

Stable pair compactification of moduli of K3 surfaces of degree 2

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Abstract. We prove that the universal family of polarized K3 surfaces of degree 2 can be extended to a flat family of stable KSBA pairs $(X, \epsilon R)$ over the toroidal compactification associated to the Coxeter fan. One-parameter degenerations of K3 surfaces in this family are described by integral-affine structures on a sphere with 24 singularities.

Contents

1. Introduction
 2. Kulikov models and IAS^2
 3. Compactifications of F_2
 4. The Coxeter fan and compactifications of F_2
 5. The mirror surfaces
 6. Family of IAS^2 over the Coxeter fan
 7. Determination of stable models
 8. Proof of main theorems
- References

1. Introduction

By the Torelli theorem [48], the coarse moduli space F_{2d} of primitively polarized K3 surfaces (X, L) of degree $L^2 = 2d$ is the quotient $F_{2d} = \Gamma \backslash \mathbb{D}$ of a 19-dimensional Hermitian symmetric domain by an arithmetic group. In its capacity as an arithmetic quotient, there are the Baily–Borel $\overline{F}_{2d}^{\text{BB}}$ [10] and infinitely many toroidal $\overline{F}_{2d}^{\text{tor}}$ [9] compactifications of F_{2d} . These were unified by the more general semitoric compactifications $\overline{F}_{2d}^{\text{semi}}$ [42] of Looijenga.

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The geometry of these Hodge-theoretic compactifications can be described explicitly. For instance, the incidence structure of the boundary strata is encoded by combinatorial information, called a semifan $\mathfrak{F}^{\text{semi}}$. But *a priori*, semitoric compactifications are not modular—the boundary points need not parameterize some geometric generalization of a K3 surface.

On the other hand, if we canonically choose for every polarized K3 surface (X, L) an effective divisor $R \in |NL|$ in a fixed multiple of the polarization, we get a geometrically meaningful compactification $F_{2d} \hookrightarrow \overline{F}_{2d}^{\text{slc}}$ by taking the closure of the space of pairs $(X, \epsilon R)$ in the moduli space of all KSBA stable pairs. These are pairs with semi-log-canonical (slc) singularities and ample log-canonical class $K_X + \epsilon R$, see e.g. [1, 3, 33, 34]. Generally, it is very hard to describe the boundary of $\overline{F}_{2d}^{\text{slc}}$ and the surfaces appearing over it.

Thus, finding compactifications of K3 moduli which are both Hodge-theoretic and algebro-geometric has been a central, and largely open, motivating question.

Question 1.1. Do $\overline{F}_{2d}^{\text{slc}}, \overline{F}_{2d}^{\text{tor}}$ coincide for appropriate choices of divisor R and fan $\mathfrak{F}$? If so, what are the fibers over the toroidal boundary strata?

For the moduli space A_g of principally polarized abelian varieties (ppavs), these questions were answered affirmatively in [2]: On a ppav (X, L) , we choose the unique theta divisor $R = \Theta \in |L|$ in the principal polarization. Then the closure of the pairs $(X, \epsilon \Theta)$ in the space of KSBA stable pairs coincides (up to normalization) with the toroidal compactification associated to the second Voronoi fan.

In this paper, we answer Question 1.1 affirmatively for the moduli space F_2 . A K3 surface (X, L) of degree 2 is canonically equipped with an involution ι , switching the sheets of $\phi_{|L|}$. So its ramification divisor $R = \text{Fix}(\iota) \in |3L|$ is uniquely determined by (X, L) . Thus, the closure of the space of pairs $(X, \epsilon R)$ gives a geometric compactification of F_2 .

On the other hand, there is a natural choice of toroidal compactification. A fan is given by an $O(N)$ -invariant polyhedral decomposition of the rational closure of the positive cone in $N := H \oplus E_8^2 \oplus A_1$ which is a hyperbolic lattice of signature $(1, 18)$. Then N is a hyperbolic root lattice, and we define the Coxeter fan $\mathfrak{F}_{\text{cox}}$ to have walls equal to the perpendiculars of the roots, i.e. vectors $r \in N$ of norm -2 . Our main result is the following.

Theorem 1.2. *There is a semifan for which $v: \overline{F}_2^{\text{semi}} \rightarrow \overline{F}_2^{\text{slc}}$ is the normalization of the KSBA compactification associated to the ramification divisor R . The Coxeter fan refines this semifan, and hence there is a family of stable pairs over the associated toroidal compactification $\overline{F}_2^{\text{tor}}$.*

The KSBA-stable surfaces over the boundary of $\overline{F}_2^{\text{tor}}$ admit completely explicit descriptions, in terms of sub-Dynkin diagrams of the Coxeter diagram for N .

For a generic K3 surface of degree 2, the quotient $Y = X/\iota$ is isomorphic to $\mathbb{P}^2$, and the double cover is branched in a sextic curve B . The pair $(X, \epsilon R)$ is stable if and only if the pair $(Y, \frac{1+\epsilon}{2}B)$ is. Hacking [30] defined and studied the stable pair compactification $\overline{M}(\mathbb{P}^2, d)$ for the pairs $(\mathbb{P}^2, \frac{3+\epsilon}{d}C_d)$, where C_d is a curve of degree d . Then the space $\overline{F}_2^{\text{slc}}$ is the special case $d = 6$. Hacking provides a complete description of $\overline{M}(\mathbb{P}^2, d)$ for $d = 4, 5$ and a fairly complete one for $3 \nmid d$. Some examples of degenerate surfaces for $d = 6$ are given in [31], but the problem of giving a complete description of $\overline{M}(\mathbb{P}^2, 6)$ remained open. Theorem 1.2 provides such a description.

A moduli space related to $\overline{F}_{2d}^{\text{slc}}$ is the compactified space $\overline{P}_{2d}$ of K3 pairs $(X, \epsilon D)$, where $D \in |L|$ is an *arbitrary* divisor in the polarization class. This space has dimension $20 + d$ versus 19 for $\overline{F}_{2d}$. Laza [40], building on the work of Shah [50] and Looijenga [42], described $\overline{P}_2$ and the degenerate pairs at the boundary. Our constructions are unrelated since the ramification divisor R lies in $|3L|$.

Our compactifications of the universal family over F_2 provide toroidal, semitoric, and stable pair compactifications for any subfamily. Among them is the Heegner divisor $F_{\text{ell}} \subset F_2$ of elliptic K3 surfaces. Theorem 1.2 directly generalizes to these subfamilies. In particular, it leads to three compactifications of F_{ell} which are discussed further in [5].

The compactification $\overline{F}_{\text{ell}}^{\text{slc}}$ induced by $\overline{F}_2^{\text{slc}}$ is for the polarizing divisor equal to the tri-section of nontrivial 2-torsion. Stable pair compactifications of F_{ell} for different choices of polarizing divisors, weighted sums of the section and fibers, were investigated by Brunyate [12], Ascher–Bejleri [8], and [5], with a description of the surfaces appearing on the boundary.

We now briefly explain our approach and features that parallel or contrast the case of principally polarized abelian varieties.

One-parameter degenerations of ppavs admit a toric description, due to Mumford [45]. Let $M \simeq \mathbb{Z}^g$ be a fixed lattice and $N = M^*$ its dual. The Voronoi fan $\mathfrak{F}^{\text{vor}}$ is supported on the rational closure $\overline{\mathcal{C}}$ of the cone of positive definite symmetric forms

$$\mathcal{C} = \{Q: M \times M \rightarrow \mathbb{R}, Q > 0\},$$

equivalently of positive symmetric maps $f_Q: M \rightarrow N_{\mathbb{R}}$. Classically, a positive semi-definite quadratic form Q defines two dual polyhedral decompositions of $M_{\mathbb{R}}$, periodic with respect to translation by M : Voronoi and Delaunay, cf. [55, 55] or [6]. As Q varies continuously, so does $\text{Vor } Q$, but the set of possible Delaunay decompositions is discrete. Locally closed cones of the fan $\mathfrak{F}^{\text{vor}}$ are precisely the subsets of $\overline{\mathcal{C}}$ where the combinatorial type of $\text{Vor } Q$ stays constant, or equivalently where $\text{Del } Q$ stays constant.

A one-parameter degeneration $(X_t, \epsilon \Theta_t)$ of ppavs with an integral monodromy vector $Q \in \mathcal{C}$ can be written as a $\mathbb{Z}^g$ -quotient of an infinite toric variety whose fan in $\mathbb{R} \oplus N_{\mathbb{R}}$ is the cone over a shifted Voronoi decomposition $(1, \ell + f_Q(\text{Vor } Q))$, see [6, Lemma 1.8]. Mikhalkin–Zharkov [43] called the quotient

$$(X_{\text{trop}}, \Theta_{\text{trop}}) = (N_{\mathbb{R}}, \ell + f_Q(\text{Vor } Q))/f_Q(M)$$

a tropical principally polarized abelian variety. It is an integral-affine torus

$$X_{\text{trop}} = N_{\mathbb{R}}/f_Q(M) \simeq (S^1)^g$$

with a tropical divisor Θ_{trop} on it. Then Θ_{trop} induces a cell decomposition of X_{trop} which is the dual complex of the singular central fiber $(X_0, \epsilon \Theta_0)$. The normalization of each component of X_0 is a toric variety, whose fan is modeled by the corresponding vertex of Θ_{trop} .

Kontsevich and Soibelman proposed in [36] that, for K3 surfaces, the real torus X_{trop} should be replaced by an integral-affine structure with 24 singular points on a sphere S^2 (let us call it an IAS^2 for short). This fits into the general framework of the Gross–Siebert program [28], which seeks to understand mirror symmetry near a maximally unipotent degeneration of Calabi–Yau varieties via tropical and integral-affine geometry.

By work of Kulikov [38], Persson–Pinkham [47], and Friedman–Miranda [23], it is understood that a triangulated two-sphere is the combinatorial model for a *Type III Kulikov*

degeneration: a K -trivial, semistable, maximally unipotent, one-parameter family $\mathcal{X} \rightarrow (C, 0)$ of degenerating K3 surfaces. In fact, the dual complex $\Gamma(\mathcal{X}_0)$ of the central fiber admits the structure of a triangulated IAS^2 , cf. [16, 26], which encodes the combinatorial information of $\mathcal{X}_0$. As for ppavs, one uses toric geometry and the triangulation to build the central fiber $\mathcal{X}_0$. The main complication for K3s is that an integral-affine structure on S^2 necessarily has singularities, whereas an integral-affine structure on $(S^1)^g$ is non-singular.

Conversely, from a triangulated IAS^2 B , one can reconstruct a surface $\mathcal{X}_0$ satisfying $\Gamma(\mathcal{X}_0) = B$, which smooths to a Type III degeneration by [20]. This “reconstruction” procedure was used in [16, 18] to study deformations and smoothings of cusp singularities via a crepant resolution of the smoothing. The key innovation in this paper is to introduce an *integral-affine divisor* on an IAS^2 : a weighted 1-dimensional subcomplex $R_{\text{IA}} \subset B$ which is balanced at its vertices. The Kulikov degenerations in [16, 18] used to study cusp singularities were only analytic—in fact non-algebraizable because the central fiber contains a Type VII surface, so there was no integral-affine divisor.

For each vector in a connected component $\mathcal{C} \subset \{\vec{a} \in N \otimes \mathbb{R} \mid \vec{a}^2 > 0\}$, we construct an IAS^2 $B(\vec{a})$ with up to 24 singularities, together with an integral-affine divisor R_{IA} . As $\vec{a} \in \mathcal{C}$ varies continuously, so does the pair $(B(\vec{a}), R_{\text{IA}})$. Dual to the polyhedral decomposition of $B(\vec{a})$ induced by R_{IA} is a discrete subdivision of S^2 with 24 singularities. The set of the dual subdivisions is discrete. Thus, the family of $(B(\vec{a}), R_{\text{IA}})$ varying continuously over $\mathcal{C}$ is the analogue of Vor Q , and the dual subdivisions are the analogues of Del Q .

This family of IAS^2 with integral-affine divisors extends over the rational closure $\overline{\mathcal{C}}$ of the positive cone. As $\vec{a}$ approaches a cusp of $\overline{\mathcal{C}}$, the sphere B collapses to a segment, which are dual complexes of Type II degenerations of K3 surfaces. The cones of the Coxeter fan are exactly the subsets of $\overline{\mathcal{C}}$ where the combinatorial type of the pair $(\text{IAS}^2(\vec{a}), R_{\text{IA}})$ is constant, respectively, where the dual subdivision is constant, in complete analogy with the second Voronoi fan for ppavs.

When the vector $\vec{a}$ is integral and satisfies a certain parity condition, a triangulation of $B(\vec{a})$ into elementary lattice triangles defines a combinatorial type of Kulikov model. By surjectivity of an appropriate period map, cf. [24], these Kulikov models describe all one-parameter degenerations of K3 surfaces with a given Picard–Lefschetz transformation, encoded in the vector $\vec{a}$. The canonical models of these Kulikov models are the stable pairs at the boundary of KSBA moduli. We describe explicitly what curves and components get contracted on the Kulikov model to produce the stable model.

Our IAS^2 are quite different from those appearing in [46]. The main difference is that our pairs $(B(\vec{a}), R_{\text{IA}})$ vary in a PL manner, and so define a polyhedral decomposition of $\overline{\mathcal{C}}$.

The plan of the paper is as follows. In Section 2, we recall the definition of Kulikov models and discuss their connection to integral-affine structures on S^2 . Using symplectic geometry, we state and prove the Monodromy Theorem, allowing one to concretely compute the monodromy invariant of a Kulikov degeneration.

In Section 3, we recall various compactifications of moduli spaces as they apply to K3 surfaces of degree 2, and prove some auxiliary results about them. Section 4 lays out the combinatorics of the Coxeter fan and the corresponding toroidal compactification $\overline{F}_2^{\text{tor}}$ in detail, along with a semitoric compactification $\overline{F}_2^{\text{semi}}$.

In Section 5, we discuss a one-dimensional family of K3 surfaces with Picard rank 19 that is mirror-symmetric to F_2 . For a general surface in this family, its nef cone is isomorphic to a fundamental chamber of the Coxeter fan.

In Section 6, we apply the general theory of polarized IAS^2 to the case at hand, building the family of pairs $(B(\vec{a}), R_{\text{IA}})$ over the Coxeter fan $\mathfrak{F}^{\text{cox}}$. We interpret an integral vector $\vec{a}$ in this fan as a combinatorial type of Kulikov model of K3 surfaces with the monodromy vector $\vec{a}$. In Section 7, we describe explicitly the resulting stable models, in terms of the ADE and $\tilde{A}\tilde{D}\tilde{E}$ surfaces of [7].

Finally, in Section 8, we prove Theorem 1.2. Throughout, we work over $\mathbb{C}$.

2. Kulikov models and IAS^2

2A. Kulikov models and anticanonical pairs. One of the first results about degenerations of K3 surfaces is the well-known theorem of Kulikov and Persson–Pinkham [38, 47].

Theorem 2.1. *Let $\mathcal{X} \rightarrow (C, 0)$ be a flat proper family over a germ of a curve such that the fibers of $\mathcal{X}^* \rightarrow C^* = C \setminus 0$ are projective K3 surfaces. Then there is a finite ramified base change $(C', 0) \rightarrow (C, 0)$ and a birational modification $\mathcal{X}' \rightarrow \mathcal{X} \times_C C'$ such that $\pi: \mathcal{X}' \rightarrow C'$ is semistable (a smooth threefold with $\mathcal{X}'_0$ a reduced normal crossing divisor) with $\omega_{\mathcal{X}'/C'} \simeq \mathcal{O}_{\mathcal{X}'}$.*

Moreover, by Shepherd-Barron [51], for a relatively nef line bundle $\mathcal{L}^*$ on $\mathcal{X}^* \rightarrow C^*$, there is a model as above to which $\mathcal{L}^*$ extends as a nef line bundle $\mathcal{L}$.

Definition 2.2. A degeneration $\mathcal{X} \rightarrow (C, 0)$ satisfying the conclusion of the theorem is a *Kulikov degeneration*, and we call the central fiber a *Kulikov surface*.

Let $\log T$ be the nilpotent logarithm of the unipotent Picard–Lefschetz transformation $T: H^2(\mathcal{X}_t, \mathbb{Z}) \rightarrow H^2(\mathcal{X}_t, \mathbb{Z})$. There are three possible cases for the order of $\log T$, called Types I, II, III.

- (I) If $\log T = 0$, then $\mathcal{X}_0$ is a smooth K3 surface.
- (II) If $(\log T)^2 = 0$ but $\log T \neq 0$, then $\mathcal{X}_0 = \bigcup_{i=1}^n V_i$ is a chain of surfaces with dual complex a segment. The ends V_1 and V_n are rational and V_i for $i \neq 1, n$ are birational to $E \times \mathbb{P}^1$ for a fixed elliptic curve E . The double curves $D_{i,i+1} := V_i \cap V_{i+1}$ are isomorphic to E , the union of the double curves lying on V_i is an anticanonical divisor, and

$$D_{i,i+1}|_{V_i}^2 + D_{i,i+1}|_{V_{i+1}}^2 = 0.$$

- (III) If $(\log T)^3 = 0$ but $(\log T)^2 \neq 0$, then $\mathcal{X}_0 = \bigcup_{i=1}^n V_i$ is a union of rational surfaces whose dual complex is a triangulation of the sphere. The union of all double curves $D_{ij} := V_i \cap V_j$ lying on (the normalization of) V_i form an anticanonical cycle of rational curves. Declaring $D_{ij} \subset V_i$ and $D_{ji} \subset V_j$ and $d_{ij} := -2p_a(D_{ij}) - D_{ij}^2$, we have

$$d_{ij} + d_{ji} = -2.$$

Note that $p_a(D_{ij}) = 0$ unless $D_{ij} \subset V_i$ is an anticanonical cycle of length 1, i.e. an irreducible nodal anticanonical divisor $D_{ij} \in |-K_{V_i}|$.

Every natural compactification of the moduli space of K3 surfaces has strata of Types I, II, III, with Types II, III on the boundary. The three cases are distinguished by the property that, for Type I, the central fiber is smooth, for Type II, the central fiber has double curves but no triple points, and for Type III, the central fiber has triple points.

Definition 2.3. An *anticanonical pair* (V, D) is a smooth rational surface V together with a cycle of smooth rational curves $D \in |-K_V|$.

Definition 2.4. Let (V, D) be an anticanonical pair, with $D = D_1 + \cdots + D_n$. The *charge* is $Q(V, D) := 12 - \sum (D_i^2 + 3)$.

Definition 2.5. A *corner blow-up* of (V, D) is the blow-up at a node of the cycle D , and an *internal blow-up* is a blow-up at a smooth point of D . In both cases, the blow-up has an anticanonical cycle mapping to D . The corner blow-up leaves the charge invariant, while the internal blow-up increases the charge by 1.

For the internal blow-up, the resulting anticanonical cycle is the strict transform of D , whereas for the corner blow-up, it is the reduced inverse image of D .

Remark 2.6. By [22, Lemma 2.7], the pair (V, D) is *toric*, in the sense that V is toric and D is the toric boundary, if and only if $Q(V, D) = 0$. Otherwise, $Q(V, D) > 0$.

We have the following proposition.

Proposition 2.7 (Conservation of charge). *Let $\mathcal{X} \rightarrow (C, 0)$ be a Type III Kulikov degeneration. Then $\sum_{i=1}^n Q(V_i, \sum_j D_{ij}) = 24$. In particular, at most 24 components of $\mathcal{X}_0$ are non-toric.*

Proof. See [23, Proposition 3.7] of Friedman–Miranda. □

As we will see in Section 2C, this proposition presaged the existence of an integral-affine structure on the dual complex $\Gamma(\mathcal{X}_0)$ of the central fiber.

The *combinatorial type* of a Kulikov degeneration is the combinatorial information of the simplicial complex $\Gamma(\mathcal{X}_0)$, together with the deformation type of each irreducible component $(V_i, \sum_j D_{ij})$, which, in particular, determines (but is not always determined by) the collection of integers d_{ij} .

The remaining data is continuous: one must choose a point in the deformation space of anticanonical pairs for each component, and choose how to glue double curves D_{ij} . These moduli are parameterized by a torus $(\mathbb{C}^*)^N$ of some large dimension, but for $\mathcal{X}_0$ to be smoothable, we must choose the gluings and moduli of V_i carefully. A theorem of Friedman [20] states that *d-semistability*

$$\text{Ext}^1(\Omega_{\mathcal{X}_0}^1, \mathcal{O}_{\mathcal{X}_0}) = 1 \in \text{Pic}^0((\mathcal{X}_0)_{\text{sing}}) \simeq (\mathbb{C}^*)^{\#\{V_i\}-1}$$

is a necessary and sufficient condition for smoothability.

By [24], the logarithm of monodromy in Types II and III is given by

$$\log T: x \mapsto (x \cdot \delta)\lambda - (x \cdot \lambda)\delta$$

for elements $\delta, \lambda \in H^2(\mathcal{X}_t, \mathbb{Z})$ satisfying

$$\delta^2 = \delta \cdot \lambda = 0, \quad \delta \text{ primitive,} \quad \text{and} \quad \lambda^2 = \#\{\text{triple points of } \mathcal{X}_0\}.$$

Thus, $\lambda^2 = 0$ if the degeneration is Type II.

Definition 2.8. Let $\mathcal{X} \rightarrow C$ be a Type III degeneration. We call $\delta \in H^2(\mathcal{X}_t, \mathbb{Z})$ the *vanishing cycle* and the vector $\lambda \in \delta^\perp / \delta$ the *monodromy invariant*. If the family $\mathcal{X} \rightarrow C$ is polarized by L , the vanishing cycle and monodromy invariant are defined similarly, but with reference to the ambient lattice $c_1(L)^\perp \subset H^2(\mathcal{X}_t, \mathbb{Z})$.

By taking a Kulikov model and setting

$$\begin{aligned} I &= \mathbb{Z}\delta && \text{if } \mathcal{X} \rightarrow C \text{ is Type III,} \\ J &= (\mathbb{Z}\delta \oplus \mathbb{Z}\lambda)^{\text{sat}} && \text{if } \mathcal{X} \rightarrow C \text{ is Type II,} \end{aligned}$$

any degeneration of K3 surfaces determines a primitive isotropic sublattice of $H^2(\mathcal{X}_t, \mathbb{Z})$.

2B. Integral-affine structures: General definitions.

Definition 2.9. An *integral-affine structure* on a real surface S is a collection of charts from S to $\mathbb{R}^2$ such that the transition functions lie in $\text{SL}_2(\mathbb{Z}) \ltimes \mathbb{R}^2$.

Definition 2.10. The *monodromy representation* $\rho: \pi_1(S, *) \rightarrow \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{R}^2$ is constructed by patching together charts along a loop $\gamma \in \pi_1(S, *)$ in the unique way such that they glue on overlaps, then comparing the chart at the end of the loop with the one at the beginning. This process of patching charts together defines the *developing map* from the universal cover $\tilde{S} \rightarrow \mathbb{R}^2$ which is equivariant with respect to ρ . Usually, we further project the monodromy to the group $\text{SL}_2(\mathbb{Z})$.

As defined, the two-sphere admits no integral-affine structures. One must introduce a reasonable class of singularities of such structures.

Definition 2.11. An I_1 *singularity* is the germ of a singular integral-affine surface isomorphic to the following basic example.

Cut from $\mathbb{R}^2 = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ the convex cone with the sides $\mathbb{R}_{\geq 0}e_2$ and $\mathbb{R}_{\geq 0}(e_2 - e_1)$, as on the left in Figure 1, and glue one boundary ray to another by a shear in the e_1 -direction, i.e. by the rule $e_1 \mapsto e_1, e_2 \mapsto -e_1 + e_2$.

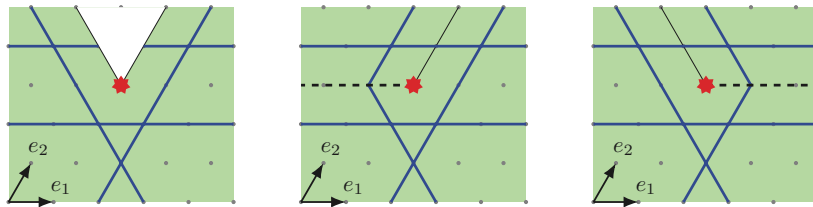


Figure 1. Three representations of the I_1 singularity.

Three straight lines in the affine structure are shown in bold blue. The second and third figures in Figure 1 also represent the I_1 singularity, with a dashed ray in the monodromy-invariant direction removed. The image of the developing map is $\mathbb{R}^2$ minus the ray. We can visualize this presentation as taking the standard affine structure on $\mathbb{R}^2$ minus the ray, then gluing across the ray by a shear.

Remark 2.12. The I_1 singularity can be presented by removing any ray emanating from the singularity. When this ray is not in a monodromy-invariant direction, the two sides of the ray separate to produce a gap as in the left-hand figure.

Definition 2.13. Let $\vec{v}_1, \dots, \vec{v}_k$ be a sequence of primitive integral vectors, ordered cyclically counterclockwise around the origin. Define an *integral-affine singularity*

$$(S, p) = I(n_1 \vec{v}_1, \dots, n_k \vec{v}_k)$$

to be the result of shearing the affine structure of $\mathbb{R}^2$ a total of n_i times along $\mathbb{R}_{\geq 0} \vec{v}_i$.

Let $M(\vec{v})$ be the unique matrix conjugate in $\mathrm{SL}_2(\mathbb{Z})$ to $(e_1, e_2) \mapsto (e_1, e_1 + e_2)$ such that $\vec{v} M(\vec{v}) = \vec{v}$, i.e. $M(\vec{v})$ is the unit shear along $\vec{v}$. Then the $\mathrm{SL}_2(\mathbb{Z})$ monodromy of a counterclockwise loop around the singularity (S, p) is the product $M(S, p) = M(\vec{v}_1)^{n_1} \dots M(\vec{v}_k)^{n_k}$.

We can view $I(n_1 \vec{v}_1, \dots, n_k \vec{v}_k)$ as the collision of $n_1 + \dots + n_k$ I_1 singularities, with monodromy invariant directions along the $\vec{v}_i$.

Definition 2.14. The *charge* of a singularity (S, p) is the number $\sum_{i=1}^k n_i$ of rays sheared to produce it, counted with multiplicity. For instance, the I_1 singularity $I(\vec{v})$ has charge one.

Definition 2.15. An integral-affine sphere B , or IAS^2 for short, is a sphere $B = S^2$ and a finite set $\{p_1, \dots, p_n\} \in B$ such that $B \setminus \{p_1, \dots, p_n\}$ has a non-singular integral-affine structure, and a neighborhood of each p_i is modeled by some integral-affine singularity

$$I(n_1 \vec{v}_1, \dots, n_k \vec{v}_k).$$

Proposition 2.16. Let B be an integral-affine structure with singularities on a compact oriented surface of genus g . Then the sum of the charges is $12(2 - 2g)$.

Proof. See [36] or [17]. □

Remark 2.17. The shearing directions $\vec{v}_i$ used to construct each singularity form part of the definition of B . Thus, two IAS^2 may not be isomorphic even if there is a homeomorphism $B_1 \rightarrow B_2$ which is an integral-affine isomorphism away from the singular sets. We discuss the appropriate equivalence relation below.

Definition 2.18. Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{Z}^2$ be three vectors so that $(\vec{u}, \vec{v})$ form an oriented basis and $\vec{u} + \vec{v} + \vec{w} = 0$. As a further shortcut, we define $I(p) = I(p\vec{u})$, called an I_p singularity. Let $I(p, q) = I(p\vec{u}, q\vec{v})$, and $I(p, q, r) = I(p\vec{u}, q\vec{v}, r\vec{w})$. Up to the action of $\mathrm{SL}_2(\mathbb{Z})$, this notation is symmetric under cyclic rotations. Finally, we set $I(p, q, r, s) = I(p\vec{u}, q\vec{v}, r(-\vec{u}), s(-\vec{v}))$, also symmetric up to cyclic rotation.

2C. Pseudo-fans and Kulikov models. In this section, we describe how to encode a deformation type of anticanonical pairs as an integral-affine surface singularity, and in turn how to encode a Type III Kulikov model as an IAS².

Definition 2.19. The *pseudo-fan* of an anticanonical pair $\mathfrak{F}(V, D)$, see [26, Section 1.2] or [16, Definition 3.8], is a triangulated integral-affine surface with boundary constructed as follows.

As a PL surface, $\mathfrak{F}(V, D)$ is the cone over the dual complex of D . The affine structure on each triangle in this cone is declared integral-affine equivalent to a lattice triangle of lattice volume 1. Two adjacent triangles are glued by the following rule. Let $\vec{e}_j$ be the directed edge of $\mathfrak{F}(V, D)$ emanating from the cone point and pointing towards the vertex corresponding to D_j . In a chart containing the union of the two adjacent triangles containing $\vec{e}_j$, we have $\vec{e}_{j-1} + \vec{e}_{j+1} = d_j \vec{e}_j$, where $d_j = -D_j^2$ if D_j is smooth and $d_j = -D_j^2 + 2$ if D_j is a rational nodal curve.

Remark 2.20. When (V, D) is a toric pair, the pseudo-fan $\mathfrak{F}(V, D)$ is a non-singular integral-affine surface with a single chart to a polygon in $\mathbb{R}^2$. The vertices of this polygon are the endpoints of the primitive integral vectors pointing along the 1-dimensional rays of the fan of (V, D) .

Remark 2.21. A *toric model* $\pi: (V, D) \rightarrow (\bar{V}, \bar{D})$ is a blow-down to a toric pair. After some corner blow-ups, every anticanonical pair admits a toric model, see [27, Proposition 1.3]. Assume that π consists only of internal blow-ups, as corner blow-ups do not affect toricity. Then [16, Proposition 3.13] implies $\mathfrak{F}(V, D)$ is the result of shearing along the rays of the fan of $(\bar{V}, \bar{D})$ corresponding to components which get blown up. Hence, by Definition 2.13, every integral-affine surface singularity is the cone point of the pseudo-fan of some anticanonical pair, and by subdividing the singularity into standard affine cones, the converse is also true.

Furthermore, the charge $Q(V, D)$ coincides with the charge of the corresponding singularity $\mathfrak{F}(V, D)$. It is the number of internal blow-ups of the toric model.

Let $\mathcal{X} \rightarrow C$ be a Type III degeneration. We label the vertices of the dual complex $\Gamma(\mathcal{X}_0)$ by v_i , the edges by e_{ij} , and the triangles by t_{ijk} , corresponding respectively to the components, double curves, and triple points of $\mathcal{X}_0$. Let $\text{star}(v_i)$ be the union of the triangles containing v_i .

Proposition 2.22. *The dual complex $\Gamma(\mathcal{X}_0)$ of a Type III degeneration of K3 surfaces admits a natural integral-affine structure such that*

$$\text{star}(v_i) = \mathfrak{F}\left(V_i, \sum_j D_{ij}\right).$$

Conversely, given an integral-affine structure B on the two-sphere with a triangulation into lattice triangles of lattice volume 1 and singularities at the vertices, there is a Type III degeneration $\mathcal{X} \rightarrow C$ such that $\Gamma(\mathcal{X}_0) = B$.

Here, *lattice volume* means twice the Euclidean area.

Proof. See [16] or [26, Remark 1.11v1]. The key point is that the pseudo-fans of the components compatibly glue to form a well-defined integral affine structure on any quadrilateral formed from two adjacent triangles of $\Gamma(\mathcal{X}_0)$. This follows from the formula $d_{ij} + d_{ji} = 2$ in (III), below Definition 2.2. $\square$

Definition 2.23. Two anticanonical pairs (V_1, D_1) and (V_2, D_2) lie in the same *corner blow-up equivalence class* (c.b.e.c.) if they are related by a sequence of corner blow-ups and blow-downs, and a topologically trivial deformation. A *toric model* of a c.b.e.c. is a representative (V, D) of the equivalence class, and a toric model $(V, D) \rightarrow (\bar{V}, \bar{D})$.

Note that all topologically trivial deformations of (V, D) are the result of deforming the points on $\bar{D}$ which are blown up.

By Remark 2.21, a toric model of an anticanonical pair (V, D) determines an integral-affine singularity at the cone point of $\mathfrak{F}(V, D)$. Corner blow-ups subdivide the pseudo-fan, which do not affect the singularity. Neither do topologically trivial deformations. We conclude that there is a bijection between presentations $I(n_1 \vec{v}_1, \dots, n_k \vec{v}_k)$ of integral-affine singularities by shears and toric models of c.b.e.c.s. We now forget the dependence on the toric model.

Definition 2.24. Two integral-affine singularities are *equivalent*,

$$(S_1, p_1) = I(n_1 \vec{v}_1, \dots, n_k \vec{v}_k) \sim I(m_1 \vec{w}_1, \dots, m_{\text{ell}} l \vec{w}_{\text{ell}} l) = (S_2, p_2),$$

if the corresponding c.b.e.c.s are equal $[(V_1, D_1)] = [(V_2, D_2)]$.

By choosing a single anticanonical pair (V, D) which admits both toric models corresponding to $\vec{v}_i$ and to $\vec{w}_j$, and building $\mathfrak{F}(V, D)$ by the recipe in Definition 2.19 (which does not use a toric model), an equivalence of integral-affine singularities provides a homeomorphism $(S_1, p_1) \rightarrow (S_2, p_2)$ which is an integral-affine isomorphism away from the p_i . But the converse is false, see [18, Example 4.13]. Such examples explain why it does not suffice to define an integral-affine singularity as purely a geometric structure—the presentation via shears (at least up to equivalence) is part of the definition.

Remark 2.25. Each toric model of the c.b.e.c. of (V, D) defines a Zariski open subset of the open Calabi–Yau $(\mathbb{C}^*)^2 \hookrightarrow V \setminus D$. One may choose a different toric model by changing exactly one exceptional curve E blown down in the toric model—to a curve F such that $E + F$ is the fiber of a toric ruling. The change-of-coordinates to the new inclusion $(\mathbb{C}^*)^2 \hookrightarrow V \setminus D$ is a birational map called a *cluster mutation*. It is almost always the case that there are infinitely many such cluster charts. Any two toric models of a c.b.e.c. are connected by a series of cluster mutations, by a theorem of Blanc [11].

Example 2.26. Start with the toric pair $(\mathbb{P}^2, L_1 + L_2 + L_3)$ and make a corner blow-up to get $(\mathbb{F}_1, s_0 + f + s_\infty + f)$, with $s_0^2 = 1$, $s_\infty^2 = -1$. Blow up one point on s_0 , then contract one exceptional curve intersecting s_∞ to obtain $\mathbb{P}^1 \times \mathbb{P}^1$. This corresponds to a single cluster mutation as in Remark 2.25. We may also blow up p points on the first copy of f , q points on the second copy of f , and one more point on s_0 . In this way, we see the equivalences

$$I(2, p, q) \sim I(1, p, 1, q) \sim I(p, 1, q, 1) \sim I(2, q, p).$$

2D. Birational modifications and base change. All Kulikov models $\mathcal{X} \rightarrow (C, 0)$ completing a punctured family $\mathcal{X}^* \rightarrow C^*$ are related by flops along smooth rational curves. The modifications which change the isomorphism type of $\mathcal{X}_0$ are the following.

- (1) *M1 modifications* are Atiyah flops along an exceptional curve $E \subset V_i$ meeting a double curve D_{ij} at a single point p . The effect on $\mathcal{X}_0$ is to blow down E on V_i and blow up V_j at p .
- (2) *M2 modifications* are Atiyah flops along an exceptional double curve $E = D_{ij} = V_i \cap V_j$. The effect on $\mathcal{X}_0$ is to blow down E on both V_i and V_j , blow up the two triple points T_{ijk} and T_{ijl} contained in E , on the components V_k and V_l , and then glue the resulting exceptional curves.

Definition 2.27. Let $(S, p) = I(n_1 \vec{v}_1, \dots, n_k \vec{v}_k)$ be an integral-affine singularity. A *nodal slide along $\vec{v}_i$ of length t* , cf. [52, Definition 6.1], is a surgery on the integral affine structure (S, p) which translates by $t \vec{v}_i$ the originating point of one shearing ray in the direction $\vec{v}_i$.

Note that nodal slides are called *moving worms* in the mirror symmetry literature, see e.g. [36] or [26].

Starting with the single singularity (S, p) , the nodal slide results in an integral-affine surface with two singularities $I(n_1 \vec{v}_1, \dots, (n_i - 1) \vec{v}_i, \dots, n_k \vec{v}_k)$ and an I_1 singularity at the endpoint of $t \vec{v}_i$. The result is an integral-affine surface which is isomorphic to the original one on the complement of the segment $t \vec{v}_i$. Thus, the operation is purely local and can be done independently of the rest of the integral affine surface. For appropriately large t , a nodal slide may result in the I_1 singularity sent off colliding into another singularity.

In fact, any integral-affine singularity can be defined as the result of colliding a collection of I_1 singularities moving along nodal slides.

Proposition 2.28 ([18, Propositions 4.5, 4.6]). *An M2 modification does not change the IAS^2 structure on $B = \Gamma(\mathcal{X}_0)$, but retriangulates B by cutting along the opposite diagonal of an integral-affine unit square.*

An M1 modification preserves the triangulation of $\Gamma(\mathcal{X}_0)$, but changes the IAS^2 B by a unit length nodal slide, moving an I_1 singularity along $\vec{e}_{ij}$ from v_i to v_j .

A sequence of M1 and M2 modifications connecting two Kulikov surfaces $\mathcal{X}_0 \dashrightarrow \mathcal{X}'_0$ is thus modeled as a sequence of retriangulations and integer length nodal slides

$$\Gamma(\mathcal{X}_0) \dashrightarrow \Gamma(\mathcal{X}'_0)$$

on the corresponding dual complexes.

Proposition 2.29 ([19]). *Let $\mathcal{X} \rightarrow (C, 0)$ be a Kulikov model, and consider the base change $\mathcal{X}' \rightarrow (C', 0)$ ramified over 0 to order N . There is a standard resolution $\mathcal{X}[N] \rightarrow \mathcal{X}'$, producing a new Kulikov model whose central fiber $\mathcal{X}_0[N]$ is the result of inserting “special bands of hexagons” of width N between all the components of $\mathcal{X}_0$. The effect on the dual complex $\Gamma(\mathcal{X}_0)$ is to take the standard refinement every triangle into N^2 triangles (see also Claim 3.15 below).*

In fact, the integral-affine structure on the dual complex $B[N] := \Gamma(\mathcal{X}_0(N))$ is the result of post-composing the integral-affine charts $U \rightarrow \mathbb{R}^2$ on $B = \Gamma(\mathcal{X}_0)$ with multiplication by N , cf. [18, Proposition 4.3]. We call this the *order N refinement* of B . Note that the base change multiplies the monodromy invariant $\lambda \mapsto N\lambda$.

2E. Integral-affine divisors. In this section, we define an integral-affine divisor on an IAS^2 . For motivation, consider a line bundle $\mathcal{L} \rightarrow \mathcal{X}$ on a Kulikov model. Let

$$L_i := \mathcal{L}|_{V_i} \in \text{Pic}(V_i).$$

These line bundles automatically satisfy a compatibility condition $L_i \cdot D_{ij} = L_j \cdot D_{ji}$. Thus, we have the following definition.

Definition 2.30. Let B be an IAS^2 . An *integral-affine divisor* R_{IA} on B consists of two pieces of data.

- (1) A weighted graph $R_{\text{IA}} \subset B$ with vertices v_i , straight line segments as edges e_{ij} , and integer labels n_{ij} on each edge.
- (2) Let $v_i \in R_{\text{IA}}$ be a vertex and (V_i, D_i) be an anticanonical pair such that $\mathfrak{F}(V_i, D_i)$ models v_i and contains all edges of e_{ij} coming into v_i . We require the data of a line bundle $L_i \in \text{Pic}(V_i)$ such that $\deg L_i \cdot D_{ij} = n_{ij}$ for the components D_{ij} of D_i corresponding to edges e_{ij} and L_i has degree zero on all other components of D_i .

Definition 2.31. Given a line bundle $\mathcal{L} \rightarrow \mathcal{X}$ on a Kulikov degeneration, the intersection numbers $n_{ij} = L_i \cdot D_{ij}$ define an integral-affine divisor $R_{\text{IA}} \subset B = \Gamma(\mathcal{X}_0)$ supported on the 1-skeleton. If $\mathcal{L}$ is nef, then R_{IA} is effective, i.e. $n_{ij} \geq 0$.

Remark 2.32. When $v_i \in R_{\text{IA}}$ is non-singular, the pair (V_i, D_i) is toric, and the labels n_{ij} uniquely determine L_i . They must satisfy a balancing condition. If v_{ij} are the primitive integral vectors in the directions e_{ij} , then one must have $\sum n_{ij} v_{ij} = 0$ for such a line bundle $L_i \rightarrow V_i$ to exist.

Similarly, if $I_1 = \mathfrak{F}(V_i, D_i) = I(\vec{v})$, i.e. (V_i, D_i) is the result of a single internal blow-up of a toric pair, the n_{ij} determine a unique line bundle L_i so long as $\sum n_{ij} v_{ij} \in \mathbb{Z}\vec{v}$. This condition is well-defined as the v_{ij} are well-defined up to shears in the $\vec{v}$ direction.

Definition 2.33. We say that a divisor on B is *polarizing* if each line bundle L_i is nef and at least one L_i is big. The *self-intersection* of an integral-affine divisor is

$$R_{\text{IA}}^2 := \sum_i L_i^2 \in \mathbb{Z}.$$

Definition 2.34. An IAS^2 is *generic* if it has 24 distinct I_1 singularities.

Remark 2.35. Let B be a lattice triangulated IAS^2 , or equivalently, $B = \Gamma(\mathcal{X}_0)$ is the dual complex of a Type III degeneration. Then B is generic if and only if $Q(V_i, D_i) \in \{0, 1\}$ for all components $V_i \subset \mathcal{X}_0$. When B is generic, an integral-affine divisor $R_{\text{IA}} \subset B$ is uniquely specified by a weighted graph satisfying the balancing conditions of Remark 2.32, so the extra data (2) of Definition 2.30 is unnecessary.

Definition 2.36. An integral-affine divisor $R_{\text{IA}} \subset B$ is *compatible* with a triangulation if every edge of R_{IA} is formed from edges of the triangulation.

If B comes with a triangulation, we require the integral-affine divisor to be compatible with it.

2F. Integral-affine structures from Lagrangian torus fibrations. The reference for this section is Symington [52]. Let (S, ω) be a smooth symplectic 4-manifold. Given a Lagrangian torus fibration $\mu: (S, \omega) \rightarrow B$ with only nodal singularities, the base B inherits a natural integral-affine structure with an I_n singularity under a necklace of n two-spheres.

Definition 2.37. Let C_α and C_β be cylinders in S fibering over a path from a fixed base point $* \in B$ to a point $p \in B$ such that the ends of the cylinders over $*$ are homologous to α and β , an oriented basis of $H_1(S_*, \mathbb{Z})$. The *induced integral affine structure* on B is the collection of charts of the form

$$p \mapsto (x(p), y(p)) = \left(\int_{C_\alpha} \omega, \int_{C_\beta} \omega \right) \in \mathbb{R}^2.$$

These charts are only defined up to monodromy in $\text{SL}(2, \mathbb{Z}) \ltimes \mathbb{R}^2$, by choosing a path in a different homotopy class and moving the base point $*$.

Let $\bar{T}$ be a complex toric surface, $\bar{L} \in \text{Pic}(\bar{T}) \otimes \mathbb{R}$ an ample class, and $\bar{\omega}$ a symplectic form with $[\bar{\omega}] = \bar{L}$. The moment map $\mu_{\bar{T}}: (\bar{T}, \bar{\omega}) \rightarrow \bar{P}$ is a Lagrangian torus fibration which induces the integral-affine structure on the moment polytope $\bar{P}$ coming from its embedding into $\mathbb{R}^2$. It degenerates over the toric boundary $\bar{D} \subset \bar{T}$ and sends the components of $\bar{D}$ to the boundary components of $\bar{P}$.

Now let $\phi: T \rightarrow \bar{T}$ be a blow-up at a smooth point of the boundary $\bar{D}$, with exceptional divisor E . Symington [52] constructed a Lagrangian torus fibration $\mu_T: (T, \omega) \rightarrow P$ satisfying $[\omega] = \phi^*[\bar{\omega}] - aE$ over a singular integral-affine disk P (a “Symington polytope”) obtained as follows.

Definition 2.38. A *Symington surgery* is the result of cutting a triangle of lattice size a (and lattice volume a^2) from the side of the moment polytope $\bar{P}$ corresponding to the component blown up, then gluing the two remaining edges, introducing an I_1 singularity $p \in P$ at the interior corner of the triangle.

The fiber over p is an irreducible nodal I_1 fiber of the torus fibration. In symplectic geometry, this procedure is called an *almost toric blow-up*. The monodromy axis of the singularity is parallel to the side of $\bar{P}$ on which the surgery triangle rests and the location of the cut on the side of $\bar{P}$ is essentially arbitrary.

Construction 2.39. Let B be a generic IAS² and let $B^\circ = B \setminus \{p_1, \dots, p_{24}\}$ be its non-singular locus. Let $\gamma = \sum_i (\gamma_i, \alpha_i) \subset B$ be a 1-chain with values in the constructible sheaf $T_{\mathbb{Z}} := i_*(T_{\mathbb{Z}} B^\circ)$, where $i: B^\circ \hookrightarrow B$ is the inclusion. This sheaf is a $\mathbb{Z}$ -local system of rank 2 on B° and has rank 1 at the I_1 singularities.

Concretely, γ is a collection of oriented paths $\gamma_i \rightarrow B$ and a (constant) integral vector field α_i on each path. There is a boundary map ∂ to 0-chains with values in $T_{\mathbb{Z}}$ gotten by taking

an oriented sum of the tangent vectors α_i at the endpoints of γ_i . We say that γ is a 1-cycle if $\partial\gamma = 0$. Some care must be taken at the singularities, where the rank of $T_{\mathbb{Z}}$ drops. Here, the condition that the boundary is zero means that $\sum \alpha_i$ is parallel to the monodromy-invariant direction of the singularity.

From such a 1-cycle γ , we may construct a PL surface $\Sigma_\gamma \subset S$ inside the symplectic 4-manifold with a Lagrangian torus fibration $\mu: (S, \omega) \rightarrow B$. We take a cylinder in S which maps to γ_i whose fibers are the circles in the torus fiber that correspond to α_i via the symplectic form. The condition that $\partial\gamma = 0$ is exactly the condition that the ends of these cylinders over the points in $\bigcup_i \partial\gamma_i$ are null-homologous in the fiber. Thus, we may glue in a (Lagrangian) 2-chain contained in the fiber over $\bigcup_i \partial\gamma_i$ and produce a closed PL surface Σ_γ .

Definition 2.40. The surfaces Σ_γ constructed as above are the *visible surfaces*.

Example 2.41. Given a path γ connecting two I_1 singularities p and q such that the monodromy-invariant directions at both p and q are parallel to α , the 1-cycle (γ, α) defines a visible surface, which we denote $E_{(\gamma, \alpha)}$. It satisfies $E_{(\gamma, \alpha)}^2 = -2$ because $E_{(\gamma, \alpha)}$ is attached to each nodal fiber S_p, S_q by a (-1) -framed 2-handle.

Note that Σ_γ is non-canonical even on the level of its homology class: there are many choices of Lagrangian 2-chains in the fibers over $\bigcup_i \partial\gamma_i$. But they all differ by some multiple of the fiber class $f = [\mu^{-1}(p)]$. Note that also $[\Sigma_\gamma] \cdot f = 0$. We do have a well-defined class $[\Sigma_\gamma] \in f^\perp/f$.

We note an important special case of the above construction.

Definition 2.42. Suppose that all γ_i 's are straight line segments e_{ij} forming a graph in the integral-affine structure on B , and that the tangent vector field is an integer multiple n_{ij} of the primitive integral tangent vector along γ_i . Then the cylinder lying over e_{ij} can be made Lagrangian and the surface Σ_γ is a PL Lagrangian surface in (S, ω) . We call the result a *Lagrangian visible surface*.

In particular, the class of a Lagrangian visible surface satisfies $[\Sigma_\gamma] \cdot [\omega] = 0$. Observe that the condition that γ is a 1-cycle is exactly the balancing condition of Remark 2.32. Thus, an integral-affine divisor R on B in the sense of Definition 2.30 corresponds to a Lagrangian visible surface Σ_R .

2G. The Monodromy Theorem. Our goal now is to understand the vanishing cycle δ , monodromy invariant λ , and polarization of a Kulikov degeneration $\mathcal{X} \rightarrow C$, see Definition 2.8, in terms of IAS² and symplectic geometry. We now prove a version of [18, Proposition 3.14], the key new ingredient being the presence of a polarizing divisor R .

Theorem 2.43. *Let B be a generic IAS², together with a triangulation into lattice triangles of lattice volume 1.*

- (1) *Let $\mathcal{X} \rightarrow C$ be a Type III Kulikov degeneration such that $\Gamma(\mathcal{X}_0) = B$.*
- (2) *Let $\mu: (S, \omega) \rightarrow B$ be a Lagrangian torus fibration over the same B .*

Then there exists a diffeomorphism $\phi: S \rightarrow \mathcal{X}_t$ to a nearby fiber $t \neq 0$ such that

- (a) $\phi_* f = \delta$,
- (b) $\phi_*[\omega] = \lambda$ in $\delta^\perp/\delta \otimes \mathbb{R}$.

Moreover, suppose that $\mathcal{L} \rightarrow \mathcal{X}$ is a line bundle, which defines the integral-affine divisor R_{IA} on B . Let $\Sigma_{R_{\text{IA}}}$ be the corresponding Lagrangian visible surface in S . Then we have

- (c) $\phi_*[\Sigma_{R_{\text{IA}}}] = c_1(\mathcal{L}_t)$ in $\delta^\perp/\delta$.

Proof. We first prove (a) and (b) following [18, Proposition 3.14] closely. There, an almost exactly analogous statement is proved for Type III degenerations of anticanonical pairs, so we only describe the minor modifications necessary. We ignore the parts of the proof in [18] which refer to D , and similarly the special component of $\mathcal{X}_0$ equal to the hyperbolic Inoue surface, instead treating all surfaces $V_i \subset \mathcal{X}_0$ on equal footing. Then the construction of ϕ proceeds the same way, by using the Clemens collapse to show that (S, ω) and $\mathcal{X}_t$ can be written as the same fiber connect-sum of 2-torus fibrations over the intersection complex $\Gamma(\mathcal{X}_0)^\vee$. Statement (a) follows immediately.

Again following [18], we consider the collection of Lagrangian visible surfaces Σ_γ which fiber over the 1-skeleton $\Gamma(\mathcal{X}_0)^{[1]}$. The images under ϕ_* of the classes $[\Sigma_\gamma]$ generate a 19-dimensional lattice in $\delta^\perp/\delta$ invariant under the Picard–Lefschetz transformation $H^2(\mathcal{X}_t; \mathbb{Z}) \rightarrow H^2(\mathcal{X}_t; \mathbb{Z})$. Since $[\omega] \cdot [\Sigma_\gamma] = 0$, we conclude that the monodromy invariant λ and $\phi_*[\omega]$ are proportional in $\delta^\perp/\delta$. By [24],

$$\lambda^2 = \#\{\text{triple points of } \mathcal{X}_0\} = \text{vol}(\Gamma(\mathcal{X}_0)) = [\omega]^2.$$

We conclude that $\lambda = \phi_*[\omega] \bmod \mathbb{Z}\delta$, i.e. (b).

Now suppose that $\mathcal{X}$ admits a line bundle $\mathcal{L}$. There is an integral-affine divisor R on $\Gamma(\mathcal{X}_0)$ whose defining line bundles $L_i \in \text{Pic}(V_i)$ are $\mathcal{L}|_{V_i}$. Since $\Gamma(\mathcal{X}_0)$ is generic, these line bundles are uniquely determined by the integer weights $n_{ij} = L_i \cdot D_{ij}$ on the edges of $\Gamma(\mathcal{X}_0)^{[1]}$. By construction, the Lagrangian visible surface $\Sigma_{R_{\text{IA}}} \subset S$ fibering over the weighted balanced graph R_{IA} is sent by ϕ to a surface whose Clemens collapse is a union of surfaces $\Sigma_i \subset V_i$ satisfying

- (1) $\Sigma_i \cap D_{ij} = \Sigma_j \cap D_{ji}$,
- (2) $\Sigma_i \cdot D_{ij} = L_i \cdot D_{ij}$.

These conditions uniquely determine the class $\phi_*[\Sigma_{R_{\text{IA}}}]$. We conclude (c). $\square$

Remark 2.44. Statements similar to Theorem 2.43 (a) and (b) have appeared in the mirror symmetry literature. For instance, [29, Theorem 5.1] computes the monodromy of the Picard–Lefschetz transform of a toric degeneration of Calabi–Yau varieties in terms of cup product with the *radiance obstruction*

$$c_B \in H^1(B, i_*(T_{\mathbb{Z}} B^o)),$$

a cohomology class canonically associated to an integral-affine structure, first studied in [25]. The class c_B is identified with $[\omega]$ via the Leray spectral sequence of the map $\mu: (S, \omega) \rightarrow B$. These monodromy formulas verify the prediction of topological SYZ mirror symmetry that the Picard–Lefschetz transformation is cup product with a section of the SYZ fibration. See also [46, Corollary 4.24].

3. Compactifications of F_2

We first recall the basics about the moduli spaces of K3 surfaces as they apply to the degree 2 case. For the Baily–Borel and toroidal compactifications, a convenient reference is [49]. Then we describe a compactification via stable pairs and prove some auxiliary results about it.

3A. Period domain and moduli space. Let $\Lambda_{K3} \simeq H^3 \oplus E_8^2$ be a fixed lattice of signature $(3, 19)$ isomorphic to $H^2(S, \mathbb{Z})$ for a K3 surface S . Here, H is the hyperbolic plane, and the lattice E_8 for convenience is negative definite. All primitive vectors of square $h^2 = 2d$ lie in the same orbit of the isometry group of Λ_{K3} . The lattice $h^\perp$ is isometric to $H^2 \oplus E_8^2 \oplus \langle -2d \rangle$. The period domain for the polarized K3 surfaces of degree $2d$ is a connected component of

$$\mathbb{D} = \mathbb{D}_{2d} := \mathbb{P}\{x \in h^\perp \otimes \mathbb{C} \mid x \cdot x = 0, x \cdot \bar{x} > 0\},$$

a Hermitian symmetric domain associated to the group $O^+(2, 19)$. On it, we have the action of the group $\Gamma = \Gamma_{2d}$ which is the spinor norm 1 subgroup of the stabilizer of h in the isometry group $O(\Lambda_{K3})$. By the Torelli theorem, the quotient space $F_{2d} = \Gamma \backslash \mathbb{D}$ is the coarse moduli space of polarized K3 surfaces (X, L) , where X is a K3 surface with ADE (Du Val) singularities, and L is an ample line bundle with $L^2 = 2d$. One has $\dim F_{2d} = \dim \mathbb{D}_{2d} = 19$.

The moduli stack $\mathcal{F}_{2d}$ of polarized K3 surfaces of degree $2d$ is a smooth DM stack. This stack and its coarse moduli space F_{2d} are incomplete, and F_{2d} is quasiprojective.

3B. Baily–Borel compactification. Let $\mathbb{D}^\vee$ denote the compact dual of $\mathbb{D}$; it is the quadric defined by dropping the condition $x \cdot \bar{x} > 0$. Let $\overline{\mathbb{D}} \subset \mathbb{D}^\vee$ be the topological closure. Let I be a primitive isotropic sublattice of $h^\perp$. Then I has rank one or two. One calls the former Type III and the latter Type II. The *boundary component* associated to I is by definition

$$F_I := \mathbb{P}\{x \in \overline{\mathbb{D}} \mid \text{span}\{\text{Re}(x), \text{Im}(x)\} = I \otimes \mathbb{R}\} \subset \mathbb{D}^\vee$$

which is either a 0-cusp, a point for Type III or a 1-cusp, a copy of $\mathbb{H}$ for Type II ($\mathbb{H}$ is the upper-half plane).

Notation 3.1. To distinguish the ranks, we henceforth use I or J for rank 1 or 2 primitive isotropic lattices, respectively.

Then the Baily–Borel compactification is, topologically,

$$\overline{F}_{2d}^{\text{BB}} := \Gamma \backslash (\mathbb{D} \cup_J F_J \cup_I F_I).$$

In $\overline{F}_2^{\text{BB}}$, the boundary consists of four curves, meeting at a single point, see [49]. The point is the Type III boundary, while the curves (minus the point) are the Type II boundary. The curves correspond to four distinct orbits of rank 2 primitive isotropic sublattices $J \subset h^\perp$. For each of them, $J^\perp/J$ contains a finite index root sublattice, which can be used as a label for this 1-cusp,

$$A_{17}, D_{10} \oplus E_7, E_8^2 \oplus A_1, \text{ and } D_{16} \oplus A_1.$$

The stabilizer $\text{Stab}_\Gamma(J) \subset \Gamma$ acts on $J \simeq \mathbb{Z}^2$ by a finite index subgroup $\Gamma_J \subset \text{SL}_2(\mathbb{Z})$, and the boundary component $\Gamma_J \backslash F_J$ is a modular curve corresponding to a Type II boundary

curve. One has a natural finite morphism $\Gamma_J \backslash F_J \rightarrow \mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H} = \mathbb{A}_J^1$ to the j -line. Thus, the boundary of the Baily–Borel compactification has codimension 18.

3C. Toroidal compactifications. Toroidal compactifications $F_{2d} \hookrightarrow \overline{F}_{2d}^{\mathfrak{F}}$ have divisorial boundary, but depend on a Γ -admissible collection of fans. This is a choice of a fan $\mathfrak{F} = \{\mathfrak{F}_I\}$ for each cusp of the Baily–Borel compactification, satisfying conditions described below. For the 1-cusps, the fans are 1-dimensional and no choice is involved; they are automatically compatible with the fans for the 0-cusps.

Each 0-cusp corresponds to a primitive isotropic line $I \subset h^\perp$. Consider the lattice

$$\Lambda_I := I^\perp / I$$

whose intersection form has signature $(1, 18)$. Let $\Gamma_I := \mathrm{Stab}_\Gamma(I) / U_I$, where $U_I \subset \mathrm{Stab}_\Gamma(I)$ is the unipotent subgroup, isomorphic to a translation subgroup of $I^\perp / I$. Let $\mathcal{C}_I$ denote the positive cone of $\Lambda_I \otimes \mathbb{R}$ and let $\overline{\mathcal{C}}_I$ denote its *rational closure*—the union of the positive cone and the rational null rays on its boundary. Then the fan $\mathfrak{F}_I = \{\tau_i\}$ is a collection of closed, convex, rational polyhedral cones in $\overline{\mathcal{C}}_I$, closed under taking intersections and faces, such that

- (1) $\mathrm{Supp} \mathfrak{F}_I = \overline{\mathcal{C}}_I$ and $\mathfrak{F}_I$ is locally finite in the positive cone $\mathcal{C}_I$,
- (2) $\mathfrak{F}_I$ is invariant under the action of Γ_I with only finitely many orbits.

Then, for each 0-cusp I , the infinite type toric variety $X(\mathfrak{F}_I)$ contains an analytic open subset V_I satisfying the following conditions.

- (1) V_I contains all toric boundary strata of $X(\mathfrak{F}_I)$ which correspond to cones of $\mathfrak{F}_I$ that intersect $\mathcal{C}_I$ (the only strata it does not fully contain are those corresponding to null rays and the origin).
- (2) V_I is Γ_I -invariant and the action of Γ_I is properly discontinuous.
- (3) The open stratum of V_I modulo Γ_I is the intersection of a neighborhood of the Type III point P_I of $\overline{F}_{2d}^{\mathrm{BB}}$ with F_{2d} .

Taking the union of F_{2d} with the open sets from (3), for all I , we get a map

$$F_{2d} \cup_I (\Gamma_I \backslash V_I) \rightarrow \overline{F}_{2d}^{\mathrm{BB}}$$

with complete fibers. It surjects onto the union of F_{2d} with an open neighborhood of the Type III boundary point. This map extends over the Type II boundary as a fibration in finite quotients of abelian varieties. More explicitly, the preimage of the Type II boundary component $\Gamma_J \backslash F_J \subset \overline{F}_{2d}^{\mathrm{BB}}$ in the toroidal compactification is the quotient by a subgroup of $O(J^\perp / J)$ of a family of abelian varieties isogenous to $J^\perp / J \otimes \mathcal{E}$, the self-product of the universal elliptic curve over $\Gamma_J \backslash F_J$.

The *toroidal compactification* $\overline{F}_{2d}^{\mathfrak{F}}$ associated to the Γ -admissible collection of fans $\mathfrak{F}$ is then the result of extending these abelian variety families from $F_{2d} \cup_I (\Gamma_I \backslash V_I)$, along all orbits of rank 2 isotropic lattices J . The toroidal compactification admits a birational morphism $\overline{F}_{2d}^{\mathfrak{F}} \rightarrow \overline{F}_{2d}^{\mathrm{BB}}$ which is an isomorphism on F_{2d} .

For degree 2 K3 surfaces, there is only one 0-cusp, and the fan for this unique 0-cusp has the support on $\overline{\mathcal{C}} = \overline{\mathcal{C}}_I$ in the vector space $N \otimes \mathbb{R}$, where $N = I^\perp / I = H \oplus E_8^2 \oplus A_1$ is a lattice of signature $(1, 18)$. The fan must be $\Gamma_I = O^+(N)$ -invariant, where $O^+(N)$ is the index 2 subgroup of the isometry group $O(N)$ preserving the positive cone $\mathcal{C}$. For us, the critical fact is the following.

Proposition 3.2. *The unipotent $U_I \backslash \mathbb{D}$ embeds into $I^\perp / I \otimes \mathbb{C}^*$ and the period map $C^* \rightarrow U_I \backslash \mathbb{D}$ of a Kulikov model $\mathcal{X} \rightarrow (C, 0)$ with monodromy invariant λ is well-approximated by a translate of the cocharacter $\lambda \otimes \mathbb{C}^*$ near $0 \in C$.*

Proof. This is a direct consequence of Schmid's nilpotent orbit theorem. $\square$

3D. Stable pair compactification. First, we recall the definitions.

Definition 3.3. A pair $(X, B = \sum b_i B_i)$ consisting of a normal variety and a $\mathbb{Q}$ -divisor with $0 \leq b_i \leq 1$, $b_i \in \mathbb{Q}$, is *log-canonical (lc)* if the divisor $K_X + B$ is $\mathbb{Q}$ -Cartier and for a resolution $f: Y \rightarrow X$ with a divisorial exceptional locus $\text{Exc}(f) = \bigcup E_j$ and normal crossing $\bigcup f_*^{-1} B_i \cup \text{Exc}(f)$, in the natural formula

$$f^*(K_X + B) = K_Y + \sum_i b_i f_*^{-1} B_i + \sum_j b_j E_j,$$

one has $b_j \leq 1$.

Definition 3.4. A pair $(X, B = \sum b_i B_i)$ consisting of a reduced variety and a $\mathbb{Q}$ -divisor is *semi-log-canonical (slc)* if X is S_2 , has at worst double crossings in codimension 1, and for the normalization $v: X^\nu \rightarrow X$ writing

$$v^*(K_X + B) = K_{X^\nu} + B^\nu,$$

the pair (X^ν, B^ν) is log-canonical. Here, $B^\nu = D + \sum b_i v^{-1}(B_i)$, and D is the double locus.

Definition 3.5. A pair (X, B) consisting of a connected projective variety X and a $\mathbb{Q}$ -divisor is *stable* if

- (1) (X, B) has semi-log-canonical singularities, in particular $K_X + B$ is $\mathbb{Q}$ -Cartier,
- (2) the $\mathbb{Q}$ -divisor $K_X + B$ is ample.

Next, we introduce the objects that we are interested in here.

Definition 3.6. For a fixed degree $e \in \mathbb{N}$ and fixed rational number $0 < \epsilon \leq 1$, a *stable K -trivial pair* of type (e, ϵ) is a pair $(X, \epsilon R)$ such that

- (1) X is a Gorenstein surface with $\omega_X \simeq \mathcal{O}_X$,
- (2) the divisor R is an ample Cartier divisor of degree $R^2 = e$,
- (3) the surface X and the pair $(X, \epsilon R)$ are slc. In particular, the pair $(X, \epsilon R)$ is stable in the sense of Definition 3.5.

Definition 3.7. A family of stable K -trivial pairs of type (e, ϵ) is a flat morphism $f: (\mathcal{X}, \epsilon \mathcal{R}) \rightarrow S$ such that $\omega_{\mathcal{X}/S} \simeq \mathcal{O}_{\mathcal{X}}$ locally on S , the divisor $\mathcal{R}$ is a relative Cartier divisor, such that every fiber is a stable K -trivial pair of type (e, ϵ) .

Lemma 3.8. *For a fixed degree e , there exists an $\epsilon_0(e) > 0$ such that, for any $0 < \epsilon \leq \epsilon_0$, the moduli stacks $\mathcal{M}^{\text{slc}}(e, \epsilon_0)$ and $\mathcal{M}^{\text{slc}}(e, \epsilon)$ coincide.*

Proof. For a fixed surface X , there exists an $0 < \epsilon_0 \ll 1$ such that the pair $(X, \epsilon_0 R)$ is slc if and only if R does not contain any centers of log-canonical singularities: images of the divisors with codiscrepancy $b_i = 1$ on a log-resolution of singularities $Y \rightarrow X^\vee \rightarrow X$ as in Definitions 3.3, 3.4. There are finitely many of such centers. Then, for any $\epsilon < \epsilon_0$, the pair $(X, \epsilon_0 R)$ is slc if and only if $(X, \epsilon R)$ is.

Now, since $\omega_X \simeq \mathcal{O}_X$ and R is ample Cartier of a fixed degree, the family of the pairs (X, R) is bounded, and the number ϵ_0 with this property can be chosen universally. $\square$

We will be interested in the moduli space M_e^{slc} of such pairs, and more precisely in the closure of F_{2d} in M_e^{slc} for a chosen *intrinsic* polarizing divisor $R \in |NL|$.

We refer to [3, 33, 34] for the existence and projectivity of the moduli space of stable pairs $(X, \sum b_i B_i)$. In general, when some coefficients b_i are $\leq \frac{1}{2}$, there are delicate problems with the definition of a family since a flat limit of divisors may happen to be a nonreduced scheme with embedded components. In our case, the situation is much easier since R is Cartier.

Definition 3.9. A family of stable K -trivial pairs of degree e is a family of type (e, ϵ_0) , where $\epsilon_0(e)$ is chosen as in Lemma 3.8. We denote the corresponding moduli functor by M_e^{slc} . For a scheme S , $M_e^{\text{slc}}(S) = \{\text{families of type } (e, \epsilon_0(e)) \text{ over } S\}$.

Proposition 3.10. *There is a Deligne–Mumford stack $\mathcal{M}_e^{\text{slc}}$ and a coarse moduli space M_e^{slc} of stable K -trivial pairs.*

Proof. The spaces $\mathcal{M}_e^{\text{slc}}$ and M_e^{slc} are constructed by standard methods, as quotients of appropriate Hilbert schemes by a PGL group action. Again, for general stable pairs, there are delicate questions of the formation of $(\omega_{\mathcal{X}/S}^{\otimes n}(nR))^{**}$ commuting with base changes. But in our case, both $\omega_{\mathcal{X}/S}$ and $\mathcal{O}_{\mathcal{X}}(\mathcal{R})$ are invertible, so these questions disappear. $\square$

We do not prove that the moduli space M_e^{slc} is proper, but we do prove below that it provides a compactification for the moduli spaces of ordinary K3 surfaces. (The components of M_e^{slc} where X is generically non-normal require additional arguments.) A related moduli space is given by the following definition.

Definition 3.11. Let $N \in \mathbb{N}$. The moduli stack $\mathcal{P}_{N,2d}$ parameterizes proper flat families of pairs (X, R) such that (X, L) is a polarized K3 surface with *ADE* singularities and a primitive ample line bundle L , $L^2 = 2d$, and $R \in |NL|$ is an arbitrary divisor. One has $R^2 = 2dN^2$. In particular, one defines $\mathcal{P}_{2d} := \mathcal{P}_{1,2d}$.

If we take $\epsilon_0(e)$ as in Lemma 3.8, then the pair $(X, \epsilon_0 R)$ is stable. Obviously, the stack $\mathcal{P}_{N,e}$ is fibered over the stack $\mathcal{F}_{2d}$ with fibers isomorphic to $\mathbb{P}^{dN^2+1}$. The automorphism groups of stable pairs are finite, and it is easy to see that the stack $\mathcal{P}_{N,2d}$ is coarsely represented by a scheme, denoted $P_{N,2d}$.

Definition 3.12. One defines $\overline{P}_{N,2d}$ to be the closure of the coarse moduli space $P_{N,2d}$ in M_e^{slc} for $e = 2dN^2$. A *canonical choice* of a divisor $R \in |NL|$ for each $(X, L) \in F_{2d}$ gives an embedding $F_{2d} \subset P_{N,2d}$. Its closure in $\overline{P}_{N,2d}$ is denoted by $\overline{F}_{2d}^{\text{slc}}$.

Theorem 3.13. $\overline{P}_{N,2d}$ and thus also $\overline{F}_{2d}^{\text{slc}}$ are proper and projective.

Proof. Properness follows from the next theorem. Projectivity follows from it by results of Fujino and Kovács–Patakfalvi [37]. $\square$

Theorem 3.14. *For a fixed degree e , every family $f^*: \mathcal{X}^* \rightarrow C^* = C \setminus 0$ over a smooth curve of K3 surfaces with ADE singularities and ample R , $R^2 = e$, can be extended to a family of stable K3 pairs $(\mathcal{X}', \epsilon_0(e)\mathcal{R}') \rightarrow C'$ of type $(e, \epsilon_0(e))$ possibly after a ramified base change $C' \rightarrow C$.*

Proof. (Cf. [40, Theorem 2.11, Remark 2.12]) The proof is achieved by modifying that of a theorem of Shepherd-Barron [51, Theorem 1]. His theorem says that if $\mathcal{X} \rightarrow C$ is a semistable model with $K_{\mathcal{X}} = 0$ and $\mathcal{L}^*$ is a relatively nef line bundle on $\mathcal{X}^*$ of positive degree, then there exists another semistable model to which $\mathcal{L}^*$ extends as a relatively nef line bundle $\mathcal{L}$. This is done over C , without a base change. Then [51, Theorem 2] says that $\mathcal{L}^n$ for $n \geq 4$ gives a contraction $\pi: \mathcal{X} \rightarrow \overline{\mathcal{X}}$ so that $\omega_{\overline{\mathcal{X}}} \simeq \mathcal{O}_{\mathcal{X}}$, with $\overline{\mathcal{L}}$ an ample line bundle on $\overline{\mathcal{X}}$ and $\mathcal{L} = \pi^*(\overline{\mathcal{L}})$.

Now let $f: (\mathcal{X}^*, \epsilon\mathcal{R}^*) \rightarrow C^*$ be a family of K3 surfaces with ADE singularities and a relatively ample Cartier divisor $\mathcal{R}$. After shrinking the base, we can simultaneously resolve the singularities to obtain a family of smooth K3s $(\mathcal{X}_1^*, \mathcal{R}_1^*)$ with a relatively big and nef effective divisor. By Theorem 2.1, after a further base change, we get a semistable model $\mathcal{X}_2 \rightarrow C$ with $K_{\mathcal{X}_2} = 0$. We are now in a situation where Shepherd-Barron's theorem applies. However, first we make another base change that exists by Claim 3.15 to obtain a Kulikov model $\mathcal{X}_3$ satisfying condition (*).

(*) The closure of $\mathcal{R}^*$ in $\mathcal{X}$ does not contain any strata (double curves or triple points) of the central fiber $\mathcal{X}_0$.

The proof of [51, Theorem 1] proceeds by starting with a divisor $\mathcal{R}_3$ which does not contain an entire component of the central fiber. One then makes it nef using flops along curves E with $\mathcal{R}_3 \cdot E < 0$. The flops are called elementary modifications. They are of three types: (0) along an interior (-2) -curve, (I) along a curve intersecting a double curve, and (II) along a double curve.

But with condition (*) achieved, the divisor $\mathcal{R}_3$ already intersects the double curves non-negatively, and the flops of type (II) are not needed. The flops of types (0) and (I) preserve (*). Thus, the end result is a model $\mathcal{X}_4 \rightarrow C'$ with an effective, relatively nef divisor $\mathcal{R}_4$ satisfying (*).

Since the central fiber $(\mathcal{X}_4)_0$ is normal crossing, for $0 < \epsilon \ll 1$, the pair $(\mathcal{X}_4, \epsilon\mathcal{R})$ is slc. Then the contraction $\mathcal{X}_4 \rightarrow \overline{\mathcal{X}}_4$ provided by [51, Theorem 2] gives a family $(\overline{\mathcal{X}}_4, \epsilon\overline{\mathcal{R}}_4)$ of stable pairs extending the original family $(\mathcal{X}^*, \epsilon\mathcal{R}^*) \rightarrow C^*$ after a base change $C' \rightarrow C$. $\square$

Claim 3.15. *For any Kulikov model $\mathcal{X} \rightarrow C$, there exists a finite base change $C' \rightarrow C$ and birational modification to a Kulikov model $\mathcal{X}' \rightarrow C'$ of $\mathcal{X} \times_C C'$ satisfying (*).*

Proof. This is a local toric computation. We give an argument for a triple point, which by simplification also applies to double curves. An obvious modification of this proof works for a semistable degeneration in any dimension.

Let the triple point be the origin with a local equation $xyz = t$. A local toric model of it is $\mathbb{A}_{x,y,z}^3$. Its fan is the cone σ that is the first octant in $\mathbb{R}^3$ with the lattice $N = \mathbb{Z}^3$. In the lattice of monomials $M = N^* \simeq \mathbb{Z}^3$, the dual cone $\sigma^\vee$ is the first octant as well. A ramified base change $t = s^d$ means choosing the new lattices

$$M' = M + \mathbb{Z} \frac{(1, 1, 1)}{d}, \quad N \supset N' = \left\{ n = (a, b, c) \mid n \cdot \frac{(1, 1, 1)}{d} \in \mathbb{Z} \right\}.$$

Choosing a Kulikov model locally at this 0-stratum is equivalent to choosing a triangulation $\mathcal{T}$ of the triangle $\sigma \cap \{a + b + c = d\}$ with the vertices $(d, 0, 0)$, $(0, d, 0)$, $(0, 0, d)$ into elementary triangles of lattice volume 1. Then the new fan is obtained by subdividing σ into the cones over these elementary triangles.

We note that an arbitrary triangulation $\mathcal{T}$ will *not* achieve condition (*). Instead, it has to be chosen carefully. Using x, y, z as local parameters, the equation of the divisor is a power series $f \in k[[x, y, z]]$. Let $\{m_j\}$ be the set of the monomials appearing in f . Let P be the convex hull of $\bigcup_j (m_j + \sigma^\vee)$. This is an infinite polyhedron, but it has only finitely many vertices, say m_j for $1 \leq j \leq r$.

Let $\text{NFan}(P)$ be the normal fan of P ; it is a refinement of the cone σ . Let X' be the toric variety, possibly singular, for this fan. We have a toric blow-up $X' \rightarrow \mathbb{A}^3$ modeling a blow-up $f: \mathcal{X}' \rightarrow \mathcal{X}$. The strict preimage of the divisor $\mathcal{R}$ has the same equation f which still makes sense for each of the standard open sets $\mathbb{A}^3$ that cover X' . The reason for taking the convex hull was this: the vertices of P correspond to the 0-dimensional strata x'_j of X' and the fact that, for each of them, the corresponding monomial has a nonzero coefficient means that the divisor does not pass through x'_j . These points are in a bijection with the maximal-dimensional cones σ'_j of $\text{NFan}(P)$. Subdividing these cones further means blowing up at the points x'_j further. The preimage of the divisor under these blow-ups will not contain any strata on the blow-up.

So the final recipe is this. From the equation of f , obtain the polyhedron P and its normal fan $\text{NFan}(P)$. It has finitely many rays $\mathbb{R}_{\geq 0}(a_i, b_i, c_i)$, where $(a_i, b_i, c_i) \in \mathbb{Z}_{\geq 0}^3$. Let $d_i = a_i + b_i + c_i$ and let d be the gcd(d_i) so that these rays are cones over some integral points of the triangle $\sigma \cap \{a + b + c = d\}$. The fan $\text{NFan}(P)$ gives a subdivision of this triangle. Refine it arbitrarily to a triangulation $\mathcal{T}$ into volume 1 triangles. This defines a Kulikov model locally. Then repeat this procedure at all the 0-strata of $\mathcal{X}$. The resulting Kulikov model satisfies condition (*). $\square$

Remark 3.16. Difficulties with the moduli spaces of stable pairs $(X, B = \sum b_i B_i)$ arise when $K_X + B$ is $\mathbb{Q}$ - or $\mathbb{R}$ -Cartier but K_X and B by themselves are not. One solution was proposed in [4, Section 1.5]: choose the coefficients b_i so that $(1, b_1, \dots, b_n)$ are $\mathbb{Q}$ -linearly independent. In the situation at hand, this means picking ϵ to be irrational. We do not need this trick for the K3 surfaces, however, since by the above the divisor R remains Cartier in the interesting part of the compactified moduli space.

Theorem 3.17. *The rational maps $(\overline{P}_{N,2d})^\vee \dashrightarrow \overline{F}_{2d}^{\text{BB}}$ and, for a canonical choice of a polarizing divisor, $(\overline{F}_{2d}^{\text{slc}})^\vee \dashrightarrow \overline{F}_{2d}^{\text{BB}}$ from the normalizations of $\overline{P}_{N,2d}$ and $\overline{F}_{2d}^{\text{slc}}$ to the Baily–Borel compactification are regular.*

Proof. We apply Lemma 3.18 with $X = (\overline{P}_{N,2d})^\vee$ resp. $X = (\overline{F}_{2d}^{\text{slc}})^\vee$, and $Y = \overline{F}_2^{\text{BB}}$. We claim that the condition of Lemma 3.18 is satisfied. Namely, for a one-parameter family of

stable K3 surfaces over $(C, 0)$, the central fiber uniquely determines if the limit in the Baily–Borel compactification is of Type II or Type III, and if it is of Type II, then the j -invariant of the elliptic curve is uniquely determined.

As in the proof of Theorem 3.14, we get a Kulikov model $\mathcal{X}$ to which a big and nef line bundle $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(\mathcal{R})$ extends and then a contraction $\mathcal{X} \rightarrow \overline{\mathcal{X}}$ to the canonical model. If $\mathcal{X}$ is of Type III, then $\overline{\mathcal{X}}$ is a union of rational surfaces with rational singularities, glued along rational curves. If $\mathcal{X}$ is of Type II, then either some components of $\overline{\mathcal{X}}$ are glued along an elliptic curve E or, if all the elliptic curves that constitute the double locus of $\mathcal{X}$ are contracted, a component of $\overline{\mathcal{X}}$ has an elliptic singularity, resolved by inserting E . So the type, and for Type II the j -invariant of E , can be recovered from the central fiber $\overline{\mathcal{X}}_0$. $\square$

Lemma 3.18. *Let X and Y be proper varieties, with X normal. Let $\varphi: X \dashrightarrow Y$ be a rational map, regular on an open dense subset $U \subset X$. Let $(C, 0)$ be a regular curve and $f: C \rightarrow X$ a morphism whose image meets U . Let $g: C \rightarrow Y$ be the unique extension of $f \circ \varphi$ which exists by the properness of Y . Assume that, for all f with the same $f(0)$, there are only finitely many possibilities for $g(0)$. Then φ can be extended uniquely to a regular morphism $X \rightarrow Y$.*

Proof. Let $Z \subset X \times Y$ be the closure of the graph of $U \rightarrow Y$. The projection $Z \rightarrow Y$ is a morphism extending φ . The morphism $Z \rightarrow X$ is birational, and the condition is that it is finite. By Zariski’s Main Theorem, $Z \rightarrow X$ is an isomorphism. $\square$

4. The Coxeter fan and compactifications of F_2

4A. The Coxeter fan. For F_2 , a toroidal compactification depends on a single fan, supported on the rational closure $\overline{\mathcal{C}}$ of the positive cone in the space $N_{\mathbb{R}}$ for the hyperbolic lattice $N = H \oplus E_8^2 \oplus A_1$. We now describe a particularly nice fan on N , cf. [49, Section 6.2].

Definition 4.1. The Coxeter fan $\mathfrak{F}^{\text{cox}}$ is obtained by cutting $\overline{\mathcal{C}}$ by the mirrors $r^\perp$ to the roots of N , i.e. the vectors $r \in N$ with $r^2 = -2$.

The Weyl group $W(N)$ generated by reflections in the roots has finite index in the isometry group $O^+(N)$, with the quotient $O^+(N)/W(N) = S_3$. Here, $O^+(N)$ is the index 2 subgroup of $O(N)$ fixing the positive cone. Reflection groups acting on hyperbolic spaces were studied by Vinberg, see e.g. [53, 54]. Note that, in those papers, a hyperbolic space has signature $(r - 1, 1)$ vs. our $(1, 18)$.

A fundamental chamber $\mathfrak{K}$ of $W(N)$ is described by a Coxeter diagram given in Figure 2. The nodes represent 24 roots r_i that generate N , with the index i given by the label in Figure 2. We have $(r_i, r_j) = 0, 1, 2, 6$ depending on whether there is no line, a single line, a doubled line, or a dashed line connecting i to j , respectively. The fundamental chamber is

$$\mathfrak{K} = \{\lambda \in \overline{\mathcal{C}} : \lambda \cdot r_i \geq 0 \text{ for } 0 \leq i \leq 23\}.$$

The group S_3 acts on the fundamental chamber by symmetries of the diagram. The projectivization $P = \mathbb{P}(\mathfrak{K})$ is a hyperbolic polytope with cusps: it has infinite vertices corresponding to null rays $v \in \mathfrak{K}$ with $v^2 = 0$. However, it has finite hyperbolic volume.

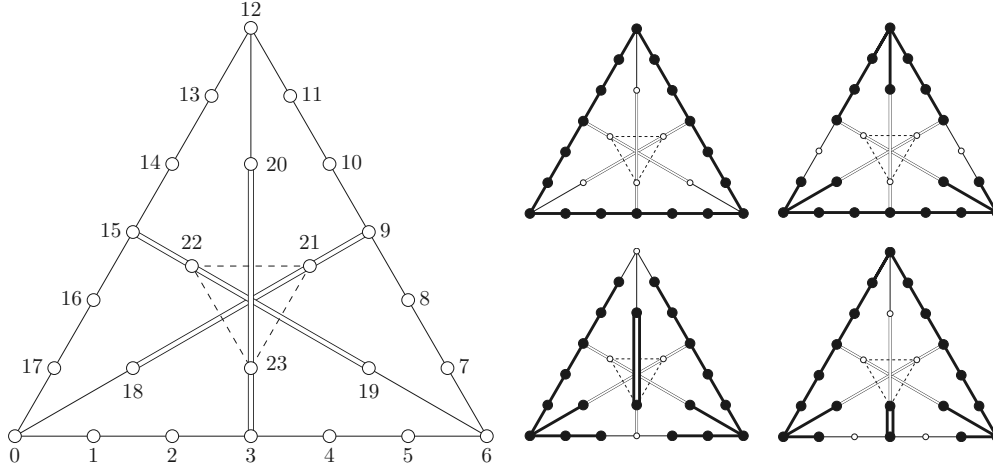


Figure 2. Coxeter diagram G_{cox} and its maximal parabolic subdiagrams $\tilde{A}_{17}$, $\tilde{D}_{10}\tilde{E}_7$, $\tilde{E}_8^2\tilde{A}_1$, $\tilde{D}_{16}\tilde{A}_1$.

Definition 4.2. A subdiagram of G_{cox} is a subgraph $G \subset G_{\text{cox}}$ induced by a subset $V \subset V(G_{\text{cox}})$ of the vertices, i.e. a subset of the 24 roots r_i . It defines a vector subspace $\mathbb{R}V = \langle r_i, i \in V \rangle \subset N_{\mathbb{R}}$.

A subdiagram is called *elliptic* if the restriction of the quadratic form of N to $\mathbb{R}V$ is negative definite. It is called *parabolic* if it is negative semi-definite. *Maximal parabolic* means maximal by inclusion among the parabolic diagrams.

Vinberg described the faces of the fundamental polytope P , see [53, Theorem 3.3]. In our situation, this gives the following theorem.

Theorem 4.3. *The correspondence*

$$F \mapsto G(F) = \{i \mid F \subset r_i^\perp\}, \quad G = \{r_i, i \in G\} \mapsto F(G) = \bigcap_{i \in I} r_i^\perp$$

defines an order reversing bijection between the faces of the fundamental chamber $\mathfrak{R}$ and the elliptic and maximal parabolic subdiagrams $G \subset G_{\text{cox}}$. The chamber itself corresponds to $G = \emptyset$.

Type III cones (meeting the interior $\mathcal{C}$) of dimension $d > 0$ correspond to elliptic subdiagrams of rank $r = 19 - d$. These are disjoint unions of Dynkin diagrams G_i of ADE type with $\sum |G_i| = r$.

Type II rays $\mathbb{R}_{\geq 0}v$ with $v^2 = 0$ correspond to maximal parabolic subdiagrams of G_{cox} . These are disjoint unions of affine Dynkin diagrams $\tilde{G}_i$ with $\sum |G_i| = 17$.

Lemma 4.4. *The cones of the Coxeter fan $\mathfrak{F}^{\text{cox}} \bmod W(N)$ are in a bijection with the faces of the fundamental chamber. The cones of $\mathfrak{F}^{\text{cox}} \bmod O^+(N)$ are in a bijection with elliptic and maximal parabolic subdiagrams of $G_{\text{cox}} \bmod S_3$.*

Proof. This follows since $\mathfrak{R}$ is a fundamental domain for the $W(N)$ -action and

$$O^+(N) = S_3 \ltimes W(N).$$

□

The following two lemmas are proved by direct enumeration.

Lemma 4.5. *Modulo S_3 , there are 4 maximal parabolic subdiagrams of G_{cox} , illustrated on the right in Figure 2.*

- (1) $\tilde{A}_{17} = [i, 0 \leq i < 18]$,
- (2) $\tilde{D}_{10}\tilde{E}_7 = [18, 17, 0, \dots, 6, 7, 19] \sqcup [9, \dots, 15, 20]$,
- (3) $\tilde{E}_8^2\tilde{A}_1 = [13, \dots, 2, 18] \sqcup [4, \dots, 11, 19] \sqcup [20, 23]$.
- (4) $\tilde{D}_{16}\tilde{A}_1 = [19, 5, 6, \dots, 0, 1, 18] \sqcup [3, 23]$.

Lemma 4.6. *Modulo S_3 , the numbers of elliptic subdiagrams of G_{cox} that have ranks $r = 1, \dots, 18$ are 6, 51, 328, 1518, 5406, 14979, 33132, 59339, 87077, 105236, 105078, 86505, 58223, 31564, 13371, 4209, 883, 99. In particular, in $\mathfrak{F}^{\text{cox}} \bmod O^+(N)$, there are $4 + 99 = 103$ rays.*

For each of the extended Dynkin diagrams $\tilde{A}_k, \tilde{D}_k, \tilde{E}_k$, there is a unique primitive positive integer combination of the roots which is null in the affine root lattice. The coefficients for the first k nodes are the fundamental weights of the corresponding Lie algebra and the coefficient of the extended node is 1. Alternatively, these are labels of the extended Dynkin diagram such that each label is half the sum of its neighbors. For example, for the first $\tilde{E}_8$ diagram in case (3) above, this vector is

$$n(\tilde{E}_8^{(1)}) = r_{13} + 2r_{14} + 3r_{15} + 4r_{16} + 5r_{17} + 6r_0 + 4r_1 + 2r_2 + 3r_{18}.$$

Lemma 4.7. *For each maximal parabolic subdiagram of G_{cox} , the square-zero vectors of its connected components coincide:*

$$n(\tilde{D}_{10}) = n(\tilde{E}_7), \quad n(\tilde{E}_8^{(1)}) = n(\tilde{E}_8^{(2)}) = n(\tilde{A}_1), \quad n(\tilde{D}_{16}) = n(\tilde{A}_1).$$

The six $\tilde{E}_8^2\tilde{A}_1$ equations generate all the relations between the 24 roots r_i . The unique syzygy between them is that the sum of the three $\tilde{E}_8^2$ differences is zero.

Proof. An easy direct check. □

4B. Connected Dynkin subdiagrams of G_{cox} . We adopt the notation of [7] for the connected subdiagrams of G_{cox} using decorated Dynkin diagrams.

Definition 4.8. The subdiagrams of G_{cox} with the vertices entirely contained in the subset $\{18, 19, 20, 21, 22, 23\}$ are called *irrelevant*. A diagram is *relevant* if it has no irrelevant connected components. For each $G \subset G_{\text{cox}}$, its *relevant content* G^{rel} is the subdiagram obtained by dropping all irrelevant connected components.

We list the connected subdiagrams of G_{cox} in Table 1. The indices $0 \leq i < 18$ are taken in $\mathbb{Z}_{18}$. We first give the elliptic subdiagrams, then parabolic, then irrelevant elliptic, and finally irrelevant parabolic. The diagrams are considered up to the S_3 -symmetry if they do not lie in the outside 18-cycle. The ones that are contained in the 18-cycle are considered up to the dihedral symmetry group D_9 .

Type	Vertices	Type	Vertices
A_{2n+1}	$2i + 1, \dots, 2i + 2n + 1, \quad n \leq 8$	$\tilde{A}_{17}$	$i, 0 \leq i < 18$
A_{2n}^-	$2i + 1, \dots, 2i + 2n, \quad n \leq 8$	$\tilde{D}_{10}$	$18, 17, 0, \dots, 6, 7, 19$
$\bar{A}_{2n+1}^-$	$2i, \dots, 2i + 2n, \quad n \leq 8$	$\tilde{E}_7$	$9, \dots, 15, 20$
A_{2n+1}'	$18, 0, 1, \dots, 2n - 1, \quad n \leq 8$	$\tilde{E}_8^-$	$13, \dots, 2, 18$
A_{2n}^-	$18, 0, 1, \dots, 2n - 2, \quad n \leq 8$	$\tilde{D}_{16}$	$19, 5, 6, \dots, 0, 1, 18$
A_9'	$18, 0, \dots, 6, 19$	$\tilde{A}_1^*$	$3, 23$
A_{15}'	$18, 0, \dots, 12, 20$		
D_{2n}	$18, 17, 0, 1, \dots, 2n - 3, \quad n \leq 8$	A_1^{irr}	18
D_{2n+1}^-	$18, 17, 0, 1, \dots, 2n - 2, \quad n \leq 8$	$\bar{A}_1^{\text{irr}}$	21
D_{10}'	$18, 17, 0, \dots, 6, 19$		
D_{16}'	$18, 17, 0, \dots, 12, 20$	$\tilde{A}_1^{\text{irr}}$	$20, 23$
$\bar{E}_6^-$	$18, 16, 17, 0, 1, 2$		
$\bar{E}_7$	$18, 16, 17, 0, 1, 2, 3$		
$\bar{E}_8^-$	$18, 16, 17, 0, 1, 2, 3, 4$		

Table 1. Connected elliptic and parabolic subdiagrams of G_{cox} .

The parabolic subdiagrams of G_{cox} are shown in Figure 2.

Definition 4.9. The *skeleton* of a diagram its intersection with the cycle $0, 1, \dots, 17$.

In the shortcut notation of Table 1, a minus or prime on the left and right implies that the clockwise and counterclockwise vertex, respectively, adjacent to the skeleton is odd. The absence of a marking implies the vertex is even. The prime indicates that an extra leaf of the subdiagram has entered the interior vertices $\{18, 19, 20, 21, 22, 23\}$ of Figure 2.

Definition 4.10. The *stable type* of an elliptic or maximal parabolic subdiagram

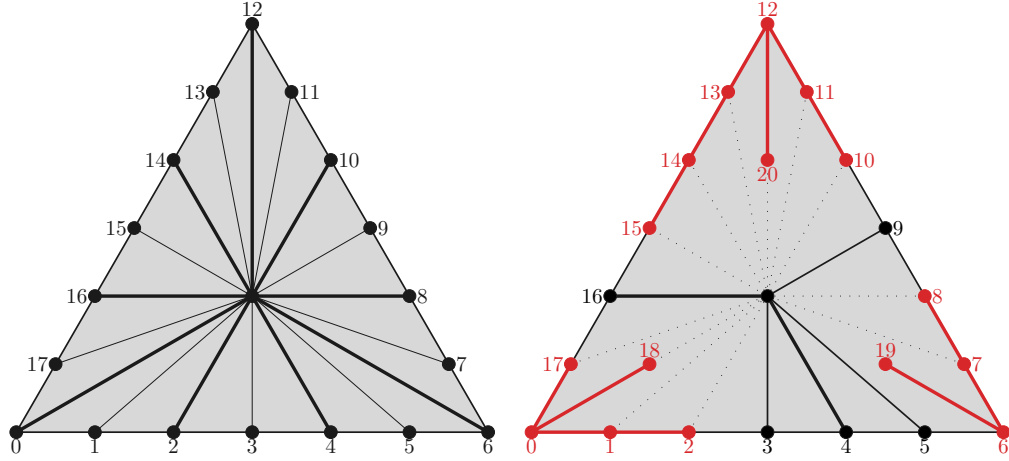
$$G = \bigsqcup G_k \subset G_{\text{cox}}$$

is its relevant content G^{rel} , with diagrams notated as in Table 1, listed in cyclic order around the 18-cycle. We introduce symbols A_0^- or $\bar{A}_0$ to indicate, respectively, that both $2i, 2i + 1$ or both $2i + 1, 2i + 2 \pmod{18}$ do not lie in the skeleton of G .

The insertion of the symbols A_0^- or $\bar{A}_0$ is necessary to determine the spacing between the relevant connected components. Two examples are shown in Figure 3. Note that the S_3 or D_9 action cyclically rotates and/or flips the diagram labels in the stable type, and orientation reversing symmetries flip which sides of a symbol are decorated with a $-$ sign.

4C. A toroidal compactification.

Definition 4.11. The toroidal compactification $\bar{F}_2^{\text{tor}} = \bar{F}_2^{\mathfrak{F}_{\text{cox}}}$ we consider in this paper is the one corresponding to the Coxeter fan $\mathfrak{F}^{\text{cox}}$.

Figure 3. Stable types $(A_0^- A_0)^9$ and $D_5^- A_0 A_0^- A_4^- E_7$.

We describe the strata of $\overline{F}_2^{\text{tor}}$ which by Theorem 4.3, Lemma 4.4 correspond to elliptic and maximal parabolic subdiagrams of G_{cox} mod S_3 .

Notation 4.12. An elliptic subdiagram $G = \bigsqcup G_k$ is a union of ADE Dynkin diagrams. We denote by R_G the corresponding root system and $W(G)$ its Weyl group. Let

$$S_G \ltimes W(G) \subset O(R_G)$$

be the extension by the symmetries $S_G \subset S_3$ of the subdiagram. A parabolic subdiagram $\tilde{G} = \bigsqcup \tilde{G}_k$ is a union of affine ADE Dynkin diagrams. In this case, let $G = \bigsqcup G_k$ be the union of the corresponding ordinary (not extended) Dynkin diagrams.

Proposition 4.13. *The Type III and II strata in $\overline{F}_2^{\text{tor}}$ are as follows.*

- (1) *Let G be an elliptic diagram. Then $\text{Str}(G)$ is the quotient by $S_G \ltimes W(G)$ of the torus $\text{Hom}(M_G, \mathbb{C}^*)$ where M_G is the saturation of the root torus R_G in $M = N^*$.*
- (2) *Let $\tilde{G}$ be a maximal parabolic diagram. Then $\text{Str}(\tilde{G})$ is the quotient by $S_G \ltimes W(G)$ of $\text{Hom}(M_G, \mathcal{E}) \simeq \mathcal{E}^{17}$, where $\mathcal{E}^{17} \rightarrow \mathbb{A}_j^1$ is the self-fiber product of the universal family of elliptic curves $\mathcal{E} \rightarrow \mathcal{M}_{1,1}$ over the moduli stack.*

Proof. The strata of Type III are contained in the fiber of $\overline{F}_2^{\text{tor}} \rightarrow \overline{F}^{\text{BB}}$ over the unique Type III point and can be described purely in terms of toric geometry. We have two lattices $N = I^\perp/I = H \oplus E_8^2 \oplus A_1$ and $M = N^*$. Using the quadratic form on N , we can present N^* as an overlattice with $N^*/N = \mathbb{Z}_2$. The lattice N is generated by the 24 roots r_i in Coxeter diagram. Thus,

$$M = N^* = \left\{ v \in \frac{1}{2}N \mid (v, r_i) \in \mathbb{Z} \right\} = N + \frac{1}{2}r_{21}.$$

For each Type III cone $\sigma = \sigma(G)$ of $\mathcal{F}^{\text{cox}}$, we have a cone $\sigma \subset N_{\mathbb{R}}$ and a toric variety U_σ with a unique closed orbit O_σ , which is a torus itself. It is standard in toric geometry that $O_\sigma = \text{Hom}(\sigma^\perp \cap M, \mathbb{C}^*)$, and we have $\sigma^\perp = M_G = R_G^{\text{sat}}$, the saturation of the root lattice R_G in M . In the toroidal compactification, we divide an infinite toric variety by $\Gamma = O^+(N)$.

The orbit T_G is divided by its stabilizer in $O^+(N)$, which is $S_G \ltimes W(G)$. The description in Type III follows.

The exact structure of a Type II boundary divisor is determined by the parabolic group $\text{Stab}_\Gamma(J)$ stabilizing the corresponding rank 2 isotropic lattice J . This parabolic group acts on the period domain $\mathbb{H} \times \mathbb{C}^{17}$ of Type II mixed Hodge structures, and the quotient is the boundary divisor. The unipotent subgroup U_J is the kernel of the map

$$\text{Stab}_\Gamma(J) \xrightarrow{q} \text{SL}(J) \times O(J^\perp/J)$$

and induces the full group of translations $J^\perp/J \otimes (\mathbb{Z} \oplus \mathbb{Z}\tau) \simeq (\mathbb{Z} \oplus \mathbb{Z}\tau)^{17}$ on the second factor $\mathbb{C}^{17}$. Quotienting by U_J first gives $J^\perp/J \otimes (\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau) \rightarrow \mathbb{H}$ on which the image of q further acts.

We claim that q is surjective. First, we show that, for any isotropic J , there is a complementary isotropic subspace J' , i.e. a lift of $h^\perp/J^\perp$ to an isotropic plane in $h^\perp$ such that the pairing between J and J' realizes $J' = \text{Hom}(J, \mathbb{Z})$. For instance, let e_1, e_2 be a basis of J . There is an isotropic f_1 such that $e_1 \cdot f_1 = 1$. Taking the perpendicular of $\{e_1, f_1\}$, we get a sublattice of $h^\perp$ isometric to N because there is a unique 0-cusp. We claim that there is an isotropic $f_2 \in N$ such that $e_2 \cdot f_2 = 1$. Observe that e_2 is primitive in N^* ; it is primitive in N and is not of the form $r_{21} + 2n$ for any $n \in N$ because the norm of any such element is nonzero. Hence there is an f_2 such that $e_2 \cdot f_2 = 1$. Since N is even, we can modify f_2 by a multiple of e_2 to ensure it too is isotropic. Then we choose $J' = \{f_1, f_2\}$.

We can now realize any element $(\gamma, g) \in \text{GL}(J) \times O(J^\perp/J)$ by an isometry of $h^\perp$: we declare the action on J to be γ , on $J' = \text{Hom}(J, \mathbb{Z})$ to be the transpose action γ^T , and the action on the lattice summand $(J \oplus J')^\perp \simeq J^\perp/J$ to be g . Thus, the Type II boundary divisor is the quotient of $\mathcal{E}^{17} \rightarrow \mathbb{H}$ by all of $\text{SL}_2(\mathbb{Z}) \times O(J^\perp/J)$; we only get $\text{SL}_2(\mathbb{Z})$ because the isometry must have spinor norm 1. $\square$

Lemma 4.14. *For the connected elliptic subdiagrams G , one has $M_G = R_G$ except for the following diagrams given up to S_3 , where the quotient M_G/R_G is*

- (1) $\mathbb{Z}_2$ for $A_1^{\text{irr}} = [23]; A'_9, A'_{15}, D'_{10}, D'_{16};$

$$A_{17} = [3, \dots, 1], \quad \bar{A}_{17} = [4, \dots, 2], \quad \text{and} \quad D_{18} = [18, 17, 0, \dots, 15];$$

- (2) $\mathbb{Z}_6$ for $A_{17} = [1, \dots, 17].$

Proof. For a vector $u \in M_{\mathbb{Q}}$, one has $u \in M \Leftrightarrow (u, v) \in \mathbb{Z}$ for all $v \in N$, i.e. if and only if $(u, r_i) \in \mathbb{Z}$ for the 24 roots r_i . Now, for each of the lattices $\Lambda = R_G$, we check the finitely many vectors in Λ^*/Λ and see for which of them all the intersection numbers with the 24 roots r_i are integral. As usual, $A_n^*/A_n = \mathbb{Z}_{n+1}$, $D_n^*/D_n = \mathbb{Z}_2^2$ or $\mathbb{Z}_4$ for n even or odd, and $E_n^*/E_n = \mathbb{Z}_{9-n}$. $\square$

Example 4.15. For the lattice A'_9 , the vector

$$u = \frac{1}{2}(r_{18} + r_1 + r_3 + r_5 + r_{19}) \in R_G \otimes \mathbb{Q}$$

in fact lies in M because $(u, r_j) \in \mathbb{Z}$ for all roots r_j . Note that $u \equiv \varpi_5 \pmod{A_9}$, the fundamental weight of the A_9 lattice for the middle node. Similarly for the A_{17} diagram in (2), the vector $u = \frac{1}{6} \sum_{i=1}^{17} i r_i$ is in $(R_G \otimes \mathbb{Q}) \cap M$.

4D. Generalized Coxeter semifan. We start with a more general situation and then specialize to our case. Let N be a hyperbolic lattice of signature $(1, r-1)$, $\mathcal{C} \subset N_{\mathbb{R}}$ the positive cone, and $\overline{\mathcal{C}}$ its rational closure. Let $W \subset O(N)$ be a discrete group generated by reflections in vectors $\{r_k \in N \mid k \in K\}$ such that $r_k^2 < 0$ and $r_k \cdot r_{k'} \geq 0$ for $k \neq k'$. Let

$$\mathfrak{R} = \{v \mid v \cdot r_k \geq 0\} \cap \overline{\mathcal{C}} = \bigcap_k H_{r_k}^+ \cap \overline{\mathcal{C}}$$

be the fundamental domain of W . Then $P = \mathbb{P}(\mathfrak{R})$ is a polytope in a hyperbolic space whose faces by Vinberg [54] admit a description similar to Theorem 4.3.

Definition 4.16. We split the set $K = I \sqcup J$ into two subsets of *active* and *inactive mirrors*. We call a face $\mathfrak{R} \cap_{j \in V} r_j^\perp$ of $\mathfrak{R}$ *irrelevant* if $V \subset J$. Let $W_J = \langle w_j, j \in J \rangle$. We define a bigger chamber $\mathfrak{L} = \bigcup_{h \in W_J} h(\mathfrak{R})$ and a *generalized Coxeter semifan* $\mathfrak{F}^{\text{semi}}$ as the one whose maximal cones are $g(\mathfrak{L})$ for $g \in W$.

Proposition 4.17. *The following statements hold.*

- (1) *One has $\mathfrak{L} = \bigcap_{i \in I, h \in W_J} H_{h(r_i)}^+$. In particular, $\mathfrak{L}$ is convex and locally finite.*
- (2) *The stabilizer group of $\mathfrak{L}$ in W is W_J .*
- (3) *The support of $\mathfrak{F}^{\text{semi}}$ is $\overline{\mathcal{C}}$.*
- (4) *The cones in $\mathfrak{F}^{\text{semi}}$ are $g(F)$ for $g \in W$ and the relevant faces F of $\mathfrak{R}$.*

Proof. Consider a single reflection w_j in a vector $r_j, j \in J$ and a neighboring chamber $w_j(\mathfrak{R}) = \bigcap_i H_{r'_i}^+$, where $r'_i = w_j(r_i)$. Then, for $i \neq j$ and $v \in \mathfrak{R}$, one has

$$r_i \cdot w_j(v) = r_i \cdot \left(v - \frac{2r_i \cdot r_j}{r_j^2} r_j \right) \geq r_i \cdot v = w_j(r_i) \cdot w_j(v).$$

A product of two generators of W_J is $w_j w_{j'} = (w_j w_{j'} w_j^{-1}) w_j$, which is the same as the reflection in the inactive mirror $r_j^\perp$ followed by the reflection in an inactive mirror of the neighboring chamber $w_j(\mathfrak{R})$. In the same way, any element $h \in W_J$ is a product of reflections in inactive mirrors in a sequence of neighboring chambers.

By induction, we get $r_i \cdot h(v) \geq h(r_i) \cdot h(v) = r_i \cdot v \geq 0$ for any $i \in I$ and $h \in W_J$. Thus, $\mathfrak{L} \subset H_{h(r_i)}^+$.

Vice versa, suppose $v \in \overline{\mathcal{C}}$ is such that $h(r_i) \cdot v \geq 0$ for all $i \in I$ and $h \in W_J$. Let $\rho \in N$ be a vector in the interior of $\mathfrak{R}$. Then $\rho \cdot r_k \in \mathbb{Z}_{>0}$ for all $k \in K$. If there exists $j \in J$ such that $v \cdot r_j < 0$, then

$$\rho \cdot w_j(v) = \rho \cdot \left(v - \frac{2v \cdot r_j}{r_j^2} r_j \right) < \rho \cdot v.$$

Both $\rho \cdot v$ and $\rho \cdot w_j(v)$ are positive integers and the set of vectors v with $0 < \rho \cdot v \leq \text{const}$ is finite. Therefore, after finitely many reflections in $h'(r_j), h' \in W_J$, we arrive at an element $h(v), h \in W_J$, such that $r_j \cdot h(v) \geq 0$ for $j \in J$. For all $i \in I$, we already have

$$r_i \cdot h(v) = h^{-1}(r_i) \cdot v \geq 0.$$

Thus, $h(v) \in \mathfrak{R}$ and $v \in h^{-1}(\mathfrak{R})$. This proves (1). Parts (2) and (3) are immediate.

For (4), clearly, each face of $\mathfrak{L}$ is of the form $g(F)$ for some face F of $\mathfrak{R}$. A face $F = \mathfrak{R} \cap_{i \in V} r_i^\perp$ is *not* a face of $\mathfrak{L}$ if the images $g(\mathfrak{R})$ for $g \in W_J$ cover its open neighborhood. This happens when $W_V \subset W_J$, i.e. $V \subset J$ and F is irrelevant. $\square$

We now apply this to our lattice $N = H \oplus E_8^2 \oplus A_1$, the 24 roots r_k , and the sets $I = \{0, \dots, 17\}$ and $J = \{18, \dots, 23\}$. In this situation, the cone $\mathfrak{L}$ has infinitely many faces and an infinite stabilizer group in $O(N) = S_3 \ltimes W(N)$. This explains the name *semifan* that we use for the generalized Coxeter semifan $\mathfrak{F}^{\text{semi}}$.

Corollary 4.18. *The semifan $\mathfrak{F}^{\text{semi}}$ is a coarsening of the Coxeter fan $\mathfrak{F}^{\text{cox}}$. For two elliptic or maximal parabolic subdiagrams $G_1, G_2 \subset G_{\text{cox}}$, the corresponding cones of $\mathfrak{F}^{\text{cox}}$ map to the same cone in $\mathfrak{F}^{\text{cox}}$ if and only if $G_1^{\text{rel}} = G_2^{\text{rel}}$.*

Remark 4.19. The same construction applies to an elliptic lattice or parabolic ambient diagram G_{cox} . When the subdiagram J is elliptic, the Weyl group W_J is finite. In this case, the resulting semifan is a fan, and it can be alternatively defined as the normal fan of a permutahedron.

The fan $\mathfrak{F}^{\text{cox}}$ itself is the normal fan of the permutahedron, that is, an infinite polyhedron $\text{Conv}(W.p)$ for a point p in the interior of $\mathfrak{R}$. If q is chosen to be on a lower-dimensional face of $\mathfrak{R}$ for an elliptic subdiagram J , with a finite Weyl group W_J , then $\mathfrak{F}^{\text{semi}}$ is again the normal fan of the permutahedron $\text{Conv}(W.q)$. This is basically the “Wythoff construction” for the uniform polytopes by Coxeter [13].

Looijenga [41, 42] has constructed a generalization of both the Baily–Borel and toroidal compactifications of an arithmetic quotients $\Gamma \backslash \mathbb{D}$ of a symmetric Hermitian domain. The starting data is a semifan supported on $\overline{\mathcal{C}}$ in which the cones are not assumed to be finitely generated or to have finite stabilizers. For example, the Baily–Borel compactification corresponds to the semifan consisting only of the cone $\overline{\mathcal{C}}$ itself and its null rays.

Definition 4.20. Let $\overline{F}_2^{\text{semi}}$ be the semi-toric compactification for the generalized Coxeter semifan $\mathfrak{F}^{\text{semi}}$.

Theorem 4.21. *There is a morphism $\overline{F}_2^{\text{tor}} \rightarrow \overline{F}_2^{\text{semi}}$, an isomorphism on $\Gamma \backslash \mathbb{D}$, whose induced map on strata is isogenous to the natural map of tori*

$$\text{Hom}(M_G, \mathbb{C}^*) \rightarrow \text{Hom}(M_{G^{\text{rel}}}, \mathbb{C}^*), \quad \text{Hom}(M_G, \mathbb{E}) \rightarrow \text{Hom}(M_{G^{\text{rel}}}, \mathbb{E})$$

in Types III, II, respectively.

Proof. This follows directly from Proposition 4.17, Corollary 4.18, and the functoriality of the semi-toric construction under refinement of semifans. $\square$

Note that $|G| - |G^{\text{rel}}| \leq 3$, with the maximum achieved when there are three A_1^{irr} diagrams in G . So the largest fiber dimension of the morphism in Theorem 4.21 is 3. For the Type II boundary strata, $G = G^{\text{rel}}$ except when $G = \tilde{E}_8^2 \tilde{A}_1$, in which case the morphism of Theorem 4.21 loses 1 dimension.

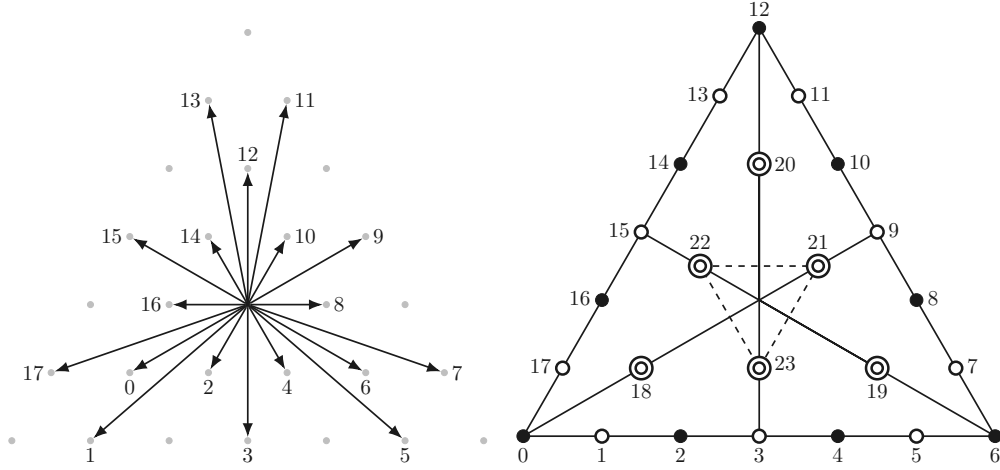


Figure 4. Fan of the toric surface $\overline{T}$ and the dual graph of negative curves on the surface $T = \text{Bl}_{p_0, p_6, p_{12}}(\overline{T})$.

5. The mirror surfaces

In Dolgachev–Nikulin–Voisin mirror symmetry for K3 surfaces [14], the 19-dimensional moduli space F_2 of polarized K3 surfaces with a rank 1 Picard lattice $\mathbb{Z}h$, $h^2 = 2$, is mirror-symmetric to the 1-dimensional moduli space of lattice-polarized K3 surfaces with a primitive rank 19 sublattice $H \oplus E_8^2 \oplus A_1 \subset \text{Pic } S$. We describe the latter explicitly and show that, for a general surface S , its nef cone can be identified with the fundamental chamber $\mathfrak{K}$ of the Coxeter fan $\mathfrak{F}^{\text{cox}}$.

The K3 surfaces in this family admit several elliptic fibrations, one of which contains an I_{18} Kodaira fiber. It turns out that they also come with an involution that fixes this I_{18} fiber, and the quotient surface $T = S/\iota$ is a non-minimal rational elliptic surface with an I_9 Kodaira fiber in its minimal form.

5A. A toric model. We begin with a toric surface $\overline{T}$ whose fan is depicted in Figure 4 on the left. It is easy to see that $\overline{T}$ is smooth and projective. For each ray, we have a boundary curve $\overline{F}_i$. One has $\overline{F}_i^2 = -3$ for $i = 0, 6, 12$, $\overline{F}_i^2 = -4$ for other even i , and $\overline{F}_i^2 = -1$ for odd i . The Picard rank is $\rho(\overline{T}) = 16$. There are three toric rulings $\overline{T} \rightarrow \mathbb{P}^1$ corresponding to the opposite pairs of rays numbered 0, 9, 6, 15, and 12, 3, respectively.

5B. A rational elliptic surface. We define T as the blow-up of $\overline{T}$ at three points $P_i \in \overline{F}_i$, $i = 0, 6, 12$, each corresponding to the identity $1 \in \mathbb{P}^1$ under the torus action. Let the exceptional divisors of this blow-up be F_{18} , F_{19} , F_{20} , and let F_i for $0 \leq i < 18$ be the strict transforms of the divisors $\overline{F}_i$ on $\overline{T}$.

The fiber over P_0 in the first ruling defined above is, after pullback, $F_{18} + F_{21}$, where F_{21} is a (-1) -curve intersecting F_9 . Similarly, the pulled back fiber of the second fibration over P_6 is $F_{19} + F_{22}$, and the pulled back fiber of the third fibration over P_{12} is $F_{20} + F_{23}$. One has $F_i^2 = -4$ for the even $0 \leq i < 18$, and -1 for all other i . The intersection graph of F_i 's is given in Figure 4 on the right. The black vertices correspond to the (-4) -curves and white vertices to the (-1) -curves. For the solid edges, one has $F_i \cdot F_j = 1$, and for the dashed edges, $F_i \cdot F_j = 3$.

The divisor

$$F = \sum_{i=0}^{17} F_i$$

satisfies $\mathcal{O}_F(F) \simeq \mathcal{O}_F$ and defines an elliptic fibration $T \rightarrow \mathbb{P}^1$. Contracting the nine (-1) -curves $F_1, F_3, \dots, F_{17}$ gives a relatively minimal elliptic fibration with an I_9 Kodaira fiber and three I_1 fibers. This is the extremal elliptic surface X_{9111} in the terminology of [44, Theorem 4.1]; it has three sections and three bisections, given by F_i for $18 \leq i < 24$. The exceptional curves not lying in the fibers are precisely the sections. Thus, F_i for $0 \leq i < 24$ are all the negative curves on T .

5C. An elliptic K3 double cover. Let $\pi: S \rightarrow T$ be the double cover ramified in the nine (-4) -curves $F_0, F_2, \dots, F_{16}$ and another fiber F' of the elliptic fibration. Since there are three special I_1 fibers, one gets a 1-parameter family of such surfaces, with three members of the family having a rational double point. For a very general choice of F' , one has $\rho(S) = 19$. A more detailed discussion of the moduli space of these mirror K3s may be found in [15, Section 5].

For the preimages E_i of the exceptional curves, one has $\pi^*(F_i) = 2E_i$ for the even $0 \leq i < 18$ and $\pi^*(F_i) = E_i$ for all other i . Then $E_i^2 = -2$ for all $0 \leq i < 24$ and the intersection graph of E_i 's is the Coxeter graph of Figure 2. Thus, E_i generate a 19-dimensional lattice $N = H \oplus E_8^2 \oplus A_1$. Since $\det N = 2$ is square-free, it follows that $\text{Pic } S = N$. Thus, S is a 2-elementary K3 surface described by Nikulin and Kondō. Note that the graph of the (-2) -curves in [35, Figure 1] is exactly our Coxeter graph.

The elliptic fibration on T induces an elliptic fibration on S with an I_{18} fiber, which is $\tilde{A}_{17}$ in Dynkin notation. The preimage of a ruling on T for the rays 0, 9 (or 6, 15 or 12, 3) gives an elliptic fibration on S with $\tilde{E}_8 \tilde{E}_8 \tilde{A}_1$ fibers. The preimage of a ruling for the rays 2, 10 (or 4, 14 or 8, 16) gives an elliptic fibration with $\tilde{D}_{10} \tilde{E}_7$ fibers. The three subdiagrams $\tilde{D}_{16} \tilde{A}_1$ give yet three more elliptic fibrations on S which also double cover rulings on T , see Section 5E.

5D. The nef cones of the rational and K3 surfaces.

Lemma 5.1. *For a surface S as above with $\rho = 19$, the nef cone is a finite, polyhedral cone equal to*

$$\text{Nef}(S) = \{\lambda \mid \lambda \cdot E_i \geq 0 \mid 24 \text{ curves } E_i\}.$$

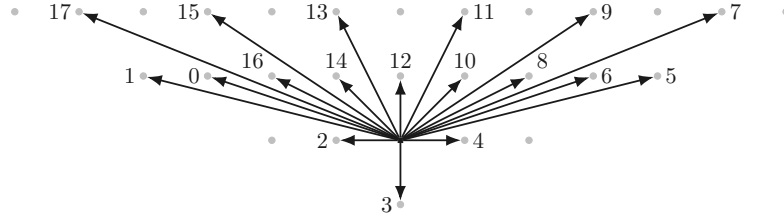
Under the identification $\text{Pic}(S) = N$, it maps isomorphically to a fundamental chamber $\mathfrak{K}$ of the Coxeter fan $\mathfrak{F}^{\text{cox}}$. The double cover defines identifications

$$\pi^*: \text{Pic}(T)_{\mathbb{Q}} \xrightarrow{\sim} \text{Pic}(S)_{\mathbb{Q}}, \quad \pi^*: \text{Nef}(T) \xrightarrow{\sim} \text{Nef}(S).$$

However, the lattice structures on $\text{Pic}(T)$ and $\text{Pic}(S)$ are different.

Proof. The nef cone of an algebraic surface is the intersection of the closure of the positive cone $\mathcal{C} = \{\lambda \mid \lambda^2 > 0, \lambda \cdot h > 0\} \subset \text{Pic}(S)_{\mathbb{R}}$ with the half spaces $\lambda \cdot E \geq 0$ for the irreducible curves E with $E^2 < 0$. By [35], the 24 curves E_i are the only negative curves on S . We thus get the same inequalities that define a fundamental chamber $\mathfrak{K}$ of $\mathfrak{F}^{\text{cox}}$.

The pullback of a negative curve is a sum of negative curves. Thus, F_i for $0 \leq i < 24$ are the only negative curves on T , and $\pi^*: \text{Nef}(T) \xrightarrow{\sim} \text{Nef}(S)$. $\square$

Figure 5. Fan of the toric surface $\overline{T}$.

5E. A second toric model. Let $T \rightarrow \overline{T}$ be the contraction of the disjoint (-1) -curves F_{19}, F_{20}, F_{21} . Just as $\overline{T}$, the surface $\overline{T}$ is also a smooth projective toric surface with $\rho(\overline{T}) = 16$. Its fan is shown in Figure 5.

6. Family of IAS^2 over the Coxeter fan

We now define a family of polarized IAS^2 over the Coxeter fan (our “Voronoi” decompositions). We motivate the construction with mirror symmetry.

As we saw in Section 4, a compactification of F_2 is governed by a fan decomposition $\mathfrak{F}$ of the rational closure of the positive cone of $N = H \oplus E_8^{\oplus 2} \oplus A_1$. Each $\lambda \in N$, $\lambda^2 \geq 0$, determines a Picard–Lefschetz transformation of a one-parameter degeneration of complex structure, whose logarithm is given by $(\log T) \cdot x = (x \cdot \delta)\lambda - (x \cdot \lambda)\delta$. Mirror symmetry dictates that the complex moduli of F_2 are interchanged with the Kähler moduli of the Dolgachev–Nikulin–Voisin mirror K3 surface S from Section 5. This is instantiated in the isomorphisms $\text{Pic}(S) = N$, $\text{Nef}(S) = \mathfrak{R}$.

To make the mirror correspondence more precise, consider some $\lambda \in N$, $\lambda^2 > 0$. The symplectic geometry of (S, ω) in Kähler class $[\omega] = \lambda$ should be interchanged with the complex geometry of a degenerating family of degree 2 surfaces, whose monodromy vector is λ . We have a mechanism for this interchange—the Monodromy Theorem of Section 2G. It states that the IAS^2 on the base of a Lagrangian torus fibration $\mu: (S, \omega) \rightarrow B$ should be identified with the dual complex $B = \Gamma(\mathcal{X}_0)$ of a Kulikov degeneration $\mathcal{X} \rightarrow (C, 0)$ whose monodromy vector is λ .

Finally, we recall the construction $S \rightarrow T$ as a double cover of a rational surface. This motivates a construction of B for any monodromy vector λ and thus any Type III degeneration: we should produce a Symington polytope P for the rational surface T , then glue two copies $B = P \cup P^{\text{op}}$ together to form an IAS^2 which is the base of a Lagrangian torus fibration $\mu: (S, \omega) \rightarrow B$ satisfying $[\omega] = \lambda$.

We also give an explicit description of the Type II degenerations corresponding to the cusps of $\mathfrak{R}$, when $\lambda^2 = 0$.

6A. Construction of IAS^2 . Let $\pi: S \rightarrow T$ be the double cover of a special K3 rational by a special K3 surface as defined in Section 5. Let $L \in \text{Pic}(T) \otimes \mathbb{R}$ be a nef class. Let $a_i = \pi^*(L) \cdot E_i$ for the (-2) -curves $E_i \subset S$, and let $b_i = L \cdot F_i$ for the (-1) - and (-4) -curves on T . Thus, $a_i = b_i$ for the even $0 \leq i < 18$, and $a_i = 2b_i$ for all other i .

Let $\phi: T \rightarrow \bar{T}$ be the blow-up of the first toric model, which contracts exceptional curves F_{18}, F_{19}, F_{20} meeting F_0, F_6, F_{12} . Set $\bar{b}_i = \bar{L} \cdot \bar{F}_i$. Then

$$L = \phi^*(\bar{L}) - b_{18}E_{18} - b_{19}E_{19} - b_{20}E_{20},$$

$$b_0 = \bar{b}_0 - b_{18}, \quad b_6 = \bar{b}_6 - b_{19}, \quad b_{12} = \bar{b}_{12} - b_{20}, \quad b_i = \bar{b}_i \text{ for other } i.$$

Construction 6.1. In Lemma 5.1, we identified the nef cones of S and T with a fundamental chamber $\mathfrak{K}$ of the Coxeter fan $\mathfrak{F}^{\text{cox}}$. So let $L = \vec{b} \in \text{Nef}(T) = \mathfrak{K}$ be a nef $\mathbb{R}$ -divisor.

First, assume that all $b_i > 0$; a fortiori, all $\bar{b}_i > 0$. Let P be the Symington polytope obtained from the moment polytope $\bar{P}$ for $\bar{L}$ by three almost toric blow-ups of size b_{18}, b_{19}, b_{20} on sides 0, 6, 12 as shown in Figure 6. This introduces three I_1 singularities in the interior of P whose monodromy-invariant lines parallel the side from which the surgery triangle was removed. So P is an integral-affine disc with 18 boundary components. The location of the cut on the sides 0, 6, 12 can be chosen arbitrarily; ultimately, choices will produce Kulikov models differing by “nodal slides” defined below, which do not affect anything. We make *the symmetric choice*: with the cut centered around the midpoint of the side.

By [18, Theorem 5.3], the class L on T is nef if and only if it is possible to fit surgery triangles of the appropriate size inside the polytope for a toric model without overlapping. In our case, this is also easy to see directly.

Define an integral-affine sphere $B := P \cup P^{\text{op}}$ by gluing together two copies of P . This requires introducing an I_1 singularity at each corner of P , whose monodromy-invariant lines are shown dashed in Figure 6. More precisely, we can take the top figure for P in Figure 6, and take its isometric reflection along the edge 3 (with edges 9 or 15, it is similar). This produces a copy of P^{op} attached to P along 3, but there is a gap between edge 4 and its reflection. This gap is closed exactly by gluing edge 4 and its reflection with a unit shear in the dotted direction. Once this gluing is made, we must introduce another singularity to glue edge 5 and its reflection. And so on for edges 0, 1, 2, 3, 4, 5, 6 (and similarly for the other edges).

The general case is obtained as a limit of the above construction by sending some of the b_i ’s to zero.

Definition 6.2. For any real vector $\vec{a} = (\lambda \cdot r_i)_{i \in \{0, \dots, 23\}}$ with $\lambda \in \mathfrak{K}$, $\lambda^2 > 0$, this construction defines an integral affine structure $B(\vec{a})$ on a sphere with 24 singularities, some of which may coalesce, an IAS² for short. We sometimes suppress the dependence on $\vec{a}$.

When all $a_i > 0$, we define an integral-affine divisor R_{IA} whose supporting graph is the equator, that is, the common boundary of P and P^{op} . The multiplicities are 2 for the even sides and 1 for the odd sides. The assumption $a_i > 0$ implies that the IAS² has 18 isolated I_1 singularities on the equator. By Remark 2.32, this suffices to define R_{IA} uniquely.

When some $a_i = 0$, the definition of R_{IA} is quite subtle. It is delayed until Section 6C, but the supporting graph is still the equator, and the multiplicities are the same values, 1 and 2, for the odd and even sides $i \in \{0, \dots, 17\}$ with $a_i \neq 0$.

The pair (B, R_{IA}) is an analogue of a Voronoi decomposition in the case of abelian varieties. As $\vec{a}$ varies continuously, so do they.

Lemma 6.3. One has $(\pi^*L)^2 = 2L^2 = \text{vol}(B)$, where the latter is the lattice volume, twice of the Euclidean one.

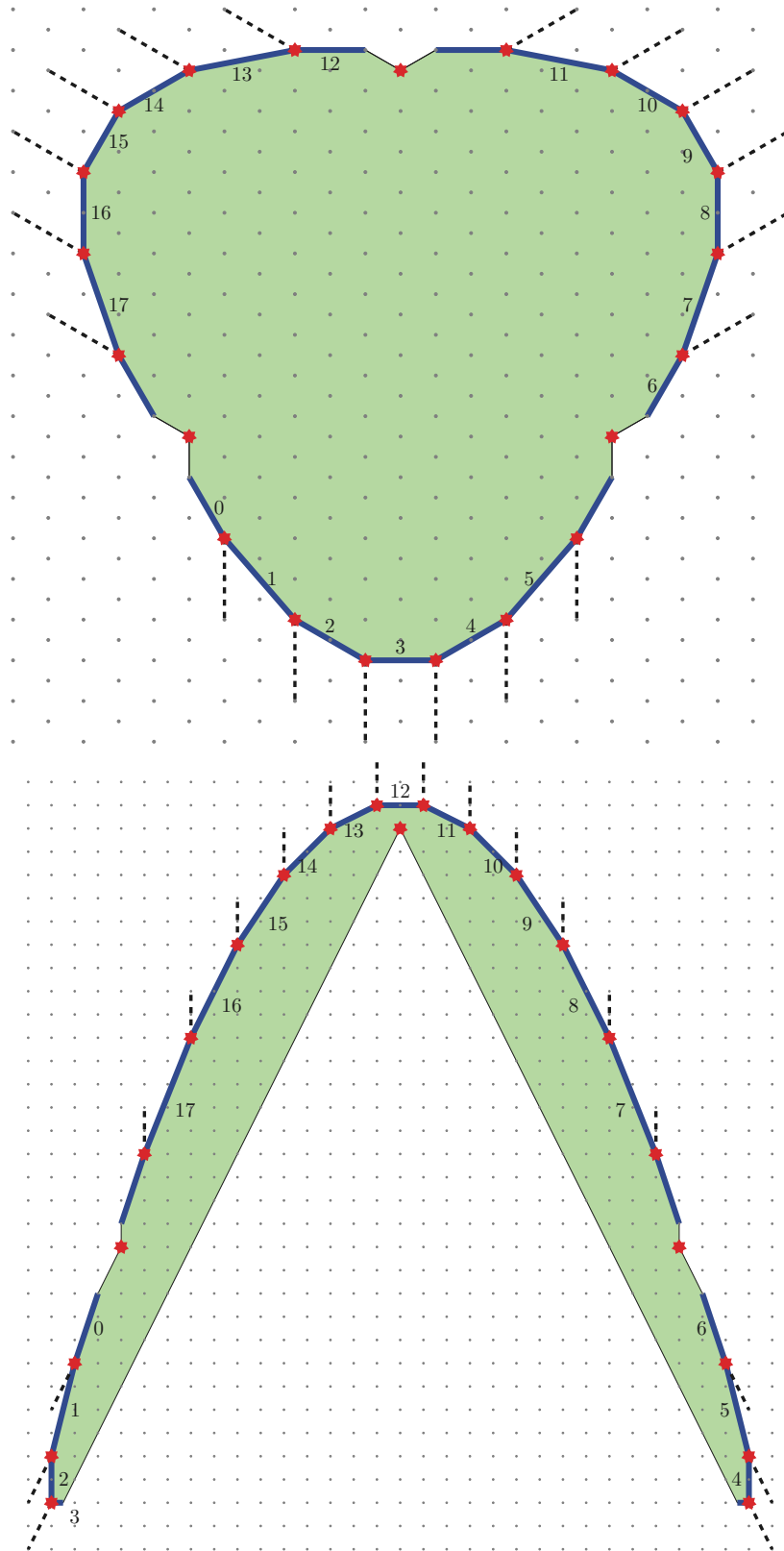


Figure 6. An example of the same IAS^2 glued in two different ways.

Proof. By definition, $\text{vol}(B) = 2 \text{vol}(P)$ and $\text{vol}(P) = \text{vol}(\overline{P}) - b_{18}^2 - b_{19}^2 - b_{20}^2$. It is easy to see that $L^2 = \overline{L}^2 - b_{18}^2 - b_{19}^2 - b_{20}^2$. For any toric variety with a nef class, its volume is the lattice volume of the moment polytope; this gives $\overline{L}^2 = \text{vol } \overline{P}$. $\square$

Remark 6.4. By definition, b_{18} is the lattice distance from the singularity to the side 0. The linear relation $n(\tilde{E}_8) = n(\tilde{A}_1)$ of Lemma 4.7 implies that b_{21} is the lattice distance to the opposite side 9. Similarly for b_{19}, b_{22} and b_{20}, b_{23} .

Example 6.5. Figure 6 shows a concrete example with

$$\begin{aligned} \bar{b}_0 = \bar{b}_6 = \bar{b}_{12} = 3, \quad b_2 = b_4 = \dots = b_{16} = 2, \quad b_1 = b_3 = \dots = b_{17} = 1, \\ b_{18} = b_{19} = b_{20} = 1, \quad b_0 = b_6 = b_{12} = 2, \quad b_{21} = b_{22} = b_{23} = 29. \end{aligned}$$

The green interior region is an open chart for the integral-affine structure on the disc P . In the a -coordinates, $a_i = 2 \cdot 1$ for $0 \leq i < 21$ and $a_{21} = a_{22} = a_{23} = 2 \cdot 29$.

The second picture gives an alternative way of presenting the same IAS^2 , using the second toric model $\overline{T}$. It is obtained by cutting a different ray emanating from the I_1 singularity, which, instead of hitting the edge 12, goes in the opposite direction, towards edge 3.

Recall that, in Lemma 4.7, we defined the vectors $n(\tilde{\Lambda})$ for the affine Dynkin diagrams $\tilde{A}_{17}, \tilde{E}_8, \tilde{A}_1^{\text{irr}}, \tilde{D}_{10}, \tilde{E}_7, \tilde{D}_{16}$ and $\tilde{A}_1^*$, using notations of Table 1.

Lemma 6.6. *The circumference in the vertical direction, that is, twice the lattice distance in P and in $\overline{P}$ between the sides 3 and 12, is $\text{ev}(n_{3,12})$, where*

$$n_{3,12} = n(\tilde{E}_8^{(1)}) = n(\tilde{E}_8^{(2)}) = n(\tilde{A}_1^{\text{irr}})$$

and $\text{ev}(r_i) = a_i$ is the evaluation map. Similarly, the circumference in the 8-16 direction is $\text{ev}(n_{8,16})$, $n_{8,16} = n(\tilde{D}_{10}) = n(\tilde{E}_7)$; the circumference in the 2-4 direction in the second presentation (i.e. around a singularity, close to the side 12) is $\text{ev}(n_{2,4})$, $n_{2,4} = n(\tilde{D}_{16}) = n(\tilde{A}_1^*)$; and the circumference along the equator is $\text{ev}(n(\tilde{A}_{17}))$.

Proof. This follows by observation using Lemma 4.7. $\square$

Example 6.7. In the example of Figure 6, all the $a_i = 2$ for $i \neq 21, 22, 23$. It follows and is indeed very amusing to observe that the projections of sides 13, 14, 15, 16, 17, 0, 1, 2, and 18 to a vertical line have lattice lengths 1, 2, 3, 4, 5, 6, 4, 2, and 3, which are the multiplicities of the simple roots in $n(\tilde{E}_8)$. Similarly, the projections of the sides 18, 17, 0, $\dots$, 6, 7, 19 to a horizontal line have lattice lengths 1, 1, 2, $\dots$, 2, 1, 1, which are the multiplicities for $\tilde{D}_{10}$, and similarly for $\tilde{E}_7$.

Corollary 6.8. *Near the rays $L^2 = 0$ of $\mathfrak{K} = \text{Nef}(S)$, the sphere B with its integral-affine structure degenerates to an interval as follows.*

- (1) $\tilde{A}_{17}$. The Symington polytope P degenerates to a segment from the boundary of P to the north pole, and B degenerates to a longitude.
- (2) $\tilde{D}_{10}\tilde{E}_7$. Both P and B degenerate to the side 8, identified with the side 16.

- (3) $\tilde{E}_8^2 \tilde{A}_1$. Both P and B degenerate to the side 3, identified with the side 12.
- (4) $\tilde{D}_{16} \tilde{A}_1$. Both P and B degenerate to the side 2, identified with the side 4.

In cases (2), (3), (4), the interval lies in the equator.

Definition 6.9. We define the family of IAS^2 over the fundamental chamber $\mathfrak{R}$ by Construction 6.1. By Lemma 6.3, this gives a family outside of the boundary rays with $L^2 = 0$, where IAS^2 degenerates to an interval.

We then extend it to a family over $\overline{\mathcal{C}}$ by reflections in the Weyl group $W(N)$. This is well defined because $\mathfrak{R}$ is a fundamental domain of the reflection group and because, on the boundary of $\mathfrak{R}$ where some $a_i = 0$, the limits of the structures from both sides coincide.

Remark 6.10. As we mentioned, the locations of the cuts on the sides 0, 6, 12 are quite arbitrary and may be moved by a “nodal slide”. Instead of the symmetric choice for the cuts, one could also make a “vertex-preferred” choice: for this choice, if b_i are integral, then the coordinates of the three internal singularities are also integral. For the symmetric choice, they are only half-integral.

This is quite similar to the case of abelian varieties where, given an integral positive-definite symmetric bilinear form $B: M \times M \rightarrow \mathbb{Z}$ on $M \simeq \mathbb{Z}^g$, the Voronoi decomposition $f_B(\text{Vor } B)$ in $N_{\mathbb{R}} = M_{\mathbb{R}}^*$ has only half-integral coordinates, but in low dimensions, there is a “vertex-preferred” linear shift ℓ so that $\ell + f_B(\text{Vor } B)$ has vertices in the lattice $N = M^*$.

6B. Collisions of singularities in IAS^2 . We now describe how the 24 singularities collide and the resulting singularities of the integral-affine structures.

Theorem 6.11. *For a big and nef class $L \in \text{Nef}(S)$, the possibilities for the collisions of the 24 singular points are in bijection with the elliptic subdiagrams G of the Coxeter diagram G_{cox} , excepting 6.12. Each collision point, excepting 6.12, is in bijection with a connected component G_k of G .*

Proof. In Lemma 5.1, we made an identification of $\text{Nef}(S)$ with the fundamental chamber $\mathfrak{R}$. Now we simply apply Theorem 4.3. With the noted insignificant exceptions, the collisions correspond to the collections of indices $\{i \mid a_i = 0\} \subset \{0, \dots, 23\}$, i.e. to the faces of $\mathfrak{R}$, by virtue of Construction 6.1. $\square$

6.12. The exceptions, which play no role in the end, occur as artifacts of the “symmetric choice” of cuts for the Symington polytope P . A collision is insignificant if a different choice of cut would get rid of the collision, for instance, when two cuts are made that have the same apex in the interior of P .

Lemma 6.13. *The singularities appearing $B(\vec{a})$ as some collection of $a_i \rightarrow 0$ for $i \in G_k$ (such subdiagrams are listed in Table 1) are exactly those listed in Table 2 with the same Dynkin label.*

Decorations “—” from Table 1 are dropped in Table 2 as they do not affect the resulting integral-affine singularity.

Definition	Name	Charge
$I(n+1)$	A_n	$n+1$
$I(2, 2, n-2)$	D_n	$n+2$
$I(2, 3, 3)$	E_6	8
$I(2, 3, 4)$	E_7	9
$I(2, 3, 5)$	E_8	10
$I(2, 3, n-3)$	D'_{n-1}	$n+2$
$I(n+1, 1)$	$'A_n$	$n+2$
$I(n, n, 2)$	$'A'_{2n-1}$	$2n+2$

 Table 2. Possible integral-affine singularities on $B(\vec{a})$ for some $\vec{a} \in \mathfrak{R}$.

Proof. A singularity resulting from a collision as $a_i \rightarrow 0$ is determined by (and in fact, is defined by, see Definition 2.13) tracking the monodromy directions of the I_1 singularities as they coalesce. This presents the coalesced singular point in the form $I(n_1 \vec{v}_1, \dots, n_k \vec{v}_k)$. The results are determined by direct geometric examination of Figure 6, and tabulated in Table 2.

For example, the E_8 diagram formed from nodes $i \in \{18, 16, 17, 0, 1, 2, 3, 4\}$ of G_{cox} corresponds to the coalescence where these lengths a_i all approach zero. This results in the collision of 10 total I_1 singularities. Choosing monodromy-invariant cut directions for each singularity in a counterclockwise fashion about the center of edge 0 (like a windmill) and letting $a_i \rightarrow 0$, we see that this collision can be presented as $I(5, 1, 3, 1) \sim I(2, 3, 5)$, which is the “ E_8 ” integral affine singularity. $\square$

6C. Polarization of the IAS². We now define the polarizing divisor R_{IA} on B , when some singularities have collided. By Definition 2.30 and Remark 2.35, the data of R_{IA} is specified by a nef line bundle L_i on an anticanonical pair (V_i, D_i) for which $\mathfrak{F}(V_i, D_i)$ models each integral-affine singularity. This line bundle is furthermore required to have intersection numbers $n_{ij} = L_i \cdot D_{ij}$ agreeing with the weighted balanced graph in B .

The graph underlying R_{IA} is supported on the equator and has exactly two nonzero weights $n_{ij} \in \{1, 2\}$ emanating from an equatorial vertex $v_i \in \mathfrak{F}(V_i, D_i)$. These weights are notationally incorporated into the decorations of the corresponding Dynkin subdiagram G_k by the $-$ and $'$ decorations, see the discussion following Table 1. For each singularity, we must give an anticanonical pair (V_i, D_i) in the c.b.e.c. describing the singularity, and the appropriate line bundle $L_i \rightarrow V_i$.

Theorem 6.14. *Let ι_{IA} be the involution of B switching the hemispheres P, P^{op} and fixing the equator pointwise. For each singularity v_i on the equator of B , consider the deformation class of “involution pairs” $(\bar{V}_i, \bar{D}_i + \epsilon \bar{R}_i)$, see [7] and Section 7A, notated in [7] by exactly the same decorated Dynkin symbol of the corresponding subdiagram, listed in Table 1.*

Let $\bar{\iota}_i$ be the involution, so $\bar{R}_i = \text{Fix}(\bar{\iota}_i)$, and let ι_i be the induced involution on the minimal resolution $\pi_i: (V_i, D_i) \rightarrow (\bar{V}_i, \bar{D}_i)$. Then $v_i = \mathfrak{F}(V_i, D_i)$ as integral-affine singularities, and ι_i induces the same the action as ι_{IA} . Furthermore, denoting $R_i = \pi_i^(\bar{R}_i)$, the nef line*

bundle $L_i := \mathcal{O}_{V_i}(R_i)$ has intersection numbers $n_{ij} = L_i \cdot D_{ij}$ which give the weighted graph on the equator described in Definition 6.2.

Proof. Essentially, the proof is by direct calculation of all the cases. We simply check that $\mathfrak{F}(V_i, D_i)$ is the correct integral-affine singularity, and $L_i \cdot D_{ij}$ are the correct values. We perform this check below for some representative examples. $\square$

Remark 6.15. The proposition should not come as a surprise—the notation for subdiagrams of G_{cox} was reverse-engineered so that Theorem 6.14 becomes true.

For notational convenience, we drop the index i .

Example 6.16 ($\bar{A}_0$ and A_0^-). $(V, D) = (\mathbb{P}^2, D_1 + D_2)$ is a projective plane with a line D_1 plus conic D_2 as anticanonical divisor. The singularity of $\mathfrak{F}(V, D)$ is an I_1 singularity, and the degrees for R_{IA} must be 1, 2 on the components D_1, D_2 corresponding to equatorial edges, respectively.

The pair (V, D) admits an involution ι fixing another line R , and an isolated point on D_1 . The line bundle $L = \mathcal{O}_V(R) = \mathcal{O}_{\mathbb{P}^2}(1)$, which gives the correct multiplicities 1, 2 on the two equatorial edges of R_{IA} . The two cases $\bar{A}_0$ and A_0^- are, respectively, distinguished by whether the line is on the left (clockwise) or right (counterclockwise) side of the equator.

Example 6.17 (A_{2n-1}). As the singularity is $I(2n) = I(n, 0, n, 0)$, see Remark 2.21, we can model the c.b.e.c. as the blow-up

$$(V, D) = (V, D_1 + D_2 + D_3 + D_4) \rightarrow (\mathbb{P}^1 \times \mathbb{P}^1, s_1 + f_1 + s_2 + f_2),$$

at n points on s_1 , then n points on s_2 . The edges corresponding to the equator of B correspond to the two fibers f_1, f_2 and are required to intersect the polarization with degree 2. There is an involution $\iota: (V, D) \rightarrow (V, D)$ preserving f and switching the strict transforms of s_1 and s_2 , which have classes

$$D_1 = s - \sum_{i=1}^n e_i \quad \text{and} \quad D_3 = s - \sum_{i=n+1}^{2n} e_i.$$

Here, e_i are the exceptional divisors.

Assuming the points blown up are chosen generically, the ramification divisor of ι is the strict transform of the divisor on $\mathbb{P}^1 \times \mathbb{P}^1$ in the linear system $|2s + nf|$ which passes through all the $2n$ points. It has the class $R = 2s + nf - \sum_{i=1}^{2n} e_i$ with $R^2 = 2n$. The line bundle $L = \mathcal{O}_V(R)$ has intersection numbers $L \cdot D_2 = L \cdot D_4 = 2$ and $L \cdot D_1 = L \cdot D_3 = 0$ with the boundary. Thus, it gives the correct intersection numbers for R_{IA} as it passes through an $I(2n)$ singularity on the equator.

Finally, the stable model $\bar{V}$ is the result of contracting D_1 and D_3 which are the only curves on which L has degree 0. The involution descends and defines the A_{2n-1} involution pair from [7].

Example 6.18 (A'_{2n-1}). As the singularity is $I(n, n, 2) = I(n, 1, n, 1)$, the c.b.e.c. is represented by blowing up an A_{2n-1} pair at one point on each of f_1, f_2 with the exceptional classes g_1, g_2 . We blow up at a point in $R \cap f_1$ and $R \cap f_2$, respectively. So the resulting pair $(V, D_1 + D_2 + D_3 + D_4)$ still admits an involution ι lifting the involution of the A_{2n-1} pair.

The boundary curves have classes

$$D_1 = s - \sum_{i=1}^n e_i, \quad D_3 = s - \sum_{i=n+1}^{2n} e_i, \quad D_2 = f - g_1, \quad D_4 = f - g_2,$$

$$R = 2s + nf - \sum_{i=1}^{2n} e_i - (g_1 + g_2)$$

The polarization is defined to be $L = \mathcal{O}_V(R)$, and note that

$$L \cdot D_2 = L \cdot D_4 = 1 \quad \text{and} \quad L \cdot D_1 = L \cdot D_3 = 0,$$

as desired. The stable model is again the result of contracting D_1 and D_3 and gives the A'_{2n-1} involution pair.

Example 6.19 (D_{2n}). The easiest model for D_{2n} is a pair $(V, D_1 + D_2)$, whose components are a fiber f and the $2n$ -fold blow-up of a bisection in class $2s + f$, on $\mathbb{P}^1 \times \mathbb{P}^1$. Taking some corner blow-ups and a toric model, one finds the pseudo-fan is

$$\mathfrak{F}(V, D) = I(2, 2, 2n - 2),$$

as desired, and that D_1 and D_2 correspond to the edges emanating from v along the equator.

There is an involution ι preserving f and switching the two sheets of the bisection. Its ramification divisor has class $R = 2s + nf - \sum_{i=1}^{2n} e_i$, $R^2 = 2n$. Setting $L = \mathcal{O}_V(R)$, one has $L \cdot D_1 = L \cdot D_2 = 2$, as desired. In this case, L is already ample, so the stable model is the same surface, and it is the D_{2n} involution pair.

Example 6.20 (D'_{2n}). The D'_{2n} surface is obtained from the D_{2n} surface by an additional blow-up at one of the two points $D_1 \cap R$. This gives the singularity $I(2, 3, 2n - 2)$, which is the same as for E_n , but in these two cases, the equator sits differently with respect to the shearing rays. The involution on the D_{2n} pair lifts to give an involution.

Example 6.21 (E_n). For E_8 , the singularity is $I(2, 3, 5) = I(5, 1, 3, 1)$, which can be represented by blowing up 5, 1, 3, 1 points, respectively, on the four sides of an anticanonical square in $\mathbb{P}^1 \times \mathbb{P}^1$. Contracting the two boundary exceptional curves gives the blow-up of a nodal cubic in $\mathbb{P}^2$ at 8 smooth points, and at the node. Then R is the fixed locus of the Bertini involution, which intersects each boundary component with degree 1, as desired. Here, $L = \mathcal{O}(R)$ is already ample.

For $\bar{E}_7$, there are 7 blow-ups on the cubic and an additional blow-up at the node, and for $\bar{E}_6^-$, there are two more blow-ups at the node. The involution is the Geiser involution. See [7, § 4.5] for more details.

Remark 6.22. In Examples 6.17, 6.18, 6.19, 6.20, 6.21, the description of R as $\text{Fix}(\iota)$ is valid only when the blow-ups are chosen generically. This is because, as we vary the blow-up points on a smooth anticanonical pair, the fixed loci $\text{Fix}(\iota)$ do not vary in a flat manner. The resolution of this issue is to work with the ADE surfaces of [7], on which $\bar{R}_i = \text{Fix}(\bar{\iota}_i)$ does vary in a flat manner. Then the pullback R_i to the minimal resolutions also varies in a flat manner.

The same phenomenon occurs even for smooth degree 2 K3 surfaces acquiring a (-2) -curve, such as the minimal resolution of a double cover of $\mathbb{P}^2$ ramified over a nodal sextic. The divisor R on such a smooth K3 is *not* $\text{Fix}(\iota)$ but rather the pullback of $\text{Fix}(\bar{\iota})$ for the involution $\bar{\iota}$ of the ADE K3 surface.

Proposition 6.23 (Reconstructing the polarization). *The line bundles L_i defining the polarization R_{IA} at a singularity $v_i = \mathfrak{F}(V_i, D_i)$ are uniquely characterized by*

- (1) *the intersection numbers $n_{ij} = L_i \cdot D_{ij} \in \{0, 1, 2\}$,*
- (2) *the ι_i -invariance of the class of L_i ,*
- (3) *$L_i^2 =$ the number of equatorial I_1 singularities involved in the collision.*

Proof. As for Theorem 6.14, this simply requires a direct check in all cases. $\square$

This completes the construction of a family $(B(\vec{a}), R_{\text{IA}})$ of polarized IAS^2 , varying over $\overline{C}$, which is combinatorially constant exactly along the cones of $\mathfrak{F}_{\text{cox}}$.

6D. Kulikov degenerations and their monodromy. The goal of this section is to verify that the monodromy invariant of a Kulikov model $\mathcal{X} \rightarrow (C, 0)$ whose central fiber satisfies $\Gamma(\mathcal{X}_0) = B(\vec{a})$ is in fact $\vec{a} \in \mathfrak{K}$.

Definition 6.24 (The parity condition). We say that $\vec{a} \in \mathbb{Z}^{24}$ *satisfies the parity condition* if $a_i \equiv 0 \pmod{2}$ for i odd, and all $i \geq 18$. Equivalently that all $b_i \in \mathbb{Z}$.

Let $N = H \oplus E_8^2 \oplus A_1$ be our standard lattice of signature $(1, 18)$ as in Section 4. For each vector $\vec{a} \in \mathbb{Z}_{\geq 0}^{24}$ coming from an integral point in $\mathfrak{K}$ and satisfying the parity condition, we define a combinatorial type of polarized Kulikov surface. Then we prove that a Kulikov degeneration with this central fiber has the monodromy λ .

Construction 6.25. Suppose that $\vec{a} \in \mathfrak{K}$ satisfies the parity condition so that $B(\vec{a})$ has singularities only at integral points. Let ι_{IA} be the orientation-reversing involution of $B(\vec{a})$ which switches P and P^{op} , fixing the equator pointwise. Choose an ι_{IA} -invariant triangulation $\mathcal{T}$ of $B(\vec{a})$ into triangles of lattice volume 1 which contains the equator R_{IA} as a set of edges.

We now apply Proposition 2.22 to produce a Kulikov surface $\mathcal{X}_0 = \bigcup_i (V_i, D_i)$ for which $\mathfrak{F}(V_i, D_i) = \text{star}(v_i)$ as an integral-affine surface, and $\Gamma(\mathcal{X}_0) = B(\vec{a})$. This specifies a unique deformation type of $\mathcal{X}_0$ but not its continuous moduli.

To choose from the continuous moduli, first, we pick an anticanonical pair (V_i, D_i) on the equator admitting an involution ι_i which induces ι_{IA} on $\mathfrak{F}(V_i, D_i)$. This is possible by Theorem 6.14. Then we glue the equatorial edges of $\mathcal{X}_0$ by ensuring that R_i glue into a Cartier divisor, i.e. $R_i \cap D_{ij} = R_j \cap D_{ji}$ as multisets. Finally, we glue the northern and southern hemispheres of $\mathcal{X}_0$ onto this equatorial band of surfaces, in an arbitrary involution-invariant manner.

The resulting Kulikov surface $\mathcal{X}_0$ admits an involution which we denote ι_0 and which acts on $\Gamma(\mathcal{X}_0)$ by ι_{IA} . Furthermore, by construction, there is a Cartier divisor $R \subset \mathcal{X}_0$ given by $R = \bigcup_i R_i$. We show in Section 6G that it is possible to glue so that this involutive surface

$\mathcal{X}_0$ is also d -semistable (see Section 2A), but for the moment, assume this. In particular, $\mathcal{X}_0$ is smoothable by [20].

Definition 6.26. We write $\mathcal{X}_0(\vec{a})$ for the Kulikov surface defined in Construction 6.25 and $\mathcal{X}(\vec{a})$ for a smoothing of it.

Theorem 6.27. *Let $\vec{a}$ satisfy the parity condition and suppose $B(\vec{a})$ is generic.*

- (1) *Let $\mathcal{X}(\vec{a}) \rightarrow (C, 0)$ be a Kulikov degeneration defined above.*
- (2) *Let $\mu: (S, \omega) \rightarrow B(\vec{a})$ be a Lagrangian torus fibration over $B(\vec{a})$.*
- (3) *Let $\phi: S \rightarrow \mathcal{X}_t$ be the diffeomorphism of Theorem 2.43.*

Define $v := \phi_[\Sigma_{\text{IA}}] \in \delta^\perp/\delta$, where $\Sigma_{\text{IA}} := \Sigma_{R_{\text{IA}}}$. Then $\{v, \delta\}^\perp/\delta$ is isometric to N and the monodromy invariant is $\lambda = \vec{a} \bmod O^+(N)$.*

Proof. By construction of ϕ , $\phi_*[\Sigma_{\text{IA}}] \in \delta^\perp/\delta$ is invariant under the Picard–Lefschetz transformation, hence perpendicular to the monodromy invariant λ . So $\lambda \in \{v, \delta\}^\perp/\delta$.

We describe a collection of 24 spheres $\{E_i\}$ of self-intersection -2 in (S, ω) , which intersect according to the Coxeter diagram G_{cox} . They are all presented as *non-Lagrangian* visible surfaces. Let γ_i for $i = 0, \dots, 17$ be the i -th edge of P . Then the monodromy-invariant vectors α_i at the two endpoints of γ_i are parallel. By Construction 2.39 and Example 2.41, there is a visible surface $E_i := \Sigma_{(\gamma_i, \alpha_i)}$ fibering over γ_i . Now, let $i = 18, 19, 20$. Define γ_i as a path which connects the singularity of P over the edge $0, 6, 12$, to the mirror singularity in P^{op} , crossing the edge $0, 6, 12$, respectively. As before, let $E_i := \Sigma_{(\gamma_i, \alpha_i)}$ where α_i is the (common) vanishing cycle at the two endpoints of γ_i . Finally, we define E_i for $i = 21, 22, 23$ similarly, but this time connecting the singularity in P to the mirror one in P^{op} via a path γ_i which crosses the edge $9, 15, 3$, respectively. It is an easy check that if the E_i are properly oriented, the intersection numbers $E_i \cdot E_j$ give exactly a system of roots as in G_{cox} .

We directly compute by perturbing and counting signed intersection points that

$$[\Sigma_{\text{IA}}] \cdot [E_i] = 0.$$

Since the classes $[E_i]$ generate a lattice of determinant 2 and rank 19, we conclude that $\phi_*[E_i]$ generate the rank 19 lattice $\{\delta, v\}^\perp/\delta$ over $\mathbb{Z}$ and that this lattice is isometric to N , with the isometry identifying $\phi_*[E_i]$ and r_i .

Finally, we wish to show λ and $\vec{a}$ define the same element of N modulo $O^+(N)$. We have the following formula for the symplectic area of a visible surface:

$$[\omega] \cdot E_i = \int_0^1 \det(\alpha_i, \gamma'_i(t)) dt = a_i$$

for all i . Hence $\lambda \cdot \phi_*[E_i] = a_i$ for all i . This shows that λ and $\vec{a}$ represent the same lattice point in $\mathfrak{R}$, i.e. $\lambda = \vec{a} \bmod O^+(N)$. $\square$

Corollary 6.28. *The vector $v \in \delta^\perp/\delta$ is imprimitive with $v = 3w$ and $w^2 = 2$.*

Proof. By taking a generic perturbation of Σ_{IA} off itself and counting signed intersections, we see that $v^2 = 18$. Also, v lies in $\text{span}\{\phi_*[E_i]\}^\perp \subset \delta^\perp/\delta$, a one-dimensional lattice of determinant 2, which is necessarily generated by a vector w with $w^2 = 2$. Hence $v = 3w$. $\square$

Theorem 6.29. *Theorem 6.27 holds, even when $B(\vec{a})$ is not generic.*

Proof. The primary issue with the proof of Theorem 6.27 in the non-generic case is that there is no smooth Lagrangian torus fibration $\mu: (S, \omega) \rightarrow B(\vec{a})$ when $B(\vec{a})$ has more complicated singularities such as A'_{2n-1} . So we cannot directly apply Theorem 2.43.

Let $\vec{a}(t)$ be a one-parameter family over $t \in [0, 1]$ such that $a_i(t) > 0$ for all $t > 0$ and $a_i(0)$ results in a collision of I_1 singularities. Let $N > 0$ be a large integer. For all t , let $B'(\vec{a}(t))$ be the result of performing nodal slides (Definition 2.27) of fixed length lying in $N^{-1}\mathbb{Z}$, to every singularity involved in a collision. Then, as $t \rightarrow 0$, the singularities no longer collide, and rather the collection of singularities of $B(\vec{a})$ are factored into I_1 singularities by the nodal slides. Let

$$\begin{aligned} \mu(t): (S(t), \omega(t)) &\rightarrow B(\vec{a}(t)) & \text{for } t \in (0, 1], \\ \mu'(t): (S'(t), \omega'(t)) &\rightarrow B'(\vec{a}(t)) & \text{for } t \in [0, 1] \end{aligned}$$

be the corresponding families of almost toric fibrations. The fibration $\mu(0)$ does not exist unless $B(\vec{a})$ has all unprimed singularities, but $\mu'(0)$ does. Define

$$\sigma(t) = [\Sigma_{\text{IA}}] \in H^2(S(t), \mathbb{Z}).$$

The $\sigma(t)$ are identified under the Gauss–Manin connection on the fiber bundle $S(t) \rightarrow (0, 1]$. Define $\sigma'(t)$ by parallel transport of $\sigma(t)$ along the nodal slide connecting $B(\vec{a}(t))$ to $B'(\vec{a}(t))$. It is easy to see that $\sigma'(t)$ is also represented by a visible surface $\Sigma'_{\text{IA}}(t)$ which fibers over $R_{\text{IA}}(t)$ and the segments along which the nodal slides were made. Since $\mu'(0)$ exists (as the I_1 singularities no longer collide), we have that $\sigma'(0) = [\Sigma'_{\text{IA}}(0)]$ is the parallel transport of $\sigma'(t)$.

As the slide parameters lie in $N^{-1}\mathbb{Z}$, these parameters are integral on the order N refinement. So $B'(a(0))[N]$ admits a triangulation into lattice triangles. By Proposition 2.22, we get a Kulikov degeneration $\mathcal{X}'[N] \rightarrow (C, 0)$ such that $\Gamma(\mathcal{X}'_0[N]) = B'(a(0))[N]$.

The nodal slides destroy the involution symmetry of $B(a(0))[N]$ and any chance of $\mathcal{X}$ having an involution. But we may apply the first half of Theorem 2.43 to $B'(a(0))[N]$ to conclude that the vanishing cycle is identified with $[\mu'(0)^{-1}(p)]$ and the monodromy invariant $\lambda'[N]$ is identified with $N[\omega'](0)$. Furthermore, the class $\phi_*\sigma'(0)$ is invariant under Picard–Lefschetz, and the conclusion of Theorem 6.27,

$$\lambda'[N] = N\vec{a}(0) \bmod O^+(N),$$

holds by continuity: we have $[\omega(t)] = [\omega'(t)]$ because nodal slides leave the class of the symplectic form invariant. Hence

$$[\omega'(0)] = \lim_{t \rightarrow 0} [\omega'(t)] = \lim_{t \rightarrow 0} [\omega(t)] = \lim_{t \rightarrow 0} a(t) = a(0).$$

Integral length nodal slides correspond to M1 modifications of $\mathcal{X}'[N]$ by Proposition 2.28. Thus, there is a sequence of such modifications after which we have a Kulikov degeneration $\mathcal{X}''[N] \rightarrow C$ satisfying $\Gamma(\mathcal{X}''_0[N]) = B(a(0))[N]$. After a series of M2 modifications corresponding to retriangulation (again Proposition 2.28), we can produce a Type III degeneration $\mathcal{X}[N] \rightarrow C$ such that the dual complex is the standard order N refinement of a triangulated $\text{IAS}^2 B(a(0))$. We conclude by Proposition 2.29 that $\mathcal{X}[N] \rightarrow C$ is in fact an order N base change of a Kulikov degeneration $\mathcal{X} \rightarrow C$ such that $\Gamma(\mathcal{X}_0) = B(\vec{a}(0))$, whose vanishing cycle is the same, and whose monodromy-invariant λ is $\vec{a}(0)$.

Furthermore, we have produced not just a class $\phi_*\sigma'(0)$ but an actual surface $\phi(\Sigma'_{\text{IA}}(0))$ on the general fiber $\mathcal{X}_t$ (note the M1 and M2 modifications act trivially on the punctured family). Under the Clemens collapsing map $\mathcal{X}_t \rightarrow \mathcal{X}_0$, the surface $\phi(\Sigma'_{\text{IA}}(0))$ is pinched at the double curves to produce a union of surfaces $R_i \subset V_i$ on the equator such that

$$R_i \cap D_{ij} = R_j \cap D_{ij}.$$

We note that the involution is restored in the limit $a_i \rightarrow 0$ when undoing the nodal slides. The class $[R_i]$ is invariant under the involution on an anticanonical pair of deformation type (V_i, D_i) . We also know the values of $[R_i]^2$ and $[R_i] \cdot D_{ij}$ by continuity, so we can apply the reconstruction proposition, Proposition 6.23, to determine R_i uniquely as the class of the ramification locus. $\square$

6E. An example: The A'_{18} ray. Consider the A'_{18} subdiagram of G_{cox} where $a_i = 0$ for $i \in [18, 0, 1, \dots, 16]$. The corresponding cone in $\mathfrak{F}^{\text{cox}}$ is a ray. Take $\vec{a}$ to be twice the integral generator so that it satisfies the parity condition, Definition 6.24. Then $a_{17} = 6$. Relations in N determine $(a_{19}, a_{20}, a_{21}, a_{22}, a_{23}) = (10, 8, 30, 14, 22)$. Recall from Section 6 that $\vec{a}$ corresponds to line bundle M in the nef cone $\text{Pic } S$ of the mirror K3 satisfying $M \cdot E_i = a_i$. Letting $\pi: S \rightarrow T$ be the double cover of the rational elliptic surface, we have $M = \pi^*L$, where $L \cdot F_i = b_i$ with $\vec{b} = (0, \dots, 0, 3, 0, 5, 4, 15, 7, 11)$. Then we may further blow down $\phi: T \rightarrow \bar{T}$ to the first “6-6-6” toric model. The values $\bar{b}_i = (\phi_*L) \cdot \bar{F}_i$ are $\bar{b}_6 = 5, \bar{b}_{12} = 4, \bar{b}_{17} = 3$ with all other $\bar{b}_i = 0$.

Take a moment polygon of $\bar{T}$ with polarization $\bar{L} = \phi_*L$ and apply two Symington surgeries of size 5 and 4 on the edges associated to $\bar{b}_6$ and $\bar{b}_{12}$, respectively, producing the green integral-affine disc $P(\vec{a})$ with blue boundary depicted in the upper part of Figure 7. We double the disc so that the blue edge becomes the equator of the $\text{IAS}^2 B(\vec{a})$.

We triangulate $B(\vec{a})$ into lattice triangles in an involution-invariant manner, respecting the blue edge. The singular red stars and non-singular black points form the vertices v_i . Finally, we interpret each vertex as the pseudo-fan $\mathfrak{F}(V_i, \sum_j D_{ij})$ an anticanonical pair and glue according to the combinatorics of the triangulation. The resulting Kulikov surface $\mathcal{X}_0$ is shown in the lower part of Figure 7, with double curves in gray, self-intersections in purple, and triple points in yellow.

The line bundles R_i are trivial on all but three components, those along the blue line. On the outer component, R_i is the fixed locus on involution on a resolution of the type A'_{18} involution pair. On the two other equatorial components, R_i is the fiber of a ruling along the equator. These glue to form the Cartier divisor R . Taking the image of a multiple of $\mathcal{O}_{\mathcal{X}_0}(R)$, we get the stable model: this contracts the northern and southern hemispheres to a point, contracts a ruling on two equatorial components, and is a birational morphism of the outer component to the polarized A'_{18} involution pair.

The resulting stable model is irreducible, and is the contraction of an anticanonical pair with cycle of self-intersections $(-10, -2, -1, -2, -10, -1)$ to a singular surface with two boundary components glued. It has two cyclic quotient singularities at the north and south poles whose resolutions are a chain of rational curves of self-intersections $(-10, -2)$ and the images of the (-1) -curves which are glued.

6F. Type II Kulikov models. It remains to determine the Kulikov models corresponding to the rational cusps of $\mathfrak{K}$.

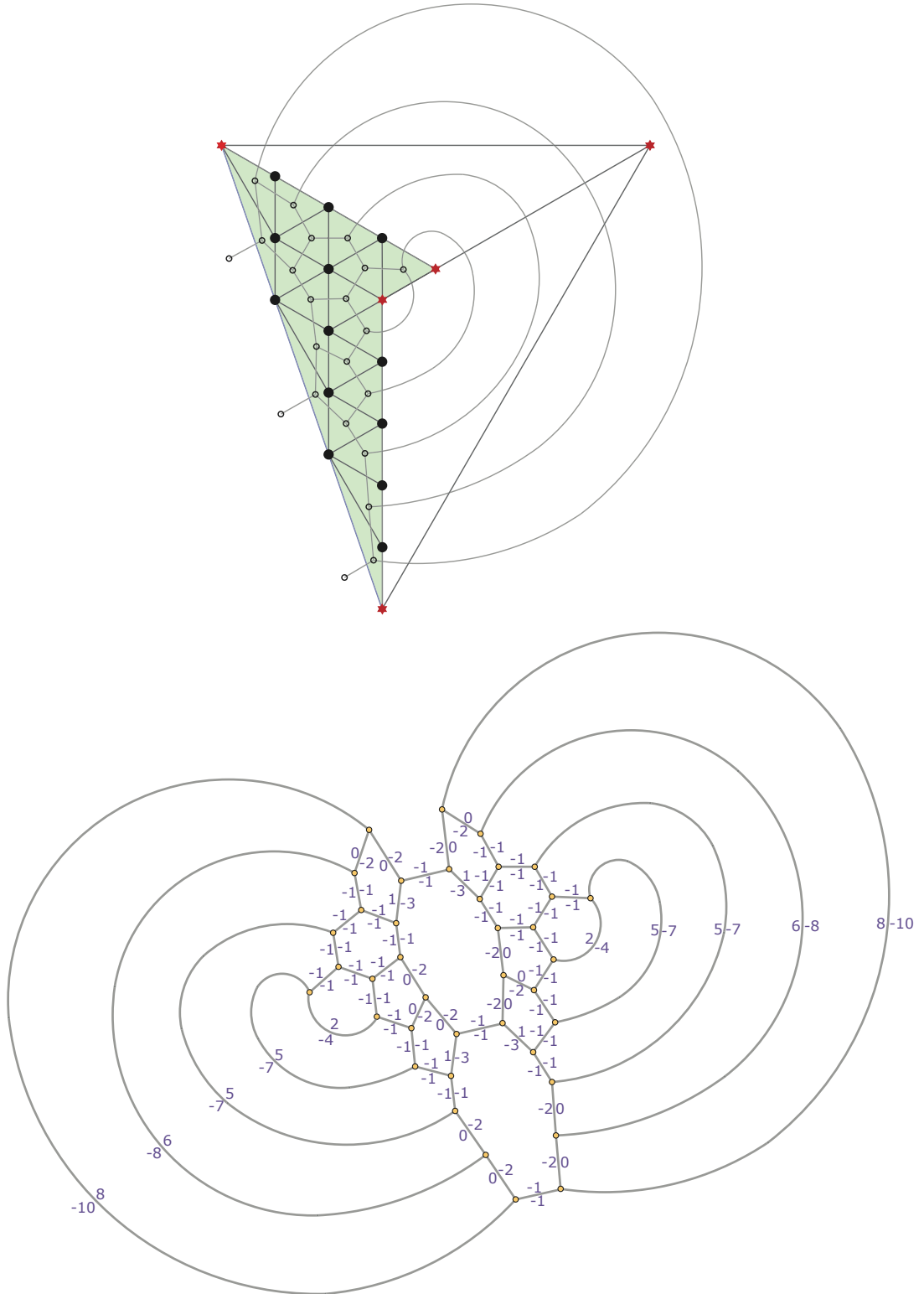


Figure 7. Top: $P(\vec{a})$ for the $\vec{a} \in \mathfrak{K}$ generating the A'_{18} ray. Bottom: A Kulikov surface $\mathcal{X}_0(\vec{a})$ with $\Gamma(\mathcal{X}_0(\vec{a})) = B(\vec{a})$.

Construction 6.30. To a vector $\vec{a} \neq 0 \in \mathbb{R}$ with $\vec{a}^2 = 0$, we associate a Type II Kulikov surface $\mathcal{X}_0(\vec{a})$ with an involution ι_0 and a stable surface $(\overline{\mathcal{X}}_0(\vec{a}), \epsilon \overline{R})$.

For the relevant connected parabolic diagrams, we have the Type II $\tilde{A}\tilde{D}\tilde{E}$ involution pairs (X_k, D_k) of [7] which glue to a stable surface $\overline{\mathcal{X}}_0(\vec{a})$ with an involution $\bar{\iota}_0$ and fixed locus $\overline{R}$. For the diagrams $\tilde{E}_8^2 \tilde{A}_1, \tilde{D}_{10} \tilde{E}_7, \tilde{D}_{16} \tilde{A}_1$ where there are two components, the elliptic curves D_k must be isomorphic to a fixed E .

Now we describe the Kulikov models. If $\vec{a} = m\vec{a}_0$ with primitive $\vec{a}_0$, then the dual complex $\Gamma(\mathcal{X}_0(\vec{a}))$ will be an interval $[0, m]$. A triangulation in this case is the subdivision into intervals of length 1.

For $\tilde{D}_{10} \tilde{E}_7$ and $\tilde{D}_{17} \tilde{A}_1$, the surface $\mathcal{X}_0(\vec{a})$ is constructed by taking the minimal resolution of each component of $\overline{\mathcal{X}}_0(\vec{a})$ and gluing these components, with a chain of $m - 1$ $\mathbb{P}^1$ -bundles over E inserted between them, like an accordion.

In the $\tilde{E}_8^2 \tilde{A}_1$ case, we assume after an order 2 base change that m is even. At the ends, we put the minimal resolutions of two $\tilde{E}_8$ involution pairs. We build a chain of surfaces as in the previous case, but on the middle component, we blow up a pair of points on the boundary of $\mathbb{P}^1 \times E$ switched by the involution. This corresponds to the $\tilde{A}_1 = \tilde{A}_1^{\text{irr}}$ diagram.

In the $\tilde{A}_{17}$ case, resolve the two simple elliptic singularities of the $\tilde{A}_{17}$ involution pair $\overline{\mathcal{X}}_0(\vec{a}) = (\overline{V}, \overline{p}_1, \overline{p}_2)$ to obtain a surface $(V, D_1 + D_2)$ which is ruled over an elliptic curve with 18 broken fibers, and whose anticanonical divisor $D_1 \sqcup D_2$ is the disjoint union of two elliptic curves E of self-intersection -9 . We again assume m is even, and put the anticanonical pair $(V, D_1 + D_2)$ at the $m/2$ vertex. We add ruled surfaces over the same elliptic curve E for the integral points $l \neq 0, m/2, m$ and cap both ends of the segment by the rational anticanonical pair $(\mathbb{P}^2, E)$.

Remark 6.31. The Type II Kulikov models can also be constructed directly from the segment, together with the data of how it degenerated from an IAS²; an analogue of the Monodromy Theorem, Theorem 2.43, likely holds. This requires first generalizing pseudo-fans to allow blow-ups of $E \times \mathbb{P}^1$, corresponding to the surfaces in the interior of the interval. The ends of the interval are anticanonical pairs (V, D) with D smooth. These obviously have no toric models, but can be constructed via *node smoothing surgeries*, cf. [16, Proposition 3.12]. For example, in the $\tilde{A}_{17}$ case, the three surgery triangles consume all of $\overline{P}$. At the north pole, this dictates three node smoothing surgeries on the anticanonical pair $(\mathbb{P}^2, L_1 + L_2 + L_3)$, giving the pair $(\mathbb{P}^2, E)$, as in Construction 6.30.

6G. Deformations of Kulikov models with involution. We now prove that the Kulikov surfaces $\mathcal{X}_0(\vec{a})$ constructed

- (1) can be made d -semistable and admit a smoothing into F_2 , and
- (2) the union $R \subset \mathcal{X}_0(\vec{a})$ of the curves $R_i \subset (V_i, D_i)$ from Theorem 6.14 is the flat limit of the ramification divisor.

Once these are demonstrated, we show that

- (3) every degeneration $C^* \rightarrow F_2$ admits a Kulikov limit of the form $\mathcal{X}_0(\vec{a})$ with R the flat limit of the ramification divisor.

We first recall the basic statements about d -semistable Kulikov surfaces from [20, 24, 27, 39]. Let $\mathcal{X}_0$ be a Type III Kulikov surface with v irreducible components V_i , e double curves

$D_{ij} = V_i \cap V_j$, and f triple points $T_{ijk} = V_i \cap V_j \cap V_k$. One defines an important lattice of “numerical Cartier divisors”

$$\tilde{\Lambda} = \ker\left(\bigoplus_i \text{Pic } V_i \rightarrow \bigoplus_{i < j} \text{Pic } D_{ij}\right)$$

with the homomorphism given by restricting line bundle and applying ± 1 signs. The map is surjective over $\mathbb{Q}$ by [24, Proposition 7.2]. The set of isomorphism classes of not necessarily d -semistable Type III surfaces of the combinatorial type $\mathcal{X}_0$ is isogenous to $\text{Hom}(\tilde{\Lambda}, \mathbb{C}^*)$.

For a given $\psi \in \text{Hom}(\tilde{\Lambda}, \mathbb{C}^*)$, the Picard group of the corresponding surface is $\ker(\psi)$. The surface is d -semistable if and only if the following v divisors are Cartier:

$$\xi_i = \sum_j D_{ij} - D_{ji} \in \tilde{\Lambda}.$$

Note that $\sum_i \xi_i = 0$. Thus, the d -semistable surfaces correspond to the points of multiplicative group $\text{Hom}(\Lambda, \mathbb{C}^*)$, where

$$\Xi = \frac{\bigoplus_i \mathbb{Z}\xi_i}{(\sum_i \xi_i)}, \quad \Lambda = \text{coker}(\Xi \rightarrow \tilde{\Lambda}).$$

By [21, 24], the Clemens–Schmid exact sequence identifies Λ as isometric to

$$\{\delta, \lambda\}^\perp / \delta = \lambda^\perp \subset I^\perp / I \quad \text{or} \quad J^\perp / J$$

in Types III or II, respectively.

The dimension of the space of the d -semistable surfaces is

$$\sum_i \rho(V_i) - e - (v - 1) = (2e - 2v + 24) - e - (v - 1) = e - 3v + 25 = 19$$

because $e - 3v = -6$ for a triangulation of a sphere.

Lemma 6.32. *For all $\vec{a}$, there is at least one d -semistable Kulikov surface $\mathcal{X}_0(\vec{a})$ which admits an involution acting by switching the hemispheres of $B(\vec{a})$, and acts in the prescribed way on the equatorial components (cf. Theorem 6.14).*

Proof. Within any deformation type, the Kulikov surface $\mathcal{X}_0$ for which $\psi = 1$ is the one for which all moduli of components and gluing data are trivial: only -1 (in toric coordinates) is blown up on a toric model of a component (V_i, D_i) , and all double curves D_{ij} and D_{ji} are identified by gluing (in toric coordinates) by -1 .

For this surface, it is automatic that the equatorial edges D_{ij} are glued in such a way that $R_i \cap D_{ij} = R_j \cap D_{ji}$. Thus, the union of the equatorial components admits an involution, and by uniqueness of this Kulikov surface, the involution extends across the two hemispheres.

Finally, since $\psi = 1$, the d -semistability condition is automatic. $\square$

Lemma 6.33. *Any d -semistable equisingular deformation of the Kulikov surface $\mathcal{X}_0(\vec{a})$ from Lemma 6.32 keeping $[R]$ Cartier smooths to a degree 2 K3 surface. The space of such deformations is isogenous to $\text{Hom}(\Lambda / \frac{1}{3}\mathbb{Z}[R], \mathbb{C}^*)$ in Type III and $\text{Hom}(\Lambda / \frac{1}{3}\mathbb{Z}[R], \mathbb{E})$ in Type II.*

Proof. To prove the second part, observe that Corollary 6.28 implies $[\Sigma_{\text{IA}}]$ is 3-divisible in $\{f, [\omega]\}^\perp/f$ and therefore $[R]$ is 3-divisible in $\{\delta, \lambda\}^\perp/\delta = \Lambda$. Since $\frac{1}{3}[R]$ is Cartier on the surface with $\psi = 1$, any deformation keeping $[R]$ Cartier also keeps $\frac{1}{3}[R]$ Cartier. Thus, any deformation keeping $[R]$ Cartier admits a line bundle L , with $L^2 = 2$.

By [20], the analytic smoothing component S of $\mathcal{X}_0$ is 20-dimensional and analytically locally isomorphic to an extension of vector spaces

$$0 \rightarrow \text{Hom}(\Lambda, \mathbb{C}) \rightarrow S \rightarrow H^0(\text{Ext}^1(\Omega_{\mathcal{X}_0}^1, \mathcal{O}_{\mathcal{X}_0})) \rightarrow 0.$$

The first space forms the tangent space to equisingular d -semistable deformations, and by d -semistability, the third space has dimension one. The hyperplane $S_{[R]} \subset S$ keeping $[R]$ Cartier fits into an exact subsequence

$$0 \rightarrow \text{Hom}(\Lambda/\frac{1}{3}\mathbb{Z}[R], \mathbb{C}) \rightarrow S_{[R]} \rightarrow \mathbb{C} \rightarrow 0.$$

and a deformation is first-order smoothing if and only if it has nonzero image in $\mathbb{C}$. So there are smoothings keeping $[R]$ Cartier and admitting a line bundle L , $L^2 = 2$. The first part of the lemma follows. $\square$

Lemma 6.34. *Any equisingular deformation as in Lemma 6.33 admits an involution ι_0 and a Cartier divisor R representing the deformation of $[R]$, realizing it as a Kulikov surface $\mathcal{X}_0(\vec{a})$ coming from Constructions 6.25, 6.30.*

Proof. It suffices to prove that the deformations which keep the class $[R]$ Cartier also admit an involution ι_0 acting in the desired way on $\mathcal{X}_0$ and are, therefore, instances of Construction 6.25 (*caveat lector*: R and $\text{Fix}(\iota_0)$ need not be equal, see Remark 6.22).

First, we suppose $B(\vec{a})$ is generic. In this case, we prove that a deformation keeps $[R]$ Cartier if and only if it deforms the involution ι_0 . The reverse implication is easy, as the Cartier divisor R can be reconstructed from ι_0 ; it is the pullback of $\bar{R}_i = \text{Fix}(\bar{\iota}_i)$ from the stable models of the equatorial components.

Next we show that the first-order d -semistable equisingular deformations of $\mathcal{X}_0$ keeping $[R]$ Cartier preserve the involution. The tangent space to the d -semistable equisingular deformations is $\text{Hom}(\Lambda, \mathbb{C})$. Here, the target vector space $\mathbb{C}$ depends on the orientation of $\Gamma(\mathcal{X}_0)$, so the involution ι_0 acts on it as (-1) . Thus, the tangent space to deformations preserving the involution is $\text{Hom}(\Lambda/\Lambda_+, \mathbb{C})$, where Λ_+ is the $(+1)$ -eigenspace of ι_0 on Λ . Obviously, $[R] \in \Lambda_+$. So all we have to show is that $(n_+, n_-) := (\text{rank } \Lambda_+, \text{rank } \Lambda_-) = (1, 18)$. We now compute the ranks of the $(+1)$ and (-1) -eigenspaces for all the lattices involved.

Let us denote by e_E, e_N the edges of the triangulation of the sphere that appear on the equator and in the northern hemisphere. One has $e = e_E + 2e_N$. Similarly, $v = v_E + 2v_N$ for the vertices and $q = q_E + 2q_N = 18 + 6$ for the charges.

For an irreducible component, the Picard rank is $\rho(V_i) = e^i + q^i - 2$, where we only count the edges and charges belonging to V_i . For a symmetric pair of surfaces in the northern and southern hemispheres, this gives $(e_N^i + q_N^i - 2, e_N^i + q_N^i - 2)$. For a surface in the equator, $(e_E^i + e_N^i + q_N^i - 1, e_N^i + q_E^i + q_N^i - 1)$. Adding up the eigenspaces for $\bigoplus \text{Pic } V_i$ gives

$$(2e_E + 2e_N - v_E - 2v_N + q_N, 2e_N - v_E - 2v_N + q_E + q_N).$$

For the lattice $\bigoplus \text{Pic } D_{ij}$, it is of course $(e_E + e_N, e_N)$, and for Ξ , it is $(v_E + v_N - 1, v_N)$. Putting this together, the ranks (n_+, n_-) are

$$\left(\frac{1}{2}(e - 3v) + \frac{1}{2}(e_E - v_E) + q_N + 1, \frac{1}{2}(e - 3v) - \frac{1}{2}(e_E - v_E) + q_E + q_N \right).$$

Using $e - 3v = -6$ and $e_E = v_E$, we have $(n_+, n_-) = (q_N - 2, q_E + q_N - 3) = (1, 18)$.

When $\Gamma(\mathcal{X}_0)$ is non-generic, the computation has an additional subtlety: the action of the involution on $\text{Pic}(V_i)$ for an equatorial component varies (see Remark 6.22) as one varies the involution pair $(\bar{V}_i, \bar{D}_i + \epsilon \bar{R}_i)$. But choosing a generic member of the space of (V_i, D_i) admitting an involution ι_i gives a specified action on $\text{Pic}(V_i)$, and for this generic choice, $(n_+, n_-) = (1, 18)$.

We now lift to higher-order deformations, noting that these higher-order lifts form a torsor over the first-order deformation space $\text{Hom}(\Lambda, \mathbb{C})$. Thus, the involution ι_0 acts on higher-order deformations by an affine-linear transformation, whose linear part fixes an 18-dimensional subspace. It follows that the involution fixes an 18-dimensional affine-linear subspace. So the involution can be lifted to higher order. Furthermore, these lifts are exactly those preserving the line bundle since the fixed locus of the involution is Cartier. We conclude that deformations over an analytic open subset of $\text{Hom}(\Lambda, \mathbb{C}^*)$ have an involution. This open subset is Zariski dense since the condition of having such an involution is algebraic.

We now specialize from this sublocus of $\text{Hom}(\Lambda, \mathbb{C}^*)$ of Kulikov surfaces with involution, observing that a limiting Kulikov surface $\mathcal{X}_0$ still admits an involution, and the limiting class $[R]$ is still Cartier. Alternatively, we can cite [7, Theorem B]—the spaces of ADE surfaces are parameterized by tori $(\mathbb{C}^*)^n$, so by varying the moduli of the equatorial components and the edge gluings, the space of $\mathcal{X}_0(\vec{a})$ fills out all of $(\mathbb{C}^*)^{18}$, as opposed to some Zariski open subset.

In the Type II case, a dimension count shows that varying moduli of the ADE surfaces and gluings from Construction 6.30, with a fixed elliptic curve E , produces an abelian variety isogenous to E^{17} . Thus, additionally varying $j(E)$ fills out the entire abelian variety fibration $\text{Hom}(\Lambda / \frac{1}{3}\mathbb{Z}[R], \mathcal{E})$ over the modular curve. $\square$

Theorem 6.35. *Let $\mathcal{X}_0(\vec{a})$ be a d -semistable Kulikov surface from Constructions 6.25, 6.30. The Cartier divisor $R \subset \mathcal{X}_0(\vec{a})$ is the flat limit of the ramification divisors $\mathcal{R}^* \subset \mathcal{X}^*$ on any smoothing $\mathcal{X} \rightarrow (C, 0)$ keeping $[R]$ Cartier.*

Furthermore, every degenerating family $(\mathcal{X}^, \mathcal{R}^*) \rightarrow C^*$ of degree 2 K3 surfaces with ramification divisor admits, after some finite base change, a Kulikov model $\mathcal{X} \rightarrow (C, 0)$ of this form.*

Proof. Let $\mathcal{X}_0(\vec{a})$ be a generic element of $\text{Hom}(\Lambda / \frac{1}{3}\mathbb{Z}[R], \mathbb{C}^*)$. Then each anticanonical pair (V_i, D_i) with involution ι_i is generic, and the involution ι_0 acts on Λ with eigenspaces of dimension $(1, 18)$. The argument of Lemma 6.34 shows that any smoothing keeping $[R]$ Cartier preserves the involution because ι_0 acts on $H^0(\mathcal{E}xt^1(\Omega_{\mathcal{X}_0}^1, \mathcal{O}_{\mathcal{X}_0}))$ by negation.

So there is an involution ι on any Kulikov model $\mathcal{X}$ smoothing $\mathcal{X}_0$ which keeps $[R]$ Cartier. This implies $\lim_{t \rightarrow 0} \text{Fix}(\iota_t) = \text{Fix}(\iota_0)$ because $\mathcal{X}$ is smooth, so $\text{Fix}(\iota)$ consists of only 0- and 2-dimensional components. In particular, each 2-dimensional component is irreducible and forms a flat family of divisors. Furthermore, since we are in the generic case, $\text{Fix}(\iota_t) = R_t$ for all t including 0.

For $\mathcal{X}_0(\vec{a})$ non-generic, i.e. having (-2) -curves in equatorial components, and a general smoothing $\mathcal{X} \rightarrow (C, 0)$, the involution ι_0 does not extend to a regular involution ι of $\mathcal{X}$. Instead, $\mathcal{X}^*$ admits an involution ι^* which only extends as a birational involution $\iota: \mathcal{X} \dashrightarrow \mathcal{X}$ whose locus of indeterminacy is the (-2) -curves in the equatorial components, and the restriction $\iota|_{\mathcal{X}_0(\vec{a})}$ extends to ι_0 .

We conclude that the flat limit of $\mathcal{R}^*$ differs from $\text{Fix}(\iota_0)$ at most along the equatorial (-2) -curves, as does R , by construction. So $\lim_{t \rightarrow 0} \mathcal{R}_t = R + \sum a_i C_i$ for C_i these (-2) -curves. On the other hand, $\mathcal{R}_t^2 = R^2$ by construction, $R \cdot C_i = 0$, and C_i span a negative-definite lattice, so we conclude that $a_i = 0$ for all i . This completes the proof of the first paragraph in the theorem.

To prove the second paragraph, we observe that, after a finite base change, any degeneration $\mathcal{X}^* \rightarrow C^*$ has unipotent monodromy and thus has some monodromy invariant $\lambda \in \mathfrak{R}$. After a further order 2 base change, we can ensure vector $\vec{a} \in \mathbb{Z}_{\geq 0}^{24}$ defined by $(\lambda \cdot r_i)_{i \in \{0, \dots, 23\}}$ satisfies the parity condition. Let $\mathcal{X}_0(\vec{a})$ be one of the corresponding Kulikov surfaces. By Theorem 6.29, the monodromy invariant of a smoothing $\mathcal{X}(\vec{a}) \rightarrow (C, 0)$ is, in fact, equal to λ .

It remains to show that we can vary the continuous moduli of $\mathcal{X}(\vec{a})$ until our given family $\mathcal{X}^* \rightarrow C^*$ agrees with $\mathcal{X}^*(\vec{a})$. By Lemmas 6.33, 6.34, the d -semistable surfaces $\mathcal{X}_0(\vec{a})$ keeping $[R]$ Cartier form a family $\mathfrak{X}_0(\vec{a})$ over (a variety isogenous to) $(\mathbb{C}^*)^{18}$ or $\mathcal{E}^{\times 17}$ in Types III and II, respectively.

A result of Friedman–Scattone [24, Lemmas 5.5, 5.6] shows that the smoothing components of the fibers of $\mathfrak{X}_0(\vec{a})$ keeping $[R]$ Cartier can be glued together, to form a family $\mathfrak{X}(\vec{a})$ of smooth and Kulikov K3 surfaces with line bundle. The base of $\mathfrak{X}(\vec{a})$ is 19-dimensional and, up to the action of a finite group, is identified with the toroidal extension $F_2 \hookrightarrow F_2^\lambda$ whose only cones are the Γ -orbits of the ray $\mathbb{R}_{\geq 0}\lambda$. The boundary divisor is exactly the base of $\mathfrak{X}_0(\vec{a})$, parameterizing the d -stable equisingular deformations of $\mathcal{X}_0(\vec{a})$ keeping $[R]$ Cartier. Proposition 3.2 implies that $\mathcal{X}^* \rightarrow C^*$ is a subfamily of $\mathfrak{X}(\vec{a})$ because the period map approximates a co-character $\lambda \otimes \mathbb{C}^*$ which is completed at 0 in F_2^λ . The theorem follows. $\square$

7. Determination of stable models

The goal of this section is to determine the KSBA stable limit of any one parameter degeneration $(\mathcal{X}^*, \epsilon \mathcal{R}^*) \rightarrow C^*$ in F_2 . We describe the components which will appear on any stable limit of degree 2 K3 pairs $(X, \epsilon R)$, and how they are glued.

7A. ADE and $\tilde{A}\tilde{D}\tilde{E}$ surfaces. In this section, we describe the classification of involution pairs of [7] in more detail.

Definition 7.1. Let X be a normal projective surface with a reduced boundary divisor D and an involution $\iota: X \rightarrow X$, $\iota(D) = D$ such that

- (1) $K_X + D \sim 0$ is a Cartier divisor linearly equivalent to 0,
- (2) the ramification divisor R is Cartier and ample, and
- (3) the pair $(X, D + \epsilon R)$ has log-canonical singularities for $0 < \epsilon \ll 1$.

Such pairs were called $(K + D)$ -trivial polarized involution pairs in [7], where they are classified in terms of the decorated ADE diagrams in Type III and extended $\tilde{A}\tilde{D}\tilde{E}$ diagrams in Type II.

The classification in [7] is done in terms of the quotients $(Y, C) = (X, D)/\iota$ and the branch divisors $B \subset Y$. The surface X is recovered as a double cover $\pi: X \rightarrow Y$ branched in B . Then $R = \pi^{-1}(B)$ and $D = \pi^{-1}(C)$.

In toric geometry, a lattice polytope P corresponds to a toric variety Y_P with an ample line bundle L_P . For many Dynkin diagrams, there exists a polytope P corresponding to it in an obvious way. Then Y is defined to be Y_P and the branch divisor B to be a divisor in the linear system $|L_P|$.

For example, there are polytopes of shapes associated to A_0^- , D_5 , E_7 in Figure 3. In general, the polytope P has the following vertices:

- (1) A_n , A_n^- for n odd and even, respectively: $(2, 2)$, $(0, 0)$, $(n + 1, 0)$.
- (2) $-A_n^-$, $-A_n$ for n odd and even, respectively: $(2, 2)$, $(1, 0)$, $(n + 2, 0)$.
- (3) D_n and D_n^- : $(2, 2)$, $(0, 2)$, $(0, 0)$, $(n - 2, 0)$.
- (4) E_n ($-E_6^-$, $-E_7^-$, $-E_8^-$): $(2, 2)$, $(0, 3)$, $(0, 0)$, $(n - 3, 0)$.
- (5) $\tilde{D}_{2n}$: $(0, 2)$, $(0, 0)$, $(2n - 4, 0)$, $(4, 2)$.
- (6) $\tilde{E}_7$: $(0, 4)$, $(0, 0)$, $(4, 0)$.
- (7) $\tilde{E}_8$: $(0, 3)$, $(0, 0)$, $(6, 0)$.

In the ADE cases, the boundary D has two components. In the $\tilde{A}\tilde{D}\tilde{E}$ cases, D is an irreducible smooth elliptic curve.

The only non-toric initial cases are $\tilde{A}_{2n-1}$ and two small exotic $\tilde{A}$ shapes.

- (8) $\tilde{A}_{2n-1}$. The surface is cone $\text{Proj}_E(\mathcal{O} \oplus \mathcal{F})$ over an elliptic curve E , where $\mathcal{F}$ is a line bundle of degree n . The boundary $C = 0$ is empty and $B \in |-2K_Y|$.
- (9) $\tilde{A}_1^*$. Here, $Y = \mathbb{P}^2$, the boundary C is a smooth conic, and the branch curve B is a possibly singular conic. If B is smooth, then $X = \mathbb{P}^1 \times \mathbb{P}^1$, and if B is two lines, then $X = \mathbb{P}(1, 1, 2)$ with R passing through the apex. Also included is a degenerate subcase when $\mathbb{P}^2$ degenerates to $Y = \mathbb{P}(1, 1, 4)$ with R not passing through the apex.
- (10) $\tilde{A}_0^-$. Here, $Y = \mathbb{P}(1, 1, 2) = \mathbb{F}_2^0$. The curve C is the image of $\tilde{C} \in |s + 3f|$ on $\mathbb{F}_2$. The branch curve is a conic section disjoint from the apex.

All other pairs are obtained from these by a process called “priming”: making up to 4 weighted $(1, 2)$ blow-ups $Y' \rightarrow Y$ at the points of intersection of the branch divisor B with the boundary C and then contracting parts of the boundary C' on which $-K_{Y'}$ is no longer ample. On the double cover $\pi: X \rightarrow Y$, this corresponds to an ordinary smooth blow-up at a point of $R \cap D$ and then contracting parts of the boundary D' on which R' is no longer ample.

These “primed” surfaces Y' are usually not toric. But they are toric in the A'_{2n-1} , A'_{2n} , A'_{2n-1} and D'_{2n} cases for which there are also lattice polytopes. The polytope for A'_n is obtained from that for A_n by cutting a corner, a triangle with lattice sides 1, 1, 2, which corresponds to the weighted $(1, 2)$ blow-up. For the A'_{2n+1} diagram, the corners on both sides are cut. For the D'_{2n} diagram, the corner on the “right” side is cut. See a concrete example of a polytope of shape A'_4 in Figure 3.

Examples 6.16, 6.17, 6.18, 6.19, 6.20, 6.21 describe explicitly the minimal resolutions of involution pairs (X, D) of shapes $A_0^-, A_{2n-1}, A'_{2n-1}, D_{2n}, D'_{2n}, E_n$ to smooth anticanonical pairs admitting an involution.

Notation 7.2. In general, the involution pairs with elliptic diagram have two boundary components, each isomorphic to $\mathbb{P}^1$, and meeting at two points to form a banana curve. We call the two nodes the *north* and *south poles*, and the two components the *left* and *right components*.

7B. All degenerations of K3 surfaces of degree 2. Recall that the *stable type* (Definition 4.10) of an elliptic or maximal parabolic subdiagram of G_{cox} was a cyclically ordered list of its equatorial diagrams, with A_0^- and A_0^- diagrams inserted in the space between diagrams.

Definition 7.3. Associated to every Type III stable type, we build a stable surface as follows. For each diagram, we take an involution pair (X_k, D_k, ι_k) with the corresponding label by [7] (see Section 7A). Then we successively glue the surfaces together,

$$(X, \iota) = \bigcup_k (X_k, D_k, \iota_k),$$

along their boundary components, identifying the right component of D_k to the left component of D_{k+1} and identifying the two north poles and the two south poles. The intersection complex of the resulting surface is a sphere, decomposed like the slices of an orange. We glue in such a way that the ramification divisors R_k glue to a Cartier, ample divisor R .

In Type II, we do something similar for stable types $\tilde{E}_8^2, \tilde{D}_{10}\tilde{E}_7, \tilde{D}_{16}\tilde{A}_1$ by gluing the two components along elliptic curves. Finally, the stable surface associated to $\tilde{A}_{17}$ is simply the $\tilde{A}_{17}$ involution pair.

The scheme-theoretic structure of the surface (X, ι) is uniquely determined by the requirement that the gluing be seminormal with SNC double locus, see [32, Proposition 5.3, Corollary 5.33].

Example 7.4. Consider the empty subdiagram of G_{cox} , corresponding to the open cell of $\mathfrak{R}$. Its stable type is $(A_0^- A_0^-)^9$, see Figure 3, and the corresponding stable surface is the result of taking 18 copies of $(\mathbb{P}^2, L + C)$ with a line and conic, and successively gluing conics to conics, and lines to lines, in such a way that the fixed divisors, which are lines in each $\mathbb{P}^2$, glue into a Cartier divisor.

This will be the unique maximal degeneration of $\overline{F}_2^{\text{slc}}$.

Theorem 7.5. *The stable limits of K3 pairs $(X, \epsilon R)$ of degree 2, polarized by the ramification divisor, are exactly the stable surfaces of Definition 7.3, formed from the union of involution pairs associated to a stable type of an elliptic or maximal parabolic subdiagram of G_{cox} . More precisely, if the monodromy-invariant λ of a one-parameter degeneration $\mathcal{X}^* \rightarrow C^*$ lies in the relative interior $\lambda \in \sigma^\circ$ of a cone $\sigma \in \mathfrak{F}_{\text{cox}}$, the stable limit is a stable surface associated to the stable type of the subdiagram defining σ .*

Proof. Let $\mathcal{X}^* \rightarrow C^*$ be a degeneration of degree 2 K3 surfaces with monodromy invariant λ . By Theorem 6.35, there is some finite base change and an extension to a Kulikov model $\mathcal{X} \rightarrow (C, 0)$ for which the central fiber $\mathcal{X}_0 = \mathcal{X}_0(\vec{a})$ arises from Constructions 6.25

and 6.30. Here, $\vec{a} \in \mathbb{Z}_{\geq 0}^{24}$ is the vector corresponding to $\lambda \in \mathfrak{K}$ via $\lambda \cdot r_i = a_i$. The flat limit of R is then a Cartier divisor on $\mathcal{X}_0(\vec{a})$ which is empty in the hemispheres of $\mathcal{X}_0(\vec{a})$ and is the pullback of $R_k = \text{Fix}(\iota_k)$ on the involution pair (X_k, D_k) which is the contraction of an equatorial component (see Theorem 6.14, but note that the involution pair is notated there as $(\overline{V}_i, \overline{D}_i)$).

In particular, $\mathcal{R} \subset \mathcal{X}$ is a relatively big and nef Cartier divisor not containing any strata of $\mathcal{X}_0$. By the proof of Theorem 3.14, the stable limit of $\mathcal{X}^* \rightarrow C^*$ can be computed as

$$\text{Proj}_C \bigoplus_{n \geq 0} \pi_* \mathcal{O}_{\mathcal{X}}(n\mathcal{R}),$$

which contracts each hemisphere of $\mathcal{X}_0$ to a single point, contracts the edges along the equator by rulings, and contracts each equatorial vertex to the involution pair (X_k, D_k) . The resulting stable surface is exactly that described in Definition 7.3.

It is worth remarking that, when a subdiagram of G_{cox} has an A_1^{irr} or $\tilde{A}_1^{\text{irr}}$ component, there is an equatorial surface in $\mathcal{X}_0$ receiving two internal blow-ups switched by the involution, but the information of the location of these blow-ups is lost on the stable model because they are contracted to points on a double curve. $\square$

7C. Moduli of stable strata. The following proposition should be compared with Proposition 4.13.

Proposition 7.6. *The strata in $\overline{F}_2^{\text{slc}}$ are as follows.*

- (1) *For a Type III stable type G^{rel} , $\text{Str}(G^{\text{rel}})$ is, up to an isogeny and a $W(G^{\text{rel}})$ action, the root torus $\text{Hom}(R_{G^{\text{rel}}}, \mathbb{C}^*)$.*
- (2) *For a Type II stable type $\tilde{G}^{\text{rel}}$, $\text{Str}(\tilde{G}^{\text{rel}})$ is, up to an isogeny and a $W(G^{\text{rel}})$ action, $\text{Hom}(R_{G^{\text{rel}}}, \mathcal{E}) \simeq \mathcal{E}^{17}$, where $\mathcal{E}^{17} \rightarrow \mathcal{M}_1$ is the self-fiber product of the universal family of elliptic curves $\mathcal{E} \rightarrow \mathcal{M}_1$ over its moduli stack.*

Proof. The parameter space for a Type III stratum is, up to a finite group, the product of the parameter spaces for the irreducible components $(X_k, D_k + \epsilon R_k)$ because the gluings of double curves which make $\bigcup_k R_k$ Cartier are, up to a finite group, unique. By [7], each of these is a quotient of torus isogenous to the root torus $\text{Hom}(R_{G_k}, \mathbb{C}^*)$ by the Weyl group $W(G_k)$. Without the additional data of an involution, this result is essentially due to Gross–Hacking–Keel [27].

The same works for Type II strata. Such a stratum is a finite quotient of the fiber product over $\mathcal{M}_1$ of the period domains of involution pairs with smooth elliptic anticanonical divisor. The *period point* of an anticanonical pair (V_k, D_k) is the element of $\text{Hom}(D^\perp, D)$ which sends $L \mapsto L|_{D_k} \in \text{Pic}^0(D_k) = D_k$. The ι -invariant period points form an abelian subvariety isogenous to $\text{Hom}(R_{G_k}, D_k)$, and the moduli space for each component is the quotient by the ι -invariant “admissible isometries” of $H^2(V_k, D_k)$, cf. [22], which in our case is the Weyl group $W(G_k)$.

Fixing an elliptic curve $D = D_k$ and taking the product of moduli of components produces the quotient by $W(G^{\text{rel}})$ of an abelian variety isogenous to $\text{Hom}(R_{G^{\text{rel}}}, D)$. Finally, we may vary the moduli of D over $\mathcal{M}_1$, giving the fiber product. $\square$

8. Proof of main theorems

We now assemble the ingredients from the above sections to prove the main theorems. In the proof of Theorem 6.35, we defined the toroidal extension $F_2 \hookrightarrow F_2^\lambda$ whose fan consists of the Γ -orbit of a ray $\mathbb{R}_{\geq 0}\lambda$, and a family $\mathcal{X}(\vec{a}) \rightarrow \mathcal{U}(\vec{a})$ of Kulikov and smooth K3 surfaces, with $\mathcal{U}(\vec{a})$ a finite cover of F_2^λ and $\vec{a} = (\lambda \cdot r_i)_{i \in \{0, \dots, 23\}}$ assumed to satisfy the parity condition. Recall that the boundary divisor of $\mathcal{U}(\vec{a})$ was isogenous to $\text{Hom}(\Lambda/\frac{1}{3}\mathbb{Z}[R], \mathbb{C}^*) \simeq (\mathbb{C}^*)^{18}$ or $\text{Hom}(\Lambda/\frac{1}{3}\mathbb{Z}[R], \mathcal{E}) \simeq \mathcal{E}^{\times 17}$, where Λ (see Section 6G) is the lattice $\lambda^\perp \subset I^\perp/I$ or $J^\perp/J$.

Theorem 8.1. *Let $\lambda \in \sigma_G^o \cap N$ lie in the relative interior of a cone of $\mathfrak{F}^{\text{cox}}$ for a subdiagram $G \subset G_{\text{cox}}$. Assume $\vec{a} = (\lambda \cdot r_i)_{i \in \{0, \dots, 23\}}$ satisfies the parity condition. Then the classifying map $\mathcal{U}(\vec{a}) \dashrightarrow \overline{F}_2^{\text{slc}}$ is a morphism, and the induced morphism on the boundary divisor is (isogenous to) the restriction map $\text{Hom}(\Lambda/\frac{1}{3}\mathbb{Z}[R], \mathbb{C}^* \text{ or } \mathcal{E}) \rightarrow \text{Hom}(R_{G^{\text{rel}}}, \mathbb{C}^* \text{ or } \mathcal{E})$ for the natural inclusion $R_{G^{\text{rel}}} \hookrightarrow R_G \hookrightarrow \Lambda/\frac{1}{3}\mathbb{Z}[R]$, followed by the quotienting by a finite group.*

Proof. The proof is essentially the same as Theorem 7.5, except we do not restrict to a one-parameter subfamily. Let $\mathfrak{R} \subset \mathcal{X}(\vec{a})$ be the universal ramification divisor and let $\mathcal{L} = \mathcal{O}_{\mathcal{X}(\vec{a})}(\mathfrak{R})$ be the corresponding line bundle, which is relatively big and nef. The family of divisors $\mathfrak{R}$ exists because the flat limit of the ramification divisor on any Kulikov model is $R \subset \mathcal{X}_0(\vec{a})$ by Theorem 6.35.

By Shepherd-Barron [51], the higher cohomology groups of $\mathcal{L}^n$ are zero on every fiber, so for $n \geq 4$, $\mathcal{L}^n$ defines a contraction to a model with an ample line bundle. Since the divisors $\mathfrak{R}$ do not contain strata on any fiber by construction, the fibers in the contracted family are stable pairs $(\mathcal{X}(\vec{a}), \epsilon \mathfrak{R}) \rightarrow \mathcal{U}(\vec{a})$, and the fibers over the boundary divisor have stable type determined by G^{rel} , by Theorem 7.5. This proves that the classifying map is a morphism.

So the classifying map induces a morphism from $\text{Hom}(\Lambda, \mathbb{C}^* \text{ or } \mathcal{E})$ to the slc stratum $\text{Str}(G^{\text{rel}})$ of Proposition 7.6. We claim that this morphism factors through the natural map of tori induced by the inclusions $R_{G^{\text{rel}}} \hookrightarrow R_G \hookrightarrow \Lambda$; note that $R_G \hookrightarrow \Lambda$ because

$$\lambda \in \sigma \implies \sigma^\perp \subset \lambda^\perp \subset I^\perp/I \implies R_G \subset \Lambda.$$

Let (V_i, D_i) be an equatorial component of $\mathcal{X}_0$, and define

$$\Lambda_i := \text{span}\{D_{ij}\}^\perp \subset H^2(V_i, \mathbb{Z}).$$

The period domain [22, 27] for anticanonical pairs (V_i, D_i) is $\text{Hom}(\Lambda_i, \mathbb{C}^*)$, while the corresponding period domain for involution pairs [7] is a torus with character lattice isogenous to $R_{G_i} \subset \Lambda_i$ (more canonically a quotient), consisting of periods of anticanonical pairs (V_i, D_i) accepting an involution ι_i . Finally, observe that there is an inclusion $\bigoplus_i \Lambda_i \hookrightarrow \Lambda$ as every class in Λ_i can be extended by 0 to a numerically Cartier class on $\mathcal{X}_0$. This map induces the inclusion $R_G \hookrightarrow \Lambda/\frac{1}{3}\mathbb{Z}[R]$. We conclude that the map on moduli $\text{Hom}(\Lambda/\frac{1}{3}\mathbb{Z}[R], \mathbb{C}^* \text{ or } \mathcal{E}) \rightarrow \text{Str}(G^{\text{rel}})$ is induced by the claimed map of lattices. $\square$

Theorem 8.2. *The rational map $\varphi: \overline{F}_2^{\text{semi}} \dashrightarrow \overline{F}_2^{\text{slc}}$ is regular and the normalization map.*

Proof. We first prove that the rational map $\varphi': \overline{F}_2^{\text{tor}} \dashrightarrow \overline{F}_2^{\text{slc}}$ is regular. For any ray $\mathbb{R}_{\geq 0}\lambda$ of the fan, the map extends over the interior of the corresponding divisor of $\overline{F}_2^{\text{tor}}$ by Theorem 8.1. So, if there is any indeterminacy locus of φ' , then it is contained in the Type III locus.

Suppose that φ' is not regular. Let $\overline{F}_2^{\text{tor}} \xleftarrow{p} Z \xrightarrow{q} \overline{F}_2^{\text{slc}}$ be a resolution of singularities of φ' . The preimage $Z_x = p^{-1}(x)$ of a point $x \in \overline{F}_2^{\text{tor}}$ is projective. By Theorem 7.5 and Proposition 7.6, one has $q(Z_x) \subset \text{Str}(G^{\text{rel}})$. But by Proposition 7.6, every Type III stratum in $\overline{F}_2^{\text{slc}}$ is affine. So the map $Z_x \rightarrow \overline{F}_2^{\text{slc}}$ is constant. We conclude by Lemma 3.18.

The morphism φ' factors through $\varphi: \overline{F}_2^{\text{semi}} \rightarrow \overline{F}_2^{\text{tor}}$. In fact, by Theorem 8.1 and Theorem 4.21, the curves contracted by $F_2^\lambda \rightarrow \overline{F}_2^{\text{semi}}$ and $F_2^\lambda \rightarrow \overline{F}_2^{\text{slc}}$ are the same, giving the claim. Then φ is a birational morphism with finite fibers, so by Zariski's Main Theorem, it is the normalization. $\square$

Corollary 8.3. *The Stein factorization of $\overline{F}_2^{\text{tor}} \rightarrow \overline{F}_2^{\text{slc}}$ is $\overline{F}_2^{\text{tor}} \rightarrow \overline{F}_2^{\text{semi}} \rightarrow \overline{F}_2^{\text{slc}}$.*

Proof. This follows from the fact that the fibers of $\overline{F}_2^{\text{tor}} \rightarrow \overline{F}_2^{\text{semi}}$ are connected. $\square$

Corollary 8.4. *There is a regular map $\overline{\mathcal{F}}_2^{\text{semi}} \rightarrow \overline{\mathcal{F}}_2^{\text{slc}}$ of Deligne–Mumford stacks, for an appropriate choice of stack structure on $\overline{F}_2^{\text{semi}}$.*

Remark 8.5. Corollary 8.4 is essentially a tautology by pulling back the stack structure, but it is subtle from the perspective of arithmetic quotients.

- (1) Even the interior is not the stack quotient $[\mathbb{D} : \Gamma]$. The Heegner divisors associated to roots $\beta \in h^\perp$ have inertia in $[\mathbb{D} : \Gamma]$ but not in F_2 .
- (2) Due to the presence of generic automorphisms on slc strata, we need a *stacky fan*: for each cone $\sigma \in \mathfrak{G}^{\text{cox}}$, we must choose a sublattice of $\text{span}(\sigma) \cap N$, which introduces inertia at the toric boundary components.

Similar to the enumeration of the strata of $\overline{F}_2^{\text{tor}}$ in Lemma 4.5 and Lemma 4.6, by looking at the subdiagrams of G_{cox} without irrelevant connected components only, mod S_3 or D_9 , one can enumerate the strata of $\overline{F}_2^{\text{semi}}$ or $\overline{F}_2^{\text{slc}}$. In particular, we have the following lemma.

Lemma 8.6. *Both in $\overline{F}_2^{\text{semi}}$ and in $\overline{F}_2^{\text{slc}}$, there are 38 boundary divisors, of which 3 are of Type II and 35 are of Type III.*

Remark 8.7. The normalization map $\overline{F}_2^{\text{semi}} \rightarrow \overline{F}_2^{\text{slc}}$ is not the identity map. For instance, when a diagram G^{rel} is entirely contained in the 18-cycle $0, \dots, 17$, the resulting stable pair stratum is the same for all diagrams in the D_9 dihedral group orbit of G^{rel} . For semi-toric strata, only diagrams related by $D_3 \simeq S_3$ are identified.

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