# NORM BOUNDS ON EISENSTEIN SERIES.

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Abstract. We study the sup-norm bound (both individually and on average) for Eisenstein series on certain arithmetic hyperbolic orbifolds producing sharp exponents for the modular surface and Picard 3-fold. The methods involve bounds for Epstein zeta functions, and counting restricted values of indefinite quadratic forms at integer points.

## 1. Introduction

For a compact Riemannian manifold, $X$, one can show that if $\phi \in L^2(X)$ has $\|\phi\|_2 = 1$ and is an eigenfunction of the Laplace-Beltrami operator with eigenvalue $\lambda$, then $\|\phi\|_\infty \ll \lambda^{\frac{\dim(X)-1}{4}}$, see [SS89, Cor 2.2]. This bound, usually referred to as the convexity bound, is sharp in general. However, when $X$ has negative curvature, it is believed that this exponent can be improved (a log savings is obtained by Bérard [B́77]) and there are some results of this nature for cusp forms on some arithmetic hyperbolic manifolds. Explicitly, for arithmetic hyperbolic surfaces it is conjectured that $\|\phi\|_\infty \ll_\epsilon \lambda^\epsilon$, and it was shown in [IS95] that $\|\phi\|_\infty \ll_\epsilon \lambda^{5/24+\epsilon}$ for $\phi$ a Hecke-Maass cusp form. In higher dimensions, the situation is more complicated, as it was shown in the work Rudnick and Sarnak [RS94] and in more detail by Milićević [Mil11], that for any $\epsilon > 0$, there exists Hecke-Maass forms on a given arithmetic hyperbolic 3-manifold for which $\|\phi\|_\infty \gg \lambda^{1/4-\epsilon}$. Nevertheless, a subconvex upper bound of order $\lambda^{5/12+\epsilon}$ was proved in [Koy95, BHM16], so the truth is somewhere in between.

This paper is concerned with the analogous problem where the cusp form is replaced by an Eisenstein series. Explicitly, given a non-uniform lattice, $\Gamma$, acting on hyperbolic $n+1$ space $\mathbb{H}^{n+1}$, for each cusp $\xi$, let $E_{\Gamma,\xi}(s,z)$ denote the Eisenstein series corresponding to this cusp with normalization such that the constant term of the Fourier expansion based at $\xi$ is of the form $y^s + c_\xi(s)y^{n-s}$. Then $E_{\Gamma,\xi}(\frac{n}{2}+it, z)$ is an almost-$L^2$ eigenfunction of the Laplacian with eigenvalue $\lambda = \frac{n^2}{4} + t^2$. Since the Eisenstein series is unbounded as $z$ moves into the cusp, in order to consider the supremum norm, we need to restrict to a compact set. We define the parameter $\nu_\infty = \nu_\infty(\Gamma)$ as the infimum over all $\nu > 0$ such that for any compact set $\Omega \subseteq \mathbb{H}^{n+1}$, and any cusp $\xi$, we have

$$\sup_{z\in\Omega} |E_{\Gamma,\xi}(\frac{n}{2}+it, z)| \ll_\Omega |t|^\nu.$$

*Remark* 1.1. There is another natural way to normalize the Eisenstein series; instead of ensuring that the constant term is of the form $y^s + c_\xi(s)y^{n-s}$, one could $L^2$-normalize (locally

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in $\Omega$, since Eisenstein series are not $L^2$), that is, ask that

$$\int_\Omega \left| E_{\Gamma,\xi}\left(\frac{n}{2}+it, z\right)\right|^2 dz = 1,$$

where $dz$ is the hyperbolic volume form. When $\Gamma$ is arithmetic, the two normalizations are not too different, but in the nonarithmetic setting, very little is known about the arising discrepancies.

For numerous applications, it suffices to understand sup norm bounds on average; towards this, we consider the quantity $\nu_2 = \nu_2(\Gamma)$ defined as the infimum of all $\nu > 0$ such that for any compact set $\Omega \subseteq \mathbb{H}^{n+1}$, and any cusp $\xi$, we have

$$\int_{-T}^{T} \left( \sup_{z\in\Omega} |E_{\Gamma,\xi}(\frac{n}{2}+it, z)| \right)^2 dt \ \ll_\Omega \ T^{1+2\nu}.$$

Then clearly $\nu_2 \le \nu_\infty$ but one expects that we can give a sharper bound for $\nu_2$.

*Remark* 1.2. One example of an application is as follows. Given a rational quadratic form $Q$ of signature $(n+1,1)$, the number $N(X)$ of primitive integer points $v \in \mathbb{Z}^{n+2}$ on the light cone $Q=0$ of norm bounded by $X$ can be estimated precisely in terms of $\nu_2 = \nu_2(\mathrm{SO}_Q(\mathbb{Z}))$, see [KY22, Prop 3.5] showing that

$$N(X) = cX^n + O(X^{n\left(1-\frac{1}{2(\nu_2+1)}\right)}).$$

For another application, see [BNRW20].

1.1. **The case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.**
The main result of this paper is a determination of $\nu_2$ for the modular group.

**Theorem 1.** *For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, we have that $\nu_2(\Gamma) = 0$. That is, for any compact set $\Omega \subset \Gamma\backslash\mathbb{H}$, there is a constant $c = c(\Omega)$ such that for all $T \ge 1$,*

$$\int_{-T}^{T} \left( \sup_{z\in\Omega} |E_\Gamma(z, \frac{1}{2}+it)| \right)^2 dt \ \le \ cT\log^4(T). \tag{1.3}$$

*Here $E_\Gamma$ is the Eisenstein series at the (unique) cusp at $\infty$.*

*Remark* 1.4. For $\Gamma \le \mathrm{SL}_2(\mathbb{Z})$ a congruence subgroup, it is believed that $\nu_2(\Gamma) = \nu_\infty(\Gamma) = 0$. We expect that modifications of our techniques would show that $\nu_2(\Gamma) = 0$ also for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. For $\nu_\infty$, the convexity bound is $\nu_\infty(\Gamma) \le \frac{1}{2}$, and the work of Young and consequently Huang [You18, Hua19] using amplification gives the sub-convex bound of $\nu_\infty(\Gamma) \le \frac{3}{8}$ while the best result to date is due to Blomer [Blo20] who used the approximate functional equation and Burgess' bound to prove that $\nu_\infty(\mathrm{SL}_2(\mathbb{Z})) \le \frac{1}{3}$.

*Remark* 1.5. We note that for a general non-arithmetic lattice acting on $\mathbb{H}^2$, nothing is known about $\nu_\infty(\Gamma)$. For $\nu_2(\Gamma)$, it is likely that the known convexity bound $\nu_2(\Gamma) \le \frac{1}{2}$ is actually sharp (with the caveat that our normalization is not $L^2$, but rather that which arises in the pre-trace formula; see Remark 1.1). As evidence for this, we show that if $\nu_2(\Gamma, z) < \frac{1}{2}$ at some point $z \in \mathbb{H}^2$, then $\Gamma$ must have infinitely many cusp forms, in contrast with the Phillips-Sarnak conjecture [PS85], see Theorem 9 below and the discussion in Section 5. (Here we denote by $\nu_2(\Gamma, z)$ the exponent for a fixed point $z$.) Again, this might not necessarily signal the existence of a large jump in the size of the Eisenstein series near the single point $z$, but could rather come from the discrepancy with $L^2$ normalization.

The bulk of the paper is devoted to proving Theorem 1. A key ingredient of independent interest is obtaining strong bounds on the Epstein zeta function (see §1.3).

1.2. **The case $\Gamma = \mathrm{SL}_2(\mathbb{Z}[i])$.**
For the Picard group $\Gamma = \mathrm{SL}_2(\mathbb{Z}[i])$ acting on hyperbolic 3-space, one can resolve the issue completely. Using Blomer's upper bounds [Blo20] on Epstein zeta function, and connections between such and Eisenstein series (which are well-known for $\mathrm{SL}_2(\mathbb{Z})$ and derived here for $\Gamma = \mathrm{SL}_2(\mathbb{Z}[i])$), one obtains the following.

**Theorem 2.** *For $\Gamma = \mathrm{SL}_2(\mathbb{Z}[i])$, any compact set $\Omega \subset \Gamma\backslash\mathbb{H}^3$, and for any $\epsilon > 0$, there is a constant $c = c(\Omega, \epsilon)$ such that*

$$\sup_{z\in\Omega} |E_\Gamma(z, 1 + iT)| \leq cT^{1/2+\epsilon}. \tag{1.6}$$

*As a consequence, we have that*

$$\nu_2(\mathrm{SL}_2(\mathbb{Z}[i])) = \nu_\infty(\mathrm{SL}_2(\mathbb{Z}[i])) = 1/2. \tag{1.7}$$

We note that the upper bound for $\nu_\infty$ implied by (1.6) matches a lower bound for $\nu_2$ in work of Kelmer-Yu [KY22] (see Remark 1.9), leading to (1.7).

*Remark* 1.8. We similarly expect that Theorem 2 can be proved along the same lines for $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$ (or a congruence subgroup thereof) with $K$ an imaginary quadratic field. When $\Gamma$ is a congruence lattice in $\mathrm{SL}_2(\mathbb{C})$, the convexity sup norm bound is $\nu_\infty(\Gamma) \leq 1$, it is commonly believed that $\nu_\infty(\Gamma) = \frac{1}{2}$, and the best currently known bound comes from the work of Assing [Ass19] who used amplification to show that $\nu_\infty(\Gamma) \leq \frac{7}{8}$ (his result is more general and deals with Eisenstein series for $SL_2$ defined over general number fields).

*Remark* 1.9. For lattices acting on hyperbolic $(n+1)$-space, even the "convexity" bound $\nu_\infty(\Gamma) \leq \frac{n}{2}$ is not known in general, not even for general arithmetic lattices. Nevertheless, in [KY22], this convexity bound was proved for a large family of arithmetic lattices. While it may be expected that $\nu_\infty = \frac{n-1}{2}$, there are no known subconvex bounds when $n \geq 3$. (Note that with a different normalization – see Remark 1.1 – an argument due to Sarnak [Sar04] shows that the $L^\infty$ norm is controlled by the $L^2$ norm on compacta, thus proving the convexity bound with this other normalization.)

For sup norm bounds on average, the convexity bound $\nu_2(\Gamma) \leq \frac{n}{2}$ is known to hold for a general non-uniform lattice [CS80, Cor 7.7]. Note that when $n \in \{1, 2, 3, 5, 7\}$, for certain arithmetic lattices, it is possible to evaluate $E(s, z_0)$ at special points as a product of zeta functions and $L$-functions (see [KY22]), from which one can conclude that $\nu_2(\Gamma, z_0) = \frac{n-1}{2}$ at these points. This gives a lower bound on what can be expected to hold in general.

1.3. **Mean square bounds on Epstein zeta functions.**

Both Theorems 1 and 2 are derived (in section 4) from the following bounds on Epstein zeta functions. Given a positive-definite quadratic form $Q$ in $m$ variables, let

$$Z_Q(s) := \sum_{v \in \mathbb{Z}^m \setminus 0} Q(v)^{-s}$$

denote the Epstein zeta function, defined originally in some half-plane $\mathfrak{Re}(s) \gg 1$, and having a well-known meromorphic continuation to $s \in \mathbb{C}$ (see [Ter73]) and functional equation relating $Z_Q(s)$ to $Z_{Q_-}(\frac{m}{2} - s)$, where $Q_-$ is the dual form. That is, if $Q$ is given by $Q(x) = x^T Z x$, then $Q_-(x) = x^T Z^{-1} x$.

**Theorem 3.** *Let $Q$ be a positive-definite quadratic form in $m$ variables, and let the associated Epstein zeta function be $Z_Q$. If $m = 2$ then*

$$\int_T^{2T} |Z_Q(\frac{1}{2} + it)|^2 dt \ \ll_Q \ T \log^2(T). \tag{1.10}$$

*Moreover, if $m \geq 3$, then for any $\varepsilon > 0$,*

$$\int_T^{2T} \left| Z_Q\left(\frac{m}{4} + it\right)\right|^2 dt \ \ll_Q \ T^{m/2+\varepsilon}. \tag{1.11}$$

*In either case, the implicit constants depend on the form $Q$, but may be taken uniform as $Q$ varies in a compact set in the space of positive-definite quadratic forms.*

We note that the above result is only new for $m = 2, 3$. While the $m = 3$ case is not related to bounds on Eisenstein series, we nevertheless include it for its intrinsic interest. For $m \geq 4$, (1.11) follows immediately from Blomer's pointwise bounds for the Epstein zeta function.

**Theorem 4** ([Blo20, Theorem 1])**.** *Let $Q$ be a positive-definite quadratic form in $m$ variables, with $m \geq 4$. Then for any $\varepsilon > 0$,*

$$\left| Z_Q\left(\frac{m}{4} + it\right)\right| \ll_{Q,\varepsilon} T^{(m-2)/4+\varepsilon}.$$

Blomer also gives pointwise bounds for $m = 2$ and $m = 3$, but these are weaker than what is needed for the $L^2$ bounds in Theorem 3.

1.4. **Values of indefinite forms at integer points.**

To prove Theorem 3, we require a result from the geometry of numbers, namely the following uniform version of [EMM98, Theorem 2.3].

**Theorem 5.** *For any $n = p + q \geq 3$ with $p \geq q \geq 1$, and any form $Q(v)$ of signature $(p, q)$ having discriminant one, there are constants $c = c(Q)$, $A_0$, and $B_0$ such that, for any $A \geq A_0$ and $B \geq B_0$, if $(p, q) \notin \{(2, 1), (2, 2)\}$, then*

$$\#\{v \in \mathbb{Z}^n : \|v\| \leq A,\ Q(v) \in (-B, B)\} \leq cBA^{n-2}.$$

*while for $(p, q) \in \{(2, 1), (2, 2)\}$,*

$$\#\{v \in \mathbb{Z}^n : \|v\| \leq A,\ Q(v) \in (-B, B)\} \leq cBA^{n-2}\log(A).$$

*The constant $c$ can be taken uniform for $Q$ ranging in a compact set.*

*Remark* 1.12. The result of [EMM98, Theorem 2.3] gives a similar upper bound to

$$\#\{v \in \mathbb{Z}^n : \|v\| \le A,\ Q(v) \in (a,b)\}$$

with the bound depending implicitly on the form $Q$ and the target interval $(a,b)$. The novelty of our result is to make the dependence on the target interval explicit.

**Outline.** In Section 2, we prove Theorem 5. Then in Section 3.1 we first focus on the case $m = 2$ of Theorem 3; and settle the case $m \ge 3$ in Section 3.2. We then show in Section 4 how to derive the bounds on the Eisenstein series from Theorem 3. Finally, in Section 5, we explicate Remark 1.5.

**Notation.** We use standard Vinogradov notation that $f \ll g$ if there is a constant $C > 0$ so that $f(x) \le Cg(x)$ for all $x$. When the implied constant depends on more than the lattice $\Gamma$ or $Q$ or $z$ varying in a compact set $\Omega$, which we think of as fixed, we denote this with a subscript.

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## 2. Counting estimate

In this section we prove Theorem 5. We first recall the main ideas in the proof of [EMM98]. Let $Q$ be a quadratic form of signature $(p,q)$ and let $g \in \mathrm{SL}_n(\mathbb{R})$ such that $Q(v) = Q_0(gv)$ with

$$Q_0(x) = x_1 x_n + \sum_{i=2}^{p} x_i^2 - \sum_{i=p+1}^{n-1} x_i^2.$$

We let $g$ (and hence $Q$) vary in a compact set, and fix a $\beta > 0$ so that $\max\{\|g\|, \|g^{-1}\|\} \le \beta$ in this compact region.

Let $H = \mathrm{SO}_{Q_0}(\mathbb{R})$, let $K = H \cap \mathrm{SO}(n)$ be a maximal compact, and let $a_t = \mathrm{diag}(e^{-t}, 1 \ldots, 1, , e^t) \in H$. For any real-valued, compactly supported function $f$ on $\mathbb{R}^n$, any $r > 0$, and any $\xi \in \mathbb{R}$, define the function

$$J_f(r,\xi) = \frac{1}{r^{n-2}} \int_{\mathbb{R}^{n-2}} f(r, x_2, \ldots, x_{n-1}, \frac{\xi - Q_0(0, x_2, \ldots, x_{n-1}, 0)}{2r}) dx_2 \ldots dx_{n-1}.$$

Then by [EMM98, Lemma 3.6], there is a constant $c_{p,q}$ and $T_0 > 1$ such that for every $t \ge \log(T_0)$ and any $v \in \mathbb{R}^n$ with $\|v\| > T_0$, we have that

$$\left| J_f(\|v\| e^{-t}, Q_0(v)) - c_{p,q} e^{(n-2)t} \int_K f(a_t k v) dm(k) \right| \le 1.$$

In particular, if we choose $f$ so that $J_f(r,\xi) \ge 2$ for all $r \in (1,2)$ and $\xi \in (a,b)$, then for any $v \in \mathbb{R}^n$ with $e^t \le \|gv\| \le 2e^t$ and $Q_0(gv) \in (a,b)$, we have that $J_f(\|gv\| e^{-t}, Q_0(gv))) \ge 2$, whence $c_{p,q} e^{(n-2)t} \int_K f(a_t k v) dm(k) \ge 1$ and

$$\begin{aligned} \#\{v \in \mathbb{Z}^n : Q_0(gv) \in (a,b), \|gv\| \in [e^t, 2e^t]\} &\le \sum_{v \in \mathbb{Z}^n} c_{p,q} e^{(n-2)t} \int_K f(a_t k g v) dm(k) \\ &= c_{p,q} e^{(n-2)t} \int_K \widehat{f}(a_t k g) dm(k) \qquad (2.1) \end{aligned}$$

where

$$\widehat{f}(g) = \sum_{0 \neq v \in \mathbb{Z}^n} f(gv)$$

is the Siegel transform. Next, using [Sch68, Lemma 2] we can bound

$$\widehat{f}(g) \leq c_f \alpha(g\mathbb{Z}^n), \tag{2.2}$$

where the function $\alpha(\Lambda)$ is the function on the space of lattices defined in [EMM98, equation (3.3)], and $c_f$ is a constant depending only on $f$. Finally, [EMM98, Theorem 3.2 and Theorem 3.3] state that

$$\int_K \alpha(a_t k g\mathbb{Z}^n) dm(k) \leq c_g,$$

is uniformly bounded when $(p, q) \notin \{(2, 2), (2, 1)\}$ and that

$$\int_K \alpha(a_t k g\mathbb{Z}^n) dm(k) \leq c_g t,$$

when $(p, q) = (2, 2)$ or $(p, q) = (2, 1)$. Here the constant $c_g$ is uniform when $g$ is taken from a compact set. Using this we get that when $(p, q) \notin \{(2, 2), (2, 1)\}$

$$\#\{v \in \mathbb{Z}^n : Q_0(gv) \in (a, b), \|gv\| \in [e^t, 2e^t]\} \leq c_{p,q} c_f c_g e^{(n-2)t},$$

and that for $(p, q) \in \{(2, 2), (2, 1)\}$

$$\#\{v \in \mathbb{Z}^n : Q_0(gv) \in (a, b), \|gv\| \in [e^t, 2e^t]\} \leq c_{p,q} c_f c_g t e^{(n-2)t}.$$

Summing over $t \leq \log(T)$ in dyadic intervals gives a bound of the form

$$\#\{v \in \mathbb{Z}^n : Q_0(gv) \in (a, b), \|gv\| \leq T\} \leq cT^{n-2},$$

when signature $(p, q) \notin \{(2, 2), (2, 1)\}$ and

$$\#\{v \in \mathbb{Z}^n : Q_0(gv) \in (a, b), \|gv\| \leq T\} \leq cT^{n-2} \log(T),$$

otherwise. Here the constant $c$ depends on the signature on the group element $g$, and on the function $f$ (and hence on the interval $(a, b)$.

Up to this point, the proof is identical to the treatment in [EMM98]. It is at this moment where we need one simple extra ingredient to make everything uniform, in the special case of a target interval $(-B, B)$. Our goal is to find a suitable function $f$ so that $J_f(r, \xi) \geq 2$ when $r \in [1, 2]$ and $|\xi| \leq B$ such that $c_f \leq cB$. We first note that [Sch68, Lemma 2] implies that for any $R \geq 1$ and any lattice $\Lambda = g\mathbb{Z}^n$, we have the bound

$$\#\{v \in \Lambda : \|v\| \leq R\} \leq cR^n \alpha(\Lambda),$$

with $c$ and absolute constant depending only on $n$. We combine this with the following simple observation.

**Lemma 6.** *For any $x \in \mathbb{R}^n$ and any lattice $\Lambda$ we can bound*

$$\#\{v \in \Lambda : \|v - x\| \leq R\} \leq \#\{v \in \Lambda : \|v\| \leq 2R\}.$$

*Proof.* If there is no $v \in \Lambda$ with $\|v - x\| \leq R$, then this is obvious. Otherwise, let $u \in \Lambda$ satisfy $\|u - x\| \leq R$; then $\|v - x\| \leq R$ implies that

$$\|v - u\| = \|v - x + x - u\| \leq 2R.$$

Since $v \in \Lambda$ if and only if $v - u \in \Lambda$, this concludes the proof. □

We now describe our choice of $f$. Assume that $B \geq (n-2)$ and let $f$ take values in $[0,2]$ and supported on $[0,3]\times[-2,2]^{n-2}\times[-2B,2B]$ such that $f(x)=2$ on $[1,2]\times[-1,1]^{n-2}\times[-B,B]$. Note that for any $r\in[1,2]$ and $\xi\in[-B,B]$ and $(x_2,\dots,x_{n-2})\in[-1,1]^{n-2}$ we have that $\frac{\xi - Q_0(0,x_2,\dots,x_{n-1},0)}{2r}\in[-B,B]$ so that

$$f(r,x_2,\dots,x_{n-1},\frac{\xi-Q_0(0,x_2,\dots,x_{n-1},0)}{2r})=2.$$

We thus get a lower bound $J_f(r,\xi)\geq \frac{2}{2^{n-2}}\int_{[-1,1]^{n-2}}dx_2\dots dx_{n-1}=2$. On the other hand, we can cover the support of $f$ by $2B+1$ balls of radius $R_n=2\sqrt{n}$ centered at the points $v_j=(1,0,\dots,0,2j)$ with $-B\leq j\leq B$. From Lemma 6, we can bound

$$\#\{v\in\Lambda:\|v-v_j\|\leq R_n\}\leq\#(\Lambda\cap 2R_n\}\leq c_n\alpha(\Lambda),$$

where the constant depends only on $n$. Using this, we can bound the Siegel transform $\widehat{f}(g)\leq 2c_nB\alpha(\Lambda)$. Plugging this estimate into (2.2) and (2.1) concludes the proof of Theorem 5.

## 3. Bounds on the Epstein zeta function

3.1. **Bounds for $m=2$.** In this section we fix $m=2$ and prove (1.10). In [SV05], the authors considered the case of an integral quadratic form $Q$ and proved an approximate functional equation as well as a formula for the mean square. While the formula for the mean square is special for a family of integral quadratic forms, the approximate functional equation [SV05, Theorem 1] holds in general. We record this result in our special case as follows.

**Theorem 7.** *For $Q$ a positive definite quadratic form of discriminant $D$ with dual form $Q_-$ and sequences $a_n$, $\lambda_n$, $b_n$, $\mu_n$ defined for $\mathfrak{Re}(s)\gg 1$ by:*

$$Z_Q(s)=\sum_{\lambda_n}\frac{a_n}{\lambda_n^s},\qquad \text{and}\qquad Z_{Q_-}(s)=\sum_{\mu_n}\frac{b_n}{\mu_n^s},$$

*we have, for $s=\frac{1}{2}+it$ with $|t|\geq 1$, that:*

$$Z_Q(s)=\sum_{\lambda_n\leq X}\frac{a_n}{\lambda_n^s}+\chi(s)\sum_{\mu_n\leq X}\frac{b_n}{\mu_n^{1-s}}+O_D(\log(|t|)),$$

*where*

$$\chi(s)=(\frac{\sqrt{D}}{\pi})^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)},$$

*and $X=X(t):=\frac{|t|\sqrt{D}}{\pi}$.*

Noting that $|\chi(s)|=1$ for $s=\frac{1}{2}+it$, we can estimate

$$|Z_Q(\frac{1}{2}+it)|^2\ll F_Q(t)+F_{Q_-}(t)+O(\log^2(t)), \tag{3.1}$$

where we have set

$$F_Q(t) := \left| \sum_{\lambda_n \le X(t)} \frac{a_n}{\lambda_n^{\frac{1}{2}+it}} \right|^2 .$$

Recall again that $X(t) = \frac{|t|\sqrt{D}}{\pi} \asymp |t|$.

Integrating (3.1) gives:

$$\int_T^{2T} |Z_Q(\frac{1}{2}+it)|^2 dt \ll \int_T^{2T} F_Q(t)dt + \int_T^{2T} F_{Q_-}(t)dt + O(T\log^2(T)). \tag{3.2}$$

These terms can be estimated as:

$$\begin{aligned} \int_T^{2T} F_Q(t)dt &= \int_T^{2T} \sum_{\substack{u,v\in\mathbb{Z}^2\setminus 0 \\ Q(u),Q(v)\le X(t)}} \frac{1}{Q(u)^{1/2+it}Q(v)^{1/2-it}} dt \\ &\ll \sum_{\substack{u,v\in\mathbb{Z}^2\setminus 0 \\ \|v\|,\|u\|\ll\sqrt{T}}} \frac{1}{\|u\|\|v\|} \left| \int_{\max\{Q(u),Q(v),T\}}^{2T} e^{it\log(\frac{Q(v)}{Q(u)})} dt \right|, \end{aligned}$$

where we used that $Q(u) \asymp \|u\|^2$.

We now break this into different regions depending on the ratio of $\frac{Q(u)}{Q(v)}$. First consider the range when $|\log(\frac{Q(u)}{Q(v)})| \ge 1$. In this range, we can bound the inner integral by 2 to get a bound of

$$\sum_{\substack{u,v\in\mathbb{Z}^2 \\ \|v\|,\|u\|\ll\sqrt{T}}} \frac{1}{\|u\|\|v\|} \ll \left( \sum_{\substack{v\in\mathbb{Z}^2 \\ \|v\|\ll\sqrt{T}}} \frac{1}{\|v\|} \right)^2 \ll T.$$

For the rest, we have that $Q(u) \asymp Q(v)$ so also $\|u\| \asymp \|v\|$, and we break the sum into dyadic intervals $A \le \|v\| \le 2A$ and $\frac{B}{T} \le |\log(\frac{Q(u)}{Q(v)})| \le \frac{2B}{T}$, with $A \le \sqrt{T}$ and $B \le T$. The contribution of each such dyadic interval is then given by

$$\begin{aligned} N_T(A,B) &= \sum_{\substack{v\in\mathbb{Z}^2 \\ A\le\|v\|\le 2A}} \sum_{\substack{u\in\mathbb{Z}^2 \\ |\log(\frac{Q(u)}{Q(v)})|\in(\frac{B}{T},\frac{2B}{T})}} \frac{1}{\|u\|\|v\|} \left| \int_{\max\{Q(u),Q(v),T\}}^{2T} e^{it\log(\frac{Q(u)}{Q(v)})} dt \right| \\ &\ll \frac{T}{A^2B} \#\{u,v\in\mathbb{Z}^2 : \|v\|\le 2A,\ |\frac{Q(u)}{Q(v)} - 1| \le \frac{2B}{T}\} \\ &\ll \frac{T}{A^2B} \#\{(u,v)\in\mathbb{Z}^4 : \|(u,v)\|\le 2A,\ |Q(u)-Q(v)| \ll \frac{A^2B}{T}\}. \end{aligned}$$

We also have the range where $|\log(\frac{Q(u)}{Q(v)})|\leq \frac{1}{T}$ that we can similarly bound by

$$\begin{aligned} N_T(A) &= \sum_{\substack{v\in\mathbb{Z}^2\\ A\leq\|v\|\leq 2A}} \sum_{\substack{u\in\mathbb{Z}^2\\ |\log(\frac{Q(u)}{Q(v)})|\in[0,\frac{1}{T})}} \frac{1}{\|u\|\|v\|}\left|\int_{\max\{Q(u),Q(v),T\}}^{2T} e^{it\log(\frac{Q(u)}{Q(v)})}dt\right| \\ &\ll \frac{T}{A^2}\#\{(u,v)\in\mathbb{Z}^4 : \|(u,v)\|\leq 2A,\ |Q(u)-Q(v)|\ll 1\}. \end{aligned}$$

Note that for $Q(v)$ a positive definite binary quadratic form, the form

$$\tilde{Q}(u,v) = Q(u) - Q(v)$$

has signature $(2,2)$. Applying Theorem 5 gives

$$\#\{v\in\mathbb{Z}^4 : \|v\|\leq A,\ \tilde{Q}(v)\in(-B,B)\} \leq cBA^2\log(A),$$

from which we can bound

$$N_T(A,B) \ll A^2\log(A) \ll A^2\log(T)$$

for $A\leq\sqrt{T}$. We also have the bound $N_T(A)\leq cT\log(T)$ for all $A\leq\sqrt{T}$. Taking $A,B$ to be powers of 2 and summing over $A\leq\sqrt{T}, B\leq T$, we get the bound

$$\int_T^{2T}|Z_Q(\frac{1}{2}+it)|^2dt \ll T\log^2(T),$$

as claimed in (1.10).

3.2. **Bounds for $m\geq 3$.** Turning now to (1.11) for $m\geq 3$, we follow the same approach, however instead of using the approximate functional equation of [SV05], we instead use the following.

**Theorem 8** ([Blo20, (2.2)]). *Recalling that $Q_-$ is the dual form to $Q=Q_+$, we have that*

$$Z_Q(\frac{m}{4}+it) \ll 1 + |t|^\varepsilon \sum_{\pm}\sum_{A}\int_{|w|\leq|t|^\varepsilon} A^{-m/4}\left|\sum_{v\neq 0}\frac{V_A(Q_\pm(v))}{Q_\pm(v)^{\pm it+iw}}\right|dw \tag{3.3}$$

*where the sum over $A$ ranges over powers of 2 less than $|t|^{1+\varepsilon}$, and $V_A$ is bounded and has compact support in $[A,3A]$.*

Now consider the $L^2$ norm and expand the square using Cauchy-Schwarz to get that

$$\begin{aligned} \int_T^{2T}|Z_Q(\frac{m}{4}+it)|^2dt &\ll \int_T^{2T}\left|1+t^\epsilon\sum_{\pm}\sum_{A}\int_{|w|\leq|t|^\varepsilon}A^{-m/4}\left|\sum_{v\neq 0}\frac{V_A(Q_\pm(v))}{Q_\pm(v)^{\pm it+iw}}\right|dw\right|^2dt \\ &\ll \int_T^{2T}\left(1+t^\epsilon\sum_{\pm}\sum_{\substack{A\leq t^{1+\epsilon}\\ \text{dyadic}}}\int_{|w|\leq t^\epsilon}A^{-m/2}\left|\sum_{v\neq 0}\frac{V_A(Q_\pm(v))}{Q_\pm(v)^{\pm it+iw}}\right|^2\right)dt \\ &\ll T + T^\varepsilon\sum_{\pm}\sum_{\substack{A\leq T^{1+\epsilon}\\ \text{dyadic}}}A^{-m/2}\int_{|w|\leq T^\epsilon}\int_{\max\{T,A,|w|^{1/\epsilon}\}}^{2T}\left|\sum_{v\neq 0}\frac{V_A(Q_\pm(v))}{Q_\pm(v)^{\pm it+iw}}\right|^2dt. \end{aligned}$$

For any fixed $A$ and $w$ in this range and for each of the choices of $Q = Q_+$ or $Q = Q_-$, we open the square and estimate the integral

$$\int_{\max\{T,A,|w|^{1/\epsilon}\}}^{2T} \left|\sum_{v\neq 0} \frac{V_A(Q(v))}{Q(v)^{it+iw}}\right|^2 dt \ll \sum_{v,u\neq 0} V_A(Q(v))V_A(Q(u)) \left|\int_{\max\{T,A,|w|^{1/\epsilon}\}-w}^{2T-w} e^{it\log\frac{Q(v)}{Q(u)}}dt\right|$$

depending on the range of $\frac{Q(v)}{Q(u)}$.

If $|\log(\frac{Q(v)}{Q(u)})|\geq 1$, the integral is bounded independently on the boundaries of integration. Since $V_A(Q(v))$ is bounded and supported on $Q(u) \leq 3A$, the sum over $\|u\|\ll Q(u)^{1/2} \ll A^{1/2}$ and $\|v\|\ll A^{1/2}$ is bounded by $O(A^m)$. In the remaining case we may assume that $\|v\|\asymp \|u\|\asymp A^{1/2}$. Now further break the summation by taking $\frac{B}{T} \leq |\log(\frac{Q(v)}{Q(u)})|\leq \frac{2B}{T}$, with $B \ll T$ dyadic. For each $B \geq 1$ we get a term of the form

$$\begin{aligned} N_T(A,B) &= \sum_{\substack{v\in\mathbb{Z}^m \\ \sqrt{A}\leq\|v\|\leq\sqrt{3A}}} \sum_{\substack{u\in\mathbb{Z}^m \\ |\log(\frac{Q(v)}{Q(u)})|\in(\frac{B}{T},\frac{2B}{T})}} \left|\int_{\max\{T,A,|w|^{1/\epsilon}\}-w}^{2T-w} e^{it\log(\frac{Q(v)}{Q(u)})}dt\right| \\ &\ll \frac{T}{B}\#\{(u,v)\in\mathbb{Z}^{2m} : \|(u,v)\|\leq\sqrt{2A},\ |Q(u)-Q(v)|\ll \frac{AB}{T}\} \end{aligned}$$

and for $B \leq 1$ we let

$$\begin{aligned} N_T(A) &= \sum_{\substack{v\in\mathbb{Z}^m \\ \sqrt{A}\leq\|v\|\leq\sqrt{3A}}} \sum_{\substack{u\in\mathbb{Z}^m \\ |\log(\frac{Q(v)}{Q(u)})|\in(0,\frac{1}{T})}} \left|\int_{\max\{T,A,|w|^{1/\epsilon}\}-w}^{2T-w} e^{it\log(\frac{Q(v)}{Q(u)})}dt\right| \\ &\ll T\#\{(u,v)\in\mathbb{Z}^{2m} : \|(u,v)\|\leq\sqrt{2A},\ |Q(u)-Q(v)|\ll 1\}. \end{aligned}$$

Again following our approach for $m = 2$, we note that $\tilde{Q}(u,v) := Q(u) - Q(v)$ is a form of signature $(m,m)$, and apply Theorem 5. It follows that

$$N_T(A,B) \ll \frac{T}{B}A^{m-1}\frac{AB}{T} = A^m. \tag{3.4}$$

Similarly, we can trivially bound $N_T(A)$ by $A^m$.

Plugging these bounds back, bounding the integral of $w$ by $O(T^\epsilon)$ and summing over $A, B$ powers of 2 we get the bound

$$\int_T^{2T} |Z_Q(1+it)|^2 dt \ll T + T^\varepsilon \sum_{\substack{A\leq T^{1+\epsilon} \\ \text{dyadic}}} A^{-m/2}\left(A^m + N_T(A) + \sum_{\substack{B\ll T \\ \text{dyadic}}} N_T(A,B)\right) \ll T^{m/2+\varepsilon},$$

as claimed in (1.11).

## 4. Bounds on Eisenstein series: Proof of Theorems 1 and 2

To move from bounds on the Epstein zeta function to the Eisenstein series, argue as follows. First note that for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ we have the following well known identity (for $\mathfrak{Res} > 1$):

$$\zeta(2s)E_\Gamma(s,z) = y^s \sum_{(c,d)\neq(0,0)} \frac{1}{((x^2+y^2)c^2+2xcd+d^2)^s} = y^s Z_Q(s),$$

where $Q(c,d) = Q_z(c,d) = (x^2+y^2)c^2 + 2xcd + d^2$ and

$$Z_Q(s) = \sum_{v \in \mathbb{Z}^2 \setminus 0} Q(v)^{-s}.$$

Applying (1.10) and the well-known bound $\zeta(1+2it) \gg \log(t)^{-1}$, we conclude (1.3). This completes the proof of Theorem 1.

Next we show that for $\Gamma = \mathrm{SL}_2(\mathbb{Z}[i])$ we can again write $E_\Gamma(s,z)$ in terms of an Epstein zeta function. Using the upper-half-space model of hyperbolic 3-space,

$$\mathbb{H}^3 = \{z = x_1 + ix_2 + jy \ : \ x_j \in \mathbb{R}, y > 0\},$$

the Eisenstein series is defined (for $\mathfrak{Re}\, s > 2$) by

$$E_\Gamma(s,z) = \sum_{\substack{c,d \in \mathbb{Z}[i] \\ \text{co-prime}}} \frac{y^s}{N(cz+d)^s},$$

where $N(z) = x_1^2 + x_2^2 + y^2$ denotes the norm on the quaternions. Since the norm is multiplicative, if we write

$$\zeta_{\mathbb{Z}[i]}(s) = \sum_{\alpha \in \mathbb{Z}[i]} \frac{1}{N(\alpha)^s},$$

then we can simplify the Eisenstein series to

$$\frac{1}{4}\zeta_{\mathbb{Z}[i]}(s) E(s,z) = \sum_{(c,d) \neq (0,0)} \frac{y^s}{N(cz+d)^s}.$$

Moreover, we can write $\zeta_{\mathbb{Z}[i]}(s) = 4\zeta(s)L(s,\chi_1)$ with $\chi_1$ the quadratic Dirichlet character modulo 4. Now if we expand the norm on the right hand side, we arrive at

$$\zeta(s)L(s,\chi_1)E(s,z) = y^s \sum_{(c,d) \neq (0,0)} \frac{1}{Q_z(c,d)^s},$$

where

$$Q_z(c,d) = N(z)c_1^2 + N(z)c_2^2 + d_1^2 + d_2^2 + 2(x_1c_1d_1 - x_2c_2d_1 + x_1c_2d_2 + x_2c_1d_2)$$

is a positive definite quaternary quadratic form. Again we have good control on $\zeta(s)$ for $s = 1 + it$ and on $L(s,\chi_1)$. Thus the problem reduces to estimating the Epstein zeta function

$$Z_{Q_z}(s) = \sum_{v \in \mathbb{Z}^4 \setminus 0} Q_z(v)^{-s},$$

and (1.6) follows easily from Theorem 4. This completes the proof of Theorem 2.

## 5. Sharpness

We now consider the case of a general non-arithmetic lattice and show that any subconvex bound for $\nu_2(\Gamma)$ implies the existance of infinitely many Maass cusp forms. Explicitly we show the following.

**Theorem 9.** *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a non uniform lattice and assume that there is some $z_0$ such that $\nu_2(\Gamma, z_0) < 1/2$. Then there are infinitely many Maass cusp forms $\varphi_j \in L^2(\Gamma\backslash\mathbb{H})$ with $\triangle\varphi_j + \lambda_j\varphi_j = 0$. Moreover, they satisfy the local Weyl law: for any test function $h(r)$ with Fourier transform smooth and compactly supported, for all sufficiently large $T$,*

$$\sum_j h(\frac{r_j}{T})|\varphi_j(z)|^2 = T^2\frac{|\Gamma_z|}{2\pi}\int_0^\infty h(r)rdr + O_{\delta,h}(T^{2-\delta})$$

*for some $\delta > 0$, where $\Gamma_z = \{\gamma \in \Gamma : \gamma z = z\}$ and $\{\varphi_j\}_{j\in\mathbb{N}}$ form an orthonormal system of Maass forms in $L^2(\Gamma\backslash\mathbb{H})$ with eigenvalue parametrized by $\lambda_j = \dfrac{1}{4} + r_j^2$.*

*Proof.* We recall some well known results on the pre-trace formula and refer to [Hej76] for more details. Given a point pair invariant $k(z, w) = k(\sinh^2(d(z, w)))$ with $d(z, w)$ the hyperbolic distance and $k \in C_c^\infty(\mathbb{R}^+)$, its spherical transform is defined as $H(s) = \int_{\mathbb{H}^2} k(z, i)\mathfrak{Im}(z)^s d\mu(z)$. By [Hej76, Proposition 4.1] the point pair invariant can be recovered from $H(s)$ as follows : Let $h(r) = H(\dfrac{1}{2} + ir)$ and let $g(u) = \frac{1}{2\pi}\int_{-\infty}^\infty h(r)e^{-iru}dr$ denote its Fourier transform, then, defining the auxiliary function $Q \in C_c^\infty(\mathbb{R}^+)$ by $g(u) = Q(\sinh^2(\frac{u}{2}))$ we have that $k(t) = -\frac{1}{\pi}\int_t^\infty \frac{dQ(r)}{\sqrt{r-t}}$. We also recall that $k(0) = \frac{1}{2\pi}\int_0^\infty h(r)r\tanh(\pi r)dr$ (see [Hej76, Proposition 6.4]).

Given any such point pair invariant we have the pre-trace formula

$$\sum_{\gamma\in\Gamma} k(z, \gamma z) = \sum_j h(r_j)|\varphi_j(z)|^2 + \sum_{i=1}^{\kappa} \frac{1}{2\pi}\int_{\mathbb{R}} h(r)|E_{\Gamma,\xi_i}(\frac{1}{2} + ir, z)|^2 dr,$$

where $\xi_1, \ldots, \xi_\kappa$ are the cusps of $\Gamma$.

Now, fix a smooth compactly supported function $g(u) \in C_c^\infty((-1, 1))$ and for any $T \geq 1$ let $g_T(u) = Tg(Tu)$ so that $h_T(r) = h(\frac{r}{T})$ and $k_T(z, w)$ the corresponding point pair invariant. Since $g_T(u)$ is supported on $(-\frac{1}{T}, \frac{1}{T})$ the point pair invariant $k(z, w)$ is supported on the set $\{(z, w)|d(z, w) \leq \frac{1}{T}\}$ with $d(z, w)$ the hyperbolic distance. Since $\Gamma$ acts properly discontinuously on $\mathbb{H}^2$ for any fixed $z$ there is $\delta = \delta(z)$ such that $d(z, \gamma z) \geq \delta$ for any $\gamma \in \Gamma$ with $\gamma z \neq z$. In particular taking $T_0 \geq \frac{1}{\delta(z)}$ for any $T \geq T_0$ we have that $k_T(z, \gamma z) = 0$ if $\gamma z \neq z$. Hence for any $T \geq T_0$ we have

$$\sum_j h(\frac{r_j}{T})|\varphi_j(z)|^2 + \sum_{i=1}^{\kappa} \frac{1}{2\pi}\int_{\mathbb{R}} h(\frac{r}{T})|E_{\Gamma,\xi_i}(\frac{1}{2} + ir, z)|^2 dr = |\Gamma_z|k(0).$$

Denote by $\nu_2 = \nu_2(\Gamma, z)$ and note that for any $\ell \geq 0$ we can bound $h(t) \ll_{\ell,h} |r|^{-\ell}$. We can thus bound the contribution of the integrals over Eisenstein series by

$$\frac{1}{2\pi}\int_{\mathbb{R}} h(\frac{r}{T})|E_{\Gamma,\xi_i}(\frac{1}{2}+ir,z_0)|^2 dr \;=\; \sum_{k\in\mathbb{Z}}\int_{kT}^{(k+1)T} h(\frac{r}{T})|E_{\Gamma,\xi_i}(\frac{1}{2}+ir,z)|^2 dr$$
$$\ll_h T^{2\nu_2+1}.$$

On the other hand, since the right hand side is

$$|\Gamma_z|k_T(0) = \frac{|\Gamma_z|}{2\pi}\int_0^\infty h(\frac{r}{T})r\tanh(\pi r)dr = T^2\frac{|\Gamma_z|}{2\pi}\int_0^\infty h(r)rdr + O_h(1),$$

we can conclude that if $\nu_2 < \frac{1}{2}$, then for any $\delta \in (0, 1-2\nu_2)$ and any $T \geq T_0$

$$\sum_j h(\frac{r_j}{T})|\varphi_j(z)|^2 \;=\; T^2\frac{|\Gamma_z|}{2\pi}\int_0^\infty h(r)rdr + O_{\delta,h}(T^{2-\delta}).$$

This completes the proof.

□

Reiterating Remark 1.5, if one believes the Phillips-Sarnak conjecture [PS85], then the convexity $L^2$ bounds on Eisenstein series should be sharp for generic lattices. So (1.3) really is relying heavily on the arithmeticity of $\mathrm{SL}_2(\mathbb{Z})$.

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