

COMPACT MODULI OF K3 SURFACES

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ABSTRACT. We construct geometric compactifications of the moduli space F_{2d} of polarized K3 surfaces, in any degree $2d$. Our construction is via KSBA theory, by considering canonical choices of divisor $R \in |nL|$ on each polarized K3 surface $(X, L) \in F_{2d}$. The main new notion is that of a *recognizable divisor* R , a choice which can be consistently extended to all central fibers of Kulikov models. We prove that any choice of recognizable divisor leads to a semitoroidal compactification of the period space, at least up to normalization. Finally, we prove that the rational curve divisor is recognizable for all degrees.

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1. INTRODUCTION

Let F_{2d} be the coarse moduli space of complex K3 surfaces X having ADE singularities with an ample line bundle L of degree $L^2 = 2d$. A well known corollary of the Torelli theorem [PSS71] is that $F_{2d} = \mathbb{D}/\Gamma$ is the quotient of a 19-dimensional symmetric type IV domain $\mathbb{D}$ by an arithmetic group $\Gamma \subset O(2, 19)$. In this capacity, F_{2d} admits a Baily-Borel $\overline{F}_{2d}^{\text{BB}}$ [BB66] and infinitely many toroidal $\overline{F}_{2d}^{\mathfrak{F}}$ [AMRT75] compactifications. An admissible fan $\mathfrak{F}$ consists of polyhedral decompositions of the positive cones of finitely many hyperbolic signature lattices (Def. 5.9). Looijenga [Loo03] simultaneously generalized the Baily-Borel and toroidal compactifications to the *semitoroidal* compactifications, where $\mathfrak{F}$ may only be locally rational polyhedral (Def. 5.11).

Toroidal compactifications enjoy a number of geometric properties by virtue of the fact that they are analytically-locally modeled at the boundary by finite quotients of toric varieties. But there are infinitely many, with seemingly no one being distinguished. An old and deep question is whether any toroidal, or semitoroidal, compactifications can be understood as moduli spaces parameterizing geometric objects—some generalized “stable” K3 surfaces, similar to the Deligne-Mumford’s compactifications $\overline{M}_{g,n}$ of stable curves.

For the moduli space A_g of principally polarized abelian varieties (PPAVs) the answer is *yes* by [Ale02]. A PPAV $(A, \lambda) \in A_g$ determines uniquely an abelian torsor $A \curvearrowright X$ together with a theta divisor $\Theta \subset X$. For pairs (X, Θ) or, even better $(X, \epsilon\Theta)$, there is a generalization of $\overline{M}_{g,n}$. It is the moduli space of KSBA stable pairs $(X, \epsilon\Theta)$ with slc singularities [KSB88, Kol23], [Ale96, Ale06]. This moduli space is projective, and gives a geometrically meaningful compactification $\overline{A}_g^{\Theta}$. Furthermore, the normalization of $\overline{A}_g^{\Theta}$ is the toroidal compactification $\overline{A}_g^{\mathfrak{F}}$ associated to the 2nd Voronoi fan, and so admits a purely period-theoretic definition. Thus, among the infinitely many toroidal compactifications, this one has a clear geometric meaning.

To extend this construction to polarized K3 surfaces (X, L) , first one needs a *canonical choice of polarizing divisor* (Def. 6.1), an effective divisor $R \in |nL|$ in a fixed multiple of the polarization. Given this choice, the general theory produces a modular compactification $\overline{F}_{2d}^R$ (Def. 6.11) via slc stable pairs $(X, \epsilon R)$, see [KX20, Kol23], [AET19, Sec. 3]. The divisor is needed because for the general theory to work, the divisor $K_X + \epsilon R$ must be ample. One can work with all divisors in $|L|$, without making a canonical choice, e.g. [Laz16], but that gives a larger moduli space P_{2d} of dimension $20 + d$.

At least two canonical choices for ample divisors on polarized K3 surfaces have been identified. About 15 years ago, Sean Keel proposed to consider, for a general polarized K3 surface (X, L) , the sum $R^{\text{rc}} = \sum C_i$ of rational curves in $|L|$. We call this the *rational curve divisor* (Def. 10.3). One has $R^{\text{rc}} \in |n_d L|$, where the multiple n_d is given by the Yau-Zaslow formula. For instance $n_1 = 324$, $n_2 = 3200$, etc. The second choice, suggested to the authors by Claire Voisin, is called the *flex divisor* R^{flex} [AE21]. It generalizes to all degrees the fixed locus $R \in |3L|$ of the involution on a K3 surface of degree 2.

By the Kulikov-Persson-Pinkham theorem [Kul77, PP81], any one-parameter degeneration of K3 surfaces $X \rightarrow (C, 0)$ admits a *Kulikov model*: a K -trivial model with smooth total space and reduced normal crossings central fiber X_0 (Def. 3.1). The key notion of this paper is a *recognizable divisor* (Def. 6.2). Heuristically, it is a canonical choice of polarizing divisor which can be extended to any such X_0 . More precisely: Given any Kulikov surface X_0 appearing as a one-parameter degeneration of polarized K3 surfaces (X_t, L_t) , the limit of the canonically chosen divisors $R_t \subset X_t$, $R_t \in |nL_t|$ is a unique curve $R_0 = \lim_{t \rightarrow 0} R_t \subset X_0$. Such a limit R_0 exists on any fixed Kulikov model, but recognizability additionally states that R_0 is independent of how X_0 gets smoothed.

Our two main theorems are:

Theorem 1 (Thm. 9.1). *Suppose that R is a recognizable choice of polarizing divisor. There is a unique semifan $\mathfrak{F}_R$ for which $\overline{F}_{2d}^{\mathfrak{F}_R}$ is the normalization of $\overline{F}_{2d}^R$.*

Theorem 2 (Thm. 10.11). *The rational curve divisor R^{rc} is recognizable for F_{2d} .*

Theorem 1 holds more generally for moduli of lattice-polarized K3 surfaces. Combined, Theorems 1 and 2 give an affirmative answer to the existence of a compactification of K3 moduli which is simultaneously geometric and period-theoretic:

Corollary 3. *For all degrees $2d$, there is a KSBA compactification of F_{2d} by slc stable pairs, whose normalization is semitoroidal.*

Theorem 1 is proven as follows: Kulikov models $X \rightarrow (C, 0)$ with a given Picard-Lefschetz transformation, encoded by a lattice vector λ , can be packaged into a 19-dimensional family of polarized surfaces $\mathcal{X} \rightarrow S_\lambda$ we call λ -families (Def. 7.14). The general fiber is smooth, and the discriminant $\Delta_\lambda \subset S_\lambda$ is a smooth divisor, isomorphic to $(\mathbb{C}^*)^{18}$ for Type III Kulikov models. The discriminant parameterizes the equisingular, quasipolarized, smoothable deformations of X_0 .

Recognizability implies that the divisor R extends over the boundary Δ_λ to give a family of pairs $(\mathcal{X}, \mathcal{R}) \rightarrow S_\lambda$ (Prop. 8.1). We modify $\mathcal{X}$ until $\mathcal{R} \subset \mathcal{X}$ is relatively nef and contains no singular strata of any fiber (Prop. 8.8, Sec. 7C). Taking the relative canonical model shows that all degenerations with a given Picard-Lefschetz transformation limit to stable pairs $(X_0, \epsilon R_0)$ of a fixed combinatorial type (Cor. 8.13). This fact, together with an argument involving resolution of indeterminacy (Lem. 9.18) and quasi-affineness of the strata of the KSBA compactification (Thm. 9.16), imply that there is a toroidal compactification

$$\overline{F}_{2d}^\mathfrak{G} \rightarrow (\overline{F}_{2d}^R)^\nu$$

dominating the normalization of the KSBA compactification. The proof concludes with a new characterization: semitoroidal compactifications are exactly those normal compactifications of F_{2d} that are dominated by a toroidal compactification, and dominating Baily-Borel (Thm. 5.14).

Theorem 2 is proven by borrowing ideas from Gromov-Witten theory and degenerations of stable maps. R^{rc} is recognizable because any limit $R_{i,0} \subset X_0$ of a family of rational curves $R_{i,t} \subset R_t^{\text{rc}} \subset X_t$ in a Kulikov model $X \rightarrow (C, 0)$ enjoys a geometric property which ensures its rigidity: $R_{i,0}$ is the image of an admissible stable genus zero map $f : T \rightarrow X_0$. Using K -triviality of X_0 and adjunction, we prove that $f(T)$ is locally constant on the Kontsevich space of admissible stable maps (Lem. 10.14).

Relation to earlier work. The notion of a recognizable divisor presented here arose from generalizing certain specific examples, for moduli of degree 2 [AET19] and elliptic K3 surfaces [ABE22]. In both of these papers, Kulikov models with divisor are constructed explicitly for all possible λ , providing the necessary input for the general theory to work.

Every degree 2 K3 surface (X, L) with ADE singularities admits an involution, and the fixed locus $R \in |3L|$ can be taken as a canonical choice of polarizing divisor. Every elliptic K3 surface admits the polarizing divisor $R := s + \sum_{i=1}^{24} f_i \in |s + 24f|$ formed from the section, plus the sum of the singular fibers counted with appropriate multiplicity.

Using the theory of integral-affine structures on the two-sphere S^2 [Eng18, EF21], one can, in both of these cases, explicitly construct a family of Kulikov surfaces $\mathcal{X}_0 \rightarrow \Delta_\lambda$ which smooths to

a λ -family $(\mathcal{X}, \mathcal{R}) \rightarrow S_\lambda$ with $\mathcal{R}$ relatively big and nef, and not containing strata of any fiber. This is achieved by building a continuously varying family of “polarized integral-affine spheres” $(\mathcal{B}, \mathcal{R}_{\text{trop}}) \rightarrow C_\delta^+$ over a cone associated to an appropriate hyperbolic lattice (Def. 5.8). The cone C_δ^+ contains all possible λ . Once triangulated, the integral-affine sphere $\mathcal{B}_\lambda = \Gamma(X_0)$ can be identified with the dual complex of X_0 for some Kulikov model $X \rightarrow (C, 0)$ with monodromy λ . The tropical divisor $\mathcal{R}_{\text{trop}, \lambda} \subset \mathcal{B}_\lambda$ describes the dual complex $\Gamma(R_0)$ of $R_0 \subset X_0$.

The upshot of these constructions is that recognizability is verified explicitly, and all degenerations with a fixed Picard-Lefschetz transformation admit a stable model of a fixed combinatorial type. Furthermore, the family $(\mathcal{B}, \mathcal{R}_{\text{trop}}) \rightarrow C_\delta^+$ determines which monodromy invariants λ give rise to degenerations into a specified stratum of slc stable pairs—it is those λ on which the family $(\mathcal{B}, \mathcal{R}_{\text{trop}})$ is combinatorially constant. In the above examples, these loci of combinatorial constancy in C_δ^+ are the cones of a semifan $\mathfrak{F}_R$. The semifan of Theorem 1 is exactly this one. In fact, we prove here that for any recognizable divisor, the cones of $\mathfrak{F}_R$ are the loci on which a well-defined “stratum” function $\mathbb{S}: C_\delta^+ \rightarrow \{\text{slc strata of } \overline{F}_{2d}^R\}$ is constant (Thm. 9.3).

For degree 2 K3 surfaces with ramification divisor, $\mathfrak{F}_R$ is a semifan but not a fan. It is a coarsening of the Coxeter fan. For elliptic K3 surfaces with divisor $s + \sum_{i=1}^{24} f_i$ it is a fan which refines the maximal cone of the Coxeter fan into 9 subcones. Theorems 1 and 2 imply the existence of a semifan $\mathfrak{F}^{\text{rc}}$ for all degrees $2d$. But unlike for A_g and the 2nd Voronoi fan, or the above two examples, we have no explicit description, because the structure of a hypothetical “tropical K3-rational curves pair” $(\mathcal{B}, \mathcal{R}_{\text{trop}}^{\text{rc}})$ is unknown. Such a description is an interesting open question. Integral-affine structures make no appearance in this paper because we do not explicitly construct $\mathfrak{F}_R$ for any given R —we prove a general existence result.

After this work appeared, the authors, with Han [AEH21], proved recognizability for fixed curves of non-symplectic automorphisms. Explicit semifans $\mathfrak{F}_R$ for the fixed divisor R were given in [AE22], for moduli spaces of K3 surfaces with nonsymplectic involution.

Summary of contents. In Section 2, we recall different notions of moduli of K3 surfaces, such as smooth analytic, M -quasipolarized, and polarized with ADE singularities. In Section 3 we study one-parameter degenerations: Kulikov models, as well as nef, divisor, and stable models, by adding nef line bundles, effective nef divisors, and by taking the canonical models of the latter, respectively. In Section 4 we define the periods of Kulikov surfaces. These sections compile known results about K3 moduli, giving a unified treatment of Type II and III degenerations.

Section 5 recalls the combinatorially defined, period-theoretic compactifications of arithmetic quotients: Baily-Borel, toroidal, and semitoroidal. A major result is Theorem 5.14 which states that, for Type IV arithmetic quotients, a semitoroidal compactification is precisely a normal compactification which is sandwiched between the Baily-Borel and some toroidal compactification. Section 6 discusses the compactifications of F_{2d} via stable pairs, associated to a canonical choice of divisor R . Here, we introduce the critical notion of recognizable divisors (Def. 6.2).

In Section 7, we construct the λ -families which appear in the proof of Theorem 1. A new result (Theorem 7.19) globalizes the main result of Friedman-Scattone [FS86]: Any two λ -families with the same values of λ^2 and imprimitivity of λ are connected by a series of birational modifications falling into three special forms. In Section 8, we prove the main properties of recognizable divisors with respect to λ -families. Theorem 8.11 summarizes equivalent formulations of recognizability.

Sections 9 and 10 contain the proofs of Theorems 1 and 2, respectively.

2. MODULI OF K3 SURFACES

2A. Analytic moduli. We begin by setting notation and reviewing fundamental results about K3 surfaces. For general references, see [Huy16] or [ast85].

Definition 2.1. A *K3 surface* X is a compact, complex surface with $h^1(X) = 0$ and $K_X = \mathcal{O}_X$.

Definition 2.2. Let $L_{K3} := \Pi_{3,19} = H^{\oplus 3} \oplus E_8^{\oplus 2}$ be a fixed copy of the unique even unimodular lattice of signature $(3, 19)$.

Endowed with the cup product, $H^2(X, \mathbb{Z})$ is isometric to L_{K3} for any K3 surface X . Let $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ denote the Kähler cone of X . It is a fundamental chamber for the group $W_X = \langle r_\beta \rangle \subset O(H^2(X, \mathbb{Z}))$ generated by reflections in the roots $\beta \in NS(X)$, $\beta^2 = -2$ acting on the positive cone of $H^{1,1}(X, \mathbb{R})$.

Theorem 2.3 ([PSS71, LP81]). *Two K3 surfaces X, X' are isomorphic if and only if they are Hodge-isometric: there is an isometry $i : H^2(X', \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$ for which $i(H^{2,0}(X')) = H^{2,0}(X)$. Furthermore, $i = f^*$ for a unique isomorphism $f : X \rightarrow X'$ if and only if $i(\mathcal{K}_{X'}) = \mathcal{K}_X$.*

Note that ± 1 and $g \in W_X$ act by Hodge isometries on $H^2(X, \mathbb{Z})$. For any Hodge isometry i between X' and X , there is a unique sign and unique element $g \in W_X$ such that $\pm g \circ i(\mathcal{K}_{X'}) = \mathcal{K}_X$. Thus, the group of Hodge isometries of X fits into a split exact sequence of groups

$$0 \rightarrow \{\pm 1\} \times W_X \rightarrow \text{HodgeIsom}(X) \rightarrow \text{Aut}(X) \rightarrow 0.$$

Definition 2.4. Let $\pi : \mathcal{X} \rightarrow S$ be a family of smooth analytic K3 surfaces over an analytic space S . A *marking* is an isometry of local systems $\sigma : R^2\pi_*\mathbb{Z} \rightarrow L_{K3}$.

Definition 2.5. The *period domain* of analytic K3 surfaces is

$$\mathbb{D} := \mathbb{P}\{x \in L_{K3} \otimes \mathbb{C} \mid x \cdot x = 0, x \cdot \bar{x} > 0\}.$$

It is an analytic open subset of a 20-dimensional quadric in $\mathbb{P}^{21}$. Let $(\mathcal{X} \rightarrow S, \sigma)$ be a marked family of K3 surfaces. The *period map* $P : S \rightarrow \mathbb{D}$ is defined by $s \mapsto \sigma(H^{2,0}(X_s))$.

By [ast85, Exp. XIII], there is a non-Hausdorff complex manifold $\mathcal{M}$ of dimension 20, forming a fine moduli space of marked K3 surfaces, together with a period map $P : \mathcal{M} \rightarrow \mathbb{D}$. For $x \in \mathbb{D}$ a period, let W_x be the group generated by reflections in roots of $x^\perp \cap L_{K3}$. Then $P^{-1}(x)$ is a torsor over $\{\pm 1\} \times W_x$ with action given by $(X, \sigma) \mapsto (X, g \circ \sigma)$.

2B. Quasipolarized moduli. We now give analogous definitions to Section 2A in the polarized case. The standard reference is [Dol96]. However, Thm. 3.1 in *ibid* is incorrect. We modify the definition in a way that this theorem remains true.

Let $J : M \hookrightarrow L_{K3}$ be a primitive hyperbolic sublattice of signature $(1, r-1)$ with $r \leq 20$. A vector $h \in M \otimes \mathbb{R}$ is *very irrational* if $h \notin M' \otimes \mathbb{R}$ for any proper sublattice $M' \subsetneq M$. We will henceforth fix one such, of positive norm $h^2 > 0$.

Definition 2.6. An *M-quasipolarized K3 surface* (X, j) is a K3 surface X , and a primitive lattice embedding $j : M \hookrightarrow NS(X)$ for which $j(h) \in \overline{\mathcal{K}}_X$ is big and nef. Two such $(X, j), (X', j')$ are *isomorphic* if there is an isomorphism $f : X \rightarrow X'$ of K3 surfaces for which $j = f^* \circ j'$.

Definition 2.7. A *marking* of (X, j) is an isometry $\sigma : H^2(X, \mathbb{Z}) \rightarrow L_{K3}$ for which $J = \sigma \circ j$.

The *M-quasipolarized period domain* is

$$\mathbb{D}_M := \mathbb{P}\{x \in M^\perp \otimes \mathbb{C} \mid x \cdot x = 0, x \cdot \bar{x} > 0\}.$$

Define the *Weyl group* of a point $x \in \mathbb{D}_M$ to be $W_x(M^\perp) := \langle r_\beta : \beta \in x^\perp \cap M^\perp \rangle$. Note that now $W_x(M^\perp)$ is finite because $x^\perp \cap M^\perp$ is negative-definite.

Theorem 2.8. *There is a non-Hausdorff complex manifold $\mathcal{M}_M \subset \mathcal{M}$ of dimension $20 - r$, admitting a universal family of marked M-quasipolarized K3 surfaces. The fiber $P^{-1}(x)$ of the period map $P : \mathcal{M}_M \rightarrow \mathbb{D}_M$ is a torsor over $W_x(M^\perp)$.*

Proof. The proof follows that of [Dol96, Thm. 3.1], which now works because of the modified Definition 2.6 for an *M-quasipolarization*. $\square$

Let $\mathcal{F}_M^q$ denote the moduli stack of (unmarked) *M-quasipolarized* K3 surfaces.

Corollary 2.9. *There is an isomorphism of stacks $\mathcal{F}_M^q = [\mathcal{M}_M : \Gamma]$ where*

$$\Gamma := \{\gamma \in O(L_{K3}) : \gamma|_M = \text{id}_M\}$$

is the group of changes-of-marking. The quotiented period map $\mathcal{M}_M/\Gamma \rightarrow \mathbb{D}_M/\Gamma$ is a bijection.

2C. ADE moduli. We now modify the above moduli problems to produce a Hausdorff moduli space. Theorem 2.11 below is well-known.

Definition 2.10. An M -polarized K3 surface $(\bar{X}, j)$ is a surface $\bar{X}$ with at worst rational double point (ADE) singularities whose minimal resolution $X \rightarrow \bar{X}$ is a smooth K3 surface, together with an isometric embedding $j : M \hookrightarrow \text{Pic}(\bar{X})$ for which $j(h)$ is ample.

Theorem 2.11. The coarse moduli space of M -polarized K3 surfaces is $F_M = \mathbb{D}_M / \Gamma$. The moduli stack $\mathcal{F}_M$ is the separated quotient of the stack $[\mathcal{M}_M : \Gamma]$.

Remark 2.12. The stack $\mathcal{F}_M$ and the quotient stack $[\mathbb{D}_M : \Gamma]$ are not equal. In the latter stack, the inertia group at $x \in \mathbb{D}_M$ is the stabilizer Γ_x . In the former stack, the Torelli Theorem 2.3 implies the inertia group is rather $\Gamma_x / W_x(M^\perp) = \text{Aut}(\bar{X}, j)$.

Consider an open neighborhood $U_x \ni x$ in $\mathbb{D}_M$ preserved by Γ_x . First, quotient U_x by $W_x(M^\perp)$. Since $W_x(M^\perp) \subset \Gamma_x$ is normal, the quotient group acts on the coarse space $U_x / W_x(M^\perp)$, which is a smooth complex manifold. The stack quotient $[U_x / W_x(M^\perp) : \Gamma_x / W_x(M^\perp)]$ defines orbifold charts for the smooth DM stack $\mathcal{F}_M$.

3. ONE-PARAMETER DEGENERATIONS

3A. Kulikov models. We now examine degenerations of K3 surfaces over a curve. Let $(C, 0)$ denote the analytic germ of a smooth curve at a point $0 \in C$ and let $C^* = C \setminus 0$. Let $X^* \rightarrow C^*$ be a family of smooth analytic K3 surfaces.

Definition 3.1. A *Kulikov model* $X \rightarrow (C, 0)$ is an extension of $X^* \rightarrow C^*$ for which X is smooth, $K_X \sim_C 0$, and X_0 has reduced normal crossings with all components Kähler. We say X is *Type I*, *II*, or *III*, respectively, depending on whether X_0 is smooth, has double curves but no triple points, or has triple points, respectively.

A key result is the theorem of Kulikov [Kul77] and Persson-Pinkham [PP81]:

Theorem 3.2. Let $Y^* \rightarrow C^*$ be a family of analytic K3 surfaces admitting an extension $Y \rightarrow (C, 0)$ for which every component of Y_0 is Kähler. There is a base change $(C', 0) \rightarrow (C, 0)$ and a sequence of bimeromorphic modifications $Y' \dashrightarrow X$ of the pullback, such that X is a Kulikov model.

Assume for notational convenience that the strata of X_0 are globally normal crossings. Let $V_i \subset X_0$ denote the irreducible components, $D_{ij} = V_i \cap V_j$ and $T_{ijk} = V_i \cap V_j \cap V_k$ the double curves and triple points, respectively. By convention, we write $D_{ij} \subset V_i$ and $D_{ji} \subset V_j$.

Proposition 3.3. Let $X \rightarrow (C, 0)$ be a Kulikov model. Let $D_i = \sum_j D_{ij}$ be the part of the double locus of X_0 lying on V_i . Then:

- (1) $D_i \in |-K_{V_i}|$ is an anticanonical cycle of rational curves in Type III, and an elliptic curve or the disjoint union of two elliptic curves in Type II.
- (2) $D_{ij}^2 + D_{ji}^2 = -2 + 2g$ where g is the arithmetic genus of D_{ij} in X_0 .
- (3) The dual complex $\Gamma(X_0)$ is a triangulation of S^2 in Type III, and a segment decomposed into subsegments in Type II.

Definition 3.4. A reduced normal crossings surface X_0 satisfying (1), (2), (3) is a *Kulikov surface*.

There is a converse to Proposition 3.3 due to Friedman [Fri83b]:

Theorem 3.5. Let X_0 be a Kulikov surface. Then, X_0 deforms to a smooth K3 surface if and only if it satisfies an additional property called *d-semistability*:

$$\mathcal{E}xt^1(\Omega_{X_0}, \mathcal{O}_{X_0}) \cong \mathcal{O}_{(X_0)_{\text{sing}}}.$$

The components V_i in Type III are rational surfaces with a nodal anticanonical cycle as the double locus. The two ends of a Type II degeneration are rational surfaces with a smooth elliptic anticanonical double curve, and the intermediate components are elliptic ruled surfaces with double locus an anticanonical disjoint union of two elliptic sections.

Definition 3.6. An *anticanonical pair* or simply *pair* (V, D) is a smooth surface V with a reduced, at worst nodal, anticanonical divisor $D \in |-K_V|$. A *toric pair* $(\bar{V}, \bar{D})$ is a smooth toric surface $\bar{V}$ with $\bar{D} \in |-K_{\bar{V}}|$ the toric boundary.

The topologically trivial deformations of X_0 consist of deforming the moduli of the pairs (V_i, D_i) , and regluing the double curves by an element of $\mathbb{C}^*$ (in Type III) or by a translation of the elliptic double curve E (in Type II). Only some of these regluing are smoothable by Theorem 3.5.

Definition 3.7. The *charge* of an anticanonical pair (V, D) is $\chi_{\text{top}}(V \setminus D)$. If $D = \sum D_j$,

$$Q(V, D) := \begin{cases} 12 + \sum(-3 - D_j^2) & \text{if } D \text{ is nodal with at least two components,} \\ 11 - D^2 & \text{if } D \text{ is nodal and irreducible,} \\ 12\chi(\mathcal{O}_V) - D^2 & \text{if } D \text{ is smooth.} \end{cases}$$

Proposition 3.8 (Conservation of Charge, [FM83a, Prop. 3.7]). *Let $X = \bigcup (V_i, D_i)$ be a Kulikov surface. Then $\sum Q(V_i, D_i) = 24$.*

Definition 3.9. A *corner blow-up* of (V, D) is the blow-up at a node of D . An *internal blow-up* is the blow-up at a smooth point of D .

Both the corner and internal blow-ups $\tilde{V} \rightarrow V$ are naturally anticanonical pairs $(\tilde{V}, \tilde{D})$. For a corner blow-up, $\tilde{D}$ is the reduced inverse image of D . For an internal blow-up, $\tilde{D}$ is the strict transform of D . The formula for charge easily implies $Q(\tilde{V}, \tilde{D}) = Q(V, D)$ for a corner blowup, while $Q(\tilde{V}, \tilde{D}) = Q(V, D) + 1$ for an internal blow-up.

Any toric pair satisfies $Q(\bar{V}, \bar{D}) = 0$. When V is rational and D has nodes, as is the case for any component in Type III, [GHK15, Prop. 1.3] proves the existence of a diagram

$$(V, D) \xleftarrow{f} (\tilde{V}, \tilde{D}) \xrightarrow{g} (\bar{V}, \bar{D})$$

where f is a sequence of corner blow-ups, g is a sequence of internal blow-downs, and $(\bar{V}, \bar{D})$ is a toric pair. We call this data a *toric model* of (V, D) . By the existence of a toric model, $Q(V, D) \geq 0$ for all (V, D) in Type III, with $Q(V, D) = 0$ if and only if (V, D) is toric. So the conservation-of-charge formula (3.8) says that X_0 is “24 steps from being toric.”

When (V, D) is an elliptic ruled component of a Type II Kulikov surface, we have $Q(V, D) = 0$ if and only if $V \cong \mathbb{P}_E(\mathcal{O} \oplus \mathcal{L})$ with D the disjoint union of the zero and infinity sections. Otherwise $Q(V, D)$ measures the number of steps from being a smooth $\mathbb{P}^1$ -bundle over an elliptic curve E .

Finally, we discuss base change, following [Fri83a]. Consider an order k base change $X' \rightarrow X$ of a Kulikov model along a branched cover $(C', 0) \rightarrow (C, 0)$. Let t be an analytic coordinate on $(C, 0)$. The smoothing of X_0 is locally $xy = t$ or $xyz = t$ near a double curve or triple point, respectively. So the base change is locally $xy = t^k$ or $xyz = t^k$. There is a locally toric, SNC resolution $X[k] \rightarrow X'$ near the singular locus of X_0 corresponding to the standard order k subdivision of the simplices of the dual complex $\Gamma(X_0)$. Each triangle decomposes into k^2 triangles, and each segment into k subsegments. All components of $X_0[k]$ not appearing in X_0 satisfy $Q = 0$.

3B. Nef, divisor, and stable models. We now describe some additional structures on a Kulikov model in the presence of a quasipolarization.

Definition 3.10. Let L^* be a line bundle on X^* , relatively nef and big over C^* . A relatively nef extension L to a Kulikov model X over C is called a *nef model*.

Definition 3.11. Let $R^* \subset X^*$ be the vanishing locus of a section of L^* as above, containing no vertical components. A *divisor model* is an extension $R \subset X$ to a relatively nef divisor $R \in |L|$ for which R_0 contains no strata of X_0 .

Definition 3.12. The *stable model* of (X^*, R^*) is

$$(\bar{X}, \bar{R}) := \text{Proj}_C \bigoplus_{n \geq 0} H^0(X, \mathcal{O}(nR))$$

for some divisor model. It is unique and independent of the choice of divisor model (X, R) by the theory of canonical models, since for $0 < \epsilon \ll 1$ the pair $(\bar{X}, \bar{X}_0 + \epsilon \bar{R})$ has log canonical singularities and the divisor $K_{\bar{X}} + \epsilon \bar{R}$ is relatively ample.

By adjunction, the central fiber $(\overline{X}_0, \epsilon \overline{R}_0)$ has semi log canonical (slc) singularities and the divisor $K_{\overline{X}_0} + \epsilon \overline{R}_0$ is ample.

The existence of a nef model is due to Shepherd-Barron [SB83], and the existence of a divisor model is proved in [Laz16, Thm. 2.11], [AET19, Thm. 3.12].

Now suppose one starts with a family $(\overline{X}^*, \overline{R}^*) \rightarrow C^*$ of K3 surfaces with ADE singularities. After a finite base change it admits a simultaneous resolution of singularities $f: X^* \rightarrow \overline{X}^*$. Let $R^* = f^*(\overline{R}^*)$. After a further finite base change, by the above we get a divisor model, whose stable model $(\overline{X}, \overline{R})$ is the *stable extension* of $(\overline{X}^*, \overline{R}^*)$ over C . It is unique and stable under base changes by a standard argument, see e.g. [Kol23, Thm. 2.47].

3C. Topology of Kulikov models. The primary reference for this section is [FS86]. Let $X \rightarrow (C, 0)$ be a Kulikov model. For convenience, denote integral singular cohomology by $H^i(-)$. Let $T: H^2(X_t) \rightarrow H^2(X_t)$ be the Picard-Lefschetz transformation along an oriented simple loop in C^* enclosing 0. Since X_0 is reduced normal crossings, T is unipotent. Let $N := \log T$ be its logarithm.

Theorem 3.13 ([FS86, Fri84]). *Let $X \rightarrow (C, 0)$ be a Kulikov model. We have that*

- if X is Type I, then $N = 0$,*
- if X is Type II, then $N^2 = 0$ but $N \neq 0$,*
- if X is Type III, then $N^3 = 0$ but $N^2 \neq 0$.*

Furthermore, N is integral, and of the form $Nx = (x \cdot \lambda)\delta - (x \cdot \delta)\lambda$ for $\delta \in H^2(X_t)$ a primitive isotropic vector, and $\lambda \in \delta^\perp/\delta$ satisfying $\lambda^2 = \#\{\text{triple points of } X_0\}$. When $\lambda^2 = 0$, its imprimitivity is the number of double curves of X_0 .

Thus, the Types I, II, III of Kulikov model are distinguished by the behavior of the *monodromy invariant* λ : either $\lambda = 0$, $\lambda^2 = 0$ but $\lambda \neq 0$, or $\lambda^2 \neq 0$ respectively.

Remark 3.14. If $X^* \rightarrow C^*$ admits a quasipolarization $M \hookrightarrow \text{Pic}(X^*)$ then $T \in O(H^2(X_t))$ lies in the subgroup Γ fixing M . In particular, $\delta \in M^\perp$ and we can consider the lattice of monodromy invariants $\lambda \in \delta^\perp/\delta$ as valued in a subquotient of $M^\perp$.

Definition 3.15. Let $I \subset H^2(X_t)$ denote the primitive isotropic lattice $\mathbb{Z}\delta$ in Type III or the saturation of $\mathbb{Z}\delta \oplus \mathbb{Z}\lambda$ in Type II.

As a simple normal crossings degeneration, there is a deformation-retraction $c: X \times [0, 1] \rightarrow X_0$ called the *Clemens collapse* [Cle69]. So we have $H^*(X_0) = H^*(X)$. In particular, the map $c_t^*: H^*(X_0) \rightarrow H^*(X_t)$ coincides with restriction from X to X_t .

The integral cohomology of a Type III Kulikov surface X_0 is computed in [FS86, Sec. 1] by the Mayer-Vietoris spectral sequence, associated to the exact sequence of sheaves

$$0 \rightarrow \underline{\mathbb{Z}}_{X_0} \rightarrow \bigoplus_i \underline{\mathbb{Z}}_{V_i} \rightarrow \bigoplus_{i \leq j} \underline{\mathbb{Z}}_{D_{ij}} \rightarrow \bigoplus_{i \leq j \leq k} \underline{\mathbb{Z}}_{T_{ijk}} \rightarrow 0.$$

It follows that there is an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow H^2(X_0) \rightarrow \tilde{\Lambda} \rightarrow 0, \text{ where}$$

Definition 3.16. The *numerically Cartier classes* on a Kulikov model X_0

$$\tilde{\Lambda} = \tilde{\Lambda}(X_0) := \ker \left(\bigoplus_i H^2(V_i) \rightarrow \bigoplus_{i \leq j} H^2(D_{ij}) \right)$$

are collections of classes (α_i) for which $n_{ij} := \alpha_i \cdot D_{ij} = \alpha_j \cdot D_{ji}$ for all double curves.

The lefthand term $\mathbb{Z}$ arises in the spectral sequence from the second simplicial cohomology $H^2(\Gamma(X_0))$ of the dual complex. Choosing an orientation on $\Gamma(X_0)$ gives a generator $1 \in \mathbb{Z}$ which satisfies $c_t^*(1) = \delta$.

Mayer-Vietoris for a Type II Kulikov surface X_0 implies that there is an analogous exact sequence $0 \rightarrow \mathbb{Z}^2 \rightarrow H^2(X_0) \rightarrow \tilde{\Lambda}(X_0) \rightarrow 0$ with the $\mathbb{Z}^2$ arising from $H^1(D_{i,i+1})$ for some double curve $D_{i,i+1}$. Here the image $c_t^*(\mathbb{Z}^2)$ is identified with the rank two lattice I .

Definition 3.17. Let I_0 denote the sublattice $\mathbb{Z} \cong H^2(\Gamma(X_0)) \subset H^2(X_0)$ in Type III or $\mathbb{Z}^2 \cong H^1(D_{i,i+1}) \subset H^2(X_0)$ in Type II arising from Mayer-Vietoris.

So for both Type II and III, $c_t^*(I_0) = I$ and $H^2(X_0)/I_0 = \tilde{\Lambda}$.

Definition 3.18. Define the intersection form $\cdot$ on $\tilde{\Lambda}$ by $(\alpha_i) \cdot (\beta_i) = \sum_i \alpha_i \cdot \beta_i$.

Definition 3.19. Define $\hat{\xi}_i := c_1(\mathcal{O}_X(V_i))|_{X_0} \in H^2(X_0)$ and let $\xi_i \in \tilde{\Lambda}$ be the image of $\hat{\xi}_i$. Then $\xi_i = \sum_j (D_{ji} - D_{ij})$ and $\sum_i \xi_i = 0$. Define $\Xi := \mathbb{Z}\text{-span}\{\xi_i\} \subset \tilde{\Lambda}$ and declare $\Lambda := \tilde{\Lambda}/\Xi$.

It is easy to check directly from property (2) of Proposition 3.3 that $\Xi \subset \tilde{\Lambda}$ is contained in the null sublattice of the intersection form.

Proposition 3.20. *The map $c_t^*: H^2(X_0) \rightarrow H^2(X_t)$ induces a surjection $\tilde{\Lambda} \rightarrow \{\delta, \lambda\}^\perp/I$ sending Ξ to zero, which thus descends to Λ . Furthermore, $\Xi = \tilde{\Lambda}^{\text{null}}$ is the null sublattice. Hence Ξ is saturated, Λ is torsion-free and the induced map $\Lambda \rightarrow \{\delta, \lambda\}^\perp/I$ is an isometry of lattices.*

Proof. [FS86, 4.13] gives an exact sequence

$$0 \rightarrow \hat{\Xi} \rightarrow H^2(X_0) \xrightarrow{c_t^*} \ker(N) = \{\delta, \lambda\}^\perp \rightarrow 0$$

where $\hat{\Xi} := \mathbb{Z}\text{-span}\{\hat{\xi}_i\}$. Noting that $c_t^*(I_0) = I$, we can quotient the second and third factors in the above exact sequence to get an exact sequence

$$0 \rightarrow \Xi \rightarrow \tilde{\Lambda} \rightarrow \{\delta, \lambda\}^\perp/I \rightarrow 0.$$

Since the third term is torsion-free, the kernel Ξ must be the saturated. It is exactly the null lattice because the target $\{\delta, \lambda\}^\perp/I$ is nondegenerate and c_t^* preserves the intersection form. $\square$

4. THE PERIOD MAP

4A. The period of a Kulikov surface. Let X_0 be a Kulikov surface, not necessarily d -semistable. The period map is a homomorphism $\tilde{\psi}$ from $\tilde{\Lambda}(X_0)$ (see Def. 3.16) to $\mathbb{C}^*$ in Type III or the elliptic double curve E in Type II, which measures the obstruction to a class being represented by a Cartier divisor. First, we consider the Type III case.

A resolution of the sheaf of non-vanishing holomorphic functions is given by

$$1 \rightarrow \mathcal{O}_{X_0}^* \rightarrow \bigoplus_i \mathcal{O}_{V_i}^* \rightarrow \bigoplus_{i \leq j} \mathcal{O}_{D_{ij}}^* \rightarrow \bigoplus_{i \leq j \leq k} \mathcal{O}_{T_{ijk}}^* \rightarrow 1.$$

Computing $\text{Pic}(X_0) = H^1(X_0, \mathcal{O}_{X_0}^*)$ via the Mayer-Vietoris spectral sequence [FS86, Sec. 3] shows that $\text{Pic}(X_0)$ is the kernel of a homomorphism

$$\ker \left(\bigoplus_i \text{Pic}(V_i) \rightarrow \bigoplus_{i \leq j} \text{Pic}(D_{ij}) \right) \rightarrow H^2(\Gamma(X_0), \mathbb{C}^*) \cong \mathbb{C}^*$$

where the latter space is identified with $\mathbb{C}^*$ by choosing an orientation on the dual complex. Note that since V_i and D_{ij} are rational, we have $\text{Pic}(V_i) = H^2(V_i)$ and $\text{Pic}(D_{ij}) = H^2(D_{ij})$ so the first term is nothing more than the lattice $\tilde{\Lambda}$ of (3.16).

Definition 4.1. The *period point* of a Type III Kulikov surface X_0 is $\tilde{\psi}_{X_0} \in \text{Hom}(\tilde{\Lambda}, \mathbb{C}^*)$.

Construction 4.2. Unwinding the maps in the spectral sequence, one can explicitly construct the homomorphism $\tilde{\psi}_{X_0}$. Let $\alpha = (\alpha_i) \in \tilde{\Lambda}$ be a numerically Cartier divisor. Then α_i determines a unique line bundle $L_i \in \text{Pic}(V_i)$ for all i . We have

$$L_i|_{D_{ij}} \cong L_j|_{D_{ji}} \cong \mathcal{O}_{\mathbb{P}^1}(n_{ij})$$

so we can extend a line bundle $L_i \rightarrow V_i$ by $L_j \rightarrow V_j$ to a line bundle on $V_i \cup V_j$. We may continue successively until only one component V_1 remains. The result is a line bundle $L \rightarrow \overline{X_0} \setminus \overline{V_1}$ and we may consider the line bundle

$$L|_{D_1} \otimes L_1^{-1}|_{D_1} =: L_\alpha \in \text{Pic}^0(D_1).$$

We have $\text{Pic}^0(D_1) = \mathbb{C}^*$ because the cycle D_1 is oriented by the choice of orientation on the dual complex $\Gamma(X_0)$. So α determines a period $\tilde{\psi}_{X_0}(\alpha) = L_\alpha \in \mathbb{C}^*$. It is independent of the choice of component V_1 and clearly obstructs α being represented by a Cartier divisor.

Construction 4.3. In analogy to Construction 4.2, we now construct a period map $\tilde{\psi}_{X_0}: \tilde{\Lambda} \rightarrow E$ in Type II. Orient the segment $\Gamma(X_0)$ so that $X_0 = V_0 \cup \dots \cup V_k$ with indices increasing with respect to the orientation. Let $\alpha = (\alpha_i) \in \tilde{\Lambda}$. Then $\alpha_0 \in H^2(V_0)$ and $\alpha_k \in H^2(V_k)$ define line bundles L_0 and L_k because the end surfaces are rational elliptic. On the other hand, the lifts of an element $\alpha_i \in H^2(V_i)$, $i \neq 0, k$ to an element $L_i \in \text{Pic}(V_i)$ form a torsor over $E = \text{Pic}^0 D_{i,i+1}$. So there is a unique lift L_1 of α_1 for which $L_0|_{D_{01}} \cong L_1|_{D_{10}}$. Take this lift to extend L_0 to $V_0 \cup V_1$ by L_1 . Continuing inductively gives a unique line bundle $L \rightarrow \overline{X_0 \setminus V_k}$. Then define

$$\tilde{\psi}_{X_0}(\alpha) := L|_{D_{k-1,k}} \otimes L_k^{-1}|_{D_{k,k-1}} \in \text{Pic}^0(D_{k-1,k}) = E.$$

The period map can also be defined from the exponential long exact sequence

$$\dots \rightarrow H^1(X_0) \rightarrow H^1(X_0, \mathcal{O}) \rightarrow \text{Pic}(X_0) \rightarrow H^2(X_0) \xrightarrow{\Psi} H^2(X_0, \mathcal{O}) \rightarrow H^2(X_0, \mathcal{O}^*) \rightarrow \dots$$

Note that $H^2(X_0, \mathcal{O}) = H^0(X_0, \omega_{X_0})^* \cong \mathbb{C}$ is one-dimensional. Quotienting by the image of $I_0 \subset H^2(X_0)$, we reproduce the period homomorphism

$$\tilde{\psi}_{X_0}: \tilde{\Lambda} \rightarrow \mathbb{C}/\Psi(I_0).$$

In Type III, we have $\mathbb{C}/\Psi(I_0) \cong \mathbb{C}^*$ while in Type II we have $\mathbb{C}/\Psi(I_0) \cong E$ for an elliptic curve E . In both cases, $\text{Pic}(X_0)$ is the kernel because $H^1(X_0, \mathcal{O}) = 0$.

Proposition 4.4. *The surface X_0 is smoothable if and only if the period point $\tilde{\psi}_{X_0} \in \text{Hom}(\tilde{\Lambda}, \mathbb{C}^*)$ or $\text{Hom}(\tilde{\Lambda}, E)$ descends to a period point $\psi_{X_0} \in \text{Hom}(\Lambda, \mathbb{C}^*)$ or $\text{Hom}(\Lambda, E)$.*

Proof. By Theorem 3.5, X_0 is smoothable if and only if it is d -semistable. But X_0 is d -semistable if and only if $\tilde{\psi}_{X_0}(\xi_i) = 1$ for all i , i.e. $\tilde{\psi}_{X_0}$ descends to $\Lambda = \tilde{\Lambda}/\Xi$. $\square$

4B. Markings of Kulikov surfaces. In this section, we define the analogues of markings for Kulikov surfaces X_0 to properly formulate results on the period map. Let $\Lambda_0 = \Lambda_0(t, k)$ denote a model for $\{\delta, \lambda\}^\perp / I$ in L_{K3} . It depends only on the even integer $\lambda^2 = t$ giving the number of triple points of X_0 and the imprimitivity k of $\lambda \in \delta^\perp / \delta$. We suppress (t, k) in the notation.

Definition 4.5. Let X_0 a d -semistable Kulikov surface. A *marking* $(\sigma, \underline{b})$ consists of:

- (1) An isometry $\sigma: \Lambda(X_0) \rightarrow \Lambda_0$ (see Def. 3.16) and
- (2) An ordered basis $\underline{b}$ of $I_0 \subset H^2(X_0)$ (see Def. 3.17).

The notion of a marking naturally extends to equisingular families $\mathcal{X} \rightarrow S$ of Kulikov surfaces using local systems. We can now define the period map:

Definition 4.6. Let $(\mathcal{X} \rightarrow S, \sigma)$ be a family of marked d -semistable Kulikov surfaces. The *period map* is defined by

$$S \rightarrow \text{Hom}(\Lambda_0, \mathbb{C}^*) \text{ or } \text{Hom}(\Lambda_0, \tilde{\mathcal{E}}), \quad s \mapsto B \circ \Psi_s \circ \sigma^{-1}.$$

Here Ψ_s comes from the exponential exact sequence as in Section 4A, $\tilde{\mathcal{E}} \rightarrow \mathbb{C} \setminus \mathbb{R}$ is the universal marked elliptic curve $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau \rightarrow \{\tau \in \mathbb{C} \setminus \mathbb{R}\}$, and B is the quotient map

$$B: H^2(X_0, \mathcal{O}_{X_0}) \rightarrow \mathbb{C}/\mathbb{Z} = \mathbb{C}^* \quad \text{or} \quad B: H^2(X_0, \mathcal{O}_{X_0}) \rightarrow \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$$

induced by the ordered basis $\underline{b}$ of I_0 (the first element b_1 of the basis is sent to $1 \in \mathbb{C}$).

Remark 4.7. In the Type II case, we could also require that $\underline{b}$ is an oriented basis, in the sense that $\tau \in \mathbb{H}$. Then the period map can be defined with target $\tilde{\mathcal{E}}^+ := \tilde{\mathcal{E}}|_{\mathbb{H}}$ instead.

4C. Partial markings of K3 surfaces. Let $X \rightarrow (C, 0)$ be a Kulikov model. We determine what information a marking of X_0 induces on the general fiber $X_t = Y$. In this subsection, we denote an analytic K3 surface by Y to distinguish it from the Kulikov model X .

Definition 4.8. A *partial marking* of Y is a distinguished primitive isotropic class $\delta \in H^2(Y)$, a distinguished vector $\lambda \in \delta^\perp/\delta$ of non-negative norm, and an isometry $\sigma: \{\delta, \lambda\}^\perp/I \rightarrow \Lambda_0$. We say the partial marking is *Type II* or *III* depending on whether $\lambda^2 = 0$ or $\lambda^2 > 0$, respectively.

Proposition 4.9. *Let $X \rightarrow (C, 0)$ be a Kulikov model. A partial marking of X_t whose distinguished classes δ, λ are the monodromy invariants determines uniquely a marking of X_0 .*

Proof. Proposition 3.20 gives an isometry $c_t^*: \Lambda(X_0) \rightarrow \{\delta, \lambda\}^\perp/I$. So a partial marking of X_t induces an isometry $\Lambda(X_0) \rightarrow \Lambda_0$ by composing with c_t^* . The class δ (and λ in Type II) determines a basis of $I_0 \subset H^2(X_0)$ via c_t^* . $\square$

Definition 4.10. The parabolic stabilizer of an isotropic lattice $I \subset L_{K3}$ fits into an exact sequence $0 \rightarrow U_I \rightarrow \text{Stab}_{O(L_{K3})}(I) \rightarrow \Gamma_I \rightarrow 0$ where U_I is the *unipotent radical*: the normal subgroup acting trivially on the graded pieces I and $I^\perp/I$. In Type III, U_I is isomorphic to the additive group $\text{Hom}(I^\perp/I, I)$. In Type II, U_I is a central $\mathbb{Z}$ -extension of $\text{Hom}(I^\perp/I, I)$. The quotient has the structure $\Gamma_I \cong O(I^\perp/I) \times GL(I)$. These exact sequences play an important role in the toroidal compactifications (Sec. 5B).

Definition 4.11. A partially marked K3 surface (Y, σ) is *admissible* if $[\Omega]: I \otimes \mathbb{R} \rightarrow \mathbb{C}$ sending $i \mapsto [\Omega] \cdot i$ is injective for any non-zero two-form Ω on Y . Similarly, define

$$\mathbb{D}^I := \{x \in \mathbb{D} \mid I \otimes \mathbb{R} \xrightarrow{x} \mathbb{C} \text{ is injective}\}.$$

Note that $\mathbb{D}^I \subset \mathbb{D}$ is an open subset. The period maps described in Sec. 4A can be understood as Carlson's extension class [Car85] for the limit mixed Hodge structure of X_0 and $\mathbb{D}^I$ is the domain of Hodge structures on L_{K3} for which I happens to also define a mixed Hodge structure.

Proposition 4.12. *There is a fine moduli space of admissible, partially marked, analytic K3 surfaces, admitting a period map to $\mathbb{D}^I/U_I$.*

Proof. The partial markings of a K3 surface Y are identified with U_I -orbits of the set of markings of Y . The fine moduli space $\mathcal{M}$ of marked analytic K3 surfaces admits a period map $P: \mathcal{M} \rightarrow \mathbb{D}$, and if the partial marking associated to a marked K3 surface (Y, σ) is admissible, then its image under the period map lies in $\mathbb{D}^I$. The action of U_I by post-composition on σ is free on $P^{-1}(\mathbb{D}^I)$ (as it is free on $\mathbb{D}^I$). The quotient is a non-Hausdorff complex manifold. By the Torelli theorem, the universal family descends to a universal family of partially marked K3 surfaces. $\square$

Proposition 4.13. *In Type III, there is an open embedding $\mathbb{D}^I/U_I \hookrightarrow I^\perp/I \otimes \mathbb{C}^*$ into a 20-dimensional algebraic torus. In Type II, there is an open embedding $\mathbb{D}^I/U_I \hookrightarrow A_I$ where $A_I \rightarrow I^\perp/I \otimes \tilde{\mathcal{E}}$ is a punctured holomorphic disk bundle.*

Sketch. Though the period domain $\mathbb{D}$ of analytic K3s is not Hermitian symmetric, these embeddings are defined in exactly the same way as the “torus embeddings” of the unipotent quotients of Hermitian symmetric domains [AMRT75]. In Type III, one realizes $\mathbb{D}^I$ as a tube domain inside $\mathbb{C}^{20}$. The translation group $U_I = \text{Hom}(I^\perp/I, I)$ acts by translations by $\mathbb{Z}^{20}$ on $\mathbb{C}^{20}$ and so the quotient $\mathbb{D}^I/U_I$ embeds into $(\mathbb{C}^*)^{20}$.

In Type II, $\mathbb{D}^I$ is contained in an upper half-plane bundle, fibered over the total space of a $\mathbb{C}^{18}$ -bundle over $\mathbb{C} \setminus \mathbb{R}$. The central $\mathbb{Z}$ acts on the upper-half plane bundle by fiberwise translation. Quotienting gives a punctured holomorphic disk bundle over the $\mathbb{C}^{18}$ -bundle. Then $\text{Hom}(I^\perp/I, I)$ further acts on the $\mathbb{C}^{18}$ -fiber over $\tau \in \mathbb{C} \setminus \mathbb{R}$ by translation by $(\mathbb{Z} \oplus \mathbb{Z}\tau)^{18}$. So the quotient $\mathbb{D}^I/U_I$ embeds into a punctured holomorphic disk bundle $A_I \rightarrow I^\perp/I \otimes \tilde{\mathcal{E}}$. $\square$

The unipotent quotient of $\mathbb{D}_M$ embeds into $\mathbb{D}^I/U_I$ for all M . Let $\mathbb{D}(I) := \mathbb{D}^I/U_I$.

Definition 4.14. Define an enlargement $\mathbb{D}(I) \hookrightarrow \mathbb{D}(I)^\lambda$ as follows: In Type III, it is the closure of $\mathbb{D}(I)$ in the toric variety T_λ extending the torus $I^\perp/I \otimes \mathbb{C}^*$ whose fan consists of the unique ray $\mathbb{R}_{\geq 0}\lambda$. In Type II, it is the holomorphic disk bundle $\overline{A}_I$ extending the punctured disk bundle A_I .

In Type III, the boundary divisor in $D(I)^\lambda$ is isomorphic to $\delta^\perp/\{\delta, \lambda\}^{\text{sat}} \otimes \mathbb{C}^*$. Since $\delta^\perp/\delta$ is unimodular, this torus can be identified with $\text{Hom}(\Lambda, \mathbb{C}^*)$. Similarly, the added boundary divisor in Type II is naturally isomorphic to the base $I^\perp/I \otimes \tilde{\mathcal{E}}$ of the disk bundle, which is identified with $\text{Hom}(\Lambda, \tilde{\mathcal{E}})$ again because $I^\perp/I$ is unimodular.

Definition 4.15. Let $\mathcal{X} \rightarrow S$ be a family of d -semistable Kulikov surfaces of Types I + II or I + III over a smooth base S . Suppose furthermore that the discriminant locus $\Delta \subset S$ is a smooth divisor. A *mixed marking* σ is a partial marking of the family $\mathcal{X}^* \rightarrow S \setminus \Delta$ of smooth fibers together with a compatible (Prop. 4.9) marking of the equisingular family $\mathcal{X}_0 \rightarrow \Delta$ of Kulikov surfaces.

Theorem 4.16. Let $(\mathcal{X} \rightarrow S, \sigma)$ be a mixed marked family of admissible surfaces as in Definition 4.15. The period map $\psi: S \setminus \Delta \rightarrow \mathbb{D}(I)$ extends to a morphism

$$\overline{\psi}: S \rightarrow \mathbb{D}(I)^\lambda$$

sending the discriminant Δ to the boundary divisor $\text{Hom}(\Lambda, \mathbb{C}^*)$ or $\text{Hom}(\Lambda, \tilde{\mathcal{E}})$. Furthermore, $\overline{\psi}|_\Delta$ is the period map for the family of marked Kulikov surfaces $\mathcal{X}_0 \rightarrow \Delta$, as in Definition 4.6.

Proof. This theorem is essentially the same as [FS86, Thm. 5.3]. The primary tool is the nilpotent orbit theorem [Sch73, Thm. 4.9]. $\square$

For an M -quasipolarized d -semistable Kulikov surface X_0 , we fix an embedding $M \hookrightarrow \ker(\psi_{X_0}) \subset \Lambda(X_0)$. In the M -quasipolarized case, the period point $\text{Hom}(\Lambda, \mathbb{C}^*$ or $E)$ descends to a period point in $\text{Hom}(\Lambda/M, \mathbb{C}^*$ or $E)$ which we will also denote ψ_{X_0} by abuse of notation. More precisely, a primitive sublattice $M \subset \Lambda \oplus I$ is the same as a not necessarily primitive sublattice $M \subset \Lambda$ plus a homomorphism $\Psi: \text{Tors}(\Lambda/M) \rightarrow \mathbb{C}^*$ or E . The period point belongs to the coset of $\text{Hom}(\Lambda/M^{\text{sat}}, \mathbb{C}^*$ or $E)$ of points with $\psi_{X_0}|_{\text{Tors}(\Lambda/M)} = \Psi$. The discussion of the period map in the above sections holds, replacing everywhere $H^2(X_t)$ with $j(M)^\perp$, L_{K3} with $M^\perp$, $\mathbb{D}^I$ with $\mathbb{D}_M^I := \mathbb{D}^I \cap \mathbb{D}_M$, U_I with $U_I \cap \Gamma$, $I^\perp$ with $I_{M^\perp}^\perp$, Λ with $\Lambda/j_0(M)$, Λ_0 with Λ_0/M . Recall that Γ is the subgroup of $O(L_{K3})$ acting by the identity on M .

Proposition 4.17. $\mathbb{D}_M^I = \mathbb{D}_M$ for any M .

Proof. An $x \in \mathbb{D}_M \setminus \mathbb{D}_M^I$ would satisfy $x \cdot i = 0$ for some nonzero isotropic vector $i \in I \otimes \mathbb{R}$. But then $\text{Re}(x)$, $\text{Im}(x)$ span a positive definite 2-plane in $i_{M^\perp}^\perp \otimes \mathbb{R}$ which is impossible since $M^\perp$ has signature $(2, 20 - r)$. $\square$

The moduli space of partially marked, M -quasipolarized K3 surfaces admits a period map to the torsion translate of a subtorus $\mathbb{D}_M(I) := \mathbb{D}_M/U_I \cap \Gamma \subset \mathbb{D}(I)$ which is generically one-to-one. A mixed marked M -quasipolarized family admits a period map to the toroidal extension

$$\mathbb{D}_M(I)^\lambda := \overline{\mathbb{D}_M(I)} \subset \mathbb{D}(I)^\lambda.$$

Notation 4.18. We henceforth write $I_{M^\perp}^\perp$ simply as $I^\perp$ whenever it is clear from context that we are working with M -quasipolarized surfaces.

5. COMPACTIFICATIONS OF ARITHMETIC QUOTIENTS

5A. Baily-Borel compactification. By Theorem 2.11, the coarse space of M -polarized (ADE) K3 surfaces F_M is the quotient of the period domain

$$\mathbb{D}_M := \mathbb{P}\{x \in M^\perp \otimes \mathbb{C} \mid x \cdot x = 0, x \cdot \bar{x} > 0\}$$

by the arithmetic group Γ . In this capacity, the space $F_M = \mathbb{D}_M/\Gamma$ has a Baily-Borel compactification [BB66], which we now describe.

Remark 5.1. Note that $\mathbb{D}_M = \mathbb{D}_M^1 \sqcup \mathbb{D}_M^2$ has two connected components and so F_M may have either 1 or 2 connected components, depending on whether or not Γ contains an element interchanging the two components. To simplify (but abuse) notation, we refer to $\mathbb{D}_M^1$ and its stabilizer $\Gamma^1 \subset \Gamma$ as $\mathbb{D}_M$ and Γ , respectively.

Definition 5.2. The *compact dual* is $\mathbb{D}_M^c := \mathbb{P}\{x \in M^\perp \otimes \mathbb{C} \mid x \cdot x = 0\}$. It is the compact hermitian symmetric domain containing $\mathbb{D}_M$ as an open subdomain.

Definition 5.3. A *boundary component* of $\mathbb{D}_M$ is a maximal connected complex submanifold of the boundary $\partial\mathbb{D}_M \subset \mathbb{D}_M^c$. The *rational boundary components* B_I are in bijection with primitive isotropic lattices $I \subset M^\perp$ via

$$B_I = \{x \in \partial\mathbb{D}_M \mid \text{span}\{\text{Re } x, \text{Im } x\} = I \otimes \mathbb{R}\}.$$

We have $B_I \cong \mathbb{H}$ when $\text{rk } I = 2$. We have $B_I \cong \{\text{pt}\}$ when $\text{rk } I = 1$. We call these *Type II and III* boundary components, respectively. The *rational closure* of $\mathbb{D}_M$ is $\mathbb{D}_M^+ := \mathbb{D}_M \cup B_I \subset \mathbb{D}_M^c$ topologized at the boundary points using horoballs as a neighborhood base.

Theorem 5.4 ([BB66]). *The quotient $\overline{F}_M^{\text{BB}} := \mathbb{D}_M^+/\Gamma$ is compact and has the structure of a projective variety, and the projective coordinate ring is the ring of modular forms for Γ .*

The image of a boundary component B_I in $\overline{F}_M^{\text{BB}}$ is isomorphic to $B_I/\text{Stab}_\Gamma(I)$ and so is either a point when $\text{rk } I = 1$ or a modular curve when $\text{rk } I = 2$.

Definition 5.5. The 0- and 1-cusps of $\overline{F}_M^{\text{BB}}$ are the zero- and one-dimensional boundary components, respectively. They are, respectively, in bijection with Γ -orbits of rank 1 and 2 primitive isotropic lattices $I \subset M^\perp$.

Proposition 5.6. *Let $X \rightarrow (C, 0)$ be an M -quasipolarized Kulikov model. The extension of the period map $C^* \rightarrow F_M$ to the Baily-Borel compactification sends 0 into the cusp associated to the monodromy lattice I . In Type II, the j -invariant $j(D_{i,i+1})$ of a double curve agrees with the j -invariant $j : B_I/\text{Stab}_\Gamma(I) \rightarrow \mathbb{H}/\text{SL}_2(\mathbb{Z}) = \mathbb{A}_j^1$ of the corresponding image point.*

Proof. This well-known fact follows directly from the asymptotics of the period map and the nilpotent orbit theorem, as in Theorem 4.16. $\square$

5B. Toroidal compactification. The original source on this subject is [AMRT75]. The reference [Nam80] in the case of Siegel space $\mathbb{D} = \mathcal{H}_g$ is particularly clear. The following well-known theorem is key to constructing toroidal compactifications:

Theorem 5.7. *Let B_I be a rational boundary component of $\mathbb{D}_M$. There exists a horoball neighborhood $\overline{N}_I \supset B_I$ preserved by $\text{Stab}_\Gamma(I)$ and an embedding*

$$N_I/\text{Stab}_\Gamma(I) \hookrightarrow \mathbb{D}_M/\Gamma \text{ where } N_I = \overline{N}_I \setminus B_I.$$

So a punctured neighborhood of a Baily-Borel cusp can be constructed locally as a quotient by the parabolic stabilizer $\text{Stab}_\Gamma(I)$. Let $0 \rightarrow U_I \rightarrow \text{Stab}_\Gamma(I) \rightarrow \Gamma_I \rightarrow 0$ be the exact sequence associated to the unipotent radical U_I (4.10). Then $N_I/U_I \hookrightarrow \mathbb{D}_M(I) = \mathbb{D}_M/U_I$ has an open embedding into the unipotent quotient. The Levi group Γ_I has a residual action on both.

Definition 5.8. Let $I = \mathbb{Z}\delta$ be a rank 1 isotropic lattice. Let $C_\delta \subset \delta^\perp/\delta \otimes \mathbb{R}$ denote a connected component of the positive norm vectors and let C_δ^+ be its *rational closure*: the union of C_δ with all rational rays on its boundary.

Definition 5.9. A Γ -*admissible collection of fans* $\mathfrak{F}$ (or for short, *fan*) is, for each $I = \mathbb{Z}\delta$, a fan $\mathfrak{F}_\delta$ with support C_δ^+ , such that the collection $\{\mathfrak{F}_\delta\}$ is Γ -invariant, with finitely many orbits of cones.

By “fan” $\mathfrak{F}_\delta$ we mean a decomposition into rational polyhedral cones, closed under taking faces and intersections, and locally finite in the positive cone C_δ . Infinitely many cones meet at rational rays on the boundary of C_δ^+ . Recall, when $\text{rk } I = 1$, there is a “torus embedding” $\mathbb{D}_M(I) \hookrightarrow \delta^\perp/\delta \otimes \mathbb{C}^*$ (4.13).

Construction 5.10. The *toroidal compactification* $\overline{F}_M^{\mathfrak{F}}$ associated to a fan $\mathfrak{F}$ is built as follows: Take the closure $N_I/U_I \hookrightarrow \overline{N_I/U_I} \subset X(\mathfrak{F}_\delta)$ in the toric variety containing $\delta^\perp/\delta \otimes \mathbb{C}^*$ associated to the fan $\mathfrak{F}_\delta$. Then quotient by Γ_I to get an analytic space $V_I := (\overline{N_I/U_I})/\Gamma_I$. This is possible by Γ -invariance of $\mathfrak{F}$. Note that V_I contains an open subset

$$(N_I/U_I)/\Gamma_I = N_I/\text{Stab}_\Gamma(I) \hookrightarrow \mathbb{D}_M/\Gamma.$$

Define the *Type III extension* to be the gluing of $F_M = \mathbb{D}_M/\Gamma$ to V_I along this open set, ranging over all Γ -orbits of rank 1 isotropic I .

If $I = \mathbb{Z}\delta \oplus \mathbb{Z}\lambda$ is isotropic of rank 2, take the closure $\overline{N_I/U_I} \subset \mathbb{D}_M(I)^\lambda$ in the projective line bundle over $I^\perp/I \otimes \tilde{\mathcal{E}}^+$ (4.13) and define V_I as above. The *Type II extension* is the gluing of F_M with V_I along their common open subset $(N_I/U_I)/\Gamma_I = N_I/\text{Stab}_\Gamma(I)$.

The toroidal compactification $\overline{F}_M^{\mathfrak{F}} := F_M \cup_I V_I$ is the gluing of the Type II and III extensions.

Let $\delta \in I = \mathbb{Z}\delta \oplus \mathbb{Z}\lambda$. The analytic structure where the corresponding Type III and II loci meet is described by the Mumford construction [Mum72] applied to a periodic, rational polyhedral tiling $\mathfrak{F}_{\delta,I}$ of $I^\perp/I \otimes \mathbb{R}$. The polyhedral tiles are defined as follows: Quotient the cones of $\mathfrak{F}_\delta$ passing through $\mathbb{R}^+\lambda \subset C_\delta^+$, viewing $I^\perp/I$ as the subquotient $\lambda^\perp/\lambda$ of $\delta^\perp/\delta$. Geometrically, $I^\perp/I \otimes \mathbb{R}$ is identified with a small horosphere through λ (minus λ) in the hyperbolic space $\mathbb{P}C_\delta$. The projectivized cones of $\mathfrak{F}_\delta$ decompose this horosphere in a $\text{Stab}_{\Gamma_\delta}(\lambda)$ -invariant manner.

5C. Semitoroidal compactification. The papers [Loo85, Loo03] are the only references for this section. Semitoroidal compactifications are determined combinatorially, unify the toroidal and Baily-Borel compactifications, and form the smallest class of compactifications closed under taking normal images of toroidal compactifications (proven in Theorem 5.14 below).

The combinatorial input is similar to toroidal compactifications, with two differences:

Definition 5.11 ([Loo03, Def. 6.1]). A *semifan* $\mathfrak{F}$ requires the same data as a fan (Def. 5.9), but we allow the cones in $\mathfrak{F}_\delta$ to be only *locally polyhedral* in C_δ .

We additionally require “compatibility” at each 1-cusp: Let $\delta \in I$ be a primitive integral vector in a rank 2 isotropic lattice and let $\mathfrak{F}_{\delta,I}$ denote the corresponding polyhedral tiling of $I^\perp/I \otimes \mathbb{R}$. The tiles of $\mathfrak{F}_{\delta,I}$ are of the form $B \times (H_{I,\delta} \otimes \mathbb{R})$ for bounded polytopes B and $H_{I,\delta} \subset I^\perp/I$ a primitive sublattice. We require that $H_{I,\delta} = H_I$ is independent of choice of δ .

Example 5.12. Any fan is a semifan. The tiles of $\mathfrak{F}_{\delta,I}$ are bounded polytopes, so $H_I = \{0\}$ for all I . At the other extreme, the semifan $\mathfrak{F}$ for which $\mathfrak{F}_\delta = \{C_\delta^+\}$ is locally finite and the compatibility condition holds: $H_{I,\delta} = I^\perp/I$ for all $\delta \in I$. The resulting compactification is $\overline{F}_M^{\mathfrak{F}} = \overline{F}_M^{\text{BB}}$.

We now compile the key results we need about semitoroidal compactifications:

Theorem 5.13. *There is a normal compactification $\overline{F}_M^{\mathfrak{F}}$ whose boundary strata are in bijection with Γ -orbits of cones of $\mathfrak{F}$ [Loo03, Thm. 6.7]. The stratum Str_σ corresponding to a cone $\sigma \subset \mathfrak{F}_\delta$ is finite quotient of $\delta^\perp/\{\delta, \sigma\} \otimes \mathbb{C}^*$ in Type III, and a finite quotient of $I^\perp/\{I, H_I\} \otimes \mathcal{E}$ in Type II [Loo03, p. 552]. For any semifan $\mathfrak{G}$ which refines $\mathfrak{F}$, there is a morphism*

$$\overline{F}_M^{\mathfrak{G}} \rightarrow \overline{F}_M^{\mathfrak{F}}$$

mapping strata to strata [Loo03, Lem. 6.6]. Given an inclusion of cones $\sigma_{\mathfrak{G}} \subset \sigma_{\mathfrak{F}}$ the map of corresponding strata is induced by the natural quotient map on tori.

Unlike for fans, a Type III cone σ of a semifan may have an infinite stabilizer $\text{Stab}_{\Gamma_\delta}\sigma$. Still, the corresponding stratum is a finite quotient of a torus.

A simple way to visualize a semifan or fan $\mathfrak{F}$ is as follows: For each Γ -orbit of isotropic vector δ , associate a cusped, real-hyperbolic orbifold $M_\delta := \mathbb{H}^{19-\text{rk } M}/\Gamma_\delta$ where $\mathbb{H}^{19-\text{rk } M} = \mathbb{P}C_\delta$ is real-hyperbolic space. The cusps of M_δ correspond to Γ_δ -orbits of isotropic rays in C_δ^+ . A semifan $\mathfrak{F}_\delta$ gives rise to a finite decomposition of M_δ (for all δ) into metrically convex, rational, locally polyhedral cells, compatible with the hyperbolic cusps. For a fan, these cells are polyhedra, while for a semifan, they may have nontrivial topology.

We now prove a key theorem characterizing semitoroidal compactifications:

Theorem 5.14. *Let F be a Type IV arithmetic quotient and let $\overline{F}$ be a normal compactification of F . The following are equivalent:*

- (1) $\overline{F}$ sits between some toroidal and the Baily-Borel compactification: $\overline{F}^{\mathfrak{G}} \xrightarrow{m} \overline{F} \rightarrow \overline{F}^{\text{BB}}$.
- (2) There exists a semifan $\mathfrak{F}$ for which $\overline{F} = \overline{F}^{\mathfrak{F}}$.

Proof. The implication (2) $\implies$ (1) follows from refining $\mathfrak{F}$ to some fan $\mathfrak{G}$.

Now we prove (1) $\implies$ (2). Define an equivalence relation $\sigma_1 \sim \sigma_2$ on maximal cones of $\mathfrak{G}$ generated by: $\sigma_1 \sim \sigma_2$ if σ_1 and σ_2 share a codimension one face τ such that the corresponding 1-dimensional boundary stratum Str_τ is contracted by m . Our strategy is to show that the curves contracted by any birational morphism $m: \overline{F}^{\mathfrak{G}} \rightarrow \overline{F}$ over $\overline{F}^{\text{BB}}$ are algebraically equivalent to a union of 1-dimensional torus orbits. So the “toroidally definable” equivalence relation $\sim$ captures everything one needs to know about the contracting morphism m .

Define a decomposition of C_δ^+ into a collection of maximal dimensional sets

$$[\sigma_0] := \bigcup_{\sigma \sim \sigma_0} \sigma.$$

We claim that the $[\sigma_0]$ form the maximal cones of some semifan $\mathfrak{F}$. The Γ -invariance is automatic, so it suffices to show that $[\sigma_0]$ satisfy the semifan axioms, including the compatibility condition (Def. 5.11) over the 1-cusps.

Begin with a Type III cone $\tau \in \mathfrak{G}_\delta$. The stratum $\overline{\text{Str}}_\tau \subset \overline{F}^{\mathfrak{G}}$ is the $\text{Stab}_{\Gamma_\delta}(\tau)$ -quotient of the toric variety $X(\mathfrak{G}_\delta/\tau)$ associated to the quotient fan. Consider the Stein factorization $X(\mathfrak{G}_\delta/\tau) \rightarrow Z_\tau \rightarrow m(\overline{\text{Str}}_\tau)$. It is proved in [BMSZ18, Lem. 2.3.4] that the target of a morphism from a proper toric variety to a normal variety (Z_τ here) is automatically toric and with the morphism also toric.

Since the maps $Z_\tau \rightarrow m(\overline{\text{Str}}_\tau)$ and $X(\mathfrak{G}_\delta/\tau) \rightarrow \overline{\text{Str}}_\tau$ are finite, the curves contracted by $\overline{\text{Str}}_\tau \rightarrow m(\overline{\text{Str}}_\tau)$ are exactly those that lift to curves contracted by $X(\mathfrak{G}_\delta/\tau) \rightarrow Z_\tau$. So the equivalence relation $\sim$ on the maximal cones containing τ is induced by the morphism of fans corresponding to the toric morphism $X(\mathfrak{G}_\delta/\tau) \rightarrow Z_\tau$. Thus the cones $[\sigma_0]$ locally form a fan in a tubular neighborhood of $\tau \in C_\delta^+$. In particular, they are locally polyhedral and convex at their boundary. So the $[\sigma_0]$ define a semifan within C_δ .

Next, we examine the Type II locus. Since $\overline{F}$ has a morphism to $\overline{F}^{\text{BB}}$, we conclude that m induces a fiberwise morphism over the modular curve 1-cusp. Let j be a point in the 1-cusp of $\overline{F}^{\text{BB}}$. The fiber over j in any toroidal compactification is a finite quotient of $I^\perp/I \otimes E_j$ so there is a morphism $I^\perp/I \otimes E_j \rightarrow \overline{F}_j$ induced by m . In analogy with the Type III case, take the Stein factorization of this morphism $I^\perp/I \otimes E_j \rightarrow Z_j \rightarrow \overline{F}_j$. Since the normal image of an abelian variety is an abelian variety, this map is the quotient by a sub-abelian variety $H_I \otimes E_j$.

The contracted curves are generated, up to algebraic equivalence, by $h \otimes E_j$ for $h \in H_I$. Taking the limit

$$j \rightarrow \text{the 0-cusp of } \overline{F}^{\text{BB}} \text{ associated to } \delta,$$

the elliptic curve $h \otimes E_j$ breaks in the Type III locus to a cycle of rational curves, according to the Mumford degeneration discussed after Construction 5.10.

Applying the torus action to the cycle of rational curves $\lim_j h \otimes E_j$ we can break it further into a cycle of contracted 1-dimensional boundary strata connecting 0-dimensional boundary strata of Type III. So the equivalence relation $\sim$ on maximal cones induces a polyhedral decomposition of $I^\perp/I \otimes \mathbb{R}$ whose tiles are fixed under translation by h , and in turn by H_I . Conversely, consider an $h \in I^\perp/I$ fixing all tiles in the polyhedral decomposition $H_{I,\delta}$ of $I^\perp/I$ induced by the cones $[\sigma_0]$. This h corresponds to a contracted cycle of rational curves, which deforms to a contracted elliptic curve $h \otimes E_j$. Hence $h \in H_I$. Thus $H_I = H_{I,\delta}$ is independent of δ . So there exists a semifan $\mathfrak{F}$ whose maximal cones are $[\sigma_0]$.

Since $\mathfrak{F}$ is a coarsening of $\mathfrak{G}$, there is a morphism $\overline{F}^{\mathfrak{G}} \xrightarrow{n} \overline{F}^{\mathfrak{F}}$ and the above arguments prove that the curves contracted by n are exactly those contracted by m . We conclude by Zariski’s main theorem that $\overline{F} = \overline{F}^{\mathfrak{F}}$, because $\overline{F}$ is normal. $\square$

6. MODULI OF STABLE SLC PAIRS

6A. Canonical choices of polarizing divisor. Let $\mathcal{F}_M^q$ be the moduli stack of M -quasipolarized K3 surfaces. Fix a class $L \in M$, not necessarily primitive, which defines a relatively big and nef line bundle $\mathcal{L} \rightarrow \mathcal{X} \rightarrow \mathcal{F}_M^q$ on the universal family, canonical up to twisting by line bundles pulled back from $\mathcal{F}_M^q$. Since $\mathcal{L}$ is big and nef on every fiber, $h^i(\mathcal{X}_s, \mathcal{L}_s) = 0$ for $i > 0$ for all $s \in \mathcal{F}_M^q$. By Cohomology and Base Change, the pushforward of $\mathcal{L}$ from the universal family defines a vector bundle of rank $2 + \frac{1}{2}L^2$ on $\mathcal{F}_M^q$, canonical up to twisting by line bundles, cf. [Kol23, Cor. 2.69]. Let $\mathbb{P}_{\mathcal{L}}$ denote its projectivization, a $\mathbb{P}^g$ -bundle over the stack, where $g = d + 1$.

Definition 6.1. A *canonical choice of polarizing divisor* is a rational section R of the projective bundle $\mathbb{P}_{\mathcal{L}}$. Alternatively, it is an ample divisor R on the generic K3 surface.

Let U be the regular locus of this rational section. The key definition of the paper is:

Definition 6.2. A canonical choice of polarizing divisor R is *recognizable for F_M* if every M -quasipolarized Kulikov surface X_0 of Type I, II, or III contains a divisor $R_0 \subset X_0$ which, for any M -quasipolarized smoothing $X \rightarrow (C, 0)$ with $C^* \subset U$, has the property that R_0 is the flat limit of $R_t \subset X_t$, $t \neq 0$, up to the action of $\text{Aut}^0(X_0)$.

Here $\text{Aut}^0(X_0)$ is the connected component of the identity of the automorphism group, which is always trivial in Type III, and is isomorphic to $(\mathbb{C}^*)^{k-1}$ where $k-1$ is the number of intermediate elliptic ruled components, in Type II.

We use the term “smoothing” to mean specifically a Kulikov model $X \rightarrow (C, 0)$. Roughly, Definition 6.2 amounts to saying that the canonical choice R can also be made on any Kulikov surface, including smooth K3s, at least up to $\text{Aut}^0(X_0)$.

Proposition 6.3. Let $(\mathcal{X}^*, \mathcal{R}^*) \rightarrow U$ be the universal family of pairs. If R is recognizable, it extends to a flat family of pairs $(\mathcal{X}, \mathcal{R}) \rightarrow \mathcal{F}_M^q$. That is, R defines a regular section of $\mathbb{P}_{\mathcal{L}} \rightarrow \mathcal{F}_M^q$.

Proof. Let $0 \in \mathcal{F}_M^q$ be in the complement of U . Choose any curve $C \subset \mathcal{F}_M^q$ containing 0 for which $C^* = C \setminus 0 \subset U$. Then $X \rightarrow C$ is a Type I Kulikov model, for which X_0 is a smooth M -quasipolarized K3 surface. By assumption, there is a divisor $R_0 \in |L|$ on X_0 which is the flat limit of the curves R_t for $t \neq 0$. We may extend the section $\mathcal{R}^* : U \rightarrow \mathbb{P}_{\mathcal{L}}|_U$ set-theoretically by declaring $\mathcal{R}(0) = R_0$. This extension is algebraic when restricted to any curve in $U \cup \{0\}$. Since $\mathcal{F}_M^q$ is normal, we conclude that the rational section $\mathcal{R}^*$ extends over 0 . $\square$

This proposition only concerns Type I Kulikov models. The properties of recognizability in Types II and III is discussed in Section 8.

6B. Compact moduli of stable pairs. We refer the reader to [Kol23] for a definitive account. An *slc* (or KSBA) *stable pair* $(X, B = \sum b_i B_i)$ consists of a projective variety and a $\mathbb{Q}$ -divisor which has semi log canonical (slc) singularities such that the divisor $K_X + B$ is ample. A particular case is a log Calabi-Yau pair $(X, \Delta + \epsilon R)$ such that Δ is reduced and log canonical, $0 < \epsilon \ll 1$, $K_X + \Delta \sim_{\mathbb{Q}} 0$ and R is ample, not containing any log centers of Δ . In our notations, R is a polarizing divisor. By [KX20], in any dimension the irreducible components of the moduli of log Calabi-Yau pairs with a polarizing divisor are projective. Sections 6.4 and 8.3 of [Kol23] are closely related to our setup.

The situation for K3 surfaces (note $\Delta = 0$) is easier because if $(X_0, \epsilon R_0)$ is the stable limit of a one-parameter family of K3 pairs $(X_t, \epsilon R_t)$ then the divisor R_0 is, perhaps surprisingly, Cartier and not merely $\mathbb{Q}$ -Cartier. Indeed, the pair $(X_0, \epsilon R_0)$ is the central fiber of the stable model of a divisor model we defined and discussed in Section 3B. We state the main theorem for the moduli functor we need in this paper. The details are given in [AET19, Sec. 3].

Definition 6.4. For a fixed degree $e \in \mathbb{N}$ and fixed rational number $0 < \epsilon \leq 1$, a *stable K -trivial pair* of type (e, ϵ) is a pair $(X, \epsilon R)$ such that

- (1) X is a Gorenstein surface with $\omega_X \simeq \mathcal{O}_X$,
- (2) The divisor R is an effective, ample Cartier divisor of degree $R^2 = e$.

(3) The pair $(X, \epsilon R)$ has semi log canonical singularities.

Definition 6.5. A family of stable K -trivial pairs of type (e, ϵ) is a flat morphism $f: (\mathcal{X}, \epsilon \mathcal{R}) \rightarrow S$ such that $\omega_{\mathcal{X}/S} \simeq \mathcal{O}_{\mathcal{X}}$ locally on S , the divisor $\mathcal{R}$ is a relative Cartier divisor, such that every fiber is a stable K -trivial pair of type (e, ϵ) .

By [AET19, Lem. 3.6], for a fixed degree e there exists an $\epsilon_0(e) > 0$ such that for any $0 < \epsilon \leq \epsilon_0$ the moduli stacks $\mathcal{M}^{\text{slc}}(e, \epsilon_0)$ and $\mathcal{M}^{\text{slc}}(e, \epsilon)$ coincide.

Definition 6.6. A family of stable K -trivial pairs of degree e is a family of type (e, ϵ_0) , with $\epsilon_0(e)$ chosen as above. We will denote the corresponding moduli functor by $\mathcal{M}_e^{\text{slc}}$. For a scheme S , $\mathcal{M}_e^{\text{slc}}(S) = \{\text{families of type } (e, \epsilon_0(e)) \text{ over } S\}$, with the equivalence relation being S -isomorphisms of the family $\mathcal{X} \rightarrow S$ preserving $\mathcal{R}$.

Proposition 6.7 ([AET19, Prop. 3.8]). $\mathcal{M}_e^{\text{slc}}$ is a Deligne-Mumford stack of stable K -trivial pairs.

We denote the coarse moduli space by M_e^{slc} .

Definition 6.8. Let $N \in \mathbb{N}$. The moduli stack $\mathcal{P}_{N,2d}$ parameterizes proper flat families of pairs (X, R) such that (X, L) is a polarized K3 surface with ADE singularities and a primitive ample line bundle L , $L^2 = 2d$, and $R \in |NL|$ is an arbitrary divisor. One has $R^2 = 2dN^2$. In particular, one defines $\mathcal{P}_{2d} := \mathcal{P}_{1,2d}$.

If we take $\epsilon_0(e)$ as above then the pair $(X, \epsilon_0 R)$ is stable. Obviously, the stack $\mathcal{P}_{N,e}$ is fibered over the stack $\mathcal{F}_{2d}$ with fibers isomorphic to $\mathbb{P}^{dN^2+1}$. The automorphism groups of stable pairs are finite, and it is easy to see that $\mathcal{P}_{N,2d}$ is coarsely represented by a scheme $P_{N,2d}$.

Definition 6.9. One defines $\overline{\mathcal{P}}_{N,2d}$ (resp. $\overline{\mathcal{P}}_{N,2d}$) to be the closure of the coarse moduli space $P_{N,2d}$ (resp. stack $\mathcal{P}_{N,2d}$) in M_e^{slc} (resp. $\mathcal{M}_e^{\text{slc}}$) for $e = 2dN^2$.

For K3 surfaces polarized by a lattice $M \subset \text{Pic } X$, choose a primitive vector $L \in M$ with $L^2 > 0$. Then the substack $\mathcal{F}_M \subset \mathcal{F}_{2d}$ parameterizing M -polarized K3 surfaces inside of $\mathbb{Z}L$ -polarized K3 surfaces has dimension $20 - \text{rank } M$. A canonical choice R of polarizing divisor over a Zariski open subset $U \subset \mathcal{F}_M$ (or equivalently $\mathcal{F}_M^q$ as in Def. 6.1) defines an embedding of $U \subset \mathcal{P}_{N,2d}$ if $R \in |NL|$.

Remark 6.10. We should choose U to avoid the non-separated locus of $\mathcal{F}_M^q$ to ensure that $U \subset \mathcal{P}_{N,2d}$. This embedding exists on the stack level. For instance, on the stack $\mathcal{F}_2^q$ of degree 2 K3 surfaces, there is a nontrivial generic inertia group $\mathbb{Z}_2$. Then R must be preserved by the involution, and defines an embedding of stacks $U \subset \mathcal{P}_{N,2d}$.

Definition 6.11. Let $\overline{\mathcal{F}}_M^R$ denote the closure of U in $\overline{\mathcal{P}}_{N,2d}$ and let $\overline{F}_M^R$ be its coarse space.

Proposition 6.12. If R is recognizable, $\overline{\mathcal{F}}_M^R$ contains $\mathcal{F}_M$ as an open substack.

Proof. By Proposition 6.3, the choice of divisor R extends to all of $\mathcal{F}_M^q$ when R is recognizable. Taking the relative stable model of the universal family of pairs $(\mathcal{X}, \mathcal{R}) \rightarrow \mathcal{F}_M^q$ gives a classifying morphism $\mathcal{F}_M^q \rightarrow \overline{\mathcal{F}}_M^R$ which necessarily factors through the separated quotient $\mathcal{F}_M$. $\square$

Theorem 6.13 ([AET19, Thm. 3.11]). $\overline{\mathcal{P}}_{N,2d}$ and thus also $\overline{F}_M^R$ are projective.

7. λ -FAMILIES

The goal of this section is to construct “ λ -families” of Kulikov models, both unpolarized and M -quasipolarized, of a fixed combinatorial type, and to describe the birational modifications which relate them. These are families of Kulikov models, which complete any one-parameter degeneration with monodromy invariant λ and play a critical role in the main theorem of [FS86]: two Kulikov models with the same λ are related by Atiyah flops and topologically trivial deformations.

Some improvements on *loc.cit.* are made: We construct families for which the boundary period mapping is an isomorphism onto the period torus $\text{Hom}(\Lambda, \mathbb{C}^* \text{ or } \mathcal{E})$, as opposed to simply an

isogeny. Also, we globalize the main theorem of [FS86]: two λ -families are related by certain global birational modifications (Thm. 7.19, Thm. 7.28). These global modifications are key to proving that different formulations of recognizability are equivalent (Sec. 8).

Unlike Kulikov models, which depend on continuous parameters, the λ -families depend only on combinatorial parameters, and thus are countable in number. Similar families of Kulikov surfaces previously appear in work of Olsson [Ols04]. See Remark 7.35 for a comparison with our version.

7A. Deformation spaces of Kulikov models. We recall the description of the universal deformation of a d -semistable Kulikov surface X_0 given in [Fri83b, Thm. 5.10], when $(X_0)_{\text{sing}}$ is connected. The deformation space $S \cup T$ has two smooth components. The component S is smooth and 20-dimensional, with a smooth, divisorial discriminant locus Δ . The general fiber over $s \in S$ is a smooth K3 surface. The other component T has large dimension $\text{rk } \tilde{\Lambda}(X_0)$, and consists of the topologically trivial deformations of X_0 . These result from deforming the gluings of double curves or the moduli of anticanonical pairs (V_i, D_i) and are generally not d -semistable. $\Delta = S \cap T$ consists of the d -semistable, topologically trivial deformations of X_0 . The universal family $\mathcal{X} \rightarrow S$ is topologically a product $\mathcal{X} \approx_{\text{diff}} \Delta \times X$ with a fixed Kulikov model $X \rightarrow (C, 0)$, and has smooth total space. In particular, $\mathcal{X} \rightarrow S$ admits a mixed marking (4.15) over a contractible S .

As for deformations of smooth K3 surfaces, the local period map on S is understood:

Theorem 7.1. *Let X_0 be a d -semistable Kulikov surface. Suppose $t > 0$, or $t = 0, k = 1$ (Sec. 4B). The period map $S \rightarrow \mathbb{D}(I)^\lambda$ is an order k cyclic cover, branched along the boundary divisor.*

Proof. In Type III, this is [FS86, Thm. 5.3]. The Type II case is similar [Fri84]. $\square$

By Theorem 7.1 we can ensure that $\mathcal{X} \rightarrow S$ is universal at all $s \in S$: A topologically trivial family $\mathcal{X}_0 \rightarrow \Delta$ of d -semistable Kulikov surfaces is a fiberwise universal deformation if and only if the period map to the boundary divisor $\text{Hom}(\Lambda, \mathbb{C}^* \text{ or } \tilde{\mathcal{E}}) \subset \mathbb{D}(I)^\lambda$ is a local isomorphism.

Remark 7.2. In the remaining case $t = 0, k > 1$ the singular locus of X_0 is disconnected, making it possible to independently smooth each double curve. The d -semistable deformations of X_0 have dimension $19 + k$ and fiber over $\mathbb{C}^k$, with each coordinate hyperplane parameterizing deformations which do not smooth a given double curve of X_0 .

In this case, we define S as the inverse image of the line $\mathbb{C}(1, \dots, 1) \subset \mathbb{C}^k$. It gives a slice transverse to the natural action of $\text{Aut}^0(X_0) \cong (\mathbb{C}^*)^{k-1}$. The discriminant locus $\Delta \subset S$ is the inverse image of $0 \in \mathbb{C}^k$ and is still the universal d -semistable topologically trivial deformation, while the general fiber is a K3 surface that simultaneously smooths all k double curves.

Proposition 7.3. *Let $\mathcal{X}_0 \rightarrow \Delta_\lambda$ be a topologically trivial family of marked Kulikov surfaces for which the period map $\Delta_\lambda \rightarrow \text{Hom}(\Lambda, \mathbb{C}^* \text{ or } \tilde{\mathcal{E}}) \subset \mathbb{D}(I)^\lambda$ is an isomorphism. There is a smoothing*

$$\begin{array}{ccc} \mathcal{X}_0 & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \Delta_\lambda & \longrightarrow & S_\lambda \end{array}$$

for which the mixed period map $S_\lambda \rightarrow \mathbb{D}(I)^\lambda$ defines an order k cyclic branched cover to an open neighborhood of the boundary divisor. The analytic germ of the family along $\Delta_\lambda \subset S_\lambda$ is unique.

Sketch. The construction parallels that of [ast85, Exp. XIII]. When $t > 0$, or $t = 0$ and $k = 1$, we glue together the 20-dimensional bases of everywhere-universal deformations of the fibers $X_0 \subset \mathcal{X}_0$. With the mixed markings, these bases either glue uniquely (when $k = 1$) or uniquely up to the order k cyclic action permuting the sheets of the period mapping (Thm. 7.1). Taking care to ensure that the glued base is Hausdorff, the resulting family $\mathcal{X} \rightarrow S_\lambda$ smooths $\mathcal{X}_0 \rightarrow \Delta_\lambda$ and the germ is unique by local universality. The $t = 0, k > 1$ case is proven in the same way, by instead gluing the slices S of the $\text{Aut}^0(X_0)$ action, see Remark 7.2. $\square$

7B. The gluing and period complexes. We now explicitly construct families of Kulikov surfaces $\mathcal{X}_0 \rightarrow \Delta_\lambda$ satisfying the hypotheses of Proposition 7.3, developing ideas in [FS86, Sec. 4]. We assume here that $X_0 = \bigcup (V_i, D_i)$ is Type III. For notational convenience, we drop the index i when analyzing an individual component.

Each component (V, D) admits a toric model $(V, D) \xleftarrow{f} (\tilde{V}, \tilde{D}) \xrightarrow{g} (\bar{V}, \bar{D})$ where f is a sequence of corner blow-ups and g is a sequence of internal blow-ups (3.9). Note that f has no moduli whereas g can be varied by moving the non-nodal points blown up on the $\bar{D}_j$. Note that unless (V, D) is itself toric, the toric model is non-unique.

Definition 7.4. An *ordered toric model* of (V, D) is an orientation of the cycle D and a toric model f, g as above, together with a factorization $g = \tau_Q \circ \dots \circ \tau_1$ into internal blow-ups. Here $Q = Q(V, D)$ is the charge (3.7). An *ordered toric model* of X_0 is an orientation of $\Gamma(X_0)$, a toric model of each component (V_i, D_i) , and a total ordering of the 24 internal blow-ups.

An ordered toric model of X_0 orients each cycle $D_i \subset V_i$ and thus gives a way to label the nodes on the component $D_{ij} \subset D_i$ as 0 and ∞ . But on the double curve $D_{ji} \subset D_j$ the corresponding nodes have the opposite label, so the non-nodal points of D_{ij} and D_{ji} are inverse torsors for $\mathbb{C}^*$.

Construction 7.5. Fix an ordered toric model of X_0 and fix copies of the toric surfaces $(\bar{V}_i, \bar{D}_i)$. For a given toric surface $(\bar{V}, \bar{D})$, construct a family

$$(\tilde{V}, \tilde{D}) \xrightarrow{\tau_Q} \dots \xrightarrow{\tau_1} (\bar{V}, \bar{D}) \times (\mathbb{C}^*)^Q$$

of anticanonical pairs over $(\mathbb{C}^*)^Q$ by freely varying the points blown up by τ_k . There exists a simultaneous contraction $(\mathcal{V}, \mathcal{D}) \rightarrow (\mathbb{C}^*)^Q$ which contracts the corner blowdowns of f fiberwise. So we have families $(\mathcal{V}_i, \mathcal{D}_i) \rightarrow (\mathbb{C}^*)^{Q_i}$ for all i .

Now, for each i , choose some fiber (V_i, D_i) of this family and glue $D_{ij} \subset D_i$ to $D_{ji} \subset D_j$ by a map identifying the appropriate nodes of D_i and D_j . The set of such gluings is a torsor over $\mathbb{C}^*$. Varying all such gluings, we get a family of Kulikov surfaces

$$\mathcal{X}_0^{\text{gig}} \rightarrow \prod_i (\mathbb{C}^*)^{Q_i} \times (\mathbb{C}^*)^E = (\mathbb{C}^*)^{24+E}$$

whose fibers are not necessarily d -semistable. Here E is the number of double curves of X_0 . We call this the *gigantic gluing family* of Kulikov surfaces associated to the ordered toric model of X_0 . It is globally topologically trivial by construction.

Choose an origin of the open torus orbit in the fixed toric surface $(\bar{V}_i, \bar{D}_i)$. This choice defines a distinguished origin point of any toric boundary component and thus defines an isomorphism of the $\mathbb{C}^*$ -torsor associated to any internal blow-up or any edge-gluing with $\mathbb{C}^*$. So such a choice identifies the base of the gigantic gluing family with $\text{Hom}(\mathcal{G}_0, \mathbb{C}^*)$ where

$$\mathcal{G}_0 := \bigoplus_{k=1}^{24} \mathbb{Z} E_{ijk} \oplus \bigoplus_{i < j} \mathbb{Z} D_{ij}$$

is a free $\mathbb{Z}$ -module encoding the blow-up points of the τ_k and the gluing maps. Here the index ijk indicates that E_{ijk} meets the component D_{ij} and is the k th internal blow-up in the ordered toric model. Note that D_{ij} range only over the curves corresponding to actual double curves appearing in X_0 and not to boundary components blown down in $(\tilde{V}_i, \tilde{D}_i)$.

Consider now automorphisms. Define $\mathcal{G}_1 := \bigoplus_i M_i$ with each $M_i \cong \mathbb{Z}^2$ the character lattice of the toric surface $(\bar{V}_i, \bar{D}_i)$. The set of choices of origin points in the open torus orbit of $(\bar{V}_i, \bar{D}_i)$ is naturally a torsor over $\text{Hom}(\mathcal{G}_1, \mathbb{C}^*)$. Fixing the family $\mathcal{X}_0^{\text{gig}}$ but varying the chosen origin point defines an equivariant action of $\text{Hom}(\mathcal{G}_1, \mathbb{C}^*)$ on $\mathcal{X}_0^{\text{gig}} \rightarrow \text{Hom}(\mathcal{G}_0, \mathbb{C}^*)$ by isomorphisms. On the base, this action is determined by a map of $\mathbb{Z}$ -modules $\mathcal{G}_0 \rightarrow \mathcal{G}_1$.

Definition 7.6. The *gluing complex* $\mathcal{G}$ is the two-step complex $\mathcal{G}_0 \xrightarrow{\partial_{\mathcal{G}}} \mathcal{G}_1$.

We describe the map $\partial_{\mathcal{G}}$ explicitly. An orientation of the cycle D_i (and thus of $\bar{D}_i$) gives a canonical identification $w_i : M_i \rightarrow N_i$ sending $v \mapsto \det(v, -)$.

Proposition 7.7. *We have*

$$\partial_{\mathcal{G}}(E_{ijk}) = w_i^{-1}(v_{ij}), \quad \partial_{\mathcal{G}}(D_{ij}) = w_i^{-1}(v_{ij}) + w_j^{-1}(v_{ji}).$$

Here $v_{ij} \in N_i$ in the cocharacter lattice of $(\bar{V}_i, \bar{D}_i)$ is the primitive integral vector in the fan of $(\bar{V}_i, \bar{D}_i)$ corresponding to the component $\bar{D}_{ij}$.

Proof. The action of a change-of-origin $c_i \in \text{Hom}(M_i, \mathbb{C}^*)$ on the induced origin point of $\bar{D}_{ij}$ is given by $c_i(w_i^{-1}(v_{ij}))$. This factor scales either the gluing parameter between D_{ij} and D_{ji} or the position of any blow-up E_{ijk} on the edge $\bar{D}_{ij}$. $\square$

Given Proposition 7.7, it is convenient to identify $\mathcal{G}_1 \cong \bigoplus_i N_i$ using the isomorphisms w_i^{-1} on each summand, so that $\partial_{\mathcal{G}}(E_{ijk}) = v_{ij}$ and $\partial_{\mathcal{G}}(D_{ij}) = v_{ij} + v_{ji}$.

Definition 7.8. The *period complex* $\mathcal{P}$ of a Kulikov model is the two-step complex $\mathcal{P}_0 \xrightarrow{\partial_{\mathcal{P}}} \mathcal{P}_1$ where $\mathcal{P}_0 = \bigoplus_i H^2(V_i)$, $\mathcal{P}_1 = \bigoplus_{i \leq j} H^2(D_{ij})$, and $\partial_{\mathcal{P}}$ is the signed restriction map with respect to an orientation of the edges of $\Gamma(X_0)$.

Theorem 7.9. *Let X_0 be a Type III Kulikov surface with an ordered toric model. The gluing and period complexes are quasi-isomorphic as complexes of $\mathbb{Z}$ -modules. In particular $H^0(\mathcal{G}) = H^0(\mathcal{P}) = \tilde{\Lambda}$ and $K := H^1(\mathcal{G}) = H^1(\mathcal{P})$.*

Proof. We first record some exact sequences of Picard groups arising from the basic results on smooth projective toric surfaces:

Lemma 7.10. *Write $(V_i, D_i = \sum_j D_{ij})$ as $(V, D = \sum D_j)$. For each component, one has the following exact sequences:*

$$\begin{aligned} 0 \rightarrow \text{Pic } \bar{V} &\rightarrow \bigoplus \mathbb{Z} \bar{D}_j \rightarrow N \rightarrow 0, & \bar{L} &\mapsto \sum (\bar{L} \cdot \bar{D}_j) \bar{D}_j \\ 0 \rightarrow \text{Pic } \tilde{V} &\rightarrow \bigoplus \mathbb{Z} \bar{D}_j \oplus \mathbb{Z} E_{jk} \rightarrow N \rightarrow 0, & \tilde{L} &\mapsto \sum (\bar{L} \cdot \bar{D}_j) \bar{D}_j + \sum (\tilde{L} \cdot E_{jk}) E_{jk} \\ 0 \rightarrow \text{Pic } V &\rightarrow \bigoplus \mathbb{Z} \bar{D}_j \oplus \mathbb{Z} E_{jk} \rightarrow N \rightarrow 0, & L &\mapsto \sum (\bar{L} \cdot \bar{D}_j) \bar{D}_j + \sum (\tilde{L} \cdot E_{jk}) E_{jk} \end{aligned}$$

where

- (1) $L, \tilde{L}, \bar{L}$ are the line bundles on $V, \tilde{V}, \bar{V}$, and $\bar{L} = g_* \tilde{L}$.
- (2) In the last line we take $\tilde{L} = f^* L$.
- (3) In the last line the sum goes only over $\bar{D}_j$ such that $D_j = f_* \tilde{D}_j \neq 0$.
- (4) $\bar{D}_j \mapsto v_j$ and $E_{jk} \mapsto v_j$.

Notationally reincorporating the dependence on i , and summing these exact sequences, we get a short exact sequence of two-term complexes:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus \text{Pic } V_i & \longrightarrow & \bigoplus_{i \leq j} \mathbb{Z} D_{ij} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_{i,j} \mathbb{Z} \bar{D}_{ij} \oplus \bigoplus_{k=1}^{24} \mathbb{Z} E_{ijk} & \longrightarrow & \bigoplus_{i \leq j} \mathbb{Z} D_{ij} \oplus N_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus N_i & \longrightarrow & \bigoplus N_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Note that:

- (1) The first column is a direct sum of sequences from the previous lemma.
- (2) In the first line $\partial_{\mathcal{P}}: L_i \mapsto \pm \sum (L_i \cdot D_{ij}) D_{ij}$ is the signed restriction map.

- (3) In the second column, the first map sends $D_{ij} \mapsto D_{ij} + v_{ij}$.
- (4) In the second line, $\overline{D}_{ij} \mapsto D_{ij} + v_{ij}$ and $\overline{D}_{ji} \mapsto -D_{ij} + v_{ji}$ if the corresponding edge is oriented from i to j . Also, for all i and j , $E_{ijk} \mapsto v_{ij}$.

The commutativity of the diagram follows from $\tilde{D}_i = f_i^*(\overline{D}_i) - \sum_{j,k} E_{ijk}$. Since the last complex is acyclic, the complex $\mathcal{P}$ is quasi-isomorphic to the second complex $\tilde{\mathcal{G}}$. There also is a quasi-isomorphism $\mathcal{G} \rightarrow \tilde{\mathcal{G}}$. On $\mathcal{G}_0$ it maps $\overline{D}_{ij} \mapsto \overline{D}_{ij} + \overline{D}_{ji}$ and $E_{ijk} \mapsto E_{ijk}$ and on $\mathcal{G}_1$ it is $(0, \text{id})$. $\square$

Proposition 7.11. *The gigantic gluing family $\mathcal{X}_0^{\text{gig}} \rightarrow \text{Hom}(\mathcal{G}_0, \mathbb{C}^*)$ descends along the canonical surjection $\text{Hom}(\mathcal{G}_0, \mathbb{C}^*) \twoheadrightarrow \text{Hom}(H^0(\mathcal{G}), \mathbb{C}^*)$. Furthermore, the isomorphism $\text{Hom}(H^0(\mathcal{G}), \mathbb{C}^*) = \text{Hom}(\tilde{\Lambda}, \mathbb{C}^*)$ induced by Theorem 7.9 is the period map of the descended family.*

Proof. As noted, $\text{Hom}(\mathcal{G}_1, \mathbb{C}^*)$ acts by automorphisms on $\mathcal{X}_0^{\text{gig}}$. The action is free and the quotient can be constructed, for instance, by restricting $\mathcal{X}_0^{\text{gig}}$ to a subtorus of $\text{Hom}(\mathcal{G}_0, \mathbb{C}^*)$ which intersects each orbit of $\text{Hom}(\mathcal{G}_1, \mathbb{C}^*)$ exactly once.

For the second statement, it suffices to show that the action of regluing on periods is described by the isomorphism $H^0(\mathcal{G}) \rightarrow \tilde{\Lambda}$. By Construction 4.2, regluing D_{ij} or moving the blow-up point of E_{ijk} by $c \in \mathbb{C}^*$ can be computed by gluing in V_i as the last component. The action is

$$\psi_{X_0}(\gamma) \mapsto c^{\gamma \cdot D_{ij}} \psi_{X_0}(\gamma) \quad \text{and} \quad \psi_{X_0}(\gamma) \mapsto c^{f_i^* \gamma \cdot E_{ijk}} \psi_{X_0}(\gamma) \quad \text{for } \gamma \in \tilde{\Lambda}.$$

This exactly corresponds to the first chain map $\mathcal{P}_0 \rightarrow \tilde{\mathcal{G}}_0$ in Theorem 7.9. Thus the map $\mathcal{P}_0 \rightarrow \tilde{\mathcal{G}}_0$ in Lemma 7.10 is the natural one, $H^0(\mathcal{P}) = H^0(\mathcal{G})$ canonically, and the proposition follows. $\square$

Definition 7.12. Let X_0 be a Type III Kulikov surface with ordered toric model. Define the *big gluing family* to be the descended family $\mathcal{X}_0^{\text{big}} \rightarrow \text{Hom}(\tilde{\Lambda}, \mathbb{C}^*)$ from Proposition 7.11, for which the period map is an isomorphism.

Now observe that $\text{Hom}(\Lambda, \mathbb{C}^*)$ is the subtorus of $\text{Hom}(\tilde{\Lambda}, \mathbb{C}^*)$ corresponding to the d -semistable Kulikov surfaces. So we define:

Definition 7.13. The *gluing family* associated to the ordered toric model of X_0 is the restriction of $\mathcal{X}_0^{\text{big}}$ to the subtorus $\mathcal{X}_0 \rightarrow \text{Hom}(\Lambda, \mathbb{C}^*) = \Delta_\lambda$. The period map is an isomorphism.

Definition 7.14. The λ -family of a Type III Kulikov surface X_0 with ordered toric model is the unique germ $\mathcal{X} \rightarrow S_\lambda$ of the universal smoothing (Prop. 7.3) of the gluing family $\mathcal{X}_0 \rightarrow \Delta_\lambda$.

We delay the construction of λ -families in Type II, as some complications arise from the non-existence of toric models.

7C. The global Friedman-Scattone theorem. We now discuss birational modifications. Let $X \rightarrow (C, 0)$ be a Kulikov model of Type I, II, or III.

Definition 7.15. An *(M0)*, *(M1)*, or *(M2)* modification of X is the flop along a curve $E \cong \mathbb{P}^1$ in the central fiber X_0 . The cases are distinguished by when $E \cap (X_0)_{\text{sing}} = \emptyset$, when $E \cap (X_0)_{\text{sing}} = \{\text{pt}\}$, or when $E \subset (X_0)_{\text{sing}}$, respectively.

We describe the effect of each modification on the central fiber X_0 :

- (M0) flops a smooth (-2) -curve in X_0 which does not deform to the general fiber. It leaves the isomorphism type of X_0 invariant.
- (M1) flops an internal exceptional (-1) -curve E on a component $V_i \subset X_0$. The effect on the central fiber is to contract $E \subset V_i$ and blow up the intersection point $E \cap D_{ij}$ on V_j .
- (M2) flops a double curve D_{ij} which is exceptional on both components on which it lies. The effect on X_0 is to contract D_{ij} on both V_i, V_j and to make corner blow-ups on the two remaining components V_ℓ, V_r which E intersects.

Notation 7.16. In the book [FM83b], M0, M1, M2 modifications are called Type 0, 1, 2 modifications, but we find this to conflict with the already existing usage of the word “Type.”

[FS86, Thm. 0.6] states that any two Kulikov models with the same (k, t) are related by M0, M1, and M2 modifications and topologically trivial deformations. We require an analogue of this statement on the level of the entire λ -family.

Definition 7.17. Let $\mathcal{X} \rightarrow S_\lambda$ be the λ -family associated to some ordered toric model of X_0 . Let $B \subset S_\lambda$ be a smooth divisor and let $\mathcal{E} \rightarrow B$ be a smooth $\mathbb{P}^1$ -fibration for which the normal bundle to $\mathcal{E}$ restricts to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on every fiber. We call the relative flop along $\mathcal{E}$:

- (GM0) if B is the closure in S_λ of a Noether-Lefschetz divisor of K3 surfaces with a (-2) -curve, and $\mathcal{E}$ is the family of (-2) -curves.
- (GM1) if $B = \Delta_\lambda$ is the discriminant, and $\mathcal{E}$ is a family of internal exceptional curves meeting a relative double curve $\mathcal{D}_{ij}$.
- (GM2) $B = \Delta_\lambda$ is the discriminant, and $\mathcal{E}$ is a family of relative double curves $\mathcal{D}_{ij}$ which is, on each fiber, exceptional on both components.

In all three cases, the divisor $B \subset S_\lambda$ is smooth. Indeed in the GM1, GM2 cases, B is the (smooth) discriminant divisor Δ , and in the GM0 case it is the closure of Noether-Lefschetz divisor, a hypersurface subtorus, in the divisorial toroidal extension S_λ .

The relative flop $\mathcal{X} \dashrightarrow \mathcal{X}'$ along $\mathcal{E}$ exists. Indeed, let $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$ be the blowup along $\mathcal{E}$. The exceptional divisor $\tilde{\mathcal{E}}$ is a $(\mathbb{P}^1 \times \mathbb{P}^1)$ -fibration over B and $\tilde{\mathcal{E}} \cdot \ell = -1$ for the lines of either ruling. By [Nak71] there exists the contraction $\tilde{\mathcal{X}} \rightarrow \mathcal{X}'$ along the second ruling so that $\tilde{\mathcal{X}}$ is the blowup along $\mathcal{E}' \subset \mathcal{X}'$, a $\mathbb{P}^1$ -fibration over B .

Example 7.18. Fix a Kulikov surface $X_0 = \bigcup V_i$ in the λ -family $\mathcal{X} \rightarrow S_\lambda$ and consider a boundary divisor $\overline{D}_{ij} \subset \overline{V}_i$ of some ordered toric model which receives two internal blow-ups E_1 and E_2 . On the sublocus of Δ_λ where the two blow-up points coincide, the first (-1) -curve E_1 breaks into the union of the (-1) -curve E_2 and a (-2) -curve with class $E_1 - E_2$. But the second exceptional curve E_2 never breaks, and thus satisfies the conditions of Definition 7.17(GM1).

Theorem 7.19. Any two λ -families $\mathcal{X} \rightarrow S_\lambda$ and $\mathcal{X}' \rightarrow S_\lambda$ with the same (k, t) are related by a series of GM0, GM1, and GM2 modifications.

Proof. Choose an arc $(C, 0)$ intersecting $\Delta_\lambda \subset S_\lambda$ transversely, mapping C^* generically into a locus of K3 surfaces with Picard group $\mathbb{Z}L$, $L^2 = 2d$. Consider the two Kulikov models $X \rightarrow (C, 0)$ and $X' \rightarrow (C, 0)$. Then the punctured families $X^*, (X')^* \rightarrow (C, 0)$ are isomorphic as families of (partially marked) K3 surfaces, and admit polarizations. So by [SB83, Cor. 3.1] there exists a sequence of M1 and M2 modifications $X \dashrightarrow X'$ connecting them. Requiring $\ker \psi_{X_0} = \mathbb{Z}L$, no modification of X supports a (-2) -curve, eliminating the need for M0 modifications.

We now seek to globalize these modifications to a sequence of GM0, GM1, GM2 modifications. There is no obstruction to globalizing an M2 modification to GM2, since the relative double curve $\mathcal{D}_{ij}$ never breaks. For GM0 and GM1 modifications, it suffices to work component-wise.

Lemma 7.20. Let $(\mathcal{V}, \mathcal{D}) \rightarrow (\mathbb{C}^*)^Q$ and $(\mathcal{V}', \mathcal{D}') \rightarrow (\mathbb{C}^*)^Q$ be families of anticanonical pairs (see 7.5) associated to two ordered toric models of a given pair (V, D) . Then, there is an isomorphism in $S_Q^\pm$ of the bases and a sequence of GM0 modifications connecting $\mathcal{V}$ and $\mathcal{V}'$.

Here $S_Q^\pm$ is the signed symmetric group, acting on $(\mathbb{C}^*)^Q$ by permuting and inverting coordinates.

Proof. First, note that M0 and GM0 modifications also make sense for anticanonical pairs, by flopping (-2) -curves in the complement $V \setminus D$. Since we will only be making birational modifications in the complement of the anticanonical cycle, and $V \setminus D = \tilde{V} \setminus \tilde{D}$, we may as well assume that $(V, D) = (\tilde{V}, \tilde{D})$ i.e. there are no corner blow-ups in the toric model.

Fix (V, D) very general, in the sense that it has no (-2) -curves disjoint from D . An ordered toric model is given by an ordered collection $(E_1, \dots, E_Q)$ of Q disjoint internal exceptional curves. It follows from a theorem of Blanc [Bla13, Thm. 1] describing the birational automorphism group of $((\mathbb{C}^*)^2, \frac{dx}{x} \wedge \frac{dy}{y})$ —see [HK20, Prop. 3.27] for the interpretation we employ—that any two such tuples are related by a series of two moves:

Definition 7.21. An *elementary mutation* replaces the first exceptional curve $(E_1, E_2, \dots, E_Q) \mapsto (E'_1, E_2, \dots, E_Q)$ where $E_1 + E'_1$ is the pullback of a fiber of a toric ruling on $(\overline{V}, \overline{D})$.

Definition 7.22. An *order switch* sends $(E_1, \dots, E_i, E_{i+1}, \dots, E_Q) \mapsto (E_1, \dots, E_{i+1}, E_i, \dots, E_Q)$.

An elementary mutation of the ordered toric model gives rise to an isomorphism $(\mathcal{V}, \mathcal{D}) \rightarrow (\mathcal{V}', \mathcal{D}')$ of the corresponding families: The construction of the family by successive blow-ups

$$(\mathcal{V}, \mathcal{D}) \xrightarrow{\tau_Q} \dots \xrightarrow{\tau_2} (\mathcal{V}_1, \mathcal{D}_1) \xrightarrow{\tau_1} (\mathcal{V}_0, \mathcal{D}_0) = (\overline{V}, \overline{D}) \times (\mathbb{C}^*)^Q$$

is unaltered when the blow-up τ_1 is replaced with the blow-up τ'_1 . The bases $B \cong (\mathbb{C}^*)^Q \cong B'$ of the two families of varying blow-ups $g = \tau_Q \circ \dots \circ \tau_2 \circ \tau_1$ and $g' = \tau_Q \circ \dots \circ \tau_2 \circ \tau'_1$ can thus be canonically identified. But with respect to this canonical identification, the point blown up by τ'_1 lives in the inverse $\mathbb{C}^*$ -torsor to the point blown up by τ_1 . We require the coordinates on $B' \cong (\mathbb{C}^*)^Q$ to be compatible with the orientation, so we must invert the first coordinate.

An order switch gives an isomorphism of the families whenever E_i and E_{i+1} meet distinct components— τ_i and τ_{i+1} commute. But when E_i and E_{i+1} meet the same component, the families are only canonically isomorphic over the locus where E_i and E_{i+1} meet distinct points. Then $(\mathcal{V}_{i+1}, \mathcal{D}_{i+1})$ and the family $(\mathcal{V}'_{i+1}, \mathcal{D}'_{i+1})$ constructed with the reverse ordering are related by a flop along the relative (-2) -curve $\mathcal{E}_{i+1} - \mathcal{E}_i \subset \mathcal{V}_{i+1}|_L$ fibering over the locus L where the blow-up points coincide. The remaining blow-ups τ_j for $j > i + 1$ do not interfere with the flop because $\mathcal{E}_{i+1} - \mathcal{E}_i$ is disjoint from the boundary. The order switch permutes two $\mathbb{C}^*$ coordinates of B .

Thus, we can connect any two families $(\mathcal{V}, \mathcal{D})$ and $(\mathcal{V}', \mathcal{D}')$ by a series of isomorphisms and GM0 modifications. The sequence of elementary mutations and order switches connecting $(E_1, \dots, E_Q)$ to $(E'_1, \dots, E'_Q)$ induces an isomorphism $B \rightarrow B'$ valued in $S_Q^\pm$. $\square$

By Lemma 7.20, two families $(\mathcal{V}_i, \mathcal{D}_i)$ of anticanonical pairs associated to ordered toric models of a given component $(V_i, D_i) \subset X_0$ are connected by GM0 modifications. To globalize an M1 modification along $E \subset V_i$ we apply Lemma 7.20 to find a sequence of isomorphisms and GM0 modifications until $E = E_Q$ is the *last* exceptional curve in the ordered toric model. Then $\mathcal{E}_Q$ never breaks in the gluing family $\mathcal{X}_0$ as it is the last blow-up performed on any given fiber. So $\mathcal{E}_Q$ can be flopped in $\mathcal{X}$. The GM0 modifications on the discriminant family $\mathcal{X}_0$ extend to the smoothing $\mathcal{X}$ because the relative (-2) -curve $\mathcal{E}_{i+1} - \mathcal{E}_i$ deforms over the Noether-Lefschetz divisor $B \subset S_\lambda$.

This proves that there exists a series of GM0, GM1, GM2 modifications of $\mathcal{X} \rightarrow S_\lambda$ to a new λ -family $\mathcal{X}'' \rightarrow S_\lambda$ for which the sequence of modifications restricts to the given sequence of birational modifications $X \dashrightarrow X'$. Again applying Lemma 7.20, perform a sequence of GM0 modifications until the ordered toric models defining $\mathcal{X}''$ and $\mathcal{X}'$ are the same. The theorem follows. $\square$

7D. Type II λ -families. We construct topologically trivial families of Type II Kulikov surfaces for which the period map is an isomorphism. It is simplest to construct a family for a single combinatorial type with $(k, t) = (k, 0)$, then just apply GM0 and GM1 modifications to it.

Proposition 7.23. *For each k , there exists a family of Type II Kulikov models $\mathcal{X}_0 \rightarrow \text{Hom}(\Lambda, \tilde{\mathcal{E}})$ for which the period map is the identity.*

Proof. It suffices restrict to the $k = 1$ as we may otherwise insert $k - 1$ intermediate components which are $\mathbb{P}^1$ -bundles over elliptic curves.

Let $\overline{D} \subset \mathbb{P}^2$ be an arbitrary smooth cubic. Take 18 points $p_1, \dots, p_9, q_1, \dots, q_9 \in \overline{D}$ satisfying the single condition $\mathcal{O}_{\overline{D}}(6) \cong \mathcal{O}_{\overline{D}}(\sum p_i + \sum q_i)$. Let D_1 denote the strict transform of $\overline{D}$ in $V_1 := \text{Bl}_{p_1, \dots, p_9} \mathbb{P}^2$ and let D_2 denote the strict transform of $\overline{D}$ in $V_2 := \text{Bl}_{q_1, \dots, q_9} \mathbb{P}^2$. Then $X_0 := (V_1, D_1) \cup (V_2, D_2)$ is a d -semistable Type II Kulikov surface, even when D_1 and D_2 are glued via an arbitrary translation. This construction produces a $1 + (18 - 1) + 1 = 19$ -dimensional space of Kulikov surfaces. Respectively, the parameters are the j -invariant of $\overline{D}$, the 18 points p_i, q_i subject to the single condition, and the translation to glue by.

There is a projective linear automorphism acting by translation on $\overline{D}$ and sending one 9-tuple to another $(p_1, \dots, p_9) \mapsto (p'_1, \dots, p'_9)$ if and only if $p'_i - p_i$ are all equal to a fixed element of $\text{Pic}^0(\overline{D})[3] \cong \mathbb{Z}_3^2$. Thus the family of Kulikov surfaces $\hat{\mathcal{X}}_0 \rightarrow \hat{S}$ gotten by varying the data of $\overline{D}$, p_i ,

q_i , and the gluing descends to a quotient $\mathcal{X}_0 \rightarrow S = \widehat{S}/(\mathbb{Z}_3^2 \times \mathbb{Z}_3^2)$. A straightforward computation of the period map on $\widehat{S}$ using Construction 4.3 shows that in fact, the fibers of the period mapping $\widehat{S} \rightarrow \text{Hom}(\Lambda, \widetilde{\mathcal{E}})$ are exactly the orbits of the $\mathbb{Z}_3^2 \times \mathbb{Z}_3^2$ -action. So $\mathcal{X}_0 \rightarrow S$ is the desired family. $\square$

Corollary 7.24. *In Type II, there is a family $\mathcal{X} \rightarrow S_\lambda$ of mixed marked surfaces for which the period map is an order k branched cover of a tubular neighborhood of the boundary divisor of $\mathbb{D}(I)^\lambda$.*

Proof. This follows from Proposition 7.3 and Proposition 7.23. $\square$

Definition 7.25. A Type II λ -family is a family of surfaces which arises from a series of GM0, GM1 modifications of the family $\mathcal{X} \rightarrow S_\lambda$ in Corollary 7.24.

Using techniques of Theorem 7.19, replacing toric models with minimal models, we can construct a Type II λ -family for any fixed combinatorial type of surface X_0 via a series of GM0, GM1 modifications of the one in Corollary 7.24.

7E. Quasipolarized λ -families.

Definition 7.26. An M -quasipolarized λ -family is the restriction of a λ -family $\mathcal{X} \rightarrow S_\lambda$ to the Noether-Lefschetz locus $\mathbb{D}_M(I)^\lambda \cap S_\lambda \subset \mathbb{D}(I)^\lambda$, such that the embedding $M \rightarrow \text{Pic}(\mathcal{X}_t)$ induced by the marking defines an M -quasipolarization on a generic fiber $\mathcal{X}_t$.

Notation 7.27. When the context is clear, we reuse symbols S_λ , Δ_λ and $\mathcal{X}$, $\mathcal{X}_0$ for the intersections $\mathbb{D}_M(I)^\lambda \cap S_\lambda$, $\mathbb{D}_M(I)^\lambda \cap \Delta_\lambda$ and the restrictions of the unpolarized λ -families $\mathcal{X}$, $\mathcal{X}_0$ to them.

The elements $L \in M$ extend to line bundles $\mathcal{L} \rightarrow \mathcal{X}$ which are unique up to twisting by the relative components $\mathcal{O}_{\mathcal{X}}(\mathcal{V}_i)$ and line bundles pulled back from the base S_λ .

Theorem 7.28. *Any two M -quasipolarized λ -families are related by a series of GM0, GM1, GM2 modifications.*

Proof. By Theorem 7.19, the two unpolarized λ -families from which they are restricted (see Def. 7.26) are related by GM0, GM1, GM2 modifications. These modifications specialize to birational modifications of the restricted family in all cases, except for a GM0 modification associated to a (-2) -curve $\beta \in M$. But in this case, the two restricted families are isomorphic before and after the modification so we simply replace the GM0 modification with this isomorphism. $\square$

We now define analogues of nef and divisor models.

Definition 7.29. A *nef* λ -family is an M -quasipolarized λ -family $\mathcal{X} \rightarrow S_\lambda$ together with an extension of $L \in M$ to a relatively big and nef line bundle $\mathcal{L} \rightarrow \mathcal{X}$.

Definition 7.30. A *divisor* λ -family $(\mathcal{X}, \mathcal{R}) \rightarrow S_\lambda$ is an M -quasipolarized nef λ -family and a relatively big and nef divisor $\mathcal{R} \in |\mathcal{L}|$ which contains no stratum of any fiber.

Proposition 7.31. *Given a nef model $L \rightarrow X$ of a Type III M -quasipolarized Kulikov model $X \rightarrow (C, 0)$, there is an ordered toric model of X_0 for which L defines a nef λ -family $\mathcal{L} \rightarrow \mathcal{X} \rightarrow S_\lambda$.*

Proof. Write $L|_{X_0} = (L_i)$ with each $L_i \in \text{Pic}(V_i)$. Note that L_i is nef for all i and at least one L_i is big. It follows from [EF21, Prop. 1.5] that there exists a toric model of V_i for which

$$f_i^* L_i = \sum a_{ij} \widetilde{D}_{ij} + \sum b_{ijk} E_{ijk}.$$

with $a_{ij}, b_{ijk} \geq 0$. We order this toric model so that $b_{ijk_1} > b_{ijk_2}$ implies that E_{ijk_2} is blown up after E_{ijk_1} . Then L_i defines a relatively nef line bundle on the family $(\mathcal{V}_i, \mathcal{D}_i)$ because the only irreducible curves which L_i could possibly intersect negatively are (-2) -curves of the form $\beta = (f_i)_*(E_{ijk_2} - E_{ijk_1})$ but

$$L_i \cdot \beta = f_i^* L_i \cdot (E_{ijk_2} - E_{ijk_1}) = b_{ijk_1} - b_{ijk_2} > 0.$$

Definition 7.32. An element $(\alpha_i) \in \widetilde{\Lambda}$ is *numerically nef* if α_i is the class of a nef line bundle on each component V_i .

We have that L_i defines a numerically nef class on every fiber of the unpolarized gluing family over $\text{Hom}(\Lambda, \mathbb{C}^*$ or $\tilde{\mathcal{E}})$. On the sublocus of the discriminant Δ_λ where (L_i) actually defines a Cartier divisor, in particular over the locus where $\psi_{X_0}(M) = 1$, we get a relatively big and nef line bundle $\mathcal{L}_0 \rightarrow X_0$. On the smoothing $\mathcal{X} \rightarrow S_\lambda$, the line bundle $\mathcal{L}_0$ extends to a relatively big and nef line bundle $\mathcal{L}$, because big and nefness is an open condition. $\square$

Proposition 7.33. *Given a nef model $L \rightarrow X$ of a Type II M -quasipolarized Kulikov model $X \rightarrow (C, 0)$, there is a nef λ -family $\mathcal{L} \rightarrow \mathcal{X} \rightarrow S_\lambda$ extending it.*

Sketch. The proof is roughly the same as Proposition 7.31, the key point being to order the exceptional curves one must successively blow down to get a minimal model of each component $V_i \subset X_0$. This ordering comes from the intersection numbers of L_i with each exceptional curve. $\square$

Definition 7.34. A *stable λ -family* $(\bar{\mathcal{X}}, \epsilon\bar{\mathcal{R}}) \rightarrow S_\lambda$ is defined as $\text{Proj}_{S_\lambda} \oplus_{n \geq 0} \pi_* \mathcal{O}(n\mathcal{R})$ for a divisor λ -family.

Cohomology and Base Change theorem [Har77, III.12.11] implies that the fibers of a stable λ -family are stable pairs $(\bar{X}, \epsilon\bar{R})$.

Remark 7.35. Olsson defined a moduli space closely related to λ -families in [Ols04]. The functor is defined by families of Kulikov surfaces together with a line bundle $\mathcal{L}$ extending a polarization, such that $\mathcal{L}^n$ for some $n > 0$ gives a morphism fiberwise contracting only finitely many curves. (Olsson uses the language of stacks and log schemes, so this description is approximate, see [Ols04] for complete details.) Our λ -families are different in a number of ways: our primary focus is a divisor $\mathcal{R}$, and the corresponding nef line bundle $\mathcal{L}^n = \mathcal{O}(n\mathcal{R})$ usually contracts irreducible components of the fibers.

8. RECOGNIZABLE DIVISORS

When a canonical choice of polarizing divisor (6.1) is recognizable (6.2), Proposition 6.3 allows us to extend $\mathcal{R}^*$ to the whole quasipolarized moduli space $\mathcal{F}_M^q$. We now generalize this to λ -families $\mathcal{X} \rightarrow S_\lambda$. Recall that $\text{Aut}^0(X_0)$ is non-trivial only when $t = 0$, $k > 1$, i.e. X_0 is of Type II with intermediate elliptic ruled components. This case for λ has a number of subtleties not present in the general case, and we delay its treatment to Proposition 8.10.

Proposition 8.1. *Let $\mathcal{X} \rightarrow S_\lambda$ be an M -quasipolarized λ -family. If R is recognizable, then the Zariski closure of $\mathcal{R}^*$ is a flat family of curves in $\mathcal{X}$. Conversely, if the canonical choice of divisor R extends to a flat family of divisors $\mathcal{R}^*$ on $\mathcal{F}_M^q$, then the existence of a further flat extension of $\mathcal{R}^*$ over any λ -family $\mathcal{X}$ implies that R is recognizable.*

Proof. Note that $\mathcal{R}^*$ extends to a flat family of curves in $\mathcal{X}$ if and only if the Zariski closure $\mathcal{R} := \overline{\mathcal{R}^*} \subset \mathcal{X}$ defines a relative curve, even over the discriminant Δ_λ . Equivalently, $\mathcal{R}$ contains no component of any singular fiber X_0 . By recognizability, there is a “candidate curve” $R_0 \subset X_0$ which enjoys the following property: if we take any curve $(C, 0)$ transverse to Δ_λ at 0, then the Zariski closure of $\mathcal{R}^*|_{C^*} \subset \mathcal{X}|_C$ intersects X_0 at R_0 . We say that R_0 is the *flat limit of $\mathcal{R}^*$ along the arc C* . This follows from recognizability because $\mathcal{X}|_C$ is Kulikov.

More generally, suppose that $(C, 0)$ is an arc passing through 0 which has intersection multiplicity k with Δ_λ . This arc defines a degenerating family $X \rightarrow (C^\nu, 0)$ with monodromy invariant $k\lambda$. Letting t be a local parameter at $0 \in C^\nu$, the local analytic equation of the smoothing is of the form $xy = t^k$ and $xyz = t^k$ near the double curves and triple points of X_0 .

Such a family admits a standard resolution (Sec. 3A) to a new Kulikov model $X[k] \rightarrow (C^\nu, 0)$ whose dual complex $\Gamma(X_0[k])$ is gotten by subdividing the triangles and segments of $\Gamma(X_0)$ into k^2 triangles and k segments. Then $X[k]$ defines a map $(C^\nu, 0) \rightarrow S_{k\lambda}$ which is transverse to $\Delta_{k\lambda}$. Here the Kulikov surfaces over the discriminant have the same combinatorial type as $X_0[k]$. The boundary divisors $\Delta_{k\lambda} = \Delta_\lambda$ are naturally isomorphic and the arcs $(C^\nu, 0)$ in both S_λ and $S_{k\lambda}$ limit to the same point under this isomorphism.

Then $X_0[k]$ contains a distinguished curve $R_0[k]$ which is the flat limit of the canonically chosen divisors over any arc transverse to $\Delta_{k\lambda}$. So the image $\overline{R}_0[k]$ under the morphism $X_0[k] \rightarrow X_0$ is equal to the flat limit of the restriction of $\mathcal{R}^*$ to any arc with tangency k to Δ_λ . So the flat limit of $\mathcal{R}^*$ over any arc $(C, 0)$ not fully contained in Δ_λ lies in the countable union of curves

$$\bigcup_{k \geq 1} \overline{R}_0[k] \subset X_0.$$

Supposing for the sake of contradiction $\mathcal{R} \cap X_0$ contained a component V_i , there would be some point $p \in \mathcal{R} \cap V_i$ avoiding the above countable union. Choose some irreducible curve contained in $\mathcal{R}$ passing through p whose projection is not contained in Δ_λ . Taking the image in S_λ gives an arc C passing through 0, possibly singular, which intersects Δ_λ with some finite multiplicity k for which the restriction $\mathcal{R}^*|_{C^*}$ contains p in its Zariski closure. Contradiction.

To prove the converse is easy: Every M -quasipolarized smoothing of X_0 corresponds to a transverse arc $(C, 0)$ in the base of the λ -family $\mathcal{X} \rightarrow S_\lambda$ and so the flat extension $\mathcal{R} \cap X_0$ defines a curve R_0 satisfying the recognizability property. $\square$

Intuitively, recognizability implies that the limits of canonically chosen curves over arcs $(C, 0)$ approaching the discriminant with tangency k are rigid, for all k . On the other hand, if the closure of $\mathcal{R}^*$ contained a surface in X_0 , there would have to be some finite tangency order k for which these limit curves moved.

Remark 8.2. Proposition 8.1 implies that any of the images $\overline{R}_0[k]$ must in fact equal R_0 . In particular, the divisor $R_0 \subset X_0$ is compatible with base change plus standard resolution.

Definition 8.3. We say that R is (resp. *weakly*) λ -recognizable if $\mathcal{R}^*$ extends to a flat family of curves in $\mathcal{X} \rightarrow S_\lambda$ for any (resp. some) ordered toric model of any (resp. some) Kulikov model with monodromy invariant λ .

Remark 8.4. The existence of an extension of $\mathcal{R}^*$ to $\mathcal{F}_M^q$ can be considered as λ -recognizability in the $\lambda = 0$ case. Then Proposition 8.1 states that R is recognizable if and only if it is λ -recognizable for all possible λ , including $\lambda = 0$.

We now show equivalence with weak recognizability:

Proposition 8.5. *R is λ -recognizable if and only if it is weakly λ -recognizable.*

Proof. λ -recognizability clearly implies weak λ -recognizability. To show the converse, apply Theorem 7.28: There exists a sequence of GM0, GM1, GM2 modifications connecting any two λ -families. The condition that the closure of $\mathcal{R}^*$ in a λ -family $\mathcal{X} \rightarrow S_\lambda$ contain no fiber component is a property invariant under all three types of modifications, because the center of any such modification contains no fiber component. Hence weak λ -recognizability implies λ -recognizability. $\square$

Proposition 8.5 shows that recognizability can be certified by finding *some* λ -family $\mathcal{X}$ for which $\mathcal{R}^*$ extends, for all λ . The following is a key statement:

Proposition 8.6. *Suppose R is recognizable, and let $X \rightarrow (C, 0)$ be a Kulikov model for which R_0 contains no strata of X_0 . Then all fibers of the flat extension $(\mathcal{X}, \mathcal{R}) \rightarrow S_\lambda$ (Prop. 8.1) enjoy the same property: $\mathcal{R} \cap X_p$ contains no strata of X_p .*

Proof. We show the Type III case; Type II works the same but easier. Assume the opposite: for some $p \in \Delta_\lambda$ the divisor $\mathcal{R} \cap X_p$ contains a triple point. Following the argument in [AET19, Claim 3.13], there is an order k base change and (possibly non-standard) simultaneous toric resolution producing a $k\lambda$ -family $\mathcal{X}' \rightarrow S_{k\lambda}$ for which the closure of $\mathcal{R}'|_{C^*}$ in $\mathcal{X}'|_{(C,p)}$ contains no strata. Here (C, p) is an arc intersecting $\Delta_{k\lambda}$ transversely at p and $\mathcal{R}' \subset \mathcal{X}'$ extends (Prop. 8.1) the canonical choice of polarizing divisor.

The discriminant family $\mathcal{X}'_0 \rightarrow \Delta_{k\lambda}$ is topologically trivial. The divisor $\mathcal{R}'$ intersects some irreducible component $V'_p \subset X'_p$ lying over the triple points of X_p but it is disjoint from the corresponding component $V'_0 \subset X'_0$. The divisor $\mathcal{R}'$ is a section of a line bundle $\mathcal{L}' = \mathcal{O}_{\mathcal{X}'}(\mathcal{R}')$. It restricts to line bundles L'_p resp. L'_0 on V'_p resp. V'_0 , with $R'_p \in |L'_p|$ and $R'_0 \in |L'_0|$.

But since L'_0 restricts to the trivial bundle on V'_0 , the topological triviality implies that L'_p is the trivial bundle on V'_p . So R'_p contains V'_p if it intersects it. Contradiction. An alternative contradiction avoiding reference to the line bundles is that $\mathcal{R}'_0 \subset \mathcal{X}'_0$ is a flat family of curves intersecting V'_p but not intersecting the corresponding component for a generic nearby fiber. This would only be possible if R'_p contained a triple point, which it does not. $\square$

Next, we study when we have the freedom to multiply λ by an integer:

Proposition 8.7. *Suppose R is $m\lambda$ -recognizable and the fibers of the flat extension $\mathcal{R} \subset \mathcal{X} \rightarrow S_{m\lambda}$ contain no strata of any fiber. Then R is $n\lambda$ -recognizable for all n . Conversely, if R is $n\lambda$ -recognizable for all $n \in \mathbb{N}$, then there is an $m \in \mathbb{N}$ for which the flat extension $\mathcal{R} \subset \mathcal{X}$ contains no strata of fibers.*

Proof. First we prove the forward direction, i.e. we have a flat family of curves $\mathcal{R} \subset \mathcal{X} \rightarrow S_{m\lambda}$ not containing strata of any fiber. Let $n = mk$, and consider the standard resolution $\mathcal{X}[k]$ of the global base change. On any fiber, the map $u : X_0[k] \rightarrow X_0$ satisfies the property that the inverse image $R_0[k] := u^{-1}(R_0)$ is still a divisor. This divisor certifies recognizability for $X_0[k]$. This would be false if R_0 contained a singular stratum of X_0 , as then $u^{-1}(R_0)$ would contain a component.

Hence R is weakly $n\lambda$ -recognizable for all $m \mid n$. By Proposition 8.5, we conclude that R is $n\lambda$ -recognizable whenever $m \mid n$. So consider the case $m \nmid n$. Supposing R were not $n\lambda$ -recognizable, the limiting divisor R_0 would vary depending on the chosen arc $(C, 0) \rightarrow S_{n\lambda}$. But taking a standard resolution and base change of order r , we would conclude that R is not $rn\lambda$ -recognizable for an $r \in \mathbb{N}$ as the base-changed arcs would also produce different limiting divisors. Taking $r = m$ gives a contradiction.

The reverse direction follows from the existence of divisor models: There exists some Kulikov model $X \rightarrow (C, 0)$ with monodromy $m\lambda$ for which the limit R_0 contains no strata of X_0 . Taking a λ -family, Proposition 8.6 shows we get a flat extension $\mathcal{R} \subset \mathcal{X}$ containing no strata of fibers. $\square$

Proposition 8.8. *R is $n\lambda$ -recognizable for all $n \in \mathbb{N}$ if and only if there exists a divisor $m\lambda$ -family $(\mathcal{X}, \mathcal{R}) \rightarrow S_{m\lambda}$ for some $m \in \mathbb{N}$.*

Proof. The existence of a divisor $m\lambda$ -family $(\mathcal{X}, \mathcal{R})$ implies that R is (weakly) $m\lambda$ -recognizable with $\mathcal{R}$ containing no strata of fibers, so Proposition 8.7 implies that R is $n\lambda$ -recognizable for all $n \in \mathbb{N}$. Conversely, choose a divisor model $(X, R) \rightarrow C$ with monodromy invariant $m\lambda$. Then Proposition 7.31 (or Proposition 7.33 for Type II) implies that we may choose an ordered toric model of X_0 for which the line bundle $\mathcal{O}_{X_0}(R_0)$ extends to a relatively big and nef line bundle $\mathcal{L} \rightarrow \mathcal{X}$ on the corresponding λ -family. By recognizability and Proposition 8.6, the closure $\mathcal{R} = \mathcal{R}^*$ is a section of $\mathcal{L}$ which doesn't contain strata. We conclude that $(\mathcal{X}, \mathcal{R})$ is a divisor $m\lambda$ -family. $\square$

We also show equivalence with a weaker condition:

Proposition 8.9. *Let $\mathcal{X} \rightarrow (C, 0) \times B$ be a family of Kulikov models over a curve B for which the discriminant family $\mathcal{X}_0 = X_0 \times B$ is constant, and the restriction of $\mathcal{X}$ to $(C, 0) \times \{b_0\}$ gives a divisor model. Then R is recognizable if and only if $R_{0,b} := \lim_{t \rightarrow 0} R_{t,b}$ is independent of b , i.e. $R_{0,b} = R_{0,b_0} \subset X_0$ for any such $\mathcal{X} \rightarrow (C, 0) \times B$.*

Proof. Certainly if R is recognizable, then $R_{0,b}$ will equal the divisor $R_0 \subset X_0$ certifying recognizability for any b . Conversely, suppose R is not recognizable. Following the proof of Proposition 8.1, there must be a one-parameter family of Kulikov models $\mathcal{X} \rightarrow (C, 0) \times B$ for which $R_{0,b}$ varies. It remains to show that we may assume these Kulikov models are divisor models. To do so, we perform a series of GM0, GM1, GM2 modifications (possibly after a global base change and standard resolution) until the restriction of the modified family $\mathcal{X}'$ to a fixed arc $(C', 0) \times \{b_0\}$ is a divisor model. These modifications do not affect the triviality of the discriminant family $\mathcal{X}'_0 = X'_0 \times B$ and the limit curves $R'_{0,b}$ still vary on X'_0 because they cover some component. $\square$

Proposition 8.10. *Suppose that $t = 0$ and $k > 1$. That is, X_0 is a Type II Kulikov surface with intermediate elliptic ruled components. Then, there exist λ -families $\mathcal{X} \rightarrow S_\lambda$ for which Propositions 8.1, 8.5, 8.6, 8.7, 8.8, 8.9 hold.*

Proof. Recall that the smoothing component of such a Kulikov surface X_0 has dimension $19+k$ and is fibered over $\mathbb{C}^k$, with the k th coordinate axis corresponding to the deformations which smooth the k th double curve. Imposing an M -quasipolarization reduces the dimension to $19+k-\text{rk } M$. Given any smooth arc $(C, 0) \hookrightarrow (\mathbb{C}^{19+k-\text{rk } M}, 0) =: (S, 0)$ whose tangent direction $T_0 C$ is transverse to all the coordinate axes under the projection to $(\mathbb{C}^k, 0)$, the restriction of the universal family to $(C, 0)$ is a Kulikov model, simultaneously smoothing all of the double curves.

The closure $\mathcal{R} = \overline{\mathcal{R}^*}$ over the full $(19+k-\text{rk } M)$ -dimensional smoothing component of such an X_0 could contain an entire intermediate elliptic ruled component. In fact, this does occur: Applying $g \in \text{Aut}^0(X_0) \cong (\mathbb{C}^*)^{k-1}$ to the arc $(C, 0)$ in the deformation space will translate the flat limit $R_0 \subset X_0$ by g . But a recognizable divisor R_0 need not be $\text{Aut}^0(X_0)$ -invariant, see the $\tilde{A}_{17}$ case in [AET19, Construction 9.27].

Fixing one arc $(C, 0) \hookrightarrow (S, 0)$ gives a flat limit $R_0 \subset X_0$ and assuming R is recognizable, the flat limit R'_0 along any other arc $(C', 0) \hookrightarrow (S, 0)$ differs from R_0 by an element $g \in \text{Aut}^0(X_0)$, i.e. $g(R'_0) = R_0$. But then, the flat limit along $g^*(C', 0)$ equals R_0 . So for any arc transverse to the coordinate axes of $(\mathbb{C}^k, 0)$, there is a representative of its $\text{Aut}^0(X_0)$ -orbit for which the flat limit is *equal* to R_0 . Thus, there exists a slice of the $\text{Aut}^0(X_0)$ -action on $(S, 0)$ for which the flat limit along the slice is always R_0 .

This procedure can be performed analytically-locally along the fibers over the equisingular locus $\Delta \subset S$. We call such a slice *well-chosen*. Summarizing, a well-chosen slice gives a local λ -family over an open set $U \subset S_\lambda$ around $0 \in \Delta_\lambda$ for which $\mathcal{R}^*$ extends to a flat family of divisors $\mathcal{R}$.

Now consider a collection $\{U_i\}$ of well-chosen slices for which $U_i \cap \Delta$ cover the equisingular deformation space Δ_λ . On the double overlaps $U_i \cap U_j$ these well-chosen slices are isomorphic, by a unique isomorphism preserving the mixed marking, because the isomorphisms on the smooth smooth fibers are unique (Prop. 4.12). Thus, when R is recognizable, we can glue to form a λ -family $(\mathcal{X}, \mathcal{R}) \rightarrow S_\lambda$ on which $\mathcal{R}$ extends to a flat family of divisors. The arguments of the above propositions apply verbatim to such a well-chosen slice. $\square$

We summarize the results proven above:

Theorem 8.11. *Let R be a canonical choice of polarizing divisor, defining a divisor $\mathcal{R}^*$ on the universal K3 surface over a Zariski open subset $U \subset \mathcal{F}_M^q$. Then the following are equivalent:*

- (1) *Any one-parameter deformation of a divisor model $(X, R) \rightarrow (C, 0)$ keeping X_0 constant in moduli gives rise to a constant limiting curve R_0 , up to $\text{Aut}^0(X_0)$.*
- (2) *R is recognizable.*
- (3) *For all primitive isotropic δ and all $\lambda \in C_\delta^+ \cap \delta^\perp / \delta$, there is some λ -family for which $\mathcal{R}^*$ extends a flat divisor $\mathcal{R} \subset \mathcal{X}$.*
- (4) *$\mathcal{R}^*$ extends to a flat divisor $\mathcal{R} \subset \mathcal{X}$ in every λ -family.*
- (5) *For every projective class $[\lambda]$, there exists some $k \in \mathbb{N}$ for which $\mathcal{R}^*$ extends to a divisor λ -family $(\mathcal{X}, \mathcal{R}) \rightarrow S_{k\lambda}$.*

If $t = 0$, $k > 1$, the above equivalences hold when the λ -family is a well-chosen slice.

Proof. Note that we are allowing the case $\lambda = 0$, which in conditions (3), (4), (5) amounts to saying that $\mathcal{R}^*$ extends to a section of the projective bundle $\mathbb{P}_{\mathcal{L}} \rightarrow \mathcal{F}_M^q$. Then (2) $\iff$ (4) by Proposition 8.1, (3) $\iff$ (4) by Proposition 8.5, and (4) $\iff$ (5) by Proposition 8.8. Finally, (1) $\iff$ (2) by Proposition 8.9. $\square$

The conditions in Theorem 8.11 are roughly in increasing order of strength. As such, we use condition (5) in the proof of Theorem 1, but use condition (1) in the proof of Theorem 2.

Definition 8.12. Let $(\overline{X}, \epsilon \overline{R}) = \bigcup_i (\overline{V}_i, \overline{D}_i, \epsilon \overline{R}_i)$ be a stable degeneration of K3 pairs. The *slc combinatorial type* is the data of:

- (1) The deformation types of the quasipolarized minimal resolutions (V_i, D_i, L_i) of each component, where $L_i = \mathcal{O}_{V_i}(R_i)$, and
- (2) the combinatorics $\Gamma(\overline{X})$ of the singular strata.

Corollary 8.13. *Suppose R is recognizable and let $(\overline{X}^*, \epsilon \overline{R}^*) \rightarrow C^*$, $\epsilon \ll 1$ be a family of stable K3 pairs over a punctured curve $C^* = C \setminus 0$. The slc combinatorial type of the unique stable limit $(\overline{X}_0, \epsilon \overline{R}_0)$ depends only on the projective class $[\lambda]$ of the monodromy invariant.*

Proof. Consider the divisor λ -family as in Theorem 8.11(5). The family of canonical models $(\overline{\mathcal{X}}, \epsilon \overline{\mathcal{R}})$, where $\overline{\mathcal{X}} = \text{Proj } \oplus_{n \geq 0} \pi_* \mathcal{O}_{\mathcal{X}}(n\mathcal{R})$, and $\overline{\mathcal{R}} = \text{im } \mathcal{R}$, is the corresponding family of stable slc pairs. Every one-parameter degeneration with monodromy invariant λ has a unique limit in this family. The combinatorial type of the discriminant family $(\mathcal{X}_0, \mathcal{R}_0)$ is fixed, with the line bundles $L_0 = \mathcal{O}_{X_0}(R_0)$ on every fiber identified by the Gauss-Manin connection because $\mathcal{X}_0$ is topologically trivial. Since the contraction $X_0 \rightarrow \overline{X}_0$ is defined only by the line bundle L_0 , the combinatorial type of the stable models is also fixed. $\square$

9. MAIN THEOREM FOR RECOGNIZABLE DIVISORS

9A. Proof of Theorem 1. We have proven in Corollary 8.13 that whenever R is recognizable, the slc combinatorial type of an M -polarized degeneration depends only on the projective class $[\lambda]$ of the monodromy invariant. This is the key input which recognizability gives us: from here we have an essentially birational-geometric argument to show that the KSBA compactifications associated to recognizable divisors are (up to normalization) semitoroidal.

Theorem 9.1. *If R is recognizable, there exists a unique semifan $\mathfrak{F}_R$ for which $\overline{F}_M^{\mathfrak{F}_R} \rightarrow \overline{F}_M^R$ is the normalization.*

Proof. Recall that $\overline{F}_M^R$ is, by Definition 6.11, the coarse space of the closure (in $\mathcal{P}_{N,2d}$) of the stack of pairs parameterized by $U \subset \mathcal{F}_M$.

We define the *interior* of $\overline{F}_M^R$ to be the locus in this closure parameterizing M -polarized ADE K3 surface pairs $(\overline{X}, \epsilon \overline{R})$. Proposition 6.12 implies that this locus is isomorphic to F_M .

Let $\mathfrak{G}$ be some regular fan (cones are standard affine) and let $u: \overline{F}_M^{\mathfrak{G}} \dashrightarrow \overline{F}_M^R$ be the birational map which is isomorphism on the interiors. Let $\sigma = \text{span}\{\lambda_1, \dots, \lambda_d\}$ be a Type III standard affine cone of $\mathfrak{G}$ of maximal dimension. Associated to this cone is an analytic, finite morphism from a tubular neighborhood $N(\sigma)$ of the toric boundary of

$$X(\sigma) = \mathbb{C}^d = \mathbb{C}\lambda_1 \oplus \dots \oplus \mathbb{C}\lambda_d$$

to a neighborhood of the boundary strata of $\overline{F}_M^{\mathfrak{G}}$ containing the 0-dimensional stratum associated σ . The finiteness arises from quotienting by the $\text{Stab}_{\Gamma_\delta}(\sigma)$ action on this toric chart.

Let $u(\sigma): N(\sigma) \dashrightarrow \overline{F}_M^R$ denote the corresponding meromorphic map. Consider an arc germ $(C, 0) \subset (\mathbb{C}^d, 0)$ with $C^* \subset (\mathbb{C}^*)^d$ contained in the open torus orbit. Since $\overline{F}_M^R$ is proper, $u(\sigma)$ extends uniquely over C^* to the origin 0. By Corollary 8.13, the combinatorial type of the stable model depends only on the orders r_i of tangency of $(C, 0)$ with the coordinate hyperplanes of $\mathbb{C}^d$, since this determines the monodromy invariant of $(C, 0)$ to be $\lambda = r_1\lambda_1 + \dots + r_d\lambda_d$.

The meromorphic map $u(\sigma): N(\sigma) \dashrightarrow \overline{F}_M^R$ thus satisfies the following conditions:

- (1) There is a stratification (by slc combinatorial type) of $\overline{F}_M^R$ for which the extension of $u(\sigma)$ over any arc $(C, 0)$ with fixed tangency orders r_i to the coordinate hyperplanes of $\mathbb{C}^d$ lies in a fixed slc stratum.
- (2) The indeterminacy locus lies in the coordinate hyperplanes, which map by $u(\sigma)$ into the union of Type III slc strata.

No Type III slc stratum contains a complete curve by Corollary 9.17. We conclude by Lemma 9.18 that there exists a toric blow-up of $X(\sigma)$ eliminating the indeterminacy of $u(\sigma)$. Further refining, we may assume this toric blow-up is given by a $\text{Stab}_{\Gamma_\delta}(\sigma)$ -invariant fan. Thus, we may refine $\mathfrak{G}$ so that u defines a morphism over the refinement of σ . Applying this argument to all Γ -orbits of maximal cones $\sigma \in \mathfrak{G}$, we may as well have assumed that $u: \overline{F}_M^{\mathfrak{G}} \dashrightarrow \overline{F}_M^R$ has no indeterminacy over the Type III extension of F_M .

In fact, there is no indeterminacy in the Type II ($\lambda^2 = 0$) locus either: By Theorem 8.11, there is a divisor λ -family $(\mathcal{X}, \mathcal{R}) \rightarrow S_\lambda$. Consider the resulting stable λ -family $(\overline{\mathcal{X}}, \epsilon \overline{\mathcal{R}}) \rightarrow S_\lambda$. The base S_λ is an order k branched cover of a tubular neighborhood of the boundary divisor in the unipotent quotient $\mathbb{D}_M(I)$. There is a natural quotient map $v: S_\lambda \rightarrow \overline{F}_M^\mathfrak{G}$ by the action of Γ_I .

The classifying morphism $S_\lambda \rightarrow \overline{F}_M^R$ for the stable λ -family must factor through v because the fibers of v not lying in the boundary give isomorphic ADE K3 surfaces with divisor. Ranging over all $I = \mathbb{Z}\delta \oplus \mathbb{Z}\lambda$, the maps v surject onto the Type II locus, so u extends to a morphism over the Type II extension of F_M .

Since the Type II and III extensions of F_M cover all of $\overline{F}_M^\mathfrak{G}$, we conclude that there is a morphism $\overline{F}_M^\mathfrak{G} \rightarrow \overline{F}_M^R$ —on the intersection of the closure of the Type II locus with the Type III locus, it is a morphism as opposed to just a set-theoretic map because $\overline{F}_M^\mathfrak{G}$ is normal.

By Lemma 9.19, we also have a morphism $(\overline{F}_M^R)^\nu \rightarrow \overline{F}_M^{\text{BB}}$. So by Theorem 5.14, the normalization of $\overline{F}_M^R$ is semitoroidal for a unique semifan $\mathfrak{F}_R$. $\square$

Corollary 9.2. *Suppose R is recognizable. The normalization map $\overline{F}_M^{\mathfrak{F}_R} \rightarrow \overline{F}_M^R$ sends semitoroidal strata to slc strata.*

Proof. Let $\sigma \in \mathfrak{F}_R$ be any cone and choose λ in the relative interior $\text{int}(\sigma)$. By Corollary 8.13, the stable limit of any degeneration with monodromy invariant λ lies in a fixed slc stratum. Since the natural map $\delta^\perp / \{\delta, \lambda\} \rightarrow \delta^\perp / \{\delta, \sigma\}$ is surjective, every point in Str_σ is the limit of some arc with monodromy invariant λ . So the combinatorial type of the slc stable model at any point in Str_σ is the same. $\square$

Corollary 9.2 implies that there is a well-defined function

$$\mathbb{S}: \{\text{cones of } \mathfrak{F}_R \text{ mod } \Gamma\} \rightarrow \left\{ \begin{array}{l} \text{combinatorial types of slc} \\ \text{strata which appear in } \overline{F}_M^R \end{array} \right\}.$$

Note that $\mathbb{S}$ may not be injective. For instance, $\mathbb{S}(\sigma) = \mathbb{S}(\tau)$ if the corresponding strata are unglued by normalizing. By abuse, let $\mathbb{S}(\lambda) := \mathbb{S}(\sigma)$ where $\lambda \in \text{int}(\sigma)$.

Theorem 9.3. *Let R be a recognizable divisor for F_M . Let D be the decomposition of monodromy invariants into loci $\{\lambda \in \coprod_\delta C_\delta^+ \cap \delta^\perp / \delta \mid \mathbb{S}(\lambda) \text{ is constant}\}$. Then maximal cones of $\mathfrak{F}_R$ and D are the same.*

A maximal cone of D is a top-dimensional, convex cone in C_δ^+ whose *integral* interior points lie in a single element of D , and which is maximal for this property.

Proof. $\mathbb{S}$ is constant on cones of $\mathfrak{F}_R$ by Corollary 9.2, so it suffices to show that $\mathbb{S}$ cannot take the same value on two maximal dimensional cones $\sigma_1, \sigma_2 \in \mathfrak{F}_R$ and a codimension 1 face $\tau \subset \sigma_1 \cap \sigma_2$ they share. If this were the case, the closed boundary stratum $\overline{\text{Str}}_\tau$ would map to a fixed slc stratum $\mathbb{S}(\sigma_1) = \mathbb{S}(\sigma_2) = \mathbb{S}(\tau)$. But the Type III slc strata contain no complete curve by Corollary 9.17. So $\overline{\text{Str}}_\tau$ would be contracted to a point, contradicting finiteness of the normalization $\overline{F}_M^{\mathfrak{F}_R} \rightarrow \overline{F}_M^R$. $\square$

Theorem 9.3 gives a method to compute the semifan $\mathfrak{F}_R$. Up to taking faces, its cones are sets of monodromy invariants λ which produce a fixed combinatorial slc type. This is how $\mathfrak{F}_R$ was computed in Examples 9.20, 9.21 below.

The semifan $\mathfrak{F}_R$ is also functorial under restriction to Type IV subdomains of F_M , i.e. Noether-Lefschetz loci. Let $M \subset M' \subset L_{K3}$ be primitive hyperbolic sublattices. Then there is a natural map of moduli stacks $\mathcal{F}_{M'}^q \rightarrow \mathcal{F}_M^q$ sending $(X, j) \mapsto (X, j|_M)$. Let $L \in M$.

Proposition 9.4. *Suppose $R \in |L|$ is recognizable for $\mathcal{F}_M^q$. Then its restriction to $\mathcal{F}_{M'}^q$ is also recognizable. Furthermore, $\mathfrak{F}_R(M')$ is the restriction of the semifan $\mathfrak{F}_R(M)$ to the appropriate linear subspaces of $C_\delta^+ \subset \delta_{M^\perp}^\perp / \delta$.*

More precisely, if $\delta \in M'^\perp \subset M^\perp$ is an isotropic vector corresponding to some 0-cusp of $F_{M'}$, we restrict the decomposition $\mathfrak{F}_{R,\delta}(M)$ to the subspace $\delta_{M'^\perp}^\perp/\delta$.

Proof. Proposition 9.4 follows from the fact that any M' -quasipolarized Kulikov model is also M -quasipolarized, plus the functoriality of the stable pair and semitoroidal constructions under restriction to Noether-Lefschetz subdomains. $\square$

9B. Moduli of anticanonical pairs. We prove here that Type III slc strata contain no complete curve by considering the periods of anticanonical pairs. A useful general reference is [Fri15].

Definition 9.5. Let (V, D) be an anticanonical pair with $D = D_1 + \dots + D_n$ an oriented, labeled cycle of rational curves. Define $\Lambda_{(V,D)} := \{D_1, \dots, D_n\}^\perp \subset H^2(V)$ and define the *period point* $\psi_{(V,D)} \in \text{Hom}(\Lambda_{(V,D)}, \mathbb{C}^*)$ to be the restriction map $\gamma \mapsto \gamma|_D \in \text{Pic}^0(D) = \mathbb{C}^*$.

Definition 9.6 ([Fri15, Def. 5.4]). The *generic ample cone* $A_{\text{gen}} \subset H^2(V)$ is the ample cone of a very general topologically trivial deformation of (V, D) .

It suffices to take a deformation for which $\ker(\psi_{(V,D)}) = 0$. This is possible because there is a local universal deformation $(\mathcal{V}, \mathcal{D}) \rightarrow S$ of pairs for which the assignment $s \mapsto \psi_{(V_s, D_s)}$ is an isomorphism to an open subset of $\text{Hom}(\Lambda_{(V,D)}, \mathbb{C}^*)$.

Definition 9.7 ([Fri15, Def. 6.5]). A *Looijenga root* $\beta \in \Lambda_{(V,D)}$ is a class of square $\beta^2 = -2$ which represents a smooth (-2) -curve on some topologically trivial deformation of (V, D) , and for which $\psi_{(V,D)}(\beta) = 1$.

Reflections in Looijenga roots act on A_{gen} . The ample cone A of (V, D) is a fundamental chamber for the action of the group $W_{(V,D)} := \langle r_\beta : \beta \text{ a Looijenga root} \rangle$ on A_{gen} . We can now recall the Torelli theorem for anticanonical pairs:

Theorem 9.8 ([Fri15, Thm. 8.7]). *Two pairs (V, D) and (V', D') (with oriented, labeled cycle) are isomorphic if and only if there exists an isometry $\phi : H^2(V) \rightarrow H^2(V')$ for which $\phi(D_j) = D'_j$, $\phi(A_{\text{gen}}) = A'_{\text{gen}}$, and $\psi_{(V,D)} = \psi_{(V',D')} \circ \phi$. Furthermore, $\phi = f^*$ is induced by an isomorphism $f : (V', D') \rightarrow (V, D)$ if and only if $\phi(A) = \phi(A')$. This isomorphism is unique up to the action of continuous automorphisms $\text{Aut}^0(V, D)$.*

So the analogue of the Torelli Theorem 2.3 holds nearly verbatim, replacing $\mathcal{C}$ with A_{gen} (which is notably not the positive cone), $\mathcal{K}$ with A (which is the Kähler cone), and W_X with $W_{(V,D)}$.

Definition 9.9. Fix a reference lattice $L_{(V,D)}$ isomorphic to $H^2(V)$. Fix classes $(D_j)^0 \in L_{(V,D)}$ and fix a cone $A_{\text{gen}}^0 \subset L_{(V,D)} \otimes \mathbb{R}$. A *marking* of (V, D) is an isometry $\sigma : H^2(V) \rightarrow L_{(V,D)}$ sending $\sigma(D_j) = (D_j)^0$ and $\sigma(A_{\text{gen}}) = A_{\text{gen}}^0$. Let $\Gamma_{(V,D)} \subset O(L_{(V,D)})$ be the subgroup fixing all this data.

Theorem 9.10 ([Fri15, Thm. 8.13]). *Assume $\text{Aut}^0(V, D)$ is trivial. There is a fine moduli space $\mathcal{M}_{(V,D)}$ of marked anticanonical pairs deformation-equivalent to (V, D) . It has a period map*

$$\mathcal{M}_{(V,D)} \rightarrow \text{Hom}(L_{(V,D)}, \mathbb{C}^*)$$

which is generically one-to-one, and whose fibers are torsors over a group isomorphic to $W_{(V,D)}$ with the action on a fiber given by $(X, \sigma) \mapsto (X, g \circ \sigma)$.

When $\text{Aut}^0(V, D)$ is non-trivial, there is still a space $\mathcal{M}_{(V,D)}$ admitting a family which defines at every point a universal deformation, and for which every isomorphism type is represented, but it is not a fine moduli space.

Definition 9.11. A *quasi-polarized triple* (V, D, L) is an anticanonical pair (V, D) and a big and nef line bundle $L \in \text{Pic}(V)$. A *polarized ADE triple* is an image $(\bar{V}, \bar{D}, \bar{L})$ of such under the linear system $\phi_{|nL|}$, $n \gg 0$ (we must add the condition that $\psi_{(V,D)}(L) = 1$ when $L \in \Lambda$). A *divisor triple* (V, D, R) is the extra data of an element $R \in |L|$ such that R contains no nodes of D . A *stable triple* $(\bar{V}, \bar{D}, \epsilon \bar{R})$ is an image of a divisor triple $(V, D, \epsilon R)$ under $\phi_{|nR|}$, $n \gg 0$.

The map $(V, D) \rightarrow (\bar{V}, \bar{D})$ contracts the components of D for which $L \cdot D_j = 0$, together with some negative-definite ADE configuration of (-2) -curves whose classes lie in $\Lambda_{(V,D)}$.

Theorem 9.12. *The coarse moduli space of polarized ADE triples $F_{(\bar{V}, \bar{D}, \bar{L})}$ of a fixed deformation type is the quotient of $\text{Hom}(L_{(V,D)}, \mathbb{C}^*)$ by the finite group $\Gamma_{(V,D,L)} := \text{Stab}_{\Gamma_{(V,D)}}(L)$.*

Proof. The result is analogous to Theorem 2.11. If $L \notin \Lambda$, take the sublocus $\mathcal{M}_{(V,D,L)} \subset \mathcal{M}_{(V,D)}$ where L defines a big and nef divisor—this surjects onto the period torus with fibers a torsor over the reflection subgroup $W_{(V,D,L)} := \text{Stab}_{W_{(V,D)}}(L)$. When $L \in \Lambda$, we restrict to the sublocus $\psi_{(V,D)}(L) = 1$. Now take the relative linear system of nL , which simultaneously contracts the ADE configuration in $\Lambda_{(V,D)} \cap L^\perp$ and some components of D .

The fibers of the period map $\mathcal{M}_{(V,D,L)} \rightarrow \text{Hom}(L_{(V,D)}, \mathbb{C}^*)$ (or $\text{Hom}(L_{(V,D)}/\mathbb{Z}L, \mathbb{C}^*)$ when $L \in \Lambda$) are identified with distinct resolutions of the contraction, and the moduli functor factors through the separated quotient of $\mathcal{M}_{(V,D,L)}$. Since we have included $\bar{L}$ as part of the data, our change-of-markings in $\Gamma_{(V,D)}$ must preserve L . The result follows.

We can even identify (when $\text{Aut}^0(V, D)$ is trivial) the moduli stack as the separated quotient of $[\mathcal{M}_{(V,D,L)} : \Gamma_{(V,D,L)}]$. Like in the K3 case (Rem. 2.12), its only difference with the quotient stack $[\text{Hom}(L_{(V,D)}, \mathbb{C}^*) : \Gamma_{(V,D,L)}]$ is that the inertia groups are locally quotiented by $W_{(V,D,L)}$. $\square$

Let $F_{(\bar{V}, \bar{D}, \bar{R})}$ denote the coarse moduli space of stable triples $(\bar{V}, \bar{D}, \epsilon \bar{R})$ with a fixed deformation type of minimal resolution. Here ϵ is a fixed small number.

Lemma 9.13. *$F_{(\bar{V}, \bar{D}, \bar{R})}$ is a (possibly non-flat) family of affine varieties over the coarse moduli space $F_{(\bar{V}, \bar{D}, \bar{L})}$.*

Proof. On a given polarized ADE triple $(\bar{V}, \bar{D}, \bar{L})$, we may choose $\bar{R} \in |\bar{L}|$ arbitrarily, subject to the condition that $\bar{R}$ not contain any nodes of $\bar{D}$. This condition is either not satisfied by any element of $|\bar{L}|$, or is the complement of a non-zero number of hyperplanes, corresponding to sections which go through some node. Thus, the set of choices of $\bar{R}$ on a fixed ADE triple forms an affine variety.

The automorphism group of $(\bar{V}, \bar{D}, \bar{L})$ acts on the set of such choices $\bar{R}$. So when this automorphism group is finite, the choices form an affine variety. If the automorphism group contains a continuous part of dimension 1 or 2, we may rigidify by requiring $\bar{R}$ to go through 1 or 2 generically chosen points of $\bar{V} \setminus \bar{D}$. Then, the coarse moduli space is a finite image of the rigidified moduli space, which is again affine by the reasoning of the first paragraph. $\square$

Corollary 9.14. *The coarse moduli space $F_{(\bar{V}, \bar{D}, \bar{R})}$ contains no complete curves.*

Proof. This follows immediately from Lemma 9.13 and $F_{(\bar{V}, \bar{D}, \bar{L})}$ being affine. $\square$

Let $F_{(\bar{X}, \bar{R})}$ denote the coarse moduli space of stable slc pairs of a fixed combinatorial type, as in Definition 8.12.

Remark 9.15. Semi log canonical singularities are seminormal. The seminormality implies that the scheme-theoretic structure of a 0-stratum of $\bar{X}$ is unique, since $\bar{X}$ is the direct limit of the diagram of strata, partially ordered by inclusion. So moduli is uniquely determined by the moduli of components and gluings of double curves.

Theorem 9.16. *Let $F_{(\bar{X}, \bar{R})}$ be a coarse moduli space of glued seminormal stable pairs containing a Type III stable pair degeneration of K3 surfaces. Then $F_{(\bar{X}, \bar{R})}$ contains no complete curve.*

Proof. We can construct the coarse moduli space as follows: First, take the product of the coarse moduli spaces of each component $\prod_i F_{(\bar{V}_i, \bar{D}_i, \bar{R}_i)}$. Let $\{\mu_{ij}\} \subset \mathbb{C}^*$ be the (possibly empty, but always finite) set of gluings of $\bar{D}_{ij}$ to $\bar{D}_{ji}$ which identify the nodes of $\bar{D}_i$ and $\bar{D}_j$ and for which $\bar{R}_i \cap \bar{D}_{ij} = \bar{R}_j \cap \bar{D}_{ji}$. The space $G_{(\bar{X}, \bar{R})}$ of such glued pairs is $\prod_i F_{(\bar{V}_i, \bar{D}_i, \bar{R}_i)} \times \prod_{i,j} \{\mu_{ij}\}$ which has a finite map to $\prod_i F_{(\bar{V}_i, \bar{D}_i, \bar{R}_i)}$. So by Corollary 9.14, $G_{(\bar{X}, \bar{R})}$ contains no complete curves.

The space $G_{(\overline{X}, \overline{R})}$ parameterizes seminormal pairs $(\overline{X}, \epsilon \overline{R})$ together with a combinatorial labeling of the dual complex $\Gamma(\overline{X})$. Consider the finite group of combinatorial self-maps of $\Gamma(\overline{X})$ preserving the combinatorial types of all stable triples. The coarse moduli space $F_{(\overline{X}, \overline{R})}$ is the quotient of $G_{(\overline{X}, \overline{R})}$ by this finite group. Since $G_{(\overline{X}, \overline{R})}$ contains no complete curve, neither does $F_{(\overline{X}, \overline{R})}$. $\square$

Corollary 9.17. *No Type III stratum of $\overline{F}_M^R$ contains a complete curve.*

Proof. A Type III stratum of $\overline{F}_M^R$ is a sublocus of the coarse moduli space of pairs $(\overline{X}, \epsilon \overline{R})$ as in Theorem 9.16. The corollary follows. $\square$

9C. Other lemmas. We prove the remaining lemmas used in Theorem 9.1. Let $(\mathbb{C}^n, B)$ denote the analytic germ of $B := \{x_1 \cdots x_n = 0\}$, the union of the coordinate hyperplanes, in $\mathbb{C}^n$.

Lemma 9.18. *Let V be an analytic variety stratified by sub-varieties V_i . Consider a meromorphic map $\phi: (\mathbb{C}^n, B) \dashrightarrow V$ with locus of indeterminacy contained in B . Assume that the image of the indeterminacy locus is contained in $\cup_{i \in I} V_i$ and that no V_i for $i \in I$ contains a complete curve.*

Assume that for any arc germ $f: (C, 0) \rightarrow (\mathbb{C}^n, 0)$ with $f(C \setminus 0) \subset (\mathbb{C}^)^n$, there exists an extension $g: (C, 0) \rightarrow V$ of $\phi \circ f$. Moreover, assume that for any such f , the stratum $V_i \ni g(0)$ depends only on the orders of tangency of C to the coordinate hyperplanes.*

Then the indeterminacy of ϕ can be resolved by toric blow-ups.

Proof. Fix the standard torus action of $T = (\mathbb{C}^*)^n$ on $\mathbb{C}^n$. By Hironaka (see Włodarczyk [Wł09] for a careful treatment of analytic spaces), there exists a sequence of blowups at smooth centers in the indeterminacy loci that resolves ϕ . Let H be the first center which is not T -invariant.

Let O be the largest T -orbit with $O \cap H \neq \emptyset$. By restricting to an open subset, we can assume that $H \subsetneq O$. Consider the toric cross-sections normal to O . These cross-sections satisfy the conditions of the Lemma, and so applying an inductive hypothesis in n , we resolve the indeterminacy of ϕ generically along O , by a series of toric blowups.

So we get a rational map $\phi': (X', B') \dashrightarrow V$ from a toric variety, a torus orbit O' , and a nontoric center of indeterminacy $H' \subset O'$ such that ϕ' is regular on an open set $U \subset X'$ intersecting O' . Then $\phi'(U \cap O')$ is contained in a single stratum V_i . The stratum containing the limit of an arc in X' again depends only on the orders of tangency with the components of B' .

Let $X' \leftarrow Z \rightarrow V$ be a resolution of singularities. Then there exists $p \in O' \cap H'$ such that for the fiber Z_p of $Z \rightarrow X'$ the morphism $Z_p \rightarrow V$ is non-constant. Since Z_p is proper and the strata V_i contain no complete curve, there exist two arcs with $f_1(0) = f_2(0) = p$ and with $g_1(0), g_2(0)$ lying in different strata V_i . But shifts of these arcs by the torus action have the same tangency conditions with the coordinate hyperplanes and satisfy $f(0) \in U$. So for them $g(0)$ lie in the same stratum of V . Contradiction. $\square$

Lemma 9.19. *There is a morphism $(\overline{F}_M^R)^\nu \rightarrow \overline{F}_M^{\text{BB}}$ for any canonical choice of polarizing divisor R (recognizable or not).*

This is proved in [AET19, Thm. 3.15] and amounts to the observation that in Type II, the j -invariant of the corresponding point in the 1-cusp of $\overline{F}_M^{\text{BB}}$ can be recovered from the stable slc pair $(X, \epsilon R)$. Indeed, either X is nonnormal and every connected component of the double locus is an elliptic curve E with this j -invariant, or X has an elliptic singularity corresponding to E .

9D. Examples. Previously known examples of recognizable divisors come from [AET19], [ABE22].

Example 9.20 (Degree 2 K3s). Let (X, L) be a quasipolarized K3 surface of degree $L^2 = 2$. Let $\overline{X}$ denote the corresponding polarized ADE K3 surface. Then $\phi|_{L|}$ defines a branched double cover $\overline{X} \rightarrow \mathbb{P}^2$ or $\overline{X} \rightarrow \mathbb{F}_4^0$ (the contraction of the Hirzebruch surface $\mathbb{F}_4$ along its negative section). Define $R \in |3L|$ to be the pullback of the ramification locus $\overline{R} \subset \overline{X}$.

There is only one Γ -orbit of primitive isotropic vector $\delta \in L^\perp$, and so a semitoroidal compactification is determined by a single Γ_δ -invariant semifan $\mathfrak{F} = \mathfrak{F}_\delta$ in the positive cone of C_δ^+ . Then

[AET19] verifies Theorem 8.11(5) directly, by constructing for each monodromy invariant λ , a divisor λ -family with monodromy invariant in the projective class $[\lambda]$.

These divisor models are constructed by ensuring the involution on the general fiber X_t of Kulikov model $X \rightarrow (C, 0)$ extends to the central fiber X_0 . Then the fixed locus $R_0 \subset X_0$ is the canonical choice of divisor certifying recognizability. The resulting semifan $\mathfrak{F}_R$ is not a fan. The lattice $\delta^\perp/\delta$ is a hyperbolic root lattice with a finite covolume Coxeter chamber $\mathfrak{K}$. There is an infinite subgroup $W \subset \Gamma_\delta$ for which $\mathfrak{F}_R$ is the Γ_δ -orbit of a single chamber $\mathfrak{L} := W \cdot \mathfrak{K}$.

Thus, Theorem 9.1 *cannot* be strengthened by replacing “semifan” with “fan.”

Example 9.21 (Elliptic K3s). Let (X, j) be an H -quasipolarized K3 surface, i.e. an elliptic K3 surface, with fiber class f and section s (so h lies in cone spanned by $f, s+2f$). Let $R = s + m \sum f_i$ be the section plus the sum of the singular fibers, with multiplicity, and weighted by m . Here $L = s + 24mf \in H$ is the relevant big and nef class. $\overline{F}_H^R$ is the same for all $m > \frac{1}{3}$.

As in the previous example, [ABE22] find Kulikov models $X \rightarrow (C, 0)$ for any monodromy invariant which preserve the existing structures on the general fiber: X_0 admits a fibration by genus 1 curves $\pi_0: X_0 \rightarrow B_0$ over a chain of rational curves B_0 , with a section s_0 . Finitely many fibers $f_{i,0} = \pi_0^{-1}(b_i)$ not contained in the double locus of X_0 have more nodes than all analytically nearby fibers. Counting $f_{i,0}$ with the correct multiplicity, the recognizable divisor on X_0 is

$$R_0 = s_0 + m \sum f_{i,0}.$$

There is a unique Γ -orbit of primitive isotropic $\delta \in H^\perp = \Pi_{2,18}$ and $\delta^\perp/\delta = \Pi_{1,17}$ is a hyperbolic root lattice with a finite covolume Coxeter chamber $\mathfrak{K}$. Then $\mathfrak{F}_R$ is the Γ_δ -orbit of a subdivision of $\mathfrak{K}$ into 9 subchambers. So $\mathfrak{F}_R$ is a fan.

Example 9.22. Any choice of divisor R when $\dim \mathbb{D}_M = 1$ is recognizable: There exists a divisor model in the neighborhood of any point $p \in \overline{F}_M$ in the unique toroidal compactification, which also equals the Baily-Borel compactification.

Remark 9.23. The necessity of normalizing $\overline{F}_M^R$ to get a semitoroidal compactification is apparent in both Examples 9.20, 9.21. [AET19, ABE22] compute the normalization map explicitly.

10. THE RATIONAL CURVE DIVISOR

Our goal is to now make a canonical choice of divisor for F_{2d} for any $d > 0$, then prove its recognizability. From this, we can conclude Corollary 3: there are KSBA compactifications of F_{2d} whose normalizations are semitoroidal, for all degrees. Our divisor is roughly the sum of all rational curves in $|L|$. Its recognizability is proven below, by showing that the image of a predeformable, stable, genus zero map to any Kulikov surface is rigid.

10A. Definition of R^{rc} . Consider the moduli space F_{2d}^q of quasipolarized K3 surfaces of degree $2d$. Let $(X, L) \in F_{2d}^q$.

Definition 10.1. We say that $G \in |L|$ is a *rational curve* if the normalization of every irreducible component of G is $\mathbb{P}^1$.

Theorem 10.2 (Yau-Zaslow formula [YZ96, Bea99, Che99, Che02]). *There is a Zariski open subset $U \subset F_{2d}^q$ for which any rational curve $G \in |L|$ for $(X, L) \in U$ is irreducible, nodal, and for which the number of such rational curves is exactly*

$$n_d := [q^d] \frac{1}{q} \prod_{k \geq 1} \frac{1}{(1 - q^k)^{24}} = [q^d] \frac{1}{\Delta(q)}$$

where $\Delta(q)$ is the modular discriminant, and $[q^d]$ denotes the q^d -coefficient.

The integer n_d is the number of 24-colored partitions of $d + 1$.

Definition 10.3. The *rational curve divisor* is the canonical choice of polarizing divisor $R^{\text{rc}} := \sum_{G \in |L| \text{ rational}} G \in |n_d L|$ defined over the open subset $U \subset F_{2d}^q$.

We now outline an alternative definition using Gromov-Witten invariants.

Definition 10.4. Let X be a smooth complete variety and let $\beta \in H_2(X, \mathbb{Z})$. The *Kontsevich space* $M_g(X, \beta)$ is the moduli space of stable maps $f : T \rightarrow X$ from a genus g nodal curve T , for which $f_*[T] = \beta$.

The Kontsevich space is a proper Deligne-Mumford stack. For a surface, $H_2(X, \mathbb{Z})$ and $H^2(X, \mathbb{Z})$ are canonically identified by Poincaré duality, so we make no distinction. We will take $L = \beta$.

There is a *virtual fundamental cycle* $[M_g(X, \beta)]^{\text{vir}} \in A_{\text{exp.dim}}(M_g(X, \beta))$ where $\text{exp.dim} = (\dim X - 3)(1 - g) + c_1(T_X) \cdot \beta$ is the expected dimension of the moduli space [BF97]. In particular, for stable genus 0 maps to a K3 surface, $\text{exp.dim} = -1$ so $[M_0(X, L)]^{\text{vir}} = 0$. Geometrically, this can be explained by the fact that GW invariants are deformation-invariant, but that a generic deformation of X has no nontrivial line bundles, so β cannot represent an algebraic curve.

For polarized K3 surfaces (X, L) , there is a *reduced virtual fundamental cycle* $[M_0(X, L)]^{\text{vir,red}} \in A_0(M_0(X, L))$, see [KT14]. Roughly, it is built to be invariant only under the deformations of X which stay in $\mathcal{F}_{2d}^q$. This decreases dimension of the obstruction space by one, increasing the expected dimension by one.

Lemma 10.5. *Let (T, f) be a stable map $f : T \rightarrow X$ with T a nodal curve of arithmetic genus 0 and X a smooth K3 surface. Under any deformation of the stable map $(T, f) \in M_0(X, L)$, the image divisor f_*T is constant.*

Proof. If the image divisor f_*T moves under a deformation of (T, f) , a restriction of f gives a dominant map $S = B \times \mathbb{P}^1 \xrightarrow{\pi} X$, for some (possibly incomplete) curve B . Letting $\text{Ram} \subset S$ be the ramification divisor of the map $S \rightarrow X$, the Riemann-Hurwitz formula gives $K_S = \text{Ram}$. This contradicts the adjunction formula, because restricting to a general fiber $F = \{b\} \times \mathbb{P}^1$ gives $-2 = 2g(F) - 2 = F \cdot (F + \text{Ram}) = F \cdot \text{Ram} \geq 0$. $\square$

Replicating this argument for a Kulikov surface is the key to proving that R^{rc} is recognizable.

Definition 10.6. Let $G \in |L|$ be a rational curve. Define $M_G(X)$ to be the union of the connected components of $M_0(X, L)$ for which $f_*T = G$. This is well-defined because f_*T is constant on any connected component by Lemma 10.5. Define

$$n_G := \deg_{M_G(X)}[M_0(X, L)]^{\text{vir,red}} \in \mathbb{Q}.$$

This quantity a priori only lies in $\mathbb{Q}$ because of stack-theoretic issues.

Proposition 10.7. *For any smooth quasipolarized K3 surface (X, L) , we have*

$$R^{\text{rc}} = \sum_{G \in |L|_{\text{rational}}} n_G G.$$

Furthermore, n_G is a non-negative integer for all rational curves $G \in |L|$.

Proof. Chen's theorem [Che02] implies that for a sufficiently general $(X, L) \in U$, we have $n_G = 1$ for all rational curves G . Fix an $(X_0, L_0) \in F_{2d}^q$ not in U and consider a 1-parameter deformation $(\mathcal{X}, \mathcal{L}) \rightarrow (C, 0)$ over an analytic disc C for which $X_t \in U$ for all $t \in C^*$. Consider the moduli space of relative stable maps $M_0(\mathcal{X}, \beta)$ where β is the class of L_0 pushed forward to $\mathcal{X}$.

There is a proper morphism $M_0(\mathcal{X}, \beta) \rightarrow C$ sending a curve to the fiber it is supported on, but the fibers of this family are in general poorly behaved. For instance, the dimension can and often does suddenly jump at $t = 0$. But by assumption, the fiber over any point $t \in C^*$ is a reduced zero-dimensional scheme consisting of exactly n_d points. The proposition follows if we can prove that the scheme-theoretic intersection $\overline{M}_0(\mathcal{X}^*, \beta) \cap M_0(X_0, L_0)$ represents the reduced virtual fundamental class, in homology.

The constancy of the reduced Gromov-Witten invariants $n_d = \deg_{M_0(X_t, \beta)}[M_0(X_t, \beta)]^{\text{vir,red}}$ as one varies t follows from the existence of a relative perfect obstruction theory [BF97, Sec. 7], [KT14, Rem. 3.1]. Without going into the details, this is a perfect two-term complex with a morphism to the relative cotangent complex, satisfying various axioms.

Now let $W \subset M_0(\mathcal{X}, \beta)$ be a connected component. The restriction of the axioms of a (relative) perfect obstruction theory still hold under restricting this two-term complex to W . Hence the constancy of reduced GW invariants still holds, i.e. $\deg_W[M_0(X_0, L_0)]^{\text{vir}, \text{red}}$ will equal the number of sheets of $M_0(X_t, L_t)$ whose closures over $t = 0$ lie in W . This implies the first statement.

Summing these integrals over the components W for which the image curve $f_*T = G$, we also see that n_G is a non-negative integer. $\square$

Remark 10.8. A priori, the contribution n_G could equal zero. Perhaps no genus 0 stable map with image G deforms to the general fiber X_t . Notably, this cannot occur when there is a component $W \subset M_G(X)$ of dimension $\dim W = 0$, see [Huy16, Ch. 13.2.3] and references therein.

Proposition 10.7 provides us with a definition of R^{rc} on all $(X, L) \in F_{2d}^q$.

Definition 10.9. A quasipolarized K3 surface (X, L) of degree $2d$ is *unigonal* if it is elliptic, with section and fiber classes s, f and $L = s + (d+1)f$.

As a Noether-Lefschetz locus of Picard rank 2, the unigonal locus forms a divisor in F_{2d}^q isomorphic to the moduli space F_H^q of elliptic K3s.

Proposition 10.10. *On a unigonal K3 surface (X, L) , the rational curve divisor is*

$$R^{\text{rc}} := n_d(s + \frac{d+1}{24} \sum f_i)$$

where f_i are the 24 singular fibers in $|f|$, counted with multiplicity.

Proof. The proposition follows immediately from Proposition 10.7 and the main result of [BL00], though historically [Che02] relies on [BL00]. $\square$

10B. Proof of Theorem 2. We now prove our second main result:

Theorem 10.11. *The rational curve divisor R^{rc} is recognizable for F_{2d} for all $d > 0$.*

Proof. Take a divisor model $(X, R) \rightarrow (C, 0)$. We verify Theorem 8.11(1) by showing that the limiting curve $R_0 \subset X_0$ satisfies some geometric property ensuring its rigidity on X_0 even as we deform the smoothing of X_0 . Take a base change and standard resolution of $(X, R) \rightarrow (C, 0)$ so that the irreducible components G_t of R_t are not permuted by monodromy. Then, Lemmas 10.12 and 10.14 imply that the limit of any individual rational curve G_t is rigid. $\square$

Lemma 10.12. *Let $X \rightarrow (C, 0)$ be a Kulikov model and let $G \subset X$ be a flat family of curves for which G_t is an irreducible rational curve for $t \neq 0$, and G_0 contains no strata. Then, after a finite base change and resolution of $X \rightarrow (C, 0)$, there is a stable map $f : T \rightarrow X_0$ from a nodal, genus 0 curve (a tree of $\mathbb{P}^1$ s) for which $f_*T = G_0$ and (T, f) is predeformable (see Def. 10.13).*

Definition 10.13. We say that (T, f) is *predeformable* [Li01, Def. 2.5] if no component of T is contracted into the double locus, and for each node $p \in T$ with $f(p) \in D_{ij}$, the two arcs (T_k, p) , (T_ℓ, p) with $f(T_k) \subset V_i$ and $f(T_\ell) \subset V_j$ satisfy

the tangency order of $f(T_k, p)$ to D_{ij} = the tangency order of $f(T_\ell, p)$ to D_{ji} .

Proof of Lemma 10.12. Because G_0 contains no strata, it maps into the complement of the triple points of X_0 , i.e. the union of the non-singular locus and the double locus. Then, the result follows from the properness over $(C, 0)$ of the space of predeformable stable maps [Li01, Thm. 3.10] to varieties with only double crossings. $\square$

Lemma 10.14. *Let $f : T \rightarrow X_0 \times B$ be a family of stable maps over a local curve B , such that T_b is a tree of $\mathbb{P}^1$ s of fixed combinatorial type for all $b \in B$, and for which (T_b, f_b) is predeformable. Then the image curves $f_*(T_b) = G_0$ are constant.*

Proof. Let $T = \cup_k T_k$ be the components of T . We have $T_k \cong \mathbb{P}^1 \times B$. Let $N_{k\ell} = T_k \cap T_\ell$ be the relative nodes over B . We label the vertices $\Gamma(T)^{[0]}$ of the dual complex $\Gamma(T)$ as follows:

(V0) T_k is contracted to a point inside a component.

(V1b) T_k is contracted along multisections to a curve inside a component.

	V0	V1b	V1f	V2
V0	v0	v0	—	v0
V1b		v0/v1/d0	v1	v0/v1/d0
V1f			v1	v1
V2				all

TABLE 1. Allowable adjacencies for the labeled dual complex $\Gamma(T)$.

- (V1f) T_k is contracted along fibers of $T_k \rightarrow B$ to a curve inside a component.
(V2) T_k maps generically finitely to a component.

These are the only possibilities, by noting that $T_k \rightarrow B$ is proper and that the image of f contains no triple points. Next, we label the edges $\Gamma(T)^{[1]}$ of the dual complex $\Gamma(T)$ as follows:

- (v0) $N_{k\ell}$ maps to a point in the interior of a component.
(v1) $N_{k\ell}$ maps to a curve in the interior of a component.
(d0) $N_{k\ell}$ maps to a point in a double curve.
(d1) $N_{k\ell}$ maps to a curve in a double curve.

Table 1 records the allowable adjacencies for the labeled dual complex $\Gamma(T)$, which can be verified from predeformability by straightforward geometric arguments.

Let $\Gamma \subset \Gamma(T)$ be a maximal subtree consisting of only V2-vertices and d1-edges. Let $T_\Gamma \subset T$ be the sub-family of curves with dual complex Γ . Consider the restricted family $f_\Gamma : T_\Gamma \rightarrow X_0 \times B$.

The fibers $T_{\Gamma,b}$ may only fail to map in a predeformable way to X_0 at the leaves of Γ which are not leaves of $\Gamma(T)$. Consider the edges emanating from such a V2 leaf which are connected to the rest of $\Gamma(T)$. Disconnecting $\Gamma(T)$ at a v-edge does not interfere with the condition of being predeformable, so consider only the d-edges. By Table 1 and maximality of Γ , such a V2 leaf of Γ must connect by a d0-edge.

So fix one V2 leaf of Γ , associated to a component $T_k \cong \mathbb{P}^1 \times B \subset T_\Gamma$ attached to the rest of $\Gamma(T)$ by d0-edges. The further restriction $f_k : T_k \rightarrow V_i$ is now a map of smooth surfaces.

Each outgoing d0-edge corresponds to a relative node $N_{k\ell}$ of T which maps under f_k to a single point $p_{k\ell} \in D_{ij}$. There is at most one remaining relative node $N \subset T_k$ which attaches T_k to the rest of T_Γ and for which $f(N)$ is a curve in one boundary component of V_i . Make an interior blow-up $\tilde{V}_i \rightarrow V_i$ at each fixed attaching point $p_{k\ell}$. Taking the strict transforms of the images of fibers of $T_k \rightarrow B$, we can lift f_k to a map $\tilde{f}_k : T_k \rightarrow \tilde{V}_i$. If $\tilde{f}_k$ still sends any $N_{k\ell}$ to a point in the new anticanonical boundary $\tilde{D}_{ij}$, we continue to blow up at the fixed attaching points, until the lifted map satisfies the property $\tilde{f}_k^{-1}(\tilde{D}_i) = N$.

Since both N and $\tilde{D}_i$ are divisors with coefficient 1, we have by Riemann-Hurwitz that

$$\omega_{T_k}(N) = \tilde{f}_k^*(\omega_{\tilde{V}_i}(\tilde{D}_i)) \otimes \mathcal{O}(\text{Ram}) \otimes \mathcal{O}(\sum a_i E_i).$$

Here $\text{Ram} \subset T_k$ is the interior ramification divisor, i.e. the ramification away from the boundary $\tilde{D}_i$, and $E_i \subset T_k$ are the contracted curves of $\tilde{f}_k$. Note that $a_i \geq 0$ because $\tilde{V}_i$ is smooth. Since $\omega_{\tilde{V}_i}(\tilde{D}_i) = \mathcal{O}$ we conclude $\omega_{T_k}(N)$ is effective, implying $-1 = \omega_{T_k}(N) \cdot \mathbb{P}^1 \geq 0$. Contradiction. $\square$

In analogy with Proposition 10.7, Lemma 10.14 allows us to define R^{rc} for Kulikov surfaces inherently, in terms of logarithmic Gromov-Witten invariants [Che14, AC14, AMW14].

10C. The rational curve semifan. We first give some general results concerning the Baily-Borel compactification of F_{2d} , following [Sca87].

The number of 0-cusps of $\bar{F}_{2d}^{\text{BB}}$ is exactly $\lfloor \frac{N+2}{2} \rfloor$ where $d = N^2 d_0$ for a square-free integer d_0 . As discussed in Section 5A, they are in bijection with the Γ -orbits of primitive isotropic lattices $I = \mathbb{Z}\delta$ in the lattice

$$L_{2d} := \langle -2d \rangle \oplus H^{\oplus 2} \oplus E_8^{\oplus 2} = v^\perp \subset L_{K3}.$$

The Γ -orbit of a generator $\delta \in I$ is determined by the following invariant: $\delta^* = \frac{\delta}{p^*(\delta)} \in \Delta_{2d} := L_{2d}^*/L_{2d}$, where p^* is by definition the imprimitivity in L_{2d}^* . Then δ^* is an isotropic vector for the quadratic form on Δ_{2d} valued in $\frac{1}{2d}\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/2\mathbb{Z}$. Identifying the source $\Delta_{2d} = \mathbb{Z}/2d\mathbb{Z}$ and the target with $\mathbb{Z}/4d\mathbb{Z}$, the quadratic form is given by $x \mapsto x^2$. We must have $\delta^* = 2qNd_0 \in \mathbb{Z}/2d\mathbb{Z}$ for $q \in \mathbb{Z}/N\mathbb{Z}$. So $I = \mathbb{Z}\delta$ is determined by $\{\pm q\}$ and we have

$$\delta^\perp/\delta = \langle \frac{-2d}{p^*(\delta)} \rangle \oplus H \oplus E_8^{\oplus 2}.$$

A semitoroidal compactification of F_{2d} is determined by a collection of Γ_δ -invariant semifans $\mathfrak{F}_\delta$ decomposing the rational closures C_δ^+ of the positive cones of each lattice $\delta^\perp/\delta$ as above, as one ranges over the $\lfloor \frac{N+2}{2} \rfloor$ possible values of $\{\pm\delta^*\}$. By Theorems 1 and 2, we may define:

Definition 10.15. Let $\mathfrak{F}^{\text{rc}}$ be the semifan for which $\nu: \overline{F}_{2d}^{\mathfrak{F}^{\text{rc}}} \rightarrow \overline{F}_{2d}^{R^{\text{rc}}}$ is the normalization map.

Some facts about the combinatorics of $\mathfrak{F}^{\text{rc}}$ can be deduced from Proposition 10.10 and [ABE22], by restricting to the locus of elliptic K3 surfaces.

Theorem 10.16. Consider a cone C_δ^+ with invariant $p^*(\delta) = 1$, that is, where δ is primitive in L_{2d}^* (all such δ are equivalent under Γ). The restriction of $\mathfrak{F}^{\text{rc}}$ to $\langle -2d \rangle^\perp = H \oplus E_8^{\oplus 2}$, or any Γ_δ -orbit of it, is a fan. Furthermore, $\langle -2d \rangle^\perp \cap C_\delta^+$ is a union of cones of $\mathfrak{F}_\delta^{\text{rc}}$.

We call such hyperplanes *unigonal*. The last statement in the theorem implies that $\mathfrak{F}^{\text{rc}}$ refines the unigonal hyperplane arrangement in C_δ^+ .

Proof. Suppose (X, L) is in the unigonal locus, so that $L = s + (d+1)f$. The inclusion $\mathbb{Z}L \hookrightarrow H$ induces an inclusion of moduli spaces $F_H \rightarrow F_{2d}$. So the restriction of R^{rc} to F_H is recognizable. Suppose that $\delta \in H^\perp$ is primitive isotropic. There is a unique isometry orbit of such and δ is primitive in $(L^\perp)^*$. So $\delta_{H^\perp}^\perp/\delta$ includes into the lattice $\delta_{L^\perp}^\perp/\delta$ corresponding an isotropic vector with invariant $\delta^* = 0$. Concretely, it is the summand inclusion $H \oplus E_8^{\oplus 2} \hookrightarrow \langle -2d \rangle \oplus H \oplus E_8^{\oplus 2}$.

By Proposition 9.4, the restriction of $\mathfrak{F}^{\text{rc}}$ to $H \oplus E_8^{\oplus 2}$ is the semifan $\mathfrak{F}_H^{\text{rc}}$ whose corresponding semitoroidal compactification normalizes $\overline{F}_H^{R^{\text{rc}}}$. By Proposition 10.10, the rational curve divisor, as in Definition 10.3, when extended to the unigonal locus, is a multiple of $R = s + m \sum f_i$ for $m = \frac{d+1}{24}$. [ABE22, Thm. 1.2] gives an explicit description of the fan $\mathfrak{F}_H^{\text{rc}}$ modulo the following caveat: The divisor models described in [ABE22, Sec. 7B] require a threshold value of $m > \frac{1}{3}$ for the divisors $R_0 \subset X_0$ constructed therein to be nef. The threshold is achieved when $\Gamma(X_0)$ has a so-called X_3 end singularity. So it is automatic that *loc.cit* describes the restriction of $\mathfrak{F}^{\text{rc}}$ to the unigonal hyperplane when $d > 7$. For $m \leq \frac{1}{3}$ or $d \leq 7$, the stable models only differ from those in *loc.cit.* in a minor way—one might contract the section on one or both end surfaces. But the stratum function $\mathbb{S}$ has the same level sets and so $\mathfrak{F}_H^{\text{rc}}$ is the same (Prop. 9.3).

The fan $\mathfrak{F}_H^{\text{rc}}$ consists of six orbits of maximal cones [ABE22, Sec. 4C]. To prove the final statement of the theorem, we must show that all six of the 18-dimensional cones $\sigma_H \in \mathfrak{F}_H^{\text{rc}}$ are themselves cones of $\mathfrak{F}^{\text{rc}}$ and not simply slices of the interior of some 19-dimensional cone $\sigma \in \mathfrak{F}^{\text{rc}}$.

A maximal cone σ_H corresponds to a 0-stratum of the stable pair compactification of elliptic K3s and hence to unique Type III elliptic stable K3 pair $(\overline{X}_0, \overline{R}_0)$. If $(\overline{X}_0, \overline{R}_0)$ deforms out of the unigonal locus as rational curve K3 pair, keeping the combinatorial type constant, then σ_H must be a cone of $\mathfrak{F}^{\text{rc}}$. But if the elliptic stable K3 pair $(\overline{X}_0, \overline{R}_0)$ is, as a rational curve K3 pair, rigid in its combinatorial type, then σ_H must be the slice of a larger dimensional cone of $\mathfrak{F}^{\text{rc}}$.

Let (X_0, R_0) be a Type III divisor model whose stable model is $(\overline{X}_0, \epsilon \overline{R}_0)$, see [ABE22, Sec. 7A] for an explicit description. Let $L_0 = \mathcal{O}_{X_0}(R_0)$. We can deform (X_0, L_0) to a non-elliptic, d -semistable Kulikov model (X'_0, L'_0) by regluing double curves so that $\psi_{X'_0}(f) \neq 1$. Concretely, comparing to [ABE22, Def. 7.10], it corresponds to when a connected chain of fibers of vertical rulings *fails to glue* to a closed cycle, destroying the elliptic fibration and the Cartierness of f .

Since R^{rc} is recognizable, the rational curve divisor on such a deformed Kulikov model is necessarily a deformation of the curve $n_d(s + \frac{d+1}{24} \sum f_{i,0})$ (see 9.21) living in the linear system $|L'_0|$.

So the resulting stable model has the same combinatorial type as the elliptic one

$$(\overline{X}'_0, \epsilon \overline{R}'_0) = \bigcup_{i=1}^r (\overline{V}_i, \overline{D}_i, \epsilon \overline{R}_i)$$

for $r = 18, 19, 20$ depending on the cone σ_H .

In the elliptic case, the intermediate components $\overline{V}_i$ for $i \neq 1, r$ are the result of gluing two sections of $\mathbb{P}^1 \times \mathbb{P}^1$ via the isomorphism provided by the vertical fibration. But when we deform X_0 out of the elliptic locus to the Kulikov surface X'_0 , the surface $\overline{V}_i$ also deforms: The gluing map between the two sections includes a shift exactly equal to $\psi_{X'_0}(f)$.

Hence $(\overline{X}_0, \epsilon \overline{R}_0)$ is *not* rigid in $\overline{F}_{2d}^{R^{\text{rc}}}$ within its slc combinatorial type. Even forgetting the divisor, the underlying surface $\overline{X}_0$ is not rigid. We conclude that σ_H is a cone of $\mathfrak{F}^{\text{rc}}$. $\square$

Remark 10.17. The results of this section hold for the *imprimitive rational curve divisor*

$$R^{\text{rc}}(m) = \sum_{G \in |mL| \text{ rational}} n_G G$$

where the coefficients n_G are defined using reduced GW invariants as in Definition 10.6. A naive version of Chen's theorem (that generically all rational curves are nodal) is false: For instance one can take mG for $G \in |L|$ rational. It is not clear whether one can recover Chen's theorem by subtracting out these and other obvious non-reduced and non-irreducible contributions to get a divisor $R^{\text{rc}}_{\text{prim}}(m)$. Regardless, the above serves as a definition of $R^{\text{rc}}(m)$ and produces a canonical choice of polarizing divisor. The proof of recognizability, Theorem 10.11 applies verbatim because the normalization of any irreducible component of $R^{\text{rc}}(m)$ is $\mathbb{P}^1$.

Thus, there are semifans $\mathfrak{F}^{\text{rc}}(m)$ for all $m \geq 1$ which give the normalization of the KSBA compactification associated to $R^{\text{rc}}(m)$.

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