

THE STEFAN PROBLEM AND FREE TARGETS OF OPTIMAL BROWNIAN MARTINGALE TRANSPORT

BY INWON C. KIM^{1,a} AND YOUNG-HEON KIM^{2,b}

¹*Department of Mathematics, University of California at Los Angeles, ^aikim@math.ucla.edu*

²*Department of Mathematics, University of British Columbia, ^byhkim@math.ubc.ca*

We formulate and solve a free target optimal Brownian stopping problem from a given distribution while the target distribution is free and is conditioned to satisfy a given density height constraint. The free target optimization problem exhibits monotonicity, from which a remarkable universality follows, in the sense that the optimal target is independent of its Lagrangian cost type. In particular, the solutions to this optimization problem generate solutions to both unstable and stable type of the Stefan problem, where the former stands for freezing of supercooled fluid (St_1) and the latter for ice melting (St_2). This unified approach to both types of the Stefan problem is new. In particular we obtain global-time existence and weak-strong uniqueness for the ill-posed freezing problem (St_1), for a given initial data and for a well-prepared class of initial domains generated from the initial data.

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1. Introduction. Given a configuration μ of agents subject to a random motion, what would happen if they try to stay under a given allowed population density height f , in particular, while trying to optimize their collective dynamics? In the present paper we consider the Brownian motion W_t in \mathbb{R}^d , and the collective dynamics is controlled only by a stopping time τ , that is, when each random particle stops. Such dynamics is called Brownian martingale. Precisely, we consider the following.

PROBLEM 1.1. *Consider a nonnegative function $f \in L^\infty(\mathbb{R}^d)$. Given an absolutely continuous, compactly supported measure μ on \mathbb{R}^d , find a stopping time τ^* and corresponding distribution $W_{\tau^*} \sim \nu^*$, that solve*

$$(1.1) \quad \mathcal{P}_f(\mu) := \inf_{(\tau, \nu)} \{ \mathcal{C}(\tau) \mid W_0 \sim \mu, W_\tau \sim \nu, \nu \leq f, \text{ & } \nu \text{ is compactly supported} \}.$$

Notice that here we do not assume that μ, ν are probability measures. So the expression “ $W_0 \sim \mu$ and $W_\tau \sim \nu$ ” should be understood as for each measurable set E ,

$$\nu[E] = \int \text{Prob}[W_\tau \in E \mid W_0 = x] d\mu(x).$$

The problem (1.1) obviously requires some condition for f because if f is too small there will be no admissible ν . We consider those cases where there is at least one admissible ν . Our model case is $f \equiv 1$ on \mathbb{R}^d , however a much more general f can be considered. We are particularly interested in the cost $\mathcal{C}(\tau)$ of the “Lagrangian form”:

$$(1.2) \quad \mathcal{C}(\tau) = \mathbb{E} \left[\int_0^\tau L(W_t, t) dt \right] \quad \text{for a continuous } L : \mathbb{R}^d \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}.$$

The problem (1.1) is in the spirit of what is studied in “constrained transport”, in particular, for analyzing congested crowd motion (see, e.g., [1, 23, 55]) and tumor growth (see, e.g., [40, 50]) which is subject to contact inhibition. In those cases, the optimal dynamics is deterministic, and it amounts to control a vector field (drift force) to guide the dynamics; additional noise to their dynamics would generate stochastic aspects. In our problem, which is a Brownian martingale version of the free target optimal transport problem with density upper bound constraint [23], the dynamics is completely random as it follows the Brownian motion and is controlled only by the choice of stopping rule.

Throughout this paper we make assumptions on the cost \mathcal{C} (1.2):

$$(1.3) \quad 0 \leq L \leq D \text{ for some constant } D \text{ and,}$$

$$(1.4) \quad t \mapsto L(\cdot, t) \text{ is either strictly increasing (Type (I)), or strictly decreasing (Type (II))}.$$

We obtain existence, uniqueness, and characterization of the optimal solution of Problem 1.1; see Theorem 7.4. Remarkably the problem exhibits universality (see Theorem 7.6), namely, the resulting optimal target ν^* does not depend on a particular choice of the function L and its type, as long as L satisfies (1.4). In fact, it turns out that ν^* coincides with the shadow of μ in f as introduced in [10, 52] (cf. [12]); see Remark 7.7. These are due to a monotonicity property of the problem regarding the optimal stopping τ^* and target ν^*

(Theorem 7.1), which also gives L^1 contraction (Theorem 7.8) and BV estimates for ν^* (Theorem 7.9). Such monotonicity is a novel feature, resulting from not fixing the target distribution ν but treating it as the solution of the problem. Another important feature of Problem 1.1 is that the optimal solution ν^* , also the flow of the distribution μ_t of W_t toward it, saturates the constraint f (Theorem 8.3) wherever active Brownian particles pass by before they stop.

The free target feature of Problem 1.1 distinguishes it from the more well studied, so-called optimal Skorokhod problem (see e.g., [8, 34–37] as well as the one-dimensional results considered by [20, 22, 31, 44]) where optimal stopping times are studied for given *fixed* initial and target measures μ, ν and *without* the density constraint f , that is,

$$(1.5) \quad \mathcal{P}(\mu, \nu) := \inf_{\tau} \{ \mathcal{C}(\tau) \mid W_0 \sim \mu, W_{\tau} \sim \nu \}.$$

Solutions to Problem (1.5) solve the Skorokhod problem [4, 6, 28, 48, 49, 51, 53, 57] for which a connection to optimal transport was hinted in the work of [39], and established in [8] who used optimal transport theory (see, e.g., the books [54, 60, 61]) and randomized stopping times to unify all the previously known solutions of the Skorokhod problem as the solutions of (1.5). We also mention that optimal Skorokhod problem is a special case of the optimal martingale problem (see e.g., [5, 9, 10, 27, 33, 34, 38, 58] and references therein) which has been recently popularized in mathematical finance.

As a main application we combine the above mentioned results for Problem 1.1 with the PDE methods developed in [35] for (1.5) to make progress in the supercooled Stefan problem, a free boundary problem of the heat equation, which has been poorly understood in the literature.

The supercooled Stefan problem describes freezing of supercooled water into ice, and can be written in divergence form in $\mathbb{R}^d \times \mathbb{R}_{\geq 0}$ as

$$(St_1) \quad (\eta - \chi_{\{\eta > 0\}})_t - \Delta \eta = 0.$$

Here $-\eta$ denotes the temperature of the supercooled water (water with below-freezing temperature), and the set $\{\eta > 0\}$ describes the region for the supercooled water. The solution of (St_1) is famously known to exhibit irregular, fractal-like interfaces that could jump discontinuously over time. This is in sharp contrast to the Stefan problem for melting ice into water,

$$(St_2) \quad (\eta + \chi_{\{\eta > 0\}})_t - \Delta \eta = 0,$$

which can be viewed as a singular nonlinear parabolic equation of the form $[\beta(\eta)]_t - \Delta \eta = 0$ with an increasing function β ; see [2] where one proves comparison and contraction properties of weak solutions based on this property.

While (St_2) is known to generate stable and regularizing solutions ([2, 3, 18]), solutions of (St_1) are shown to develop discontinuity and nonuniqueness in finite time, even in one space dimension and with smooth initial data ([16, 56]). For higher dimensions, the global existence of weak solutions for (St_1) has stayed largely open.

It is natural to consider these problems as generated by a particle system, which diffuses but stops its motion when it hits the interface between water and ice. In one space dimension there have been several works which construct solutions of (St_1) as a continuum limit of such particle systems, including [15, 17, 24, 25, 42]. In higher dimensions the global-time existence of solutions for (St_1) has remained open, due to the difficulty of obtaining a stable deterministic limit [47]. Such global-time existence, as well as a “weak-strong uniqueness” theorem is what we establish in this paper. Similar to aforementioned works, we also consider solutions based on Brownian particles with hitting times, but with optimization. Indeed a novelty of our approach lies in the optimization structure given by Problem (1.1), for the choice of stopping time τ^* and the final target distribution ν^* . The monotonicity types (1.4)

of the cost, (I), (II), correspond to (St_1) , (St_2) , respectively. The optimality gives a certain “stability” to (St_1) when its initial domain is given by the solution of Problem (1.1). Roughly speaking, we will show the following:

For any compactly supported $\eta_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, there is a class of domains E that include the support of η_0 such that, for each choice of E , there exists a unique global-time solution η of (St_1) with initial data η_0 and initial domain $E := \limsup_{t \rightarrow 0^+} \{\eta(\cdot, t) > 0\}$.

The initial domains E are obtained by our optimization problem (1.1), giving an initial “regularization” of the support of η_0 , by means of an obstacle problem; see Section 9. In particular, our initial domains E are strictly larger than the support of η_0 . The point is that they are to be chosen appropriately, not arbitrarily given. This limitation of our approach is natural though, given the stability of our solutions and the ill-posed nature of (St_1) . For instance, we conjecture that the class of solutions we find in this paper do not include those with jump discontinuities observed in the aforementioned literature. In this sense our class of solutions can be understood as the “strong” solutions of (St_1) which allow uniqueness results, featuring weak-strong uniqueness. Indeed we will show that, when we do not choose the initial domain E carefully, there can be infinitely many weak solutions of (St_1) , even in the radially symmetric case: see Proposition 6.3. Regularity of our solutions, and potential extension of our approach to general, discontinuous solutions, remain to be exciting open questions.

If we further require certain finite-time vanishing conditions on the solution, then the corresponding initial domain E is uniquely determined, so is the solution. Below are particular examples of our solutions which are generated by specific choices of initial domain E :

(a) (Theorem 9.3 and Corollary 9.9) For compactly supported η_0 uniformly larger than 1 in its support, there exists a unique weak solution of (St_1) with initial data η_0 that vanishes in finite time.

(b) (Theorem 9.4 and Corollary 9.10) For any compactly supported η_0 , there exists a unique weak solution of (St_1) with initial data η_0 that vanishes in finite time outside the support of η_0 and never vanishes in the support of η_0 .

Similar results hold for (St_2) (Theorem 9.2). These results rely on a rigorous connection that we establish between the weak solution and the optimal Brownian stopping (see Theorems 5.4 and 5.6).

While the cost types (I) and (II) in (1.4) generate different optimal stopping times, and solutions to (St_1) and (St_2) , respectively, we prove that their optimal target measure v^* is independent of the cost, or even of its monotonicity type (Theorem 7.6). This remarkable *universality* allows us to connect the two quite different problems (St_1) and (St_2) together; in particular, the solutions to the problem (St_2) , give us the initial trace of the solutions of (St_1) given in (a) and (b) above: we refer to Section 9 for further discussions.

Understanding further regularity of the free boundary for the corresponding supercooled Stefan problem (St_1) , is a wide open problem. In this direction, we prove a strict monotonicity result for the barrier function s , a version of comparison principle, which may shed a light on the challenging question; see Theorems 10.1 and 10.2. This is a novel feature of Problem 1.1 as it is a consequence of the optimization of the free target distribution.

As a byproduct of our approach we define the notion of *subharmonically generated sets* in Section 6, which connects the Brownian stopping time and existence of the solution to the supercooled Stefan problem (St_1) . It is a pair (Σ, E) of sets determined by a certain type of Brownian stopping time from an initial measure μ ; Definition 6.1. The solvability of (St_1) with the initial data (μ, E) is determined by whether there is Σ such that (Σ, E) is subharmonically generated from μ ; Theorem 6.2. This notion is used to describe unique solvability of (St_1) and also to characterize vanishing in finite time solution of it in Section 9; Theorem 9.7 and Corollaries 9.9 and 9.10.

1.1. *The connection between the Stefan problem, Problem (1.5), and Problem 1.1.* The present paper relies on the previous works on the optimal Skorokhod problem (1.5). Notice that after getting the optimal target v^* in Problem 1.1, finding the optimal stopping time τ^* is reduced to solving (1.5) with fixed μ and v^* . It is shown first [8] then [35] that the solution to (1.5) is the first hitting time of (W_t, t) to a space-time barrier R^* , which enjoys time-monotonicity. Such a barrier set can be defined by a barrier function s^* , such that $R^* := \{(x, t) | s^*(t) \leq t\}$ for type (I) and $R^* := \{(x, t) | s^*(t) \geq t\}$ for type (II). Our results above imply that the free boundary of the Stefan problem, at time t , corresponds to the boundary of the time t -slice of R^* , that is, $\{x | (x, t) \in R^*\}$. In fact, to derive our well-posedness of the Stefan problem (St_1) , (St_2) , it is crucial for us to verify that R^* is a closed set, so its complement $(R^*)^c$ is open. This is handled by Theorem 4.15, which connects the barrier set of τ^* to a closed set generated by the potential function U_{μ_t} of the mass flow of W_t under τ^* . Interestingly its proof essentially uses the Eulerian method in [35]. We remark that in the type (I) case, Theorem 4.15 can be derived even in a sharper form, based on [32]; they showed that the hitting time to a type (I) barrier set coincides with the stopping time generated by the corresponding potential flow, for a general class of stochastic processes [32].

The PDE methods of [35] is based on the correspondence between stopping time τ and its Eulerian flow (η, ρ) given on the space-time, that describes the distribution η of (W_t, t) before stopping by τ , and the stopped distribution of mass ρ of (W_τ, τ) . This (η, ρ) solves weakly

$$\eta_t - \frac{1}{2} \Delta \eta = -\rho.$$

Moreover, under the assumptions (1.3)–(1.4) the (η, ρ) resulting from a solution of $\mathcal{P}(\mu, \nu)$ (1.5) yields that ρ is formally supported on the boundary of the active region $\{\eta > 0\}$ [35]. Assuming sufficient regularity, ρ is concentrated on the boundary of the barrier set, which is also the boundary of the active region $\{\eta > 0\}$. Thus $\rho = v(\chi_{\{\eta>0\}})_t$ or $\rho = -v(\chi_{\{\eta>0\}})_t$ depending on the monotonicity of the set $\{\eta > 0\}$ over time, and we end up with the weighted Stefan problem,

$$(1.6) \quad (\eta \pm v \chi_{\{\eta>0\}})_t - \frac{1}{2} \Delta \eta = 0,$$

where v is the target measure for the optimal stopping time. This equation has been formally considered for a given fixed target measure v in [35], and our Theorem 5.4 and 5.6 provide a rigorous justification.

For the connection between Problem 1.1 and the original Stefan problem (St_1) , (St_2) , it is crucial to have the saturation result (Theorem 8.3) that gives $v = 1$ (when $f \equiv 1$) in the active region; it reduces the above equation (1.6) to (St_1) , (St_2) for type (I), (II), respectively.

1.2. *Remarks on future work.* It should be mentioned that most of the methods in this paper carry over to diffusions with uniformly elliptic generators, thus more general type Stefan problems; Brownian motion corresponds to the Laplacian, and more generally uniformly elliptic operators generate continuous Markov processes. For instance our results can be extended to Riemannian manifolds and Laplace–Beltrami operators. More interestingly, our method is robust enough that in an ongoing work in progress [19] we treat the nonlocal Stefan problem, extending the Laplacian to the fractional Laplacian, where even the more stable problem (St_2) is not well understood.

The main results in this paper appear rather surprising, given the lack of understanding in the literature on multi-dimensional solutions of (St_1) . In a subsequent work, we will further develop our method to extend well-posedness theory for (St_1) to more general initial boundaries. Also, the connection (Remark 9.6) between (St_1) and (St_2) that we find in this paper

through the universality result (Theorem 7.6) seems to be unexpected, and understanding this connection at a more physical level, is an interesting open problem. It would also be interesting to understand the physics behind why type (I) corresponds to the supercooled unstable Stefan and type (II) to the usual stable Stefan problem.

In this paper our method for finding the optimal target ν is solely based on optimizing with probability measures. Here we do not consider duality which is a powerful tool for understanding optimal Brownian stopping, though we heavily rely on the results from [35] that are obtained via solving the dual problem of the fixed target case, to find the barrier set of the optimal stopping time. Our free target problem gives a new aspect to the duality that we will discuss in a subsequent work.

2. Preliminaries. Throughout the paper, measurability means Borel measurability. In this section we assume for simplicity, that the measures μ and ν are compactly supported in \mathbb{R}^d . Much of the discussion (and notation) in this section is borrowed from [35] which gives foundational results for the development given in this paper. Note that the purpose of this section is to collect the precise definitions and results necessary for the discussion in the subsequent sections. We try to include the least amount of information.

2.1. Notation.

- W_t denotes the Brownian motion, while W_t^y denotes the Brownian motion with $W_0 = y$.
- $B_r(x)$ denotes the ball of radius r centered at x .
- $a \wedge b := \min[a, b]$.
- For each Borel measurable set $S \subset \mathbb{R}^d$, $|S|$ denotes its d -dimensional Lebesgue measure.
- For each (nonnegative) Borel measure μ on \mathbb{R}^d , $|\mu|$ denotes $\mu(\mathbb{R}^d)$.
- LSC = the set of lower semi-continuous functions on \mathbb{R}^d .
- $H_0^1 = H_0^1(\mathbb{R}^d)$.
- $\Omega = C(\mathbb{R}_{\geq 0}; \mathbb{R}^d)$ the space of continuous curves.
- \mathbb{P}^μ = the Wiener measure on Ω , with initial distribution μ ; this means for our Brownian motion W_t , we have $W_0 \sim \mu$.
- $\{\mathcal{F}_t\}_{t \in \mathbb{R}_{\geq 0}}$ = the natural filtration of the Brownian motion.
- $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}_{\geq 0}}, \mathbb{P}^\mu)$ = the filtered probability space over Ω , with the filtered σ -algebra \mathcal{F} and the probability measure \mathbb{P}^μ .
- $\mathcal{M}(\mathbb{R}_{\geq 0})$ = the space of Radon measures on $\mathbb{R}_{\geq 0}$.

2.2. Randomized stopping times and optimal Skorokhod problem in \mathbb{R}^d . Related to Problem 1.1, it is important to understand the notion of stopping times and randomized stopping times.

2.2.1. Stopping times and randomized stopping times. Consider the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}_{\geq 0}}, \mathbb{P}^\mu)$. A *randomized stopping time* α [7, 46] is a measure-valued random variable $\alpha : \Omega \ni \gamma \mapsto \alpha_\gamma \in \mathcal{M}(\mathbb{R}_{\geq 0})$ such that as a measure on $\mathbb{R}_{\geq 0}$,

$$\alpha_\gamma \geq 0, \quad \alpha_\gamma(\mathbb{R}_{\geq 0}) = 1, \quad \text{and} \quad \text{the map } \gamma \mapsto \alpha_\gamma([0, t]) \text{ is } \mathcal{F}_t\text{-measurable } \forall t.$$

One can view α as a certain type of probability measure over $\Omega \times \mathbb{R}_{\geq 0}$ whose marginal on Ω is \mathbb{P}^μ . Along each Brownian path, mass is dropped according to the distribution on $\mathbb{R}_{\geq 0}$ determined by the randomized stopping time. A *stopping time* $\tau : \Omega \rightarrow \mathbb{R}_{\geq 0}$ is a random variable such that for each t , $\tau \wedge t$ is \mathcal{F}_t -measurable; it can be understood as a special case of randomized stopping time, where the measure α_γ on each path γ is a Dirac mass at the value $\tau(\gamma)$ in $\mathbb{R}_{\geq 0}$. To distinguish the difference between randomized stopping time and

stopping time, we often call the latter “nonrandomized”. However, we often abuse notation and use τ for either stopping times or randomized stopping times, whenever its meaning is clear from the context; in the latter case we will also use the notation α for the actual measure corresponding to the randomized stopping time τ . In fact, one may identify a randomized stopping time given as the measure α with a nonrandomized stopping time, by extending the probability space Ω to $\Omega \times [0, 1]$, augmented by the unit interval $[0, 1]$ and its Lebesgue measure $Leb_{[0,1]}$. Here α induces a nonrandomized stopping time ρ , given as

$$(2.1) \quad \rho(\gamma, u) = \inf\{t \geq 0 | \alpha_\gamma([0, t] \geq u)\} \quad \text{for each } (\gamma, u) \in \Omega \times [0, 1];$$

see [8], Section 3.2. This point of view allows us to treat randomized stopping times as usual (nonrandomized) stopping times, and it enables us to treat expressions about probability of randomized stopping times as those about nonrandomized stopping times.

We will say that a subset $Q \subset \Omega \times \mathbb{R}_{\geq 0}$ occurs almost surely for a randomized stopping time α if $\mathbb{E}[\int_{\mathbb{R}_{\geq 0}} \mathbf{1}\{(\gamma, t) \notin Q\} d\alpha_\gamma(t)] = 0$; here the expectation \mathbb{E} is with respect to the probability measure \mathbb{P}^μ , that is, $\mathbb{E} = \mathbb{E}^{\mathbb{P}^\mu}$, and $\mathbf{1}$ is the indicator function of the set. From the correspondence $\alpha \mapsto \rho$ as above, this is equivalent to $\mathbb{E}[\mathbf{1}\{(\gamma, \rho(\gamma, u)) \notin Q\}] = 0$ where the expected value is with respect to $\mathbb{P}^\mu \otimes Leb_{[0,1]}$.

For a (randomized) stopping time τ and a fixed time t , we will often use the expression

$$t \leq \tau \quad \text{almost surely (a.s.)},$$

which means that for the distribution α of τ ,

$$\mathbb{E}\left[\int_{\mathbb{R}_{\geq 0}} \mathbf{1}\{(\gamma, s) | t \not\leq s\} d\alpha_\gamma(s)\right] = 0,$$

equivalently,

$$\text{Prob}[t > \rho(\gamma, u)] = 0.$$

We also say ν is the distribution of the stopped Brownian motion at the (randomized) stopping time τ , that is, $W_\tau \sim \nu$, if

$$\mathbb{E}[g(W_\tau)] = \int g(z) d\nu(z) \quad \forall \text{ continuous } g : \mathbb{R}^d \rightarrow \mathbb{R}.$$

Here, when τ is a randomized stopping time with the distribution α , the expected value $\mathbb{E}[g(W_\tau)]$ is defined as

$$\mathbb{E}[g(W_\tau)] := \mathbb{E}\left[\int_{\mathbb{R}_{\geq 0}} g(\gamma(s)) d\alpha_\gamma(s)\right],$$

equivalently,

$$\mathbb{E}[g(W_\tau)] := \mathbb{E}[g(\rho(\gamma, u))] = \int_{\Omega \times [0, 1]} g(\rho(\gamma, u)) d\mathbb{P}^\mu \otimes Leb_{[0,1]}.$$

2.2.2. Subharmonic order and optimal Skorokhod problem. Subharmonic order relates two measures from the perspective of the Brownian motion. First, f is a subharmonic function on an open set O in \mathbb{R}^d if it is upper-semicontinuous with values on $\mathbb{R} \cup \{-\infty\}$, and for every $x \in O$ and every closed ball B in O with center at x , it satisfies

$$f(x) \leq \frac{1}{|B|} \int_B f(y) dy.$$

When f is C^2 , the latter condition is equivalent to $\Delta f \geq 0$ on O , where $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplacian. The reason why we specify the domain O is because subharmonic functions in O may not necessarily be extended to the whole set \mathbb{R}^d .

Two measures μ, ν on \mathbb{R}^d with $|\mu| = |\nu|$ are said to be in *subharmonic order* $\mu \leq_{\text{SH}} \nu$, if for each open set O containing $\text{supp}(\mu + \nu)$,

$$\int \varphi(x) d\mu(x) \leq \int \varphi(x) d\nu(x) \quad \text{for every smooth subharmonic function on } O.$$

See, for example, [34], Definition 1.2. It is known (see, e.g., [34], Theorem 1.5) that for compactly supported μ and ν as we assume throughout the paper, we have $\mu \leq_{\text{SH}} \nu$ if and only if there exists a randomized stopping time τ for the Brownian motion such that $W_0 \sim \mu$ and $W_\tau \sim \nu$ with $\mathbb{E}[\tau] < \infty$. For such μ and ν , for any lower semicontinuous cost functional \mathcal{C} for randomized stopping times, one can find an optimal randomized time between μ and ν using the compactness of the space of randomized stopping times; this has been established by Beigleböck, Cox, and Huesmann [8]. In fact, it is a nontrivial problem to see that such optimal randomized stopping time is indeed a stopping time (nonrandomized) to solve the problem $\mathcal{P}(\mu, \nu)$ (1.5). This has been proved under the assumptions on the cost (1.3) and (1.4) [8, 35]; especially see [35], Theorem 4.7. (For other types of cost, for e.g., the one based on the distance, $\mathcal{C}(\tau) = \mathbb{E}[|W_0 - W_\tau|]$, similar results have been proved in [36]; see also [37].)

THEOREM 2.1 (Existence/uniqueness of optimal Skorokhod Problem [8, 35], Theorem 4.7). *Given compactly supported measures μ, ν on \mathbb{R}^d with $\mu \leq_{\text{SH}} \nu$ and $\mu, \nu \ll \text{Leb}$, the optimal Skorokhod problem $\mathcal{P}(\mu, \nu)$ (1.5) with the cost \mathcal{C} in (1.2) under the assumption (1.3) and (1.4), has a unique optimal stopping time τ^* , which is not randomized in the type (I) case, and in the type (II) case, randomized only at $t = 0$; in the latter case $W_0 = x$ and $\tau^* = 0$ occurs with probability $\nu(x)/\mu(x)$. Moreover, there exists a space-time barrier set $R^* \in \mathbb{R}^d \times \mathbb{R}_{\geq 0}$ such that τ^* is given by the hitting time to R^* , that is,*

$$\tau^* = \inf\{t > 0 | (W_t, t) \in R^*\} \quad (\text{in type (II) case, it holds for those paths with } \tau^* > 0).$$

For our purpose it is important to understand the barrier set R^* . It can be characterized by using the dual solutions of $\mathcal{P}(\mu, \nu)$. In [35] they used the dynamic programming principle to establish the duality: $\mathcal{P}(\mu, \nu) = \mathcal{D}(\mu, \nu)$ where

$$\mathcal{D}(\mu, \nu) := \sup_{\psi \in \text{LSC}} \left\{ \int_{\mathbb{R}^d} \psi(z) d\nu(z) - \int_{\mathbb{R}^d} J_\psi(x, 0) d\mu(x) \right\}.$$

Here, the function $(x, t) \mapsto J_\psi(x, t)$ is called the “value function” which is the result of dynamic programming:

$$J_\psi(y, t) := \sup_{\sigma} \left\{ \mathbb{E} \left[\psi(W_\sigma^y) - \int_0^\sigma L(t+s, W_s^y) ds \right] \right\},$$

where σ are randomized stopping times. The strict monotonicity assumption on L (1.4) for type (I) and (II) implies (forward or backward) time-monotonicity of the function J_ψ . Also, it is shown [35] that under the assumptions in Theorem 2.1 (in fact for more general μ and ν), there exists an optimal dual function $\psi^* \in \text{LSC} \cap H_0^1$. Then the barrier set, the barrier function s^* , and the optimal stopping time τ^* of (1.5) are given by

$$\begin{aligned} R^* &= \{(x, t) | t \geq s^*(x)\} && \text{for type (I),} \\ R^* &= \{(x, t) | t \leq s^*(x)\} && \text{for type (II);} \\ s^*(x) &= \inf\{t \in \mathbb{R}_{\geq 0} | J_{\psi^*}(x, t) = \psi^*(x)\} && \& \\ \tau^* &= \inf\{t > 0 | t \geq s^*(W_t)\} && \text{for type (I),} \\ s^*(x) &= \sup\{t \in \mathbb{R}_{\geq 0} | J_{\psi^*}(x, t) = \psi^*(x)\} && \& \\ \tau^* &= \inf\{t > 0 | t \leq s^*(W_t)\} && \text{for type (II).} \end{aligned} \tag{2.2}$$

2.3. Eulerian formulation. It is important for us to relate the Brownian motion with stopping time τ , with its corresponding mass flow, which we call Eulerian flow; considering such Eulerian formulation and effectively using it to analyze optimal stopping times is one of the main innovations of [35]. An Eulerian flow is a pair of measures (η, ρ) on $\mathbb{R}^d \times \mathbb{R}_{\geq 0}$ such that in the weak sense,

$$(2.3) \quad \begin{aligned} \rho(x, t) + \partial_t \eta(x, t) &= \frac{1}{2} \Delta \eta(x, t), \\ \int_{\mathbb{R}_{\geq 0}} d\rho &= \nu, \quad \eta(x, 0) = \mu(x). \end{aligned}$$

For a more precise description, let $O \subset \mathbb{R}^d$ be a bounded open convex set that contains the supports of μ and ν . Pick $\gamma < \lambda$ the Poincaré constant of O . We consider

$$C_{-\gamma}(\overline{O} \times \mathbb{R}_{\geq 0}) := \{w \in C(\overline{O} \times \mathbb{R}_{\geq 0}) \mid e^{-\gamma t} w(x, t) \rightarrow 0 \text{ as } t \rightarrow \infty, \text{ uniformly in } x\}.$$

Its dual space $\mathcal{M}_\gamma(\overline{O} \times \mathbb{R}_{\geq 0})$ is the finite Radon measures with γ -exponential decay in time. We let $C_{-\gamma}^{1,2}(\overline{O} \times \mathbb{R}_{\geq 0})$ denote the functions whose first derivative in time and second derivatives in space lie in $C_{-\gamma}(\overline{O} \times \mathbb{R}_{\geq 0})$. Then, following [35] we shall say that (η, ρ) is an *admissible pair*, provided $\eta, \rho \in \mathcal{M}_\gamma(\overline{O} \times \mathbb{R}_{\geq 0})$ and $\eta, \rho \geq 0$, and they satisfy the following two equations:

$$(2.4) \quad \int_{\overline{O}} u(z) d\nu(z) = \int_{\overline{O}} \int_{\mathbb{R}_{\geq 0}} u(x) d\rho(x, t) \quad \text{for all } u \in C(\overline{O}).$$

$$(2.5) \quad \begin{aligned} - \int_{\overline{O}} w(y, 0) d\mu(y) &= \int_{\overline{O}} \int_{\mathbb{R}_{\geq 0}} \left[\frac{\partial}{\partial t} w(x, t) + \frac{1}{2} \Delta w(x, t) \right] d\eta(x, t) \\ &\quad - \int_{\overline{O}} \int_{\mathbb{R}_{\geq 0}} w(x, t) d\rho(x, t) \quad \text{for all } w \in C_{-\gamma}^{1,2}(\overline{O} \times \mathbb{R}_{\geq 0}). \end{aligned}$$

Let us translate the Brownian stopping into Eulerian flow:

PROPOSITION 2.2 (see [35] Proposition 2.2). *Given μ and ν compactly supported with $\mu, \nu \ll \text{Leb}$. Suppose τ is a (randomized) stopping time with $W_0 \sim \mu$, $W_\tau \sim \nu$. Then, there is an admissible pair (η, ρ) of measures on $\mathbb{R}^d \times \mathbb{R}_{\geq 0}$, such that for every $g \in C_c(\mathbb{R}^d \times \mathbb{R}_{\geq 0})$*

$$\mathbb{E}[g(W_\tau, \tau)] = \int_{\overline{O}} \int_{\mathbb{R}_{\geq 0}} g(x, t) d\rho(t, x),$$

and

$$\mathbb{E}\left[\int_0^\tau g(W_t, t) dt\right] = \int_{\overline{O}} \int_{\mathbb{R}_{\geq 0}} g(x, t) d\eta(x, t).$$

This gives equivalence between the optimization problem for the stopping time and the one for the Eulerian flow [35]:

$$\mathcal{P}(\mu, \nu) = \mathcal{P}_1(\mu, \nu) := \inf_{(\eta, \rho)} \int_{\mathbb{R}^d} \int_{\mathbb{R}_{\geq 0}} L(x, t) d\eta(x, t)$$

where (η, ρ) 's are admissible pairs.

We close this section with an essential lemma for us to consider the Eulerian flow in the context of the barrier of the stopping time. It will be later used in Section 4.

LEMMA 2.3 ([35] Lemma 4.5). *Suppose $\mu \ll \text{Leb}$ is a compactly supported measure and $R \subset \mathbb{R}^d \times \mathbb{R}_{\geq 0}$. Let (η, ρ) be admissible with the condition $\eta(R) = 0$ and $\rho(R) = 1$.*

- We suppose R is a measurable forward-barrier, namely, $(x, r) \in R$ whenever $(x, t) \in R$ with $t \leq r$, and $(x, t) \in R$ if there is $(x, t_i) \in R$ with $t_i \rightarrow t$. Then (η, ρ) is unique.
- Suppose instead that R is a measurable backward-barrier, namely, $(x, r) \in R$ whenever $(x, t) \in R$ with $t \geq r$, and $(x, t) \in R$ if there is $(x, t_i) \in R$ with $t_i \rightarrow t$. Then (η, ρ) is uniquely determined by the restriction of ρ on the set $\{(x, 0) | s(x) = 0\}$ where s is given by $s(x) = \sup\{t | (x, t) \in R\}$.

REMARK 2.4. Let us comment on why the additional condition $\eta(R) = 0$, $\rho(R) = 1$ to the corresponding PDE (2.5) is sufficient to determine the unique solution. Because $\rho(R) = 1$ for the probability measure ρ , we have $\rho(R^c) = 0$. Therefore, the (2.5) behaves like the heat equation for η in R^c , with the Dirichlet condition which is due to $\eta(R) = 0$; therefore the uniqueness is not surprising, especially if R is closed (so R^c is open). In fact we will show that we can take R as a closed set when (η, ρ) comes from the optimal solution in Theorem 2.1; see Section 4, Theorem 4.15.

3. Properties of stopping times. In this section we prove several basic but important results for the properties of hitting times as well as optimal stopping times. We handle the general case where the barriers are merely measurable sets, say, not open or closed. This may be of its own interest. For the present paper, because of Section 4 (see Thereom 4.15), it suffices to consider only the case where the barrier set is closed, in which case some proofs of this section can possibly be simplified. Results in this section may possibly be known to experts, however, we were unable to find a suitable reference. In this section, we focus on nonrandomized stopping times, and for a measurable set $G \subset \mathbb{R}^d$, we let τ_G denote the first hitting time to G , that is,

$$\tau_G := \inf\{t > 0 | W_t \in G\}.$$

3.1. Properties of stopping times with respect to a barrier set. We first recall the following result from potential theory.

LEMMA 3.1. *Let σ denote the surface measure of the round sphere ∂B_r of radius r centered at 0. For $z \in \mathbb{R}^d$ consider the Brownian motion W_t^z starting from z , and the first hitting time $\tau_{\partial B_r}$ to the round sphere ∂B_r . Let μ_z be the distribution of $W_{\tau_{\partial B_r}}^z$ along ∂B_r , that is, $W_0 \sim \delta_z$, $W_{\tau_{\partial B_r}} \sim \mu_z$. Suppose $|z| \geq 2r$ or $|z| \leq \frac{1}{2}r$. Then, there exists a universal constant $C > 0$ depending only on the dimension, such that the Radom–Nikodym derivative satisfies*

$$\frac{d\mu_z}{d\sigma} \leq \frac{C}{\sigma[\partial B_r]}.$$

In particular, we have

$$\text{Prob}[W_{\tau_{\partial B_r}}^z \in E] \leq C \frac{\sigma[E]}{\sigma[\partial B_r]} \quad \text{for any } E \subset \partial B_r.$$

PROOF. Notice that μ_z is nothing but the harmonic measure along ∂B_r , generated by the point z . Therefore, the desired upper bound is a standard result of potential theory. This is a consequence of Dahlberg's theorem [21]. \square

Using this lemma we now prove a key lemma, which takes care of a subtle possibility that Brownian paths blocked by a barrier set may possibly drop mass in the barrier set; the lemma says it does not happen when the final distribution is absolutely continuous.

Below we will consider the following assumptions on μ , ν and τ :

$$(3.1) \quad \mu, \nu \text{ are probability measures such that } \mu \ll \text{Leb} \text{ and } \nu \in L^\infty;$$

$$(3.2) \quad \tau \text{ is a (nonrandomized) stopping time with } W_0 \sim \mu \text{ and } W_\tau \sim \nu.$$

LEMMA 3.2. *Let μ, ν, τ satisfy (3.1) (3.2). Let G be a measurable set, and suppose $\tau \leq \tau_G$ and $\mu \wedge (\nu|_G) = 0$. Then $\nu[G] = 0$.*

REMARK 3.3. Notice that the condition $\nu \in L^\infty$, especially $\nu \ll Leb$, is essential in Lemma 3.2. For example, the distribution $\nu \sim W_\tau$ of the Brownian motion of the hitting time τ to a sphere G , does not satisfy the result of this lemma.

PROOF. Suppose $\nu[G] > 0$ for contradiction. Then, since $\nu \ll Leb$ by (3.1), there exists $\delta > 0$ such that the set

$$G_\delta := \left\{ z \in G \mid \frac{d\nu}{dx} \geq \delta \right\} \quad \text{has } |G_\delta| > 0.$$

Use the Lebesgue differentiation theorem and find a point $x \in G_\delta$ of G_δ , which is a Lebesgue point for both χ_{G_δ} and the function $\frac{d\mu}{dx}$ (because almost every point x are Lebesgue points of both functions). Then, from Lebesgue density property and the condition $\mu \wedge (\nu|_G) = 0$, we see that for each small $\epsilon > 0$, there exists $\bar{r} > 0$ (depending on ϵ) such that

$$\begin{aligned} \nu[B_r(x)] &\geq \delta/2 |B_r(x)|, & |G_\delta \cap B_r(x)| &\geq (1 - \epsilon) |B_r(x)|, \\ \text{and } \mu[B_r(x)] &\leq \epsilon |B_r(x)| & \text{for all } 0 < r \leq \bar{r}. \end{aligned}$$

From Fubini's theorem, there are $0 < r_1 < r_2 < \bar{r}$ with $8r_1 < r_2 < 10r_1$ such that

$$\mathcal{H}^{d-1}[G_\delta \cap \partial B_{r_i}(x)] \geq (1 - 5\epsilon) \mathcal{H}^{d-1}[\partial B_{r_i}(x)], \quad i = 1, 2,$$

for the $d - 1$ -dimensional Hausdorff measure \mathcal{H}^{d-1} .

From Lemma 3.1, for a dimensional constant C we have

$$\text{Prob}[W_\tau \notin \overline{B_{r_2}(x)} \mid W_t \in B_{4r_1}(x) \text{ for some } t \leq \tau] \leq C\epsilon,$$

where the $\overline{B_{r_2}(x)}$ means the closure. This is because for the Brownian path from a point inside $B_{4r_1}(x)$ to go outside $B_{r_2}(x)$, it has to go through $\partial B_{r_2}(x)$ but avoiding hitting G , thus G_δ ; here the condition $\tau \leq \tau_G$ is crucial. To give details, since the ball $B_{4r_1}(x)$ is an open set and Brownian paths are continuous almost surely, the above probability is the same as

$$\text{Prob}[W_\tau \notin \overline{B_{r_2}(x)} \mid W_t \in B_{4r_1}(x) \text{ for some } t \in \mathbb{Q} \text{ with } t < \tau].$$

Now, give an ordering to \mathbb{Q} as $\mathbb{Q} = \{q_1, q_2, \dots\}$. Then, the event

$$\text{“}W_t \in B_{4r_1}(x) \text{ for some } t \in \mathbb{Q} \text{ with } t < \tau\text{”}$$

can be decomposed into the disjoint E_j , $j = 1, 2, \dots$ defined as follows. Let $Q(t)$ be the statement “ $W_t \in B_{4r_1}(x)$ and $t < \tau$ ”. Then define E_j 's inductively,

$$\begin{aligned} E_1 &:= \text{“}Q(q_1) \text{ holds.}\text{”,} & E_2 &:= \text{“}Q(q_2) \text{ holds but not } Q(q_1).\text{”,} & \dots \\ \dots & & E_k &:= \text{“}Q(q_k) \text{ holds but none of } Q(q_1), \dots, Q(q_{k-1}).\text{”,} & \dots \end{aligned}$$

Then, $\text{Prob}[W_\tau \notin \overline{B_{r_2}(x)} \mid E_k] \leq C\epsilon$ for each k . Therefore,

$$\begin{aligned} \text{Prob}[W_\tau \notin \overline{B_{r_2}(x)} \mid W_t \in B_{4r_1}(x) \text{ for some } t \in \mathbb{Q} \text{ with } t < \tau] \\ = \frac{\sum_{k=1}^{\infty} \text{Prob}[W_\tau \notin \overline{B_{r_2}(x)} \mid E_k] \text{Prob}[E_k]}{\sum_{k=1}^{\infty} \text{Prob}[E_k]} \\ \leq C\epsilon \quad \text{as desired.} \end{aligned}$$

From this estimate we get

$$\text{Prob}[W_\tau \in \overline{B_{r_2}(x)} \mid W_t \in B_{4r_1}(x) \text{ for some } t \leq \tau] \geq 1 - C\epsilon.$$

By a similar argument of using Lemma 3.1, we get

$$\text{Prob}[W_\tau \in B_{r_1}(x) \mid W_t \in B_{4r_1}(x) \setminus \overline{B_{2r_1}(x)} \text{ for some } t \leq \tau] \leq C\epsilon.$$

Now let us consider

$$\begin{aligned} v[B_{r_1}(x)] &= \text{Prob}[W_\tau \in B_{r_1}(x)] \\ &\leq \text{Prob}[W_0 \in \overline{B_{2r_1}(x)}] + \text{Prob}[W_\tau \in B_{r_1}(x) \& W_0 \notin \overline{B_{2r_1}(x)}] \\ &= I + II. \end{aligned}$$

Notice that from our choice of x and $\mu \ll Leb$,

$$I = \mu[\overline{B_{2r_1}(x)}] = \mu[B_{2r_1}(x)] \leq \epsilon |B_{2r_1}(x)|.$$

On the other hand, from continuity of Brownian paths, for the Brownian path from a point outside $\overline{B_{2r_1}(x)}$ to arrive in $B_{r_1}(x)$, it first needs to arrive in $B_{4r_1}(x) \setminus \overline{B_{2r_1}(x)}$. Therefore,

$$\begin{aligned} II &\leq \text{Prob}[W_\tau \in B_{r_1}(x) \& W_t \in B_{4r_1}(x) \setminus \overline{B_{2r_1}(x)} \text{ for some } t \leq \tau] \\ &= \text{Prob}[W_\tau \in B_{r_1}(x) \mid W_t \in B_{4r_1}(x) \setminus \overline{B_{2r_1}(x)} \text{ for some } t \leq \tau] \\ &\quad \times \text{Prob}[W_t \in B_{4r_1}(x) \setminus \overline{B_{2r_1}(x)} \text{ for some } t \leq \tau]. \end{aligned}$$

Also, notice that

$$\begin{aligned} v[\overline{B_{r_2}(x)}] &= \text{Prob}[W_\tau \in \overline{B_{r_2}(x)}] \geq \text{Prob}[W_\tau \in \overline{B_{r_2}(x)} \mid W_t \in B_{4r_1}(x) \text{ for some } t \leq \tau] \\ &\quad \times \text{Prob}[W_t \in B_{4r_1}(x) \text{ for some } t \leq \tau]. \end{aligned}$$

Combining all these with the previous probability estimates with ϵ , we see that

$$II \leq \frac{C\epsilon}{1 - C\epsilon} v[\overline{B_{r_2}(x)}]$$

which implies

$$v[B_{r_1}(x)] \leq I + II \leq \epsilon |B_{2r_1}(x)| + \frac{C\epsilon}{1 - C\epsilon} v[\overline{B_{r_2}(x)}].$$

Recall that $v[B_{r_1}(x)] \geq \delta |B_{r_1}(x)|$, v is bounded, and that the volumes $|B_{r_1}(x)|$, $|B_{2r_1}(x)|$, and $|\overline{B_{r_2}(x)}|$ are all comparable up to a constant factor depending only on the dimension. Then, we get a contradiction by letting $\epsilon \rightarrow 0$. This completes the proof. \square

There are several important consequences of Lemma 3.2. An immediate consequence is this.

COROLLARY 3.4. *Let μ, v, τ satisfy (3.1) (3.2). In addition, suppose that τ is given as the first hitting time to a barrier function s (which is a measurable function $s : \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$) such that*

$$\begin{aligned} \tau &= \inf\{t > 0 \mid t \geq s(W_t)\} \quad \text{in type (I) case,} \\ \tau &= \inf\{t > 0 \mid t \leq s(W_t)\} \quad \text{in type (II), respectively.} \end{aligned}$$

Let $G := \{x \mid s(x) = 0\}$ in type (I) case, and $G := \{x \mid s(x) = \infty\}$ in type (II) case, respectively. If $v \wedge \mu = 0$, then $v[G] = 0$.

PROOF. Notice that $\tau \leq \tau_G$ from the definition of τ and τ_G . Therefore, the result follows from Lemma 3.2. \square

Later in this paper (e.g., Section 8) we will use another technical fact regarding stopping times.

LEMMA 3.5. *Let τ be a stopping time of the Brownian motion W_t . Suppose that G is a (Borel) measurable set in \mathbb{R}^d with $|G| > 0$, and every point of G has Lebesgue density larger than a constant $a > 0$. Suppose there is a countable and dense subset D of $(0, \infty)$ such that*

$$(3.3) \quad \text{Prob}[W_{\bar{t}} \in G \text{ and } \bar{t} < \tau] = 0 \quad \text{for each fixed } \bar{t} \in D.$$

Then, $\tau \leq \tau_G$ almost surely.

PROOF. **Step 1.** We first prove

$$\text{Prob}[\exists t'' \in D \text{ such that } \tau_G < t'' < \tau \text{ and } W_{t''} \in G | \tau_G < \tau] = 1.$$

To see this, observe from the definition of the hitting time τ_G , that for each $0 < \delta$,

$$(*) \quad \dots \dots \quad \text{Prob}[\exists t' \text{ such that } t \leq t' < t + \delta/2 \text{ and } W_{t'} \in G | (W_{\tau_G}, \tau_G) = (x, t)] = 1.$$

(Notice that $(*)$ may not hold if we require $t < t'$ instead of $t \leq t'$.)

Moreover, at each $y \in G$, there is $r_y > 0$ such that for every $0 < r < r_y$,

$$|B_r(y) \cap G| \geq a/2 |B_r(y)|.$$

For each fixed $t'' > t'$, and for given $y = W_{t'}$, the probability density σ for $W_{t''}$ is nothing but the heat kernel at y with time $t'' - t'$, that is,

$$\sigma(z) = \frac{C_d}{(t'' - t')^{d/2}} e^{-|z-y|^2/(t'' - t')}.$$

Therefore, by choosing $t'' \in D$ close to t' (possible because of denseness of D) that $t'' - t' = r^2$ for $r \ll r_y$, we see that for $y \in G$,

$$\text{Prob}[W_{t''} \in G | W_{t'} = y] \geq C,$$

for $C = C(d, a) > 0$; this is for each $r \ll r_y$. This means that, for each $\delta > 0$,

$$\text{Prob}[\exists t'' \in D \text{ such that } t' < t'' < t' + \delta/2 \text{ and } W_{t''} \in G | W_{t'} = y] \geq C.$$

Then by applying the 0-1 law, we get for each $\delta > 0$,

$$\text{Prob}[\exists t'' \in D \text{ such that } t' < t'' < t' + \delta/2 \text{ and } W_{t''} \in G | W_{t'} = y] = 1.$$

Combining this with the above conditional probability $(*)$, we get for each $\delta > 0$,

$$\text{Prob}[\exists t'' \in D \text{ such that } t < t'' < t + \delta \text{ and } W_{t''} \in G | (W_{\tau_G}, \tau_G) = (x, t)] = 1.$$

By the strong Markov property, this implies, for each $\delta > 0$,

$$\text{Prob}[\exists t'' \in D \text{ such that } t < t'' < t + \delta \text{ and } W_{t''} \in G | (W_{\tau_G}, \tau_G) = (x, t) \text{ and } \tau_G < \tau] = 1.$$

Notice that

$$\text{Prob}[\tau > t + \delta | (W_{\tau_G}, \tau_G) = (x, t) \text{ and } \tau_G < \tau] \rightarrow 1 \quad \text{as } \delta \rightarrow 0 \text{ for } \delta > 0.$$

Therefore,

$$\text{Prob}[\exists t'' \in D \text{ such that } t < t'' < \tau \text{ and } W_{t''} \in G | (W_{\tau_G}, \tau_G) = (x, t) \text{ and } \tau_G < \tau] = 1,$$

which implies

$$\text{Prob}[\exists t'' \in D \text{ such that } \tau_G < t'' < \tau \text{ and } W_{t''} \in G | \tau_G < \tau] = 1.$$

Step 2. From Step 1, we get

$$\text{Prob}[\exists t'' \in D \text{ such that } \tau_G < t'' < \tau \text{ and } W_{t''} \in G] = \text{Prob}[\tau > \tau_G].$$

On the other hand, since D is countable, we have

$$\text{Prob}[\exists t'' \in D \text{ such that } \tau_G < t'' < \tau \text{ & } W_{t''} \in G] \leq \sum_{\bar{t} \in D} \text{Prob}[W_{\bar{t}} \in G \text{ & } \bar{t} < \tau].$$

Therefore, if $\text{Prob}[\tau > \tau_G] > 0$, then there exists $\bar{t} \in D$ such that

$$\text{Prob}[W_{\bar{t}} \in G \text{ & } \bar{t} < \tau] > 0,$$

contradicting (3.3). Therefore $\text{Prob}[\tau > \tau_G] = 0$ completing the proof. \square

REMARK 3.6. In Lemma 3.5 the assumption that G consists of its Lebesgue point (of positive density $> a$) is essential. For example, let $d = 1$ and $G_1 := [-4, -3] \cup [3, 4]$, and consider $G := \{0\} \cup G_1$. Let μ be the uniform distribution on the interval $[-1/2, 1/2]$. One can find a stopping time τ with $W_0 \sim \mu$, $W_\tau \sim \nu$ such that ν is supported on the small intervals around the points $-3/2$ and $3/2$. Then, because of the support of ν , we have that $\tau \leq \tau_{G_1}$. Therefore, for each $t > 0$, the probability $\text{Prob}[W_t \in G \text{ & } t < \tau] = \text{Prob}[W_t = 0 \text{ & } t < \tau] = 0$. However, on the other hand, $\tau \not\leq \tau_G$, as Brownian paths will pass through 0 before τ , with positive probability.

Lemma 3.5 together with Lemma 3.2 give the following useful result.

COROLLARY 3.7. *Let μ, ν, τ satisfy (3.1) (3.2), and let G be a measurable subset of \mathbb{R}^d . Let $D \subset (0, \infty)$ be a countable dense subset such that (3.3) holds. Then either $\mu \wedge (\nu|_G) > 0$ or $\nu[G] = 0$.*

PROOF. The case $|G| = 0$ is obvious since $\nu \ll \text{Leb}$. Let \tilde{G} be the set of Lebesgue points of G . Suppose $|G| > 0$ and $\mu \wedge (\nu|_G) = 0$. Then, $|\tilde{G}| > 0$ and in $\mu \wedge (\nu|_{\tilde{G}}) = 0$. We will show that $\nu[G] = 0$. Since (3.3) remains valid for \tilde{G} , Lemma 3.5 yields $\tau \leq \tau_{\tilde{G}}$ (almost surely), which then implies $\nu[\tilde{G}] = 0$ by Lemma 3.2 as $\mu \wedge (\nu|_{\tilde{G}}) = 0$. Since $\nu[G] = \nu[\tilde{G}]$ for $\nu \ll \text{Leb}$, this completes the proof. \square

REMARK 3.8. Note that the condition $\nu \in L^\infty$, especially $\nu \ll \text{Leb}$, is essential in Corollary 3.7. In particular, if $|G| = 0$, then $\text{Prob}[W_t \in G \text{ & } t < \tau] = 0$ for each constant $t > 0$ when $W_0 \sim \mu \ll \text{Leb}$. So, if a singular measure ν supported on such a zero Lebesgue measure set, for example, the distribution $\nu \sim W_\tau$ for the hitting time τ to a sphere, does not satisfy the result of this Corollary.

REMARK 3.9. This corollary will be used in Section 8, where we prove a saturation result of the density upper bound constraint for optimal target measures, in Theorem 8.3. However, one may bypass using this, by using the Eulerian formulation and closedness of the barrier set for the optimal stopping times in Section 4; there Proposition 3.10, which does not use this corollary or Lemma 3.5 but Lemma 3.2, is used crucially. We still think Lemma 3.5 and Corollary 3.7 are of their own interest.

We now focus on stopping times that are optimal solutions in Theorem 2.1. Such a stopping time is given as the hitting time to a barrier function $s^* : \mathbb{R}^d \rightarrow [0, \infty]$ that is determined by the dual optimal solution. Lemma 3.2 and its consequences give a useful characterization for such a stopping time, which will play a key role later in the paper (see, for instance Theorems 4.15 and 5.4 and Section 10). Especially, the following proposition is crucial, which roughly speaking, endows a certain regularity to the set $\{(x, t) | t = s^*(x)\}$ from the condition (3.1), by ensuring that the Brownian particles stop only at the set given as the graph of s^* , nor above or below (in t), for instance. Such a property does not hold in general, and in our

case it comes from optimality that the barrier set and the corresponding stopping time are determined by solving (1.5).

PROPOSITION 3.10. *Let μ, ν satisfy (3.1), and let τ^*, R^*, s^* be as in Theorem 2.1 and (2.2) with the same assumptions (on μ, ν), the optimal stopping time and the corresponding barrier set R^* and the barrier function. In the (II) case further assume that $\mu \wedge \nu = 0$. Then,*

$$(3.4) \quad \tau^* = s^*(W_{\tau^*}) \quad \text{almost surely.}$$

PROOF. First recall that, by [35], Corollary 3.6, the optimal stopping occurs only in the barrier set which is given by the optimal solution to the dual problem. More precisely, in [35], Corollary 3.6, for the dual optimal solution ψ^* , it is shown that $J_{\psi^*}(W_{\tau^*}, \tau^*) = \psi^*(W_{\tau^*})$ almost surely, equivalently, we get the following key property for us:

$$(W_{\tau^*}, \tau^*) \in R^* \quad \text{almost surely.}$$

Thus, if (3.4) is not true, then

$$\begin{aligned} \text{Prob}[\tau^* > s^*(W_{\tau^*})] &> 0 \quad \text{in the (I) case,} \\ \text{Prob}[\tau^* < s^*(W_{\tau^*})] &> 0 \quad \text{in the (II) case, respectively,} \end{aligned}$$

so there exists $\mathbb{Q} \ni \bar{t} > 0$ such that

$$\begin{aligned} \text{Prob}[\tau^* > \bar{t} > s^*(W_{\tau^*})] &> 0 \quad \text{in the (I) case,} \\ \text{Prob}[\tau^* < \bar{t} < s^*(W_{\tau^*})] &> 0 \quad \text{in the (II) case, respectively.} \end{aligned}$$

Let us separate the (I) and (II) cases.

1. (I) case. Consider the set $S := \{x | s^*(x) < \bar{t}\}$ and the distribution $\nu_{\bar{t}}$ of $W_{\bar{t} \wedge \tau^*}$. Note that $W_{\bar{t}} \in S$ implies that $\tau^* \leq \bar{t}$. Therefore, $W_{\bar{t} \wedge \tau^*} \in S$ is equivalent to “ $W_{\tau^*} \in S$ and $\tau^* \leq \bar{t}$ ”, which is then equivalent to “ $s^*(W_{\tau^*}) < \bar{t}$ and $\tau^* \leq \bar{t}$ ”, whose probability is strictly < 1 due to $\text{Prob}[\tau^* > \bar{t} > s^*(W_{\tau^*})] > 0$. Therefore, $\nu_{\bar{t}}[S] < 1$, thus the measure $\mu_{\bar{t}} := \nu_{\bar{t}} - \nu_{\bar{t}}|_S$ has a positive total mass $|\mu_{\bar{t}}| > 0$.

We then start the Brownian motion from the distribution $\mu_{\bar{t}}$ and initial time \bar{t} ; we immediately stop if $\tau^* \leq \bar{t}$, otherwise we continue the Brownian motion until τ^* ; we call this stopping time $\bar{\tau}$. Let $\bar{\nu}$ denote that distribution of $W_{\bar{\tau}}$, then, from the condition $\nu \ll \text{Leb}$, we also have $\bar{\nu} \ll \text{Leb}$. Moreover, from the definition of S and the fact $\text{Prob}[\tau^* > \bar{t} > s^*(W_{\tau^*})] > 0$, we have

$$\bar{\nu}[S] > 0.$$

On the other hand, let τ_S be the first hitting time to the set S for the Brownian motion starting from the time \bar{t} with the initial distribution $\mu_{\bar{t}}$. Notice that if $W_t \in S$ (with $t \geq \bar{t}$) then $t \geq \bar{t} > s^*(W_t)$ by the definition of S , therefore $\tau^* \leq t$, thus $\bar{\tau} \leq t$ from the construction of the stopping time $\bar{\tau}$. This implies that $\bar{\tau} \leq \tau_S$. From the construction of $\mu_{\bar{t}}$ we see that $\mu_{\bar{t}} \ll \text{Leb}$ and $\mu_{\bar{t}} \wedge \bar{\nu}|_S = 0$. Therefore from Lemma 3.2 we see that $\bar{\nu}[S] = 0$, a contradiction. This completes the proof in the (I) case.

2. (II) case. Let $S = \{x | s^*(x) > \bar{t}\}$. Let $\bar{\tau} = \bar{t} \wedge \tau^*$ and $\tau_S := \inf\{t | W_t \in S\}$. Notice that obviously if $t \geq \bar{t}$, then $t \geq \bar{\tau}$. On the other hand, if $t < \bar{t}$ and $W_t \in S$ then $s^*(W_t) > t$ so $t \geq \tau^*$. These imply that $\bar{\tau} \leq \tau_S$. Moreover, let $\bar{\nu}$ be the distribution of $\bar{\tau}$, that is, $W_{\bar{\tau}} \sim \bar{\nu}$. Then from $\text{Prob}[\tau^* < \bar{t} < s^*(W_{\tau^*})] > 0$ and the assumption $\mu \wedge \nu = 0$ in this (II) case, there exists $S_1 \subset S$ such that $\mu[S_1] = 0$ and $\bar{\nu}[S_1] > 0$. Since $\mu \wedge (\bar{\nu}|_{S_1}) = 0$, and $\bar{\tau} \leq \tau_S \leq \tau_{S_1}$, we have from Lemma 3.2, that $\bar{\nu}[S_1] = 0$, a contradiction. This completes the proof. \square

4. Optimal stopping times and potential flows. For each stopping time τ we can consider the time flow of distribution μ_t of $W_{\tau \wedge t}$, $0 \leq t \leq \infty$, which then defines potential functions by solving the corresponding Poisson equation. These potential functions can be used back to study the stopping times, which is especially beneficial because they have regularity properties coming from the elliptic regularity and Ito's formula. Using potential functions to study stopping times has been considered in the literature starting from [6] for existence of Brownian stopping times (Skorokhod problem), and more recently for analyzing optimal stopping problems (see [20, 22, 31], for one space dimension, and [32, 35] for higher dimensions).

The main result of this section (Theorem 4.15) shows that when τ is the optimal stopping time in Theorem 2.1, given as the hitting time to a barrier of type (I) or (II), the barrier is a closed set; this is a consequence from the correspondence between the optimal stopping time τ and its potential flow. We remark that in the type (I) case, such correspondence (a sharper form than ours) is proved in [32] even for a much more general class of stochastic processes. Their proof does not use optimality while ours that covers both type (I) and (II), uses the optimality assumption in an essential way; in particular, we use Proposition 3.10.

Closednesss of the barrier set is a key fact that will be used in Section 5 where we establish the consistency of our probabilistic formulation of the Stefan problem with the PDE formulation.

4.1. Potential flows, definition, and regularity. In this subsection we define the potential flow U_{μ_t} and establish its space-time continuity (Corollary 4.9).

Let $N(y)$ be the Newtonian potential function on \mathbb{R}^d , namely

$$N(y) := \begin{cases} \frac{1}{2}|y| & d = 1, \\ 2\pi \log |y| & d = 2, \\ \frac{1}{d(2-d)\omega_d} |y|^{2-d} & \text{otherwise,} \end{cases}$$

where ω_d is the volume of the unit ball in \mathbb{R}^d so that

$$\Delta N(y) = \delta_0.$$

Given a measure μ , define the potential U_μ as

$$U_\mu(x) := \int N(x - y) d\mu(y).$$

Since $\Delta U_\mu = \mu$, elliptic regularity yields the following.

LEMMA 4.1 (Spatial continuity). *Assume that there are constants $M, R > 0$ such that $\mu \leq M$ and $\mu = 0$ outside B_R . Then $\|U_\mu\|_{C^{1,\alpha}(\mathbb{R}^d)} \leq C$ for any $0 < \alpha < 1$, with $C = C(\alpha, M, R)$.*

REMARK 4.2. Our main focus is on the stopping times τ generating a compactly supported ν , $W_\tau \sim \nu$, with the upper density constraint $\nu \leq f \in L^\infty$. For such τ , the measure μ_t , that is, the distribution of $W_{\tau \wedge t} \sim \mu_t$, is uniformly bounded with respect to t , as μ_t is bounded by the greater of the solution to the heat equation (with initial value μ) and f . Also, the support of μ_t is contained in the convex hull of the support of ν ; one can easily verify this from the subharmonic order $\mu_t \leq_{SH} \nu$ by using linear functions as test functions ϕ for $\int \phi d\mu_t \leq \int \phi d\nu$. In what follows we thus focus on the case where the result of Lemma 4.1 holds.

We now recall a simple consequence of Ito's formula. For each $0 \leq g \in C_c^1(\mathbb{R}^d)$, consider a subharmonic function $u \in C^2$ such that $\Delta u = g$. Note that one can find such $u \in C^{2,\alpha}$ from elliptic regularity. Using Ito's formula, for each stopping time $\sigma \geq 0$ we have

$$\begin{aligned}\mathbb{E}[u(W_\sigma) | W_0 = y] - u(y) &= \mathbb{E}\left[\int_0^\sigma \frac{1}{2} \Delta u(W_t) dt | W_0 = y\right] \\ &= \mathbb{E}\left[\int_0^\sigma \frac{1}{2} g(W_t) dt | W_0 = y\right].\end{aligned}$$

In particular, we can show

LEMMA 4.3. *Suppose that μ is a probability measure on \mathbb{R}^d and $W_0 \sim \mu$ and $W_{\tau_i} \sim \nu_i$, $i = 1, 2$ and that the potentials U_{ν_1} , U_{ν_2} are measurable functions. Assume that $\tau_1 \leq \tau_2$ almost surely. Then, for each closed or open set $E \subset \mathbb{R}^d$, we have for the characteristic function χ_E that*

$$(4.1) \quad \int_E (U_{\nu_2} - U_{\nu_1})(y) dy = \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2} \chi_E(W_t) dt\right].$$

In general, for a measurable E , there exists a monotonically increasing sequence of compact sets $K_n \subseteq E$ with $\lim_{n \rightarrow \infty} |E \setminus K_n| = 0$ and monotonically decreasing sequence of open sets with $E \subseteq O_n$ and $\lim_{n \rightarrow \infty} |O_n \setminus E| = 0$ such that

$$\int_E (U_{\nu_2} - U_{\nu_1})(y) dy = \lim_{n \rightarrow \infty} \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2} \chi_{K_n}(W_t) dt\right] = \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2} \lim_{n \rightarrow \infty} \chi_{K_n}(W_t) dt\right],$$

$$\int_E (U_{\nu_2} - U_{\nu_1})(y) dy = \lim_{n \rightarrow \infty} \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2} \chi_{O_n}(W_t) dt\right] = \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2} \lim_{n \rightarrow \infty} \chi_{O_n}(W_t) dt\right].$$

PROOF. **Step 1.** First let us consider the case where E is compact or open and bounded. For a compact set $E = K$, there exists a monotonically decreasing sequence $0 \leq g_k \in C_c^\infty(\mathbb{R}^d)$, point-wisely converging to χ_K , that is, $\chi_K = \inf_k g_k$. Indeed, take a continuous function h_k that equals $d(x, K)$ if $d(x, K) \leq 1/k$, 1 if $d(x, K) \geq 2/k$, and the linear interpolation, $(k-1)(d(x, K) - 1/k) + 1/k$, for $1/k \leq d(x, K) \leq 2/k$. After mollifying, this generates a monotone increasing sequence of C^1 functions that converges to $1 - \chi_K$. When $E = O$ is open and bounded, we can modify the above construction for $K = (\mathbb{R}^d \setminus O) \cap B_r$ for $r \gg 1$ to find the corresponding $\{g_k\}_k$ converging to χ_O .

Now, consider the subharmonic functions $u_k \in C^\infty$ such that $\Delta u_k = g_k$; because $g_k \in C_c^\infty$ we have $u_k \in C^\infty$ from elliptic regularity. We have

$$\begin{aligned}\int_E (U_{\nu_2} - U_{\nu_1})(y) dy &= \lim_{k \rightarrow \infty} \int (U_{\nu_2} - U_{\nu_1})(y) g_k(y) dy \\ &\quad (\text{by the monotone convergence theorem}) \\ &= \lim_{k \rightarrow \infty} \int (\nu_2 - \nu_1)(y) u_k(y) dy \quad (\text{integration by parts}) \\ &= \lim_{k \rightarrow \infty} \mathbb{E}[u_k(W_{\tau_2}) - u_k(W_{\tau_1})] \\ &= \lim_{k \rightarrow \infty} \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2} g_k(W_t) dt\right] \quad (\text{by Ito's formula}),\end{aligned}$$

where by the monotone convergence theorem (applied to the Wiener measure on the path space as the monotone convergence $g_k \rightarrow \chi_E$ can be extended to the path space), the last line is the same as

$$\mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2} \chi_E(W_t) dt\right].$$

Step 2. Next for open or closed E , consider $E_r = E \cap B_r$. Notice $\chi_{E_r} \rightarrow \chi_E$ monotonically as $r \rightarrow \infty$. We get (4.1) for E_r from Step 1, and applying the monotone convergence theorem as $r \rightarrow \infty$, we get it for E .

Step 3. Now for a measurable E , one can find a sequence of increasing compact sets $K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots$ contained in E such that $\lim_{n \rightarrow \infty} |E \setminus K_n| = 0$ and monotonically decreasing sequence of open sets with $E \subseteq O_n$, $O_1 \supseteq O_2 \supseteq O_3 \supseteq \dots$, and $\lim_{n \rightarrow \infty} |O_n \setminus E| = 0$. Let $\{E_n\}$ denote either the sequence $\{K_n\}$ or $\{O_n\}$. Then, from the monotone convergence theorem applied to χ_{E_n} , we have

$$\int_E (U_{v_2} - U_{v_1})(y) dy = \lim_{n \rightarrow \infty} \int_{E_n} (U_{v_2} - U_{v_1})(y) dy.$$

Since for the sets $\int_{E_n} (U_{v_2} - U_{v_1})(y) dy = \mathbb{E}[\int_{\tau_1}^{\tau_2} \frac{1}{2} \chi_{E_n}(W_t) dt]$ from the previous cases, we get

$$\int_E (U_{v_2} - U_{v_1})(y) dy = \lim_{n \rightarrow \infty} \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2} \chi_{E_n}(W_t) dt\right] = \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2} \lim_{n \rightarrow \infty} \chi_{E_n}(W_t) dt\right],$$

where for the last equality we applied the monotone convergence theorem to the Wiener measure on the path space as the monotone sequence χ_{E_n} can be extended to the path space. This completes the proof. \square

The following is a standard fact, but we provide its proof for clarity.

COROLLARY 4.4. *Suppose μ is a bounded, compactly supported measure on \mathbb{R}^d and that τ_1, τ_2 are stopping times with the given initial distribution $W_0 \sim \mu$ and with the respective final distribution $W_{\tau_i} \sim \nu_i$, $i = 1, 2$. Then*

$$\text{if } \tau_1 \leq \tau_2 \text{ almost surely, then } U_{v_1} \leq U_{v_2} \text{ a.e.}$$

Moreover, $U_{v_1} = U_{v_2}$ a.e. if and only if $\tau_1 = \tau_2$ almost surely.

PROOF. The first statement follows immediately from Lemma 4.3. Next suppose $U_{v_1} = U_{v_2}$ a.e. Apply Lemma 4.3 to $E = \mathbb{R}^d$ and obtain

$$0 = \int_{\mathbb{R}^d} (U_{v_2} - U_{v_1})(y) dy = \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2} dt\right] = \frac{1}{2} \mathbb{E}[\tau_2 - \tau_1].$$

Hence $\mathbb{E}[\tau_2 - \tau_1] = 0$. Since $\tau_2 \geq \tau_1$, we conclude that $\tau_2 = \tau_1$ almost surely. \square

REMARK 4.5. Note that there are many stopping times with the same final distribution. Therefore, $U_{v_1} = U_{v_2}$ without an assumption like $\tau_1 \leq \tau_2$ does not give much information.

Now we define our potential flow.

DEFINITION 4.6 (Potential flow). For compactly supported measures $\mu, \nu \in L^\infty(\mathbb{R}^d)$ with $|\mu| = |\nu|$, let τ be a stopping time between μ and ν , that is, $W_0 \sim \mu$ and $W_\tau \sim \nu$. Let μ_t be the distribution of the Brownian stopping time $\tau \wedge t$, that is, $W_{\tau \wedge t} \sim \mu_t$. We then call $\{U_{\mu_t}\}_{t > 0}$ the *potential flow* associated with τ .

From Corollary 4.4 we immediately have

COROLLARY 4.7. *Under the assumptions of Definition 4.6,*

- (a) U_{μ_t} is monotonically nondecreasing in t ;
- (b) $U_\mu \leq U_{\mu_t} \leq U_\nu$ for all t .

PROOF. The items (a) and (b) are obvious from Corollary 4.4, for under our assumptions and Lemma 4.1, the functions U_{μ_t} , U_μ , U_v are all continuous (in fact, $C^{1,\alpha}$ for some $\alpha > 0$). \square

We now prove continuity of the potential flow in time.

LEMMA 4.8 (Time continuity). *Suppose $C := \|\mu_t\|_{L^\infty(\mathbb{R}^d \times [0, \infty))} < \infty$, and that the support of μ_t is bounded uniformly in t . Then U_{μ_t} is Lipschitz in time, more precisely, for each x ,*

$$0 \leq U_{\mu_{t'}}(x) - U_{\mu_t}(x) \leq \frac{C}{2}[t' - t] \quad \text{for all } 0 \leq t < t'.$$

PROOF. Recall from Lemma 4.3 that for each open set E ,

$$\int_E (U_{\mu_{t'}} - U_{\mu_t})(y) dy = \mathbb{E} \left[\int_{\tau \wedge t}^{\tau \wedge t'} \frac{1}{2} \chi_E(W_r) dr \right].$$

The first inequality then follows from spatial continuity of U_{μ_t} given in Lemma 4.1. For the second inequality, apply Fubini in the above equality to obtain

$$(4.2) \quad \int_E (U_{\mu_{t'}} - U_{\mu_t})(y) dy = \frac{1}{2} \int_t^{t'} \text{Prob}[W_r \in E \text{ & } r < \tau] dr.$$

Notice that for each $r > 0$, by the definition of μ_t ,

$$\text{Prob}[W_r \in E \text{ & } r < \tau] \leq \mu_r[E].$$

Since $\mu_r \leq C$, this and (4.2) implies

$$\frac{1}{|E|} \int_E (U_{\mu_{t'}} - U_{\mu_t})(y) dy \leq \frac{C}{2}(t' - t).$$

Now set $E = B_r(x)$ and let $r \rightarrow 0^+$, which completes the proof. \square

From Lemma 4.1 and Lemma 4.8, the following holds.

COROLLARY 4.9 (Space-time Lipschitzness of the potential). *Assume that there exists a constant $C > 0$ such that $\|\mu_t\|_\infty \leq C$ for all $t \geq 0$, and that the support of μ_t is bounded uniformly in t . Then, U_{μ_t} is uniformly $C^{1,\alpha}$ in space and Lipschitz in time in $\mathbb{R}^d \times [0, \infty)$.*

4.2. *Hitting times to monotone barriers and potential functions.* In this section we investigate the relation between the stopping time τ and the hitting time to a barrier set generated by the potential U_{μ_t} .

DEFINITION 4.10. For the potential flow $\{U_{\mu_t}\}_{t \geq 0}$ given in Definition 4.6, we define the forward/backward stopping times

$$\tau^{U,f} := \inf\{t | U_{\mu_t}(W_t) = U_v(W_t)\}, \quad \tau^{U,b} := \inf\{t > 0 | U_{\mu_t}(W_t) = U_\mu(W_t)\}.$$

(Notice that the condition $t > 0$ for $\tau^{U,b}$ is necessary: otherwise $\tau^{U,b} \equiv 0$.) We also define the corresponding barrier functions and the barrier sets:

$$s^{U,f}(x) := \inf\{t | U_{\mu_t}(x) = U_v(x)\}, \quad R^{U,f} := \{(x, t) | t \geq s^{U,f}(x)\}$$

and

$$s^{U,b}(x) := \sup\{t | U_{\mu_t}(x) = U_\mu(x)\}, \quad R^{U,b} := \{(x, t) | t \leq s^{U,b}(x)\}.$$

It is important for us to observe the following.

LEMMA 4.11. *Assume μ_t and its support are uniformly bounded for all $0 \leq t \leq \infty$. Then*

- (a) $s^{U,f}$ and $-s^{U,b}$ are lower semicontinuous. Equivalently, the sets $R^{U,f}$ and $R^{U,b}$ are closed in $\mathbb{R}^d \times [0, \infty]$;
- (b) we have

$$\tau^{U,f} = \inf\{t | s^{U,f}(W_t) \leq t\}, \quad \tau^{U,b} = \inf\{t | s^{U,b}(W_t) \geq t > 0\}.$$

PROOF. These follow from Corollary 4.9 and time monotonicity of U_{μ_t} in Corollary 4.7(a)(b). \square

The potential flow U_{μ_t} can be defined for any stopping time, even for randomized stopping times, so the hitting times $\tau^{U,f}$, $\tau^{U,b}$ can be associated to any (randomized) stopping time τ . On the other hand it is easy to see that τ and either of $\tau^{U,f}$, $\tau^{U,b}$ do not coincide in general; even for a hitting time τ , it will not be equal to $\tau^{U,f}$, $\tau^{U,b}$ when its barrier is not monotone in time. For the type (I) case, from the result of [32], we have $\tau = \tau^{U,f}$. To our knowledge a similar result in such a generality is not available for the type (II) case. When $\tau = \tau^*$ is an optimal stopping time (as in Theorem 2.1), we verify that it is equal to $\tau^{U,f}$, $\tau^{U,b}$, for type (I), (II), respectively; this is proved in Theorem 4.15 below. It will then imply that the sets $R^{U,f}$ and $R^{U,b}$, in type (I) and (II), respectively, give the barrier set of τ^* . Closedness of these barrier sets (Lemma 4.11(a)) allows us to apply the maximum principle in the PDE formulation in Section 5.

Note that the equivalence between τ and $\tau^{U,f}$ or $\tau^{U,b}$, is hinted in the formula (4.1) which is a consequence of Ito's formula. One sees there that spending more time for the Brownian motion increases the potential function, which is precisely controlled but, only in integral/expected value sense; it is not obvious how to use such a control on average quantities to get information for each individual Brownian path. We overcome this difficulty by optimality of τ , in particular, using Proposition 3.10.

Towards the main result of this section, in the next proposition we show that the barrier set R of τ (e.g., the set R^* for the optimal stopping time τ^* as in Section 2) is contained, except a measure zero set, in the barrier set $R^{U,f}$, $R^{U,b}$, for type (I),(II), respectively. This is basically a result of the Ito's formula via Lemma 4.3.

PROPOSITION 4.12. *Let μ , ν , and τ be as given in Definition 4.6. Suppose τ is characterized as the hitting time $\tau = \inf\{t | t \geq s(W_t)\}$ for type (I), $\tau = \inf\{t | 0 < t \leq s(W_t)\}$ for type (II), for a measurable function $s : \mathbb{R}^d \rightarrow [0, \infty]$. Then we have*

$$|\{x \in \mathbb{R}^d | s(x) < s^{U,f}(x)\}| = 0 \quad \text{for (I) and} \quad |\{x \in \mathbb{R}^d | s(x) > s^{U,b}(x)\}| = 0 \quad \text{for (II).}$$

In particular, since $\nu \ll \text{Leb}$,

$$\nu[\{x \in \mathbb{R}^d | s(x) < s^{U,f}(x)\}] = 0 \quad \text{for (I) and} \quad \nu[\{x \in \mathbb{R}^d | s(x) > s^{U,b}(x)\}] = 0 \quad \text{for (II).}$$

PROOF. We will prove for costs of type (II), since the argument is parallel for the other case.

Suppose not, that is, $|\{s > s^{U,b}\}| > 0$. We can then find a constant $\bar{t} \in [0, \infty)$ such that

$$|E| > 0 \text{ where } E := \{x | s(x) > \bar{t} > s^{U,b}(x)\}.$$

Let us consider $w(x, t) := U_{\mu_t}(x) - U_\mu(x)$. From the definition of $s^{U,b}$ and E we have $w(\cdot, \bar{t}) > 0$ on E , yielding

$$\int_E w(y, \bar{t}) dy > 0.$$

From Lemma 4.3, there exists a monotonically increasing sequence of compact sets $K_n \subseteq E$ such that

$$\int_E w(y, \bar{t}) dy = \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^{\bar{t} \wedge \tau} \frac{1}{2} \chi_{K_n}(W_t) dt \right].$$

By Fubini's theorem, we thus can write

$$\int_E w(y, \bar{t}) dy = \lim_{n \rightarrow \infty} \frac{1}{2} \int_0^{\bar{t}} \text{Prob}[W_r \in K_n \& r < \tau] dr \leq \frac{1}{2} \int_0^{\bar{t}} \text{Prob}[W_r \in E \& r < \tau] dr.$$

On the other hand, recall that $\tau = \inf\{t \mid 0 < t \leq s(W_t)\}$. Hence for $r \leq \bar{t}$, if $W_r \in E$, then $s(W_r) > \bar{t} \geq r$ and thus $\tau \leq r$. Therefore

$$\text{Prob}[W_r \in E \& r < \tau] = 0 \quad \text{for } r \leq \bar{t}.$$

Back to the previous inequalities of integrals, we get

$$\int_E w(y, \bar{t}) dy \leq 0,$$

a contradiction, completing the proof. \square

DEFINITION 4.13. We let τ^U commonly denote $\tau^{U,f}$ and $\tau^{U,b}$ depending on whether μ_t is generated from type (I) or (II) stopping time τ . Likewise we use R^U for the barrier sets $R^{U,f}$ and $R^{U,b}$, and s^U for $s^{U,f}$ and $s^{U,b}$.

REMARK 4.14. Proposition 4.12 by itself may not give much information on the relation between τ and τ^U , as the stopped distribution of (W_τ, τ) in general can even be outside the barrier set R , in particular, if the barrier function has discontinuity in a dense subset of \mathbb{R}^d , of the Hausdorff dimension $d - 1$.

We are now ready to prove our main result of this section, which is the equivalence between τ and τ^U . Such a result is known for type (I) [32]. We prove the equivalence for both type (I) and (II) by employing the connection between the Brownian motion with stopping time and a parabolic flow, assuming optimality of the stopping time τ . In Section 5 consistency results will be proved using a slightly weaker result, namely, that the set $\{\eta > 0\}$, for the Eulerian variable (η, ρ) , is open.

THEOREM 4.15 (see also [32] for type (I)). *Use the same assumptions as in Propositions 3.10 and 4.12; in particular, $\tau = \tau^*$ is an optimal stopping time of Theorem 2.1, and in the type (II) case $\mu \wedge \nu = 0$. Then,*

(a) *for the Eulerian variable (η, ρ) that corresponds to τ^* we have*

$$\{\eta = 0\} = R^U \quad \text{in particular, the set } \{\eta > 0\} \text{ is open;}$$

(b) *$\tau^* = \tau^U$ almost surely, in particular, the set R^U gives a barrier for τ^* as a hitting time to it.*

PROOF. Because of Proposition 4.12, we have

$$s^*(W_{\tau^*}) \geq s^U(W_{\tau^*}) \quad \text{a.s. for type (I)}, \quad s^*(W_{\tau^*}) \leq s^U(W_{\tau^*}) \quad \text{a.s. for type (II)}.$$

This implies

$$\tau^* \geq \tau^U \quad \& \quad (W_{\tau^*}, \tau^*) \in R^U \cap R^* \quad \text{a.s.},$$

since $\tau^* = s^*(W_{\tau^*})$ a.s. from Proposition 3.10. In particular, we can find a barrier set R of τ^* such that $R \subseteq R^U$. To see this, let $R = R^* \cap R^U$ and let $\tilde{\tau}$ be its hitting time. Since $(W_{\tau^*}, \tau^*) \in R^U \cap R^*$ we have $\tilde{\tau} \leq \tau^*$ almost surely. On the other hand from the definition of the hitting time we have $\tilde{\tau} \geq \max[\tau^*, \tau^U]$ while the latter is nothing but τ^* from $\tau^* \geq \tau^U$, so we conclude that τ^* is the hitting time $\tilde{\tau}$ to $R = R^* \cap R^U$. Below we use this R .

From Proposition 2.2, the stopping time τ^* induces its Eulerian flow (η, ρ) which satisfies (2.3) with $\eta[R] = 0$ and $\rho[R] = 1$; in particular, ρ is the distribution of (W_{τ^*}, τ^*) . Furthermore from [35] we have $\eta \in L^2(\mathbb{R}_{\geq 0}; H_0^1(\mathbb{R}^d))$.

Let us denote $w := U_v - U_{\mu_t}$ for (I), and $w := U_{\mu_t} - U_\mu$ for (II). For the rest of the proof we focus on type (II), as (I) follows a parallel proof, and also because for (I) a different proof is available in [32].

Since $\Delta w = \mu_t - \mu$ at each $t \geq 0$, from the definition of μ_t and (2.3) we have

$$(4.3) \quad \Delta w(x) = \eta(t, x) + \int_0^t \rho(dr, x) - \mu \quad \text{in } \mathbb{R}^d, \text{ for type (II)}.$$

Now, for $g \in C(\mathbb{R}_{\geq 0} \times \mathbb{R}^d)$ with $g(t, \cdot) \in C_c(\mathbb{R}^d)$ for each $t > 0$, consider φ solving $\Delta\varphi(t, \cdot) = g(t, \cdot)$ for each $t > 0$, with decay at infinity. Using (2.3) for (η, ρ) and (4.3), we integrate by parts to obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}_{\geq 0}} g(x, t) \frac{\partial}{\partial t} w(x) dt dx &= \int_{\mathbb{R}^d} \int_{\mathbb{R}_{\geq 0}} \varphi(\partial_t \eta + \rho) dt dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}_{\geq 0}} \varphi\left(\frac{1}{2} \Delta \eta\right) dt dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}_{\geq 0}} \frac{1}{2} g(x, t) \eta(x, t) dt dx. \end{aligned}$$

Therefore we have

$$(4.4) \quad \partial_t w = \frac{1}{2} \eta \quad (\text{in type (I), the sign is opposite}).$$

From (4.4) and the fact that $w(x, 0) = 0$, it follows that

$$(4.5) \quad w(x, t) = \frac{1}{2} \int_0^t \eta(x, r) dr.$$

Since η is nonnegative, it follows that the set $(R^U)^c = \{w > 0\}$ includes $\{\eta > 0\}$, namely

$$\{\eta > 0\} \subseteq (R^U)^c.$$

As ρ is the distribution of (W_{τ^*}, τ^*) , Proposition 3.10 and Proposition 4.12 together imply that

$$\rho((R^U)^c) = 0,$$

and thus η solves the heat equation in $(R^U)^c$. Therefore η is positive in the connected open component \mathcal{R} of $(R^U)^c$ that contains $\{\mu > 0\} \times \{t = 0\}$, and η is zero in any other component of $(R^U)^c$, yielding that $\{\eta > 0\} = \mathcal{R}$. It follows that the set $\{\eta > 0\}$ increases in time, and from

(4.5) it follows that $\{w > 0\} = \mathcal{R}$. We now conclude from the definition of w that $\mathcal{R} = (R^U)^c$ and thus shows the item (a). The set is open due to Lemma 4.11(a).

Now let us prove the item (b). Let (η^U, ρ^U) be the Eulerian flow of τ^U that is, the hitting time to R^U . From the definition of the hitting time we see $\eta^U(R^U) = 0$. Also, since R^U is a closed set, we have that $\rho^U(R^U) = 1$. From the previous paragraph, $\eta(R^U) = 0$, $\rho(R^U) = 1$, thus, by applying the uniqueness in Lemma 2.3, we get $(\eta, \rho) = (\eta^U, \rho^U)$. In particular, this implies that $W_{\tau^U} \sim \nu$. Since we already have $\tau^* \geq \tau^U$ a.s. and $W_{\tau^*} \sim \nu$, we conclude $\tau^U = \tau^*$ a.s. verifying the item (b). \square

REMARK 4.16. The simple but powerful argument of part (b) is borrowed from [19]. In the older argument of a previous version of this paper, we used an obstacle PDE (4.6) to derive a weaker result (only $s^* = s^U$ ν -a.e with an additional assumption). In [19], the results of Theorem 4.15 are extended to nonlocal case (jump processes), especially in the type (II) case; type (I) case on the other hand is already a result of [32] even for the nonlocal case.

Theorem 4.15 states that if $\tau = \tau^*$ is the optimal hitting time obtained in Theorem 2.1, then the barrier can be given by R^U . As the potential flow is a natural object associated to a stopping time, we regard R^U as the canonical barrier associated to $\tau = \tau^*$, μ , and ν . From now on we mean R^U whenever we say a monotone barrier R for an optimal stopping time τ^* . Using closedness of R^U we can in fact show that R^U (so R in our convention) is not dependent on τ^* (more precisely, the cost function it optimizes) but only on μ and ν , as long as the optimal stopping time is given by the hitting time to a monotone barrier. In particular this yields uniqueness of the optimal stopping time τ^* (hitting a forward or backward barrier set) for given μ and ν , depending only on the type, not on a particular cost.

LEMMA 4.17 (See also Remark 6.19 of [8]). *Let μ, ν satisfy the assumptions of Definition 4.6 and Theorem 2.1. Then, there exists unique optimal stopping time τ^* between them for a type (I) cost, the same optimal τ^* for any choice of type (I) cost; in particular, the barrier set R^U for an optimal τ^* is uniquely determined by μ and ν . A same result holds for type (II) case, further assuming $\mu \wedge \nu = 0$.*

PROOF. As the barrier R^U is closed, it follows from the argument of Loynes [43] that the barrier set is unique; for example, follow the argument of [8], Remarks 2.3 and 6.19, using the closedness. Since R^U determines τ^* from Theorem 4.15, uniqueness of τ^* follows as well. \square

REMARK 4.18. A first result about uniqueness of the barrier set for a stopping time between two given measures has been made by Loynes [43] by introducing the notion of regular barriers in the one-dimensional case with $\mu = \delta_0$, which then was extended to more general measures (in 1-D) by Gassiat, Oberhauser, and dos Reis [31], Section 3.3. Lemma 2, who introduced (μ, ν) -regular barriers. Their notion coincides with R^U in our case.

DEFINITION 4.19. For μ, ν as given in Lemma 4.17, $(R, s(x))$ is each *the barrier set* and *the barrier function* associated to (μ, ν) for the cost type (I) or the cost type (II), if $R = R^U$, $s = s^U$ is the barrier set and the barrier function, uniquely given in Lemma 4.17, either in the form of $R := \{(x, t) : t \geq s(x)\}$ for cost (I), or $R := \{(x, t) : 0 < t \leq s(x)\}$ for cost (II).

For a given barrier R , the corresponding Eulerian flow is uniquely determined, due to Lemma 2.3. This justifies the following definition.

DEFINITION 4.20. We say the pair (η, ρ) is the *Eulerian variables associated with* (μ, ν) for the cost type (I) (or (II)) if it is the unique pair (η, ρ) solving (2.3) in the weak sense with the property $\eta \in L^2([0, \infty); H_0^1(\overline{O}))$, with $\eta(R) = 0$ and $\rho(R) = 1$ for the unique barrier $R = R^U$ determined by μ, ν . Here, $O \subset \mathbb{R}^d$ is a bounded open convex set that contains the supports of μ and ν .

4.3. *Remarks on the potential approach and the parabolic obstacle problem.* One may prove Lemma 4.17 by using a comparison principle, viewing the functions $w(x, t) := U_\nu - U_{\mu_t}$ for type (I), $w := U_{\mu_t} - U_\mu$ for (II) as solutions to parabolic obstacle problems:

$$(4.6) \quad \begin{aligned} \min \left[\partial_t w - \frac{1}{2} \Delta w + \frac{1}{2} \nu, w \right] &= 0, \quad w(\cdot, 0) = U_\nu - U_\mu \text{ for type (I);} \\ \min \left[\partial_t w - \frac{1}{2} \Delta w + \frac{1}{2} \nu - \frac{1}{2} \mu, w \right] &= 0, \quad w(\cdot, 0) = 0, \text{ for type (II).} \end{aligned}$$

Such problems have been actively studied in the literature. In fact, this connection between the parabolic obstacle problems and the supercooled Stefan problem has been used in [26] to introduce a notion of weak solution solely based on (4.6) for a regularized version of (St_1) . However, even ignoring the regularization, such a notion of weak solution has its limitations due to a lack of sufficient regularity of the w -variable to track the problem back to (St_1) . We refer to [13] and [42] for available results on regularity and singularity for solutions of (4.6). We point out that the low regularity of w for type (I), thus (St_1) , is related to the fact that $\partial_t w$ is nonpositive; see (4.4). When $\partial_t w$ is nonnegative, which corresponds to the costs of type (II) and thus (St_2) , much stronger regularity results holds for the parabolic obstacle problem: see, for example, [30].

5. Consistency with the Stefan problem. Let (η, ρ) be the Eulerian variables associated with (μ, ν) , given in Definition 4.20. In this section we will show that η solves the Stefan problem, with initial distribution μ and weight ν , and vice versa. This connection has been indicated in [35] with formal analysis.

Let us define weak solutions of the (weighted) Stefan problems $(St_1)_\nu$ and $(St_2)_\nu$ with initial density η_0 and initial domain E :

$$(5.1) \quad \begin{aligned} (\eta \mp \nu \chi_{\{\eta > 0\}})_t - \frac{1}{2} \Delta \eta &= 0, \quad \eta(\cdot, 0) = \eta_0 \in L^1(\mathbb{R}^d), \\ E &:= \limsup_{t \rightarrow 0^+} \{\eta(\cdot, t) > 0\}. \end{aligned}$$

DEFINITION 5.1. A nonnegative function $\eta \in L^1(\mathbb{R}^d \times [0, \infty))$ is a weak solution of $(St_1)_\nu$ (or $(St_2)_\nu$) with initial data (η_0, E) with $\eta_0 \in L^1(\mathbb{R}^d)$ if

- (a) the set $\{\eta(\cdot, t) > 0\}$ decreases (or increases) in t ;
- (b) $E = \limsup_{t \rightarrow 0^+} \{\eta(\cdot, t) > 0\}$;
- (c) for any test function $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, \infty))$,

$$(5.2) \quad \begin{aligned} \int_0^\infty \int_{\mathbb{R}^d} \left[(\eta - (\text{or } +) \nu \chi_{\{\eta > 0\}}) \varphi_t + \frac{1}{2} \eta \Delta \varphi \right] dx dt \\ = \int_{\mathbb{R}^d} [(\eta_0 + (\text{or } -) \nu \chi_E) \varphi](x, 0) dx. \end{aligned}$$

We say that η is a weak solutions of (St_i) if it is for $(St_i)_\nu$ with $\nu = 1$, for $i = 1, 2$.

REMARK 5.2. Let us point out that $(St_i)_v$, $i = 1, 2$ are characterized by only the value of v restricted on the trace of the interface (free boundary) within times in $(0, \infty)$. In other words, a weak solution η of $(St_i)_{v_1}$ is also a solution of $(St_i)_{v_2}$, $i = 1, 2$, if $v_1 = v_2$ in the set $\{x : 0 < s(x) < \infty\}$ where $s(x) := \sup\{t > 0 : \eta(x, t) > 0\}$. To see this, note that

$$\int_0^\infty \int v \chi_{\{\eta>0\}} \varphi_t dx dt = \int_0^\infty v(x) \int_0^{s(x)} \varphi_t dt dx = \int_0^\infty v(x) (\varphi(x, s(x)) - \varphi(x, 0)) dt dx.$$

So the only value of v that matters is at x with $0 < s(x) < \infty$.

REMARK 5.3. Note that the weak solution for $(St_1)_v$ or $(St_2)_v$ requires specifying not only the initial data η_0 but the initial domain E to be solved as an initial-value problem. With this information we can find a unique solution of (St_2) by comparison principle, for instance see [2]. On the other hand, even with specified η_0 and E , (St_1) can exhibit a high degree of nonuniqueness, as we will see in Section 6.1.

THEOREM 5.4. *Let μ, v be compactly supported and satisfy (3.1). Let (η, ρ) be the associated Eulerian flow given in Definition 4.20 for either (I) or (II). If the optimal stopping time τ associated with (μ, v) is strictly positive, the following hold:*

- (a) *For (I), η is a weak solution $(St_1)_v$ with initial data (μ, E) , where E is a set containing the support of v .*
- (b) *For (II), η is a weak solution $(St_2)_v$ with initial data (μ, E) where E contains the support of μ .*

PROOF. Let us remark that in the type (II) case, optimal stopping time being positive is equivalent to $\mu \wedge v = 0$ as seen in Theorem 2.1.

Let $R = R^U$ and $s(x)$ be the barrier set and function associated with (μ, v) for either the cost of type (I) and (II) given in Definition 4.19; these are associated with the optimal stopping time (c.f. Lemma 4.17). The following holds from Proposition 3.10 and Theorem 4.15:

- (i) ρ is supported on $\{t = s(x)\}$;
- (ii) $\{\eta > 0\} = R^C = \{0 < t < s(x)\}$ (or $\{t > s(x)\}$) for (I) (or for (II)).

For the next set of computations we focus on (I). Let $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, \infty))$. Recall that (η, ρ) then satisfies (2.3), namely

$$\int_0^\infty \int \eta \left(\varphi_t + \frac{1}{2} \Delta \varphi \right) dx dt = \int_0^\infty \int \rho \varphi dx dt + \int \mu \varphi(x, 0) dx.$$

We would like to see that η satisfies the weak equation for $(St_1)_v$. To this end observe that

$$\int_0^\infty \int \rho \varphi dx dt = \int \left(\int_0^\infty \rho(x, t) \varphi(x, t) dt \right) dx = \int_{\{s(x) > 0\}} v(x) \varphi(x, s(x)) dx,$$

where the second equality comes from (i) and the fact that $\int_0^\infty \rho(x, t) dt = v(x)$ a.e. x , and the fact that the optimal stopping time is strictly positive so the set $\{s(x) > 0\}$ contains the set $\{v(x) > 0\}$.

Next, from (ii) we have $\{\eta > 0\} = \{t < s(x)\}$ and $E = \limsup_{t \rightarrow 0^+} \{\eta(\cdot, t) > 0\} = \{s(x) > 0\}$. Thus by Fubini's theorem

$$\begin{aligned} \int_{\{s(x) > 0\}} v(x) \varphi(x, s(x)) dx &= \int v(x) \int_0^{s(x)} \varphi_t(x, t) dt dx + \int_{\{s(x) > 0\}} v(x) \varphi(x, 0) dx \\ &= \int_{\{\eta > 0\}} v(x) \varphi_t(x, t) dt dx + \int v(x) \varphi(x, 0) \chi_E(x) dx. \end{aligned}$$

Hence (2.3) can be written as

$$\begin{aligned} \int_0^\infty \int \eta \left(\varphi_t + \frac{1}{2} \Delta \varphi \right) dx dt &= \int v(x) \chi_{\{\eta > 0\}} \varphi_t(x, t) dx dt \\ &\quad + \int [\mu(x) + v(x)] \chi_E(x) \varphi(x, 0) dx. \end{aligned}$$

Hence η is a weak solution $\eta \in L^1(\mathbb{R}^d \times [0, \infty))$ of $(St_1)_v$, with initial data $\eta_0 = \mu$ and the initial trace $E = \{s(x) > 0\}$; notice that as $R = R^U$, this set is an open set. Note that E contains the set $\{v(x) > 0\}$ and as it is an open set it thus contains the support of v .

For (II), since $\{\eta(\cdot, t) > 0\} = \{s(x) < t\}$, it follows from our assumption $\tau > 0$ and (i), that $v = 0$ in E ; also, $E = \lim_{t \rightarrow 0^+} \{s(x) < t\} = \{s(x) = 0\}$. Moreover, since $\tau > 0$ almost surely, for μ -a.e. x , it holds that $s(x) > 0$, therefore, the set E contains the set $\{\mu(x) > 0\}$. As E is a closed set, it contains the support of μ . Now, by Fubini's theorem

$$(5.3) \quad \int v(x) \varphi(x, s(x)) dx = - \int v(x) \int_{s(x)}^\infty \varphi_t(x, t) dt dx = - \int_{\{\eta > 0\}} v(x) \varphi_t(x, t) dt dx.$$

Hence (2.3) for case (II) can then be written as

$$\int_0^\infty \int \eta \left(\varphi_t + \frac{1}{2} \Delta \varphi \right) dx dt = - \int_{\{\eta > 0\}} v(x) \varphi_t(x, t) dx dt + \int \mu(x) \varphi(x, 0) dx,$$

which is the weak expression for $(St_2)_v$ with initial data μ and $E \subset \{v = 0\}$. \square

REMARK 5.5. In general the theorem will hold with the revised initial data $\mu - \mu_0$ and revised target measure $v - \mu_0$, where μ_0 is the portion of μ with $\tau = 0$, namely

$$\mu_0(x) := \text{Prob}[\tau = 0 | W_0 = x] \mu(x).$$

Next we consider the reverse direction, starting with $(St_1)_v$.

THEOREM 5.6. *Let μ, v satisfy (3.1) and compactly supported. Suppose η is a weak solution of $(St_1)_v$ with initial data (μ, E) , where E is bounded. Let us define s and ρ by*

$$(5.4) \quad \begin{aligned} s(x) &:= \sup\{t > 0 : \eta(x, t) > 0\} \quad \text{and} \\ \int \int \rho(x) \varphi(x, t) dx dt &= \int v(x) \varphi(x, s(x)) dx \end{aligned}$$

for any test function $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, \infty))$. Then $\mu \leq_{\text{SH}} \tilde{v} := v \chi_{\{s(x) < \infty\}}$. Moreover (η, ρ) is the Eulerian variables between $\eta_0 = \mu$ and \tilde{v} , generated by the optimal stopping time with costs of type (I).

PROOF. By definition of $s(x)$, $\{\eta > 0\} = \{t < s(x)\}$. This allows us to write, for instance in the case of (St_1) ,

$$\begin{aligned} \int v(x) \varphi(s(x), x) dx - \int v(x) \varphi(0, x) dx &= \int v(x) \int_0^{s(x)} \varphi_t(t, x) dt dx \\ &= \int \int_{\{\eta > 0\}} v(x) \varphi_t(t, x) dt dx. \end{aligned}$$

Arguing as in the proof of Theorem 5.4, we can then verify that (η, ρ) satisfies the equation (2.3). Moreover, we have $\eta(R) = 0$ for the barrier set $R := \{(x, t) : t \geq s(x)\}$ by the definition

of $s(x)$. Observe also that

$$\rho(R) = \int \rho(x, t) dx dt = \int \eta(x, 0) dx = \mu(\mathbb{R}^d) = 1,$$

where the first equality holds since ρ is supported in R , and the second is due to the mass preserving property of the heat equation.

Let us define $\xi := \int_t^\infty \eta(x, t) dt$. Integrating in time the weak equation for $(St_1)_v$, we see that ξ solves the parabolic obstacle problem

$$\xi_t - \frac{1}{2} \Delta \xi = -\frac{1}{2} v \chi_{\{\xi > 0\}}.$$

On the other hand, since we have $\eta_t - \Delta \eta = -\rho$ in the distribution sense, and since η vanishes as time tends to infinity, it follows that

$$\mu = \eta(\cdot, 0) = \int_0^\infty (-\eta_t) = \int_0^\infty (-\Delta \eta + \rho) = -\Delta \left(\int_0^\infty \eta \right) + \tilde{v}.$$

Hence $\mu - \tilde{v} = -\Delta \xi(\cdot, 0)$. Thus, by uniqueness of the parabolic obstacle problem,

$$\xi(x, t) = w(x, t) = \int_t^\infty \tilde{\eta}(x, s) ds,$$

where $\tilde{\eta}$ is the Eulerian variable generated by the optimal stopping time between μ and \tilde{v} . In particular $\{\tilde{\eta} > 0\} = \{\eta > 0\}$.

To conclude, note that $\eta \in L^2([0, \infty), H_0^1(\mathbb{R}^n))$, due to the fact that η is a subsolution of the heat equation with compactly supported initial data. Hence taking their zero set as R (where clearly both ρ and ρ_0 are supported), we can apply Lemma 2.3 to conclude that $\eta = \tilde{\eta}$. \square

We finish this section with the corresponding statement for $(St_2)_v$. The proof is parallel to $(St_1)_v$.

THEOREM 5.7. *Let μ, v satisfy (3.1) and compactly supported, and let η be a weak solution of $(St_2)_v$ and initial data (μ, E) , where E is bounded. Let us define s and ρ by*

$$s(x) := \inf\{t > 0 : \eta(x, t) > 0\}$$

and (5.4). Then (η, ρ) is the Eulerian variables between μ and v generated by the optimal stopping time with costs of type (II), where

$$\mu = \eta_0 \quad \text{and} \quad \tilde{v} = v \chi_{E^c}.$$

In the next section, we will discuss a specific class of the target measures, from which solutions to the classical Stefan problems (St_1) and (St_2) are generated.

6. Subharmonically generated sets. This section defines a central notion of the present paper, which characterizes a pair of sets by existence of a certain stopping time. This notion then is connected to solvability of the supercooled Stefan problem (St_1) .

DEFINITION 6.1. We say the pair (Σ, E) is *subharmonically generated* by μ , if there is a stopping time τ with the corresponding Eulerian flow (η, ρ) (2.3), such that $\tau > 0$ almost surely, $W_0 \sim \mu$ and $W_\tau \sim v = \chi_\Sigma$ and $E = \{x | \eta(x, t) > 0 \text{ for some } t > 0\}$.

This definition is motivated by the following equivalence result.

THEOREM 6.2. *Consider a compactly supported measure $\mu \ll \text{Leb}$ on \mathbb{R}^d . Then for a given bounded set E , the following are equivalent:*

- (a) *There exists a weak solution η of (St_1) with initial data (μ, E) .*
- (b) *There exists a measurable set Σ such that (Σ, E) is subharmonically generated by μ .*

Moreover, $\Sigma = \{z(x) < \infty\}$ with $z(x) := \sup\{t > 0 : \eta(x, t) > 0\}$.

PROOF. If (a) holds, then (b) follows from Theorem 5.6, with $\Sigma := \{s(x) < \infty\}$ for $s(x) := \sup\{t > 0 : \eta(x, t) > 0\}$. Let us remark that, since the set $\{\eta(\cdot, t) > 0\}$ decreases as t increases, we have

$$\limsup_{t \rightarrow 0^+} \{\eta(\cdot, t) > 0\} = \bigcup_{t > 0} \{\eta(\cdot, t) > 0\} = E.$$

Similarly, if (b) holds, then (a) follows from Theorem 5.4. \square

We mention that obtaining ν as a characteristic function χ_Σ does not guarantee a corresponding solution for (St_1) unless the stopping time is strictly positive. Even with the stopping time strictly positive, for a given μ and E there can be many Σ such that (Σ, E) is subharmonically generated by μ . Below we will discuss an example that illustrates these points. In Section 7 we will present results concerning the optimization problem (1.1) for the target measure ν . Together with the results in Section 8 this optimization problem uniquely yields a subharmonically generated pair (E, E) for each given μ ; see Theorem 9.7. In Section 9, we will see that our optimal target scheme provides a mechanism to construct solutions of (St_1) in a stable manner, by choosing an initial domain that is strictly larger than the support of the initial data and yields an instant regularizing effect for the evolution, at the initial time.

6.1. Example for nonuniqueness for (St_1) . The weak solution of (St_1) is known to have nonuniqueness with given initial data. Here we give an example that yields infinitely many weak solutions to (St_1) with the same initial data (μ, E) : these are solutions that do not vanish in finite time.

PROPOSITION 6.3. *Let $\mu := \frac{1}{|B_\epsilon|} \chi_{B_\epsilon}$ for some $0 < \epsilon < 1/2$ and $E = \{|x| \leq 1\}$. Then for a sufficiently small ϵ , there are infinitely many weak solutions to (St_1) with initial data (μ, E) .*

PROOF. Consider the set $\Sigma := A_1 \cup A_2 \subset \mathbb{R}^d$ consisting of the union of two annuli,

$$A_1 := \{2\epsilon \leq |x| \leq r\}, \quad A_2 := \{r' \leq |x| \leq 1\} \quad \text{where } 0 < 2\epsilon < r < r' < 1.$$

Assume $|\Sigma| = 1$ by choosing appropriate ϵ, r , and r' . Let $\nu = \chi_\Sigma$. We will prove

Claim: there exists a randomized stopping time τ , with $W_0 \sim \mu$ and $W_\tau \sim \nu$.

After verifying this claim, we can find the optimal stopping time τ^* for the optimal Skorokhod problem $\mathcal{P}(\mu, \nu)$ (1.5). Then, the corresponding Eulerian flow (η, ρ) as given in Proposition 2.2 gives a solution to $(St_1)_\nu$ via Theorem 5.4, with the initial set

$$E = \limsup_{t \rightarrow 0^+} \{\eta(\cdot, t) > 0\} = \{|x| \leq 1\}.$$

This is because, to reach the outer annulus A_2 , the Eulerian variable η should be active in the whole set $\{|x| \leq 1\}$. From Remark 5.2, the solution we constructed here for $(St_1)_\nu$, for $\nu = \chi_\Sigma$, actually solves the original supercooled Stefan (St_1) . Hence, by choosing different combinations of r, r' , but still making $|\Sigma| = 1$, we can generate infinitely many solutions

to (St_1) with the above (μ, E) . In this case, $\{s(x) = \infty\} = \{|x| \leq 2\epsilon\} \cup \{r \leq |x| \leq r'\}$, the outside part of the two annuli in $E = \{|x| \leq 1\}$.

It remains to prove the claim. We first show that there exists a randomized stopping time τ' with $W_0 \sim \delta_{\{x=0\}}$, such that $W_{\tau'} \sim \nu$. To construct such stopping time, let τ_a be the first hitting time to $S(a) := \{|x| = a\}$. Then its distribution ν_a is a uniform $d-1$ -dimensional measure along the set $S(a)$, that is, $\nu_a = C_a \delta_{S(a)}$ with some constant $C_a > 0$. Thus we can randomize at $t = 0$ to find a stopping time τ_A with $W_0 \sim \delta_{\{x=0\}}$ whose distribution ν_A is given by $\nu_A = \int_{r_1}^{r_2} \nu_a f(a) da$ for some weight $f(a)$ that can be controlled by the randomization at $t = 0$. In particular, we can find such a randomization at $t = 0$ so that ν_A becomes a uniform measure on the annulus $A = \{r_1 \leq |x| \leq r_2\}$. Now, for such τ_{A_1} and τ_{A_2} we can consider the randomized stopping time τ' given as

$$\tau' = \begin{cases} \tau_{A_1} & \text{with probability } \frac{|A_1|}{|A_1| + |A_2|}, \\ \tau_{A_2} & \text{with probability } \frac{|A_2|}{|A_1| + |A_2|}. \end{cases}$$

Then it follows that $W_{\tau'} \sim \nu$. Moreover, by the Markov property of the Brownian motion, we can change the randomization at $x = 0$ to a randomization at $S(\epsilon)$, that is, when the Brownian motion from the origin first hits $S(\epsilon)$ we make a probabilistic choice how to move from there.

One can find a randomized stopping time τ'' as in the above, with $W_0 \sim \delta_{\{x=0\}}$ and $W_{\tau''} \sim \frac{1}{|B_\epsilon|} \chi_{B_\epsilon}$ for the ball B_ϵ . Note that $\tau' > \tau_\epsilon$ almost surely. Clearly $\tau'' \leq \tau_\epsilon$, so $\tau'' \leq \tau'$. This means that for the measure $\mu = \frac{1}{|B_\epsilon|} \chi_{B_\epsilon}$ there exists a randomized stopping time τ with $W_0 \sim \mu$ and $W_\tau \sim \nu$, verifying the claim. \square

7. Optimal target problem. We now consider the optimal target problem (1.1) proposed in our introduction. Namely we solve for the optimal target $\bar{\nu}$ minimizing the cost function $\mathcal{C}(\tau)$ under the upper density constraint

$$(7.1) \quad \nu \leq f, \text{ where a given function } f \text{ satisfies } 0 \leq f \in L^\infty(\mathbb{R}^d) \text{ and } |\{f > 0\}| > 0.$$

One can view this as a projection problem in the space of probability measures: Given $\mu \in P(\mathbb{R}^d)$, what is the closest measure ν in the constraint set $\{\nu \leq f\}$, under the subharmonic order condition $\mu \leq_{SH} \nu$? Here, the closeness between the two measures is measured by the cost $C(\tau)$ for an optimal stopping time τ between them. Once we have the optimal target ν then we can apply Theorem 2.1, obtaining the optimal stopping time τ that is given by the hitting time to a barrier given by a barrier function s .

Let us point out that it is not obvious how to explicitly construct the optimal target even in simple cases. For instance when $f \equiv 1$ and $\mu = \delta_{x=0}$, one may guess that the optimal stopping time τ is given by the constant time, when the heat kernel $K(t, y)$ becomes $K \leq 1$. This is not the case as we see below, for the either cost type (I) or (II).

For the rest of the paper we will see that the optimal target comes with many interesting characteristics, based on the two main features: monotonicity and saturation property. These two properties allow us to connect the optimal target problem (1.1) with global solvability of the Stefan problem; see Section 9.

We begin our discussion with the monotonicity property.

7.1. Monotonicity. The following theorem demonstrates an ordering principle for optimal targets of Problem (1.1) and the corresponding optimal stopping times.

THEOREM 7.1 (monotonicity). *Let f is as given in (7.1), and let the compactly supported measures (μ_i, v_i) , $i = 1, 2$ satisfy (3.1) (except that they are not necessarily probability measures) with $v_i \leq f$, as well as $\mu_i \leq_{\text{SH}} v_i$. Last, let τ_i be the optimal stopping times with the initial and target distributions μ_i and v_i , as given in Theorem 2.1, with the barrier functions s_i . If $\mu_1 \leq \mu_2$, then the following hold:*

(1) *Suppose that v_1 is a solution to the optimal free target problem (1.1) for μ_1 . Then, as the stopping time from μ_1 , we have*

$$\tau_1 \leq \tau_2 \quad \text{almost surely.}$$

(Notice that from $\mu_1 \leq \mu_2$, the stopping time τ_2 from μ_2 can be restricted to the initial distribution μ_1 .) Here, we do not necessarily assume that v_2 is a solution to (1.1).

(2) *Suppose that both v_i solve the optimal free target problem (1.1) for μ_i . Then*

$$v_1 \leq v_2.$$

In particular there is at most one solution to (1.1).

REMARK 7.2. As mentioned above, for (1) we only need τ_1 to be optimal for (1.1), not τ_2 . This in particular characterizes τ_1 as the smallest stopping time of barrier type that starts from μ_1 among eligible target distributions with the constraint f .

REMARK 7.3. Note that $\tau_1 \leq \tau_2$ should be understood to hold for those Brownian paths starting from the initial points x distributed as the common mass $\mu_1 = \mu_1 \wedge \mu_2$ for $\mu_1 \leq \mu_2$. Due to the Markov property of Brownian motion, the stopping time τ_1 can be applied to the initial mass of μ_2 in the region where $\mu_1 > 0$. From the region $\mu_1 = 0$, it should be understood that the stopping time $\tau_1 = 0$ because, from where $\mu_1 = 0$ there is no motion for τ_1 . In this sense, the inequality $\tau_1 \leq \tau_2$ can be applied to all Brownian paths from the initial distribution μ_2 not only from μ_1 .

PROOF OF THEOREM 7.1. To show (1), we first prove that $\tau_1 \leq \tau_2$, when τ_2 is restricted to the initial distribution μ_1 . For this it only needs optimality of v_1 .

Type (I) case. In this case τ_i , $i = 1, 2$, are the first hitting times to the set

$$R_i := \{(x, t) | t \geq s_i(x)\}; \quad \tau_i = \inf\{t > 0 | (W_t, t) \in R_i\}.$$

Let $\bar{\tau} := \tau_1 \wedge \tau_2$. Recall that from Proposition 3.10 we have $\tau_i = s_i(W_{\tau_i})$ almost surely, $i = 1, 2$. Therefore $\bar{\tau} = \min[s_1(W_{\bar{\tau}}), s_2(W_{\bar{\tau}})]$ almost surely. Now, for a Brownian path, if $W_{\bar{\tau}} \in \{x | s_1(x) \leq s_2(x)\}$ then $\bar{\tau} \geq \tau_1$ so $\bar{\tau} = \tau_1$. Similarly, if $W_{\bar{\tau}} \in \{x | s_2(x) \leq s_1(x)\}$ then $\bar{\tau} = \tau_2$.

Therefore, \bar{v} denoting the distribution of $\bar{\tau}$, we have

$$\bar{v}|_{\{s_1 \leq s_2\}} \leq v_1|_{\{s_1 \leq s_2\}} \leq f \quad \text{and} \quad \bar{v}|_{\{s_2 \leq s_1\}} \leq v_2|_{\{s_2 \leq s_1\}} \leq f;$$

so $\bar{v} \leq f$ and $\bar{\tau}$ is admissible.

Now recall that τ_1 is the minimizer for \mathcal{C} , among admissible stopping times for (1.1), which implies $\bar{\tau} = \tau_1$ almost surely, as otherwise if $\text{Prob}[\bar{\tau} < \tau_1] > 0$ then from the strict monotonicity of \mathcal{C} , we see $\mathcal{C}(\bar{\tau}) < \mathcal{C}(\tau_1)$, a contradiction.

This immediately implies that $\tau_1 \leq \tau_2$, where we understand τ_2 as the stopping time restricted to the initial distribution μ_1 .

Type (II) case. In this case, from Theorem 2.1 τ_i , $i = 1, 2$, are randomized at the initial time to give the distribution $v_i \wedge \mu_i$, $i = 1, 2$, and the rest of the Brownian particles stop when they first hit the set $R_i := \{(x, t) | t \leq s_i(x)\}$. This gives the corresponding initial and final distribution $\tilde{\mu}_i = \mu_i - v_i \wedge \mu_i$ $\tilde{v}_i = v_i - v_i \wedge \mu_i$ for those Brownian paths with $\tau_i > 0$. Observe that $\tilde{\mu}_i \wedge \tilde{v}_i = 0$. Since τ_i , v_i 's are optimal to (1.1), it follows that \tilde{v}_i is the optimal

solution to (1.1) for the initial $\tilde{\mu}_i$ and upper bound $\tilde{f}_i = f - \nu_i \wedge \mu_i$. If $\tilde{\mu}_i \wedge \tilde{f}_i \neq 0$, then it is possible for τ_i to stop more mass initially (than $\nu_i \wedge \mu_i$, so increasing $\tilde{\mu}_i$ but decreasing $\tilde{\nu}_i$) still satisfying the upper-bound constraint of f . The resulting stopping time will have strictly less cost than τ_i due to strict monotonicity of \mathcal{C} . This contradicts optimality of τ_i , and as a result, we see that $\tilde{\mu}_i \wedge \tilde{f}_i = 0$, namely, $\tilde{\mu}_i = \mu_i - f \wedge \mu_i$ and $\tilde{f}_i = f - f \wedge \mu_i$, and it follows that

$$(7.2) \quad \nu_i \wedge \mu_i = f \wedge \mu_i.$$

Therefore, $\tilde{\nu}_i = \nu_i - f \wedge \mu_i$, thus $\tilde{\nu}_i \leq f - f \wedge \mu_i$, $i = 1, 2$. Notice that $f - f \wedge \mu_2 \leq f - f \wedge \mu_1$ due to $\mu_1 \leq \mu_2$.

We now define a randomized stopping time $\bar{\tau}$ as follows: For the portion $\mu_1 \wedge f$ stop immediately, and for the rest $\mu_1 - \mu_1 \wedge f$, follow the Brownian motion until $\tau_1 \wedge \tau_2$. Notice that $\bar{\tau} \leq \tau_1$ from its construction. The rest of the proof is similar to the (I) case with additional consideration for stopping at the initial time. Details follow.

We show that $\bar{\tau}$ is admissible, namely, if we let \bar{v} be its distribution (with the initial distribution μ_1), $W_{\bar{\tau}} \sim \bar{v}$, then $\bar{v} \leq f$. To see this, first write $\bar{v} = \mu_1 \wedge f + \bar{v}^1$ where \bar{v}^1 is the distribution of the rest. For Brownian particles with positive time $\bar{\tau} > 0$, that is, those accounting for \bar{v}^1 , if $W_{\bar{\tau}} \in \{x | s_1(x) \geq s_2(x)\}$ then $\bar{\tau} \geq \tau_1$ so $\bar{\tau} = \tau_1$. Similarly, if $W_{\bar{\tau}} \in \{x | s_2(x) \geq s_1(x)\}$ then $\bar{\tau} \geq \tau_2$ so $\bar{\tau} = \tau_2$. Therefore,

$$\bar{v}^1|_{\{s_1 \geq s_2\}} \leq (\nu_1 - f \wedge \mu_1)|_{\{s_1 \geq s_2\}} \leq (f - f \wedge \mu_1)|_{\{s_2 \geq s_1\}}$$

as well as

$$\bar{v}^1|_{\{s_2 \geq s_1\}} \leq (\nu_2 - f \wedge \mu_2)|_{\{s_2 \geq s_1\}} \leq (f - f \wedge \mu_2)|_{\{s_2 \geq s_1\}} \leq (f - f \wedge \mu_1)|_{\{s_2 \geq s_1\}}.$$

From these, we have

$$\bar{v} = \mu_1 \wedge f + \bar{v}^1 \leq \mu_1 \wedge f + (f - f \wedge \mu_1) = f \quad \text{verifying admissibility of } \bar{\tau}.$$

Now recall that τ_1 is the minimizer for \mathcal{C} , which implies $\bar{\tau} = \tau_1$ almost surely, as otherwise if $\text{Prob}[\bar{\tau} < \tau] > 0$ then from the strict monotonicity of \mathcal{C} and from $\bar{\tau} \leq \tau_1$, we see $\mathcal{C}(\bar{\tau}) < \mathcal{C}(\tau_1)$ for admissible $\bar{\tau}$, a contradiction. This immediately implies that $\tau_1 \leq \tau_2$ as desired.

We now show (2): this requires optimality of both ν_1 and ν_2 .

Suppose not, that is, $\int_{\{\nu_1 > \nu_2\}} f(x) dx > 0$. For a small $\epsilon > 0$, define

$$E_\epsilon := \{x | \nu_1(x) > \nu_2(x) + 2\epsilon \nu_1(x)\}$$

and $\mu_2^\epsilon := \mu_2 - \epsilon \mu_1 \geq 0$. Define an auxiliary randomized stopping time $\tilde{\tau}$, from the initial distribution μ_2 as follows:

$$\tilde{\tau} := \begin{cases} \tau_2 & \text{from } \mu_2^\epsilon, \\ \tau_1 & \text{if } W_{\tau_1} \in E_\epsilon \text{ from } \epsilon \mu_1, \\ \tau_2 & \text{if } W_{\tau_1} \notin E_\epsilon \text{ from } \epsilon \mu_1. \end{cases}$$

Let us explain the meaning of this. Here the wording “from μ_2^ϵ ” or “from $\epsilon \mu_1$ ” should be understood as that at the initial time the Brownian particles belong to the distribution μ_2^ϵ or $\epsilon \mu_1$, respectively. Notice that such a decomposition in the initial distribution is allowed for the randomized stopping time as it is the same as randomizing at the initial time. Notice that we already proved in (1) that $\tau_1 \leq \tau_2$ for Brownian motion starting from μ_1 , so the second and third lines in the definition of $\tilde{\tau}$ are well defined.

Now let us show that $\tilde{\tau}$ is admissible. Define the distribution \tilde{v} of $\tilde{\tau}$, that is, $W_{\tilde{\tau}} \sim \tilde{v}$. From the definition of $\tilde{\tau}$ we see that

$$\tilde{v}|_{E_\epsilon^c} = \text{the distribution of } W_{\tau_2} \text{ from } \mu_2^\epsilon \text{ and from } \epsilon \mu_1, \text{ restricted to the set } E_\epsilon^c.$$

Therefore $\tilde{\nu}|_{E_\epsilon^c} \leq \nu_2$.

On the other hand,

$$\begin{aligned}
 \tilde{\nu}|_{E_\epsilon} &\leq \text{the distribution of } W_{\tau_1} \text{ from } \epsilon\mu_1 \\
 &\quad + \text{the distribution of } W_{\tau_2} \text{ from } \mu_2 = \mu_2^\epsilon + \epsilon\mu_1, \text{ restricted to the set } E_\epsilon \\
 &\leq \epsilon\nu_1 + \nu_2 \quad \text{on } E_\epsilon \\
 &\leq \nu_1 \quad \text{on } E_\epsilon \text{ by the definition of } E_\epsilon \\
 &\leq f.
 \end{aligned}$$

We thus have verified that $\tilde{\nu} \leq f$, therefore, $\tilde{\tau}$ is admissible as the stopping time from the initial distribution μ_2 .

Now notice that by the construction $\tilde{\tau} \leq \tau_2$, where the latter is optimal. From the strict monotonicity of the cost C , we see that $\tilde{\tau} = \tau_2$.

To see why this leads to a contradiction, notice that it implies that for those Brownian particles starting from $\epsilon\mu_1$ with $W_{\tau_1} \in E_\epsilon$ we have $\tau_1 = \tilde{\tau} = \tau_2$. Since $\epsilon\nu_1$ is the final distribution for τ_1 starting from $\epsilon\mu_1$ we have

$$\begin{aligned}
 \epsilon\nu_1|_{E_\epsilon} &\leq \text{the distribution for } \tau_2 \text{ starting from } \epsilon\mu_1, \text{ restricted to } E_\epsilon. \\
 &\leq \text{the distribution for } \tau_2 \text{ starting from } \epsilon\mu_2, \text{ restricted to } E_\epsilon \text{ (because } \mu_1 \leq \mu_2\text{)} \\
 &= \epsilon\nu_2|_{E_\epsilon}.
 \end{aligned}$$

Note that this is a contradiction to the definition of E_ϵ , thus proves the claim that $\nu_1 \leq \nu_2$. \square

The rest of the section discusses important consequences of the monotonicity property.

7.2. Well-posedness.

THEOREM 7.4. *Let f be as given in (7.1), and let $\mu \ll \text{Leb}$ have bounded, compactly supported density. Suppose for a given $0 \leq f \in L^\infty(\mathbb{R}^d)$, the set*

$$\mathcal{A} := \{\nu : \mu \leq_{\text{SH}} \nu, \nu \text{ is compactly supported and } \nu \leq f\} \quad \text{is nonempty.}$$

Then (1.1) yields a unique optimal target measure $\nu^ = \nu(\mu, f, \mathcal{C})$, which depends only on μ , f , and the cost \mathcal{C} .*

PROOF. Once existence is established, uniqueness is a direct consequence of Theorem 7.1.

To show existence, let us first consider the case where f is compactly supported. In this case $\nu \in \mathcal{A}$ is compactly supported, and for any Brownian stopping time τ with $W_0 \sim \mu$, $W_\tau \sim \nu$, with finite $\mathbb{E}[\tau]$, we see that for any t , the distribution ν_t of $W_{t \wedge \tau}$, that is, $W_{t \wedge \tau} \sim \nu_t$, has support inside the convex hull of the support of ν . This follows because $\tau \wedge t \leq \tau$ so $\nu_t \leq_{\text{SH}} \nu$, and for the convex order the smaller measure has support inside the convex hull of the support of the bigger (in convex order) measure. This implies that the relevant domain both for the measures $\nu \in \mathcal{A}$ and the Brownian motion (with stopping time) between μ and ν is compact, in particular, it is a subset of the convex hull of $\text{supp } f$. Therefore existence of a minimizer ν follows easily from the weak-* compactness of the minimizing sequence, since that the condition $\nu \leq f$ is closed with the weak-* convergence.

Now, consider the general case, and let $f_R = f \wedge 1_{B_R}$. Then, for $R \gg 1$, the corresponding admissibility set \mathcal{A}_R with f_R is nonempty. (It is because we assumed \mathcal{A} is nonempty, so there

exists a compactly supported admissible target $v \in \mathcal{A}$.) Therefore, for the initial measure μ , and for each $1 \ll R < R'$, applying the compactly supported case, we find $v_R, v_{R'}$ to be the optimal solutions with the constraint f_R and $f_{R'}$, respectively. Let $\tau_R, \tau_{R'}$ be the corresponding optimal stopping times. Notice that v_R also satisfies the density constraint $f_{R'} (\geq f_R)$. We can then apply the monotonicity, Theorem 7.1(1), and get $\tau_{R'} \leq \tau_R$. It implies that the support of $v_{R'}$ is contained in the convex hull of the support of v_R , in particular, in the ball B_R . In the ball B_R we have $f_R = f_{R'}$, so, $v_{R'}$ should be an optimal solution for the constraint f_R , because for the cost we have $\mathcal{C}(\tau_{R'}) \leq \mathcal{C}(\tau_R)$. From uniqueness of optimal solution in the compactly supported constraint case, we have $v_R = v_{R'}$. This implies that the optimal v_R is independent of R as long as R is sufficiently large. That $v_R, R \gg 1$, is the optimal target for f . To see this notice that for any compactly supported target measure $v' \leq f$, there exists $R > 0$ such that $\text{supp } v' \subset B_R$, so $v' \leq f_R$ and the cost for v_R is less than or equal to that of v' . \square

REMARK 7.5. We note that those f with $\mathcal{A} \neq \emptyset$ are plenty. For example, $f \equiv 1$ on \mathbb{R}^d , or any $f \geq 0$ which has a positive lower bound on a ball B_R , $R \gg 1$, will work. Or any $f \geq 0$ that has a positive lower bound in the annulus $\{R_1 \leq |x| \leq R_2\}$, with $1 \ll R_2 < R_2 \leq \infty$ works. Of course, here how large R, R_1, R_2 are depends on μ . These are some particular types of f we focus on in this paper, especially in Section 9.

7.3. Universality of the optimal target. We prove in this section a remarkable consequence of the monotonicity property, namely, that for a given initial measure μ and the constraint f , the optimal target v is unique independent of the choice of cost function \mathcal{C} , as long as it is either type (I) or (II). For the same type of costs we have Lemma 4.17, which shows a universality of optimal stopping time for a fixed target. What is surprising here is that the universality holds even across the different types for which the corresponding optimal stopping times behave very differently. For its proof we exploit the fact that Theorem 7.1(1) requires optimality only for one of the targets. This universality result will have interesting consequences later in the context of Stefan problem (see Section 9).

THEOREM 7.6 (Universality). *Let μ and f be as given in Theorem 7.4. Let us consider the optimal solutions of (1.1), $v_i := v(\mu, f, \mathcal{C}_i)$, $i = 1, 2$, for the cost functions \mathcal{C}_i of either type (I) or (II), same or different types. Then $v_1 = v_2$.*

PROOF. Let τ_i , $i = 1, 2$ are the corresponding optimal stopping times, for \mathcal{C}_i , $i = 1, 2$, respectively. For the cost \mathcal{C}_1 consider the optimal stopping time τ'_2 for the given target v_2 . Theorem 7.1(1) and optimality of τ_1 now yields $\tau_1 \leq \tau'_2$. This implies that $v_1 \leq_{\text{SH}} v_2$. Similarly, for the cost \mathcal{C}_2 , consider the optimal stopping time τ'_1 for the given target v_1 . Theorem 7.1(1) and optimality of τ_2 now yields $\tau_2 \leq \tau'_1$. This implies that $v_2 \leq_{\text{SH}} v_1$. The two subharmonic orders imply $v_2 = v_1$, completing the proof. \square

REMARK 7.7. In fact, what we have shown in Theorem 7.6 using the monotonicity in Theorem 7.1(1), is the following: For our solution $v^* = v(\mu, f, \mathcal{C})$ of Problem (1.1) (for any cost \mathcal{C} of either type (I) or (II)), for a measure \tilde{v} ,

$$\text{if } \tilde{v} \leq f \text{ and } \mu \leq_{\text{SH}} \tilde{v} \text{ then } v^* \leq_{\text{SH}} \tilde{v}.$$

So, our solution $v^* = v(\mu, f, \mathcal{C})$ coincides with the notion of the shadow $S^f(\mu)$ of the measure μ in f as introduced in [10, 52] (see [10], Lemma 4.6); such a notion was first considered in [52] for general cases, and was named *shadow* in [10] in the one-dimensional case. It is then further considered in a recent paper [12]. We thank the anonymous referee for drawing our attention to this connection.

7.4. *L^1 contraction and BV estimate.* Next we show the L^1 contraction, which is a consequence of the monotonicity and the fact that the total mass of μ is the same as that of ν .

THEOREM 7.8 (L^1 -Contraction). *Let μ_i , $i = 1, 2$, and f be as given in Theorem 7.4, and consider the optimal solutions of (1.1), $\nu_i := \nu(\mu_i, f, \mathcal{C})$ with a cost \mathcal{C} of either type (I) or (II). Then*

$$\|(\nu_1 - \nu_2)_+\|_{L^1} \leq \|(\mu_1 - \mu_2)_+\|_{L^1}.$$

PROOF. Let $\tilde{\mu} = \mu_1 \wedge \mu_2$; notice that $(\mu_1 - \mu_2)_+ = \mu_1 - \tilde{\mu}$. Let $\tilde{\nu}$ be an optimal solution of the corresponding problem (1.1) with the initial distribution $\tilde{\mu}$. As $\tilde{\mu} \leq \mu_i$, $i = 1, 2$, we have from the monotonicity (Theorem 7.1) that

$$\tilde{\nu} \leq \nu_i, \quad i = 1, 2.$$

Now, let $E_+ = \{x | \nu_1(x) - \nu_2(x) \geq 0\}$. Then,

$$(\nu_1 - \nu_2)_+ = (\nu_1 - \nu_2)\chi_{E_+} \leq (\nu_1 - \tilde{\nu})\chi_{E_+} \leq \nu_1 - \tilde{\nu}.$$

Therefore,

$$\|(\nu_1 - \nu_2)_+\|_{L^1} \leq \|\nu_1 - \tilde{\nu}\|_{L^1} = \|\mu_1 - \tilde{\mu}\|_{L^1} = \|(\mu_1 - \mu_2)_+\|_{L^1}$$

as desired. Notice that the first equality follows from the fact that the problem preserves the total mass, that is, $\|\mu_1\|_{L^1} = \|\nu_1\|_{L^1}$, $\|\tilde{\mu}\|_{L^1} = \|\tilde{\nu}\|_{L^1}$. \square

The BV estimate follows as an immediate corollary of the L^1 contraction: such consequence is well-known for problems that are translation invariant; it is the case for (1.1) if $f \equiv 1$. For (St_2) that corresponds to type (II), such a result is shown by Meirmanov [45].

THEOREM 7.9 (BV estimate). *Let ν be the solution of (1.1) as given in Theorem 7.4 with μ and $f \equiv 1$. Then,*

$$\|\nu\|_{BV} \leq \|\mu\|_{BV}.$$

PROOF. Let us first consider the case $\mu \in C^1$. Let us define ν^δ to be a smooth approximation to ν by making convolution with a mollifier supported in a small ball B_δ . Since ν is L^1 and ν is compactly supported, ν^δ uniformly converges to ν in L^1 . From lower-semicontinuity of BV norm under L^1 convergence it is enough to prove that

$$\|\nu^\delta\|_{BV} \leq \|\mu\|_{BV} \quad \text{for small } \delta > 0.$$

Observe that for each $y \in B_1$, $\epsilon > 0$, we have

$$\int |\nu^\delta(x + \epsilon y) - \nu^\delta(x)| dx \leq \int |\nu(w + \epsilon y) - \nu(w)| dw.$$

Notice that $f \equiv 1$ is translation invariant, therefore from the uniqueness result, $\nu(\cdot + \epsilon y)$ is the optimal solution of (1.1) for the initial measure $\mu(\cdot + \epsilon y)$. We apply then the L^1 contraction (Theorem 7.8) and get for any $\epsilon > 0$,

$$\int \frac{|\nu^\delta(x + \epsilon y) - \nu^\delta(x)|}{\epsilon} dx \leq \int \frac{|\mu(x + \epsilon y) - \mu(x)|}{\epsilon} dx.$$

Now for a fixed δ , ν^δ and μ are C^1 with compact supports, and so by sending $\epsilon \rightarrow 0$ in above expression we obtain

$$\int |D\nu^\delta(x) \cdot y| dx \leq \int |D\mu(x) \cdot y| dx.$$

We then integrate both sides in $y \in B_1$,

$$\int_{B_1} \int |Dv^\delta(x) \cdot y| dx dy \leq \int_{B_1} \int |D\mu(x) \cdot y| dy$$

and apply Fubini's theorem,

$$\int \int_{B_1} |Dv^\delta(x) \cdot y| dy dx \leq \int \int_{B_1} |D\mu(x) \cdot y| dy dx.$$

Observe that for any unit vector e_1 , we have

$$\int_{B_1} |Dv^\delta(x) \cdot y| dy = |Dv^\delta(x)| \int_{B_1} |e_1 \cdot y| dy$$

and similarly for μ , which leads to

$$\int |Dv^\delta(x)| \left(\int_{B_1} |e_1 \cdot y| dy \right) dx \leq \int |D\mu(x)| \left(\int_{B_1} |e_1 \cdot y| dy \right) dx.$$

Therefore, we get the inequality

$$\int |Dv^\delta(x)| dx \leq \int |D\mu(x)| dx.$$

Hence we showed that $\|v^\delta\|_{BV} \leq \|\mu\|_{BV}$ as desired.

Let us now consider the general μ in BV . We consider a sequence of measures $\mu_k \in C^1$ that converges as $k \rightarrow \infty$, to μ in BV (also in L^1); in particular $\|\mu_k\|_{BV} \rightarrow \|\mu\|_{BV}$. Consider the corresponding v_k , namely, the solution of (1.1) with the initial μ_k . Apply the previous step, and get

$$\|v_k\|_{BV} \leq \|\mu_k\|_{BV}.$$

Now notice that by the L^1 contraction v_k converges, as $k \rightarrow \infty$, to v in L^1 , therefore, from the lower-semicontinuity of the BV -norm in L^1 convergence, we get

$$\|v\|_{BV} \leq \|\mu\|_{BV}.$$

This completes the proof. \square

8. Saturation property of the optimal target. Here we prove that the optimal target measure of (1.1), when obtained with positive stopping time, saturates up to the upper density limit f . We first show a converse to Corollary 3.7.

LEMMA 8.1. *Let μ , f and $v = v^*$ be as given in Theorem 7.4. Let τ be the optimal stopping time with $W_0 \sim \mu$ and $W_\tau \sim v$. Let G be a measurable set such that $f|_G > 0$. Suppose that there exists a constant $t_1 > 0$ such that*

$$\text{Prob}[W_{t_1} \in G \ \& \ t_1 < \tau] > 0.$$

Then, $v[G] > 0$.

PROOF. Suppose for contradiction, that $v|_G = 0$. We define a randomized stopping time $\bar{\tau}$ (starting from the distribution μ) as follows.

- For those Brownian trajectories with $\tau \leq t_1$: stop at τ .
- For those Brownian trajectories with $\tau > t_1$: If $W_{t_1} \notin G$ just proceed until τ . But, if $W_{t_1} \in G$ then drop a portion of mass at the time t_1 , such that the resulting mass is positive but has density $\leq f$ over the set G , and then proceed until τ ; note that this is possible because $\text{Prob}[W_{t_1} \in G \ \& \ t_1 < \tau] > 0$ as well as $f|_G > 0$ for the function $f \in L^\infty$.

- To be more precise on this second point, notice that the density at x of the distribution of W_{t_1} for those Brownian particles with $t_1 < \tau$, is bounded from above by $K_{t_1}(x)$ of the solution to the heat equation with initial data μ , thus at each $x \in G$ when $W_{t_1} = x \in G$, one can stop the mass with the probability

$$\frac{\min[f(x), K_{t_1}(x)]}{K_{t_1}(x)} > 0,$$

and we get the positive stopped density $\leq f(x)$.

- Note that because $\nu|_G = 0$, the resulting distribution of $W_{\bar{\tau}}$ on G is only added from the second point above, thus is $\leq f$. On other parts, the distribution is \leq the distribution of W_τ by the construction.

Therefore this randomized stopping time $\bar{\tau}$ has distribution $W_{\bar{\tau}} \sim \bar{\nu}$ with $\bar{\nu} \leq f$. Moreover, notice that $\bar{\tau} \leq \tau$ for each random path almost surely, and $\text{Prob}[\bar{\tau} < \tau] > 0$ because of the second point above. Then, the cost $\mathcal{C}(\bar{\tau}) < \mathcal{C}(\tau)$ from the strictness of the cost functional \mathcal{C} . This contradicts optimality of τ , thus completing the proof. \square

As an immediate corollary of this lemma and Lemma 3.5, we see that for the optimally stopped Brownian motion for our problem (1.1), any set G with zero mass of the final distribution blocks off the Brownian motion.

COROLLARY 8.2. *Let μ , ν , and τ be as given in Lemma 8.1. Let G be a measurable set with positive measure, and suppose that every point of G has positive Lebesgue density. Assume that $f|_G > 0$ and $\nu[G] = 0$. Then, $\tau \leq \tau_G$ almost surely.*

PROOF. This is a direct consequence of Lemma 8.1 and Lemma 3.5. \square

We can combine ideas used in the proof of Lemma 8.1 and the result of Corollary 3.7 (see also Remark 3.9) to prove that the optimal solution ν saturates the density upper bound.

THEOREM 8.3 (saturation). *Let μ , f and $\nu = \nu^*$ be as given in Theorem 7.4, and τ be the corresponding optimal stopping time. If $\tau > 0$ almost surely, then ν is given in the form*

$$\nu = f \chi_E \quad \text{for some set } E.$$

More generally, without assuming $\tau > 0$, we have the following results:

(a) *For costs of type (I), we have*

$$\nu = f \chi_E + \mu|_F \quad \text{for some set } E \text{ and } F \text{ with } |E \cap F| = 0,$$

where in E the Brownian motion does not stop immediately, that is, $s(x) > 0$ for a.e. $x \in E$, and in F the Brownian paths stop immediately, that is, $s(x) = 0$, for a.e. $x \in F$.

(b) *For costs of type (II), we have*

$$\nu = \tilde{\nu} + f \wedge \mu,$$

where $\tilde{\nu}$ is the optimal solution from the initial measure $\tilde{\mu} = \mu - f \wedge \mu$, while the upper bound constraint is given by $\tilde{f} = f - f \wedge \mu$; here $\tilde{\nu}$ is given in the form

$$\tilde{\nu} = \tilde{f} \chi_E \quad \text{for some set } E$$

and the corresponding optimal stopping time $\tilde{\tau}$ is positive almost surely. Moreover

$$(f \wedge \mu)(x) = \text{Prob}[W_0 = x, \tau = 0]\mu(x).$$

REMARK 8.4. For costs of type (II), Theorem 8.3(b) implies that the set $\{\nu < f\}$ is nothing but where $\nu = f\mu \wedge f$, equivalently $\tilde{\nu} = 0$, and it belongs to the set where $\tau = 0$.

Before proceeding to the proof of above theorem, we state the following characterization of the instantly stopped portion for each cost types using the potential flow.

COROLLARY 8.5. *For costs of type (I) and (II), and E and F as given in Theorem 8.3 we have*

$$E = \{w > 0\} \quad \text{and} \quad F = \{w = 0\},$$

where $w \geq 0$ is the continuous solution of the obstacle problem

$$(8.1) \quad -\Delta w + (f - \mu)\chi_{\{w>0\}} = 0.$$

PROOF. First, for type (I), set $w := U_\nu - U_\mu$. Then from Theorem 8.3(a), $E = \{w > 0\}$ and $F = \{w = 0\}$. To see this, for example, apply Lemma 4.3 and continuity of U_ν and U_μ . Therefore, we see that

$$\Delta w = \nu - \mu = f\chi_{\{w>0\}} + \mu\chi_{\{w=0\}} - \mu = (f - \mu)\chi_{\{w>0\}}.$$

In the type (II) case, set $w := U_{\tilde{\nu}} - U_{\tilde{\mu}}$, with $\tilde{\mu} = \mu - f \wedge \mu$ and $\tilde{\nu} = \nu - f \wedge \mu$. Then from Theorem 8.3(b), $E = \{w > 0\}$ and $\text{supp } \tilde{\mu} \subset \{w > 0\}$; this is due to the fact that $\tilde{\tau} > 0$. Therefore, we see that

$$\Delta w = \tilde{\nu} - \tilde{\mu} = \tilde{f}\chi_{\{w>0\}} - \tilde{\mu}\chi_{\{w>0\}} = (f - f \wedge \mu - \mu + f \wedge \mu)\chi_{\{w>0\}} = (f - \mu)\chi_{\{w>0\}}.$$

This completes the proof. \square

As in this corollary, the set E in Theorem 8.3 where the target density ν saturates the constraint f , is determined by the solution w of (8.1), and, it is contained in the convex hull of the support ν . Therefore, it is compact, because the optimal target measure ν is compactly supported, in particular in a large ball B_R centred at the origin. Now, w is characterized as the unique solution of the well-known elliptic obstacle problem. Namely w minimizes

$$(8.2) \quad \int_{B_R} (|D\varphi|^2 + 2(f - \mu)\varphi) dx$$

among nonnegative functions $\varphi \in H_0^1(B_R)$, see [11, 14]. The regularity property of the free boundary $\partial E = \partial\{w > 0\}$ has been actively studied in the literature, which sensitively depends on the regularity of f ; for instance, when f is at least Hölder continuous (C^α), ∂E is $C^{1,\alpha}$ except at singular points with explicit and unique asymptotic blow-up profiles. In dimensions $d = 2, 3$, it is expected that ∂E has no singular points when f is sufficiently smooth for a generic choice of f , as suggested by the particular discussion in [29]. Thus in the type (I) case the set E exhibits an initial expansion of the support of μ to a larger and (mostly) smooth set; it suggests an instant regularization of the initial data which helps in our case, establishing a well-posed evolution for (St_1) , a problem that is well known to be ill-posed in general. For computational methods to obtain E , through the obstacle problem, see for instance [59] and the references therein.

PROOF OF THEOREM 8.3. We proceed in several steps.

1. When $\tau > 0$ a.s. For $0 < \delta < 1$ define

$$H_\delta := \left\{ x \mid \delta \leq \frac{d\nu(x)}{dx} < f(x) - \delta \right\}.$$

We claim that $\nu[H_\delta] = 0$ for each sufficiently small δ . Suppose $\nu[H_\delta] > 0$. Due to Corollary 3.7 applied to $G = H_\delta$, either

- (i) $|\mu \wedge (\nu|_{H_\delta})| > 0$, or
- (ii) there exists a constant $\bar{t} > 0$ such that

$$\text{Prob}[W_{\bar{t}} \in H_\delta \text{ \& } \bar{t} < \tau] > 0.$$

We will show that neither case is possible.

If (i) holds, then first observe that, from the fact that H_δ increases as δ decreases, we have the set $S := \{x : \frac{d\mu(x)}{dx} \geq \delta\} \cap \nu|_{H_\delta}$ has positive measure for sufficiently small $\delta > 0$. We then define a cost-decreasing randomized stopping time $\tilde{\tau}$ (starting from the distribution μ) as follows. First, $\tilde{\tau}$ is randomized at the initial time for those Brownian particles from S , in such a way that the distribution of initially stopped particles from S has positive mass with density δ . For the remaining particles it follows τ . Then the resulting terminal distribution by $\tilde{\tau}$, say $\tilde{\nu}$, has density less than f on S , by the definition of the set H_δ . Outside S , $\tilde{\nu} \leq \nu$, therefore overall $\tilde{\nu} \leq f$. Moreover, from the construction of $\tilde{\tau}$ and the strict monotonicity of \mathcal{C} we have $\mathcal{C}(\tilde{\tau}) < \mathcal{C}(\tau)$. This contradicts optimality of τ , and thus (i) does not happen.

For (ii), we define $\tilde{\tau}$ in such a way that for those Brownian trajectories with $\tau < \bar{t}$, stop at τ . For those Brownian trajectories with $\tau > \bar{t}$, if $W_{\bar{t}} \notin H_\delta$ just proceed until τ , but, if $W_{\bar{t}} \in H_\delta$ drop a portion of mass at time \bar{t} , such that the resulting mass is positive but has density $\leq \delta$ over the set H_δ , then proceed until τ ; note that this is possible because the density at x of the distribution of $W_{\bar{t}}$ is bounded from above by $K_{\bar{t}}(x)$ of the solution to the heat equation with initial data μ , thus at each $x \in H_\delta$ when $W_{\bar{t}} = x \in H_\delta$, one can stop the mass with the probability

$$\frac{\min[\delta, K_{\bar{t}}(x)]}{K_{\bar{t}}(x)} > 0,$$

and we get the positive stopped density with the total mass $\leq \delta$. Therefore this randomized stopping time $\tilde{\tau}$ has distribution $\tilde{\nu}$ with $\tilde{\nu} \leq f$. Moreover, notice that $\tilde{\tau} \leq \tau$ for each random path almost surely, and $\text{Prob}[\tilde{\tau} < \tau] > 0$. Then, the cost $\mathcal{C}(\tilde{\tau}) < \mathcal{C}(\tau)$ from the strictness of the cost functional \mathcal{C} . This contradicts optimality of τ , thus completing the proof for the case “ $\tau > 0$ almost surely”: we verified that $\nu = f \chi_E$ for some set E for both (I) and (II).

We now consider the general case.

2. (I) case. Recall that in this case $\tau = \inf\{t > 0 | t \geq s(W_t)\}$ (see Theorem 2.1 and (2.2)). Notice that the set $F := \{x | s(x) = 0\}$ gives a barrier set for W_t , $t \leq \tau$. That is, any Brownian path starting from a point in F with the stopping time τ stops immediately, equivalently, $\tau \leq \tau_F$. As a consequence the resulting final distribution with τ starting from $\mu|_F$ is $\mu|_F$. On the other hand, denote $\bar{\mu} := \mu|_{F^c}$ and apply the stopping time τ for those Brownian particles starting from $\bar{\mu}$. We let $\bar{\tau}$ denote such a restriction of τ . Notice that $\bar{\tau} > 0$ almost surely because the set $F^c = \{s > 0\} = R^c = (R^U)^c$ is an open set (see Theorem 4.15). Moreover $\bar{\tau}$ is the optimal stopping time for $\bar{\mu}$ and $\bar{f} = f - \mu|_F$; otherwise, τ will not be optimal. Let $\bar{\nu}$ be the resulting final distribution from the stopping time $\bar{\tau}$. Then from Step 1, we have that

$$\bar{\nu} = \bar{f} \chi_E \quad \text{for some set } E \text{ where } \bar{f}|_E > 0.$$

Moreover, notice that $\bar{\mu} \wedge (\bar{\nu}|_F) = 0$ from the definition $\bar{\mu} = \mu|_{F^c}$. Therefore, from $\tau \leq \tau_F$ (since F is a part of the barrier) and Lemma 3.2 we get $\bar{\nu}[F] = 0$. This also implies that $|E \cap F| = 0$, therefore, $\bar{f} \chi_E = f \chi_E$. In summary, we have

$$\nu = \bar{\nu} + \mu|_F = f \chi_E + \mu|_F \quad \text{as desired.}$$

This completes the proof for (I) in part (a).

3. (II) case. In this case the stopping time τ , which is the optimal stopping time between μ and ν , is randomized at $t = 0$ to give $f \wedge \mu$ and proceeds with positive time starting from the rest $\tilde{\mu} = \mu - \mu \wedge \nu = \mu - f \wedge \mu$, which gives the additional final distribution

$\tilde{v} = v - \mu \wedge v = v - f \wedge \mu$; for explanation (in particular why $\mu \wedge v = f \wedge \mu$) see the discussion around (7.2). Let $\tilde{\tau}$ denote the stopping time τ restricted to the initial distribution $\tilde{\mu}$. Then its final distribution \tilde{v} gives an optimal solution to (1.1) with the upper bound constraint $\tilde{f} = f - f \wedge \mu$. Notice that $\tilde{\tau} > 0$ almost surely as $\tilde{\mu} \wedge \tilde{v} = 0$. Thus from Case 1, we have $\tilde{v} = \tilde{f} \chi_E$ for a set E . Notice that $v = \tilde{v} + f \wedge \mu$. All things combined we verified the result for type (II) in part (b). This completes the proof. \square

8.1. *Upper bound for the optimal stopping time in the (I) case.* One of the applications of the saturation result is the following.

THEOREM 8.6. *Let μ, f and $v = v^*$ be as given in Theorem 7.4, and τ be the corresponding optimal stopping time. In addition assume that $f \geq \delta$ for a constant $\delta > 0$. Then for costs of type (I), there is a constant $\bar{T} = \bar{T}(\delta, \|\mu\|_{L^1})$ such that*

$$\tau \leq \bar{T} \quad \text{almost surely.}$$

REMARK 8.7. It is interesting that the extinction time \bar{T} does not depend on the size of the support of μ .

PROOF. Recall that τ has a barrier function s such that $\tau = \inf\{t > 0 | t \geq s(W_t)\}$. Consider the optimal target v and the set E and F from Theorem 8.3(a) (so in the (I) case) which satisfies that for each random path,

$$\begin{aligned} \text{“}\tau > 0\text{”} &\text{ implies } \text{“}W_\tau \in E\text{”} \quad \text{almost surely, and} \\ \text{“}\tau = 0\text{”} &\text{ implies } \text{“}W_0 \in F\text{”} \quad \text{almost surely,} \end{aligned}$$

and that for any measurable set S ,

$$\text{Prob}[W_\tau \in S \& \tau > 0 | W_0 \sim \mu] = v[S \cap E] = \int_{S \cap E} df \geq \delta |S \cap E|,$$

where the last inequality is from the assumption that $f \geq \delta$. Since $v \ll Leb$, we may assume, without loss of generality, that E^c consists of its Lebesgue points, by adding the non-Lebesgue density portion of E^c , which has zero Lebesgue measure, to E . Then, from Corollary 8.2, we see that

$$(8.3) \quad \text{if } \tau > 0 \quad \text{then almost surely } \tau \leq \tau_{E^c}.$$

Now consider for each $T > 0$, the set

$$Z_T := \{x | s(x) > T\}.$$

Notice that $v[Z_T] = v[Z_T \cap E]$.

It suffices to show that there exists \bar{T} such that $v[Z_{\bar{T}}] = 0$ because from Proposition 3.10 $\tau = s(\tau)$ almost surely.

Now Proposition 3.10 implies, almost surely, $W_\tau \in Z_T$ if and only if $\tau > T$. Also note that $\tau > T$ implies $s(W_T) > T$ so $W_T \in Z_T$. Also, (8.3) implies $W_T \in E$ when $\tau > T$. Therefore, almost surely,

$$W_\tau \in Z_T \quad \text{implies} \quad W_T \in Z_T \cap E \& \tau > T.$$

In particular, we have

$$\text{Prob}[W_\tau \in Z_T | W_0 \sim \mu] \leq \text{Prob}[W_T \in Z_T \cap E \& \tau > T | W_0 \sim \mu].$$

On the other hand, let ρ_T be the distribution of the heat flow in \mathbb{R}^d at time T with the initial condition $\rho_0 = \mu$. Notice that $\rho_T \leq C_T$ for some constant C_T , depending only on $\|\mu\|_{L^1}$ and T , decaying exponentially to zero as $T \rightarrow \infty$. Consider

$$\begin{aligned} C_T |Z_T \cap E| &\geq \int_{Z_T \cap E} d\rho_T = \text{Prob}[W_T \in Z_T \cap E | W_0 \sim \mu] \\ &\geq \text{Prob}[W_T \in Z_T \cap E \ \& \ \tau > T | W_0 \sim \mu] \\ &\geq \text{Prob}[W_\tau \in Z_T | W_0 \sim \mu] \\ &= \nu[Z_T] = \nu[Z_T \cap E] \\ &\geq \delta |Z_T \cap E|. \end{aligned}$$

By examining the limit $T \rightarrow \infty$ we see that there exists $\bar{T} = \bar{T}(\delta, \|\mu\|_{L^1}) > 0$ such that $|Z_{\bar{T}} \cap E| = 0$. Thus we conclude that $\nu[Z_{\bar{T}}] = 0$, completing the proof. \square

Note that a similar result does not hold for costs of type (II), as easily seen in the following example.

EXAMPLE 8.8. Let $B_r := \{x \in \mathbb{R}^d : |x| < r\}$, and solve (1.1) for a cost of type (II) with

$$\mu = 2^d \chi_{B_1} \quad \text{and} \quad f = 1.$$

Then the optimal measure is given by $\nu^* = \chi_{B_2}$ as seen from the symmetry of the problem and the saturation result Theorem 8.3(b). The corresponding barrier function s is radially symmetric and can be given by a function of the form

$$s(x) = \begin{cases} 0 & \text{for } |x| \leq 1, \\ g(|x|) & \text{for } 1 \leq |x| < 2, \\ +\infty & \text{for } |x| \geq 2, \end{cases}$$

where $g : [1, 2) \rightarrow \mathbb{R}$ is an increasing function with $g(1) = 0$ and $g(2^-) = \infty$. Here the optimal stopping time τ , which is the hitting time to such a barrier, has no upper bound.

9. Global-time existence of supercooled Stefan problem. Translating the saturation result (Theorem 8.3) into a PDE formulation (Theorem 5.4), we can now derive global-time existence results for both (St_1) (Theorems 9.3 and 9.4) and (St_2) (Theorem 9.2). Our emphasis will be on (St_1) which has not been well understood in the literature beyond one space dimension. We will see that different choices of f (i.e., different characteristic functions) yield different global-time solutions of (St_1) . For instance with $f = 1$ (on \mathbb{R}^d) we find the unique solution that vanishes in finite time (Theorem 9.3), for a certain class of initial data (μ, E) . Another choice of f yields a solution that does not vanish in finite time (Theorem 9.4). Let us point out that the initial domain E needs to be chosen carefully to have such solutions. For instance E can be obtained from a variational problem depending on μ and f as given in (8.2) through (8.1).

We first show that the optimal target ν is positive in the support of the Eulerian variable η as long as f is. Together with the saturation theorem it follows that $\nu = f$ in the active region if $\tau > 0$.

LEMMA 9.1. *Let $\mu, f, \nu = \nu^*$ be as given in Theorem 7.4, and let η be the Eulerian variable associated with (μ, ν) . Then we have*

$$\nu > 0 \quad \text{on the set } \bigcup_{t>0} \{\eta(\cdot, t) > 0\} \cap \{f > 0\}.$$

PROOF. We can apply Lemma 8.1 to the set $G := \{\eta(\cdot, t_1) > 0\}$ for each $t_1 > 0$. \square

9.1. *Stable Stefan problem.* Let us first briefly discuss global well-posedness for (St_2) . Let us consider (1.1) for cost type (II) with $f = 1$ and $\mu > 1$ in its support.

THEOREM 9.2 (Global-time well-posedness for (St_2)). *Consider $\mu := (1 + \eta_0)\chi_\Sigma$ with a positive, bounded function η_0 and a compact set Σ . Let $f = 1$, and let v solve (1.1) with μ and f for cost type (II). Then the corresponding η -variable for (μ, v) is the unique weak solution of the stable Stefan problem*

$$(St_2) \quad (\eta + \chi_{\{\eta > 0\}})_t - \frac{1}{2}\Delta\eta = 0,$$

with the initial data $(\eta_0\chi_\Sigma, \Sigma)$.

PROOF. Theorem 8.3(b) yields the decomposition $\mu = \tilde{\mu} + \chi_\Sigma$ and $v = \tilde{v} + \chi_\Sigma$. Moreover the “actively accumulated target” \tilde{v} solves (1.1) with $\tilde{f} = \chi_{\Sigma^c}$ and $\tilde{\mu} = \eta_0\chi_\Sigma$. Observe that the η -variable for (μ, v) is the same as that of $(\tilde{\mu}, \tilde{v})$, because χ_Σ presents precisely the distribution at $\tau = 0$. In particular the optimal stopping time $\tilde{\tau}$ between \tilde{v} and $\tilde{\mu}$ is positive, allowing us to apply Theorem 5.4 for $\tilde{\mu}$ and \tilde{v} .

Before doing so, let us evaluate \tilde{v} in the set $\{0 < s(x) < \infty\}$ where s is the barrier function for $\tilde{\tau}$. From Lemma 9.1 and the saturation (Theorem 8.3) we conclude that

$$\tilde{v} = 1 \quad \text{in } \bigcup_{t>0} \{\eta(\cdot, t) > 0\} \setminus \Sigma.$$

Recall from Theorem 4.15 we have $\{\eta > 0\} = \{s(x) < t\}$. Thus we have

$$E = \lim_{t \rightarrow 0^+} \{\eta(\cdot, t) > 0\} = \{s(x) = 0\},$$

which equals Σ . It now follows that $\tilde{v} = 1$ in the region $\{0 < s(x) < \infty\}$. Theorem 5.4(b) and Remark 5.2 finally yield that η solves (St_2) uniquely with the initial data (η_0, Σ) . \square

Note that for the above theorem the initial set for the active region coincides with the support Σ of the initial distribution μ . This is due to the nature for the type (II) costs whose barrier sets in the space-time for the optimal stopping are time-backward monotone, in contrast to the type (I) costs that gives time-forward monotone barrier sets.

For the supercooled (St_1) case, not every initial set Σ has a global-time weak solution, as we expect from the instability of the problem. This will be apparent in the theorems below for the supercooled case.

9.2. *Supercooled Stefan problem.* Next let us discuss existence results for (St_1) . First we apply a parallel reasoning to Theorem 9.2 but now applied to the type (I) costs, to derive a finite-time vanishing solution.

THEOREM 9.3 (A global-time existence for (St_1)). *Let μ and f be as given in Theorem 9.2, and let v solve (1.1) with μ and f for cost type (I). Then $v = \chi_E$, and the Eulerian variable η for (μ, v) solves the supercooled Stefan problem*

$$(St_1) \quad (\eta - \chi_{\{\eta > 0\}})_t - \frac{1}{2}\Delta\eta = 0$$

with initial data (μ, E) . Moreover E can be characterized as the positive set of w , where w solves the obstacle problem

$$(9.1) \quad \chi_{\{w>0\}} - \Delta w = \mu, \quad w \geq 0.$$

In particular E contains Σ . Last, η vanishes to zero in finite time.

PROOF. From Theorem 8.3(a) and the choice of μ it follows that $\tau > 0$ and $\nu = \chi_E$. Due to Theorem 5.4 (a) it follows that η solves (St_1) with initial data (μ, \tilde{E}) , where the initial active region \tilde{E} contains E . Notice that $\tilde{E} = \bigcup_{t>0} \{\eta(\cdot, t) > 0\}$ because the set $\{\eta(\cdot, t) > 0\}$ decreases monotonically in t . From Lemma 9.1 and the fact that $f = 1$, it follows that $\nu > 0$ in \tilde{E} . In particular, $\tilde{E} \subseteq E$, therefore $\tilde{E} = E$. This also means that E contains the support of μ , since there is no instantly frozen part of μ its support is contained in the initial active region \tilde{E} which is nothing but E as we saw.

Last, note that the characterization of E via (9.1) is due to Corollary 8.5 and the fact that E contains the support of μ . Last, again since $f = 1$, Theorem 8.6 yields that η turns to zero in finite time. \square

In the next subsection (see Corollary 9.9) we will show that η in Theorem 9.3 is indeed the unique solution for the initial data μ that vanishes in finite time.

On the other hand, we note that one can generate solutions of (St_1) from a wider class of initial data by making use of Theorem 8.3 and Theorem 5.4. For instance, one can choose $f = \chi_K$ where K and the support of μ are disjoint to avoid instant stopping of particles. In this case we obtain the following.

THEOREM 9.4 (A global-time existence for (St_1) , second version). *Let $\mu \geq 0$ be a compactly supported function in $L^\infty(\mathbb{R}^d)$, and let $f := \chi_{K^c}$, where K is a compact set that includes the support of μ . Let ν solve (1.1) with μ and f for cost of type (I). Then the Eulerian variable η for (μ, ν) solves (St_1) with the initial data (μ, E) , and $\nu = \chi_{E \cap K^c}$. Here E is the positive set of w , where w solves the obstacle problem*

$$\Delta w = (f - \mu) \chi_{\{w>0\}} = \chi_{\{w>0\} \cap K^c} - \mu, \quad w \geq 0.$$

PROOF. In this setting, to travel from μ to ν the optimal stopping time τ must be positive. Theorem 8.3 now yields that $\nu = \chi_\Sigma$, where $\Sigma \subset \{0 < s(x) < \infty\} \subset K^C$. (In fact, $\Sigma = \{0 < s(x) < \infty\}$ as one can see from the similar reasoning in the second half of the proof of Theorem 9.2, by using Lemma 9.1.) Hence the consistency theorem (Theorem 5.4(a)) yields that η solves (St_1) with the initial data (μ, E) , where the initial active region E contains Σ . On the other hand $\Sigma \cup K$ contains E due to Lemma 9.1. Now we can conclude by characterizing E in terms of w by Corollary 8.5. \square

Note that in the case of Theorem 9.4 the Eulerian variable η does not vanish in finite time. It is because the Brownian motion will travel forever in $E \cap K$, where no stopping happens there due to the fact $f|_K = 0$.

REMARK 9.5. Uniqueness is in general not true for (St_1) with initial data μ , even with the same initial domain E (see the example in Section 6). On the other hand, the optimal target problem with f yields a unique η -variable corresponding to the optimal stopping time. Thus our approach yields a variationally oriented criteria for a unique solution of (St_1) with a given initial distribution μ .

REMARK 9.6 (Relation between (St_1) and (St_2)). Due to universality (Theorem 7.6) we have the same optimal target measure ν for type (I) and (II), for given μ and f . Therefore the solution of stable Stefan problem (St_2) in Theorem 9.2 determines the initial set E that yields global-time solution for (St_1) .

◦ *A discussion on the initial expansion.* An interesting feature of our solutions of (St_1) is the initial trace E of the active region, instantly expanded from the support of the initial data μ . It is well-understood that the local-in-time solutions to (St_1) develops jump discontinuity when the solution is “overloaded”, namely when the average density goes beyond 1 (see, for instance, [16]). When $\mu > 1$ in its support, as given in Theorem 9.3, μ is entirely overloaded and thus it is natural that one must expand its support to activate the supercooling process. On the other hand when μ has small density as allowed in Theorem 9.4, such expansion follows from our choice of f , which describes the particular scenario where the freezing stimulus is only available outside of the initial supercooled region. The general criteria for μ and E to generate a global solution of (St_1) , in terms of the size of μ and the geometry of E , remains open.

Our characterization of the initial domain E via (8.1) is also reminiscent of the classical paper by Dibenedetto and Friedman [26], which considers a time integrated version of (St_1) . See the discussion in [26], Section 4.3.

9.3. *Existence of finite-time vanishing solution characterizes the initial set.* In this section, we show that the choice of the initial set E in Theorem 9.3 is necessary for a finite-time vanishing solution to exist. We begin by connecting finite-time vanishing solutions to the notion of subharmonically generated sets introduced in Section 6.

THEOREM 9.7. *For a given compactly supported $0 \leq \mu \in L^\infty(\mathbb{R}^d)$, let η be a weak solution of (St_1) with the initial data (μ, E) . Then η vanishes in finite time if and only if (E, E) is subharmonically generated by μ and η is the corresponding Eulerian variable to (μ, χ_E) .*

PROOF. The “if” direction is a direct consequence of Theorem 6.2 and Theorem 8.6. The “only if” direction follows from Theorem 6.2 and the fact that $s(x) := \sup\{t : \eta(x, t) > 0\} \leq T$ for some finite T . \square

THEOREM 9.8 (Necessary condition for finite time vanishing solution). *For a given compactly supported $0 \leq \mu \in L^\infty(\mathbb{R}^d)$, suppose there is a solution to (St_1) with the initial data (μ, E) that vanishes in finite time. Then, the set E is determined by the optimal free target problem (1.1) of type (I) cost, for μ and $f \equiv 1$, where the optimal target is $v^* = \chi_E$.*

Before proving this theorem, we point out that to have such an initial active region E which is also the support of v^* at the same time, we must have the corresponding optimal time $\tau > 0$ almost surely, so E contains the support of μ . In particular Corollary 8.5 yields that such a set E is determined by the obstacle problem (9.1). Hence we arrive at the following conclusion that uniquely characterizes the solution in Theorem 9.3.

COROLLARY 9.9 (Unique characterization). *Let μ be as given in Theorem 9.3. Then there is a unique solution η of (St_1) with the initial data μ that vanishes in finite time. In this case, the initial domain E is given by (9.1), and η corresponds to the Eulerian variable given by (μ, χ_E) .*

PROOF OF THEOREM 9.8. For this result, we apply Theorem 9.7 and Theorem 7.1.

To have a vanishing in finite time solution to (St_1) , from Theorem 9.7 it is necessary that (E, E) has to be subharmonically generated by μ . By definition, there is a corresponding type (I) optimal stopping time, say, $\tau > 0$, solving the optimal Skorokhod problem $\mathcal{P}(\mu, \chi_E)$. Moreover, since $\tau > 0$ and we have $E = \{s(x) > 0\}$ where $s(x)$ is the barrier function. From

Theorem 4.15 it follows that $\{\eta > 0\} = \{t < s(x)\}$, and thus $E = \limsup_{t \rightarrow 0^+} \{\eta(\cdot, t) > 0\}$ for the corresponding Eulerian flow (η, ρ) .

On the other hand, we have the optimal stopping time τ^* for the free target problem (1.1) for type (I) cost, from the measure μ , with $f \equiv 1$, with the optimal target ν^* with its density $\nu^* \leq 1$ everywhere.

We claim that $\nu^* \leq \chi_E$. This will imply $\nu^* = \chi_E$ because they have the same mass.

To prove the claim, first note that the monotonicity result, Theorem 7.1 part (1), does not require optimality of one of the targets, therefore, we have $\tau^* \leq \tau$. Therefore, the Eulerian flow η^* for τ^* is supported inside the support of the Eulerian flow η of τ . In particular the initial domain for the Eulerian flow η^* is a subset of the initial domain for η , that is, E . Moreover, even when there is instant stopping $\tau^* = 0$ for a set S , since $\tau > 0$ almost surely, we have $S \subset E$. From these considerations and the fact that $\nu^* \leq 1$, we have $\nu^* \leq \chi_E$ as desired. This completes the proof. \square

From a parallel reasoning we can uniquely characterize η from Theorem 9.4 as well:

COROLLARY 9.10 (Unique characterization). *Let μ, K, η , and E be as given in Theorem 9.4. Then η is the unique weak solution of (St_1) such that $\bigcap_{\{t > 0\}} \{\eta(\cdot, t) > 0\} = K$. In other words, E is the unique set for which $(E \cap K^c, E)$ is subharmonically generated.*

10. Monotonicity of the optimal barrier functions. We show in this section that the barrier function $s(x)$ for the optimal stopping time τ (the solution to the optimal free target problem (1.1)) enjoys a monotonicity property in the spirit of Theorem 7.1. Namely, the barrier functions s_i are ordered if the initial distributions μ_i are ordered (Theorem 10.1), and such an order is strict (Theorem 10.2). This ordering property yields a *strict* comparison principle for the Stefan problems, where a local strict ordering of μ_i implies global strict ordering on s_i 's; this local-to-global strictness is new, to our best knowledge, even for (St_2) , where only global-global strict comparison was proven before [41]. The proof is entirely probabilistic and is based on the properties of the optimal free target problem. It thus provides new insights into the Stefan problems, in particular for the supercooled Stefan problem (St_1) . Especially, we hope it may shed a light on understanding regularity of the barrier function $s(x)$. Understanding regularity of $s(x)$, so regularity of the free boundaries of the Stefan problem, is a wide open problem in (St_1) for any class of solutions beyond one space dimension.

Our results in Theorem 4.15 show that for the optimal stopping time, the space-time barrier set R is nothing but R^U , thus is closed, and the barrier function $s = s^U$ is lower-semicontinuous for type (I) and upper semicontinuous for type (II). This then can be translated, via the results in Section 9, to corresponding properties for the free boundaries of the Stefan problems (St_1) and (St_2) . To our best knowledge even this very mild regularity result for the free boundary is new for (St_1) ; even though it was for an appropriately chosen initial domain E . However, for such semicontinuity we only use the optimality of the stopping time for the optimal Skorokhod problem $\mathcal{P}(\mu, \nu)$ (1.5). Our monotonicity of the barrier function below, which is a result of optimality of the target measure, may lead to a nicer regularity result.

In view of Theorem 4.15(b) and for clarity, we let s^U denote the barrier function from now on.

THEOREM 10.1 (Monotonicity for optimal barrier functions). *Assume $0 \leq \mu_1 \leq \mu_2 \in L^\infty(\mathbb{R}^d)$ be compactly supported and let $0 \leq f \in L^\infty(\mathbb{R}^d)$. Let ν_i , $i = 1, 2$, be the optimal solutions of the Problem (1.1) with μ_i , $i = 1, 2$, respectively. Let s_i^U be the barrier functions*

associated with (μ_i, v_i) as in Definition 4.19. In the type (II) case assume that $\tau_i > 0, i = 1, 2$. Then we have

$$\begin{aligned} s_1^U &\leq s_2^U \quad \text{for type (I) cost, and} \\ s_1^U &\geq s_2^U \quad \text{for type (II) cost.} \end{aligned}$$

PROOF. Due to Theorem 4.15(b), the corresponding optimal stopping time τ_i 's, the barriers, and the barrier functions for (μ_i, v_i) can be represented as τ_i^U, R_i^U , and s_i^U , respectively. We utilize the potential flow formulation in Section 4 as well as the monotonicity result for the optimal target problem (Theorem 7.1). In what is below for ease of notation we use the notation τ_i 's for the optimal stopping time τ_i^U , but, we will keep the superscript U for R_i^U and s_i^U .

Let $\mu_t^i, i = 1, 2$, be the distribution of $W_{\tau_i \wedge t}$, that is, $W_{\tau_i \wedge t} \sim \mu_t^i$. Below we treat type (I) and (II) separately.

We derive the results by examining Ito's formula given in Lemma 4.3.

Type (I). Define the functions

$$U_1(x, t) := U_{\nu_1}(x) - U_{\mu_t^1}(x) \quad \text{and} \quad U_2(x, t) := U_{\nu_2}(x) - U_{\mu_t^2}(x).$$

Notice that they are all nonnegative, continuous, and monotone decreasing in time, due to Corollary 4.9 and Corollary 4.7. Fix t and x . Consider an ϵ -ball $B_\epsilon(x)$ around x . Then, Lemma 4.3 yields that

$$\begin{aligned} \int_{B_\epsilon(x)} U_1(y, t) dy &= \int_{B_\epsilon(x)} (U_{\nu_1} - U_{\mu_t^1})(y) dy = \mathbb{E} \left[\int_{\tau_1 \wedge t}^{\tau_1} \frac{1}{2} \chi_{B_\epsilon(w)}(W_t) dt \right], \\ \int_{B_\epsilon(x)} U_2(y, t) dy &= \int_{B_\epsilon(x)} (U_{\nu_2} - U_{\mu_t^2})(y) dy = \mathbb{E} \left[\int_{\tau_2 \wedge t}^{\tau_2} \frac{1}{2} \chi_{B_\epsilon(x)}(W_t) dt \right]. \end{aligned}$$

For the expected value $\mathbb{E}[\int_{\tau_1 \wedge t}^{\tau_1} \frac{1}{2} \chi_{B_\epsilon(w)}(W_t) dt]$, exactly those Brownian paths (staring from the distribution μ_1) for which $t < \tau_1$ contributes, while for $\mathbb{E}[\int_{\tau_2 \wedge t}^{\tau_2} \frac{1}{2} \chi_{B_\epsilon(x)}(W_t) dt]$ exactly those paths (starting from the distribution μ_2) for which $t < \tau_2$ contribute. Now, recall that $\mu_1 \leq \mu_2$ and $\tau_1 \leq \tau_2$ (due to Theorem 7.1(1), for the restriction of τ_2 on those paths starting from the distribution μ_1). Therefore, the set of paths that contribute to the expected value $\mathbb{E}[\int_{\tau_1 \wedge t}^{\tau_1} \frac{1}{2} \chi_{B_\epsilon(w)}(W_t) dt]$, is contained in the set of paths that contribute to the expected value $\mathbb{E}[\int_{\tau_2 \wedge t}^{\tau_2} \frac{1}{2} \chi_{B_\epsilon(x)}(W_t) dt]$. As a result, we have

$$\mathbb{E} \left[\int_{\tau_1 \wedge t}^{\tau_1} \frac{1}{2} \chi_{B_\epsilon(w)}(W_t) dt \right] \leq \mathbb{E} \left[\int_{\tau_2 \wedge t}^{\tau_2} \frac{1}{2} \chi_{B_\epsilon(x)}(W_t) dt \right]$$

which implies

$$\frac{1}{B_\epsilon(x)} \int_{B_\epsilon(x)} U_1(y, t) dy \leq \frac{1}{B_\epsilon(x)} \int_{B_\epsilon(x)} U_2(y, t) dy.$$

However, as $\epsilon \rightarrow 0$, due to continuity of U_1 and U_2 , we get $U_1(x, t) \leq U_2(x, t)$. Note that from the definition

$$s_i^U(x) := \inf \{t | U_i(x, t) = 0\}.$$

Therefore, $s_1^U \leq s_2^U$ as desired.

Type (II). Define the functions

$$U_1(x, t) := U_{\mu_t^1}(x) - U_{\mu_1}(x) \quad \text{and} \quad U_2(x, t) := U_{\mu_t^2}(x) - U_{\mu_2}(x).$$

Similar to the type (I) case apply Lemma 4.3, and get for each (x, t) and $\epsilon > 0$,

$$\int_{B_\epsilon(x)} U_1(y, t) dy = \int_{B_\epsilon(x)} (U_{\mu_1^t} - U_{\mu_1})(y) dy = \mathbb{E} \left[\int_0^{\tau_1 \wedge t} \frac{1}{2} \chi_{B_\epsilon(w)}(W_t) dt \right],$$

$$\int_{B_\epsilon(x)} U_2(y, t) dy = \int_{B_\epsilon(x)} (U_{\mu_2^t} - U_{\mu_2})(y) dy = \mathbb{E} \left[\int_0^{\tau_2 \wedge t} \frac{1}{2} \chi_{B_\epsilon(x)}(W_t) dt \right].$$

Since $\mu_1 \leq \mu_2$ and $\tau_1 \leq \tau_2$ (due to Theorem 7.1(1), for τ_2 restricted to μ_1), as in the type (I) case above, we can compare the set of Brownian paths that contribute to the expected values. Then we get $\mathbb{E}[\int_0^{\tau_1 \wedge t} \frac{1}{2} \chi_{B_\epsilon(w)}(W_t) dt] \leq \mathbb{E}[\int_0^{\tau_2 \wedge t} \frac{1}{2} \chi_{B_\epsilon(x)}(W_t) dt]$. Thus using continuity of U_1 and U_2 , we get $U_1(x, t) \leq U_2(x, t)$. Because

$$s_i^U(x) := \sup \{t \mid U_i(x, t) = 0\},$$

this implies $s_1^U \geq s_2^U$ as desired. This completes the proof. \square

We now prove *strict* monotonicity for s^U , which is derived by combining the above theorem with the monotonicity of the stopping time (Theorem 7.1), Proposition 3.10, and the saturation result (Theorem 8.3). In the discussion below we use the assumption given in Theorem 10.1.

As a preparation of the statement of the result, let us recall that the barrier sets $R_i = R_i^U$, $i = 1, 2$ for the stopping times in Theorem 10.1 are closed, therefore their complements are open. Define for $i = 1, 2$, by using the Eulerian variable (η_i, ρ_i) of τ_i ,

$$(10.1) \quad \begin{aligned} \text{in type (I), } E_i &:= \limsup_{t \rightarrow 0^+} \{x \mid \eta_i(x, t) > 0\}, \\ \text{in type (II), } E_i &:= \limsup_{t \rightarrow \infty} \{x \mid \eta_i(x, t) > 0\}. \end{aligned}$$

Note that due to the time monotonicity of R_i , $E_i = R_i^c = \bigcup_{t>0} \{\eta_i(\cdot, t) > 0\}$, thus they are open. Note also that as E_i 's are connected to the active regions, the Brownian paths that start from E_i 's have $\tau_i > 0$ almost surely, except those in type (II) that may immediately stop; see Theorem 8.3, where the mass that immediately stops in type (II) is characterized by the initial data as $\mu \wedge f$. Thus in the type (II) case, by taking the initial data as $\mu - \mu \wedge f$ we may assume $\tau_i > 0$ in E_i 's. The complement E_2^c is the set where there is immediate and complete stopping for τ_2 , so τ_2 is zero, and so is $\tau_1 = 0$ from monotonicity (Theorem 7.1); in E_2^c the corresponding s_i^U 's are 0 in type (I) case, ∞ in type (II) case. Moreover, in type (I) case, E_2^c is exactly the set where the immediate stopping occurs for τ_2 so also for τ_1 . This justifies that for comparison between s_i^U we can without loss of generality assume that $\tau_i > 0$ in both type (I) and (II), and compare s_i^U 's only over E_2 . Also assuming $f > 0$ everywhere does not cost much generality for our purpose of comparing s_i^U 's, because in the region $f = 0$ no stopping to accumulate mass occurs, so the value s_i^U can only be either ∞ or 0. With these considerations we see that the following theorem essentially covers the general case and the whole domain for the strict monotonicity of s^U .

THEOREM 10.2 (Strict monotonicity for optimal barrier functions). *Let f, μ_i, τ_i , and s_i^U be as given in Theorem 10.1, in particular, with $\mu_1 \leq \mu_2$ and $s_1^U \leq s_2^U$. Assume that $f > 0$ everywhere and that $\tau_i > 0$, $i = 1, 2$, almost surely. Let O be a path connected component of E_2 given in (10.1). Suppose $\mu_2 > \mu_1$ on a subset $G \subset O$ with $|G| > 0$. Then $s_1^U(x) \neq s_2^U(x)$ for a.e. $x \in O$.*

PROOF. From the continuity of Brownian paths, almost surely any Brownian path starting from O should stay inside the corresponding path-connected component of the active region, say $A \subset \mathbb{R}^d \times \mathbb{R}_{\geq 0}$, that is connected to O , that is,

$$A = \{(x, t) | x \in O, t < s_2^U(x)\} \quad \text{in type (I),}$$

$$A = \{(x, t) | x \in O, t > s_2^U(x)\} \quad \text{in type (II).}$$

From this and the condition $\mu_1 \leq \mu_2$, we may assume that $O = E_2$, namely, we consider only those mass distributions, Brownian motion, and the barriers, associated to O ; for example, we assume $\mu_i|_E = \mu_i|_O$, $v_i|_E = v_i|_O$. Notice that A is a connected and an open set and $\eta_2 > 0$ on A . From this we also see that

$$v_2 > 0 \quad \text{on } O.$$

(To see this, we can for instance apply Lemma 8.1, which uses optimality of v_2 with respect to Problem (1.1).) Therefore, it suffices to show that $s_1^U(x) \neq s_2^U(x)$ for v_2 -a.e. x .

Also Theorem 8.3 yields that $v_i = f$ on its support, since we assume that $\tau_i > 0$. Therefore to prove the theorem it suffices to show that

$$(10.2) \quad v_2[S] = 0,$$

where

$$S := \{x | s_1^U(x) = s_2^U(x) < \infty\} \cap \{x | v_1(x) = v_2(x) = f(x)\} \cap O.$$

For the rest of the proof, we verify this claim.

Recall that $\tau_1 \leq \tau_2$ and $v_1 \leq v_2$ from Theorem 7.1. Because of the Markov property of Brownian motion, the inequality $\tau_1 \leq \tau_2$ can be applied to all Brownian paths from the initial distribution μ_2 not only from μ_1 (see Remark 7.3).

From optimality of τ_i 's we can apply Proposition 3.10, to get $s_i^U(W_{\tau_i}) = \tau_i$, $i = 1, 2$ a.s. Now if $s_1^U(W_{\tau_1}) = s_2^U(W_{\tau_1})$ then almost surely $\tau_1 = s_2^U(W_{\tau_1})$, so from the hitting time characterisation of τ_2 , with the barrier s_2^U , we have $\tau_1 \geq \tau_2$. Since we already have $\tau_1 \leq \tau_2$ a.e., we therefore have,

$$(10.3) \quad \text{almost surely, if } s_1^U(W_{\tau_1}) = s_2^U(W_{\tau_1}) \text{ then } \tau_2 = \tau_1.$$

We have from $W_0 \sim \mu_i|_O$, $W_{\tau_i} \sim v_i|_O$ that

$$v_i[S] = \int_O \text{Prob}[W_{\tau_i} \in S | W_0 = x] d\mu_i(x) \quad \text{for } i = 1, 2.$$

Let G be as given in the statement of this theorem, and observe that $\mu_2[G] > 0$. Then, for $i = 1, 2$,

$$(10.4) \quad v_i[S] = \int_G \text{Prob}[W_{\tau_i} \in S | W_0 = x] d\mu_i(x) + \int_{O \setminus G} \text{Prob}[W_{\tau_i} \in S | W_0 = x] d\mu_i(x).$$

On the other hand,

$$\begin{aligned} & \text{Prob}[W_{\tau_2} \in S | W_0 = x] \\ &= \text{Prob}[W_{\tau_2} \in S \& \tau_2 = \tau_1 | W_0 = x] + \text{Prob}[W_{\tau_2} \in S \& \tau_2 > \tau_1 | W_0 = x] \\ &\geq \text{Prob}[W_{\tau_2} \in S \& \tau_2 = \tau_1 | W_0 = x] \\ &= \text{Prob}[W_{\tau_1} \in S \& \tau_2 = \tau_1 | W_0 = x]. \end{aligned}$$

From (10.3) it holds that for μ_2 -a.e. x ,

$$\text{Prob}[W_{\tau_1} \in S \& \tau_2 = \tau_1 | W_0 = x] = \text{Prob}[W_{\tau_1} \in S | W_0 = x].$$

Therefore, from the previous inequality we have for μ_2 -a.e. x ,

$$\text{Prob}[W_{\tau_2} \in S | W_0 = x] \geq \text{Prob}[W_{\tau_1} \in S | W_0 = x].$$

Apply this to the above integrals (10.4) and get

$$\begin{aligned} \nu_2[S] - \nu_1[S] &\geq \int_G \text{Prob}[W_{\tau_2} \in S | W_0 = x] d\mu_2(x) - \int_G \text{Prob}[W_{\tau_1} \in S | W_0 = x] d\mu_1(x) \\ &\geq \int_G \text{Prob}[W_{\tau_1} \in S | W_0 = x] d\mu_2(x) - \int_G \text{Prob}[W_{\tau_1} \in S | W_0 = x] d\mu_1(x) \\ &= \int_G \text{Prob}[W_{\tau_1} \in S | W_0 = x] d(\mu_2 - \mu_1)(x) \\ &\geq 0. \end{aligned}$$

On the other hand, due to the definition of S , $\nu_1(S) = \nu_2(S)$. This and the condition $\mu_2 > \mu_1$ on G yields

$$(10.5) \quad \text{Prob}[W_{\tau_2} \in S | W_0 = x] = \text{Prob}[W_{\tau_1} \in S | W_0 = x] = 0 \quad \text{for } \mu_2\text{-a.e. } x \text{ in } G.$$

Let $\tilde{\tau}_2$ be the restriction of τ_2 to the initial distribution $\mu_2|_G$ and consider its Eulerian flow $\tilde{\eta}_2$. Since the active set A is a path-connected open set, we have $\tilde{\eta}_2 > 0$ everywhere in A . Recall from above that $\nu_2 > 0$ on O and that $\eta_2 > 0$ on A for the Eulerian flow η_2 of τ_2 with μ_2 . In other words, $\tilde{\eta}_2 > 0$ wherever $\eta_2 > 0$. Therefore, from the Markov property of the Brownian motion, the resulting target distribution $\tilde{\nu}_2$ of $\tilde{\tau}_2$ has $\tilde{\nu}_2 > 0$ on O , because it has to be positive wherever $\nu_2 > 0$. Now, (10.5) implies that $\tilde{\nu}_2[S] = 0$, so we can conclude with (10.2). This completes the proof. \square

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REFERENCES

- [1] ALEXANDER, D., KIM, I. and YAO, Y. (2014). Quasi-static evolution and congested crowd transport. *Non-linearity* **27** 823–858. MR3190322 <https://doi.org/10.1088/0951-7715/27/4/823>
- [2] ALT, H. W. and LUCKHAUS, S. (1983). Quasilinear elliptic-parabolic differential equations. *Math. Z.* **183** 311–341. MR0706391 <https://doi.org/10.1007/BF01176474>

- [3] ATHANASOPOULOS, I., CAFFARELLI, L. and SALSA, S. (1996). Regularity of the free boundary in parabolic phase-transition problems. *Acta Math.* **176** 245–282. MR1397563 <https://doi.org/10.1007/BF02551583>
- [4] AZÉMA, J. and YOR, M. (1979). Une solution simple au problème de Skorokhod. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)*. *Lecture Notes in Math.* **721** 90–115. Springer, Berlin. MR0544782
- [5] BACKHOFF-VERAGUAS, J., BEIGLBÖCK, M., HUESMANN, M. and KÄLLBLAD, S. (2020). Martingale Benamou–Brenier: A probabilistic perspective. *Ann. Probab.* **48** 2258–2289. MR4152642 <https://doi.org/10.1214/20-AOP1422>
- [6] BAXTER, J. R. and CHACON, R. V. (1974). Potentials of stopped distributions. *Illinois J. Math.* **18** 649–656. MR0358960
- [7] BAXTER, J. R. and CHACON, R. V. (1977). Compactness of stopping times. *Z. Wahrscheinlichkeitstheorie Verw. Gebiete* **40** 169–181. MR0517871 <https://doi.org/10.1007/BF00736045>
- [8] BEIGLBÖCK, M., COX, A. M. G. and HUESMANN, M. (2017). Optimal transport and Skorokhod embedding. *Invent. Math.* **208** 327–400. MR3639595 <https://doi.org/10.1007/s00222-016-0692-2>
- [9] BEIGLBÖCK, M., HENRY-LABORDÈRE, P. and PENKNER, F. (2013). Model-independent bounds for option prices—a mass transport approach. *Finance Stoch.* **17** 477–501. MR3066985 <https://doi.org/10.1007/s00780-013-0205-8>
- [10] BEIGLBÖCK, M. and JUILLET, N. (2016). On a problem of optimal transport under marginal martingale constraints. *Ann. Probab.* **44** 42–106. MR3456332 <https://doi.org/10.1214/14-AOP966>
- [11] BLANK, I. A. (2000). *Sharp Results for the Regularity and Stability of the Free Boundary in the Obstacle Problem*. ProQuest LLC, Ann Arbor, MI. MR2700210
- [12] BRÜCKERHOFF, M. and HUESMANN, M. (2021). Shadows and Barriers. arXiv preprint. Available at [arXiv:2103.03620](https://arxiv.org/abs/2103.03620).
- [13] CAFFARELLI, L., PETROSYAN, A. and SHAHGHOLIAN, H. (2004). Regularity of a free boundary in parabolic potential theory. *J. Amer. Math. Soc.* **17** 827–869. MR2083469 <https://doi.org/10.1090/S0894-0347-04-00466-7>
- [14] CAFFARELLI, L. A. (1998). The obstacle problem revisited. *J. Fourier Anal. Appl.* **4** 383–402. MR1658612 <https://doi.org/10.1007/BF02498216>
- [15] CHAYES, L. and KIM, I. C. (2008). A two-sided contracting Stefan problem. *Comm. Partial Differential Equations* **33** 2225–2256. MR2479285 <https://doi.org/10.1080/03605300802553906>
- [16] CHAYES, L. and KIM, I. C. (2012). The supercooled Stefan problem in one dimension. *Commun. Pure Appl. Anal.* **11** 845–859. MR2861812 <https://doi.org/10.3934/cpaa.2012.11.845>
- [17] CHAYES, L. and SWINDLE, G. (1996). Hydrodynamic limits for one-dimensional particle systems with moving boundaries. *Ann. Probab.* **24** 559–598. MR1404521 <https://doi.org/10.1214/aop/1039639355>
- [18] CHOI, S. and KIM, I. C. (2010). Regularity of one-phase Stefan problem near Lipschitz initial data. *Amer. J. Math.* **132** 1693–1727. MR2766502
- [19] CHU, R., KIM, I. C., KIM, Y.-H. and NAM, K. (2023). The Nonlocal Stefan Problem via a Martingale Transport. arXiv preprint. Available at [arXiv:2310.04640](https://arxiv.org/abs/2310.04640).
- [20] COX, A. M. G. and WANG, J. (2013). Root’s barrier: Construction, optimality and applications to variance options. *Ann. Appl. Probab.* **23** 859–894. MR3076672 <https://doi.org/10.1214/12-AAP857>
- [21] DAHLBERG, B. E. J. (1979). Harmonic functions in Lipschitz domains. In *Harmonic Analysis in Euclidean Spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1*. *Proc. Sympos. Pure Math.*, XXXV, Part 1 313–322. Amer. Math. Soc., Providence, RI. MR0545271
- [22] DE ANGELIS, T. (2018). From optimal stopping boundaries to Rost’s reversed barriers and the Skorokhod embedding. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 1098–1133. MR3795078 <https://doi.org/10.1214/17-AIHP833>
- [23] DE PHILIPPIS, G., MÉSZÁROS, A. R., SANTAMBROGIO, F. and VELICHKOV, B. (2016). BV estimates in optimal transportation and applications. *Arch. Ration. Mech. Anal.* **219** 829–860. MR3437864 <https://doi.org/10.1007/s00205-015-0909-3>
- [24] DELARUE, F., INGLIS, J., RUBENTHALER, S. and TANRÉ, E. (2015). Global solvability of a networked integrate-and-fire model of McKean–Vlasov type. *Ann. Appl. Probab.* **25** 2096–2133. MR3349003 <https://doi.org/10.1214/14-AAP1044>
- [25] DELARUE, F., NADTOCHIY, S. and SHKOLNIKOV, M. (2022). Global solutions to the supercooled Stefan problem with blow-ups: Regularity and uniqueness. *Probab. Math. Phys.* **3** 171–213. MR4420299 <https://doi.org/10.2140/pmp.2022.3.171>
- [26] DIBENEDETTO, E. and FRIEDMAN, A. (1984). The ill-posed Hele–Shaw model and the Stefan problem for supercooled water. *Trans. Amer. Math. Soc.* **282** 183–204. MR0728709 <https://doi.org/10.2307/1999584>

- [27] DOLINSKY, Y. and SONER, H. M. (2014). Martingale optimal transport and robust hedging in continuous time. *Probab. Theory Related Fields* **160** 391–427. [MR3256817](https://doi.org/10.1007/s00440-013-0531-y) <https://doi.org/10.1007/s00440-013-0531-y>
- [28] FALKNER, N. (1980). On Skorohod embedding in n -dimensional Brownian motion by means of natural stopping times. In *Seminar on Probability, XIV (Paris, 1978/1979) (French)*. *Lecture Notes in Math.* **784** 357–391. Springer, Berlin. [MR0580142](https://doi.org/10.1007/BF00580142)
- [29] FIGALLI, A., ROS-OTON, X. and SERRA, J. (2020). Generic regularity of free boundaries for the obstacle problem. *Publ. Math. Inst. Hautes Études Sci.* **132** 181–292. [MR4179834](https://doi.org/10.1007/s10240-020-00119-9) <https://doi.org/10.1007/s10240-020-00119-9>
- [30] FIGALLI, A., ROS-OTON, X. and SERRA, J. (2023). The singular set in the Stefan problem. *J. Amer. Math. Soc.* <https://doi.org/10.1090/jams/1026>
- [31] GASSIAT, P., OBERHAUSER, H. and DOS REIS, G. (2015). Root’s barrier, viscosity solutions of obstacle problems and reflected FBSDEs. *Stochastic Process. Appl.* **125** 4601–4631. [MR3406597](https://doi.org/10.1016/j.spa.2015.07.010) <https://doi.org/10.1016/j.spa.2015.07.010>
- [32] GASSIAT, P., OBERHAUSER, H. and ZOU, C. Z. (2021). A free boundary characterisation of the Root barrier for Markov processes. *Probab. Theory Related Fields* **180** 33–69. [MR4265017](https://doi.org/10.1007/s00440-021-01052-6) <https://doi.org/10.1007/s00440-021-01052-6>
- [33] GHOUSSOUB, N., KIM, Y.-H. and LIM, T. (2019). Structure of optimal martingale transport plans in general dimensions. *Ann. Probab.* **47** 109–164. [MR3909967](https://doi.org/10.1214/18-AOP1258) <https://doi.org/10.1214/18-AOP1258>
- [34] GHOUSSOUB, N., KIM, Y.-H. and LIM, T. (2020). Optimal Brownian stopping when the source and target are radially symmetric distributions. *SIAM J. Control Optim.* **58** 2765–2789. [MR4149541](https://doi.org/10.1137/19M1270513) <https://doi.org/10.1137/19M1270513>
- [35] GHOUSSOUB, N., KIM, Y.-H. and PALMER, A. Z. (2019). PDE methods for optimal Skorokhod embeddings. *Calc. Var. Partial Differential Equations* **58** Paper No. 113, 31. [MR3959935](https://doi.org/10.1007/s00526-019-1563-7) <https://doi.org/10.1007/s00526-019-1563-7>
- [36] GHOUSSOUB, N., KIM, Y.-H. and PALMER, A. Z. (2021). A solution to the Monge transport problem for Brownian martingales. *Ann. Probab.* **49** 877–907. [MR4255133](https://doi.org/10.1214/20-aop1462) <https://doi.org/10.1214/20-aop1462>
- [37] GHOUSSOUB, N., KIM, Y.-H. and PALMER, A. Z. (2021). Optimal stopping of stochastic transport minimizing submartingale costs. *Trans. Amer. Math. Soc.* **374** 6963–6989. [MR4315594](https://doi.org/10.1090/tran/8458) <https://doi.org/10.1090/tran/8458>
- [38] GUO, G., TAN, X. and TOUZI, N. (2016). On the monotonicity principle of optimal Skorokhod embedding problem. *SIAM J. Control Optim.* **54** 2478–2489. [MR3549873](https://doi.org/10.1137/15M1025268) <https://doi.org/10.1137/15M1025268>
- [39] HOBSON, D. (2011). The Skorokhod embedding problem and model-independent bounds for option prices. In *Paris–Princeton Lectures on Mathematical Finance 2010. Lecture Notes in Math.* **2003** 267–318. Springer, Berlin. [MR2762363](https://doi.org/10.1007/978-3-642-14660-2_4) https://doi.org/10.1007/978-3-642-14660-2_4
- [40] JACOBS, M., KIM, I. and TONG, J. (2021). Darcy’s law with a source term. *Arch. Ration. Mech. Anal.* **239** 1349–1393. [MR4215195](https://doi.org/10.1007/s00205-020-01595-3) <https://doi.org/10.1007/s00205-020-01595-3>
- [41] KIM, I. C. (2003). Uniqueness and existence results on the Hele–Shaw and the Stefan problems. *Arch. Ration. Mech. Anal.* **168** 299–328. [MR1994745](https://doi.org/10.1007/s00205-003-0251-z) <https://doi.org/10.1007/s00205-003-0251-z>
- [42] LINDGREN, E. and MONNEAU, R. (2015). Pointwise regularity of the free boundary for the parabolic obstacle problem. *Calc. Var. Partial Differential Equations* **54** 299–347. [MR3385162](https://doi.org/10.1007/s00526-014-0787-9) <https://doi.org/10.1007/s00526-014-0787-9>
- [43] LOYNES, R. M. (1970). Stopping times on Brownian motion: Some properties of Root’s construction. *Z. Wahrscheinlichkeitstheorie Verw. Gebiete* **16** 211–218. [MR0292170](https://doi.org/10.1007/BF00534597) <https://doi.org/10.1007/BF00534597>
- [44] MCCONNELL, T. R. (1991). The two-sided Stefan problem with a spatially dependent latent heat. *Trans. Amer. Math. Soc.* **326** 669–699. [MR1008699](https://doi.org/10.2307/2001779) <https://doi.org/10.2307/2001779>
- [45] MEIRMANOV, A. M. (1992). *The Stefan Problem. De Gruyter Expositions in Mathematics* **3**. de Gruyter, Berlin. Translated from the Russian by Marek Niezgódka and Anna Crowley. [MR1154310](https://doi.org/10.1515/9783110846720.245) <https://doi.org/10.1515/9783110846720.245>
- [46] MEYER, P.-A. (1978). Convergence faible et compacité des temps d’arrêt d’après Baxter et Chacon. In *Séminaire de Probabilités, XII (Univ. Strasbourg, Strasbourg, 1976/1977)*. *Lecture Notes in Math.* **649** 411–423. Springer, Berlin. [MR0520015](https://doi.org/10.1007/BF00534597)
- [47] NADTOCHIY, S., SHKOLNIKOV, M. and ZHANG, X. (2021). Scaling limits of external multi-particle DLA on the plane and the supercooled Stefan problem. arXiv preprint. Available at <https://doi.org/10.4236/ojs.202109040>
- [48] OBŁÓJ, J. (2004). The Skorokhod embedding problem and its offspring. *Probab. Surv.* **1** 321–392. [MR2068476](https://doi.org/10.1214/154957804100000060) <https://doi.org/10.1214/154957804100000060>
- [49] PERKINS, E. (1986). The Cereteli–Davis solution to the H^1 -embedding problem and an optimal embedding in Brownian motion. In *Seminar on Stochastic Processes, 1985 (Gainesville, Fla., 1985)*. *Progr. Probab. Statist.* **12** 172–223. Birkhäuser, Boston, MA. [MR0896743](https://doi.org/10.1007/978-1-4612-4973-8_6)

[50] PERTHAME, B., QUIRÓS, F. and VÁZQUEZ, J. L. (2014). The Hele–Shaw asymptotics for mechanical models of tumor growth. *Arch. Ration. Mech. Anal.* **212** 93–127. MR3162474 <https://doi.org/10.1007/s00205-013-0704-y>

[51] ROOT, D. H. (1969). The existence of certain stopping times on Brownian motion. *Ann. Math. Stat.* **40** 715–718. MR0238394 <https://doi.org/10.1214/aoms/1177697749>

[52] ROST, H. (1971). The stopping distributions of a Markov Process. *Invent. Math.* **14** 1–16. MR0346920 <https://doi.org/10.1007/BF01418740>

[53] ROST, H. (1976). Skorokhod stopping times of minimal variance. In *Séminaire de Probabilités, X. Lecture Notes in Math.*, Vol. 511 194–208. Springer, Berlin. (Première partie, Univ Strasbourg, Strasbourg, année universitaire 1974/1975). MR0445600

[54] SANTAMBROGIO, F. (2015). *Optimal Transport for Applied Mathematicians: Calculus of Variations, PDEs, and Modeling. Progress in Nonlinear Differential Equations and Their Applications* **87**. Birkhäuser, Cham. MR3409718 <https://doi.org/10.1007/978-3-319-20828-2>

[55] SANTAMBROGIO, F. (2018). Crowd motion and evolution PDEs under density constraints. In *SMAI 2017—8^E Biennale Française des Mathématiques Appliquées et Industrielles. ESAIM Proc. Surveys* **64** 137–157. EDP Sci., Les Ulis. MR3883985 <https://doi.org/10.1051/proc/201864137>

[56] SHERMAN, B. (1970). A general one-phase Stefan problem. *Quart. Appl. Math.* **28** 377–382. MR0282082 <https://doi.org/10.1090/qam/282082>

[57] SKOROKHOD, A. V. (1965). *Studies in the Theory of Random Processes*. Addison-Wesley, Reading, MA. Translated from the Russian by Scripta Technica, Inc. MR0185620

[58] STRASSEN, V. (1965). The existence of probability measures with given marginals. *Ann. Math. Stat.* **36** 423–439. MR0177430 <https://doi.org/10.1214/aoms/1177700153>

[59] TRAN, G., SCHAEFFER, H., FELDMAN, W. M. and OSHER, S. J. (2015). An L^1 penalty method for general obstacle problems. *SIAM J. Appl. Math.* **75** 1424–1444. MR3366915 <https://doi.org/10.1137/140963303>

[60] VILLANI, C. (2003). *Topics in Optimal Transportation. Graduate Studies in Mathematics* **58**. Amer. Math. Soc., Providence, RI. MR1964483 <https://doi.org/10.1090/gsm/058>

[61] VILLANI, C. (2009). *Optimal Transport: Old and New. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **338**. Springer, Berlin. MR2459454 <https://doi.org/10.1007/978-3-540-71050-9>