by themselves is ambiguous in this context.

Next, consider a Nguyen-Tanizaki-Ünsal universe. Expanding (3.54) in the zero-area limit, we find

$$\begin{aligned} Z_R^+(0,p,N) &= \frac{N^{2n}}{n!}\delta\left((\Omega_n)^2 P_r\right), & (3.59)\\ &= \frac{N^{2n}}{n!}\delta\left((1)P_r \,+\, 2\left(\frac{1}{N}\right)vP_r \,+\, \left(\frac{1}{N}\right)^2 v^2 P_r\right), & (3.60)\\ &= \frac{N^4}{2!}\delta(P_r) \,+\, 2\frac{N^3}{2!}\delta(vP_r) \,+\, \frac{N^2}{2!}\delta(v^2P_r), & (3.61)\end{aligned}$$

where in this case, $P_r = (1/2)(1 \pm v)$. The first and last terms look essentially the same as in the ordinary Gross-Taylor case, but we also have a new term, not present previously, namely the term linear in v. This term can survive here, whereas previously it did not, because of the P_r.

Expanding out P_r, we have

$$Z_R^+(0,p,N) \;=\; \frac{N^4}{4} \pm \frac{N^3}{2} + \frac{N^2}{4}. \qquad (3.62)$$

Adding two such contributions together, for each choice of projector P_r, recovers the original Gross-Taylor result (3.57).

The first term can be interpreted as before, in terms of maps $\mathbb{P}^1 \coprod \mathbb{P}^1 \to \mathbb{P}^1$, which is consistent with the fact that $\chi(\mathbb{P}^1 \coprod \mathbb{P}^2) = 2\chi(\mathbb{P}^1) = 4$.

The second term, the new term, is more interesting. The Euler characteristic of any smooth closed Riemann surface is even, and here we need something of Euler characteristic 3. It is constructed from a Dijkgraaf-Witten correlation function with a single v, plus a projector.

Later in section 4 we will discuss these two issues – the construction of a theory that restricts to single instanton sectors, and the interpretation of the extra terms. Briefly, to address the first issue, we will propose a new constraint on the Gross-Taylor sigma model, that it possesses a symmetry making such a localization meaningful, and for the second issue, we will propose that the extra terms arise from stacky worldsheets Σ_W.

3.4.2 $p = 0, n = 3$

Next, we again consider the case of Yang-Mills theory on $\Sigma_T = \mathbb{P}^1$, and restrict to maps of degree $n = 3$. Here, $|S_3| = 3! = 6$, so we write

$$\Omega_{n=3} \;=\; 1 \;+\; \sum_{v \neq 1} \left(\frac{1}{N} \right)^{n-K_v} v. \tag{3.63}$$

For reference, the six elements of S_3 can be characterized as

$$(1)(2)(3), \quad (12)(3), (13)(2), (23)(1), \quad (123), (132). \tag{3.64}$$

These six elements form three conjugacy classes, essentially labelled by the orders of the cycles.

As before, we begin by considering the ordinary Gross-Taylor expansion in this case (at fixed degree $n = 3$). Expanding (3.52) in the zero-area limit, we find

$$\frac{N^{2n}}{n!} \delta \left((\Omega_n)^2 \right) \;=\; \frac{N^{2n}}{n!} \delta \left(1 \;+\; \sum_{ij} \left(\frac{1}{N} \right)^{2n - K_{v_1} - K_{v_2}} v_i v_j \right). \tag{3.65}$$

As before, each term can be interpreted as a 3-fold cover of $\Sigma_T = \mathbb{P}^1$.

The first term can be interpreted as a disjoint union of three copies of $\mathbb{P}^1$. As a consistency check, note that

$$\chi(\mathbb{P}^1 \coprod \mathbb{P}^1 \coprod \mathbb{P}^1) \;=\; 6, \tag{3.66}$$

which matches the power of N in that term.

Next, consider the second term. There are two cases that contribute to the sum:

- If both v_1 and v_2 have order 2, then $K_{v_1} = K_{v_2} = 2$, so the term is of the form

  $$\frac{N^6}{3!} \left(\frac{1}{N} \right)^{6-2-2} \delta(v_1 v_2) \;=\; \frac{N^4}{3!} \delta(v_1 v_2), \tag{3.67}$$

  and $\Sigma_W = \mathbb{P}^1 \coprod \mathbb{P}^1$, with one copy of $\mathbb{P}^1$ a double cover of Σ_W, branched over two points, and the second $\mathbb{P}^1$ a degree-one cover. Note that

  $$\chi(\mathbb{P}^1 \coprod \mathbb{P}^1) \;=\; 4, \tag{3.68}$$

  which matches the exponent of N.

- The other case is that v_1 and v_2 have order 3.

  In this case, from the Riemann-Hurwitz formula (3.45, we see that Σ_W has genus 0, consistent with the fact that this term is proportional to N^2. Hence, this term describes degree-three maps $\Sigma_W \to \mathbb{P}^1$ for $\Sigma_W = \mathbb{P}^1$.

No other cases can arise in this sum, given the delta function.

Next, we turn to the Nguyen-Tanizaki-Ünsal decomposition, meaning we incorporate the projector, and interpret the expression above as the entire partition function, instead of just one term in a larger function. Here, from (3.54), we have in the zero-area limit that

$$
\begin{aligned}
Z_R^+(0,p,N) &= \frac{N^{2n}}{n!}\delta\left((\Omega_n)^2 P_r\right), \\
&= \frac{N^{2n}}{n!}\delta\left(P_r \,+\, 2\sum_v \left(\frac{1}{N}\right)^{n-K_v} vP_r \,+\, \sum_{ij}\left(\frac{1}{N}\right)^{2n-K_{v_1}-K_{v_2}} v_i v_j\right),
\end{aligned} \tag{3.69}
$$

for $n = 3$ here. The first and third terms can be interpreted as before, modulo the addition of P_r. P_r itself depends upon the representation r of S_n, itself determined by the representation R of $(S)U(N)$, but our discussion below will apply to all cases.

It remains to discuss the middle term, which does not appear in the original Gross-Taylor expansion. Depending upon the order of v, there are two cases appearing in the sum.

- First, consider the case that $K_v = 2$, for example if $v = (12)(3)$. Here, the term takes the form
$$\frac{2}{3!}N^5\delta\left(\sum_v vP_r\right). \tag{3.70}$$

- Next, consider the case that $K_v = 1$, for example if $v = (123)$. Here, the term above takes the form
$$\frac{2}{3!}N^4\delta\left(\sum_v vP_r\right). \tag{3.71}$$

Both terms can be nonzero (because of the projector P_r). The first would describe a worldsheet with odd Euler characteristic, not possible for a smooth oriented closed Riemann surface.

3.4.3 $p = 1$

Next, we turn to the case that $\Sigma_T = T^2$. and consider degree n covering maps. In (3.52), in the zero-area limit, this corresponds to the terms

$$\frac{N^0}{n!} \sum_{s,t \in S_n} \delta\left((\Omega_n)^0 sts^{-1}t^{-1} \right). \tag{3.72}$$

In [1–6], these describe n-fold covers of T^2, with no branch points, for which, from the Riemann-Hurwitz formula (3.45), the covering spaces are $\Sigma_W = T^2$, consistent with the power of N.

To be clear, these covering spaces Σ_W are not necessarily connected. Whether the cover is one T^2 or several depends upon s, t. For example, if s and t each factorize suitably, the resulting n-fold cover can be, for example, a disjoint union of a k-fold cover and a $(n-k)$-fold cover. Thus, this description encompasses both connected and disconnected covers.

Next, we turn to the Nguyen-Tanizaki-Ünsal decomposition, meaning we incorporate the projector. Here, from (3.54), in the zero-area limit, we have

$$Z_R^+(0, p, N) \;\; = \;\; 1\frac{N^0}{n!} \sum_{s,t \in S_n} \delta\left((\Omega_n)^0 sts^{-1}t^{-1} P_r \right). \tag{3.73}$$

First, consider the case $n = 2$. Since S_2 is abelian, there are no extra terms arising from P_r, as the commutator $[s, t] = 1$ for all $s, t \in S_2$. The terms can be interpreted as follows:

- $s = t = 1$: Here, we get a disjoint union of two copies of T^2.
- all other s, t: Here, we get a single T^2, which is an unbranched double cover of T^2.

For this case ($n = 2$, $p = 1$), and only this case, the Nguyen-Tanizaki-Ünsal universe appears to be a restriction to maps of a single degree, without additional orbifold contributions. For higher n, S_n is nonabelian, and so there will be projector contributions multiplying the commutator.

Now, let us turn to the Nguyen-Tanizaki-Ünsal decomposition, and add projectors. In this case, from (3.54), the partition function is

$$Z_R^+(0, p, N) \; = \; \frac{N^0}{n!} \sum_{s,t \in S_n} \delta\left((\Omega_n)^0 sts^{-1}t^{-1} P_r \right) \tag{3.74}$$

for r an irreducible representation of S_n. This can admit additional contributions. For example, consider the case $n = 3$. Since S_3 is nonabelian, there are additional terms involving the projector P_r, corresponding to cases in which the commutator is different from the identity, and instead has either order 2 or 3.

In section 4.3.3 we will interpret the terms of the expansion above, as a combination of a new symmetry in the Gross-Taylor sigma model (to realize the restriction to specific degrees) and by adding stacky worldsheets (to encompass the additional cases not present previously).

3.4.4 $p = 2, n = 2$

Next, we turn to the case of double covers of a genus-two Riemann surface Σ_T. As before, we begin with the Gross-Taylor expansion at fixed degree $n = 2$, for which the relevant term in the full partition function in the zero-area limit is

$$\frac{N^{-2n}}{n!} \sum_{s_i, t_i \in S_n} \delta \left((\Omega_n)^{-2} \prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) \tag{3.75}$$

Now,

$$\Omega_2 \; = \; 1 + \left(\frac{1}{N} \right) v \tag{3.76}$$

for v the nontrivial element of $S_2 = \mathbb{Z}_2$, and Ω_2^{-1} is the inverse element in the group algebra, which one can quickly verify is

$$\Omega_2^{-1} \; = \; \frac{1}{1 - (1/N^2)} \left(1 - \frac{v}{N} \right). \tag{3.77}$$

Thus, expanding, the relevant terms are

$$\begin{aligned} & \frac{N^{-4}}{2!} \sum_{s_i, t_i \in S_n} \delta \left((\Omega_n)^{-2} \prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) \\ & = \; \frac{N^{-4}}{2!} \sum_{s_i, t_i \in S_n} \left(\frac{1}{1 - (1/N^2)} \right)^2 \delta \left(\left(1 - \frac{v}{N} \right)^2 \prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right), \qquad (3.78) \\ & = \; \frac{N^{-4}}{2!} \sum_{s_i, t_i \in S_n} \left(\frac{1}{1 - (1/N^2)} \right)^2 \delta \left(\left(1 \, - \, \frac{2}{N} v \, + \, \frac{1}{N^2} v^2 \right) \prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right), \qquad (3.79) \end{aligned}$$

Let us interpret the three terms above systematically:

- First, consider the term

$$\frac{N^{-4}}{2!} \sum_{s_i,t_i \in S_n} \left(\frac{1}{1-(1/N^2)} \right)^2 \delta \left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right)$$
$$= \sum_{k=0}^{\infty} \frac{N^{-4}}{2!} \left(\sum_{k=0}^{\infty} N^{-2k} \right)^2 \sum_{s_i,t_i \in S_n} \delta \left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) \tag{3.80}$$

  To leading order in N, this term goes like N^4, and describes an unbranched double ($n = 2$) cover of the genus-two Riemann surface Σ_T. From the Riemann-Hurwitz formula (3.45), that double cover Σ_T must be a genus 3 surface, which is consistent with the power of N.

- Next, consider the term

$$-\frac{N^{-4}}{2!} \sum_{s_i,t_i \in S_n} \left(\frac{1}{1-(1/N^2)} \right)^2 \delta \left(\frac{2}{N} v \prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right). \tag{3.81}$$

  This term describes a branched double cover of Σ_T. However, it does not make a nonzero contribution:

  - Physically, it is straightforward to check that the δ above vanishes. Since S_2 is abelian,

$$\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \;=\; 1, \tag{3.82}$$

    and as $v \neq 1$, the argument of the delta function is not the identity.
  - Mathematically, from the Riemann-Hurwitz formula (3.45),

$$\chi(\Sigma_W) \;=\; 2\chi(\Sigma_T) - 1 \;=\; -5, \tag{3.83}$$

    which is odd, not consistent with the Euler characteristic of a smooth curve.

  When we add a projector, this term will reappear in the Nguyen-Tanizaki-Ünsal decomposition.

- Finally, consider the term

$$\frac{N^{-4}}{2!} \sum_{s_i,t_i \in S_n} \left(\frac{1}{1-(1/N^2)} \right)^2 \delta \left(\frac{1}{N^2} v^2 \prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right). \tag{3.84}$$

  This term describes a branched double cover of Σ_T, branched over two points. From the Riemann-Hurwitz formula (3.45), we see that Σ_W is a genus-four Riemann surface, which is consistent with the factor N^{-6}.

Now, we turn to the Nguyen-Tanizaki-Ünsal universes. We can add projectors and revisit these arguments to interpret (3.54) for $n = 2$ in the zero-area limit, namely

$$
\begin{aligned}
Z_R^+(0, p, N) &= \frac{N^{-4}}{2!} \sum_{s_i, t_i \in S_2} \delta \left((\Omega_2)^{-2} \left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right), & (3.85) \\
&= \sum_{k=0}^{\infty} \frac{N^{-4}}{2!} \left(\frac{1}{1 - (1/N^2)} \right)^2 \sum_{s_i, t_i \in S_n} \delta \left(\left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right) & (3.86) \\
&\quad - \frac{N^{-4}}{2!} \sum_{s_i, t_i \in S_n} \left(\frac{1}{1 - (1/N^2)} \right)^2 \delta \left(\frac{2}{N} v \left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right) \\
&\quad + \frac{N^{-4}}{2!} \sum_{s_i, t_i \in S_n} \left(\frac{1}{1 - (1/N^2)} \right)^2 \delta \left(\frac{1}{N^2} v^2 \left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right).
\end{aligned}
$$

We have already described how to interpret the first and third terms as a disjoint union of stacky versions of Σ_T; it merely remains to discuss the middle term.

Much as in the $p = 0$ case, one effect of adding a projector is to add a term not present previously, namely the middle term (3.81) that we discarded above:

$$
-\frac{N^{-4}}{2!} \sum_{s_i, t_i \in S_n} \left(\frac{1}{1 - (1/N^2)} \right)^2 \delta \left(\frac{2}{N} v \left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right). \tag{3.87}
$$

Previously, this term vanished, because the delta function could not be nonzero. Here, on the other hand, thanks to the insertion of P_r, this term can be nonzero.

We will discuss our proposed interpretation of these terms in section 4.3.4. Briefly, we will propose that the Gross-Taylor sigma model has a new symmetry that allows for a localization on maps of a fixed degree, and also introduce additional (stacky) worldsheets to describe the extra terms that appear in the presence of the projector P_r.

4 Interpretation of the terms

In this section we discuss the interpretation of the Gross-Taylor series expansion of the partition functions of the distinct Nguyen-Tanizaki-Ünsal universes. We now identify the universes with[12] coupled representation $R\overline{S}$, and as outlined earlier, factorize the partition

[12]In relating the Gross-Taylor combinatorics to individual Nguyen-Tanizaki-Ünsal universes, we are implicitly assuming that a coupled representation $R\overline{S}$ uniquely determines R and S separately, which is believed to hold in the large N limit, but not at finite N.

function in the form

$$\left(\dim R\overline{S}\right)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R\overline{S})\right) \;=\; Z_R^+(A,p,N)\, Z_S^-(A,p,N) \;+\; \text{subleading}. \quad (4.1)$$

For simplicity, we focus on the chiral partition functions $Z_R^+(A,p,N)$ for fixed representation R, given in equation (3.32), which we repeat below:

$$\begin{aligned} Z_R^+(0,p,N) \;&=\; N^{n(2-2p)} \sum_{s_i,t_i\in S_n} \sum_{L=0}^{\infty} \sum_{v_1,\cdots,v_L\in S_n}{}' N^{\sum_j (K_{v_j}-n)} \frac{d(2-2p,L)}{n!} \\ &\qquad \cdot \delta\left(v_1\cdots v_L \left(\prod_{i=1}^{p}[s_i,t_i]\right) P_r \right) \exp\left(-\frac{A}{\alpha'_{GT}} n\right). \quad (4.2) \end{aligned}$$

This expression differs from the chiral partition function $Z^+(0,p,N)$ of the Gross-Taylor string in two important ways:

- First, there is no sum over degrees n, the expression above for $Z_R^+(A,p,N)$ references one fixed degree (equal to the number of boxes in the Young tableau for R). Physically in the Gross-Taylor string, this corresponds to a restriction to maps of fixed degree, which is analogous in a gauge theory to restricting to sectors of fixed instanton number. Restrictions on instantons to those of instanton degree satisfying a divisibility criterion is common in decomposition, but a restriction to a single instanton degree is novel (and at least sometimes problematic).
- Second, for maps of any one degree n, the delta function includes a projector P_r, not present in the original chiral Gross-Taylor partition function $Z^+(A,p,N)$, determined as in (A.8) by an irreducible representation r of S_n, itself determined by R. We will see that the projector P_r in the expression above is going to result in new terms not present in the original analysis, which can not be interpreted in terms of ordinary covering maps $\Sigma_W \to \Sigma_T$. (Existence of extra contributions in individual universes which cancel out when the universes are summed over is a typical feature of decomposition, as we review in section 4.2.)

  Such projectors have previously appeared in analyses of finite N contributions e.g. [77, 83], where they are interpreted as (nonperturbative) open string contributions, but here they arise in contributions in the large N limit, where nonperturbative contributions (in the string coupling, as opposed to α') would not be expected.

In this section we will propose resolutions for both of these issues.

1. In subsection 4.1, we observe that a restriction to individual instanton sectors is consistent if the theory admits a symmetry whose Noether current couples to the instanton

degrees, as either a 2-form or its Hodge dual. In other words, we propose that the Gross-Taylor sigma model must admit either a global 1-form symmetry or a (-1)-form symmetry coupling to the instanton degree. We discuss a prototypical example (two-dimensional pure Maxwell theory), and also walk through a number of possible alternatives.

2. In subsection 4.2, we propose an interpretion of the extra contributions as arising from worldsheets that have orbifold points (meaning, technically, certain kinds of smooth stacks), in the special case that $\Sigma_T = \mathbb{P}^1$.

In subsection 4.3 we illustrate these ideas in examples.

4.1 Restrictions on worldsheet instanton (map) degrees

Our computations earlier in this paper have suggested that, if we have correctly tracked through the series expansion, then the Nguyen-Tanizaki-Ünsal universes are described by Gross-Taylor sigma models in which maps are restricted to a single degree (i.e. fixed worldsheet instanton number).

From a field theory perspective, this seems particularly troubling. Ordinarily we sum over all instantons. Labelling field configurations by instanton number is typically just an artifact of a semiclassical expansion, and does not have an intrinsic meaning in quantum field theory.

Concretely, there is a standard old argument of Weinberg (see e.g. [111, section 23.6]) linking cluster decomposition to instanton sums in semiclassical expansions. The idea is as follows. Imagine trying to restrict to instantons of a single degree, then add a field configuration consisting of a closely-spaced instanton-antiinstanton pair. Now, move the centers of that pair far apart. If cluster decomposition holds, then asymptotically one has theories with different numbers of instantons. Any restriction on instantons can only arise via a violation of cluster decomposition.

Now, there is a loophole in this argument: if the restriction arises via a disjoint union of theories, each separately summing over all instantons, with theta angles arranged so that some instantons cancel out in the sum, one can arrange for a restriction on instanton numbers, at the cost of violating cluster decomposition in a very mild controllable fashion (as any disjoint union violates cluster decomposition). This is often exploited in decomposition [10]. For example, theories with restrictions to instanton degree divisible by an integer have been discussed in e.g. [11,112–114], but the resulting theories are believed to be consistent only by virtue of decomposition, in the sense that they are the result of superimposing similar theories with slightly different theta angles, as outlined above. The different theories (universes) each separately sum over all instanton degrees; only in the sum does one see an apparent restriction on instanton degrees (in the semiclassical expansion).

To restrict to a single instanton degree, as we appear to see here, and not just to instanton degrees satisfying a divisibility criterion, would be considerably stronger than the examples above discuss. Formally, the partition function of a single instanton sector is given by integrating over values of the theta angle, analogously to the ensemble[13] averaging discussed in e.g. [115–119], essentially Fourier-transforming along the theta angle to pick out the contribution from a single degree. To interpret this directly as a a decomposition would require an uncountable infinity of universes, parametrized by a nondynamical theta angle, which we find unlikely.

We emphasize that a restriction to a single instanton sector will not always be possible, even with a summation over countably infinitely many universes, in an arbitrary quantum field theory. For example, consider ordinary orbifolds. As they are finite gauge theories, the instanton sectors are precisely the twisted sectors, which are enumerated by equivalence classes of bundles. On T^2, for example, modular invariance tightly constrains possible theories, and the only individual instanton sector consistent with modular invariance is the untwisted sector. Restrictions to subsets of twisted sectors do frequently arise in decomposition, but such restrictions are always to modular-invariant subsets, never to a single nontrivial twisted sector.

In this section, we will describe several different potential proposals for possible resolutions of this puzzle, but ultimately we will observe that there is (at least) one set of circumstances where this would be consistent: when the theory has a symmetry whose Noether current couples to either the pullback of the Kähler form or its dual (at least perturbatively in the string coupling constant), as the integral of that pullback is the worldsheet instanton number. This would require the Gross-Taylor sigma model to admit either a global 1-form symmetry or a (-1)-form symmetry. In the former case, the Gross-Taylor sigma model would decompose, as we will discuss later.

4.1.1 Proposal

Before making our proposal, we shall first walk through several possibilities.

Later in section 5 we will argue that the branched covers Σ_W can be replaced, for at least some purposes, with disjoint unions, so one might suspect that perhaps one can deform Σ_W to a corresponding disjoint union. Since the Gross-Taylor theory is a topological (string) theory, it should be invariant under smooth deformations, suggesting that perhaps the Gross-

[13]It should be noted that an ensemble is not the same as a decomposition, which becomes visible in QFTs on spacetimes with multiple connected components. (In the former case, there is only one summand/integral over the ensemble, whereas in the second, there are as many as connected components.)

Taylor string on a branched cover is equivalent to the Gross-Taylor string on a disjoint union, potentially shedding light on the present question. Unfortunately, we will see in section 5.2 that although such deformations exist, the disjoint unions are not in the closure of the Hurwitz moduli space, one must instead pass a finite distance through a larger moduli space, which invalidates this potential argument.

Another option one might consider is that potentially the extra contributions of section 4.2 might, conceivably, cancel out some of the contributions from the regular curves, rendering the sum of contributions trivial. Unfortunately, the contributions from section 4.2 are weighted with different factors of N, so such a cancellation can not take place.

In appendix D we will outline another proposal, which is to interpret the single-instanton sectors as different QFTs, obtained from localizing the original QFT onto the desired instanton sectors by adding (analogues of) BF terms. This is straightforward to describe in the bosonic case, but as we describe in appendix D, we run into a subtlety with the supersymmetrization which appears to obstruct its application in cohomological field theories.

Having walked through several possibilities, we now turn to our proposal.

Briefly, we are proposing that the Gross-Taylor sigma model has a suitable symmetry which enables one to meaningfully select worldsheet instanton sectors. In other contexts, for ordinary (invertible zero-form) symmetries, one would require that there be a conserved charge, so as to make the corresponding quantity meaningful in the full quantum field theory, not just in some semiclassical expansion.

Here, we propose[14] that the Gross-Taylor sigma model admit a symmetry whose Noether current is either the pullback of the Kähler form or its Hodge dual (since the integral of that pullback is the worldsheet instanton number), as these seem the most conservative options. Given that the Kähler form is a two-form, this suggests that the theory either has

- a global 1-form symmetry, or
- a global (-1)-form symmetry,

(related to the worldsheet instanton degree). If the (two-dimensional) theory has a 1-form symmetry, then it decomposes. That said, the Gross-Taylor string theory is also expected to couple to worldsheet gravity, so any such decomposition would not yield completely disjoint universes, but rather the universes would still communicate via gravitational interactions. We discuss gravitational couplings and decomposition in appendix C.

We list two possibilities above, but these two possibilities are closely linked. Recall that in d spacetime dimensions, if a theory has a global $(d-1)$-form symmetry, it decomposes. If one

[14] We would like to thank R. Plesser for suggesting this direction.

gauges that symmetry, the resulting theory is one of the universes of the decomposition [120], which has a (-1)-form quantum symmetry, corresponding to spacetime-filling defects. If one then gauges that quantum symmetry, one recovers the original theory (as expected for a quantum symmetry). Since the quantum symmetry is a (-1)-form symmetry, gauging it corresponds to summing over spacetimes, and so one explicitly recovers the original theory as sum over universes. It is natural to speculate that in the present circumstances, the Gross-Taylor string associated to the whole two-dimensional Yang-Mills theory may have a global 1-form symmetry and so decomposes (as $d-1=1$ here), in which case the string associated to one Nguyen-Tanizaki-"Unsal universe of the Yang-Mills decomposition is obtained by gauging that 1-form symmetry, and so has a global (-1)-form symmetry.

This proposal may sound exotic, but in the next section we will describe another two-dimensional theory where precisely this takes place.

Furthermore, in hindsight, existence of such a global symmetry in the Gross-Taylor sigma model should be expected from standard yoga. The restriction to a Nguyen-Tanizaki-ünsal universe in the decomposition of two-dimensional pure Yang-Mills should be equivalent to gauging the corresponding one-form symmetry [120], and as is well-known, target-space gauge symmetries correspond to worldsheet global symmetries. (For example, in a conventional string theory, the Noether current for an ordinary worldsheet symmetry forms part of the vertex operator for a target-space gauge field.) Thus, at least in general terms, in hindsight, it should not be surprising that the Gross-Taylor expansion for a single Nguyen-Tanizaki-Ünsal universe should possess a global symmetry. (That said, it is not clear how yoga alone would predict coupling of that symmetry to worldsheet instanton degree, which is our prediction here.)

Now, we are aware of two separate proposals for a worldsheet theory of the Gross-Taylor string, namely the Cordes-Moore-Ramgoolam proposal [4–6], describing a sigma model localizing on holomorphic maps, and the Horava proposal [8, 9], describing a sigma model localizing on harmonic maps, which are both proofs-of-principle of the possible existence of a Gross-Taylor sigma model. We are not aware of a linearly-realized symmetry of the form proposed above in either[15] of these proposals. One possibility is that the symmetry is present, but nonobvious (perhaps realized noninvertibly, much as in e.g. [47,48,62]). Another possibility is that there exists a third possible worldsheet theory for the Gross-Taylor sigma model. We leave this issue for future work.

We should add that although we believe the property above is a necessary condition, it might not be sufficient. Here we have in mind the example of electric charge, which is conserved and defines a superselection rule (see e.g. [121]), in which in the superselection sectors, the total charge is fixed. Similarly, here, the Gross-Taylor expansion of the

[15] In the Horava proposal, it is tempting to consider an analogue of a $BU(1)$ symmetry acting as $d\phi \mapsto d\phi + \Lambda$, which naively is a symmetry of the kinetic terms for constant metric, but we do not understand how this would be consistent with nontrivial metrics or curvature terms.

Nguyen-Tanizaki-Ünsal universes produces maps of one[16] fixed degree. In any event, to be consistent, more than a superselection rule is needed here – the Nguyen-Tanizaki-Ünsal universes are separately well-defined physical theories, and so too should the restrictions of the Gross-Taylor sigma model to maps of total fixed degree be consistent. For example, if the Gross-Taylor sigma model has a 1-form symmetry (one of the options we list), then it also has a decomposition, which is certainly stronger than just superselection, and would yield separately well-defined theories (modulo its gravitational coupling, see appendix C). We leave this question for future work.

In passing, we observe that two-dimensional cohomological field theories (such as existing proposals for the Gross-Taylor string) typically have semisimple operator algebras, and so decompose – but that is a decomposition only of the topological subsector, not necessarily of the entire theory, and more to the point, it need not have any connection to worldsheet instanton degrees. As a result, it is unfortunately not relevant for our purposes here.

4.1.2 Prototype: two-dimensional pure Maxwell theory

In this section we will study two-dimensional pure Maxwell theory, as a theory whose symmetries are a prototype for those we propose for the Gross-Taylor string. Specifically, it has a (one-form) symmetry with Noether current related to the $U(1)$ bundle degree, also known as the $U(1)$ monopole[17] number (meaning, the first Chern class of the bundle). Because it has a one-form symmetry, it decomposes, and we shall see explicitly that the universes of the decomposition are indexed by an integer which is 'Poisson dual' to the $U(1)$ monopole number, reflecting the symmetries of the theory. At the end of this section, we will also discuss an analogue of the Witten effect in dyons in four dimensions, here interchanging universes under rotations of the theta angle.

The pure Maxwell theory in any dimension has a global $BU(1)$ symmetry, given explicitly by shifts $A \mapsto A + \Lambda$, with Noether current $J^e = *F$, associated to [122, section 4.1] an operator $U_\alpha(p) = \exp(i\alpha * F(p))$. There is also a magnetic symmetry with current $J^m = F$, associated with an operator $U_\beta(\Sigma) = \exp(i\beta \int_\Sigma F)$ corresponding to a (-1)-form symmetry [122, foonote 10, section 4.1]. (Noether currents of this form, or their Hodge duals,

[16] In examples in which Σ_W has multiple connected components, that fixed number arises as the sum of the degrees of the maps from the components of Σ_W to Σ_T.

[17] We shall use the terms monopole number and vortex number interchangeably in two-dimensional abelian theories to describe the first Chern class of a $U(1)$ bundle. We are avoiding referring to 'instanton numbers' in two-dimensional gauge theories in this section, as in two dimensions this is sometimes used to refer to the value of

$$\int_\Sigma \mathrm{Tr}\, F \wedge *F. \tag{4.3}$$

Unlike four dimensions, in two dimensions this is not a topological invariant, and indeed has a nonzero area dependence, but in the literature the term instanton number is often associated with such a term, even in two-dimensional theories, see e.g. [127].

are essentially our prediction for the Gross-Taylor sigma model.)

As has been argued elsewhere (see e.g. [10, 46]), a d-dimensional theory with a global $(d-1)$-form symmetry should decompose. Here, since two-dimensional pure Maxwell theory has a $BU(1)$ symmetry, one should expect that it decomposes into countably many universes, indexed by irreducible representations of $U(1)$ (i.e. integers). Indeed, in [18], it was argued that two-dimensional pure Maxwell theory is equivalent to a disjoint union of invertible theories, in much the same way that [47, 48] later argued that two-dimensional pure Yang-Mills theories decompose.

Given the fact that the Noether current for the global 1-form symmetry is related to F, it is natural to expect some sort of correlation between the universes (labelled by $U(1)$ representations) and and the $U(1)$ monopole numbers (meaning, values of $\int F$), and indeed, it was argued in e.g. [123–125] that they are Poisson dual to one another. As this will be important for our analysis, and is slightly subtle, we will review the relationship carefully.

Following section 2 and [58, section 2], [57] write the partition function for two-dimensional pure Maxwell theory on a Riemann surface Σ in the form

$$Z(\Sigma) \;=\; \sum_R (\dim R)^{\chi(\Sigma)} \exp\left[-g_{YM}^2 A C_2(R)\right], \tag{4.4}$$

$$\;=\; \sum_{q\in\mathbb{Z}} \exp\left[-g_{YM}^2 A q^2\right], \tag{4.5}$$

using the fact that for $G = U(1)$ [126, chapter 7, table 7.1]

$$C_2(R) \;=\; q^2. \tag{4.6}$$

In the expression above, since all irreducible representations R of $U(1)$ are one-dimensional, we have replaced representations R with charges q. The universes of the decomposition are then indexed by q, and the qth universe has partition function

$$\exp\left[-g_{YM}^2 A q^2\right]. \tag{4.7}$$

Now, let us carefully study the relationship between the $U(1)$ charge q and $U(1)$ monopole numbers $\int F$. First, note that the area dependence of the exponent is linear in the area A, but the kinetic term for a gauge field of monopole number q should[18] be inversely proportional to the area, so the $U(1)$ charge q cannot be precisely the same as the monopole number.

Let us take a moment to check this carefully, as it is important for our analysis, For $U(1)$ monopole number n, we claim that the Maxwell kinetic term

$$\frac{1}{g_{YM}^2}\int_\Sigma F^{\mu\nu}F_{\mu\nu} \;\propto\; \frac{n^2}{g_{YM}^2 \mathrm{Area}}, \tag{4.8}$$

[18]We would like to thank R. Szabo for making this observation, and for pointing out the role of the Poisson resummation that will appear momentarily.

a completely different area dependence from that of the partition function for the qth universe (4.7). Explicitly, in the special case of $\Sigma = S^2$, following [127, section 3], a classical gauge field configuration with $U(1)$ monopole number n is given by

$$A_\theta \;=\; 0, \;\; A_\phi \;=\; n\,\frac{1-\cos\theta}{2}, \tag{4.9}$$

hence

$$F_{\theta\phi} \;=\; \frac{n}{2}\sin\theta, \;\;\; F^{\theta\phi} \;=\; \frac{1}{2r^4\sin\theta}, \tag{4.10}$$

using

$$ds^2 \;=\; r^2\left(d\theta^2 \;+\; \sin^2\theta\, d\phi^2\right), \tag{4.11}$$

so that

$$\sqrt{\det g} \;=\; r^2\sin\theta, \tag{4.12}$$

and

$$\int_{S^2}\sqrt{\det g}\,F^{\theta\phi}F_{\theta\phi} \;=\; \frac{\pi n^2}{r^2} \;\propto\; \frac{n^2}{\text{Area}}. \tag{4.13}$$

Thus, we see that the area dependence of the partition function for the qth universe (4.7) is not consistent with what one would expect if the universes were indexed by the $U(1)$ monopole number. Instead, we shall argue that they are indexed by a Poisson-dual number.

To understand the role of $U(1)$ monopoles and the two-dimensional θ angle, we will outline the derivation of the exact expression for the partition function, following the analysis of [56], but including a θ angle term. We will see explicitly that the irreducible representation and the $U(1)$ monopole number are related by a Poisson[19] resummation. Schematically, working on $\Sigma = S^2$ for simplicity, for the two-dimensional pure Maxwell theory with action

$$S \;=\; \frac{1}{g_{YM}^2}\int_\Sigma F^{\mu\nu}F_{\mu\nu} \;+\; \int_\Sigma i\theta F, \tag{4.14}$$

the partition function can be expressed as a sum over contributions from $U(1)$ monopoles (nontrivial $U(1)$ bundles) of charge n, in the form

$$Z(\Sigma_g) \;\;=\;\; \sum_{n=-\infty}^{\infty}\left(\pi g_{YM}^2 A\right)^{-1/2}\exp\left(-\frac{n^2}{g_{YM}^2 A} \;+\; i\theta n\right), \tag{4.15}$$

(with overall factors $\left(\pi g_{YM}^2 A\right)^{-1/2}$ chosen in hindsight to make the result clean). We can

[19] See also e.g. [55,127–129] for related discussions of Poisson resummation in the context of two-dimensional pure Yang-Mills theories, and also e.g. [130, 131] for more detailed computations.

Poisson resum this expression as follows:

$$Z(\Sigma_g) \;=\; \int_{-\infty}^{\infty} d\lambda \sum_{n=-\infty}^{\infty} \delta(\lambda - n) \left(\pi g_{YM}^2 A\right)^{-1/2} \exp\left(- \frac{\lambda^2}{g_{YM}^2 A} \,+\, i\theta\lambda \right), \tag{4.16}$$

$$= \int_{-\infty}^{\infty} d\lambda \sum_{m=-\infty}^{\infty} \exp(2\pi i m\lambda) \left(\pi g_{YM}^2 A\right)^{-1/2} \exp\left(- \frac{\lambda^2}{g_{YM}^2 A} \,+\, i\theta\lambda \right), \tag{4.17}$$

$$= \int_{-\infty}^{\infty} d\lambda' \sum_{m=-\infty}^{\infty} \left(\pi g_{YM}^2 A\right)^{-1/2} \exp\left[- \frac{(\lambda')^2}{g_{YM}^2 A} \,-\, \frac{g_{YM}^2 A}{4} (\theta + 2\pi m)^2 \right], \tag{4.18}$$

$$= \sum_{m=-\infty}^{\infty} \exp\left[- \frac{g_{YM}^2 A}{4} (\theta + 2\pi m)^2 \right], \tag{4.19}$$

where

$$\lambda' \;=\; \lambda \,-\, i \frac{g_{YM}^2 A}{2} \left(\theta + 2\pi m\right)^2. \tag{4.20}$$

For simplicity we specialized to $\Sigma = S^2$; for other Riemann surfaces, in principle one would need to take into account e.g. moduli of flat connections in the analysis. However, the form of the result for the exact partition function, as reviewed in section 2, is universal.

Including the theta angle, we see that the partition function of the $q = m$-th universe is

$$\exp\left[- \frac{g_{YM}^2 A}{4} (\theta + 2\pi m)^2 \right]. \tag{4.21}$$

When $\theta = 0$, we see that this recovers the exact expression (4.5), identifying the $U(1)$ charge q with m, and absorbing factors of 2 and π into g_{YM}^2. For nonzero θ, we can treat the θ^2 term as a contribution to an overall multiplicative factor, leaving us just with two terms: one quadratic in m (corresponding to the Casimir $C_2(R)$) and one linear in m (corresponding to the Casimir $C_1(R)$). This form of the exact expression, taking into account θ dependence, has also been discussed in [58, section 2], [57], albeit with a different normalization on the θ term.

In particular, comparing the area-dependence of the m^2 term to the exact result for the partition function (4.19), we emphasize that the irreducible $U(1)$ representation, the charge q, corresponds to m, the Poisson-dual to the $U(1)$ monopole number, which was the n in equation (4.15), recovering the result in [123–125]. That said, although the universes of the decomposition are not indexed by $U(1)$ monopole number, since the universes are all invertible field theories with one-dimensional Fock spaces, the difference between fixed $U(1)$ monopole number and fixed $U(1)$ charge is just a Fourier transform, so it might still be meaningful to speak of universes associated to fixed $U(1)$ monopole number.

The θ dependence of the partition function (4.21) suggests further interesting physics, which we will return to shortly. For the moment, we confirm the result above by giving an alternative computation. We can quickly outline a confirmation of this result[20] for the partition function as a function of $m = q$, by semiclassically gauging the $BU(1)$ symmetry, which in principle [120] should select particular universes. Start with the pure Maxwell action

$$S \;=\; \frac{1}{g_{YM}^2} \int_\Sigma F^{\mu\nu} F_{\mu\nu} \;+\; \int_\Sigma i\theta F, \tag{4.22}$$

and gauge $BU(1)$. This means we add a dynamical two-form tensor field potential B, the gauge field for the gauged $BU(1)$, whose gauge transformations couple to the gauge field A as follows:

$$A \;\mapsto\; A - \Lambda, \tag{4.23}$$

$$B \;\mapsto\; B + d\Lambda, \tag{4.24}$$

for Λ any one-form. The action of the pure Maxwell theory with the gauged $BU(1)$ then takes the form[21]

$$S' \;=\; \frac{1}{g_{YM}^2} \int_\Sigma (F+B) \wedge *(F+B) \;+\; \int_\Sigma i\theta(F+B). \tag{4.25}$$

Now, in principle, to select one particular universe, we add a theta angle term for the $BU(1)$ symmetry, parametrized by irreducible representations of $U(1)$, namely $\mathbb{Z}$. This is similar in principle to the procedure described in [120, section 8], where a $B\mathbb{Z}_k$ symmetry was gauged. Explicitly, here, we add a term proportional to qB, for $q \in \mathbb{Z}$ corresponding to an irreducible representation of $U(1)$:

$$S'' \;=\; \frac{1}{g_{YM}^2} \int_\Sigma (F+B) \wedge *(F+B) \;+\; \int_\Sigma \left(i\theta(F+B) + 2\pi i q B \right). \tag{4.26}$$

We absorb F into B via the affine gauge transformation $A \mapsto A - \Lambda$, to write

$$S'' \;=\; \frac{1}{g_{YM}^2} \int_\Sigma B \wedge *B \;+\; \int_\Sigma i\left(\theta + 2\pi q\right) B. \tag{4.27}$$

Integrating out B (and glossing over operator determinants, see e.g. [132, section 2.2] for a more complete analysis), we get

$$S'' \;=\; \frac{1}{4} g_{YM}^2 A \left(\theta + 2\pi q\right)^2 \tag{4.28}$$

[20] We would like to thank C. Closset for a discussion of this computation.

[21] A close analogue of this procedure was used in four-dimensional pure Maxwell theory in [132, section 2.2], [133, section 2.4], to implement S-duality. There, it was noted that just gauging $BU(1)$ left a trivial theory, and so a dual gauge field was added, which coupled via a topological term, a four-dimensional analogue of the $\int B$ term we introduce above. Here, by contrast, our goal is to generate a trivial theory – one of the universes of decomposition, itself an invertible field theory in this case – so no additional fields are needed.

matching the exponent of the partition function of the theory (4.21) in any one fixed universe, determined by q (up to factors of 2, π), which can be identified with the m in (4.21).

In passing, we observe that the form of the exact result for the partition function (4.21) for pure Maxwell theory with a theta angle, suggests the existence of a two-dimensional decomposition analogue of the four-dimensional Witten effect in dyons [134]. Recall that, under a rotation $\theta \mapsto \theta + 2\pi$ of the four-dimensional theta angle, dyon charges also rotate. (The complete physical theory is invariant, as this just interchanges existing dyon charges.) Here, judging from the exact partition function (4.21), a rotation $\theta \mapsto \theta + 2\pi$ of the two-dimensional theta angle is equivalent to shifting $q \mapsto q + 1$, meaning that under a rotation of the theta angle, the universe shifts.

We can also see the same result from the perspective of the Hilbert spaces. Recall that, at least for vanishing theta angle, the Hilbert space of two-dimensional pure Yang-Mills is given by the class functions on the Lie group G, which has a basis of characters χ_R associated to irreducible representations. Because of the theta angle, a particle moving along a closed noncontractible loop will pick up a phase, or more formally,

$$f(gz) \;=\; \lambda(z) f(g). \tag{4.29}$$

(This is discussed in e.g. [12, section 2.4], [135] for discrete theta angles; the present case is similar. In essence, in two dimensions, the θ angle acts as an electric field [136], hence it modifies wavefunctions by phases.) Here, for a fixed irreducible representation R corresponding to $U(1)$ charge q, and identifying $g \in U(1)$ with a phase $\alpha = \exp(i\beta) \in \mathbb{C}^*$, $|\alpha|^2 = 1$, we can write

$$f(g) \;=\; \exp(i(\theta/2\pi)\beta)\exp(i\beta q) \;=\; \alpha^{q+\theta/(2\pi)}. \tag{4.30}$$

Explicitly, rotating $\theta \mapsto \theta + 2\pi$ is equivalent to incrementing q, thereby shifting the universe, just as we saw in partition functions. (See also [137] for related remarks.)

So far we have discussed two-dimensional pure Maxwell theory as a possible prototype for expected symmetries and properties of the Gross-Taylor string. It should be noted in addition that, at least morally, two-dimensional pure Yang-Mills theory is also similar, in the sense that it has a global 1-form symmetry (realized noninvertibly [47, 48]), and the gauge instantons of the theory are roughly Poisson dual to the representations (and hence universes) [127–129].

4.2 The extra terms: stacky worldsheets

In the Gross-Taylor expansion of Nguyen-Tanizaki-Ünsal universes in section 3, we came across two puzzles:

- terms that appear to correspond to contributions to sigma models from maps of fixed degree,

- terms that appear to arise from additional worldsheets, often with powers of N that could not be realized from smooth orientable Riemann surfaces.

In the previous subsection, we proposed a solution for the first puzzle. In this subsection, we will discuss the second puzzle, and in the special case that $\Sigma_T = S^2$, propose a possible interpretation of the extra terms.

Before going on, we should observe that such extra terms are common in decomposition. As a simple prototypical example, let us consider the example of a two-dimensional $SU(2)$ gauge theory with center-invariant matter. This is equivalent to (decomposes into) a pair of $SO(3)$ theories with different discrete theta angles, schematically,

$$SU(2) \; = \; SO(3)_+ \coprod SO(3)_- . \tag{4.31}$$

(See e.g. [46, section 2] and references therein.) Each of the $SO(3)$ theories has nonperturbative sectors ($SO(3)$ bundles) not possessed by the $SU(2)$ theory. However, because of their different weightings, those $SO(3)$-specific bundles cancel out. Essentially the same thing is happening here: we have extra worldsheets arising in the Gross-Taylor expansion of a single Nguyen-Tanizaki-Ünsal universe (analogous to the non-$SU(2)$ $SO(3)$ bundles), whose contributions cancel out when one sums over all universes.

A context that is more relevant for us is decomposition of sigma models. We describe some analogous cases here:

- First, consider two-dimensional orbifolds with trivially-acting subgroups. A prototypical example is $[X/D_4]$, where D_4 denotes the eight-element dihedral group with center $\mathbb{Z}_2$. If that center acts trivially, then [10, section 5.2]

$$[X/D_4] \; = \; [X/\mathbb{Z}_2 \times \mathbb{Z}_2] \coprod [X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\rm d.t.}, \tag{4.32}$$

  where each of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds has twisted sectors not present in the D_4 orbifold (analogous to the extra contributions present here). Those additional sectors are weighted differently by discrete torsion, and cancel out when the universes are summed together, just as the extra contributions do here.

- A related example is WZW models arising on boundaries of three-dimensional Chern-Simons theories. If one gauges a trivially-acting one-form symmetry in the bulk three-dimensional theory, the boundary sees an orbifold of a WZW model by a trivially-acting ordinary group symmetry, see [15]. Those WZW models are precisely sigma models (whose targets are Lie groups, with background H flux), and so the story here is closely analogous.

- Sigma models whose targets are gerbes, which can be realized by two-dimensional gauged linear sigma models with gauge groups in which a subgroup acts trivially [11, 112, 113]. Physically, these are equivalent to gauge theories with a restriction on gauge instantons. These also decompose into disjoint unions of ordinary sigma models on underlying spaces (realized via ordinary gauged linear sigma models). The constituent universes are described by gauge theories with no restriction on gauge instantons – so again, there are extra contributions not present in the original theory – which are weighted differently by B fields / theta angles.

As the Gross-Taylor string is believed to be a sigma model, these examples are more directly relevant, albeit they all only refer to cases involving finitely many universes.

Thus, existence of extra terms in the Gross-Taylor expansion of individual Nguyen-Tanizaki-Ünsal universes is not surprising. However, their interpretation in the language of string field theories of sigma models needs to be addressed, and we turn to this next.

Technically, the extra terms are the result of including a projection operator, and such a projection operator has been previously discussed, in the context of finite N corrections, in e.g. [83], [77, section 3]. Reference [83] describes the projection operator as giving rise to a "projection point," an analogue of the Ω-points (interpreted in terms of branched covers in [4]), and [77, section 3] breaks the resulting contributions up into a "perturbative" sector (with even powers of N) and a "residual" sector (which contains terms with odd powers of N), which they suggest should be attributed to open string worldsheets. (See also e.g. [123, section 3.2], which also interprets odd powers of N in $U(N)$ theories in terms of open strings.) As their focus is on finite N corrections, which should incorporate nonperturbative corrections, it is natural for them to assume open strings are involved in their analysis.

Here, we are seeing projection operators arising in the large N limit, not just in finite N corrections, which distinguishes this case from the analyses in e.g. [77, 83]. At least naively, we would not expect nonperturbative[22] corrections to a large N limit, and hence, although it is still a potential interpretation, we would not expect open string contributions in our case.

We should quickly add that another places where odd powers of N arises is in the Gross-Taylor expansion of $SO(N)$ and $Sp(N)$ gauge theories, see e.g. [71–75], where it is said to reflect nonorientable Riemann surfaces.

Briefly, one other interpretation of these terms is in terms of singular branched covers, obtained as limits of smooth branched covers in which the branch points collide. We will describe this in examples, and one might speculate that perhaps this is a reflection of some subtle differences in contact terms arising in the Gross-Taylor sigma model for the individual Nguyen-Tanizaki-Ünsal universes. We do not exclude this possibility. In any event, in this section we will float a different proposal, which will have another application later in section 5.

[22]Nonperturbative in the string dilaton, of course, as opposed to α'_{GT}.

In this subsection, we suggest another potential interpretation of those extra terms arising in the Gross-Taylor expansion of the Nguyen-Tanizaki-Ünsal universes (and potentially also the "projection points" of [83]), different from either open strings or nonorientable worldsheets, in the special case $\Sigma_T = S^2$. We propose that they may arise from sigma models with stacky worldsheets, and will merely provide a 'proof of principle' for such a description, but, we emphasize, we are only making a proposal, a suggestion, not a definitive statement.

Our proposal is that the extra contributions may rise from worldsheets $\hat{\Sigma}_W$ that are disjoint unions of n stacky copies S_i of Σ_T:

$$\hat{\Sigma}_W \;=\; \coprod_{i=1}^{n} S_i \tag{4.33}$$

(the same n of S_n, meaning that previously the worldsheet was an n-fold cover of Σ_T). We will explicitly describe a construction of stacks S_i with matching combinatorial description, covering degree, and Euler characteristic of $\hat{\Sigma}_W$ correctly matches the power of N. (We will discuss orbifold Euler characteristics of Hurwitz moduli spaces shortly.) However, there will be multiple disjoint unions of this form which satisfy those constraints, as we shall see. Nevertheless, worldsheets of this form represent our current best guess at an interpretation of these extra terms, at least in the case $\Sigma_T = S^2$.

Stacks may sound exotic, but in many ways they are simple generalizations of spaces. They admit metrics, spinors, gauge fields, and so forth, and can be dealt with using the usual tools of differential geometry. See for example [138–143], [144, lecture 3] for some introductory material. In any case, we will only require stacks[23] that take the form of Riemann surfaces with local orbifold points, which will simplify the discussion. We collect some relevant facts about stacks in appendix B.

For these reasons, it is natural to conjecture that it may be possible to define sigma models with stacky worldsheets, not just stacky target spaces (as has been discussed in e.g. [11, 112, 113]). We shall assume that this is the case in the remainder of this subsection, and turn to the construction of the stacky worldsheets from the Gross-Taylor combinatorics.

Now, let us turn to the construction of our new worldsheets $\hat{\Sigma}_W$, which are disjoint unions of stacky copies S_i of the Riemann surface Σ_T,

$$\hat{\Sigma}_W \;=\; \coprod_{i=1}^{n} S_i, \tag{4.34}$$

The stacks S_i are constructed from the Gross-Taylor combinatorics as follows. Write each $v_a \in S_n$ as a product of cycles, of the form $(1 \cdots n_1)(n_2 \cdots n_3) \cdots$. If the integer i is an element of a cycle of v_a of length k, put a $\mathbb{Z}_k$ orbifold point on Σ_T, corresponding to v_i.

[23] Technically, we are working with smooth Deligne-Mumford stacks, and the subset of those which are of the form of local orbifolds on Riemann surfaces. We will not use, for example, gerbes on curves.

As a simple example, suppose $n = 6$ and $v = (12)(345)(6) \in S_6$. Then, of the six copies of Σ_T appearing in the disjoint union, two would have $\mathbb{Z}_2$ orbifold points, three would have $\mathbb{Z}_3$ orbifold points, and one would have no orbifold point at all.

The reader should note that even this prescription is not unique; we could interchange which sheets S_i receive which specific orbifold points. Such interchanges will preserve the Euler characteristic, as will be clear momentarily. In section 5, we will apply similar ideas to replace branched cover interpretations, and we will give a more systematic construction of such disjoint unions from branched covers, which will eliminate such ambiguity.

Now, let us compute the Euler characteristic of the disjoint union (4.34) above. The idea is to start with the Euler characteristic of a disjoint union of n copies of Σ_T, then subtract the Euler characteristic of nL disks, and add back the contribution from disks containing single orbifold points. In doing so, since a disk with a single $\mathbb{Z}_k$ orbifold point has Euler characteristic $1/k$ and appears k times, a collection of k such disks will contribute 1. In the prescription above, the Euler characteristic contributed by all of the disks determined by a single v is therefore equal to the number of cycles.

Assembling these pieces, we then see that

$$\begin{aligned} \chi\left(\coprod_{i=1}^{n} S_i \right) &= n\chi(\Sigma_T) \; - \; nL \; + \; \sum_{j=1}^{L} K_{v_j}, && (4.35) \\ &= n(2-2p) \; + \; \sum_{j=1}^{L} \left(K_{v_j} - n \right), && (4.36) \end{aligned}$$

which matches the exponent of N in $Z_R^+(A, p, N)$. The ambiguity mentioned above, namely redefining the S_i by moving orbifold points between different sheets, clearly preserves $\chi(\coprod_i S_i)$.

It is important to note that this ansatz will only generate the correct power of N in the case of $\Sigma_T = S^2$, so that $p = 0$. For $p > 0$, p the genus of Σ_T, there are additional group-theoretic factors of the form

$$\prod_{i=1}^{p} [s_i, t_i] \, . \qquad (4.37)$$

Ordinarily these have to close up to the v_i insertions, but, in the presence of a projector P_r, they no longer need close, and also do not come with any ameliorating factors of N. In examples, we will see that they do not appear to have a natural stacky interpretation, at least not following the ansatz above. For this reason, we only apply our ansatz to the case $p = 0$.

So far we have verified that the exponent of N matches the Euler characteristic of $\hat{\Sigma}_W$. Ideally, to thoroughly check this proposal, one would also compute the Euler characteristics of the corresponding Hurwitz moduli spaces, essentially to verify the detailed prediction [4, equ'n (1.1)]

$$Z^{+}(0,N,p) \;=\; \exp\left[\sum_{h=0}^{\infty} N^{2-2h}\chi\left(\overline{H(h,p)}\right)\right], \tag{4.38}$$

where $H(h,p)$ denotes the Hurwtiz moduli space of maps from a connected worldsheet of genus h to one of genus p. Here, we are enlarging the number of possible worldsheets through the addition of stacky points, and it is not entirely clear how the different numbers of stacky points should be weighted, to extend the expression above. For example, formally, possible extensions of (4.38) include the form

$$Z^{+}(0,N,p) \;=\; \exp\left[\sum_{h=0}^{\infty} N^{2-2h}\left(\prod_{k=2}^{\infty}\left(\sum_{n_k=0}^{\infty} f_k(n_k) N^{(-1+1/k)n_k}\right)\right)\chi\left(\overline{H(h,p,n_2,n_3,\cdots)}\right)\right],$$

where $H(h,p,n_2,n_3,\cdots)$ denotes the Hurwitz moduli space with n_k $\mathbb{Z}_k$ orbifold points for each k, and for unknown functions $f_k(n_k)$, defining relative multiplicities of $\mathbb{Z}_k$ orbifold points, (For example, $f_k(n) = 1$, $f_k(n) = 1/n!$, and $f_k(n) = 1/(k!n!)$ all naively seem equally plausible.) To understand this proposal at the level of orbifold Euler characteristics of Hurwitz moduli spaces, we would need a proposal for those functions. Furthermore, as previously discussed, there are multiple possible interpretations as stacky worldsheets for the extra terms. We will discuss possibilities in examples, but, for the reasons above and because we are only attempting to provide a proof of principle, not a definitive answer, we leave a detailed analysis of Hurwitz moduli spaces for future work.

For terms in which a smooth branched cover Σ_W exists, as in the Gross-Taylor interpretation, there exists a(t least one) disjoint union $\coprod_i S_i$ of the form above, to which Σ_W can be deformed (albeit not without leaving the Hurwitz moduli space). The details of this construction are described in section 5.1.

Another aspect of those projectors is that they weight the contributions by phases. In the same spirit as the rest of this section, we propose that those phases be interpreted as resulting from B fields on Σ_W, as we will elaborate in examples.

Later in section 5, we will utilize such disjoint unions to give a different geometric picture of the Gross-Taylor expansion of a single Nguyen-Tanizaki-Ünsal universe. Each component of the disjoint union, each stacky copy of Σ_T, maps to Σ_T with degree one, and we identify each such contribution with an invertible field theory. The projector P_r then contributes weights which can be interpreted in terms of flat B fields on the various components of the disjoint union replacing Σ_W. We will propose to interpret those (stacky) copies of Σ_T, replacing Σ_W, as a reflection of a decoposition of the Gross-Taylor string to invertible field theories.

In passing, we find it interesting that stacky worldsheets appear in this construction. Certainly stacks have previously been described as target spaces of strings, see in particular [11, 112, 113], but there, the worldsheets were ordinary Riemann surfaces, whereas here, the worldsheets are stacks. On the other hand, since maps from stacks factor through the underlying spaces, which is an important part of the proposed dictionary, it is not entirely clear to us how much weight should be ascribed to the difference between a stacky copy S of Σ_T and Σ_T itself. We will leave this for future work.

We will re-use this same proposal to a different end in section 5, where we give an alternative geometric interpretation of the terms in the Gross-Taylor expansion, which better reflects the fact that the Nguyen-Tanizaki-Ünsal universes are invertible field theories.

4.3 Examples

In this section we will walk through the same examples as in section 3.4, this time demonstrating how to describe the extra contributions (arising from presence of projectors P_r) in terms of stacky worldsheets. As in section 3.4, in examples we will restrict to the zero-area limit, to simplify computations.

4.3.1 $p = 0, n = 2$

In this section we consider the special case $p = 0$ (so that $\Sigma_T = \mathbb{P}^1$), and a representation R described by a Young tableau with $n = 2$ boxes. From section 3.4.1, the Gross-Taylor expansion of the Nguyen-Tanizaki-Ünsal universe has the form

$$
\begin{aligned}
Z_R^+(0,p,N) &= \frac{N^{2n}}{n!}\delta\left((\Omega_n)^2 P_r\right), && (4.39)\\
&= \frac{N^{2n}}{n!}\delta\left((1)P_r \,+\, 2\left(\frac{1}{N}\right) vP_r \,+\, \left(\frac{1}{N}\right)^2 v^2 P_r\right), && (4.40)\\
&= \frac{N^4}{2!}\delta(P_r) \,+\, 2\frac{N^3}{2!}\delta(vP_r) \,+\, \frac{N^2}{2!}\delta(v^2 P_r), && (4.41)\\
&= \frac{N^4}{4} \pm \frac{N^3}{2} + \frac{N^2}{4}. && (4.42)
\end{aligned}
$$

As discussed in section 3.4.1, the first and third terms above can be interpreted as contributions to the original Gross-Taylor sigma model, from maps of fixed degree $n = 2$. We interpret these as reflecting the presence of a suitable symmetry in the Gross-Taylor sigma model.

The middle term is novel to the expansion of a Nguyen-Tanizaki-Ünsal universe. The exponent of N is odd, and so this term cannot correspond to a map from a smooth curve Σ_W to $\Sigma_T = \mathbb{P}^1$. It is constructed from a Dijkgraaf-Witten correlation function with a single v, plus a projector.

Now, how should this term be interpreted?

Suppose we try to interpret this term by expanding out the projector P_r, picking out the v term in P_r. Then, this term is

$$\pm 2 \frac{N^3}{2!} \delta(vv). \tag{4.43}$$

Following the usual prescription, this would naively appear to be a branched double cover of $\mathbb{P}^1$, branched over two points, which is another $\mathbb{P}^1$. However, that has the wrong Euler characteristic to match the power of N.

If we ignore the projector, this term appears to describe a double cover of $\mathbb{P}^1$, branched over a single point, which does not exist as a smooth manifold.

Instead, we propose to interpret this term as describing a stack. Following the discussion in section 4.2, given the single factor of v, we propose to interpret this in terms of a curve with a $\mathbb{Z}_2$ orbifold point, and in fact, as $n = 2$ copies of Σ_T, each with a $\mathbb{Z}_2$ orbifold point, again restricted to maps of total degree $n = 2$ as above.

To check this proposal, we check Euler characteristics. Consider a stacky $\mathbb{P}^1$ with a single $\mathbb{Z}_2$ orbifold point, explicitly the weighted projective stack $\mathbb{P}^1_{[1,2]}$. Euler characteristics are additive, and this is the union of a disk and a single $\mathbb{Z}_2$ orbifold, so we can write

$$\chi(\text{stacky } \mathbb{P}^1) \; = \; \chi(\text{disk}) \, + \, \chi(B\mathbb{Z}_2) \; = \; 1 \, + \, 1/2 \; = \; 3/2, \tag{4.44}$$

using the fact that $\chi(B\mathbb{Z}_2) = 1/2$. (See for example appendix B.1 for more details on Euler characteristic computations in stacky curves.)

Now, this is just a single cover of $\Sigma_T = \mathbb{P}^1$, but we can create a double cover by taking two copies. Note that the Euler characteristic of two copies is given by

$$\chi(\mathbb{P}^1_{[1,2]} \coprod \mathbb{P}^1_{[1,2]}) \; = \; (2)(3/2) \; = \; 3, \tag{4.45}$$

which precisely matches the power of N appearing in the middle term. Furthermore, each stack separately has a projector to $\mathbb{P}^1$, so there certainly exists at least one map $\mathbb{P}^1_{[1,2]} \coprod \mathbb{P}^1_{[1,2]} \to \mathbb{P}^1$.

The two choices of sign on the middle term of (4.42) arise from the two possible projectors P_r, for r an irreducible representation of $S_2 = \mathbb{Z}_2$. One might interpret these as choices of B fields on each of the two copies of $\mathbb{P}^1$. For example, perhaps in one universe, the two copies have the same B field (trivial), and in the other universe, one copy has trivial B field whereas the other has $\int B = -1$. An alternate possible interpretation is that one picks the same B field on both elements of the disjoint union: $+1$ for one r, -1 for the other. (To be clear, ordinarily in a sigma model, the B field is defined on the target Σ_T, not the worldsheet, whereas here we are fixing a closed 2-form on the worldsheet.)

In principle, there is another stack that is a double cover of $\mathbb{P}^1$, with related combinatorics and matching Euler characteristics. Specifically, consider the disjoint union of one ordinary $\mathbb{P}^1$ and one $\mathbb{P}^1$ with two $\mathbb{Z}_2$ orbifolds points. The ordinary $\mathbb{P}^1$ has Euler characteristic two, and a $\mathbb{P}^1$ with two orbifold points has Euler characteristic one, so the Euler characteristic of the disjoint union is three, also matching the power of N. (The existence of this second disjoint union can be interpreted as an ambiguity in the disjoint union construction, in which we have moved one of the $\mathbb{Z}_2$ orbifold points to another sheet in the cover.)

Now, let us briefly comment on orbifold Euler characteristics of Hurwitz moduli spaces. From appendix B.2, a map from a stacky curve to Σ_T is equivalent to an ordinary map from the underlying curve to Σ_T, so it is natural to suspect that the orbifold Euler characteristic of a Hurwitz moduli space of maps from a curve with stacky points is the same as that of maps from a curve with the stacky points omitted. (This might possibly ignore sublteties in the compactification, however.) In the present case, we use the fact that the orbifold Euler characteristic of the Hurwitz moduli space of unbranched maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree one is one. We discussed two potential interpretations of the stacky curves above:

- $S_1 \coprod S_1$, where S_1 denotes a $\mathbb{P}^1$ with a single $\mathbb{Z}_2$ orbifold point. Assuming that one sums over all possible orbifold point insertions with the same weighting, it is natural to speculate that the pertinent factor would be $1/2!$ (from expanding an exponential, to get a disjoint union, in the form of (4.38). Here, however, the extra terms have magnitude 1.
- $\mathbb{P}^1 \coprod S_2$, where S_2 denotes a $\mathbb{P}^1$ with two $\mathbb{Z}_2$ orbifold points inserted. Here, there would not be a symmetry factor as appeared in the previous example, so it is natural to speculate that the pertinent factor would be 1, which does match the magnitude of the extra terms.

We emphasize that this is not a definitive conclusion.

In passing, there is another interpretation of this term, which we will not utilize here. Specifically, there exist a singular branched double cover of $\mathbb{P}^1$. This can be obtained as a limit of a branched double cover of $\mathbb{P}^1$ which is branched over two points, by taking a limit that the two points approach one another. Since the resulting curve can be constructed from two caps and a point, its Euler characteristic is 3, matching the exponent of N.

4.3.2 $p = 0, n = 3$

Next, we consider the case that $\Sigma_T = \mathbb{P}^1$ (so that $p = 0$), and the case that the representation R is described by a Young tableau with $n = 3$ boxes. From section 3.4.2, the Gross-Taylor expansion of the Nguyen-Tanizaki-Ünsal universe is

$$
\begin{aligned}
Z_R^+(0,p,N) &= \frac{N^{2n}}{n!}\delta\left((\Omega_n)^2 P_r\right), \qquad (4.46)\\
&= \frac{N^{2n}}{n!}\delta\left(P_r \,+\, 2\sum_v \left(\frac{1}{N}\right)^{n-K_v} vP_r \,+\, \sum_{ij}\left(\frac{1}{N}\right)^{2n-K_{v_1}-K_{v_2}} v_i v_j P_r\right),
\end{aligned}
$$

The first and third terms can be interpreted as in section 3.4.2. In particular, in neither case does the P_r make any difference, as only the identity element of the projector contributes to the sum. Both of these terms describe smooth covers Σ_W describing degree-three maps to the target $\Sigma_T = \mathbb{P}^1$.

The interpretation of the middle terms is more interesting, as here the projector plays an important role. Depending upon the order of v, there are two cases appearing in the sum.

- First, consider the case that $K_v = 2$, for example if $v = (12)(3)$. Here, the term takes the form
$$
\frac{2}{3!}N^5\delta\left(\sum_v vP_r\right). \qquad (4.47)
$$
Following the prescription of section 4.2, this describes a disjoint union of a single $\mathbb{P}^1$ (mapped to itself by v) and two copies of $\mathbb{P}^1_{[1,2]}$, hence $\Sigma_W = \mathbb{P}^1 \coprod \mathbb{P}^1_{[1,2]} \coprod \mathbb{P}^1_{[1,2]}$, and since $\chi(\mathbb{P}^1_{[1,2]}) = 3/2$, as discussed previously, we see $\chi(\Sigma_W) = 2 + 3/2 + 3/2 = 5$, also matching the exponent of N.

  The disjoint union description is not unique; we can get another such description by moving orbifold points between sheets of the cover. For example, another possible interpretation is as the disjoint union $\mathbb{P}^1 \coprod \mathbb{P}^1 \coprod S$, where S is $\mathbb{P}^1$ with two $\mathbb{Z}_2$ orbifold points. This has Euler characteristic $2 + 2 + 1 = 5$, again matching the exponent of N.

  (Alternatively, we could interpret this as a singular curve, a 3-cover branched over a single point, but we will not utilize that description in this paper.)

- Next, consider the case that $K_v = 1$, for example if $v = (123)$. Here, the term above takes the form
$$
\frac{2}{3!}N^4\delta\left(\sum_v vP_r\right). \qquad (4.48)
$$
Following the prescription of section 4.2, we interpret this as a disjoint union $\Sigma_W = \mathbb{P}^1_{[1,3]} \coprod \mathbb{P}^1_{[1,3]} \coprod \mathbb{P}^1_{[1,3]}$, a disjoint union of three stacky copies of $\Sigma_T = \mathbb{P}^1$, each with a single $\mathbb{Z}_3$ orbifold point. A disk with one $\mathbb{Z}_3$ orbifold has Euler characteristic $1/3$, so $\chi(\mathbb{P}^1_{[1,3]}) = 4/3$, hence Σ_W has Euler characteristic 4, matching the exponent of N.

  As before, the prescription of section 4.2 is not unique, and we can get other valid disjoint unions by moving orbifold points between sheets. Here, such possibilities are

– a disjoint union $\mathbb{P}^1 \coprod \mathbb{P}^1 \coprod S'$ where S' is a $\mathbb{P}^1$ with three $\mathbb{Z}_3$ orbifold points. The latter summand has Euler characteristic zero, so the Euler characteristic of the disjoint union is $2 + 2 + 0 = 4$, matching the exponent of N,
– a disjoint union with three summands: one ordinary $\mathbb{P}^1$, one $\mathbb{P}^1$ with one $\mathbb{Z}_3$ orbifold point, and one $\mathbb{P}^1$ with two $\mathbb{Z}_3$ orbifold points. The latter summand has Euler characteristic $2/3$, so the Euler characteristic of the disjoint union is $2 + 4/3 + 2/3 = 4$, matching the exponent of N.

(Alternatively, we could also interpret this in terms of a singular Σ_W, a 3-cover branched over a single point, where all three sheets meet. This curve can also be constructed from three caps and a point, so its Euler characteristic is 4, which matches the exponent of N. As before, we will not utilize that interpretation here.)

Next, we outline a preliminary check of orbifold Euler characteristics of Hurwitz moduli spaces, to compare amongst these possibilities. As in the previous section, we use the fact that maps from curves with orbifold points factor uniquely through maps without orbifold points, and ignore potential compactification subtleties.

- First, consider the case of extra terms with v such that $K_v = 2$. Since there are three such $v \in S_3$, the magnitude of the numerical factor appearing in the extra terms is

$$\frac{2}{3!}(3) \;=\; 1. \tag{4.49}$$

  We described two possibilities:

  – $\mathbb{P}^1 \coprod S_1 \coprod S_1$, where S_1 denotes a $\mathbb{P}^1$ with a single $\mathbb{Z}_2$ orbifold point. Proceeding as in the previous example, naively we expect a $1/2!$ (as a symmetry factor, from expanding out an exponential).
  – $\mathbb{P}^1 \coprod \mathbb{P}^1 \coprod S_2$, where S_2 denotes a $\mathbb{P}^1$ with two $\mathbb{Z}_2$ orbifold points. Proceeding as in the previous example, naively we expect another $1/2!$, again as a symmetry factor, this time relating the two copies of $\mathbb{P}^1$.

  Neither separately has the right numerical factor, though we do observe that the sum of these two possibilities does add up correctly.

- Next, consider the case of extra terms with v such that $K_v = 1$. Since there are two such $v \in S_3$, the magnitude of the numerical factor appearing in the extra terms is

$$\frac{2}{3!}(2) \;=\; \frac{2}{3}. \tag{4.50}$$

  We described three possibilities:

- $\tilde{S}_1 \coprod \tilde{S}_1 \coprod \tilde{S}_1$, where $\tilde{S}_1$ denotes a $\mathbb{P}^1$ with one $\mathbb{Z}_3$ point. Here, from the same reasoning as before, we naively expect a $1/3!$, as a symmetry factor (from expanding an exponential).
- $\mathbb{P}^1 \coprod \mathbb{P}^1 \coprod \tilde{S}_3$, where $\tilde{S}_3$ denotes a $\mathbb{P}^1$ with three $\mathbb{Z}_3$ points. Here, we expect a $1/2!$.
- $\mathbb{P}^1 \coprod \tilde{S}_1 \coprod \tilde{S}_2$, where $\tilde{S}_2$ denotes a $\mathbb{P}^1$ with two $\mathbb{Z}_3$ orbifold points. Here, from the same reasoning, we naively expect 1, as there is no symmetry between the terms (unless the presence of two orbifold points contributes a symmetry factor).

As before, none of these terms separately duplicates the whole expected factor, though it is possible a sum might, at least if the terms receive additional analogues of symmetry factors due to the presence of multiple orbifold points.

As there are multiple possibilities, we leave a detailed examination of orbifold Euler characteristics of Hurwitz moduli spaces for other work.

4.3.3 $p = 1$

From section 3.4.3, the Gross-Taylor expansion of the Nguyen-Tanizaki-Ünsal universe is

$$Z_R^+(0,p,N) \;=\; 1\frac{N^0}{n!}\sum_{s,t\in S_n} \delta\left((\Omega_n)^0 sts^{-1}t^{-1}P_r \right), \tag{4.51}$$

for r an irreducible representation of S_n.

If the commutator

$$[s,t] \;=\; sts^{-1}t^{-1} \tag{4.52}$$

equals the identity, then the projector P_r is irrelevant, and one recovers the same terms as in a Gross-Taylor expansion restricted to degree n, namely, unbranched n-fold covers of T^2.

If the commutator is different from the identity, then unfortunately our ansatz does not apply in general. For example, suppose that $n = 3$, so that $s, t \in S_3$. Now, S_3 is a nonabelian group with three elements of order 2 and two elements of order 3. The commutator takes values in the commutator subgroup, consisting of even permutations, denoted A_3, which has order 3 (and also contains order-3 elements of S_3, as it happens). If we try to interpret the result as a disjoint union $S_1 \coprod S_2 \coprod S_3$ where the S_3 are copies of T^2 with orbifold points, we run into the issue that $\chi(S_i) \neq 0$ (unless $S_i = T_2$), and so if there are any stacky points present at all, the resulting disjoint union has nonzero Euler characteristic, which does not match the power of N.

Examples of this form are the reason we restrict our ansatz for the extra contributions to the special case $p = 0$ ($\Sigma_T = S^2$).

Now, let us turn to the terms that can be interpreted merely as the restriction to maps of a single degree. Here, these are maps $T^2 \to T^2$. Now, sigma models $T^2 \to T^2$ have, of course, been extensively studied in the literature, as simple computable examples of CFTs. Using existing results, one can show that the partition function of an A-twisted sigma model on T^2 with target T^2 remains modular invariant even after restriction to maps of a single degree. Specifically, recall from [145, section 1] that the partition function on T^2 of the A model with target T^2 is

$$\begin{aligned}\mathrm{Tr}\,(-)^F F_L F_R q^{L_0} \overline{q}^{\overline{L}_0} \;&=\; \frac{t+\overline{t}}{4\pi\tau_2} \sum_{m,n,r,s} \exp\left[-\frac{t}{4\tau_2\rho_2} \left| (m+r\rho) - \overline{\tau}(n+s\rho) \right|^2 \right. \\ &\qquad \left. - \frac{\overline{t}}{4\tau_2\rho_2} \left| (m+r\rho) - \tau(n+s\rho) \right|^2 \right]\end{aligned} \tag{4.53}$$

where ρ is the complex modulus of the target T^2, and the (not necessarily invertible) matrix

$$R \;=\; \left[\begin{array}{cc} r & m \\ s & n \end{array} \right] \tag{4.54}$$

encodes windings, with the degree of the map $T^2 \to T^2$ given by $|\det R\,|$, as explained in [145, section 1]. Now, it is straightforward to check that a modular transformation of

$$F_1^{\mathrm{top}} \;=\; \int \frac{d^2\tau}{\tau_2} \mathrm{Tr}\,(-)^F F_L F_R q^{L_0} \overline{q}^{\overline{L}_0} \tag{4.55}$$

by

$$\tau \;\mapsto\; \frac{a\tau+b}{c\tau+d}, \quad \rho \mapsto \frac{e\rho+f}{g\rho+h} \tag{4.56}$$

for

$$\mathcal{A} \;=\; \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \;\in\; SL(2,\mathbb{Z}), \quad \mathcal{B} \;=\; \left[\begin{array}{cc} e & f \\ g & h \end{array} \right] \;\in\; SL(2,\mathbb{Z}), \tag{4.57}$$

can be absorbed into a redefinition of R as

$$R \;\mapsto\; \mathcal{A}^{-1} R \mathcal{B}. \tag{4.58}$$

However, since $\mathcal{A}$ and $\mathcal{B}$ both have unit determinant, $|\det R\,|$ is invariant, and so restricting to maps of a single degree does not break modular invariance.

4.3.4 $p = 2, n = 2$

From section 3.4.4, the Gross-Taylor expansion of the Nguyen-Tanizaki-Ünsal universe is

$$Z_R^+(0,p,N) = \frac{N^{-4}}{2!} \sum_{s_i,t_i \in S_2} \delta \left((\Omega_2)^{-2} \left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right), \tag{4.59}$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \frac{N^{-4}}{2!} \left(\frac{1}{1-(1/N^2)} \right)^2 \sum_{s_i,t_i \in S_n} \delta \left(\left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right) \\ &\quad - \frac{N^{-4}}{2!} \sum_{s_i,t_i \in S_n} \left(\frac{1}{1-(1/N^2)} \right)^2 \delta \left(\frac{2}{N} v \left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right) \\ &\quad + \frac{N^{-4}}{2!} \sum_{s_i,t_i \in S_n} \left(\frac{1}{1-(1/N^2)} \right)^2 \delta \left(\frac{1}{N^2} v^2 \left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right). \end{aligned} \tag{4.60}$$

Terms in the first and third lines we interpret as in section 3.4.4, in terms of maps from smooth branched covers Σ_W of fixed degree $n = 2$. We propose that the restriction to fixed degree be implemented as described previously.

In the special case that the product of the two commutators, we could interpret the middle term as describing $\Sigma_W = C \coprod C$, where each stack C is Σ_T with a single $\mathbb{Z}_2$ singularity. The curve C has $\chi(C) = -2 - 1 + 1/2 = -5/2$, so the disjoint union has $\chi(\Sigma_W = C \coprod C) = -5$, matching the leading power of N. (Alternatively, we could interpret Σ_W as a singular curve, a double cover of the genus-two curve Σ_T branched over a single point. Since $\chi(\Sigma_T) = 2 - 2p = -2$, removing a disk yields $\chi = -3$, and two copies with one disk added then have $\chi = (2)(-3) + 1 = -5$, matching the exponent of N.) In any event, for reasons previously described, we only suggest this stacky interpretation in the case $p = 0$, so the present example is not applicable.

5 Alternative geometric interpretation of the decomposition

In this section we give an alternative geometric interpretation of the terms in the Gross-Taylor expansion of a Nguyen-Tanizaki-Ünsal universe, in the special case that $\Sigma_T = S^2$, motivated by the fact that the Nguyen-Tanizaki-Ünsal universes are trivial quantum field theories, with a one-dimensional state space. Specifically, instead of interpreting the terms in terms of sigma model maps from branched covers of Σ_T, we instead interpret them in terms of a counting problem, counting disjoint unions

$$\coprod_i S_i \tag{5.1}$$

of stacky copies S_i of Σ_T. The idea is that the Gross-Taylor expansion of a single universe is really just reproducing a single numerical factor, which can be reproduced by a suitable counting problem, hamely counting (stacky) copies of Σ_T.

In the case $\Sigma_T = S^2$, we will provide a systematic procedure for reinterpreting Dijkgraaf-Witten partition functions in terms of disjoint unions of stacks instead of branched covers, and will check that the result is consistent with powers of N, in the sense that the Euler characteristic of the stack one obtains matches the Euler characteristic of the branched cover.

We will not attempt to interpret the numerical factors in terms of automorphisms of covering maps (or orbifold Euler characteristics of Hurwitz moduli spaces), as in this section we are not associating a nontrivial sigma model. Instead, we interpret the numerical factors merely as defining counterterms, and as such provide no further consistency tests.

That said, in principle, one could also imagine interpreting these terms in terms of a Gross-Taylor string, restricted to degree one maps. As discussed in appendix B.2, maps from a stacky curve to a space factor through an underlying smooth curve. In the present case, maps $S_i \to \Sigma_T$ factor through the projection to the underlying ordinary curve Σ_T:

$$S_i \stackrel{\pi}{\longrightarrow} \Sigma_T \longrightarrow \Sigma_T, \tag{5.2}$$

hence degree one maps $S_i \to \Sigma_T$ can be identified with degree one maps $\Sigma_T \to \Sigma_T$.

This is another perspective on identifying each stacky curve S_i with a copy of an invertible field theory on Σ_T. Put another way, roughly speaking, we are constructing the Nguyen-Tanizaki-Ünsal universes on a two-dimensional space Σ_T by summing over copies of degree one maps $\Sigma_T \to \Sigma_T$.

In any event, our perspective in this section is merely to provide a geometric counterpoint to the idea that the Nguyen-Tanizaki-Ünsal universes are invertible field theories, by rethinking the combinatorics as describing counting copies of Σ_T, rather than in terms of a path integral for a sigma model.

We emphasize that the existence of a decomposition of the Gross-Taylor string is not in question – it is automatic for any unitary two-dimensional topological field theory (with semisimple local operator algebra). What is more interesting is that there exists a structure in the combinatorics used to justify the existence of the Gross-Taylor string, which appears to reflect the presence invertible field theories.

We also note that the stacks we consider in this paper are all smooth Deligne-Mumford stacks, which are specifically of the form of Riemann surfaces with isolated local orbifold points. More general Deligne-Mumford stacks describing e.g. gerbe structures will not appear in this paper.

5.1 Disjoint unions instead of branched covers

As previously discussed, the chiral partition function of the Gross-Taylor expansion of a single Nguyen-Tanizaki-Ünsal universe has the form (3.32), namely

$$
\begin{aligned}
Z_R^+(A,p,N) \quad &= \quad N^{n(2-2p)} \sum_{s_i,t_i \in S_n} \sum_{L=0}^{\infty} \sum_{v_1,\cdots,v_L \in S_n} {}' N^{\sum_j (K_{v_j}-n)} \frac{d(2-2p,L)}{n!} \\
& \qquad \cdot \delta\left(v_1 \cdots v_L \left(\prod_{i=1}^{p} [s_i,t_i] \right) P_r \right) \exp\left(-\frac{A}{\alpha'_{GT}} n \right),
\end{aligned}
\tag{5.3}
$$

where n is the number of boxes in the Young tableau for R.

Previously we have discussed how this can be interpreted in terms of

1. a sum over branched n-fold covers $\Sigma_W \to \Sigma_T$,
2. plus some extra contributions, arising from the presence of the projector P_r.

In the special case $\Sigma_T = S^2$, we have described how the extra contributions can be interpreted as disjoint unions of stacky copies of Σ_T. Here, for the case $\Sigma_T = S^2$, we extend that alternative interpretation to include terms previously interpreted as branched covers. In other words, in this section we will describe how all of the terms can be interpreted as a disjoint union of n stacky copies of Σ_T, with Euler characteristic matching each power of N.

To be clear, in this proposal in this section, we are not interpreting the terms physically as a sigma model from that disjoint union; instead, we are setting up a counting problem, which seems more nearly appropriate to the fact that the Nguyen-Tanizaki-Ünsal universes are trivial (invertible) field theories.

Since we are discarding the sigma model interpretation in this section, we will not attempt to compare orders of automorphism groups, as in this alternative interpretation, we are merely giving a geometric counterpoint to the description of invertible field theories.

For the remainder of this section, we will focus on understanding the terms previously interpreted as branched covers in this language.

Previously, the elements $v_1, \cdots, v_L \in S_n$ defined monodromies about L branch points in the base curve Σ_T. In the present interpretation, the orders of the cycles in each monodromy v_i define orders of orbifold points on copies of the Riemann surface Σ_T.

We can describe the construction systematically as follows. Let B be the branch locus on Σ_T, and p some point (to base loops). We can associate either branched n-covers or disjoint unions $\hat{\Sigma}_W$ of n stacky copies S_i of Σ_T to homomorphisms $f : \pi_1(\Sigma_T - B, p) \to S_n$. We have already discussed branched covers; we construct the disjoint union of n stacky copies of Σ_T as follows. Pick a set of nonintersecting paths emanating from the fixed point p, each winding around a point in the branch locus B. For each point b in the branch locus B, let v_b be the image of the corresponding path under the homomorphism f. We construct a set of stacky curves S_i, where each S_i is a copy of Σ_T with orbifold structures at each $b \in B$. If $d(i,b)$ is the length of a cycle containing i in the cycle decomposition of v_b, then the orbifold structure on S_i at $b \in B$ is $\mathbb{Z}_{d(i,b)}$.

Later in this section we also describe a construction of $\hat{\Sigma}_W$ directly from Σ_W.

To make this more clear, let us update our previous example. Specifically, consider $\Sigma_T = \mathbb{P}^1$ with two insertions at positions denoted A, B, and let $p \in \mathbb{P}^1$, as illustrated in the figure below.

Shown on the left is a schematic illustration of $\mathbb{P}^1$ with the two points A, B, and the basepoint p for paths. On the right is the same illustration with two nonintersecting paths from p marked.

Let the monodromies about the two points be denoted v_A, v_B. Suppose that $n = 3$, so that $v_A, v_B, v_C \in S_3$, and take

$$v_A \;=\; (12)(3) \;=\; v_B. \tag{5.4}$$

It is straightforward to check that the product

$$v_A v_B \;=\; 1, \tag{5.5}$$

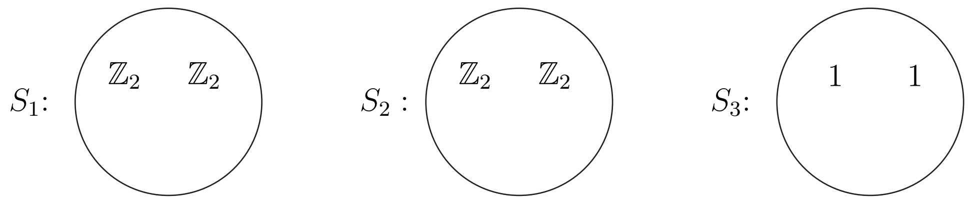
so the delta function is nonzero.

We have already seen in section 3.3 how this translates into branching data for branched 3-covers. Briefly, locally near the points A and B, two of the sheets collide, but the third remains disjoint. This results in a branched cover of the form

$$\Sigma_W \;=\; \mathbb{P}^1 \coprod \mathbb{P}^1, \tag{5.6}$$

as already discussed, and $\chi(\Sigma_W) = 4$.

In terms of disjoint unions of stacky curves, the three stacky copies $S_{1,2,3}$ of $\mathbb{P}^1$ have local orbifolds as illustrated in the diagram below:



Using the methods in appendix B.1, it is straightforward to check

$$\chi(S_1) \;=\; 1 \;=\; \chi(S_2), \quad \chi(S_3) \;=\; 2, \tag{5.7}$$

hence

$$\chi\left(S_1 \coprod S_2 \coprod S_3 \right) \;=\; 4, \tag{5.8}$$

matching the Euler characteristic of $\Sigma_W = \mathbb{P}^1 \times \mathbb{P}^1$.

Next, we shall give a systematic description of how to replace n-fold covers of Σ_T with disjoint unions of n copies of Σ_T with orbifold points, of matching Euler characteristic.

We will replace any $\sigma \in S_n$ by a product of cyclic orbifolds defined by the permutation structure, in the obvious way. We list examples in the table below:

n	$\sigma \in S_n$	Orbifold
2	$1 = (1)(2)$	1
2	(12)	$\mathbb{Z}_2$
3	$(12)(3)$	$\mathbb{Z}_2$
3	(123)	$\mathbb{Z}_3$
4	$(12)(34)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
4	$(123)(4)$	$\mathbb{Z}_3$
4	(1234)	$\mathbb{Z}_4$

The construction is as follows. Given a smooth branched n-cover $\pi : \Sigma_W \to \Sigma_T$, let $B \subset \Sigma_T$ denote the branch locus. Let $p \in \Sigma_T$, $p \not\in B$, and choose a set of nonintersecting paths emanating from p, each winding around a point in the branch locus B. For each $b \in B$, let $v_b \in S_n$ denote the monodromy about b. Now, we construct a series of stacky curves S_i, where each S_i is associated to an element of $\pi^{-1}(p)$, and is a copy of Σ_T, with orbifold structures at each $b \in B \subset \Sigma_T$. If $d(i,b)$ is the length of a cycle containing i in the cycle decomposition of v_b, then the orbifold structure on S_i at $b \in B$ is $\mathbb{Z}_{d(i,b)}$.

We then define $\hat{\Sigma}_W = \coprod_i S_i$. Note that $\hat{\Sigma}_W$ itself depends upon a choice of paths; different paths will change the S_i and hence $\hat{\Sigma}_W$. (As a result, since there is a braid group action on the paths, there is a braid group action on the possible choices of $\hat{\Sigma}_W$. We will see an example later in which the braid group can change the disjoint union $\hat{\Sigma}_W$, for a fixed branched cover Σ_W.)

We claim that

$$\chi(\Sigma_W) \;=\; \chi(\hat{\Sigma}_W). \tag{5.9}$$

We can see this as follows. First, from results on orbifold Euler characteristics of stacky curves in appendix B.1, note that

$$\chi(S_i) \;=\; \chi(\Sigma_T) \;+\; \sum_{b\in B} \left[\frac{1}{d(i,b)} - 1 \right]. \tag{5.10}$$

Then,

$$\begin{aligned}
\chi(\hat{\Sigma}_W) &= \chi\left(\coprod_i S_i \right) \;=\; \sum_{i=1}^{n} \chi(S_i), && (5.11)\\
&= n\chi(\Sigma_T) \;+\; \sum_{i=1}^{n} \sum_{b\in B} \left[\frac{1}{d(i,b)} - 1 \right], && (5.12)\\
&= n\chi(\Sigma_T) \;-\; n|B| \;+\; \sum_{i=1}^{n} \sum_{b\in B} \frac{1}{d(i,b)}. && (5.13)
\end{aligned}$$

However,

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{b\in B} \frac{1}{d(i,b)} &= \sum_{b\in B} (\text{number of cycles in } v_b)\,, && (5.14)\\
&= \sum_{b\in B} (\text{number of ramification points over } b)\,, && (5.15)\\
&= \text{degree of ramification divisor}, && (5.16)
\end{aligned}$$

and so from the Riemann-Hurwitz formula,

$$\chi(\Sigma_W) \;=\; n\chi(\Sigma_T) \;-\; \sum_{P\in\Sigma_W} (e_P - 1)\,, \tag{5.17}$$

where Σ_W is a branched n-fold cover of Σ_T and e_P is the ramification index at P, we have that the Euler characteristic of the disjoint union $\hat{\Sigma}_W$ matches that of the smooth branched cover Σ_W:

$$\chi(\hat{\Sigma}_W) \;=\; \chi(\Sigma_W). \tag{5.18}$$

In passing, the construction above applies to Σ_T of any genus. We restrict to genus zero to accomodate interpretations of the additional terms of section 4.2, which we only know how to interpret in terms of disjoint unions of stacks in the special case of genus zero.

Below we list some examples.

- If Σ_W is an unbranched n-sheeted cover of Σ_T, we replace Σ_W with a disjoint union of n copies of Σ_T. From the Riemann-Hurwitz formula (5.17),

$$\chi(\Sigma_W) \;=\; \chi\left(\coprod_n \Sigma_T \right). \tag{5.19}$$

- Suppose Σ_W is a branched double cover of Σ_T, branched over k points. We replace Σ_W by a disjoint union of two copies of a stack S, where S is Σ_T with k $\mathbb{Z}_2$ orbifold points.

  Let us check that these have the same Euler characteristic. First, from Riemann-Hurwitz (5.17),

$$\chi(\Sigma_W) \;=\; 2\chi(\Sigma_T) - k. \tag{5.20}$$

  To compute $\chi(S)$, remove k disks from Σ_T and replace each with a disk containing a $\mathbb{Z}_2$ orbifold. This modifies the Euler characteristic as

$$\chi(S) \;=\; \chi(\Sigma_T) \;-\; k(1) \;+\; k(1/2), \tag{5.21}$$

  using the fact that the Euler characteristic of an ordinary disk is 1, and that of a disk containing a $\mathbb{Z}_m$ orbifold is $1/m$. It follows immediately that

$$\chi(\Sigma_W) \;=\; \chi(S \coprod S) \;=\; 2\chi(S). \tag{5.22}$$

- Let Σ_W be a 3-fold cover of Σ_T, branched along two points, with branching at each point described by elements of the conjugacy class (12)(3). Then, Σ_W is a disjoint union of one copy of Σ_T and one branched double-cover of Σ_T, branched over two points, hence

$$\chi(\Sigma_W) \;=\; \chi(\Sigma_T) \;+\; 2\chi(\Sigma_T) - 2. \tag{5.23}$$

  The corresponding stack is a disjoint union $\Sigma_T \coprod S \coprod S$, where S is Σ_T with a pair of $\mathbb{Z}_2$ orbifold points. As a result, from the analysis above, $\chi(S) = \chi(\Sigma_T) - 1$, hence

$$\chi(\Sigma_T \coprod S \coprod S) \;=\; 3\chi(\Sigma_T) - 2, \tag{5.24}$$

  matching $\chi(\Sigma_W)$.

- Suppose there are two branch points. If the monodromy about one is $v \in S_n$, then the monodromy about the other is v^{-1}. Assume v has m cycles, in which the ith cycle has k_i elements. (The same will be true of v^{-1}.) Define S_i to be a $\mathbb{P}^1$ with two $\mathbb{Z}_{k_i}$ orbifolds. Then,

$$\hat{\Sigma}_W \;=\; \coprod_{j=1}^{m} \left(\coprod_{k_j} S_j \right). \tag{5.25}$$

  Now, let us compare Euler characteristics. For a single curve,

$$\chi(S_i) \;=\; \chi(\mathbb{P}^1) - 2 + 2/k_i, \tag{5.26}$$

hence

$$\chi\left(\hat{\Sigma}_W\right) = \sum_{j=1}^{m} k_j \chi(S_j), \tag{5.27}$$

$$= \sum_{j=1}^{m} k_j \left(\chi(\mathbb{P}^1) - 2 + 2/k_j\right), \tag{5.28}$$

$$= n\chi(\mathbb{P}^1) - 2n + 2m, \tag{5.29}$$

using the fact that

$$\sum_{j=1}^{m} k_j = n. \tag{5.30}$$

By comparison,

$$\chi(\Sigma_W) = n\chi(\mathbb{P}^1) - \sum_{i=1}^{2}(n-m), \tag{5.31}$$

which matches.

- Let Σ_W be a 3-fold cover of Σ_T, branched along three points, with branching at each point described by elements of the conjugacy class (123). Then,

$$\chi(\Sigma_W) = 3\chi(\Sigma_T) - (3)(2). \tag{5.32}$$

  The corresponding stack is a disjoint union $S \coprod S \coprod S$, where each S is a copy of Σ_T with three $\mathbb{Z}_3$ orbifold points. It is straightforward to compute

$$\chi(S) = \chi(\Sigma_T) - 3 + (3)(1/3) = \chi(\Sigma_T) - 2, \tag{5.33}$$

  hence

$$\chi(S \coprod S \coprod S) = 3\chi(\Sigma_T) - 6, \tag{5.34}$$

  matching Σ_W.

- Let Σ_W be a 3-fold cover of Σ_T, branched along three points, with branching at the three points described by (12)(3), (13)(2), and (23)(1). (In other words, at each branch point, two of the three sheets intersect, but a different pair at each one.) In this case,

$$\chi(\Sigma_W) = 3\chi(\Sigma_T) - 3(3-2) = 3\chi(\Sigma_T) - 3. \tag{5.35}$$

  We can replace Σ_W by a disjoint union of three copies of the stack S, where S is Σ_T with two $\mathbb{Z}_2$ orbifold points. It is straightforward to compute

$$\chi(S) = \chi(\Sigma_T) - 2 + 2(1/2) = \chi(\Sigma_T) - 1 \tag{5.36}$$

  (by omitting two disks and replacing them with disks with $\mathbb{Z}_2$ orbifolds), hence

$$\chi(S \coprod S \coprod S) = 3\chi(\Sigma_T) - 3, \tag{5.37}$$

  matching $\chi(\Sigma_W)$.

- Finally, we consider an example which will illustrate the path-dependence of the prescription, and demonstrate that in general one will get different disjoint unions from different choices of path.

  Begin with an $n = 4$ cover of $\Sigma_T = \mathbb{P}^1$, branched over five points. As before, fix a base point and pick a set of paths around the five branch points b_{1-5} with monodromies

$$\begin{aligned} v_1 &= (12)(34), && (5.38)\\ v_2 &= (13)(24), && (5.39)\\ v_3 &= (14)(23), && (5.40)\\ v_4 &= (13), && (5.41)\\ v_5 &= (13). && (5.42) \end{aligned}$$

  It is straightforward to check that the product of the v_i is 1, so this is well-defined on $\mathbb{P}^1$, and from the Riemann-Hurwitz formula, the genus of the branched cover is 1. As a disjoint union of stacks, our prescription gives a disjoint union of four S_i, each a stacky version of $\mathbb{P}^1$, given as follows:

  - S_1 has five $\mathbb{Z}_2$ orbifold points, one at each of the original branch points.
  - S_2 has three $\mathbb{Z}_2$ orbifold points, at the branch points, b_{1-3}.
  - S_3 has five $\mathbb{Z}_2$ orbifold points, one at each of the original branch points.
  - S_4 has three $\mathbb{Z}_2$ orbifold points, at the branch points, b_{1-3}.

  Now, consider a braid group action that maps the monodromies above to

$$\begin{aligned} v_1' &= v_1 = (12)(34), && (5.43)\\ v_2' &= v_2 = (13)(24), && (5.44)\\ v_3' &= v_4^{-1} v_3 v_4^{-1} = (12)(34), && (5.45)\\ v_4' &= v_4^{-1} v_3^{-1} v_4 v_3 v_4 = (24), && (5.46)\\ v_5' &= v_5 = (13). && (5.47) \end{aligned}$$

  The branched cover is the same, but the disjoint union of stacks differs. Here, we have a disjoint union of four stacky copies S_i' of $\mathbb{P}^1$, each with four $\mathbb{Z}_2$ orbifold points:

  - S_1' has $\mathbb{Z}_2$ orbifold points at b_{1-3}, b_5.
  - S_2' has $\mathbb{Z}_2$ orbifold points at b_{1-4}.
  - S_3' has $\mathbb{Z}_2$ orbifold points at b_{1-3}, b_5.
  - S_4' has $\mathbb{Z}_2$ orbifold points at b_{1-4}.

  Thus, we see here explicitly that the path dependence means one can obtain different disjoint unions of stacks from the same branched cover.

5.2 Deforming branched covers to disjoint unions

Intuitively, one might expect these disjoint unions $\hat{\Sigma}_W$ constructed from branched covers to be limits of deformations of the branched covers Σ_W. However, matters are more complex. Let us walk through this carefully.

If we fix the complex structure of the base Σ_T and the location of the branch points, then there are only finitely many covers for fixed n. Varying the complex structure on Σ_T and the location of the branch points gives the Hurwitz moduli space. If that Hurwitz moduli space is connected, then the disjoint union $\hat{\Sigma}_W$ cannot be a limit of points on that moduli space, because of Zariski's main theorem, which says that if a proper family of varieties (here, curves) has a connected general fiber, then every fiber must be connected. So the disjoint unions $\hat{\Sigma}_W$ automatically describe points in a different component of the Hurwitz moduli space.

Similarly, there are no metric deformations (Ricci-flat or otherwise) relating the branched cover Σ_W to the disjoint union $\hat{\Sigma}_W$. A simple example involves unbranched covers of T^2, which are also themselves T^2. A metric deformation would deform one T^2 (the cover) to a disjoint union of multiple T^2's; however, as a Calabi-Yau, the space of Ricci-flat metrics on T^2 is well-known, and no such deformation through Ricci-flat metrics exists.

One could consider more general metric deformations, not necessarily through Ricci-flat metrics, but the same problem arises. All metric deformations induce conformal deformations, or put another way, each conformal class contains a metric of constant scalar curvature, and so Zariski's connectedness argument above applies again.

That said, there are (at least) two options for larger moduli spaces which can link the covers to disjoint unions. (Neither of these is a metric deformation, however.)

One option is to view the covers and disjoint unions as points in a larger moduli space, distinct from the Hurwitz moduli space. To that end, we can view the Hurwitz moduli space as parametrizing tuples (C, B, ρ, p) where C is a curve, B the branch locus, $p \in C - B$, and $\rho : \pi_1(C - B, p) \to S_n$ is a homomorphism, and then to get a larger moduli space, embed S_n into some continuous group such as $GL(n, \mathbb{C})$. Let λ denote the composition of ρ with the embedding of S_n into a larger group. If one fixes the conjugacy classes of the images of the loops around the points of B (but as conjugacy classes in $GL(n, \mathbb{C})$ instead of S_n), then this space is connected and all of the Hurwitz moduli spaces with local monodromies in these conjugacy classes will embed as closed subvarieties (but will be disjoint inside it).

A second option exists, which changes the genus of the cover at intermediate points between the Hurwitz moduli space and the disjoint union. For example, given an n-sheeted etale cover of an elliptic curve E, there is a moduli space M of ramified n-sheeted covers of deformations of E such that general points of M represent smooth genus n ramified covers of a deformation of E. You can also arrange for M to contain two disjoint special closed subvarieties N_c and N_d parametrizing singular ramified n-sheeted covers of deformations of E (of arithmetic genus n). Furthermore, starting from the universal family of curves over N_c, the family of normalizations of the fibers is a family of smooth and unramified connected covers of deformations of E, i.e. of smooth curves of genus one. Similarly, passing to the normalizations of the fibers of the universal family over N_d gives a family of n disconnected copies of deformations of E. In any event, this is not a metric deformation.

5.3 Examples

5.3.1 $p = 0, n = 2$

First, consider the case that the Yang-Mills theory lives on $\Sigma_T = \mathbb{P}^1$ (so that $p = 0$), and restrict to maps of degree $n = 2$, as previously discussed in sections 3.4.1, 4.3.1. The Gross-Taylor expansion of the Nguyen-Tanizaki-Ünsal universe is

$$Z_R^+(0, p, N) \;=\; \frac{N^4}{4} \pm \frac{N^3}{2} + \frac{N^2}{4}. \tag{5.48}$$

Adding two such contributions together, for each choice of projector P_r, recovers the original Gross-Taylor result (3.57).

The first term can be interpreted as before, in terms of maps $\mathbb{P}^1 \coprod \mathbb{P}^1 \to \mathbb{P}^1$, which is consistent with the fact that $\chi(\mathbb{P}^1 \coprod \mathbb{P}^2) = 2\chi(\mathbb{P}^1) = 4$.

The middle term we interpret as previously in section 4.3.1, as a disjoint union of two copies of a stacky $\mathbb{P}^1$, each copy with a single $\mathbb{Z}_2$ orbifold point.

Finally, we turn to the third term in (5.48). Previously, we interpreted this term as describing maps from a branched double cover Σ_W of $\mathbb{P}^1$, branched over two points (the locations of each v). (In fact, Σ_W is itself another $\mathbb{P}^1$.) Here, we interpret the last term as describing a stacky Σ_W, given by a disjoint union of two stacky $\mathbb{P}^1$, each with two $\mathbb{Z}_2$ orbifold points.

As a consistency check, let us compute Euler characteristics. A stacky $\mathbb{P}^1$ with two $\mathbb{Z}_2$ orbifold points can be described as a cylinder with two $\mathbb{Z}_2$ orbifolds. Since the Euler characteristic is additive, we can write it as

$$\chi(\text{cylinder}) \;+\; 2\chi(B\mathbb{Z}_2) \;=\; 0 \;+\; 2(1/2) \;=\; 1. \tag{5.49}$$

So, any one stacky $\mathbb{P}^1$ with two $\mathbb{Z}_2$ orbifold points has Euler characteristic 1, half of the Euler characteristic of a $\mathbb{P}^1$. A disjoint union of two such stacky $\mathbb{P}^1$'s has the same Euler characteristic as a single ordinary $\mathbb{P}^1$, and so, this has the same Euler characteristic as the branched double cover utilized in [1–6]. (See appendix B.1 for more information on Euler characteristics of the stacky curves appearing in this paper.)

To summarize, we have interpreted the terms in the Gross-Taylor expansion of the Nguyen-Tanizaki-Ünsal universe (3.54) as disjoint unions $\coprod_i S_i$ of stacky copies S_i of Σ_T, such that each component S_i of the disjoint union maps (at degree one) to Σ_T, with the property that the Euler characteristic of the disjoint union $\coprod_i S_i$ matches the power of N. Each of the maps $S_i \to \Sigma_T$ factors through the base Σ_T,

$$S_i \xrightarrow{\pi} \Sigma_T \longrightarrow \Sigma_T, \tag{5.50}$$

and we identify them combinatorially with copies of invertible field theories on Σ_T.

5.3.2 $p = 0, n = 3$

Next, we again consider the case of Yang-Mills theory on $\Sigma_T = \mathbb{P}^1$, and restrict to maps of degree $n = 3$, as discussed previously in sections 3.4.2, 4.3.2. Here, $|S_3| = 3! = 6$, so we write

$$\Omega_{n=3} \;=\; 1 \;+\; \sum_{v \neq 1} \left(\frac{1}{N} \right)^{n-K_v} v. \tag{5.51}$$

For reference, the six elements of S_3 can be characterized as

$$(1)(2)(3), \quad (12)(3), (13)(2), (23)(1), \quad (123), (132). \tag{5.52}$$

These six elements form three conjugacy classes, essentially labelled by the orders of the cycles.

From (3.54), we have

$$\begin{aligned} Z_R^+(0,p,N) \;&=\; \frac{N^{2n}}{n!} \delta\left((\Omega_n)^2 P_r \right), \\ &=\; \frac{N^{2n}}{n!} \delta \left(P_r \;+\; 2 \sum_v \left(\frac{1}{N} \right)^{n-K_v} v P_r \;+\; \sum_{ij} \left(\frac{1}{N} \right)^{2n-K_{v_1}-K_{v_2}} v_i v_j \right), \end{aligned} \tag{5.53}$$

for $n = 3$ here.

Previously, we interpreted the first term as a disjoint union of three copies of $\mathbb{P}^1$, which also holds in the present setting.

Next, we discuss the middle term. We interpret this in the same fashion described earlier in section 4.3.2, in terms of stacky copies of $\Sigma_T = \mathbb{P}^1$. Specifically,

- for $K_v = 2$, we interpret $\Sigma_W = \mathbb{P}^1 \coprod \mathbb{P}^1_{[1,2]} \coprod \mathbb{P}^1_{[1,2]}$, in other words three copies of $\mathbb{P}^1$, two each with one $\mathbb{Z}_2$ orbifold point.

- for $K_v = 1$, we interpret Σ_W as a disjoint union of three copies of $\mathbb{P}^1_{[1,3]}$, meaning three copies of $\mathbb{P}^1$ each with a single $\mathbb{Z}_3$ orbifold point.

This is consistent with the displayed combinatorics and also has the same Euler characteristic as the exponent of N.

Now, we turn to the last term. There are two cases that contribute to the sum:

- If both v_1 and v_2 have order 2, then $K_{v_1} = K_{v_2} = 2$, so the term is of the form
$$\frac{N^6}{3!} \left(\frac{1}{N} \right)^{6-2-2} \delta(v_1 v_2) \;=\; \frac{N^4}{3!} \delta(v_1 v_2), \tag{5.54}$$
Previously we interpreted $\Sigma_W = \mathbb{P}^1 \coprod \mathbb{P}^1$, with one copy of $\mathbb{P}^1$ a double cover of Σ_W, branched over two points, and the second $\mathbb{P}^1$ a degree-one cover. Here, we interpret $\Sigma_W = \mathbb{P}^1 \coprod S \coprod S$, where each S is a $\mathbb{P}^1$ with two $\mathbb{Z}_2$ orbifold points, as will be relevant later. It is straightforward to see that $\chi(S) = (2)(1/2) = 1$, so $\chi(\Sigma_W) = 2+1+1 = 4$, matching that of the interpretation above.

- The other case is that v_1 and v_2 have order 3.

  Previously, we interpreted $\Sigma_W = \mathbb{P}^1$, a three-fold cover of $\mathbb{P}^1$ branched over three points. Here, we interpret $\Sigma_W = S' \coprod S' \coprod S'$, where each S' is a stacky $\mathbb{P}^1$ with two $\mathbb{Z}_3$ orbifold points. It is straightforward to see that $\chi(S') = (2)(1/3) = 2/3$, hence $\chi(\Sigma_W) = (3)(2/3) = 2$, matching that of the interpretation above.

5.3.3 $p = 1$

Our proposal is only intended for the case $p = 0$, because, we do not always have a stacky interpretation of all of the extra terms of section 4.2.

Nevertheless, it may help the reader to understand the construction to examine some examples of ordinary terms (interpretable as branched covers) at higher genus, so in this and the next section, we discuss some examples.

Consider the case that $\Sigma_T = T^2$. and consider degree n covering maps, as previously discussed in sections 3.4.3, 4.3.3. Here, from (3.54), we have
$$Z_R^+(0, p, N) \;=\; 1\frac{N^0}{n!} \sum_{s,t \in S_n} \delta\left((\Omega_n)^0 sts^{-1}t^{-1} P_r \right). \tag{5.55}$$

First, consider the case $n = 2$. Since S_2 is abelian, there are no extra terms arising from P_r, as the commutator $[s,t] = 1$ for all $s, t \in S_2$. There are four terms, corresponding to two cases:

- $s = t = 1$: Previously, we interpreted this as a disjoint union of two copies of T^2.
- $s \neq 1$ or $t \neq 1$: Previously, we interpreted this as a single T^2, which is an unbranched double cover of T^2.

Each of these cases is interpreted here as a disjoint union of 2 copies of $\Sigma_T = T^2$, without any orbifold points. The reader should note that this Σ_W has the same Euler characteristic as the Σ_W of the Gross-Taylor expansion, so powers of N match.

5.3.4 $p = 2, n = 2$

Again, our construction is only intended to apply to the case $p = 0$, but it may help the reader to see how terms interpretable as branched covers can be reinterpreted as disjoint unions of stacks, at higher genus. (We do not have an interpretation of the extra terms arising from the presence of a projector P_r, but our description does apply to the ordinary terms arising in the branched cover description.)

Consider the case of double covers of a genus-two Riemann surface Σ_T, as previously discussed in sections 3.4.4, 4.3.4. Here, from equation (3.54), the Gross-Taylor expansion of the Nguyen-Tanizaki-Ünsal universe is

$$
\begin{aligned}
Z_R^+(0,p,N) &= \frac{N^{-4}}{2!} \sum_{s_i,t_i \in S_2} \delta\left((\Omega_2)^{-2} \left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right), \qquad (5.56)\\
&= \sum_{k=0}^{\infty} \frac{N^{-4}}{2!} \left(\frac{1}{1-(1/N^2)} \right)^2 \sum_{s_i,t_i \in S_n} \delta\left(\left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right) \qquad (5.57)\\
&\quad - \frac{N^{-4}}{2!} \sum_{s_i,t_i \in S_n} \left(\frac{1}{1-(1/N^2)} \right)^2 \delta\left(\frac{2}{N} v \left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right)\\
&\quad + \frac{N^{-4}}{2!} \sum_{s_i,t_i \in S_n} \left(\frac{1}{1-(1/N^2)} \right)^2 \delta\left(\frac{1}{N^2} v^2 \left(\prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right).
\end{aligned}
$$

The first term above was intepreted previously as giving Σ_W as a unbranched double cover of the genus-two Riemann surface Σ_T, which means Σ_W has genus 3. Here, we can interpret this term in terms of $\Sigma_W = S \coprod S$, where S is an ordinary Riemann surface of genus 2, and hence $\chi(S) = 2 - (2)(2) = -2$. As a result, $\chi(\Sigma_W) = -4$, matching that of the genus 3 surface in the interpretation above.

We interpret the middle term as in section 4.3.4, as describing $\Sigma_W = C \coprod C$, where each stack C is Σ_T with a single $\mathbb{Z}_2$ singularity.

Finally, consider the term

$$\frac{N^{-4}}{2!} \sum_{s_i, t_i \in S_n} \left(\frac{1}{1 - (1/N^2)} \right)^2 \delta \left(\frac{1}{N^2} v^2 \prod_{i=1}^{2} s_i t_i s_i^{-1} t_i^{-1} \right). \tag{5.58}$$

Previously, we took this term to describe Σ_W as a branched double cover of Σ_T, branched over two points, which from Riemann-Hurwitz implies that Σ_W is a genus-four Riemann surface. Here, we take $\Sigma_W = S' \coprod S'$, where S' is a stacky genus-two surface with two $\mathbb{Z}_2$ orbifold points. It is straightforward to check that $\chi(S') = -2 - 2 + (2)(1/2) = -3$ (cutting out two ordinary disks and replacing each with a $\mathbb{Z}_2$ orbifold), so $\chi(\Sigma_W) = -6$, the same as the Euler characteristic in the first interpretation, as a genus-four Riemann surface.

6 Conclusions

In this paper we have discussed the Gross-Taylor expansion of universes of the decomposition of two-dimensional pure Yang-Mills.

One open problem is to find a (large N) interpretation of the extra terms arising from the presence of the projector P_r, at genus $p > 0$. We have suggested one possible interpretation, in terms of stacky worldsheets, that at least has some desired properties, but until we can understand all cases, we consider that proposal to be merely tentative.

We also find it intriguing that in our proposal, stacky worldsheets arise. Now, in string theory we are very used to working with orbifolds of target spaces, and in computations in target space orbifolds, one sometimes considers orbifold structures on worldsheets, but we are not aware of previous work in which the worldsheet theories were defined on two-dimensional worldsheet stacks (though see [146] for an analogous special case in three-dimensional Chern-Simons theories). We leave this as a consideration for the future.

We have focused on two-dimensional pure Yang-Mills with gauge group $SU(N)$ for large N; however, the Gross-Taylor expansion has also been computed for $SO(N)$, $Sp(N)$ groups, see e.g. [71–75]. We expect similar results will hold there, but leave the details for future work.

We have also focused on the large N limit. It would be interesting to understand finite N corrections to the Gross-Taylor expansion of the Nguyen-Tanizaki-Ünsal universes. Finite N corrections to the original Gross-Taylor expansion are discussed in e.g. [82, section 3], [58, 77, 83, 84]. We leave that for future work.

In passing, we should also mention that two-dimensional pure Yang-Mills partition functions have been related to black holes and higher-dimensional topological strings in e.g. [57, 58, 82, 147–149]. It is tempting to ask whether decomposition of two-dimensional pure Yang-Mills [47, 48] may be applied to those results, perhaps for example to the fragmentation processes described in [150]. We leave this for future work.

7 Acknowledgements

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A Some identities

To make this paper self-contained, we include here a handful of pertinent character identities.

A.1 Orthogonality relations

For Γ a finite group, [98, section 2], [103, chapter V, section 31.1], [102, section 7.3], [104, chapter 2.1],

$$\frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_r(ag) \chi_s(g^{-1}b) \; = \; \frac{\delta_{r,s}}{\dim r} \chi_r(ab), \tag{A.1}$$

$$\frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_r(agbg^{-1}) \; = \; \frac{1}{\dim r} \chi_r(a) \chi_r(b), \tag{A.2}$$

$$\sum_r \chi_r(g) \chi_r(h^{-1}) \; = \; \left\{ \begin{array}{cl} 0 & g, h \mbox{ not conjugate}, \\ \frac{|G|}{|[g]|} & g, h \mbox{ conjugate}, \end{array} \right. \tag{A.3}$$

where the sum is over all (honest, non-projective) irreducible representations of Γ. (Analogues exist for projective representations, but such will not be used in this paper, and so are omitted for brevity.)

For example, if b is in the center of the group algebra, then

$$\chi_r(a)\chi_r(b) = \frac{\dim r}{|\Gamma|}\sum_{g\in\Gamma}\chi_r(agbg^{-1}), \tag{A.4}$$

$$= \frac{\dim r}{|\Gamma|}\sum_{g\in\Gamma}\chi_r(ab) = (\dim r)\chi_r(ab). \tag{A.5}$$

Similarly, the results above imply [86, equ'n (B.10)]

$$\sum_r \frac{(\dim r)}{|\Gamma|}\chi_r(g) = \delta(g). \tag{A.6}$$

A.2 Delta functions and projectors

In this section, we shall demonstrate that

$$\delta(gP_r) = \frac{\dim r}{|\Gamma|}\chi_r(g), \tag{A.7}$$

where P_r is the projector [86, equ'n (2.17)]

$$P_r = \frac{\dim r}{|\Gamma|}\sum_{g\in\Gamma}\chi_r(g^{-1})\tau_g. \tag{A.8}$$

To that end, first note that

$$\chi_s(hP_r) = \frac{\dim r}{|\Gamma|}\sum_g \chi_r(g^{-1})\chi_s(hg), \tag{A.9}$$

$$= \frac{\dim r}{|\Gamma|}\frac{|\Gamma|}{\dim r}\,\delta_{r,s}\,\chi_r(h), \tag{A.10}$$

$$= \delta_{r,s}\,\chi_r(h), \tag{A.11}$$

where we used the identity [86, equ'n (B.6)]

$$\frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_r(ag)\chi_s(g^{-1}b) = \frac{\delta_{r,s}}{\dim r}\,\chi_r(ab) \tag{A.12}$$

for untwisted finite group representations.

Next, using the identity (A.6), we expand

$$\delta(gP_r) = \sum_s \frac{\dim s}{|\Gamma|} \chi_s(gP_r), \tag{A.13}$$

$$= \sum_s \frac{\dim s}{|\Gamma|} \delta_{r,s} \chi_r(g), \tag{A.14}$$

$$= \frac{\dim r}{|\Gamma|} \chi_r(g). \tag{A.15}$$

As a consistency check, note that summing both sides of the identity above over irreducible representations r yields $\delta(g)$, on the left because of completeness of the projectors, and on the right from (A.6).

B Some basics of stacks

In this appendix we collect a few facts about stacks that will be important in the main discussion. Introductions to topological and smooth stacks include [138–143], [144, lecture 3]. Other details are given in sections 4.2 and 5. For example, as discussed there, for many purposes, smooth stacks can be treated as if they were smooth manifolds, in that they have metrics, spinors, and all the other structures needed to define quantum field theories. (See also [11, 112, 113], where stacks were discussed as targets of sigma models, instead of worldsheets.)

B.1 Euler characteristics of stacky curves

In this paper we will compute and utilize orbifold Euler characteristics of stacky curves, which we will briefly outline in this appendix.

First, our stacky curves will all be Deligne-Mumford stacks, with generic stabilizer 1 (hence, not gerbes), and isolated points of nontrivial stabilizer, which in this paper will always be local[24] $\mathbb{Z}_n$ orbifolds.

Next, we will use additivity and the fact that the orbifold Euler characteristic of a disk with a single $\mathbb{Z}_n$ orbifold in its interior is $1/n$.

Since the Euler characteristic is additive, the Euler characteristic of any curve with a finite number of orbifold points can be computed by adding the Euler characteristic of the curve with disks about the orbifold points excised, to the Euler characteristics of the disks.

[24]In passing, as varieties, $\mathbb{C}/\mathbb{Z}_n = \mathbb{C}$; however, as stacks, $[\mathbb{C}/\mathbb{Z}_n] \neq \mathbb{C}$.

For example, consider a genus g curve Σ with k $\mathbb{Z}_n$ orbifold points. Now, the Euler characteristic of a smooth genus g curve is $2-2g$, and if we excise k disks,

$$\chi\,(\text{genus } g \text{ curve minus } k \text{ disks}) \;=\; 2-2g-k. \tag{B.1}$$

We can then add back the Euler characteristics of each of the disks with orbifold points, which in this case means adding $(k)(1/n)$, to get the Euler characteristic of the stacky curve Σ:

$$\chi(\Sigma) \;=\; 2-2g-k+k(1/n). \tag{B.2}$$

For example, an S^2 with four $\mathbb{Z}_2$ orbifold points has $\chi=0$, same as an ordinary torus T^2.

B.2 Maps from stacky curves

In the text, we sometimes discuss maps from stacky curves to ordinary curves. There is a one-to-one correspondence between maps $S \to X$, for S a stacky curve, and $M \to X$, where M is the moduli space of S, and X is a space. This correspondence revolves around the canonical projection map $\pi : S \to X$, as follows:

- First, consider a map $f : S \to X$. Since X is a space, f must send any nontrivial stabilizers on S to the trivial stabilizers on X, hence f factors through π:

$$f \;=\; g \circ \pi \tag{B.3}$$

  for $g : M \to X$ some map.
- Conversely, given a map $g : M \to X$, we can compose with π to get $f = g \circ \pi : S \to X$.

C Gravitational coupling and decomposition

We have argued that the Gross-Taylor string sigma model should admit either a (-1)-form or 1-form global symmetry, the latter of which would imply a decomposition. Since the Gross-Taylor string is also coupled to worldsheet gravity, in this appendix we outline basics of decomposition in the presence[25] of gravitational couplings.

Briefly, given a decomposing two-dimensional theory, gravitational couplings will result in the universes communicating gravitationally, but we do not expect any non-gravitational interactions between the universes. We will study this in examples.

[25]Decomposition is a property of theories in d spacetime dimensions with a global $(d-1)$-form symmetry; however, as is now well-known, it is believed that quantum gravity in dimensions greater than two cannot have ungauged global symmetries. We restrict to two dimensions here.

Consider first abelian BF theory in two dimensions at level k (not yet coupled to worldsheet gravity). As a unitary two-dimensional topological field theory with a semisimple local operator algebra, it decomposes [40, 60–62], in this case to a disjoint union of k distinct invertible field theories. The projection operators are linear combinations of the local operators

$$\mathcal{O}_m \; = : \exp(imB) :, \tag{C.1}$$

which have clock-shift commutation relations with the Wilson lines.

Now, consider coupling[26] BF theory to worldsheet gravity. The BF theory itself does not involve the worldsheet metric at all, hence the 'coupling' is trivial: at least naively, this is just a disjoint union of an ordinary BF theory and a pure worldsheet gravity theory. The BF theory itself still decomposes, but there is only one worldsheet gravity sector. One might describe this as a limiting case of a mostly-disjoint union, with universes that interact with one another only gravitationally, and which become a true disjoint union in the limit that the gravitational sector is unbound.

In the same spirit, two-dimensional Dijkgraaf-Witten theory also decomposes, as was discussed earlier in section 3.2. One could similarly 'couple' Dijkgraaf-Witten theory to worldsheet gravity, though again, one would expect that the coupling is trivial, yielding a disjoint union of a Dijkgraaf-Witten theory (which decomposes) and one copy of worldsheet gravity.

Now, to further confuse matters, the Gross-Taylor sigma model is described in current proposals [4–6, 8, 9] as a cohomological field theory, a topologically-twisted supersymmetric theory with a topological subsector. In such a theory, although the topological subsector may formally decompose [40, 60–62], the entire QFT does not necessarily decompose.

As a prototype for such details, consider the A model with target $\mathbb{P}^1$, coupled to topological gravity. This theory has been considered in e.g. [155, 156]. The puncture operators and gravitational descendants obey recursion relations (see e.g. [157, equ'ns (5.4)-(5.5)]). For example, in the A model with target $\mathbb{P}^1$ coupled to gravity, the puncture operator P and the operator Q generating ordinary A model correlation functions obey [155, equ'n (2.26)], [156]

$$\langle \sigma_n(\Phi) XY \rangle \; = \; n \langle \sigma_{n-1}(\Phi) P \rangle \langle QXY \rangle \; + \; n \langle \sigma_{n-1}(\Phi) Q \rangle \langle PXY \rangle \tag{C.2}$$

for Φ either P or Q, and where σ_n denotes gravitational descendants. For example,

$$\langle \sigma_1(P) PP \rangle \;\; = \;\; \langle PP \rangle \langle QPP \rangle \; + \; \langle PQ \rangle \langle PPP \rangle, \tag{C.3}$$

$$\langle \sigma_1(P) PQ \rangle \;\; = \;\; \langle PP \rangle \langle QPQ \rangle \; + \; \langle PQ \rangle \langle PPQ \rangle. \tag{C.4}$$

If we restrict to the topological subsector of the A model with target $\mathbb{P}^1$, so as to get a

[26] It will not be relevant here, but nonabelian BF theory, for gauge group $PSL(2, \mathbb{R})$, is itself a model of two-dimensional gravity, see e.g. [151, section 6.2.8], [152–154].

decomposition[27], then the projecction operators are of the form

$$\Pi_\pm \;=\; \frac{1}{2\sqrt{q}}\left(Q \pm \sqrt{q}\right), \tag{C.5}$$

where $Q^2 = q$. From the expressions above,

$$\begin{aligned}\langle \sigma_n \Pi_\pm \Pi_\pm \rangle &= 2n\langle \sigma_{n-1}(\Phi) P\rangle \langle Q \Pi_\pm \Pi_\pm \rangle \;+\; 2n \langle \sigma_{n-1}(\Phi) Q \rangle \langle P \Pi_\pm \Pi_\pm \rangle, && \text{(C.6)}\\ &= \langle \sigma_n(\Phi) \Pi_\pm \rangle, && \text{(C.7)}\end{aligned}$$

trivially. Formally, this behavior of the topological subsector is consistent with earlier observations about BF theory coupled to worldsheet gravity: on on the face of it, if a decomposition arises in the theory without gravity, then after coupling to gravity, one has a partial decomposition in which the different universes can (only) interact gravitationally.

D Potential alternative interpretations

As possible alternative interpretations of the localization onto sectors of distinct instanton number in section 4.1, we outline here two proposals for constructions of distinct quantum field theories which localize onto specific instanton sectors.

D.1 Single instanton restriction as a limit

In the spirit of [17], consider[28] a sigma model with a restriction to instantons of degree divisible by $k \in \mathbb{Z}$. This restriction can be accomplished in a local action as follows. Begin with a standard nonlinear sigma model action S_0, and add two new fields (a circle-valued scalar $\tilde{\varphi}$ and a $U(1)$ gauge field A), and terms

$$\int_\Sigma \tilde{\varphi}\left(\phi^* \omega - kF\right) \tag{D.1}$$

where ω is the Kähler form on the target space, ϕ is the map into the target space, and $F = dA$ is the curvature of a worldsheet $U(1)$ gauge field.

Integrating out $\tilde{\phi}$ gives the constraint

$$\phi^* \omega = kF \tag{D.2}$$

[27] But only of the topological subsector, not the entire theory.

[28] We would like to thank Y. Tanizaki and M. Ünsal for a discussion of such instanton restrictions in their model.

so that $\phi^*\omega$ is constrained to be k times an integer. (This accomplishes the restriction on instanton degrees.)

Integrating out A gives the constraint

$$k d\tilde{\varphi} \;=\; 0, \tag{D.3}$$

forcing $\tilde{\varphi}$ to be a constant taking values (on connected components of Σ) in kth roots of unity. If we proceed down this road, the path integral is written as a sum over (constant) values of $\tilde{\varphi}$. Assuming for simplicity and without loss of generality that Σ is connected, the path integral can be written in the form

$$\begin{aligned} Z &= \sum_{\tilde{\varphi}} \int [D\phi] \exp(-S_0) \exp\left(- \int_\Sigma \phi^*(\tilde{\varphi}\omega) \right), && \text{(D.4)} \\ &= \sum_{n=0}^{k-1} \int [D\phi] \exp(-S_0) \exp\left(-\frac{n}{k} \int_\Sigma \phi^*\omega \right), && \text{(D.5)} \end{aligned}$$

which matches the path integral for a sum of universes (indexed by $\tilde{\varphi}, n$) with variable B fields (given on universe n by $(n/k)\omega$). As is typical in examples of this form, the sum over universes enforces the restriction on instanton degree: instantons of the wrong degree cancel out of the sum, leaving only instantons of degree divisible by k.

So far, we have generated a restriction to instantons satisfying a divisibility criterion, which (as expected from decomposition) can be described via a sum over universes. We want something stronger – a restriction to instantons of a single possible degree. On the face of it, there are two natural ways one might try to get such a restriction on instantons. One way is to take $k \to 0$; however, this limit is ill-behaved [65].

Instead, one could try to interpret[29] the limit $k \to \infty$ as defining a restriction to a single instanton sector. Formally, this would correspond to a sum over infinitely many universes.

Fourier analysis suggests a related interpretation. Write a partition function as

$$Z(\theta) \;=\; \sum_{n \in \mathbb{Z}} Z(n) \exp(in\theta). \tag{D.6}$$

Then,

$$Z(n) \;=\; \int_0^{2\pi} Z(\theta) \exp(-i\theta n) \frac{d\theta}{2\pi}. \tag{D.7}$$

Formally, this would naively suggest an interpretation as a sum over uncountably many universes, indexed by θ angles, though as that would also appear to imply an uncountably infinite number of dimension-zero operators, which we cannot reconcile with other results at this time, we will not pursue such an interpretation here..

[29]This conclusion was previously reached by Y. Tanizaki and M. Ünsal [65].

In passing, we observe that similar integrals have arisen in ensemble averaging, see e.g. [115–119]. It should be noted that an ensemble is not the same as a decomposition, which becomes visible in QFTs on spacetimes with multiple connected components. (In the former case, there is only one summand/integral over the ensemble, whereas in the second, there are as many as connected components.)

D.2 Direct single instanton restriction

In this subsection we outline a proposal for a direct restriction to single instantons. It is well-defined in bosonic theories, but does not quite apply to cohomological field theories, as we shall discuss.

In brief, this alternative proposal is to understand the restriction to single instanton degrees via a construction of countably infinitely many new quantum field theories, separately consistent, which localize onto sectors of the original theory. The idea is to promote the theta angle to an axion, albeit with a kinetic term.

As a prototype, consider two-dimensional pure Maxwell theory. This theory has classical action

$$S \;=\; \int_{\Sigma} F \wedge *F. \tag{D.8}$$

Since it is a $U(1)$ gauge theory, it admits nontrivial $U(1)$ bundles, which are classified by their first Chern classes, which on a connected two-dimensional surface Σ are elements of $H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$.

Now, corresponding to the values n of the first Chern class, we consider a countable family of theories with local[30] actions

$$S_n \;=\; \int_{\Sigma} \left(F \wedge *F \;+\; B \left(F - n \, \mathrm{vol} \right) \right), \tag{D.10}$$

where n vol denotes a harmonic representative of $[n] \in H^2(\Sigma, \mathbb{Z})$, and B is a circle-valued field. (The theory is well-defined under $B \mapsto B + 2\pi$ because $\exp(2\pi i n) = 1$.)

[30] A nonlocal alternative would be to add a term of the form

$$\int_{\Sigma} B \left[\left(\int_{\Sigma} F \right) - n \right], \tag{D.9}$$

for B a Lagrange multiplier. This would clearly select out contributions from $U(1)$ instanton number n. However, as a nonlocal theory, it is unclear to us to what extent it can be renormalized in general, and hence it is unclear to what extent it would exist as a quantum theory. Instead, we shall work with a local action, in which we use a Lagrange multiplier to force F itself (not its integral) to match a harmonic representative of the desired cohomology class. As every cohomology class has a unique harmonic representative, this should accomplish the same goal, while retaining locality.

In passing, this is similar in spirit to the Duistermaat-Heckman localization as described in [55], where one has a moment map $\mu \propto F$ [55, equ'n (1.9)], so that $\mu = 0$ is a critical locus of the Yang-Mills action.

As B is a dynamical field acting as a Lagrange multiplier, each theory S_n effectively localizes onto gauge field configurations of first Chern class n. Furthermore, since the BF terms are relevant deformations, we expect that they dominate in the IR. (Put another way, B is an axion without kinetic terms, a dynamical theta angle, hence its zero mode integral recovers an average of the form discussed in [116].)

For $n = 0$, the theory is effectively then, in the IR, ordinary level 1 BF theory. For $n \neq 0$, pick a representative gauge field $\tilde{A}$ such that $\tilde{F} = n\mathrm{vol}$, and write the action as

$$S_n \;=\; \int_\Sigma \left(F \wedge *F \;+\; B\left(F - \tilde{F} \right) \right), \tag{D.11}$$

By defining a new gauge field to be the difference $A - \tilde{A}$, we see that the $B(F - \tilde{F})$ theory effectively reduces to another BF theory at level 1, in the sense that it has the same operators and OPEs (though in principle the partition function would be slightly different).

The theories defined by the S_n form a countably infinite set of theories which localize onto single instanton sectors of the original (pure Maxwell) theory. It is natural to speculate that these pieces are Poisson-dual to the universes of the decomposition, following the analysis in section 4.1.2.

This construction is analogous to a Fourier series. To that end, recall that the central identity there is Poisson resummation, which can be expressed as

$$\sum_n \exp(inx) \;=\; 2\pi \sum_n \delta(x - 2\pi n). \tag{D.12}$$

Here, an analogous expression in path integrals would be

$$\sum_n \exp\left(i \int B n\mathrm{vol} \right) \;=\; \sum_n \delta\left[B - 2\pi n \right], \tag{D.13}$$

Applying this analogue naively, one finds that

$$\sum_n \int [DB] \exp\left(i \int B\left(F - n\mathrm{vol} \right) \right) \;=\; \int [DB] \exp\left(i \int BF \right) \delta[B], \tag{D.14}$$

$$=\; 1. \tag{D.15}$$

In general, we would not expect such a naive computation to necessarily survive quantum corrections, but it does suggest that in special cases, it may be possible to recover the original theory through some sort of sum over the pieces S_n. In the next section, we will discuss how this may happen in this particular example, two-dimensional pure Maxwell theory.

To be relevant for us, we would need a supersymmetrized version of this procedure. We shall next outline how that could be accomplished, and the puzzles that result.

Recall for example from [158, section 2] that in two-dimensional (2,2) supersymmetric gauge theories, the theta angle can be encoded in a twisted superpotential. That references defines Σ to be a twisted chiral superfield given as superderivatives of the vector superfield, with components [158, equ'n (2.16)]

$$\Sigma \; = \; \sigma - i\sqrt{2}\theta^+\overline{\lambda}_+ - i\sqrt{2}\overline{\theta}^- \lambda_- + \sqrt{2}\theta^+\overline{\theta}^- (D - iF_{01}) + \cdots \, , \tag{D.16}$$

from which one computes [158, equ'n (2.26)]

$$\int d\theta^+ d\overline{\theta}^- \Sigma|_{\theta^- = \overline{\theta}^+ = 0} \;\; = \;\; \sqrt{2}\,(D - iF_{01})\,, \tag{D.17}$$

$$\int d\theta^- d\overline{\theta}^+ \overline{\Sigma}|_{\theta^+ = \overline{\theta}^- = 0} \;\; = \;\; \sqrt{2}\,(D + iF_{01})\,, \tag{D.18}$$

hence if one defines

$$t \; = \; ir + \frac{\theta}{2\pi}, \tag{D.19}$$

where r is the FI parameter and θ the theta angle, then [158, equ'n (2.27)]

$$\frac{it}{2\sqrt{2}} \int d\theta^+ d\overline{\theta}^- \Sigma|_{\theta^- = \overline{\theta}^+ = 0} \; - \; \frac{i\overline{t}}{2\sqrt{2}} \int d\theta^- d\overline{\theta}^+ \overline{\Sigma}|_{\theta^+ = \overline{\theta}^- = 0} \; = \; -rD + \frac{\theta}{2\pi} F_{01}. \tag{D.20}$$

With this in mind, we can define a supersymmetric axion in two dimensions by promoting t to a twisted chiral superfield T, with components

$$T \; = \; t - i\sqrt{2}\theta^+\overline{\gamma}_+ - i\sqrt{2}\overline{\theta}^- \gamma_- + \sqrt{2}\theta^+\overline{\theta}^- F + \cdots \, , \tag{D.21}$$

and then consider a twisted superpotential

$$\begin{aligned} & \frac{i}{2\sqrt{2}} \int d\theta^+ d\overline{\theta}^- (T\Sigma)|_{\theta^- = \overline{\theta}^+ = 0} \; - \; \frac{i}{2\sqrt{2}} \int d\theta^- d\overline{\theta}^+ (\overline{T\Sigma})|_{\theta^+ = \overline{\theta}^- = 0} \\ & = \;\; -rD + \frac{\theta}{2\pi} F_{01} \; + \; \frac{i}{2} \left[\sigma F - \overline{\sigma}\overline{F} - \sqrt{2}\overline{\gamma}_+ \lambda_- + \sqrt{2}\gamma_- \overline{\lambda}_+ + \sqrt{2}\gamma_+ \overline{\lambda}_- - \sqrt{2}\overline{\gamma}_- \lambda_+ \right] \end{aligned} \tag{D.22}$$

(in the conventions of [158]).

In passing, this is closely analogous to axion couplings in four dimensions, which are given by superpotential terms $\mathcal{A} W_\alpha W^\alpha$, for $\mathcal{A}$ the superfield containing the axion, see for example [159, equ'n (7)].

It remains to supersymmetrize the coupling of the axion to a nontrivial cohomology class. We will encounter a fatal flaw when we try to do so. For completeness, we outline the analysis in the remainder of this appendix.

Let ω_0 denote a harmonic representative of the desired cohomology class. Since ω_0 is a differential form on the worldsheet, by itself it is a constant in superspace. A twisted superpotential term $T\omega_0$ will yield a bosonic term $F\omega_0$, instead of the desired $t\omega_0$.

To get the desired bosonic $t\omega_0$ term, we can instead think of ω_0 as the auxiliary field component of a superfield whose other components vanish. In the present case, in terms of twisted chiral superfields, that means we imagine a superfield

$$\Omega_0 \;=\; -i2\sqrt{2}\theta^+\overline{\theta}^-(\omega_0)_{01}. \tag{D.23}$$

Given this superfield, we can add the twisted superpotential term $T\Omega_0$, which generates the desired bosonic term $t\omega_0$.

For example, [120, section 8.1] gives a twisted chiral multiplet for a dynamical two-form field, which arises in the $\theta^+\overline{\theta}^-$ component of the multiplet. If we reduce to vevs, so that the two-form field is nondynamical and there are no other components, this twisted chiral multiplet reduces to the Ω_0 above.

Unfortunately, this method has the disadvantage that it breaks supersymmetry, as for example we have given a vev to the F term. More explicitly, the twisted superpotential term $T\Omega_0$ gives the desired bosonic term $t\omega_0$ but with no other fermionic partners, which will not be closed under supersymmetry.

However, for purposes of understanding a cohomological field theory, our requirements are weaker. All we really need is for the BRST symmetry to be preserved, which in the untwisted theory is only half of the (2,2) supersymmetry transformations. Furthermore, from [160, equ'n (3.49)], the scalar component of Σ is BRST-closed under the A-twist discussed in that reference. Given the same twist here, we see that a Lagrangian term

$$t(\omega_0)_{01}, \tag{D.24}$$

though it would be closed under only half of supersymmetry, would be BRST-closed under a (suitably chosen) A-twist, and so could be consistently added to a cohomological field theory.

We emphasize that the key feature of a cohomological field theory is the existence of a nilpotent scalar charge – the BRST operator. Existence of a BRST symmetry is what enables a topological field theory to be studied semiclassically. (The same symmetry is at the heart of gauge-fixing in Yang-Mills theories, and is the reason for the shared name.) We ordinarily obtain cohomological field theories by topologically twisting a supersymmetric theory – but the key outcome of that process is a BRST symmetry, which follows from only half of the supersymmetry. Thus, breaking half of the supersymmetry but retaining the BRST symmetry is sufficient for our purposes.

The reader should note that although the twisted superpotential term $T\Omega_0$ will be BRST closed, its hermitian conjugate would not. (For example, in the A-twisted GLSM of [160], σ

is BRST-closed but $\overline{\sigma}$ is not.) Hence, one can only add a $T\Omega_0$ term, and not its hermitian conjugate.

To summarize, our proposal to implement the Lagrange multiplier restriction in an A-twisted GLSM is to add the terms

$$\frac{i}{2\sqrt{2}} \int d\theta^+ d\overline{\theta}^- \left(T(\Sigma - \Omega_0)\right)|_{\theta^-=\overline{\theta}^+=0} \;-\; \frac{i}{2\sqrt{2}} \int d\theta^- d\overline{\theta}^+ (\overline{T\Sigma})|_{\theta^+=\overline{\theta}^-=0} \tag{D.25}$$

$$= \; -rD + \frac{\theta}{2\pi} F_{01} \;+\; \frac{i}{2}\left[\sigma F - \overline{\sigma}\overline{F} - \sqrt{2}\overline{\gamma}_+\lambda_- + \sqrt{2}\gamma_-\overline{\lambda}_+ + \sqrt{2}\gamma_+\overline{\lambda}_- - \sqrt{2}\overline{\gamma}_-\lambda_+\right] - t(\omega_0)_{01}, \tag{D.26}$$

$$= \; r\left(-D - i(\omega_0)_{01}\right) + \frac{\theta}{2\pi}\left(F_{01} - (\omega_0)_{01}\right) + \frac{i}{2}\left[\sigma F - \overline{\sigma}\overline{F} - \sqrt{2}\overline{\gamma}_+\lambda_- + \sqrt{2}\gamma_-\overline{\lambda}_+ + \sqrt{2}\gamma_+\overline{\lambda}_- - \sqrt{2}\overline{\gamma}_-\lambda_+\right]. \tag{D.27}$$

The resulting theory will still be BRST closed under an A-twist of the form of [160]. The reader should note that the dynamical circle-valued field θ acts as the desired Lagrange multiplier above, forcing $F_{01} = \omega_0$. The remaining terms are the result of supersymmetrization.

As a consistency test, in a GLSM without (untwisted) superpotential, in the notation of [158], the A-twist of [160] twists the supersymmetry parameters ϵ_+, $\overline{\epsilon}_-$ to scalars, hence using the supersymmetry transformations [158, equ'n (2.12)], we have BRST transformations including

$$\delta\sigma \;=\; 0 \;=\; \delta t, \tag{D.28}$$

and we see that $\overline{\lambda}_-$, $\overline{\gamma}_-$, λ_+, and γ_+ are twisted to scalars. The fermion bilinears in (D.25) are all either scalars or 2-forms, so we see explicitly that promoting the theta angle to an axion is compatible with the topological twist.

Unfortunately, at this point we now encounter a basic problem we have not been able to solve, arising from the

$$r\left(-D - i\omega_0\right) \tag{D.29}$$

terms. Since the superfield T is dynamical, the field r is a Lagrange multiplier, forcing

$$D \;=\; -i(\omega_0)_{01}. \tag{D.30}$$

However, D is real (by construction), and $i(\omega_0)_{01}$ is pure imaginary, so this has no solutions. For this reason, we do not utilize this supersymmetrized constraint framework to try to understand the restriction on map degrees arising in the Gross-Taylor expansion of Nguyen-Tanizaki-Ünsal universes.

In passing, promoting the FI parameter to a dynamical field often arises in constructions of H flux in GLSMs, see for example [161–172]. For example, GLSMs with (2,2) supersymmetry, a single $U(1)$, and a gauged FI parameter with a nonzero kinetic term are discussed in [172, section 3.1.1]. The result of gauging the FI parameter in the $\mathbb{P}^n$ model, is described in [172] as a $\mathbb{P}^n$ fibered over a cylinder parametrized by the FI parameter, giving altogether what is described there as a trumpet geometry, with nonzero H flux (roughly, the wedge product of the Fubini-Study form on the projective space and a one-form along the cylinder). (Since the geometry is realized by a mix of ordinary chiral and twisted chiral multiplets, the geometry is necessarily an example of a generalized geometry in the sense of [173,174].) This theory is also described as having a nontrivial IR limit. It also has a kinetic term for the axion. The case of no kinetic term is described in [172, equ'n (3.23)] as the limit $b_\alpha \to \infty$, which the authors describe as the "no squashing" limit.

In the case of the Gross-Taylor sigma model, similar ideas would apply, as well as the same fatal flaw. For completeness, we outline the details here.

At heart, the Cordes-Moore-Ramgoolam picture of the Gross-Taylor sigma model is a (supersymmetric, topologically-twisted) sigma model, so the F of two-dimensional Maxwell theory is replaced by $\phi^*\omega$, for $\phi : \Sigma_W \to \Sigma_T$ the map between worldsheets, and ω the Kähler form on the target Σ_T. The analogue of the procedure above would be accomplished by adding a periodic scalar φ (without kinetic term, i.e. a Lagrange multiplier) and local[31] bosonic terms in the Lagrangian

$$\varphi \left(\phi^*\omega - \omega_0 \right), \tag{D.32}$$

where ω_0 is a fixed harmonic two-form on the worldsheet Σ_W, and ω is the Kähler form on the target. The two-form ω_0 is a fixed harmonic two-form on the worldsheet Σ_W, which should also capture the area of the worldsheet. A natural candidate is the worldsheet Kähler form, as on any Kähler manifold, the Kähler form is harmonic. (This ultimately follows from the fact that $[L, \Delta_d] = 0$ for L the Lefschetz operator [175, section 0.7], [176].)

As the Gross-Taylor theory is a topologically-twisted supersymmetric theory, one would need a supersymmetric version of the terms above. We do not expect this to result in a well-defined theory, for the same reasons as above, but for completeness we outline a few details. Following [173, equ'n (23)], [177, section 4.2], and in close analogy with our analysis of topologically twisted gauge theories in the previous subsection, we can write the supersymmetric extension of the terms above as a twisted chiral superpotential

$$\int d\theta^+ d\overline{\theta}^- \, \tilde{\Phi} \left(\omega_{i\overline{\jmath}} D_- \Phi^i \overline{D}_+ \overline{\Phi}^{\overline{\jmath}} \, - \, \Omega_0 \right) \, + \, \int d\theta^- d\overline{\theta}^+ \overline{\tilde{\Phi}} \left(\omega_{i\overline{\jmath}} D_+ \Phi^i \overline{D}_- \overline{\Phi}^{\overline{\jmath}} \right) \tag{D.33}$$

[31] As discussed earlier, one could imagine adding a nonlocal term to the Lagrangian, of the form

$$\varphi \int \left(\phi^*\omega - \omega_0 \right). \tag{D.31}$$

The nonlocal certain would certainly force the cohomology classes to match, not just the representatives, but would be nonlocal, hence renormalizability of the resulting theory is unclear. On the other hand, since ω_0 is harmonic and every cohomology class has a unique harmonic representative, the local proposal above should accomplish the same goal.

where $\tilde{\Phi}$ is a twisted chiral multiplet whose scalar component includes θ, the Lagrange multiplier theta angle, the scalar component of Φ^μ is ϕ^μ, and $\Omega_0 \propto \theta^+ \overline{\theta}^- \omega_0$ was defined in the previous section.. In passing, note that

$$\omega_{i\overline{\jmath}} D_+ \Phi^i \overline{D}_- \Phi^{\jmath} \tag{D.34}$$

will be a twisted chiral multiplet precisely when ω is a closed (1,1) form. Just as in GLSMs, the $\tilde{\Phi}\Omega_0$ term in the twisted superpotential breaks supersymmetry, but is compatible with an A-twist, in the sense that the twisted theory still possesses the BRST symmetry. However, the same fatal flaw will arise here as arose in the previous supersymmetric example, so we do not advocate this approach as a means of understanding the Gross-Taylor expansion of Nguyen-Tanizaki-Ünsal universes.

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