Annals of Mathematics **198** (2023), 1013–1046
https://doi.org/10.4007/annals.2023.198.3.2

# Regularity of minimal surfaces near quadratic cones

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## Abstract

Hardt-Simon proved that every area-minimizing hypercone $\mathbf{C}$ having only an isolated singularity fits into a foliation of $\mathbb{R}^{n+1}$ by smooth, area-minimizing hypersurfaces asymptotic to $\mathbf{C}$. In this paper we prove that if a stationary integral $n$-varifold $M$ in the unit ball $B_1 \subset \mathbb{R}^{n+1}$ lies sufficiently close to a minimizing quadratic cone (for example, the Simons' cone $\mathbf{C}^{3,3}$), then $\operatorname{spt} M \cap B_{1/2}$ is a $C^{1,\alpha}$ perturbation of either the cone itself, or some leaf of its associated foliation. In particular, we show that singularities modeled on these cones determine the local structure not only of $M$, but of any nearby minimal surface. Our result also implies the Bernstein-type result of Simon-Solomon, which characterizes area-minimizing hypersurfaces asymptotic to a quadratic cone as either the cone itself, or some leaf of the foliation.

## Contents



Keywords: minimal surface, regularity, quadratic cone, foliation

AMS Classification: Primary: 49Q05, 49Q20, 53A10, 35J93.

N.E. was supported in part by NSF grants DMS-1606492 and DMS-2204301. L.S. was partially supported by NSF grants DMS-1810645 and Career DMS-2044954.

## 1. Introduction

In this paper we are interested in the following question:

*Question* 1.1. Suppose $M_i$ is a sequence of minimal $n$-dimensional surfaces, converging to some $\tilde{M}$ with multiplicity one in the unit ball in $\mathbb{R}^{n+1}$. How does the singular structure of $\tilde{M}$ determine the singular or regular structure of the $M_i$ in $B_{1/2}$?

This question and its variants underlie a significant amount of research in the field of minimal surfaces, and other variational problems. Question 1.1 arises when attempting to study the singularity structure or compactness properties of some class of surfaces. For example, often the $M_i$ form some kind of "blow-up" sequence for a singularity, and the resulting $\tilde{M}$ is a singularity model. In the very important special case when $M_i$ are dilations around a fixed point of a given minimal $M$, i.e., when

$$M_i = \lambda_i(M - x), \quad \lambda_i \to \infty,$$

then $\tilde{M}$ is dilation invariant, and any such $\tilde{M}$ arising in this fashion is called a tangent cone of $M$ at $x$.

Question 1.1 is entirely answered when $\tilde{M}$ is smooth; in this case Allard's theorem [1] implies that for $i$ large, the $M_i \cap B_{1/2}$ must be smooth also (in fact must be $C^{1,\alpha}$ perturbations of $\tilde{M}$).

For singular $\tilde{M}$, Question 1.1 has been answered under some structural assumptions. When $\tilde{M}$ has at most a "strongly isolated" singularity,[1] *and* each $M_i$ has at least one singularity of the same type (i.e., with the same or greater density), then profound work of [2], [11] shows that the $M_i \cap B_{1/2}$ must be $C^1$ perturbations of $\tilde{M}$ for large $i$. This situation naturally occurs when $\tilde{M}$ is a tangent cone of some minimal surface, and $M_i$ are dilations around a fixed singular point.

In certain particular cases one can use the topology of $\tilde{M}$ to deduce the singular structure on $M_i$. For example, when $\tilde{M}$ is a union of three half-planes, then [12] showed that $M_i \cap B_{1/2}$ are a $C^{1,\alpha}$ perturbation of $\tilde{M}$. Similar results hold if $\tilde{M}$ has tetrahedral singularities and the $M_i$ have an associated "orientation structure" ([6]), or when $\tilde{M}$ is a union of two planes and the $M_i$ are 2-valued graphs ([3]).

[1]Strongly isolated here means that some tangent cone of $M$ at the singularity is a multiplicity-one cone with an isolated singularity at 0.

Notice that in these theorems, either by assumption or by the nature of $\tilde{M}$, all the $M_i$ have the same regular or singular structure as $\tilde{M}$.

Question 1.1 becomes more subtle when one does not assume anything about the singular nature of the $M_i$. In this case it is possible for a sequence of smooth minimal (even area-minimizing!) surfaces to limit to a singular one. For example, Bombieri, De Giorgi, and Giusti [4] have constructed a foliation by smooth minimal surfaces of the complement of the Simons's cone

$$\mathbf{C}^{3,3} = \{(x, y) \in \mathbb{R}^8 : |x| = |y|\}.$$

More precisely, the authors in [4] showed that there exist two smooth, area-minimizing hypersurfaces $S_\pm$, contained respectively in the connected components $E_\pm = \{(x, y) \in \mathbb{R}^8 : \pm|x| > |y|\}$ of $\mathbb{R}^8 \setminus \mathbf{C}^{3,3}$, which are smoothly asymptotic to $\mathbf{C}^{3,3}$ at infinity, and whose dilations $\cup_{\lambda>0}\lambda S_\pm$ foliate $E_\pm$. Similar $S_\pm$ satisfying the same properties were constructed by Hardt and Simon [9] for any area-minimizing cone $\mathbf{C}^n \subset \mathbb{R}^{n+1}$ having an isolated singularity (see Section 2.3), and often the associated family of dilations $\cup_\lambda \lambda S_\pm$ is called the Hardt-Simon foliation of $\mathbf{C}$.

In the above circumstances the $M_i$ need not have the same singularity structure as $\tilde{M}$. More generally, it is possible that singularities of one type can limit to a singularity of a different type. Or, even worse, it is possible that multiple singularities of various types or dimensions could collapse into an isolated singularity of some other type.

In this paper we answer Question 1.1 in the case when $\tilde{M}$ has singularities modeled on area-minimizing quadratic cones $\mathbf{C}^{p,q}$, i.e., so-called minimizing $(p, q)$-singularities (see Definition 1.1.1). One of the main results of this paper is that the foliation collapsing into $\mathbf{C}^{p,q}$ is the "worst" behavior one will ever see. In particular, in *any minimal surface* which is sufficiently nearby a minimizing $(p, q)$-singularity, there is at most one singularity, and necessarily of the same type.

We contrast our result with recent constructions of [14], [17], which give examples of multiple singularities of different types/dimensions collapsing into cylindrical-type singularities like $\mathbf{C}^{3,3} \times \mathbb{R}$ (provided one allows the ambient metric to change smoothly also). One can also of course build many toy examples out of geodesic nets. It would be very interesting to understand whether this behavior can occur for smooth, multiplicity-one or area-minimizing hypercones of dimension $> 1$. (See [7] for related work when $n = 7$.)

It is important to note that, due to the presence of multiplicity, many examples of aforementioned pathology can also occur for area-minimizing currents in higher codimension, or stationary varifolds in arbitrary codimension. For example, the sequence of area-minimizing currents

$$\Gamma_k := \{(z, w) \in \mathbb{C}^2 \,:\, w^3 = (z - z_k)^2 \, z^2\} \subset \mathbb{C}^2,$$

for a sequence $(z_k)_k$ converging to 0, is such that each $\Gamma_k$ has two singular points of branching type at 0 and $z_k$, while the limit

$$\Gamma := \{(z, w) \in \mathbb{C}^2 : w^3 = z^4\} \subset \mathbb{C}^2$$

is singular only at 0. The tangent cones at all these singularities are planes with multiplicity $> 1$. Similar examples can be constructed for codimension one stationary varifolds using Weierstrass representation.

*Definition* 1.1.1. Following [15], define the $\mathbf{C}^{p,q}$ quadratic minimal cone as the hypersurface

$$\mathbf{C}^{p,q} = \{(x, y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} : q|x|^2 = p|y|^2\} \subset \mathbb{R}^{n+1},$$

where $p, q > 0$ are integers and $p + q = n - 1$. Some people refer to the cones $\mathbf{C}^{p,q}$ as generalized Simons' cones.

Given a surface or varifold $M$, we say that $x \in \mathrm{spt} M$ is a $(p, q)$-singularity if some tangent cone of $M$ at $x$ is (up to rotation) equal to $\mathbf{C}^{p,q}$ with multiplicity-one. We say $x$ is a minimizing $(p, q)$-singularity if the associated cone $\mathbf{C}^{p,q}$ is area-minimizing.

Some remarks are in order.

*Remark* 1.2. An easy computation shows that each $\mathbf{C}^{p,q}$ is minimal. However, there are no non-flat area-minimizing hypercones in $\mathbb{R}^{n+1}$ for $n < 7$, and of course by dimension-reducing there are no singular area-minimizing hypersurfaces in these dimensions either. When $n = 7$, the cones $\mathbf{C}^{3,3}$ and $\mathbf{C}^{2,4}$ are area-minimizing, and in fact up to rigid motions these are the only known area-minimizing hypercones in $\mathbb{R}^8$. When $n > 7$, then every $\mathbf{C}^{p,q}$ cone is area-minimizing. See, e.g., [10] and the references therein. Thus our results are motivated by, and most relevant to, the regularity theory for area-minimizing hypersurfaces.

*Remark* 1.3. By work of [2], if $x$ is a $(p, q)$-singularity of an $M$ which is stationary (or has $L^p$ mean curvature, for $p > n$), then nearby $x$, $\mathrm{spt} M$ is a $C^{1,\alpha}$ perturbation of $\mathbf{C}^{p,q}$.

As a corollary to our main Theorem 3.1, we obtain the following answer to Question 1.1.

THEOREM 1.4. *Let $M$ be a multiplicity-one, stationary integral $n$-varifold in $B_1$, which is regular away from $0$ and has a minimizing $(p, q)$-singularity at $0$. Let $M_i$ be a sequence of stationary, integral varifold in $B_1$, so that $M_i \to M$ as varifolds. Then for each $i$ sufficiently large, $M_i \llcorner B_{1/2}$ has either an isolated singularity of the same type $(p, q)$, or is entirely regular.*

*Furthermore, if we let $M_\pm$ be the smooth manifolds obtained by replacing a small neighborhood of $0$ in $M_0 := \mathrm{spt} M$ (in which $M_0$ is a $C^{1,\alpha}$ perturbation of*

$\mathbf{C}^{p,q}$) *with a suitable dilation of* $S_\pm$, *then, for i sufficiently large*, $\mathrm{spt} M_i \cap B_{1/2}$ *is a* $C^{1,\alpha}$ *deformation of one of* $M_0$, $M_+$, $M_-$.

We remark that, only from the information that $M_i \to M$, one cannot distinguish a priori whether each $M_i$ is regular or singular. See Section 3 for a more detailed statement of our main regularity theorem, and for other corollaries.

The main novelty of Theorem 3.1 is that, unlike previous regularity results for minimal surfaces near isolated singularities (e.g., [2], [11]), we do not prescribe a priori the density of the $M_i$ at any point; that is, we do not impose them to be singular at the origin, nor at any other point. As a consequence, even if a minimal surface is close at scale 1 to a cone with an isolated singularity, the surface itself may be entirely smooth.

We can give a further characterization of the $M_i$ when they are singular; in this case the $M_i$ must be one of the examples of minimal surfaces as constructed in [5]. Finally, we can use our regularity theorem to reprove the rigidity result of [15], which characterize complete minimal surfaces asymptotic to quadratic cones.

It would be interesting to know whether our results carry over to other (area-minimizing) singularity models. Unfortunately, the only other known smooth, area-minimizing hypercones are of so-called isoparametric type, and for these other examples, our techniques do not seem to work. See Section 4 of [15] for further discussions.

## 2. **Notation and preliminaries**

We work in $\mathbb{R}^{n+1}$. We denote by $\mathcal{H}^n$ the $n$-dimensional Hausdorff measure. Given a subset $A \subset \mathbb{R}^{n+1}$, we let $d_A(x) = \inf_{a\in A}|x-a|$ be the Euclidean distance to $A$, and given $r>0$, we write

$$B_r(A) := \{x \in \mathbb{R}^{n+1} : d_A(x) < r\}$$

for the (open) $r$-tubular neighborhood of $A$. Similarly, $B_r(x)$ is the open $r$-ball centered at $x \in \mathbb{R}$. If $x=0$, we may sometimes just write $B_r$. We write $\overline{B_r(A)}$ for the closed tubular neighborhood or $r$-ball. We set $\omega_n = \mathcal{H}^n(\mathbb{R}^n \cap B_1)$ to be the volume of the $n$-dimensional unit ball. We define the translation/dilation map $\eta_{x,r}(z) := (z-x)/r$.

We may occasionally use the notation of Cheeger: we denote by $\Psi(\epsilon_1, \dots, \epsilon_k | c_1, \dots, c_N)$ a non-negative function, which for any fixed $c_1, \dots, c_N$, satisfies

$$\lim_{\epsilon_1,\dots,\epsilon_k \to 0} \Psi(\epsilon_1, \dots, \epsilon_k | c_n, \dots, c_N) = 0.$$

We use the convention that $c_i$ always denotes constants $\geq 1$, while Greek letters (such as $\epsilon_i, \delta_i, \Lambda_i$) denote constants $\leq 1$.

We shall always treat graphing functions as scalars. Given oriented hypersurfaces $M, N$, and an open subset $U \subset N$, if we write

$$M = \mathrm{graph}_N(u),$$

we mean that $M = \{x + u(x)\nu_N(x) : x \in U \subset N\}$, where $\nu_N$ is the unit normal of $N$.

Given an (oriented) hypersurface $N$, a function $u : N \to \mathbb{R}^n$, and a $\beta \in (0,1)$, we define the Holder semi-norm

$$[u]_{\beta,r} = \sup_{x \neq y \in N \cap B_r \setminus B_{r/2}} \frac{|u(x) - u(y)|}{|x - y|}$$

and, given an integer $k \geq 0$, the Holder norm

$$|u|_{k,\beta,r} = [D^k u]_{\beta,r} + \sum_{i=0}^{k} r^{1-k} \sup_{x \in N \cap B_r \setminus B_{r/2}} |\nabla^i u|.$$

Typically when we use these norms, $N$ will be conical or nearly conical.

2.1. *Varifolds and first variation.* We are concerned with integral $n$-varifolds that are stationary, or have $L^p$ mean curvature. Recall that an $n$-varifold $M$ is integral if it has the following structure: there is a countably $n$-rectifiable set $\tilde{M}$, and there is an $\mathcal{H}^n \llcorner \tilde{M}$-integrable, $\mathbb{N}$-valued function $\theta$, so that

$$M(\phi(x, V)) = \int_{\tilde{M}} \phi(x, T_x \tilde{M}) \theta(x)\, d\mathcal{H}^n(x) \quad \forall \phi \in C^0_c(\mathbb{R}^{n+1} \times \mathrm{Gr}(n, n+1)).$$

Here $\mathrm{Gr}(n, n+1)$ is the Grassmannian bundle, i.e., the space of unoriented $n$-planes in $\mathbb{R}^{n+1}$. If $N$ is an $n$-manifold, then $N$ induces a natural $n$-varifold in the obvious fashion, which we write as as $[N]$. Given a proper, $C^1$ map $\eta$, we write $\eta_\sharp M$ for the pushforward of $M$.

We write $\mu_M$ for the mass measure of $M$. The first variation of $M$ in an open subset $U \subset \mathbb{R}^{n+1}$ is the linear functional

$$\delta M(X) = \int \mathrm{div}_M(X) d\mu_M, \quad X \in C^1_c(U, \mathbb{R}^{n+1}),$$

where $\mathrm{div}_M(X)$ is defined for $\mu_M$-a.e. $x$ as follows: if $e_i$ is an orthonormal basis for the tangent space $T_x M$, then

$$\mathrm{div}_M(X) = \sum_i \langle e_i, D_{e_i} X \rangle.$$

An integral $n$-varifold $M$ (or surface) is stationary in $U \subset \mathbb{R}^{n+1}$ if $\delta M(X) = 0$ for all $X$ compactly supported in $U$. We say that $M$ has generalized mean curvature $H_M$ and zero generalized boundary in $U$ if

$$\delta M(X) = -\int H_M \cdot X d\mu_M \quad \forall X \in C^1_c(U, \mathbb{R}^{n+1}),$$

where $H_M$ is some $\mu_M$-integrable vector field.

Let $M$ have generalized mean curvature $H_M$ in $B_1$, and zero generalized boundary, and suppose $||H_M||_{L^p(B_1;\mu_M)} \leq \Lambda < \infty$ for some $p > n$. Then $M$ admits the area monotonicity (see [1])

$$\left(\frac{\mu_M(B_s(x))}{s^n}\right)^{1/p} \leq \frac{\Lambda}{p-n}(r^{1-n/p} - s^{1-n/p}) + \left(\frac{\mu_M(B_r(x))}{r^n}\right)^{1/p} \tag{1}$$

for any $x \in B_1$, and $0 < s < r < 1 - |x|$. Of course if $M$ is stationary, then $r^{-n}\mu_M(B_r(x))$ is increasing for all $r < 1 - |x|$.

Further, Allard's theorem [1] implies that, as in the previous paragraph, $M$ admits the following regularity: there is a $\delta(n,p) > 0$ so that if for some $n$-plane $V$ we have

$$\int_{B_1} d_V^2 \, d\mu_M + ||H_M||^2_{L^p(B_1;\mu_M)} \leq E \leq \delta^2,$$

$$\mu_M(B_1) \leq \frac{3}{2}\omega_n \quad \text{and} \quad \mu_M(\overline{B_{1/10}}) \geq \frac{1}{2}\omega_n(1/10)^n,$$

then there is a $C^{1,1-n/p}$ function $u : V \cap B_{1/2} \to V^\perp$, so that

$$\mathrm{spt}M \cap B_{1/2} = \mathrm{graph}(u) \cap B_{1/2}, \quad |u|_{C^{1,1-n/p}} \leq c(n)E^{1/2}.$$

2.2. *Jacobi fields.* Given a smooth, oriented minimal hypersurface $N^n$, let us write $\mathcal{M}_N$ for the mean curvature operator on graphs over $N$, i.e., so that given $u : U \subset N \to \mathbb{R}$, and $x \in U$, then $\mathcal{M}_N(u)(x)$ denotes the mean curvature of $\mathrm{graph}_U(u)$ at the point $x + u(x)\nu_N(x)$. Equivalently, $-\mathcal{M}_N$ is the Euler-Lagrange operator for the area functional on graphs over $N$. The operator $\mathcal{M}_N$ is a second-order, quasi-linear elliptic operator

$$\mathcal{M}_N(u) = a_N(x,u,\nabla u)^{ij}\nabla^2_{ij}u + b_N(x,u,\nabla u),$$

whose coefficients $a_N(x,z,p)$, $b_N(x,z,p)$ depend smoothly on $x, z, p$ and the submanifold $N$.

Write $\mathcal{L}_N$ for the linearization of $\mathcal{M}_N$ at $u = 0$. We call $\mathcal{L}_N$ the Jacobi operator, and we call any solution $w$ to $\mathcal{L}_N(w) = 0$ a Jacobi field. The operator $\mathcal{L}_N$ is a linear, elliptic operator:

$$\mathcal{L}_N = \Delta_N + |A_N|^2.$$

Here $A_N$ is the second fundamental form of $N \subset \mathbb{R}^{n+1}$, and $\Delta_N$ is the connection Laplacian. If $\phi_t$ is a family of compactly supported diffeomorphisms of $\mathbb{R}^{n+1}$, and $\partial_t\phi_t|_{t=0,x\in N} = f\nu_N$ on $N$, then

$$\frac{d^2}{dt^2}\Big|_{t=0}\mathrm{vol}(\phi_t(N)) = -\int_N f\mathcal{L}_N f \, d\mathcal{H}^n.$$

The hypersurface $N$ is called stable if $\mathcal{L}_N \leq 0$ when restricted to any compact subset of $N$.

When $N = \mathbf{C} \setminus \{0\}$ is a cone, with smooth, compact cross section $\Sigma = \mathbf{C} \cap S^n$, then we can further decompose

$$\mathcal{L}_N = \partial_r^2 + (n-1)r^{-1}\partial_r + r^{-2}\mathcal{L}_\Sigma, \quad \mathcal{L}_\Sigma = \Delta_\Sigma + |A_\Sigma|^2,$$

where $r = |x|$ is the radial distance, $\omega = x/|x|$, and $\Delta_\Sigma$, $A_\Sigma$ are the connection Laplacian, second fundamental form (resp.) of $\Sigma \subset S^n$.

Since $\Sigma$ is compact, there is $L^2(\Sigma)$-orthonormal basis of eigenfunctions $\phi_i$ of $-\mathcal{L}_\Sigma$, with corresponding eigenvalues $\mu_1 < \mu_2 \leq \cdots \to \infty$:

$$\mathcal{L}_\Sigma \phi_i + \mu_i \phi_i = 0, \quad \int_\Sigma \phi_i \phi_j d\mathcal{H}^{n-1} = \delta_{ij}.$$

By the Rayleigh quotient, $\mu_1 \leq -(n-1)$. On the other hand, when $\mathbf{C}$ is stable, we have $\mu_1 \geq -\left(\frac{n-2}{2}\right)^2$, and we have strict inequality when $\mathbf{C}$ is strictly stable (see [5]). If we define

$$\gamma_i^\pm = -\frac{n-2}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu_i},$$

then for any solution $w$ to $\mathcal{L}_\mathbf{C}(w) = 0$, with $\mathbf{C}$ being strictly stable, we can expand in $L^2_{\mathrm{loc}}(\mathbf{C})$:

$$w(r\omega) = \sum_{i=1}^\infty (a_i^+ r^{\gamma_i^+} + a_i^- r^{\gamma_i^-})\phi_i(\omega),$$

where for each $r$ the sum is $L^2(\Sigma)$ orthogonal.

2.3. *Hardt-Simon foliation.* Taking $\mathbf{C}$, $\Sigma$ as above, then $\mathbf{C}$ divides $\mathbb{R}^{n+1}$ into two connected, open, disjoint regions $E_+$ and $E_-$. We can choose an oriented unit normal $\nu_\mathbf{C}$ for $\mathbf{C}$, so that $\nu_\mathbf{C}$ points into $E_+$.

When $\mathbf{C}$ is area-minimizing, in the sense of currents, then Hardt and Simon [9] have shown there are *smooth*, area-minimizing hypersurfaces $S_\pm \subset E_\pm$ which are asymptotic to $\mathbf{C}$. Moreover, the $S_\pm$ are radial graphs, and hence the collection of dilations $\lambda S_\pm$ ($\lambda > 0$) form a foliation of $E_\pm$ by smooth, area-minimizing hypersurfaces, sometimes called the Hardt-Simon foliation. Let us orient $S_\pm$ with unit normals $\nu_{S_\pm}$ compatible with $\mathbf{C}$, so that as $|x| \to \infty$, $\nu_{S_\pm} \to \nu_\mathbf{C}$.

When $\mathbf{C}$ is *strictly* minimizing, then $S_\pm$ decays to $\mathbf{C}$ like the larger homogeneity $r^{\gamma_1^+}$. In particular, after a normalization as necessary, there is a radius $R_0 \geq 1$ and $\alpha_0 > 0$ so that

$$S_\pm \setminus B_{R_0} = \mathrm{graph}_\mathbf{C}(v_\pm), \tag{2}$$

where $v_\pm : \mathbf{C} \setminus B_{R_0/2} \to \mathbb{R}$ is a smooth function satisfying

$$v_\pm(r\omega) = \pm r^{\gamma_1^+} f_\pm(r\omega), \quad \sum_{k=0}^2 r^k |\nabla^k (f_\pm - \phi_1)| = O(r^{-\alpha_0}). \tag{3}$$

For shorthand, we will set $\gamma = \gamma_1^+$. See [9] for details about strictly minimizing.

Given $\lambda \in \mathbb{R}$, define

$$S_\lambda = \begin{cases} \lambda S_+ & \lambda > 0, \\ \mathbf{C} & \lambda = 0, \\ |\lambda| S_- & \lambda < 0, \end{cases} \qquad v_\lambda(r\omega) = \begin{cases} \lambda v_+(r/\lambda) & \lambda > 0, \\ 0 & \lambda = 0, \\ |\lambda| v_-(r/|\lambda|) & \lambda < 0 \end{cases} \tag{4}$$

so that

$$S_\lambda \setminus B_{\lambda R_0} = \mathrm{graph}_{\mathbf{C}}(v_\lambda). \tag{5}$$

Let $S_\lambda$ have the same orientation as $S_{\mathrm{sign}(\lambda)}$. Observe that

$$v_\lambda(r) = \mathrm{sign}(\lambda)|\lambda|^{1-\gamma} r^\gamma f_{\mathrm{sign}(\lambda)}(r/|\lambda|). \tag{6}$$

For shorthand, we will often write $\lambda^\alpha := \mathrm{sign}(\lambda)|\lambda|^\alpha$.

The following straightforward lemma will be useful.

LEMMA 2.4. *Provided* $R_0(\mathbf{C})$ *is sufficiently large, and* $|\mu|, |\lambda| \leq 1$, $r \geq \max\{|\mu|, |\lambda|\} R_0$, *we have*

$$\frac{1}{4}|\mu^{1-\gamma} - \lambda^{1-\gamma}|^2 r^{2\gamma} \leq |v_\mu(r) - v_\lambda(r)|^2 \leq 4|\mu^{1-\gamma} - \lambda^{1-\gamma}|^2 r^{2\gamma}. \tag{7}$$

*Proof.* If $\lambda \neq 0$, $|\lambda| \leq 1$, and $r \geq |\lambda| R$, then we have

$$\begin{aligned} \frac{d}{d\lambda} v_\lambda(r) &= (1-\gamma)|\lambda|^{-\gamma} r^\gamma f_{\mathrm{sign}(\lambda)}(r/|\lambda|) - |\lambda|^{-1-\gamma} r^{1+\gamma} f'_{\mathrm{sign}(\lambda)}(r/|\lambda|) \\ &= (1-\gamma)|\lambda|^{-\gamma} r^\gamma (1 + O(|\lambda|^{\alpha_0} R^{-\alpha_0})). \end{aligned}$$

If $\mu\lambda \geq 0$, then the required result follows from the above and the fundamental theorem of calculus.

If $\mu\lambda < 0$, $|\lambda| \geq |\mu| > 0$, then we have (recalling our shorthand $\mu^\beta = \mathrm{sign}(\mu)|\mu|^\beta$)

$$\begin{aligned} |v_\mu(r) - v_\lambda(r)|^2 &= (|\mu|^{1-\gamma} + |\lambda|^{1-\gamma})^2 r^{2\gamma} (1 + O(|\lambda|^{\alpha_0} R^{-\alpha_0})) \\ &= |\mu^{1-\gamma} - \lambda^{1-\gamma}|^2 r^{2\gamma} (1 + O(|\lambda|^{\alpha_0} R^{-\alpha_0})). \end{aligned}$$ □

2.5. *Minimizing quadratic cones.* Take $\mathbf{C} = \mathbf{C}^{p,q}$ to be an area-minimizing quadratic cone. There are two key properties of $\mathbf{C}$ which we shall need. These are proven in [15, Prop. 2.7].

(1) $\mathbf{C}$ is *strictly-minimizing*, so that the foliation decays like $r^{\gamma_1^+}$. In fact, since $\mathbf{C}$ is rotationally symmetric, we have that

$$v_\pm(r\omega) = \pm r^{\gamma_1^+} f_\pm(r), \quad \sum_{k=0}^{2} r^k |\nabla^k (f_\pm - 1)| = O(r^{-\alpha_0}). \tag{8}$$

Recall that for shorthand we write $\gamma = \gamma_1^+$.

(2) $\mathbf{C}$ is *strongly integrable*, in the following sense: any solution of $\mathcal{L}_{\mathbf{C}}(w) = 0$ can be written

$$w(x = r\omega) = \sum_{i\geq 1} a_i^- r^{\gamma_i^-}\phi_i(\omega) + er^{\gamma_1^+} + (b + Ax)\cdot \nu_{\mathbf{C}}(\omega) + \sum_{i\geq 4} a_i^+ r^{\gamma_i^+}\phi_i(\omega),$$

where $e \in \mathbb{R}$, $b \in \mathbb{R}^{n+1}$, and $A$ is a skew-symmetric $(n+1)\times(n+1)$ matrix. In other words, $\gamma_2^+ = 0$, $\gamma_3^+ = 1$, and the eigenfunctions $\phi_2$, $\phi_3$ are generated by translations/rotations.

Every result in our paper holds for any area-minimizing hypercone satisfying the above two conditions. Rotational symmetry as in (8) simplifies our computations slightly, but has no bearing on our proof. We write all our results for quadratic cones because these are the only area-minimizing cones which we can verify as "strongly integrable."

## 3. Main theorems

*For the duration of this paper*, we fix $\mathbf{C} = \mathbf{C}^{p,q}$ to be an area-minimizing quadratic cone, $S_\lambda$ the Hardt-Simon foliation, and we use the notation associated to $\mathbf{C}$, $S_\lambda$ as introduced in Section 2.3.

Our main theorem is the following, from which Theorem 1.4 follows directly.

THEOREM 3.1. *There are positive constants $\delta_1(\mathbf{C})$, $\Lambda_1(\mathbf{C})$, $c_1(\mathbf{C})$, $\beta(\mathbf{C})$ so that the following holds. Take $|\lambda| \leq \Lambda_1$, and let $M$ be a stationary integral varifold in $B_1$, satisfying*

$$\int_{B_1} d^2_{S_\lambda}\, d\mu_M \leq E \leq \delta_1^2, \tag{9}$$
$$\mu_M(B_1) \leq \frac{3}{2}\mu_{\mathbf{C}}(B_1), \quad \text{and} \quad \mu_M(\overline{B_{1/10}}) \geq \frac{1}{2}\mu_{\mathbf{C}}(B_{1/10}).$$

*Then there are $a \in \mathbb{R}^{n+1}$, $q \in \mathrm{SO}(n+1)$, $\lambda' \in \mathbb{R}$, with*

$$|a| + |q - \mathrm{Id}| + \left|\mathrm{sign}(\lambda')|\lambda'|^{1-\gamma} - \mathrm{sign}(\lambda)|\lambda|^{1-\gamma}\right| \leq c_1 E^{1/2}, \tag{10}$$

*and there is a $C^{1,\beta}$ function $u : (a + q(S_{\lambda'})) \cap B_{1/2} \to \mathbb{R}$, so that*

$$\mathrm{spt} M \cap B_{1/2} = \mathrm{graph}_{a+q(S_{\lambda'})}(u) \cap B_{1/2},$$

*and $u$ satisfies the estimates*

$$r^{-1}|u|_{C^0(B_r(a))} + |\nabla u|_{C^0(B_r(a))} + r^\beta[\nabla u]_{\beta, B_r(a)} \leq c_1 r^\beta E^{1/2} \quad \forall r \leq \frac{1}{2}. \tag{11}$$

*In particular, $M \llcorner B_{1/2}$ is either smooth, or has an isolated singularity modeled on $\mathbf{C}$.*

*Remark* 3.2. The precise form of the lower bound on $\mu_M(B_{1/10})$ in (9) is of no consequence, nor is the precise ball radius $1/10$. One could easily assume (for example) $\mu_M(B_{1/10}) \geq v > 0$ and obtain the same conclusions, except that the constants $\delta_1$ and $\Lambda_1$ would depend on the choice of $v$ also. The upper bound on $\mu_M(B_1)$ is more important; we require it to be strictly less than $2\mu_{\mathbf{C}}(B_1)$.

A further characterization is possible in the case when $M$ as in Theorem 3.1 is singular. Caffarelli, Hardt, and Simon [5] have constructed a large class of examples of minimal surfaces in $B_1$, which are singular perturbations of a given minimal cone (see Section 9.1). In fact, in a sufficiently small neighborhood, these are the only minimal surfaces which are graphical over $\mathbf{C}$. It would be interesting to know whether examples like those in [5] exist as perturbations over a foliate $S_\lambda$.

Proposition 3.3. *Let $M$, $\lambda'$ be as in Theorem 3.1. If $\lambda' = 0$, and $E \leq \delta_2(\mathbf{C})$ is sufficiently small, then* $\operatorname{spt} M \cap B_{1/4}$ *coincides with one of the graphical solutions as constructed in* [5].

The most interesting consequence of Theorem 3.1 is that singularities modeled on (minimizing) Simons's cones propagate out their structure not only to a neighborhood of the original surface, but also of *nearby* surfaces. If these nearby surfaces are not minimal, but instead of $L^p$ mean curvature, then essentially the same structure holds, but with slightly less regularity. In this sense the minimizing Simons's singularities can be thought of as "very strongly isolated."

Corollary 3.4. *Given any $p > n$, there are positive constants $\delta_3(\mathbf{C})$, $\epsilon_3(p, \mathbf{C})$, $\Lambda_3(\mathbf{C})$, $c_3(\mathbf{C})$ so that the following holds. Let $M$ be an integral $n$-varifold in $B_1$ with generalized mean curvature $H_M$, zero generalized boundary, satisfying* (9) *with $\delta_3^2$ in place of $\delta_1^2$, and*

$$\left( \int_{B_1} |H_M|^p d\mu_M \right)^{1/p} \leq \epsilon_3.$$

*Then there are $a \in B_{1/4}$, $\lambda' \in \mathbb{R}$ so that*

- *either $\lambda' = 0$, in which case* $\operatorname{spt} M \cap B_{1/2}$ *is a $C^{1,\beta}$ perturbation of $\mathbf{C}$, and $M \llcorner B_{1/2}$ is regular away from an isolated singularity modeled on $\mathbf{C}$;*
- *or $\lambda' \neq 0$, in which case $M \llcorner B_{1/2}$ is entirely regular, and for every $0 < r \leq \frac{1}{2}$, we can find a $q_r \in \mathrm{SO}(n+1)$, so that* $\operatorname{spt} M \cap B_r(a) \setminus B_{r/100}(a)$ *is a $C^{1,\beta}$ graph over $a + q_r(S_{\lambda'})$.*

*Example* 3.5. This corollary rules out many possible examples of singularity formation. For example, in an 8-dimensional manifold this rules out the possibility that $S^3 \times S^3$ singularities are collapsing into an $S^2 \times S^4$ singularity, or even worse, that multiple types of isolated singularities are collapsing into and single $S^3 \times S^3$ or $S^2 \times S^4$ singularity.

*Remark* 3.6. We cannot obtain directly that the $q_r$ have a limit as $r \to 0$. If $\lambda' = 0$, then we can use [2] to deduce a posteriori that $M \cap B_{1/2}$ is a $C^{1,\alpha}$ perturbation of $\mathbf{C}$. If $\lambda' \neq 0$, then we do not need to worry about the limiting behavior of the $q_r$ to deduce that $M \cap B_{1/2}$ is some $C^{1,\alpha}$ perturbation of $S_{\lambda'}$. However, because we have no control over $q_r$ as $r \to 0$, we cannot obtain any effective estimates on the $C^{1,\alpha}$ map in question. It would be interesting to resolve this.

Another direct consequence of our regularity theorem is the following rigidity theorem for area-minimizing surfaces asymptotic to Simons's cones, which was originally proven by Simon-Solomon:

Corollary 3.7 ([15]). *Let $M$ be an area-minimizing hypersurface in $\mathbb{R}^{n+1}$, and suppose there is a sequence of radii $R_i \to \infty$ so that*

$$R_i^{-1} M \to \mathbf{C}$$

*in the flat distance. Then up to translation, rotation, and dilation, $M = \mathbf{C}, S_1$, or $S_{-1}$.*

*Remark* 3.8. We remark that our result is much stronger than the characterization of [15]. As illustrated by the examples of [5], being close to the Simons's cone at scale 1 is much weaker than being close on all of $\mathbb{R}^{n+1}$; in particular, the latter precludes any modes growing faster than 1-homogeneous. One can think of [15] as a Bernstein-type theorem, while our theorem can be thought of as an Allard-type regularity theorem.

## 4. **Outline of proof**

Our strategy is to prove the following excess-decay type estimate (Proposition 6.1): provided both $\lambda$ and $\int_{B_1} d^2_{S_\lambda} d\mu_M$ are sufficiently small (plus some restrictions on the mass of $M$), then we have a decay estimate of the following form:

$$\theta^{-n-2} \int_{B_\theta(a')} d^2_{a'+q'(S_{\lambda'})} d\mu_M \leq \frac{1}{2} \int_{B_1} d^2_{S_\lambda} d\mu_M. \tag{12}$$

That is, provided $S_\lambda$ is sufficiently close to the cone $\mathbf{C}$, and we are sufficiently $L^2$ close to $S_\lambda$, then after a translation/rotation/dilation as necessary (depending on $M$), our $L^2$ distance to the foliation improves at a smaller scale $\theta$ (depending only on the cone $\mathbf{C}$).

We can continue iterating (12) while $S_\lambda$ is scale-invariantly close to $\mathbf{C}$, and we obtain a decay of the form: there is an $a'' \in \mathbb{R}^{n+1}$, $q'' \in \mathrm{SO}(n+1)$, and

$\lambda'' \in \mathbb{R}$ so that

$$r^{-n-2}\int_{B_r(a'')} d^2_{a''+q''(S_{\lambda''})} d\mu_M \leq C r^{2\beta} \quad \forall c|\lambda''| \leq r \leq 1. \tag{13}$$

By two straightforward contradiction arguments (one for $M$ close to $S_\lambda \cap B_1 \setminus B_{1/100}$ with $\lambda$ small, and one for $M$ close to $S_{\pm 1} \cap B_c$), we can use [1] with (13) to deduce that $\mathrm{spt}M \cap B_{1/2}$ is graphical over $a'' + q''(S_{\lambda''})$.

We would like to prove (12) by contradiction,with an argument that loosely resembles the original "excess decay" proof due to De Giorgi and, as implemented in a fashion closer to our style, [2], [12], [13]. Briefly, we would like to suppose (12) fails for some sequence $M_i$ and $\lambda_i \to 0$, $E_i = \int_{B_1} d^2_{S_{\lambda_i}} d\mu_{M_i} \to 0$. Then over larger and larger annuli $B_{1/2} \setminus B_{\tau_i}$ ($\tau_i \to 0$) we can write $\mathrm{spt}M = \mathrm{graph}_{S_{\lambda_i}}(u_i)$. If we rescale $v_i = E_i^{-1/2} u_i$, then the $v_i$ have uniformly bounded $||v_i||_{L^2(B_1)}$, and after passing to a subsequence, we get convergence

$$v_i \to w \quad \text{with} \quad \mathcal{L}_{\mathbf{C}}(w) = 0\,.$$

The idea now, in vague terms, is to use good decay properties for solutions to the linearized problem $\mathcal{L}_{\mathbf{C}}(w) = 0$, to prove good decay for solutions to the non-linear problem $\mathcal{M}_{\mathbf{C}}(u) = 0$; that is, we would like to arrange so that $w = O(r^{1+\epsilon})$ and then use this to deduce $L^2$ decay of the $u_i$ as in (12).

To ensure this argument works we need to

(i) ensure the decaying norm for the non-linear problem is comparable to the linear one (also known as non-concentration of $L^2$ norm at singularities) — that is for any $\rho$ small,

$$E_i^{-1}\int_{B_\rho} d^2_{S_{\lambda_i}} d\mu_{M_i} \to ||w||^2_{L^2(\mathbf{C}\cap B_\rho)};$$

(ii) prove good decay for the linear problem (a.k.a. killing bad homogeneities through integrability) — that is $w = O(r^{1+\epsilon})$.

The latter issue is where the concept of *integrability* arises. A minimal cone $\mathbf{C}$ is called integrable if every 1-homogeneous Jacobi field arises from a 1-parameter family of minimal cones. The idea of [1], [2] is that, under suitable density assumptions in the argument above, one can typically show that $w = O(r)$, but one needs it to grow like $r^{1+\epsilon}$. For a cone with an isolated singularity, the homogeneities are discrete, and so provided $\mathbf{C}$ is integrable, one can rewrite the minimal surface as a graph over a slightly adjusted cone, chosen to cancel the $r$ term in the Fourier expansion of $w$.

The main novelty in our approach is in treating the foliation $S_\lambda$ as a direction of integrability. In other words, we are relaxing the original notion of integrability, as a movement through cones, to allow one to push off the cone into families of entirely smooth hypersurfaces and, in particular, we are allowing for a notion of integrability in which the singularity behavior changes. In order to handle this we require new decay and non-concentration estimates for minimal surfaces near an arbitrary foliate $S_\lambda$ without any structural assumption on $M$. This is the content of Theorem 5.1.

More precisely, the key observation is that the foliation is generated by a positive Jacobi field of the form

$$v(r) = r^\gamma, \quad 0 > \gamma > -\frac{n-2}{2},$$

and this Jacobi field has itself good $L^2$ decay:

$$\int_{B_\rho \cap \mathbf{C}} v^2 d\mathcal{H}^n \leq c\rho^2 \int_{B_1 \cap \mathbf{C}} v^2 d\mathcal{H}^n. \tag{14}$$

To deal with point (i), we use the maximum principle to "trap" $M$ between two foliates, and thereby show that $M$ cannot diverge from a given $S_\lambda$ any faster than the foliation itself. This allows us to prove that $\int_{B_\rho} d^2_{S_\lambda} d\mu_M$ has a decay similar to (14), and hence no $L^2$ norm can accumulate near the non-graphical region. (Away from 0 we of course have strong $L^2$ convergence since the $v_i$ converge smoothly there.)

To deal with point (ii), we can prove that the $v_i$, and hence the resulting Jacobi field $w$, grow at least as fast as $v(r) = r^\gamma$ as $r$ increases. Using the strongly integrable nature of $\mathbf{C}$, we can then deduce that $w$ looks like

$$w(x = r\omega) = er^\gamma + (b + Ax) \cdot \nu_{\mathbf{C}} + O(r^{1+\epsilon})$$

for $f \in \mathbb{R}$, $b \in \mathbb{R}^{n+1}$, $A$ skew-symmetric. In other words, $w$ has the growth we require except for terms generated by moving into the foliation, translation, and rotation. By replacing the $S_{\lambda_i}$ with a new sequence of foliates $a_i + q_i(S_{\lambda'_i})$ and repeating the above contradiction argument with this new sequence, we can arrange so that these three lower homogeneities disappear, and thereby deduce $w = O(r^{1+\epsilon})$.

## 5. Non-concentration of $L^2$-excess

Our main theorem of this section is the following. It is spiritually similar to Theorem 2.1 in [13], except we are proving non-concentration with respect to an arbitrary foliate $S_\lambda$ instead of just $\mathbf{C}$, and we additionally obtain a pointwise decay estimate on the graphing function. Recall the shorthand $\gamma = \gamma_1^+$.

THEOREM 5.1. *For every $0 < \tau < \frac{1}{4}$, $0 < \beta$, there are positive $\Lambda_4(\mathbf{C}, \tau)$, $\epsilon_4(\mathbf{C}, \beta, \tau)$, $c_4(\mathbf{C})$ so that the following holds: if $|\lambda| \leq \Lambda_4$ and $M$ is a stationary integral $n$-varifold in $B_1$ satisfying*

$$\int_{B_1} d^2_{S_\lambda} d\mu_M \leq \epsilon_4^2, \quad \mu_M(B_1) \leq \frac{7}{4}\mu_{\mathbf{C}}(B_1), \quad \mu_M(\overline{B_{1/10}}) \geq \frac{1}{2}\mu_{\mathbf{C}}(B_{1/10}), \tag{15}$$

*then there is a smooth function $u : S_\lambda \cap B_{1/2} \setminus B_{\tau/2} \to \mathbb{R}$ so that*

$$\mathrm{spt} M \cap B_{1/2} \setminus B_\tau = \mathrm{graph}_{S_\lambda}(u) \cap B_{1/2} \setminus B_\tau, \quad \sum_{k=0}^{3} r^{k-1} |\nabla^k u| \leq \beta. \tag{16}$$

*For every* $\tau \leq \rho \leq \frac{1}{4}$, *we have*

$$\int_{B_\rho \setminus B_\tau} u^2 d\mu_{\mathbf{C}} + \int_{B_\rho} d^2_{S_\lambda} d\mu_M \leq c_4 \rho^2 \int_{B_1} d^2_{S_\lambda} d\mu_M. \tag{17}$$

*Moreover*, $u$ *has the following* $L^\infty$ *decay bound*:

$$\sup_{S_\lambda \cap \partial B_r} u^2 \leq c_4 r^{2\gamma} \int_{B_1} d^2_{S_\lambda} \qquad \forall r \in (\tau, 1/4). \tag{18}$$

*Proof.* Let $R_0(\mathbf{C})$ be as in Lemma 2.4, and recall the definitions of $v_\lambda$, $R_0$ in (2)–(5) from Section 2.3. Taking $\Lambda_4(\mathbf{C}, \tau)$ sufficiently small, we can assume $\Lambda_4 R_0 < \tau/100$ and

$$\sum_{k=0}^{2} r^{k-1} |\nabla^k v_\lambda| \leq \beta \quad \forall r \in (\tau/100, 1). \tag{19}$$

We claim that, provided $\epsilon_4(\mathbf{C}, \Lambda_4, \tau, \beta)$ is sufficiently small, we have

$$\mathrm{spt} M \cap B_{3/4} \setminus B_{\tau/10} = \mathrm{graph}_{S_\lambda}(u), \quad \sum_{k=0}^{3} r^{k-1} |\nabla^k u| \leq \beta \tag{20}$$

for $u : S_\lambda \cap B_{3/4} \setminus B_{\tau/20} \to \mathbb{R}$ smooth.

Indeed, otherwise there is a sequence of stationary integral $n$-varifolds $M_i$, and numbers $\epsilon_i \to 0$, $\lambda_i \in [-\Lambda_4, \Lambda_4]$, for which (15) holds but (20) fails. We can without loss assume $\lambda_i \to \lambda$ for some $|\lambda| \leq \Lambda_4$. By compactness of stationary varifolds with bounded mass, we can pass to a subsequence (also denoted $i$) and get varifold convergence $M_i \to M$ for some stationary integral $n$-varifold $M$ in $B_1$. The resulting $M$ satisfies

$$\int_{B_1} d^2_{S_\lambda} d\mu_M = 0, \quad 0 < \mu_M(B_1) \leq \frac{7}{4} \mu_{\mathbf{C}}(B_1). \tag{21}$$

The constancy theorem implies that $M = k[S_\lambda]$ for some natural number $k$. The lower bound of (21) implies $k \geq 1$. Ensuring $\Lambda_4(\mathbf{C})$ is sufficiently small, the upper bound in (21) implies $k \leq 1$. So in fact $M_i \to [S_\lambda]$, and hence by Allard's theorem convergence is smooth on compact subsets of $B_1 \setminus \{0\}$. This proves our claim.

It will be more convenient in this proof to work with graphs over $\mathbf{C}$. By a similar contradiction argument as above, we have (again taking $\epsilon_4(\mathbf{C}, \tau, \beta)$, $\Lambda_4(\mathbf{C}, \tau, \beta)$ sufficiently small)

$$\mathrm{spt} M \cap B_{3/4} \setminus B_{\tau/10} = \mathrm{graph}_{\mathbf{C}}(h), \quad \sum_{k=0}^{3} r^{k-1} |\nabla^k h| \leq \beta \tag{22}$$

for $h : \mathbf{C} \cap B_{3/4} \setminus B_{\tau/20} \to \mathbb{R}$ smooth.

Ensuring $\beta(\mathbf{C})$ is sufficiently small, $u$ is effectively equivalent to $h - v_\lambda$. Precisely, if $r\omega \in \mathbf{C} \cap B_{1/2} \setminus B_{\tau/4}$ and $x = r\omega + v_\lambda(r)\nu_{\mathbf{C}}(r\omega) \in S_\lambda$, then

$$|u(x)\nu_{S_\lambda}(x) - (h(r\omega) - v_\lambda(r))\nu_{\mathbf{C}}(r\omega)| \leq c(\mathbf{C})\beta|u(x)|. \tag{23}$$

To see this, observe that if $\zeta$ is the nearest-point projection to $\mathbf{C}$, then provided $\beta(\mathbf{C})$ is sufficiently small, the map $F(r\omega) = \zeta(r\omega + v_\lambda(r)\nu_{\mathbf{C}}(r\omega) + u(x)v_{S_\lambda}(x))$ as a map $\mathbf{C} \to \mathbf{C}$ is smooth, well-defined, and satisfies

$$|F(r\omega) - r\omega| \leq c(\mathbf{C})\beta|u(x)| \leq c\beta^2 r. \tag{24}$$

Hence we can write

$$r\omega + v_\lambda(r)\nu_{\mathbf{C}}(r\omega) + u(x)\nu_{S_\lambda}(x) = F(r\omega) + h(F(r\omega))\nu_{\mathbf{C}}(F(r\omega)). \tag{25}$$

Estimate (23) then follows from (25), (24), and (22).

Choosing a possibly smaller $\beta(\mathbf{C})$, (23) implies

$$\sup_{S_\lambda \cap \partial B_r} |u| \leq 2 \sup_{\mathbf{C} \cap B_{2r} \setminus B_{r/2}} |h - v_\lambda| \qquad \forall r \in (\tau/2, 1/4) \tag{26}$$

and

$$\int_{B_r \setminus B_{\tau/2}} u^2 d\mu_{S_\lambda} \leq 2 \int_{B_{2r} \setminus B_{\tau/4}} |h - v_\lambda|^2 d\mu_{\mathbf{C}}, \tag{27}$$

$$\int_{B_r \setminus B_{\tau/2}} |h - v_\lambda|^2 d\mu_{\mathbf{C}} \leq 2 \int_{B_{2r} \setminus B_{\tau/4}} d_{S_\lambda}^2 d\mu_M, \tag{28}$$

$$\int_{B_r \setminus B_{\tau/2}} d_{S_\lambda}^2 d\mu_M \leq 2 \int_{B_{2r} \setminus B_{\tau/4}} u^2 d\mu_{S_\lambda}. \tag{29}$$

So we can prove the required estimates for $h - v_\lambda$ instead of $u$.

For $\rho \in (\tau/10, 3/4)$, define

$$\lambda_\rho^+ = \inf\{\mu : v_\mu(\rho) \geq h(\rho\omega) \quad \forall \omega \in \Sigma\},$$
$$\lambda_\rho^- = \sup\{\mu : v_\mu(\rho) \leq h(\rho\omega) \quad \forall \omega \in \Sigma\}.$$

Of course all sups/infs above are actually maxs/mins, so that

$$v_{\lambda_\rho^+}(\rho) = \max_\omega h(\rho\omega), \quad v_{\lambda_\rho^-}(\rho) = \min_\omega h(\rho\omega). \tag{30}$$

From (6) and (22), we have $|\lambda_\rho^\pm| = \Psi(\beta|\mathbf{C}, \tau)$, and so ensuring $\beta(\mathbf{C}, \tau)$ is sufficiently small, $\lambda_\rho^\pm R_0 < \tau/10$ for all $\rho \in (\tau/10, 3/4)$. By the maximum principle ([16]), we have that $\mathrm{spt} M \cap B_\rho$ is trapped between $S_{\lambda_\rho^-}$ and $S_{\lambda_\rho^+}$. This implies that $\lambda_\rho^+$ is increasing in $\rho$, while $\lambda_\rho^-$ is decreasing in $\rho$, and

$$v_{\lambda_\rho^-}(r) \leq h(r\omega) \leq v_{\lambda_\rho^+}(r) \quad \forall r \in (\tau/2, \rho). \tag{31}$$

We note, however, that $\lambda$ *need not* be trapped between $\lambda_\rho^\pm$.

Both $h$ and $v_\lambda$ solve the minimal surface equation over $\mathbf{C}$, with uniform bounds (19) and (22), and so provided $\beta(\tau, \mathbf{C})$ is sufficiently small the difference

$h - v_\lambda$ solves a linear, second order, uniformly elliptic operator on $\mathbf{C} \cap B_{1/2} \setminus B_{\tau/10}$ (see, e.g., [8, §§10.1, 8.6]). Standard iteration techniques at scale $r$ imply that

(32)
$$\sup_\omega |h(r\omega) - v_\lambda(r)|^2 \leq c(\mathbf{C}) r^{-n} \int_{B_{2r} \setminus B_{r/2}} |h - v_\lambda|^2 d\mu_{\mathbf{C}} \quad \forall r \in (\tau/5, 1/4).$$

Since $\lambda_\rho^+$ is increasing in $\rho$, $\lambda_\rho^-$ is decreasing in $\rho$, and $\lambda_\rho^- \leq \lambda_\rho^+$, we get that

$$\max\{|(\lambda_\rho^+)^{1-\gamma} - \lambda^{1-\gamma}|^2, |(\lambda_\rho^-)^{1-\gamma} - \lambda^{1-\gamma}|^2\} \quad \text{is increasing in } \rho. \tag{33}$$

For any $\tau/5 < r < \rho < 1/4$, we have by (31), Lemma 2.4, (33), (30) and (32) (in order of usage):

(34)
$$\begin{aligned}
|h(r\omega) - v_\lambda(r)|^2 &\leq 2 \max\{|v_{\lambda_\rho^+}(r) - v_\lambda(r)|^2, |v_{\lambda_\rho^-}(r) - v_\lambda(r)|^2\} \\
&\leq cr^{2\gamma} \max\{|(\lambda_\rho^+)^{1-\gamma} - \lambda^{1-\gamma}|^2, |(\lambda_\rho^-)^{1-\gamma} - \lambda^{1-\gamma}|^2\} \\
&\leq cr^{2\gamma} \max\{|(\lambda_{1/4}^+)^{1-\gamma} - \lambda^{1-\gamma}|^2, |(\lambda_{1/4}^-)^{1-\gamma} - \lambda^{1-\gamma}|^2\} \\
&\leq cr^{2\gamma} \max\{|v_{\lambda_{1/4}^+}(1/4) - v_\lambda(1/4)|^2, |v_{\lambda_{1/4}^-}(1/4) - v_\lambda(1/4)|^2\} \\
&= cr^{2\gamma} \max\{|\sup_\omega h(\omega/4) - v_\lambda(1/4)|^2, |\inf_\omega h(\omega/4) - v_\lambda(1/4)|^2\} \\
&= cr^{2\gamma} \sup_\omega |h(\omega/4) - v_\lambda(1/4)|^2 \\
&\leq cr^{2\gamma} \int_{B_{1/2} \setminus B_{1/8}} |h - v_\lambda|^2 d\mu_{\mathbf{C}},
\end{aligned}$$

where $c = c(\mathbf{C})$. Combined with (26) and (28), we obtain estimate (18).

Integrating (34) in $r \in (\tau/2, \rho)$ and $\omega \in \Sigma$, in combination with (28), gives

$$\begin{aligned}
\int_{B_\rho \setminus B_{\tau/2}} |h - v_\lambda|^2 d\mu_{\mathbf{C}} &\leq \frac{c(\mathbf{C})}{n + 2\gamma} \rho^{n+2\gamma} \int_{B_1} d_{S_\lambda}^2 d\mu_M \\
&\leq c(\mathbf{C}) \rho^2 \int_{B_1} d_{S_\lambda}^2 d\mu_M,
\end{aligned}$$

since $2\gamma + n \geq -(n-2) + n \geq 2$. Using the bounds (20), (22) and (26), as in the proof of (27) and (29), we have

$$\int_{B_\rho \setminus B_\tau} d_{S_\lambda}^2 d\mu_M \leq c\rho^2 \int_{B_1} d_{S_\lambda}^2 d\mu_M. \tag{35}$$

We focus now on proving the final part of (17), i.e., bounding the excess in the ball $B_\tau$. Since $S$ is graphical over $\mathbf{C}$ near $\partial B_{R_0}$, we have

$$d_H(S_\lambda \cap B_{\max\{|\lambda|,|\mu|\} R_0}, S_\mu \cap B_{\max\{|\lambda|,|\mu|\} R_0}) \leq c(\mathbf{C}) |\lambda - \mu|.$$

Let $\lambda_\tau = \max\{|\lambda_\tau^+|, |\lambda_\tau^-|\}$. Then, recalling how $\mathrm{spt} M \cap B_\tau$ is trapped between $S_{\lambda_\tau^+}$ and $S_{\lambda_\tau^-}$, and ensuring $\Lambda_4(\tau, \mathbf{C})$ is sufficiently small, we get

$$
\begin{aligned}
&\int_{B_{\lambda_\tau R_0}} d_{S_\lambda}^2 d\mu_M \\
&\leq \max\{d_H(S_{\lambda_\tau^+} \cap B_{\lambda_\tau R_0}, S_\lambda \cap B_{\lambda_\tau R_0})^2, \\
&\qquad\qquad d_H(S_{\lambda_\tau^-} \cap B_{\lambda_\tau R_0}, S_\lambda \cap B_{\lambda_\tau R_0})^2\}\mu_M(B_{\lambda_\tau R_0}) \\
&\leq c(\mathbf{C}) \max\{|\lambda_\tau^+ - \lambda|^2, |\lambda_\tau^- - \lambda|^2\}\lambda_\tau^n R_0^n \\
&\leq c(\mathbf{C})\tau^{n+2\gamma} \max\{|(\lambda_\tau^+)^{1-\gamma} - \lambda^{1-\gamma}|^2, |(\lambda_\tau^-)^{1-\gamma} - \lambda^{1-\gamma}|^2\}.
\end{aligned} \tag{36}
$$

The last line follows because there is a constant $c(n)$ so that whenever $|\mu|$, $|\lambda| \leq 1$, we have (recall $\gamma < 0$)

$$|\mu^{1-\gamma} - \lambda^{1-\gamma}|^2 \geq \frac{1}{c(n)} \max\{|\mu|, |\lambda|\}^{-2\gamma}|\mu - \lambda|^2.$$

Choose $I$ so that $2^I \lambda_\tau R_0 \leq \tau < 2^{I+1}\lambda_\tau R_0$. We compute

$$
\begin{aligned}
&\int_{B_\tau \setminus B_{\lambda_\tau R_0}} d_{S_\lambda}^2 d\mu_M \\
&\leq \sum_{i=0}^{I} \int_{B_{2^{i+1}\lambda_\tau R_0} \setminus B_{2^i \lambda_\tau R_0}} d_{S_\lambda}^2 d\mu_M \\
&\leq \sum_{i=0}^{I} \sup_{2^i\lambda_\tau R_0 \leq r \leq 2^{i+1}\lambda_\tau R_0} \max\{|v_{\lambda_\tau^+}(r) - v_\lambda(r)|^2, \\
&\qquad\qquad |v_{\lambda_\tau^-}(r) - v_\lambda(r)|^2\}\mu_M(B_{2^{i+1}\lambda_\tau R_0}) \\
&\leq \sum_{i=0}^{I} c(\mathbf{C})(2^i\lambda_\tau R_0)^{2\gamma} \max\{|(\lambda_\tau^+)^{1-\gamma} - \lambda^{1-\gamma}|^2, |(\lambda_\tau^-)^{1-\gamma} - \lambda^{1-\gamma}|^2\}(2^i\lambda_\tau R_0)^n \\
&\leq c(\mathbf{C})\tau^{n+2\gamma} \max\{|(\lambda_\tau^+)^{1-\gamma} - \lambda^{1-\gamma}|^2, |(\lambda_\tau^-)^{1-\gamma} - \lambda^{1-\gamma}|^2\},
\end{aligned} \tag{37}
$$

where in the third inequality we used (7) in Lemma 2.4, while the last inequality follows since $n + 2\gamma \geq 2$.

Combining (36) and (37) with the computations of (34), we obtain

$$
\begin{aligned}
\int_{B_\tau} d_{S_\lambda}^2 d\mu_M &\leq c(\mathbf{C})\tau^{n+2\gamma} \max\{|(\lambda_\tau^+)^{1-\gamma} - \lambda^{1-\gamma}|^2, |(\lambda_\tau^-)^{1-\gamma} - \lambda^{1-\gamma}|^2\} \\
&\leq c(\mathbf{C})\tau^2 \int_{B_1} d_{S_\lambda}^2 d\mu_M.
\end{aligned}
$$

Together with (35), this gives the required estimate (17). ☐

The following corollary will also be useful.

Corollary 5.2. *There is a positive* $\Lambda_5(\mathbf{C})$ *so that if* $|\lambda|, |\lambda'| \leq \Lambda_5$, *and* $M$ *satisfies the hypotheses of Theorem 5.1, then*

$$\int_{B_1} d^2_{S_{\lambda'}} d\mu_M \leq c(\mathbf{C}) \int_{B_1} d^2_{S_\lambda} d\mu_M + c(\mathbf{C})|\lambda^{1-\gamma} - (\lambda')^{1-\gamma}|^2.$$

*Proof.* The computations of Theorem 5.1 show that, provided $|\lambda|, |\lambda'| \leq \Lambda_4(\mathbf{C})$, we have

$$\begin{aligned}\int_{B_{1/4}} d^2_{S_{\lambda'}} d\mu_M &\leq c(\mathbf{C}) \max\{|(\lambda^+_{1/4})^{1-\gamma} - (\lambda')^{1-\gamma}|^2, |(\lambda^-_{1/4})^{1-\gamma} - (\lambda')^{1-\gamma}|^2\} \\ &\leq c(\mathbf{C}) \int_{B_1} d^2_{S_\lambda} d\mu_M + c(\mathbf{C})|\lambda^{1-\gamma} - (\lambda')^{1-\gamma}|^2.\end{aligned}$$

It remains only to control the annular region $B_1 \setminus B_{1/4}$.

Choose $\epsilon(\Lambda_4, \mathbf{C})$ sufficiently small so that if $|\lambda| \leq \Lambda_4$, then the nearest point projection from $B_\epsilon(\mathbf{C}) \cap B_1 \setminus B_{1/4}$ onto $S_\lambda$ is smooth and lies in $S_\lambda \cap B_2 \setminus B_{1/8}$. Ensure that $\Lambda_5(\epsilon, \Lambda_4, \mathbf{C}) \leq \Lambda_4$ is sufficiently small, so that $|\lambda| \leq \Lambda_5$ implies $S_\lambda \cap B_2 \setminus B_{1/8} \subset B_{\epsilon/2}(\mathbf{C})$.

Given $x \in B_1 \setminus B_{1/4}$, first assume that $x \notin B_\epsilon(\mathbf{C})$. In this case

$$\epsilon/2 \leq d(x, S_\lambda) \leq 2,$$

and hence we have

$$\int_{(B_1\setminus B_{1/4})\setminus B_\epsilon(\mathbf{C})} d^2_{S_{\lambda'}} d\mu_M \leq 4\mu_M(B_1) \leq (16/\epsilon^2)c(\mathbf{C}) \int_{(B_1\setminus B_{1/4})\setminus B_\epsilon(\mathbf{C})} d^2_{S_\lambda} d\mu_M.$$

Now assume $x \in B_\epsilon(\mathbf{C})$. Let $x'$ be the nearest point projection to $\mathbf{C}$, and let $u(x') = x - x'$. Since $|v_\lambda| + |\nabla v_\lambda| \leq c(\mathbf{C})|\Lambda_5|^{-\gamma}$ on $B_2 \setminus B_{1/8}$, we have

$$d(x, S_\lambda) = (1 + \psi(\Lambda_5|\mathbf{C}))|x - v_\lambda(x')|.$$

In particular, ensuring that $\Lambda_5(\mathbf{C})$ is sufficiently small and using Lemma 2.4, we get

$$\begin{aligned} d(x, S_{\lambda'}) &\leq 2|x - v_{\lambda'}(x')| \\ &\leq 2|x - v_\lambda(x)| + 2|v_{\lambda'}(x) - v_\lambda(x)| \\ &\leq 4d(x, S_\lambda) + c(\mathbf{C})|(\lambda')^{1-\gamma} - \lambda^{1-\gamma}|. \end{aligned}$$

Integrating $d\mu_M$ over $B_1 \cap B_\epsilon(\mathbf{C}) \setminus B_{1/4}$ gives the required result. □

## 6. $L^2$-excess decay

In this section we work towards the following decay estimate.

Proposition 6.1 (Decay Lemma). *Given any* $0 < \theta \leq \frac{1}{8}$, *there are positive constants* $\delta_6(\mathbf{C}, \theta)$, $\Lambda_6(\mathbf{C}, \theta)$, $c_6(\mathbf{C})$, *and* $\alpha(\mathbf{C})$, *so that the following holds*:

*If $|\lambda| \leq \Lambda_6$, and $M$ is a stationary integral $n$-varifold in $B_1$, satisfying*

(38)
$$\int_{B_1} d^2_{S_\lambda} d\mu_M \leq E \leq \delta_6^2, \quad \mu_M(B_1) \leq \frac{7}{4}\mu_{\mathbf{C}}(B_1), \quad \mu_M(\overline{B_{1/10}}) \geq \frac{1}{2}\mu_{\mathbf{C}}(B_{1/10}),$$

*then we can find $a \in \mathbb{R}^{n+1}$, $q \in \mathrm{SO}(n+1)$, and $\lambda' \in \mathbb{R}$ with*

(39)
$$|a| + |q - \mathrm{Id}| + |(\lambda')^{1-\gamma} - \lambda^{1-\gamma}| \leq c_6 E^{1/2}$$

*so that*

(40)
$$\theta^{-n-2} \int_{B_\theta(a)} d^2_{a+q(S_{\lambda'})} d\mu_M \leq c_6 \theta^{2\alpha} E, \quad \mu_M(\overline{B_{\theta/10}(a)}) \geq \frac{3}{4}\mu_{\mathbf{C}}(B_{\theta/10}).$$

We first define a general notion of blow-up sequence and show how any blow-up sequence gives rise to a Jacobi field, i.e., a solution of the linearized problem $\mathcal{L}_{\mathbf{C}}(w) = 0$.

*Definition* 6.1.1. Consider the sequences $a_i \in \mathbb{R}^{n+1}$, $\lambda_i \in \mathbb{R}$, $q_i \in \mathrm{SO}(n+1)$, $M_i$ stationary integral $n$-varifolds in $B_1 \subset \mathbb{R}^{n+1}$, and $E_i \in \mathbb{R}$. We say the collection $(M_i, E_i, a_i, \lambda_i, q_i)$ is a blow-up sequence if

(1) $a_i \to 0$, $\lambda_i \to 0$, $q_i \to \mathrm{Id}$, $E_i \to 0$;
(2) $\mu_{M_i}(B_1) \leq \frac{7}{4}\mu_{\mathbf{C}}(B_1)$, $\mu_{M_i}(\overline{B_{1/10}}) \geq \frac{1}{2}\mu_{\mathbf{C}}(B_{1/10})$;
(3) $\limsup_i E_i^{-1} \int_{B_1} d^2_{a_i+q_i(S_{\lambda_i})} d\mu_{M_i} < \infty$.

PROPOSITION 6.2. *Let $(M_i, \beta_i, a_i, \lambda_i, q_i)$ be a blow-up sequence. From Theorem 5.1, there is a sequence of radii $\tau_i \to 0$ so that*

$$M \cap B_{1/2} \setminus B_{\tau_i} = \mathrm{graph}_{a_i+q_i(S_{\lambda_i})}(u_i), \quad \sum_{k=0}^{3} r^{k-1}|\nabla^k u_i| \to 0$$

*and*

$$(a_i + q_i(S_{\lambda_i})) \setminus B_{\tau_i} = \mathrm{graph}_{\mathbf{C}}(\phi_i), \quad \sum_{k=0}^{3} r^{k-1}|\nabla^k \phi_i| \to 0.$$

*Write $\Phi_i(x) = x + \phi_i(x)\nu_{\mathbf{C}}$ for the graphing function associated to $\phi_i$.*

*There is a subsequence, also denoted $i$, and there is a solution $w : \mathbf{C} \cap B_{1/4} \to \mathbb{R}$ to $\mathcal{L}_{\mathbf{C}}(w) = 0$ satisfying the following:*

(1) *smooth convergence: $E_i^{-1/2} u_i \circ \Phi_i \to w$ smoothly on compact subsets of $\mathbf{C} \cap B_{1/4} \setminus \{0\}$;*
(2) *$L^\infty$ decay: for all $r < 1/4$:*
$$w(r\omega)^2 \leq c(\mathbf{C}) r^{2\gamma} \left( \limsup_i E_i^{-1} \int_{B_1} d^2_{a_i+q_i(S_{\lambda_i})} d\mu_{M_i} \right);$$
(3) *strong $L^2$ convergence:*
$$E_i^{-1} \int_{B_r} d^2_{a_i+q_i(S_{\lambda_i})} d\mu_{M_i} \to \int_{B_r} w^2 d\mu_{\mathbf{C}} \quad \forall r \leq \frac{1}{4}.$$

*Remark* 6.3. For shorthand, we will often say $E_i^{-1/2}u_i$ converges smoothly to $w$ to indicate convergence as in Proposition 6.2(1).

*Proof.* Part (1) is a fairly standard argument (see, e.g., [12]). Parts (2) and (3) follow directly from Theorem 5.1. We outline the argument of part (1). Since $a_i + q_i(S_{\lambda_i})$ converges smoothly to $\mathbf{C}$ on compact subsets of $B_1 \setminus \{0\}$, the coefficients of $\mathcal{M}_{a_i+q_i(S_{\lambda_i})}$, $\mathcal{L}_{a_i+q_i(S_{\lambda_i})}$ converge locally smoothly to those of $\mathcal{M}_{\mathbf{C}}$, $\mathcal{L}_{\mathbf{C}}$. Using this and standard elliptic estimates, we have for any compact $K \subset (a_i + q_i(S_{\lambda_i})) \cap B_{1/2} \setminus \{0\}$, and $l = 0, 1, 2 \ldots$, uniform estimates of the form

$$\begin{aligned}\sup_K |\nabla^l u_i| &\leq c(K,l) \left( \int_{(a_i+q_i(S_{\lambda_i}))\cap B_{1/2}} u_i^2 d\mathcal{H}^n \right)^{1/2} \\ &\leq c(K,l) E_i^{1/2} \left( \limsup_i E_i^{-1} \int_{B_1} d^2_{a_i+q_i(S_{\lambda_i})} d\mu_{M_i} \right)^{1/2}.\end{aligned} \tag{41}$$

Of course by assumption, $\Phi_i \to 0$ in $C^\infty_{\mathrm{loc}}(\mathbf{C} \cap B_1 \setminus \{0\})$. After passing to a subsequence, we deduce $C^\infty_{\mathrm{loc}}(\mathbf{C} \cap B_{1/4} \setminus \{0\})$ convergence of the functions $E_i^{-1/2} u_i \circ \Phi_i$ to some $w \in C^\infty(\mathbf{C} \cap B_{1/4})$.

Now we can write

$$0 = \mathcal{M}_{a_i+q_i(S_{\lambda_i})}(u_i) = \mathcal{L}_i(u_i) + \mathcal{E}_i(u_i),$$

where $\mathcal{L}_i \equiv \mathcal{L}_{a_i+q_i(S_{\lambda_i})}$ converges smoothly away from 0 to the operator $\mathcal{L}_{\mathbf{C}}$, and where

$$\sup_K |\mathcal{E}_{S_{\lambda_i}}(u_i)| \leq C(K) |u|^2_{C^2(K)} = o(1) E_i^{1/2}.$$

It follows easily that $w$ solves the Jacobi operator $\mathcal{L}_{\mathbf{C}}(w) = 0$. $\square$

*Proof of Proposition* 6.1. Choose $\alpha(\mathbf{C})$ so that $\gamma_3^+ = 1 < 1 + \alpha \leq \gamma_4^+$. Fix $0 < \theta \leq 1/8$. We first prove the decay estimate. Suppose, towards a contradiction, that there are sequences of numbers $\delta_i \to 0$, $\lambda_i \to 0$, $E_i \to 0$, and stationary integral varifolds $M_i$ in $B_1$ which satisfy

$$\int_{B_1} d^2_{S_{\lambda_i}} d\mu_{M_i} \leq E_i \leq \delta_i, \quad \mu_{M_i}(B_1) \leq \frac{7}{4}\mu_{\mathbf{C}}(B_1), \quad \mu_{M_i}(\overline{B_{1/10}}) \geq \frac{1}{2}\mu_{\mathbf{C}}(B_{1/10}),$$

but for which

$$\theta^{-n-2} \int_{B_\theta(a)} d^2_{a+q(S_{\lambda'})} d\mu_{M_i} \geq c_6 \theta^{2\alpha} E_i$$

for any $a \in \mathbb{R}^{n+1}$, $q \in \mathrm{SO}(n+1)$, $\lambda' \in \mathbb{R}$ satisfying

$$|a| + |q - \mathrm{Id}| + |\lambda' - \lambda_i|^{1-\gamma} \leq c_6 E_i^{1/2},$$

and $c_6(\mathbf{C})$ to be fixed shortly.

For any sequence $\tau_i$, $\beta_i$ tending to 0 sufficiently slowly, by Theorem 5.1 we can write

$$\mathrm{spt} M_i \cap B_{1/2} \setminus B_{\tau_i} = \mathrm{graph}_{S_{\lambda_i}}(u_i), \quad \sum_{k=0}^{3} r^{k-1}|\nabla^k u_i| \leq \beta_i.$$

By definition, $(M_i, E_i, 0, \lambda_i, \mathrm{Id})$ is a blow-up sequence, and so by Proposition 6.2 there is a solution $w : \mathbf{C} \cap B_{1/4} \to \mathbb{R}$ to $Lw = 0$ satisfying:

$$\int_{B_{1/4}} w^2 d\mu_{\mathbf{C}} \leq 1, \quad |w(r\omega)| \leq c(\mathbf{C}) r^{\gamma}, \tag{42}$$

so that, after passing to a subsequence (also denoted $i$),

$$E_i^{-1/2} u_i \to w$$

smoothly on compact subsets of $\mathbf{C} \cap B_{1/4} \setminus \{0\}$, and

$$E_i^{-1} \int_{B_\rho} d^2_{S_{\lambda_i}} d\mu_{M_i} \to \int_{B_\rho} w^2 d\mu_{\mathbf{C}} \quad \forall \rho \leq \frac{1}{4}.$$

Using the pointwise bound (42) combined with [15, (2.10) Lemma] to kill the modes $\gamma_i^-$, $i \in \mathbb{N}$, and the strongly integrable nature of $\mathbf{C}$, there are $e \in \mathbb{R}$, $b \in \mathbb{R}^{n+1}$, and $A$ a skew-symmetric $(n+1) \times (n+1)$ matrix, so that we can expand $w$ in $L^2(\mathbf{C} \cap B_{1/4})$ as

$$w(r\omega = x) = er^{\gamma} + \nu_{\mathbf{C}}(r\omega) \cdot (b + Ax) + \sum_{j : \gamma_j^+ \geq 1+\alpha} r^{\gamma_j^+} z_j(\omega),$$

where the sum is $L^2(\Sigma)$-orthogonal for each fixed $r$. In particular, using the $L^2$ bound (42) and an appropriate choice of $r \in (\frac{1}{8}, \frac{1}{4})$, we get

$$|e| + |b| + |A| \leq c_7(\mathbf{C}),$$

and by $L^2(\Sigma)$-orthogonality,

$$\sum_{j : \gamma_j^+ \geq 1+\alpha} \frac{(1/4)^{2\gamma_j^+ + n}}{2\gamma_j^+ + n} \int_{\Sigma} z_j^2 \leq 1. \tag{43}$$

We shall demonstrate that by choosing appropriate $\lambda_i', a_i, q_i$, we can arrange so that $e = 0$, $b = 0$, $A = 0$. Define

$$\lambda_i' = (eE_i^{1/2} + \lambda_i^{1-\gamma})^{1/(1-\gamma)}, \tag{44}$$

$$a_i = bE_i^{1/2}, \tag{45}$$

$$q_i = \exp(A_i) \quad \text{for } A_i = AE_i^{1/2}. \tag{46}$$

We first show that by replacing $\lambda_i$ with $\lambda_i'$, we can get the same Jacobi field except with $e = 0$. Trivially $\lambda_i' \to 0$, and Corollary 5.2 implies that $(M_i, E_i, 0, \lambda_i', \mathrm{Id})$ is a blow-up sequence also. Our choice (44) implies that

$$v_{\lambda_i'} - v_{\lambda_i} = (1 + o(1))((\lambda_i')^{1-\gamma} - \lambda_i^{1-\gamma})r^\gamma = (1 + o(1))eE_i^{1/2}r^\gamma,$$

where we write $o(1)$ to signify any function which tends to 0 as $i \to \infty$. We can write

$$S_{\lambda_i'} \cap B_1 \setminus B_{\tau_i} = \mathrm{graph}_{S_{\lambda_i}}(v_i),$$

where setting $x = r\omega + v_{\lambda_i}(r\omega)\nu_{\mathbf{C}}(r\omega)$,

$$|v_i(x) - (v_{\lambda_i'} - v_{\lambda_i})(x)| \leq o(1)|(v_{\lambda_i'} - v_{\lambda_i})(x)|.$$

In other words,

$$v_i(x) = (1 + o(1))eE_i^{1/2}r^\gamma.$$

Write

$$\mathrm{spt}M_i \cap B_{1/2} \setminus B_{\tau_i} = \mathrm{graph}_{S_{\lambda_i'}}(\tilde{u}_i).$$

If we set $y = x + v_i(x)\nu_{S_{\lambda_i}}(x)$ (for $x$ as above), then

$$\tilde{u}_i(y) = (1 + o(1))(u_i(x) - v_i(x)) = (1 + o(1))u_i(x) - (1 + o(1))eE_i^{1/2}r^\gamma.$$

Applying Proposition 6.2 to the blow-up sequence $(M_i, E_i, 0, \lambda_i', \mathrm{Id})$, we deduce that, after passing to a further subsequence, $E_i^{-1/2}\tilde{u}_i$ converges smoothly on compact subsets to

$$w - er^\gamma \equiv \nu_{\mathbf{C}}(r\omega) \cdot (b + Ax) + \sum_{j:\gamma_j^+ \geq 1+\alpha} r^{\gamma_j^+} z_j(\omega),$$

and we have strong $L^2$ convergence

$$E_i^{-1} \int_{B_\rho} d^2_{S_{\lambda_i'}} d\mu_{M_i} \to \int_{B_\rho} (w - er^\gamma)^2 d\mu_{\mathbf{C}} \quad \forall \rho \leq \frac{1}{4}.$$

We now show how our choice $a_i, q_i$ as defined in (45) and (46) yields the same Jacobi field except with $b = 0$, $A = 0$ (in addition to $e = 0$). This is more standard and essentially follows the usual "integrability through rotations" argument.

It is easy to check that $|a_i| + |q_i - \mathrm{Id}| \leq c(\mathbf{C})E_i^{1/2}$. Since

$$d_H(S_{\lambda_i'} \cap B_2, (a_i + q_i(S_{\lambda_i'}) \cap B_2) \leq c(\mathbf{C})E_i^{1/2},$$

it follows that $(M_i, E_i, a_i, \lambda_i', q_i)$ is a blow-up sequence also. We can write

$$(a_i + q_i(S_{\lambda_i'})) \cap B_1 \setminus B_{\tau_i} = \mathrm{graph}_{S_{\lambda_i'}}(v_i)$$

(for $\tau_i \to 0$ sufficiently slowly), where

$$v_i(x) = (1 + o(1))\nu_{S_{\lambda_i'}}(x) \cdot (a_i + A_i(x)).$$

So now if $\tilde{u}_i$ is the graphing function of $M_i$ over $a_i + q_i(S_{\lambda_i'})$, and $u_i$ is the graphing function of $M_i$ over $S_{\lambda_i'}$, then we have

$$|\tilde{u}_i(x + v_i(x)\nu_{S_{\lambda_i}}(x)) - (u_i(x) - v_i(x))| \leq o(1)|u_i(x) - v_i(x)|.$$

This implies that

$$\begin{aligned}\tilde{u}_i(y) &= (1 + o(1))(u_i(x) - v_i(x)) \\ &= (1 + o(1))u_i(x) - (1 + o(1))\nu_{S_{\lambda_i'}}(x) \cdot (b + A(x))E_i^{1/2}.\end{aligned}$$

Applying Proposition 6.2 to this new blow-up sequence, we deduce that (after passing to a further subsequence) $E_i^{-1/2}\tilde{u}_i$ converges smoothly on compact subsets to

$$w - er^\gamma - \nu_{\mathbf{C}}(r\omega) \cdot (b + Ax) \equiv \sum_{j:\gamma_j^+ \geq 1+\alpha} r^{\gamma_j^+} z_j(\omega),$$

and we have strong $L^2$ convergence

$$E_i^{-1} \int_{B_\rho} d^2_{a_i + q_i(S_{\lambda_i'})} d\mu_{M_i} \to \int_{B_\rho} (w - er^\gamma - \nu_{\mathbf{C}} \cdot (b + Ax))^2 d\mu_{\mathbf{C}} \quad \forall \rho \leq \frac{1}{4}.$$

We have demonstrated that by judiciously choosing our $a_i, q_i, \lambda_i'$, we can arrange so that

$$w = \sum_{j:\gamma_j^+ \geq 1+\alpha} r^{\gamma_j^+} z_j(\omega),$$

where $z_j$ continue to satisfy the bound (43). Using (43) and the fact that $4\theta \leq 1$, we compute

$$\begin{aligned}\int_{B_{2\theta}} w^2 d\mu_{\mathbf{C}} &= \sum_{j:\gamma_j^+ \geq 1+\alpha} \frac{(2\theta)^{2\gamma_j^+ + n}}{2\gamma_j^+ + n} \int_\Sigma z_j^2(\omega) d\omega \\ &\leq \max_{j:\gamma_j^+ \geq 1+\alpha} (8\theta)^{2\gamma_j^+ + n} \\ &\leq (8\theta)^{n+2+2\alpha}.\end{aligned}$$

So by the strong $L^2$ convergence, for sufficiently large $i$, we must have

$$\int_{B_{2\theta}} d^2_{a_i + q_i(S_{\lambda_i'})} d\mu_{M_i} \leq 16^{n+2+2\alpha}\theta^{n+2+2\alpha} E_i.$$

To recenter, we simply observe that for $i$ sufficiently large, we have

$$B_\theta(a_i) \subset B_{2\theta},$$

and hence we get

$$\theta^{-n-2} \int_{B_\theta(a_i)} d^2_{a_i + q_i(S_{\lambda_i'})} d\mu_{M_i} \leq 32^{n+4}\theta^{2\alpha} E_i,$$

which is a contradiction for sufficiently large $i$.

Finally let us establish the lower volume bound (40). This is a straightforward proof by contradiction. Suppose otherwise: there is a sequence $\delta_i \to 0$, $\lambda_i \to 0$, and $M_i$ satisfying the hypotheses (38), and the decay of (39) and (40), but for which

$$\mu_M(\overline{B_{\theta/10}(a_i)}) < \frac{3}{4}\mu_{\mathbf{C}}(B_{\theta/10})$$

for all $i$.

By compactness of stationary varifolds, we can pass to a subsequence (also denoted $i$) so that $M_i \to M$ in $B_1$ for some integral stationary $n$-varifold $M$ in $B_1$. Since

$$\int_{B_1} d_{\mathbf{C}}^2 d\mu_{M_i} \to 0,$$

by the constancy theorem, $M = k[\mathbf{C}]$ for some integer $k$. Since $\mu_M(\overline{B_{1/10}}) \geq \frac{1}{2}\mu_{\mathbf{C}}(B_{1/10})$, we must have $k \geq 1$. Since $\mu_M(B_1) \leq \frac{7}{4}\mu_{\mathbf{C}}(B_1)$, we must have $k \leq 1$. So in fact $k = 1$, and we deduce that $M_i$ varifold converge to $[\mathbf{C}]$ in $B_1$.

Since $|a_i| \to 0$, for any $0 < \epsilon < 1$ and sufficiently large $i$, we have

$$\mu_{M_i}(\overline{B_{\theta/10}(a_i)}) \geq \mu_{M_i}(B_{(1-\epsilon)\theta/10}) \to \mu_{\mathbf{C}}(B_{(1-\epsilon)\theta/10}) = (1-\epsilon)^n \mu_{\mathbf{C}}(B_{\theta/10}).$$

Choosing $\epsilon(n)$ sufficiently small, so that $(1-\epsilon)^n \geq \frac{7}{8}$, we obtain a contradiction for large $i$. This completes the proof of Proposition 6.1 $\square$

## 7. Regularity

The key idea to obtain regularity is to iterate the (scale-invariant) Proposition 6.1 at decreasing scales $r_i = \theta^i$, obtaining for each scale an $a_i, q_i, \lambda_i'$, until we reach a scale at which $r_i \approx |\lambda_i'|$. This is the radius at which we start to "see" the foliation as separate from the cone, and this is the radius at which we stop. If no such radius exists, we keep iterating until radius 0 to deduce regularity over the cone. Note that, from only the information we start with, we have no way of predetermining how large this radius is.

Proposition 7.1. *There are positive constants $\beta(\mathbf{C})$, $\delta_7(\mathbf{C})$, $\Lambda_7(\mathbf{C})$, and $c_7(\mathbf{C})$ so that the following holds. Take $|\lambda| \leq \Lambda_7$, and let $M$ be a stationary integral varifold in $B_1$ satisfying*

(47)
$$\int_{B_1} d_{S_\lambda}^2 d\mu_M \leq E \leq \delta_7^2, \quad \mu_M(B_1) \leq \frac{3}{2}\mu_{\mathbf{C}}(B_1), \quad \mu_M(\overline{B_{1/10}}) \geq \frac{1}{2}\mu_{\mathbf{C}}(B_{1/10}).$$

*Then there are $a \in \mathbb{R}^{n+1}$, $q \in \mathrm{SO}(n+1)$, and $\lambda' \in \mathbb{R}$ with*

$$|a| + |q - \mathrm{Id}| + |(\lambda')^{1-\gamma} - \lambda^{1-\gamma}| \leq c_7 E^{1/2}, \tag{48}$$

*so that for all $1 \geq r > c_7|\lambda'|$, we have the decay*

$$r^{-n-2}\int_{B_r(a)} d_{a+q(S_{\lambda'})}^2 d\mu_M \leq c_7 r^{2\beta} E \tag{49}$$

*and the volume bounds*

$$\mu_M(B_r(a)) \leq \frac{7}{4}\mu_{\mathbf{C}}(B_r), \quad \mu_M(\overline{B_{r/10}(a)}) \geq \frac{1}{c_7}\mu_{\mathbf{C}}(B_{r/10}). \tag{50}$$

*Remark* 7.2. In fact one can take $\beta$ to be anything in the interval $(0, \alpha)$, except of course in this case the various constants $\delta_7$, $c_7$ will depend on the choice of $\beta$ also.

*Proof.* Ensure $\Lambda_7 \leq \min\{\Lambda_5, \Lambda_6\}$ (the constants from Corollary 5.2, Proposition 6.1). Fix $\theta(\mathbf{C}) \leq 1/4$ sufficiently small so that $c_6\theta^{2\alpha} \leq 1/4$. Set $r_i = \theta^i$. We claim that we can find an integer $I \leq \infty$ and sequences $a_i \in \mathbb{R}^{n+1}$, $q_i \in \mathrm{SO}(n+1)$, and $\lambda_i \in \mathbb{R}$, $(i = 0, 1, \ldots, I)$, so that for all $i < I$ we have

$$a_0 = 0, \quad q_0 = \mathrm{Id}, \quad \lambda_0 = \lambda, \quad r_0 = 1, \tag{51}$$

$$\begin{aligned} &r_i^{-1}|a_{i+1} - a_i| + |q_{i+1} - q_i| + r_i^{-1+\gamma}|(\lambda_{i+1})^{1-\gamma} - (\lambda_i)^{1-\gamma}| \\ &\quad \leq c_6 2^{-i} E^{1/2}, \end{aligned} \tag{52}$$

$$|\lambda_i| \leq \Lambda_7 r_i, \tag{53}$$

the decay

$$r_i^{-n-2}\int_{B_{r_i}(a_i)} d^2_{a_i + q_i(S_{\lambda_i})} d\mu_M \leq 4^{-i}E, \tag{54}$$

and the volume bounds

$$\mu_M(B_{r_i}(a_i)) \leq \frac{7}{4}\mu_{\mathbf{C}}(B_{r_i}), \quad \mu_M(\overline{B_{r_i/10}(a_i)}) \geq \frac{1}{2}\mu_{\mathbf{C}}(B_{r_i/10}). \tag{55}$$

Moreover, if $I < \infty$, then

$$|\lambda_I| > \Lambda_7 r_I.$$

Let us prove this by induction. Let us first show how the upper volume bound of (55) follows from (52). If we have $a_0, \ldots, a_i$, satisfying (52), then

$$|a_i| \leq \sum_{j=0}^{i-1} |a_{j+1} - a_j| \leq c_6 E^{1/2} \sum_{j=0}^{i-1} r_i \leq 2c_6\delta_7.$$

Therefore, provided $\delta_7(\mathbf{C})$ is sufficiently small, by volume monotonicity we have

$$\begin{aligned} \mu_M(B_{r_i}(a_i)) &\leq (1 - 2c_6\delta_7)^{-n}\mu_M(B_1) r_i^n \\ &\leq (1 - 2c_6\delta_7)^{-n}(3/2)\mu_{\mathbf{C}}(B_1) r_i^n \\ &\leq (7/4)\mu_{\mathbf{C}}(B_{r_i}). \end{aligned}$$

It remains to show the existence of the $a_i, q_i, \lambda_i$. We set $a_0 = 0$, $q_0 = \mathrm{Id}$, $\lambda_0 = \lambda$ as in (51). Since $|\lambda_0| = |\lambda| \leq \Lambda_7$, we can apply Proposition 6.1 to $M$ to obtain $a_1, q_1, \lambda_1$, which satisfy the required estimates. If $|\lambda_1| > \Lambda_7 r_1$, we set $I = 1$ and stop. This proves the base case of our induction.

Suppose, by inductive hypothesis, that we have found $a_i, q_i, \lambda_i$ satisfying (52), (53), (54) and (55), and for which $|\lambda_i| \leq \Lambda_7 r_i$. By inductive hypotheses, we can apply Proposition 6.1 to the varifold $M_i = (q_i^{-1})_\sharp(\eta_{a_i,r_i})_\sharp M$ and foliate $S_{r_i^{-1}\lambda_i}$ to obtain $\tilde{a}_{i+1}, \tilde{q}_{i+1}, \tilde{\lambda}_{i+1}$ satisfying (39) and (40). If we let

$$a_{i+1} = q_i(r_i\tilde{a}_{i+1}) + a_i, \quad q_{i+1} = q_i \circ \tilde{q}_{i+1}, \quad \lambda_{i+1} = r_i\tilde{\lambda}_{i+1},$$

then it follows by scaling that this $a_{i+1}, q_{i+1}, \lambda_{i+1}$ satisfies the requirement estimates. If $|\lambda_{i+1}| > \Lambda_7 r_{i+1}$, we stop and set $I = i+1$. Otherwise, continue. By mathematical induction this proves the existence of the sequences.

If $I < \infty$, then let $a = a_I, q = q_I, \lambda' = \lambda_I$. Otherwise, observe that (52) implies that $a_i$, $q_i$ form a Cauchy sequence, and hence we take $a = \lim_i a_i$, $q = \lim_i q_i$, $\lambda' = 0 = \lim_i \lambda_i$.

From (52), for every $i < I$, we have

$$r_i^{-1}|a - a_i| + |q - q_i| + r_i^{-1+\gamma}|(\lambda')^{1-\gamma} - (\lambda_i)^{1-\gamma}| \leq 6c_6 2^{-i} E^{1/2}.$$

In particular, taking $i = 0$ gives (48).

Given any $r_I \leq r < 1$, choose integer $i \leq I$ so that $r_{i+1} \leq r < r_i$. Then, ensuring that $\Lambda_7(\mathbf{C})$, $\delta_7(\mathbf{C})$ are sufficiently small and using Corollary 5.2, we have

$$\begin{aligned}
&r^{-n-2}\int_{B_r(a)} d^2_{a+q(S_{\lambda'})} d\mu_M \\
&\leq cr_i^{-2}|a - a_i|^2 + c|q - q_i|^2 + cr_i^{-2+2\gamma}|(\lambda')^{1-\gamma} - (\lambda_i)^{1-\gamma}|^2 \\
&\quad + cr_i^{-n-2}\int_{B_{r_i}(a_i)} d^2_{a_i+q_i(S_{\lambda_i})} d\mu_M \\
&\leq c(\mathbf{C})4^{-i}E \\
&\leq c(\mathbf{C})r^{2\beta}E,
\end{aligned} \tag{56}$$

where $\beta = \log(1/2)/\log(\theta) > 0$. Finally, observe that (52) and (53) imply

$$\Lambda_7 r_I < |\lambda'| \leq \Lambda_7 r_{I-1} + (c_6\delta_7)^{1/(1-\gamma)} r_{I-1} \leq c(\mathbf{C}) r_I.$$

Therefore, after enlarging our constant $c(\mathbf{C})$, we can take $r \geq |\lambda'|$ in (56). The volume bounds follow directly from (55) and (52), and monotonicity. This finishes the proof of Proposition 7.1. □

The proof of Theorem 3.1 is now essentially a straightforward application of Allard's theorem.

*Proof of Theorem 3.1*. Ensure that $\delta_1 \leq \delta_7$, and take $\Lambda_1 = \Lambda_7$. Apply Proposition 7.1 to obtain $a, q, \lambda'$. For every $c_7|\lambda'| \leq r \leq 1$, a straightforward contradiction argument as in the proof of Theorem 5.1 implies that

$$\mathrm{spt}M \cap B_{r/2}(a) \setminus B_{r/100}(a) = \mathrm{graph}_{a+q(S_{\lambda'})}(u),$$
$$r^{-1}|u| + |\nabla u| + [\nabla u]_{\beta,r} \leq c(\mathbf{C}) r^{\beta} E^{1/2}$$

for some $u : (a + q(S_{\lambda'})) \cap B_{r/2}(a) \setminus B_{r/200}(a) \to \mathbb{R}$. If $\lambda' = 0$, we are done.

Suppose $\lambda' \neq 0$. After scaling up by $|\lambda'|$, it suffices to show that there is a $u : (a + q(S_{\mathrm{sign}(\lambda')})) \cap B_{c_7/2}(a) \to \mathbb{R}$ so that

$$\mathrm{spt}M \cap B_{c_7/2}(a) = \mathrm{graph}_{a+q(S_{\mathrm{sign}(\lambda')})}(u), \quad |u| + |\nabla u| + [\nabla u]_{\beta} \leq c(\mathbf{C}) E^{1/2},$$

provided

$$c_7^{-n-2} \int_{B_{c_7}(a)} d^2_{a+q(S_{\mathrm{sign}(\lambda')})} d\mu_M \leq c_7 E$$

and

$$\mu_M(B_{c_7}(a)) \leq \frac{7}{4}\mu_{\mathbf{C}}(B_{c_7}), \quad \mu_M(\overline{B_{c_7/10}(a)}) \geq \frac{1}{c_7}\mu_{\mathbf{C}}(B_{c_7/10}).$$

However because $S_{\mathrm{sign}(\lambda')} \cap B_{c_7}$ is smooth with bounded geometry, the required statement follows by Allard regularity and an easy contradiction argument, taking $E \to 0$. □

## 8. Mean curvature

When $M$ has non-zero mean curvature, then we cannot use the maximum principle to conclude as in Proposition 6.1. However, provided the mean curvature is sufficiently small and $L^2$ excess sufficiently large, then we can still get decay to one scale (Proposition 8.1), by a straightforward contradiction argument, which suffices to characterize the singular nature, if not the precise local structure.

Iterating this gives scale-invariant smallness of the excess (Proposition 8.2), rather than decay. At each scale we can deduce closeness to some rotate/translate of the cone $\mathbf{C}$ (and hence foliate $S_\lambda$), but we cannot deduce that the rotations form a Cauchy sequence as we progress in scale. If at some scale $\lambda$ becomes big, we stop, and we can deduce that $M \cap B_{1/2}$ is a $C^{1,\alpha}$ deformation of $S_1$ (*without* effective estimates). If we continue all the way to scale 0, then we deduce that $M \cap B_{1/2}$ has an isolated singularity modeled on $\mathbf{C}$, and a posteriori by [2] we can deduce that $M \cap B_{1/2}$ is a $C^{1,\alpha}$ graph over $\mathbf{C}$.

PROPOSITION 8.1. *For any $\theta \in (0, \frac{1}{4})$, $E_0 \in (0, \delta_6^2)$, there is a positive $\epsilon_8(\mathbf{C}, \theta, E_0)$ so that the following holds. Assume that $M$ is an integral $n$-varifold in $B_1$, satisfying $||\delta M||(B_1) \leq \epsilon_8$, and for some $E \in [E_0, \delta_6^2]$, $|\lambda| \leq \Lambda_6$, we have*

$$\int_{B_1} d^2_{S_\lambda} d\mu_M \leq E, \quad \mu_M(B_1) \leq \frac{7}{4}\mu_{\mathbf{C}}(B_1), \quad \mu_M(\overline{B_{1/10}}) \geq \frac{1}{2}\mu_{\mathbf{C}}(B_{1/10}). \tag{57}$$

*Then there is an* $a \in \mathbb{R}^{n+1}$, $q \in \mathrm{SO}(n+1)$, $\lambda' \in \mathbb{R}$, *satisfying*

$$|a| + |q - \mathrm{Id}| + |(\lambda')^{1-\gamma} - \lambda^{1-\gamma}| \leq 2c_6 E^{1/2}, \tag{58}$$

*so that*

$$\theta^{-n-2} \int_{B_\theta(a)} d^2_{a+q(S_{\lambda'})} d\mu_M \leq 2c_6 \theta^{2\alpha} E, \quad \mu_M(\overline{B_{\theta/10}(a)}) \geq \frac{1}{2}\mu_{\mathbf{C}}(B_{\theta/10}). \tag{59}$$

*Here* $\delta_6(\mathbf{C}, \theta)$, $c_6(\mathbf{C})$ *are the constants from Proposition 6.1.*

*Proof.* Suppose, towards a contradiction, there are sequences of integral $n$-varifolds $M_i$, and numbers $\epsilon_i \to 0$, $E_i \in [E_0, \delta_6^2]$, so that $M_i$ satisfy (57) and $||\delta M_i||(B_1) \leq \epsilon_i$, but for which (59) fails for all $a, q, \lambda'$ satisfying (58).

We can pass to a subsequence, also denoted $i$, and obtain varifold converge $M_i \to M$ and convergence $E_i \to E \in [E_0, \delta_6^2]$. The resulting $n$-varifold $M$ is stationary in $B_1$, continues to satisfy (57), but fails (59) for all $a, q, \lambda'$ satisfying (58). However, $M$ satisfies the hypotheses of Proposition 6.1, contradicting the conclusions of Proposition 6.1. This proves Proposition 8.1. □

PROPOSITION 8.2. *There are* $\delta_9(\mathbf{C})$, $c_9(\mathbf{C})$ *positive so that for any* $E \in (0, \delta_9]$ *and* $p > n$, *we can find an* $\epsilon_9(\mathbf{C}, E, p) > 0$ *for which the following holds. Let* $M$ *is an integral* $n$*-varifold in* $B_1$ *with generalized mean curvature* $H_M$ *and zero generalized boundary, and suppose* $M$ *satisfies*

$$\int_{B_1} d^2_{S_\lambda} d\mu_M \leq E, \quad \mu_M(B_1) \leq \frac{3}{2}\mu_{\mathbf{C}}(B_1), \quad \mu_M(\overline{B_{1/10}}) \geq \frac{1}{2}\mu_{\mathbf{C}}(B_{1/10}),$$

*and*

$$\left( \int_{B_1} |H_M|^p d\mu_M \right)^{1/p} \leq \epsilon_9,$$

*for some* $|\lambda| \leq \Lambda_7$ (*the constant from Proposition 7.1*).

*Then we can find* $a \in \mathbb{R}^{n+1}$, $\lambda' \in \mathbb{R}$, *and* $q_r \in SO(n+1)$ *for each* $1 \geq r \geq c_9|\lambda'|$, *which together satisfy*

$$|a| + |\log(r)|^{-1}|q_r - Id| + |(\lambda')^{1-\gamma} - \lambda^{1-\gamma}| \leq c_9 E^{1/2},$$

*and for which we have at every radius* $1 \geq r \geq c_9|\lambda'|$ *the smallness*

$$r^{-n-2} \int_{B_r(a)} d^2_{a+q_r(S_{\lambda'})} d\mu_M \leq c_9 E,$$

*and the volume bounds*

$$\mu_M(B_r(a)) \leq \frac{7}{4}\mu_{\mathbf{C}}(B_r), \quad \mu_M(\overline{B_{r/10}(a)}) \geq \frac{1}{c_9}\mu_{\mathbf{C}}(B_{r/10}).$$

*Proof.* The proof is almost verbatim to Proposition 7.1, except we use Proposition 8.1 in place of Proposition 6.1. Notice that Proposition 8.1 requires a lower bound on $E$, and so we cannot deduce decay of the $L^2$ excess, only smallness. Choose $\theta(\mathbf{C})$ as in Proposition 8.1, and set $r_i = \theta^i$. We claim that we can find an integer $I \leq \infty$ and sequences $a_i \in \mathbb{R}^{n+1}$, $q_i \in \mathrm{SO}(n+1)$, $\lambda_i \in \mathbb{R}$ $(i = 0, \ldots, I)$ so that for all $i < I$, we have

$$a_0 = 0, \quad q_0 = \mathrm{Id}, \quad \lambda_0 = 0, \tag{60}$$

$$r_i^{-1}|a_{i+1} - a_i| + |q_{i+1} - q_i| + r_i^{-1+\gamma}|(\lambda_{i+1})^{1-\gamma} - (\lambda_i)^{1-\gamma}| \leq 2c_6 E^{1/2}, \tag{61}$$

$$|\lambda_i| \leq \Lambda_7 r_i, \tag{62}$$

the smallness

$$r_i^{-n-2} \int_{B_{r_i}(a_i)} d^2_{a_i+q_i(S_{\lambda_i})} d\mu_M \leq E, \quad r_i^{1-n/p} \left( \int_{B_{r_i}(a_i)} |H_M|^p d\mu_M \right)^{1/p} \leq \epsilon_9, \tag{63}$$

and the volume bounds

$$\mu_M(B_{r_i}(a_i)) \leq \frac{7}{4}\mu_{\mathbf{C}}(B_{r_i}), \quad \mu_M(\overline{B_{r_i/10}(a_i)}) \geq \frac{1}{2}\mu_{\mathbf{C}}(B_{r_i/10}). \tag{64}$$

We proceed by induction. Ensure that $\delta_9 \leq \delta_7(\mathbf{C})$. As in the proof of Proposition 7.1, given $a_0, \ldots, a_i$ satisfying (61), we have $|a_i| \leq 4c_6\delta_9$, and hence by volume monotonicity (1),

$$\mu_M(B_{r_i}(a_i)) \leq (1 + c(p, \mathbf{C})\epsilon_9)(1 - 4c_6\delta_9)^{-n}\mu_M(B_1)r_i^n \leq (7/4)\mu_{\mathbf{C}}(B_{r_i})$$

provided $\delta_9(\mathbf{C})$ and $\epsilon_9(\mathbf{C}, p)$ are sufficiently small. This proves the upper volume bounds (64). The mean curvature bound in (63) follows trivially from $p > n$.

To obtain the $a_i, q_i, \lambda_i$, we first observe that, given any $a_i, q_i$ above, if we set $M_i = (q_i^{-1})_\sharp(\eta_{a_i,r_i})_\sharp M$, then

$$\begin{aligned} ||\delta M_i||(B_1) &= r_i^{-n+1} \int_{B_{r_i}(a_i)} |H_M| d\mu_M \\ &\leq r_i^{-n+1} \left( \int_{B_{r_i}(a_i)} |H_M|^p d\mu_M \right)^{1/p} \mu_M(B_{r_i}(a_i))^{1-1/p} \\ &\leq c(\mathbf{C})\epsilon_9. \end{aligned}$$

In particular, ensuring $\epsilon_9(\mathbf{C}, p)$ is sufficiently small, we can ensure $\|\delta M_i\|(B_1) \leq \epsilon_8(\mathbf{C}, p)$. We can therefore proceed as in the proof of Proposition 7.1, using Proposition 8.1 in place of Proposition 6.1. □

*Proof of Corollary* 3.4. Ensuring $\delta_3 < \delta_9$, $\epsilon_3 < \epsilon_9(\delta, p)$, then we can obtain $a, \lambda'$ and $q_r$ as in Proposition 8.1. A straightforward argument by contradiction, like in the proof of Theorem 3.1, shows that for every $1 \geq r \geq c_9|\lambda'|$,

$$\mathrm{spt} M \cap B_{r/2}(a) \setminus B_{r/100}(a) = \mathrm{graph}_{a+q_r(S_{\lambda'})}(u), \quad r^{-1}|u| + |\nabla u| + [\nabla u]_{\beta,r} \leq c(\mathbf{C})\delta,$$

(provided $\delta_3, \epsilon_3$ are sufficiently small, depending on $\mathbf{C}$, $p$). If $\lambda' = 0$, this shows that any tangent cone at $a$ is rotation of $\mathbf{C}$. Therefore, shrinking $\delta_3, \epsilon_3$ as necessary, [2] implies $\mathrm{spt} M \cap B_{1/2}$ is a $C^{1,\beta}$ perturbation of $\mathbf{C}$.

If $\lambda' \neq 0$, then arguing as in the proof of Theorem 3.1, we get that

$$\mathrm{spt} M \cap B_{c_9|\lambda'|/2}(a) = \mathrm{graph}_{a+q_{c_9|\lambda'|}(S_{\lambda'})}(u), \quad r^{-1}|u| + |\nabla u| + [\nabla u]_\beta \leq c(\mathbf{C})\delta.$$

In particular, $M \llcorner B_{1/2}$ is entirely regular. $\square$

## 9. Corollaries, related results

In this section we give an alternate proof of [15, Th. 0.3] (i.e., Corollary 3.7 of this paper) using our main regularity Theorem 3.1. We also show how the work of [5] implies a uniqueness result for minimal graphs over $\mathbf{C}$ (Proposition 3.3).

*Proof of Corollary 3.7.* Let

$$E_i = R_i^{-n-2} \int_{B_{R_i}} d_{\mathbf{C}}^2 d\mu_M.$$

By hypothesis, $E_i \to 0$. For all $i$ sufficiently large, we can apply Theorem 3.1 to deduce the existence of $a_i \in \mathbb{R}^{n+1}, q_i \in \mathrm{SO}(n+1), \lambda_i \in \mathbb{R}$, and $u_i : (a_i + q_i(S_{\lambda_i})) \cap B_{R_i}(a_i) \to \mathbb{R}$, so that for all $c_7|\lambda_i| \leq r \leq R_i$,

$$\mathrm{spt} M \cap B_r(a_i) = \mathrm{graph}_{a_i+q_i(S_{\lambda_i})}(u_i), \quad r^{-1}|u_i| + |\nabla u_i| \leq c(\mathbf{C})E_i^{1/2}. \tag{65}$$

Suppose, towards a contradiction, that $a_i \to \infty$. Let $U_i(x) = x + u_i(x)\nu_{a_i+q_i(S_{\lambda_i})}(x)$ be the graphing function associated to $u_i$. From (65) we have that $|U_i(x) - x| \leq o(1)|x - a_i|$.

Fix any $\rho > 0$. Then by the previous paragraph, we have

$$U_i^{-1}(\mathrm{spt} M \cap B_\rho) \subset (a_i + q_i(S_{\lambda_i})) \cap B_{2|a_i|}(a_i) \setminus B_{|a_i|/2}(a_i)$$

for $i$ sufficiently large. Now the curvature of $(a_i+q_i(S_{\lambda_i}))\cap B_{2|a_i|}(a_i)\setminus B_{|a_i|/2}(a_i)$ tends to zero uniformly as $i \to \infty$, and $|\nabla u_i| = o(1)$, so we must have that $\mathrm{spt} M \cap B_\rho$ is contained in a plane. Taking $\rho \to \infty$, we deduce $\mathrm{spt} M$ is planar, and hence $\mathbf{C}$ is planar also. This is a contradiction, and so $a_i$ must be bounded.

By (65), we have

$$d(0, \mathrm{spt} M) = d(0, a_i + q_i(S_{\lambda_i})) + o(1)|a_i| \geq \frac{1}{c(\mathbf{C})}|\lambda_i| - c(\mathbf{C})|a_i|$$

for $i$ large. Since $a_i$ is bounded, we must have that $\lambda_i$ is bounded also. We can pass to a subsequence, also denoted $i$, so that $a_i \to 0$, $\lambda_i \to \lambda$, and $q_i \to q \in \mathrm{SO}(n+1)$. We get smooth convergence on compact subsets of $\mathbb{R}^{n+1} \setminus \{a\}$,

$$a_i + q_i(S_{\lambda_i}) \to a + q(S_\lambda),$$

and $C^1_{\mathrm{loc}}$ convergence $u_i \to 0$. We deduce that $\mathrm{spt}M = a + q(S_\lambda)$, which is the desired conclusion. $\square$

9.1. *Uniqueness of graphs over* $\mathbf{C}$. In [5], Caffarelli, Hardt, and Simon prove the following theorem constructing a plethora of examples of minimal surfaces in $B_1$, which are perturbations of a given minimal cone. To state their theorem properly we need some notation. Given $J \in \mathbb{N}$, define the projection mapping $\Pi_J : L^2(\Sigma) \to L^2(\Sigma)$

$$\Pi_J(g)(r\omega) = \sum_{j \geq J+1} < g, \phi_j >_{L^2(\Sigma)} r^{\gamma_j^+} \phi_j(\omega).$$

For shorthand, write $\mathbf{C}_1 = \mathbf{C} \cap \overline{B_1}$.

THEOREM 9.2 ([5]). *Take $m > 1$, $\alpha \in (0,1)$, and $J \in \mathbb{N}$ so that $\gamma_J^+ \leq m < \gamma_{J+1}^+$. There are $\epsilon(\mathbf{C}, m, \alpha)$, $\Lambda(\mathbf{C}, m, \alpha)$ positive so that given any $g \in C^{2,\alpha}$ satisfying $|g|_{2,\alpha} \leq \epsilon$, and any $\lambda \in (0, \Lambda)$, then there is a solution $u_\lambda \in C^{2,\alpha}(\mathbf{C}_1)$ to the problem*

$$M_{\mathbf{C}}(u) = 0 \ \text{on } \mathbf{C}_1, \quad \Pi_J(u_\lambda) = \lambda \Pi_J(g) \ \text{on } \Sigma,$$

*satisfying*

$$r^{-m}|u|_{2,\alpha,r} \leq c(\mathbf{C}, m, \alpha)\lambda |g|_{2,\alpha} \quad \forall 0 < r \leq 1.$$

Loosely speaking, the authors of [5] are solving a boundary-value-type problem, where one is allowed to specify the decay rate at $r = 0$, and the Fourier modes at $r = 1$ which decay *faster* than the prescribed rate. Though they do not comment on it, implicit in their work is the uniqueness statement of Proposition 3.3. The basic idea is that solutions to the minimal surface operator can be written as fixed points to a contraction mapping, provided the solutions decay like $r^{1+\epsilon}$ and have boundary data sufficiently small. We illustrate this below.

*Proof of Proposition* 3.3. We use the notation of Theorem 3.1. Fix an $\alpha \in (0,1)$, and given $w \in C^{2,\alpha}(\mathbf{C}_1)$, define the norm

$$|w|_B = \sup_{0<r\leq 1} r^{-\beta}|u|_{2,\alpha,r}. \tag{66}$$

Pick $J$ so that $\gamma_J^+ = 1$, and recall that $1 + \beta \in (\gamma_J^+, \gamma_{J+1}^+)$.

By assumption, we have $\mathrm{spt}M \cap B_{1/2} = \mathrm{graph}_{U \subset a + q(\mathbf{C})}(u)$, where $u$ satisfies the estimates (11). Since we are concerned with $M \cap B_{1/4}$, ensuring $\delta_2(\mathbf{C})$ is small, there is no loss in assuming (after scaling, translating, rotating) that $\mathrm{spt}M = \mathrm{graph}_{\mathbf{C}_1}(u)$. By standard interior elliptic estimates and decay (11), we can assume that $u$ is smooth and satisfies

$$|u|_B \leq c(\mathbf{C}, \alpha)\delta_2.$$

We can write the mean curvature operator as

$$\mathcal{M}_{\mathbf{C}}(u) = \mathcal{L}_{\mathbf{C}}(u) + \mathcal{E}(u),$$

where the non-linear error part $\mathcal{E}(u)$ satisfies certain, relatively standard scale-invariant structure conditions (see [5]). Given $g \in C^{2,\alpha}(\Sigma)$, in [5] it is shown that there are numbers $\epsilon_{10} < \delta_{10}$, depending only on $\mathbf{C}, \alpha$, so that provided $|g|_{2,\alpha} < \epsilon_{10}$, then for every $w$ in the convex space

$$\mathcal{B} := \{w \in C^{2,\alpha}(\mathbf{C}_1) : \quad \Pi_J(w_{r=1}) = \Pi_J(g), \quad |w|_B \leq \delta_{10}\},$$

there is a unique solution $v =: \mathcal{U}(w) \in \mathcal{B}$ to the linear problem

$$\mathcal{L}_{\mathbf{C}}(v) = -\mathcal{E}(w) \text{ on } \mathbf{C}_1, \quad \Pi_J(v_{r=1}) = \Pi_J(g) \text{ on } \Sigma,$$
$$\sup_{0<r\leq 1} r^{-\beta} \left( \int_\Sigma v(r\omega)^2 d\omega \right)^{1/2} < \infty,$$

and *moreover*, that $\mathcal{U}$ is a contraction mapping on $\mathcal{B}$.

To prove Proposition 3.3 it therefore suffices to show that, with $g = u|_{r=1}$, then $u \in \mathcal{B}$ and $|g|_{2,\alpha} \leq \epsilon_{10}$. However, both of these follow from (66) by ensuring $\delta_2(\mathbf{C}, \alpha)$ is sufficiently small. □

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(Received: October 4, 2019)
(Revised: October 12, 2021)

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