



The Uniqueness of Phase Retrieval of Analytic Signals from Very Few STFT Measurements

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Received: 28 June 2023 / Revised: 18 November 2023 / Accepted: 24 November 2023

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Abstract

Analytic signals constitute a class of signals that are widely applied in time–frequency analysis such as extracting instantaneous frequency (IF) or phase derivative in the characterization of ultrashort laser pulse. The purpose of this paper is to investigate the phase retrieval (PR) problem for analytic signals in $\mathbb{C}^N$ by short-time Fourier transform (STFT) measurements since such measurements enjoy some nice structures. Since generic analytic signals are not sparse in the time domain, the existing PR results for sparse (in time domain) signals do not apply to analytic signals. In this paper, the windows are required to be bandlimited such that they usually have the full support length N and allow us to get much better resolutions on low frequencies. By exploiting the structure of the STFT associated with bandlimited windows for analytic signals, we prove that the STFT-based phase retrieval (STFT-PR for short) of generic analytic signals can be achieved by their $(3\lfloor \frac{N}{2} \rfloor + 1)$ measurements, where $\lfloor x \rfloor$ denotes the largest integer that is not larger than x . Since the generic analytic signals are $(\lfloor \frac{N}{2} \rfloor + 1)$ -sparse in the Fourier domain, such a number of measurements is lower than $4N + O(1)$ and $O(k^3)$ which are required in the literature for STFT-PR of all signals and of k^2 -sparse (in the Fourier domain) signals in $\mathbb{C}^{N^2}$, respectively. Moreover, we also prove that if the length N is even and the windows are also analytic, then the number of measurements can be reduced to $(\frac{3N}{2} - 1)$. As an application of this, we get that the instantaneous frequency (IF) of a generic analytic signal can be exactly recovered from the STFT measurements.

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Keywords Phase retrieval · Short-time Fourier transform (STFT) measurement · Analytic signal · Instantaneous frequency

1 Introduction

Phase retrieval (PR) is a nonlinear sampling problem (c.f. [3, 4, 12, 21, 45, 48, 49]) that asks to recover a signal $\mathbf{z} \in \mathbb{C}^N$, *up to the potential ambiguity*, from the magnitude measurements

$$|\langle \mathbf{z}, \mathbf{a}_k \rangle|, \quad k \in \Gamma,$$

where $\mathbf{a}_k \in \mathbb{C}^N$ is referred to as a measurement vector. PR problem is of great interest since it has been widely applied in many applications including coherent diffraction imaging (CDI) ([38, 43]), quantum tomography ([28]) and holography ([32]). The most classical PR problem is to recover a signal by its Fourier transform measurements ([16, 21]).

Associated with a window $\mathbf{w} \in \mathbb{C}^N$ and a separation parameter $0 < L < N$, the short-time Fourier transform (STFT) or Gabor transform of a signal $\mathbf{z} \in \mathbb{C}^N$ at (k, m) is defined as (c.f. [7, 33]):

$$\hat{y}_{k,m}^{\mathbf{w}} = \sum_{n=0}^{N-1} \mathbf{z}_n \mathbf{w}_{mL-n} e^{-2\pi i k n / N}, \quad (1)$$

where $k = 0, 1, \dots, N-1$ and $m = 0, 1, \dots, \lceil N/L \rceil - 1$. Compared with the Fourier transform, STFT is more effective for time–frequency localization since its associated window enjoys great flexibility (c.f. [13, 19, 26, 44]), and many deeper theoretical results related to Gabor frame analysis have been established in the literature (c.f. [15, 22, 23]).

Finding the required number of measurements to do phase retrieval is always a fundamental issue, especially for practical applications including quantum tomography (c.f. [28]). For STFT-PR we refer to, e.g., [1, 5–7, 27, 33, 34, 42] for many recent results on this issue. Bojarovska and Flinth [10] characterized all the windows when N^2 -number of STFT measurements can recover all the signals in $\mathbb{C}^N$. With the help of graph theory, Pfander and Salanevich [40] proved that the recovery of any signal in $\mathbb{C}^N$ can be achieved by $O(N \log N)$ STFT measurements. From the perspective of frame theory (c.f. [27]), the STFT-PR is essentially the PR problem by the frame measurement vectors in $\mathbb{C}^N$. There exist many phase retrievable frames of $4N + O(1)$ -length (e.g., [2, 29, 46]). As for the STFT-PR, the recovery can be also achieved by $4N + O(1)$ measurements (c.f. [2, 9, 30, 31]). Note that the above mentioned results hold for all the signals in $\mathbb{C}^N$. By appropriately choosing the window $\mathbf{w}$ and the separation parameter L , Jaganathan, Eldar and Hassibi [31] proved that almost all nonvanishing signals in $\mathbb{C}^N$ can be determined by their $3N + O(1)$ number of STFT measurements. Recently, the STFT-PR for structured signals has attracted much attention (e.g., [10, 20, 31]). In particular, it was proved in [10] that a k^2 -sparse (in the Fourier domain) signal can be

recovered by $O(k^3)$ number of STFT measurements. In this paper, we will investigate the phase retrieval problem for analytic signals that appear in many important applications such as time–frequency analysis ([14]), instantaneous frequency (IF) extracting in holography (e.g., [25]) and the characterization of a changing pulse frequency (e.g., [24]). As a proper subset of $\mathbb{C}^N$, we are interested in finding fewer number of STFT measurements than the above mentioned number to guarantee the recovery of any generic analytic signal.

The definition of an analytic signal was given by Marple [36]. As in [36] the space $\mathbb{C}^N$ is supposed to consist of N -periodic and complex-valued signals $\mathbf{z} = (\mathbf{z}_0, \dots, \mathbf{z}_{N-1})$ such that the subscripts are considered modulo N . For a real-valued signal $\mathbf{x} \in \mathbb{R}^N$, its analytic signal $A(\mathbf{x}) = (A(\mathbf{x})_0, \dots, A(\mathbf{x})_{N-1})$ is defined through its discrete Fourier transform (DFT) $\widehat{A(\mathbf{x})} = ((\widehat{A(\mathbf{x})})_0, \dots, (\widehat{A(\mathbf{x})})_{N-1})$, where for even length N ,

$$(\widehat{A(\mathbf{x})})_k = \begin{cases} \widehat{\mathbf{x}}_0, & k = 0, \\ 2\widehat{\mathbf{x}}_k, & 1 \leq k \leq \frac{N}{2} - 1, \\ \widehat{\mathbf{x}}_{\frac{N}{2}}, & k = \frac{N}{2}, \\ 0, & \frac{N}{2} + 1 \leq k \leq N - 1, \end{cases} \quad (2)$$

and for odd length N ,

$$(\widehat{A(\mathbf{x})})_k = \begin{cases} \widehat{\mathbf{x}}_0, & k = 0, \\ 2\widehat{\mathbf{x}}_k, & 1 \leq k \leq \frac{N-1}{2}, \\ 0, & \frac{N+1}{2} \leq k \leq N - 1. \end{cases} \quad (3)$$

From now on, the entire set of analytic signals on $\mathbb{C}^N$ is denoted by $\mathbb{C}_A^N$. For $\mathbf{x} \in \mathbb{R}^N$, its discrete Hilbert transform $H\mathbf{x}$ is defined to be

$$\frac{(A(\mathbf{x}) - \mathbf{x})}{\mathbf{i}}. \quad (4)$$

Remark 1 (1) By [36], we know that $\mathbb{C}_A^N = \{\mathbf{x} + \mathbf{i}H\mathbf{x} : \mathbf{x} \in \mathbb{R}^N\}$, i.e., the real part $\Re(A(\mathbf{x})) = \mathbf{x}$ and the imaginary part $\Im(A(\mathbf{x}))$ is $H\mathbf{x}$, where H is the discrete Hilbert transform. (2) For a real-valued signal $\mathbf{x} = (x_0, \dots, x_{N-1}) \in \mathbb{R}^N$, it is commonly accepted that its phase $(\theta_0, \dots, \theta_{N-1})$ is defined via its analytic signal $\mathbf{z} = \mathbf{x} + \mathbf{i}H\mathbf{x} = (z_0, \dots, z_{N-1}) = (z_{0,\Re} + \mathbf{i}z_{0,\Im}, \dots, z_{N-1,\Re} + \mathbf{i}z_{N-1,\Im}) \in \mathbb{C}^N$ such that $\theta_k = \arctan \frac{z_{k,\Im}}{z_{k,\Re}}$ (cf. [17, 18]).

We say that $\mathbf{z} \in \mathbb{C}^N$ is B -bandlimited if its DFT contains $N - B$ consecutive zeros. For $\mathbf{0} \neq \mathbf{z} = (\mathbf{z}_0, \dots, \mathbf{z}_{N-1}) \in \mathbb{C}^N$, its support is defined to be $\Xi = \{i \in \{0, \dots, N - 1\} : \mathbf{z}_i \neq \mathbf{0}\}$. Then the support length of $\mathbf{z}$ is defined as the cardinality $\#\Xi$. We also say that $\mathbf{z}$ is $\#\Xi$ -sparse. For a polynomial f in N (real or complex) variables, its vanishing locus is $V(f) = \{(x_0, \dots, x_{N-1}) \in \mathbb{R}^N \text{ (resp. } \mathbb{C}^N) : f(x_0, \dots, x_{N-1}) = 0\}$. The complement of $V(f)$ in $\mathbb{R}^N$ (resp. $\mathbb{C}^N$) is dense (c.f. [7]). For $x \in \mathbb{R}$, we will use the notation $\lceil x \rceil$ (respectively, $\lfloor x \rfloor$) to denote the smallest (respectively, largest) integer that is not smaller (respectively, larger) than x .

1.1 Main Result

We start with the definition of a generic analytic signal in $\mathbb{C}^N$.

Definition 1 When saying that a generic analytic signal is uniquely determined by a collection of polynomial measurements, we mean that the analytic signals which cannot be determined by these measurements lie in the vanishing locus of a nonzero polynomial on $\mathbb{C}^N$.

The main results will be stated in Theorems 2, 3 and 4 which can be summarized as follows.

Theorem 1 Suppose that $\mathbf{w}_l \in \mathbb{C}^N, l = 1, \dots, M$ are the structured B -bandlimited windows for STFT such that their bandlimits $2 \leq B \leq \lceil \frac{N}{2} \rceil + 1$. Moreover, the STFT separation parameter $0 < L < N$ satisfies $\lceil N/L \rceil \geq 3$. Then for a generic analytic signal $\mathbf{z} \in \mathbb{C}^N$, it can be recovered (up to a sign) by its $(3\lceil \frac{N}{2} \rceil + 1)$ number of STFT measurements. Moreover, if the length N is even and the windows are analytic then the above number of measurements can be reduced to $(\frac{3N}{2} - 1)$.

The STFT in Theorem 1 requires multiple bandlimited windows. The following concerns the application background of such a type of STFT.

Remark 2 (1) The window $\mathbf{w}_l$ in Theorem 1 is bandlimited. By the discrete uncertainty principle (c.f. [10, section 3.2]) its support length is generally N . That is, $\mathbf{w}_l$ is generally a long window. By [37, 41], longer windows on low frequencies allow getting better frequency resolution, and they have been used in some STFT-PR approaches (e.g., [39]). (2) Multiple-window measurements were used in Theorem 1, and such a type of STFT measurements were also used for STFT-PR in [33]. We point out that multiple-window approach is particularly useful in coded diffraction patterns (c.f. [11]).

Remark 3 (1) Since any generic analytic signal is $(\lfloor \frac{N}{2} \rfloor + 1)$ -sparse in the Fourier domain, as mentioned previously it is generally not sparse in the time domain. Therefore, the existing PR results for sparse (in time domain) signals do not hold for analytic signals. (2) A generic analytic signal is $(\lfloor \frac{N}{2} \rfloor + 1)$ -sparse (in the Fourier domain) or equivalently has bandlimit $B = \lfloor \frac{N}{2} \rfloor + 1$. Theorem 1 implies that it can be determined (up to a sign) by its $(3B - 2)$ STFT measurements. When the windows are analytic, such a required number of measurements can be reduced to $(3B - 4)$.

An immediate consequence of Theorem 1 is the exact recovery of instantaneous frequency (IF) for generic analytic signals. Given an analytic signal $\mathbf{z} = (\mathbf{z}_0, \dots, \mathbf{z}_{N-1})$, denote its element $\mathbf{z}_k$ by $|\mathbf{z}_k|e^{i\arg(\mathbf{z}_k)}$ with $\arg(\mathbf{z}_k) \in [0, 2\pi)$. Define

$$\varphi^*(k) := (\arg(\mathbf{z}_k) - \arg(\mathbf{z}_{k-1})) \mod 2\pi. \quad (5)$$

Then $\mathbf{z}_\varphi := (\varphi^*(0), \dots, \varphi^*(N-1))$ is referred to as the phase derivative (PD) or IF of $\mathbf{z}$ (c.f. [18]).

Proposition 1 Suppose that $\mathbf{z} \in \mathbb{C}^N$ is a generic analytic signal. Then its IF can be exactly recovered from the same number of STFT measurements as specified in Theorem 1.

Proof By Theorem 1, we get $\mathbf{z}$ or $-\mathbf{z}$. Since the k -th element of $-\mathbf{z}$ is expressed as $|\mathbf{z}_k|e^{i((\arg(\mathbf{z}_k)-\pi) \bmod 2\pi)}$, $[(\arg(\mathbf{z}_k) - \pi) \bmod 2\pi - (\arg(\mathbf{z}_{k-1}) - \pi) \bmod 2\pi] \bmod 2\pi = (\arg(\mathbf{z}_k) - \arg(\mathbf{z}_{k-1})) \bmod 2\pi$. That is, the IF of $-\mathbf{z}$ is identical to that of $\mathbf{z}$. This completes the proof. $\square$

1.2 Comparisons with the Existing Results

In this subsection, we make some comparisons between Theorem 1 and the results in [10, 35].

measurement number	It was proved by [10] that the STFT-PR of a k^2 -sparse (in the Fourier domain) signal in $\mathbb{C}^{N^2}$ can be achieved by $O(k^3)$ measurements. Theorem 1 implies that its STFT-PR can be achieved by using only $3\lfloor \frac{N}{2} \rfloor + O(1)$ number of measurements.
condition	The main result in [35] only applies to the case when N is even and the separation parameter $0 < L < N$ is odd with the property that $\lceil N/L \rceil \geq 5$. However, Theorem 1 only requires $\lceil N/L \rceil \geq 3$ in this case, and the oddity of L , N is not required.
ambiguities	The ambiguity for FROG-PR in [35] is different from that in Theorem 1 since it additionally contains shift and reflection.
application	The IFs of only a few analytic signals can be extracted from FROG-PR ([35, section 4]). However, Proposition 1 applies to every generic analytic signal.

Remark 4 As mentioned previously, the required number of measurements for PR (or the uniqueness problem) is fundamental not only for the mathematical theory ([2–4]) but also for practical applications including quantum tomography ([28]). Compared with the existing result for PR of all signals in $\mathbb{C}^N$ (cf. [2, 9, 30, 31]) or that for the (Fourier transform) sparse signals in $\mathbb{C}^N$ ([10]), our required number of measurements for generic analytic signals is reduced approximately by $\frac{62.5}{100}$ and $1 - \frac{4}{9N^2}$.

2 Preliminary

A complex number $0 \neq z \in \mathbb{C}$ is traditionally denoted by $|z|e^{i\arg(z)}$, where i , $|z|$ and $\arg(z)$ are the imaginary unit, modulus and phase, respectively. The real and imaginary parts, and conjugation of z are denoted by $\Re(z)$, $\Im(z)$ and $\bar{z}$, respectively. For N sets $\mathcal{S}_1, \dots, \mathcal{S}_N$ their Cartesian product is defined to be

$$\mathcal{S}_1 \times \dots \times \mathcal{S}_N = \{(s_1, \dots, s_N) : s_k \in \mathcal{S}_k\}.$$

The discrete Fourier transform (DFT) of $\mathbf{z} \in \mathbb{C}^N$ is defined by $\widehat{\mathbf{z}} := (\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \dots, \widehat{\mathbf{z}}_{N-1})$ such that $\widehat{\mathbf{z}}_k = \sum_{n=0}^{N-1} \mathbf{z}_n e^{-2\pi i k n / N}$. The inverse discrete Fourier transform (IDFT) admits the formula

$$\mathbf{z}_k = \frac{1}{N} \sum_{n=0}^{N-1} \widehat{\mathbf{z}}_n e^{2\pi i k n / N}. \quad (6)$$

By the IDFTs of $\mathbf{z}$ and $\mathbf{w}$, the STFT $\widehat{y}_{k,m}^{\mathbf{w}}$ in (1) can be expressed as

$$\widehat{y}_{k,m}^{\mathbf{w}} = \frac{1}{N} \sum_{l=0}^{N-1} \widehat{z}_{k+l} \widehat{\mathbf{w}}_l \omega^{lm}, \quad (7)$$

where $\omega = e^{\frac{2\pi i l}{N}}$. The following is a characterization of the DFT structure for an analytic signal.

Proposition 2 (c.f. [35]) Suppose that $\mathbf{z} \in \mathbb{C}^N$. Denote the Cartesian product of sets by $\times$. Then $\mathbf{z}$ is analytic if and only if the following two items holds:

- (i) for even length N , $\widehat{\mathbf{z}} \in \mathbb{R} \times \mathbb{C}^{\frac{N}{2}-1} \times \mathbb{R} \times \overbrace{\{0\} \times \dots \times \{0\}}^{(\frac{N}{2}-1)\text{copies}};$
- (ii) for odd length N , $\widehat{\mathbf{z}} \in \mathbb{R} \times \mathbb{C}^{\frac{N-1}{2}} \times \overbrace{\{0\} \times \dots \times \{0\}}^{\frac{N-1}{2}\text{copies}}.$

The following gives a characterization of the generic analytic signals.

Proposition 3 Let Θ be a set of generic analytic signals in $\mathbb{C}^N$ such that any signal in Θ can be determined by a collection of polynomial STFT measurements. Meanwhile, all the signals in the complement $\mathbb{C}_A^N \setminus \Theta$ cannot be determined by these measurements and they lies in the vanishing locus of a nonzero polynomial f on $\mathbb{C}^N$:

$$V(f) = \{(x_0, \dots, x_{N-1}) \in \mathbb{C}^N : f(x_0, \dots, x_{N-1}) = 0\}.$$

Denote $\mathbb{C}_A^N \setminus \Theta$ by $\{A(\mathbf{x}) = \mathbf{x} + \mathbf{i}H\mathbf{x} : \mathbf{x} \in \Lambda \subseteq \mathbb{R}^N\}$ where $H\mathbf{x}$ defined in (4) is the discrete Hilbert transform of $\mathbf{x}$. Then Λ lies in the vanishing locus of a nonzero polynomial g on $\mathbb{R}^N$.

Proof For any analytic signal $A(\mathbf{x})$, it follows from Remark 1 that $H\mathbf{x} = \frac{A(\mathbf{x}) - \mathbf{x}}{\mathbf{i}}$. Consequently, $\widehat{H\mathbf{x}} = \frac{\widehat{A(\mathbf{x})} - \widehat{\mathbf{x}}}{\mathbf{i}}$. By (2) and (3), for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ we have $H(\widehat{\mathbf{x}} + \widehat{\mathbf{y}}) = \widehat{H\mathbf{x}} + \widehat{H\mathbf{y}}$. From this and the linearity of IDFT we have that $H(\mathbf{x} + \mathbf{y}) = H\mathbf{x} + H\mathbf{y}$. That is, the discrete Hilbert transform H is linear. Then there exists a matrix $H_{\mathbf{e}} \in \mathbb{R}^{N \times N}$ such that $H\mathbf{x} = H_{\mathbf{e}}\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^N$. Consequently,

$$A(\mathbf{x}) = (I + \mathbf{i}H_{\mathbf{e}})\mathbf{x}, \quad (8)$$

where I is the identity matrix. For any $\mathbf{x} + \mathbf{i}H\mathbf{x} \in \mathbb{C}_A^N \setminus \Theta$ it follows from (8) that $f(\mathbf{x} + \mathbf{i}H\mathbf{x}) = f((I + \mathbf{i}H_{\mathbf{e}})\mathbf{x}) = 0$. By choosing $g(\mathbf{x}) := f((I + \mathbf{i}H_{\mathbf{e}})\mathbf{x})$, the proof is completed. $\square$

3 Main Results

3.1 Several Lemmas

This section starts with an auxiliary result from [8, Lemma 3.2].

Lemma 1 Consider an equation system w.r.t $z \in \mathbb{C}$:

$$\begin{cases} |z + v_1| = n_1, \\ |z + v_2| = n_2, \\ |z + v_3| = n_3, \end{cases} \quad (9)$$

where $v_1, v_2, v_3 \in \mathbb{C}$ are distinct. If there exists a solution $\tilde{z} = a + ib$ to the above system and $\Im\left(\frac{v_1 - v_2}{v_1 - v_3}\right) \neq 0$, then it is the unique one. Moreover, it is given by

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2} \begin{pmatrix} c & d \\ e & f \end{pmatrix}^{-1} \begin{pmatrix} n_1^2 - n_2^2 + |v_2|^2 - |v_1|^2 \\ n_1^2 - n_3^2 + |v_3|^2 - |v_1|^2 \end{pmatrix},$$

where $c = \Re(v_1 - v_2)$, $d = \Im(v_1 - v_2)$, $e = \Re(v_1 - v_3)$ and $f = \Im(v_1 - v_3)$.

The following two lemmas will be needed in the proofs of Theorems 2 and 3.

Lemma 2 Suppose that a rational function $f(z) = \frac{az + b}{cz + d}$, $z = x + iy \in \mathbb{C}$ satisfies the conditions $ad - bc \neq 0$, $ac \neq 0$, $a, b, c, d \in \mathbb{C}$. Then the set $\{(x, y) \in \mathbb{R}^2 : \Im(f(z)) = 0, z = x + iy\}$ lies in the vanishing locus of a nonzero polynomial on $\mathbb{R}^2$.

Proof Let $z = x + iy$. We have

$$\begin{aligned} \Im(f(z)) &= \Im\left(\frac{(az + b)(\bar{c}\bar{z} + \bar{d})}{(cz + d)(\bar{c}\bar{z} + \bar{d})}\right) = \Im\left(\frac{a\bar{c}|z|^2 + a\bar{d}z + b\bar{c}\bar{z} + b\bar{d}}{|cz + d|^2}\right) \\ &= \frac{\Im(a\bar{c})(x^2 + y^2) + (\Im(a\bar{d}) + \Im(b\bar{c}))x + (\Re(a\bar{d}) - \Re(b\bar{c}))y + \Im(b\bar{d})}{|cz + d|^2}. \end{aligned} \quad (10)$$

Now the proof can be completed by choosing the polynomial $F(x, y) := \Im(a\bar{c})(x^2 + y^2) + (\Im(a\bar{d}) + \Im(b\bar{c}))x + (\Re(a\bar{d}) - \Re(b\bar{c}))y + \Im(b\bar{d})$. $\square$

Lemma 3 Let L and N be such that $0 < L < N$ and $\lceil N/L \rceil \geq 3$. If $0 \leq m_1, m_2, m_3 \leq \lceil N/L \rceil - 1$ are distinct, then $\Im\left(\frac{\omega^{m_1} - \omega^{m_2}}{\omega^{m_1} - \omega^{m_3}}\right) \neq 0$ where $\omega = e^{\frac{2\pi i L}{N}}$.

Proof Since

$$\frac{\omega^{m_1} - \omega^{m_2}}{\omega^{m_1} - \omega^{m_3}} = \frac{(\omega^{m_1} - \omega^{m_2})(\overline{\omega^{m_1} - \omega^{m_3}})}{|\omega^{m_1} - \omega^{m_3}|^2} = \frac{1 - \omega^{m_1-m_3} - \omega^{m_2-m_1} + \omega^{m_2-m_3}}{|\omega^{m_1} - \omega^{m_3}|^2},$$

we get that the condition $\Im\left(\frac{\omega^{m_1} - \omega^{m_2}}{\omega^{m_1} - \omega^{m_3}}\right) \neq 0$ is equivalent to the condition

$$\begin{aligned} &\Im(1 - \omega^{m_1-m_3} - \omega^{m_2-m_1} + \omega^{m_2-m_3}) \\ &= -\sin\left(\frac{2\pi(m_1-m_3)L}{N}\right) - \sin\left(\frac{2\pi(m_2-m_1)L}{N}\right) + \sin\left(\frac{2\pi(m_2-m_3)L}{N}\right) \neq 0. \end{aligned} \quad (11)$$

Assume to the contrary that $\Im(\frac{\omega^{m_1} - \omega^{m_2}}{\omega^{m_1} - \omega^{m_3}}) = 0$. Then we have

$$\sin\left(\frac{2\pi(m_2 - m_3)L}{N}\right) = \sin\left(\frac{2\pi(m_1 - m_3)L}{N}\right) + \sin\left(\frac{2\pi(m_2 - m_1)L}{N}\right),$$

which implies that

$$\sin\left(\frac{\pi(m_2 - m_3)L}{N}\right)\cos\left(\frac{\pi(m_2 - m_3)L}{N}\right) = \sin\left(\frac{\pi(m_2 - m_3)L}{N}\right)\cos\left(\frac{\pi(2m_1 - m_3 - m_2)L}{N}\right).$$

Since $m_2 \neq m_3$ and $0 \leq m_2, m_3 \leq \lceil N/L \rceil - 1$, we get that $\sin\left(\frac{\pi(m_2 - m_3)L}{N}\right) \neq 0$ and hence

$$\cos\left(\frac{\pi(m_2 - m_3)L}{N}\right) = \cos\left(\frac{\pi(2m_1 - m_3 - m_2)L}{N}\right).$$

This implies that $\frac{\pi(m_2 - m_3)L}{N} = \frac{\pi(2m_1 - m_3 - m_2)L}{N}$ or $\frac{\pi(m_2 - m_3)L}{N} = -\frac{\pi(2m_1 - m_3 - m_2)L}{N}$. Thus we have either $m_1 = m_2$ or $m_1 = m_3$, which leads to a contradiction. The proof is completed. $\square$

3.2 The First Main Result: Window Bandlimit $2 \leq B \leq \lceil \frac{N}{2} \rceil$ Case

Suppose that the window $\mathbf{w} \in \mathbb{C}^N$ is B -bandlimited such that $2 \leq B \leq \lceil \frac{N}{2} \rceil$. Consequently, there exists $i \in \{0, \dots, N-1\}$ such that

$$\widehat{\mathbf{w}}_i = \dots = \widehat{\mathbf{w}}_{i+N-B-1} = 0, \widehat{\mathbf{w}}_{i+N-B} \neq 0. \quad (12)$$

For such a subscript i , we consider the following measurements

$$|\widehat{y}_{\lfloor \frac{N}{2} \rfloor - (i+N-B) - n + 1, m}^{\mathbf{w}}| = \frac{1}{N} \left| \sum_{l=0}^{N-1} \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor - (i+N-B) - n + 1 + l} \widehat{\mathbf{w}}_l \omega^{lm} \right|, \quad (13)$$

where $n = 1, \dots, \lfloor \frac{N}{2} \rfloor + 1$.

The following is a 2-bandlimited window in $\mathbb{C}^{48}$ such that (12) holds with $i = 2$.

Example 1 For $N = 48$, we design a window $\mathbf{w}$ such that its bandlimit $B = 2$. Its real and imaginary parts are plotted in Fig. 1 (a) while the real and imaginary parts of $\widehat{\mathbf{w}}$ are plotted in Fig. 1 (b).

The following example on the summation in (13) for the case $N = 6$ and $B = 3$ exhibits the structure of $|\widehat{y}_{\lfloor \frac{N}{2} \rfloor - (i+N-B) - n + 1, m}^{\mathbf{w}}|$ for general N and B .

Example 2 Let $N = 6$, $B = \lceil \frac{N}{2} \rceil = 3$. For an analytic signal $\mathbf{z} \in \mathbb{C}^6$, it follows from Proposition 2 (i) that its DFT $\widehat{\mathbf{z}} = (\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \widehat{\mathbf{z}}_2, \widehat{\mathbf{z}}_3, 0, 0)$. Choose a 3-bandlimited window $\mathbf{w} \in \mathbb{C}^6$ such that $\widehat{\mathbf{w}} = (\widehat{\mathbf{w}}_0, \widehat{\mathbf{w}}_1, \widehat{\mathbf{w}}_2, 0, 0, 0)$ and correspondingly $i = 3$ in

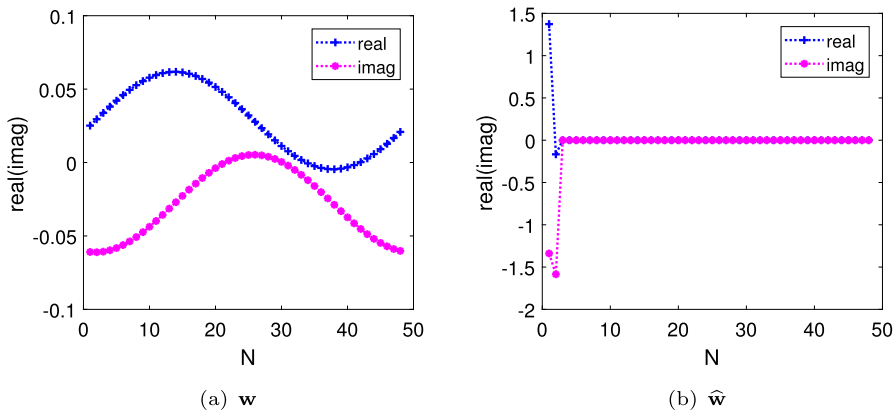


Fig. 1 (a) Graph of the real and imaginary parts of $\mathbf{w}$; (b) graph of the real and imaginary parts of $\hat{\mathbf{w}}$

(12). For $n = 1, \dots, 4$, $|\hat{y}_{4-n,0}^{\mathbf{w}}|$ in (13) are expressed as: $|\hat{y}_{3,0}^{\mathbf{w}}| = \frac{1}{6}|\hat{\mathbf{z}}_3\hat{\mathbf{w}}_0|$, $|\hat{y}_{2,0}^{\mathbf{w}}| = \frac{1}{6}|\hat{\mathbf{z}}_2\hat{\mathbf{w}}_0 + \hat{\mathbf{z}}_3\hat{\mathbf{w}}_1|$, $|\hat{y}_{1,0}^{\mathbf{w}}| = \frac{1}{6}|\hat{\mathbf{z}}_1\hat{\mathbf{w}}_0 + \hat{\mathbf{z}}_2\hat{\mathbf{w}}_1 + \hat{\mathbf{z}}_3\hat{\mathbf{w}}_2|$ and $|\hat{y}_{0,0}^{\mathbf{w}}| = \frac{1}{6}|\hat{\mathbf{z}}_0\hat{\mathbf{w}}_0 + \hat{\mathbf{z}}_1\hat{\mathbf{w}}_1 + \hat{\mathbf{z}}_2\hat{\mathbf{w}}_2|$. The terms $\hat{\mathbf{z}}_k\hat{\mathbf{w}}_l$ on which $\hat{y}_{4-n,0}^{\mathbf{w}}$ is dependent are arranged as follows,

$$\begin{array}{ll}
 n = 1 & \hat{\mathbf{z}}_3\hat{\mathbf{w}}_0 \\
 n = 2 & \hat{\mathbf{z}}_2\hat{\mathbf{w}}_0 \quad \hat{\mathbf{z}}_3\hat{\mathbf{w}}_1 \\
 n = 3 & \hat{\mathbf{z}}_1\hat{\mathbf{w}}_0 \quad \hat{\mathbf{z}}_2\hat{\mathbf{w}}_1 \quad \hat{\mathbf{z}}_3\hat{\mathbf{w}}_2 \\
 n = 4 & \hat{\mathbf{z}}_0\hat{\mathbf{w}}_0 \quad \hat{\mathbf{z}}_1\hat{\mathbf{w}}_1 \quad \hat{\mathbf{z}}_2\hat{\mathbf{w}}_2
 \end{array} \quad (14)$$

Based on (14), the terms $\hat{\mathbf{z}}_k\hat{\mathbf{w}}_l\omega^{lm}$ for $\hat{y}_{4-n,m}^{\mathbf{w}}$ can be arranged similarly.

For the general case, similar to (14), it follows from $2 \leq B \leq \lceil \frac{N}{2} \rceil$, (12) and Proposition 2 that the terms $\hat{\mathbf{z}}_k\hat{\mathbf{w}}_l$ on which $|\hat{y}_{\lfloor \frac{N}{2} \rfloor - (i+N-B) - n + 1, 0}^{\mathbf{w}}|$ in (13) is dependent are arranged as follows,

$$\begin{array}{ccccccc}
 & & & & & \hat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \hat{\mathbf{w}}_{i+N-B} & \\
 & & & & & \hat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor - 1} \hat{\mathbf{w}}_{i+N-B} & \hat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \hat{\mathbf{w}}_{i+N-B+1} \\
 & & & & & \vdots & \\
 & & & & & \vdots & \\
 & & & & \hat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor + 1 - B} \hat{\mathbf{w}}_{i+N-B} & \hat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor + 2 - B} \hat{\mathbf{w}}_{i+N-B+1} & \cdots & \hat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \hat{\mathbf{w}}_{i+N-1} \\
 & \cdots & & & & & & \\
 \hat{\mathbf{z}}_0 \hat{\mathbf{w}}_{i+N-B} & \hat{\mathbf{z}}_1 \hat{\mathbf{w}}_{i+N-B+1} & \cdots & & \hat{\mathbf{z}}_{B-1} \hat{\mathbf{w}}_{i+N-1} & & &
 \end{array} \quad (15)$$

For $n = 1$, as implied on the first row of (15) the corresponding measurement $|\hat{y}_{\lfloor \frac{N}{2} \rfloor - (i+N-B), 0}^{\mathbf{w}}|$ is involved with only the term $\hat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \hat{\mathbf{w}}_{i+N-B}$. An observation on (13) gives us that, for $|\hat{y}_{\lfloor \frac{N}{2} \rfloor - (i+N-B) - n + 1, m}^{\mathbf{w}}|$ the related terms $\hat{\mathbf{z}}_k\hat{\mathbf{w}}_l\omega^{lm}$ are arranged as in (15). The following is on the determination of $\hat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor}$.

Lemma 4 Suppose that the window $\mathbf{w} \in \mathbb{C}^N$ is B -bandlimited such that $2 \leq B \leq \lceil \frac{N}{2} \rceil$. Consequently, there exists $i \in \{0, \dots, N-1\}$ such that $\hat{\mathbf{w}}_i = \dots =$

$\widehat{\mathbf{w}}_{i+N-B-1} = 0$ and $\widehat{\mathbf{w}}_{i+N-B} \neq 0$. Then for any analytic signal $\mathbf{z} \in \mathbb{C}^N$ with DFT $\widehat{\mathbf{z}} = (\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \dots, \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor}, 0, \dots, 0)$, we have the following:

Case I: If N is even, then the component $\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor}$ can be determined (up to a sign) by the measurement $|\widehat{\mathbf{y}}_{\lfloor \frac{N}{2} \rfloor - (i+N-B), 0}^{\mathbf{w}}|$.

Case II: If N is odd, then the component $\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor}$ can be determined (up to a unimodular scalar) by $|\widehat{\mathbf{y}}_{\lfloor \frac{N}{2} \rfloor - (i+N-B), 0}^{\mathbf{w}}|$.

Proof It follows from (12) and (13) that $|\widehat{\mathbf{y}}_{\lfloor \frac{N}{2} \rfloor - (i+N-B), 0}^{\mathbf{w}}| = \frac{1}{N} |\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor}| |\widehat{\mathbf{w}}_{i+N-B}|$. Then $\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} = \frac{N |\widehat{\mathbf{y}}_{\lfloor \frac{N}{2} \rfloor - (i+N-B), 0}^{\mathbf{w}}|}{|\widehat{\mathbf{w}}_{i+N-B}|} e^{i\theta_0}$. The proof for the odd case is completed. By Proposition 2 for N being even, we have $\widehat{\mathbf{z}}_{\frac{N}{2}} \in \mathbb{R}$. Then $\widehat{\mathbf{z}}_{\frac{N}{2}} = \epsilon \frac{N |\widehat{\mathbf{y}}_{\frac{N}{2} - (i+N-B), 0}^{\mathbf{w}}|}{|\widehat{\mathbf{w}}_{i+N-B}|}$ with $\epsilon \in \{1, -1\}$. This completes the proof for the even case. $\square$

Now it is ready to establish the first main theorem.

Theorem 2 Suppose that the window $\mathbf{w} \in \mathbb{C}^N$ is B -bandlimited such that $2 \leq B \leq \lceil \frac{N}{2} \rceil$ and there exists $i \in \{0, \dots, N-1\}$ such that (12) holds and $\widehat{\mathbf{w}}_{i+N-B+1} \neq 0$. Moreover, we assume that the STFT separation parameter $0 < L < N$ satisfies $\lceil N/L \rceil \geq 3$, and choose any three distinct numbers m_1, m_2, m_3 from $\{0, 1, \dots, \lceil N/L \rceil - 1\}$. Then any generic analytic signal $\mathbf{z} \in \mathbb{C}^N$ can be determined, up to a global sign, by its $(3 \lfloor \frac{N}{2} \rfloor + 1)$ number of STFT measurements

$$\{|\widehat{\mathbf{y}}_{\lfloor \frac{N}{2} \rfloor - (i+N-B), 0}^{\mathbf{w}}|, |\widehat{\mathbf{y}}_{k - (i+N-B), m_j}^{\mathbf{w}}| : k = 0, \dots, \lfloor \frac{N}{2} \rfloor - 1, j = 1, 2, 3\}. \quad (16)$$

Proof We mainly prove for the case when N is even since the proof for the odd N case is very similar. We will complete it by induction. By Lemma 4, the component $\widehat{\mathbf{z}}_{\frac{N}{2}}$ can be determined up to a sign by the measurement $|\widehat{\mathbf{y}}_{\frac{N}{2} - (i+N-B), 0}^{\mathbf{w}}|$. Denote such a determination result by $\epsilon \widehat{\mathbf{z}}_{\frac{N}{2}}$ with $\epsilon \in \{1, -1\}$. In what follows, we discuss how to recover other components $\widehat{\mathbf{z}}_0, \dots, \widehat{\mathbf{z}}_{\frac{N}{2}-1}$.

We first address the recovery of $\widehat{\mathbf{z}}_{\frac{N}{2}-1}$ by the STFT measurements $\{|\widehat{\mathbf{y}}_{\frac{N}{2}-1 - (i+N-B), m_j}^{\mathbf{w}}| : j = 1, 2, 3\}$. Consider the equation system w.r.t $\widehat{\mathbf{z}}_{\frac{N}{2}-1}$:

$$|\widehat{\mathbf{y}}_{\frac{N}{2}-1 - (i+N-B), m_j}^{\mathbf{w}}| = \frac{1}{N} |\widehat{\mathbf{z}}_{\frac{N}{2}-1}| |\widehat{\mathbf{w}}_{i+N-B} \omega^{(i+N-B)m_j} + \epsilon \widehat{\mathbf{z}}_{\frac{N}{2}} |\widehat{\mathbf{w}}_{i+N-B+1} \omega^{(i+N-B+1)m_j}|, \quad j = 1, 2, 3. \quad (17)$$

Note that (17) is equivalent to

$$\frac{N |\widehat{\mathbf{y}}_{\frac{N}{2}-1 - (i+N-B), m_j}^{\mathbf{w}}|}{|\widehat{\mathbf{w}}_{i+N-B} \omega^{(i+N-B)m_j}|} = |\widehat{\mathbf{z}}_{\frac{N}{2}-1} + v_{j, \frac{N}{2}-1}|, \quad j = 1, 2, 3, \quad (18)$$

where

$$v_{j, \frac{N}{2}-1} := \frac{\epsilon \widehat{\mathbf{z}}_{\frac{N}{2}} \widehat{\mathbf{w}}_{i+N-B+1} \omega^{(i+N-B+1)m_j}}{\widehat{\mathbf{w}}_{i+N-B} \omega^{(i+N-B)m_j}}. \quad (19)$$

For the generic analytic signal $\mathbf{z}$, we have $\widehat{\mathbf{z}}_{\frac{N}{2}} \neq 0$. Therefore, for $v_{j, \frac{N}{2}-1}$ in (19) we have

$$\frac{v_{1, \frac{N}{2}-1} - v_{2, \frac{N}{2}-1}}{v_{1, \frac{N}{2}-1} - v_{3, \frac{N}{2}-1}} = \frac{\omega^{m_1} - \omega^{m_2}}{\omega^{m_1} - \omega^{m_3}}. \quad (20)$$

By (20) and Lemma 3, we have $\Im\left(\frac{v_{1, \frac{N}{2}-1} - v_{2, \frac{N}{2}-1}}{v_{1, \frac{N}{2}-1} - v_{3, \frac{N}{2}-1}}\right) \neq 0$. Then it follows from Lemma 1 that there exists a unique solution to the equation system (18) w.r.t $\widehat{\mathbf{z}}_{\frac{N}{2}-1}$. Clearly, $\epsilon \widehat{\mathbf{z}}_{\frac{N}{2}-1}$ is a solution. Then it is the unique one. In what follows, we address how to recover the other components $\widehat{\mathbf{z}}_{\frac{N}{2}-2}, \dots, \widehat{\mathbf{z}}_0$. Suppose that $\epsilon \widehat{\mathbf{z}}_k$ has been obtained for any $k \in \{\frac{N}{2}, \frac{N}{2} - 1, \dots, k_0\}$ with $k_0 \in \{\frac{N}{2}, \frac{N}{2} - 1, \dots, 1\}$ by the measurements

$$\{|\widehat{\mathbf{y}}_{\frac{N}{2}-(i+N-B),0}^{\mathbf{w}}|, |\widehat{\mathbf{y}}_{\ell-(i+N-B),m_j}^{\mathbf{w}}| : \ell = \frac{N}{2} - 1, \dots, k_0, j = 1, 2, 3\}.$$

Now we discuss how to recover $\widehat{\mathbf{z}}_{k_0-1}$. Consider the equation system w.r.t $\widehat{\mathbf{z}}_{k_0-1}$:

$$\begin{aligned} |\widehat{\mathbf{y}}_{k_0-1-(i+N-B),m_j}^{\mathbf{w}}| &= \frac{1}{N} |\widehat{\mathbf{z}}_{k_0-1} \widehat{\mathbf{w}}_{i+N-B} \omega^{(i+N-B)m_j} \\ &\quad + \sum_{l=1}^{\frac{N}{2}-k_0+1} \epsilon \widehat{\mathbf{z}}_{k_0-1+l} \widehat{\mathbf{w}}_{i+N-B+l} \omega^{(i+N-B+l)m_j}|, j = 1, 2, 3. \end{aligned} \quad (21)$$

Note that (21) is equivalent to

$$\frac{N |\widehat{\mathbf{y}}_{k_0-1-(i+N-B),m_j}^{\mathbf{w}}|}{|\widehat{\mathbf{w}}_{i+N-B} \omega^{(i+N-B)m_j}|} = |\widehat{\mathbf{z}}_{k_0-1} + v_{j,k_0-1}|, j = 1, 2, 3, \quad (22)$$

where

$$v_{j,k_0-1} := \frac{\epsilon \widehat{\mathbf{z}}_{k_0} \widehat{\mathbf{w}}_{i+N-B+1} \omega^{(i+N-B+1)m_j} + \sum_{l=2}^{\frac{N}{2}-k_0+1} \epsilon \widehat{\mathbf{z}}_{k_0-1+l} \widehat{\mathbf{w}}_{i+N-B+l} \omega^{(i+N-B+l)m_j}}{\widehat{\mathbf{w}}_{i+N-B} \omega^{(i+N-B)m_j}}. \quad (23)$$

Motivated by Lemma 1, define

$$f(\widehat{\mathbf{z}}_{k_0}) := \frac{v_{1,k_0-1} - v_{2,k_0-1}}{v_{1,k_0-1} - v_{3,k_0-1}}$$

$$= \frac{a\widehat{\mathbf{z}}_{k_0} + b}{c\widehat{\mathbf{z}}_{k_0} + d}, \quad (24)$$

where

$$\begin{cases} a = \epsilon \widehat{\mathbf{w}}_{i+N-B+1} (\omega^{m_1} - \omega^{m_2}), \\ b = \sum_{l=2}^{\frac{N}{2}-k_0+1} \epsilon \widehat{\mathbf{z}}_{k_0-1+l} \widehat{\mathbf{w}}_{i+N-B+l} (\omega^{lm_1} - \omega^{lm_2}), \\ c = \epsilon \widehat{\mathbf{w}}_{i+N-B+1} (\omega^{m_1} - \omega^{m_3}), \\ d = \sum_{l=2}^{\frac{N}{2}-k_0+1} \epsilon \widehat{\mathbf{z}}_{k_0-1+l} \widehat{\mathbf{w}}_{i+N-B+l} (\omega^{lm_1} - \omega^{lm_3}). \end{cases} \quad (25)$$

Recall that $\widehat{\mathbf{w}}_{i+N-B+1} \neq 0$ and m_1, m_2, m_3 are distinct. Then $ac \neq 0$. For the generic analytic signal $\mathbf{z}$, we have $ad - bc \neq 0$. That is, $f(\widehat{\mathbf{z}}_{k_0})$ meets the requirements in Lemma 2. Then $\Im[f(\widehat{\mathbf{z}}_{k_0})] \neq 0$. Therefore, by Lemma 1 the component $\epsilon \widehat{\mathbf{z}}_{k_0-1}$ can be determined by the equation system (22). Through the induction procedures, the proof can be completed.

For N being odd, as in the even case the recovery starts with $\widehat{\mathbf{z}}_{\frac{N-1}{2}}$. Suppose that what we get is $\widehat{\mathbf{z}}_{\frac{N-1}{2}} e^{i\hat{\theta}_0}$. Through the similar recursive procedures as in (22), what we get is $e^{i\hat{\theta}_0}(\widehat{\mathbf{z}}_0, \dots, \widehat{\mathbf{z}}_{\frac{N-1}{2}})$. Recall that $\widehat{\mathbf{z}}_0$ is real. Then one needs to choose a phase $\hat{\theta}$ such that $e^{i\hat{\theta}} e^{i\hat{\theta}_0} \widehat{\mathbf{z}}_0$ is real. That is, what we get is $\epsilon(\widehat{\mathbf{z}}_0, \dots, \widehat{\mathbf{z}}_{\frac{N-1}{2}})$ with $\epsilon \in \{1, -1\}$. This completes the proof. $\square$

Remark 5 (1) In Theorem 2, it is required that $\widehat{\mathbf{w}}_{i+N-B+1} \neq 0$. Such a requirement is crucial for the determination of $\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor - 1}$. If it is not satisfied, then the equation system w.r.t $\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor - 1}$:

$$\frac{N|\widehat{\mathbf{y}}_{\lfloor \frac{N}{2} \rfloor - 1 - (i+N-B), m_j}^{\mathbf{w}}|}{|\widehat{\mathbf{w}}_{i+N-B} \omega^{(i+N-B)m_j}|} = \left| \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor - 1} + \frac{\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \widehat{\mathbf{w}}_{i+N-B+1} \omega^{(i+N-B+1)m_j}}{\widehat{\mathbf{w}}_{i+N-B} \omega^{(i+N-B)m_j}} \right|, j = 1, 2, 3$$

degenerates to

$$\frac{N|\widehat{\mathbf{y}}_{\lfloor \frac{N}{2} \rfloor - 1 - (i+N-B), m_j}^{\mathbf{w}}|}{|\widehat{\mathbf{w}}_{i+N-B} \omega^{(i+N-B)m_j}|} = \left| \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor - 1} \right|, j = 1, 2, 3.$$

Clearly, the above system is underdetermined and $\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor - 1}$ cannot be determined. (2) The overall requirements for the window $\mathbf{w}$ in Theorem 2 are that

$$\widehat{\mathbf{w}}_i = \dots = \widehat{\mathbf{w}}_{i+N-B-1} = 0, \widehat{\mathbf{w}}_{i+N-B} \neq 0, \widehat{\mathbf{w}}_{i+N-B+1} \neq 0. \quad (26)$$

From this, using the DFT it is easy to construct the required window.

3.3 The Second Main Result: Window Bandlimit $B = \lceil \frac{N}{2} \rceil + 1$ Case

Suppose that the window $\mathbf{w} \in \mathbb{C}^N$ is $(\lceil \frac{N}{2} \rceil + 1)$ -bandlimited. Consequently, there exists $i \in \{0, \dots, N-1\}$ such that

$$\widehat{\mathbf{w}}_i = \dots = \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor - 2} = 0, \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor - 1} \neq 0. \quad (27)$$

We are interested in the STFT measurements at $(2-i+N-n, m)$:

$$|\widehat{y}_{2-i+N-n,m}^{\mathbf{w}}| = \frac{1}{N} \left| \sum_{l=0}^{N-1} \widehat{\mathbf{z}}_{2-i+N-n+l} \widehat{\mathbf{w}}_l \omega^{lm} \right|, \quad (28)$$

where $n = 1, \dots, \lfloor \frac{N}{2} \rfloor$.

Again the following is a motivation example for the structure of the summation in (28).

Example 3 Let $N = 6$ and the window bandlimit $B = \lceil \frac{N}{2} \rceil + 1 = 4$. For an analytic signal $\mathbf{z} \in \mathbb{C}^6$, it follows from Proposition 2 (i) that its DFT $\widehat{\mathbf{z}} = (\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \widehat{\mathbf{z}}_2, \widehat{\mathbf{z}}_3, 0, 0)$. Choose a 4-bandlimited window $\mathbf{w} \in \mathbb{C}^6$ such that $\widehat{\mathbf{w}} = (\widehat{\mathbf{w}}_0, \widehat{\mathbf{w}}_1, \widehat{\mathbf{w}}_2, \widehat{\mathbf{w}}_3, 0, 0)$ and correspondingly $i = 4$ in (27). For $n = 1, 2, 3$, $|\widehat{y}_{4-n,0}^{\mathbf{w}}|$ in (28) are expressed as: $|\widehat{y}_{3,0}^{\mathbf{w}}| = \frac{1}{6} |\widehat{\mathbf{z}}_3 \widehat{\mathbf{w}}_0 + \widehat{\mathbf{z}}_0 \widehat{\mathbf{w}}_3|$, $|\widehat{y}_{2,0}^{\mathbf{w}}| = \frac{1}{6} |\widehat{\mathbf{z}}_2 \widehat{\mathbf{w}}_0 + \widehat{\mathbf{z}}_3 \widehat{\mathbf{w}}_1|$, $|\widehat{y}_{1,0}^{\mathbf{w}}| = \frac{1}{6} |\widehat{\mathbf{z}}_1 \widehat{\mathbf{w}}_0 + \widehat{\mathbf{z}}_2 \widehat{\mathbf{w}}_1 + \widehat{\mathbf{z}}_3 \widehat{\mathbf{w}}_2|$. The terms $\widehat{\mathbf{z}}_k \widehat{\mathbf{w}}_l$ on which $\widehat{y}_{4-n,0}^{\mathbf{w}}$ is dependent are arranged as follows,

$$\begin{array}{ll} n = 1 & \widehat{\mathbf{z}}_0 \widehat{\mathbf{w}}_3 \quad \quad \quad \widehat{\mathbf{z}}_3 \widehat{\mathbf{w}}_0 \\ n = 2 & \quad \quad \quad \widehat{\mathbf{z}}_2 \widehat{\mathbf{w}}_0 \quad \widehat{\mathbf{z}}_3 \widehat{\mathbf{w}}_1 \\ n = 3 & \quad \quad \quad \widehat{\mathbf{z}}_1 \widehat{\mathbf{w}}_0 \quad \widehat{\mathbf{z}}_2 \widehat{\mathbf{w}}_1 \quad \widehat{\mathbf{z}}_3 \widehat{\mathbf{w}}_2 \end{array} \quad (29)$$

Based on (29), the terms $\widehat{\mathbf{z}}_k \widehat{\mathbf{w}}_l \omega^{lm}$ of $\widehat{y}_{4-n,m}^{\mathbf{w}}$ can be arranged similarly.

For the general case when the window bandlimit $B = \lceil \frac{N}{2} \rceil + 1$, as in (29), it follows from (27) and Proposition 2 that the terms $\widehat{\mathbf{z}}_k \widehat{\mathbf{w}}_l$ on which $|\widehat{y}_{2-i+N-n,0}^{\mathbf{w}}|$ in (28) is dependent are arranged as follows,

$$\begin{array}{ccccccc} \widehat{\mathbf{z}}_0 \widehat{\mathbf{w}}_{i-1+N} & & & & & \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor - 1} & \\ & & & & \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor - 1} \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor - 1} & \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor} & \\ & & \dots & & & \vdots & \\ & \widehat{\mathbf{z}}_1 \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor - 1} & \widehat{\mathbf{z}}_2 \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor} & \dots & \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \widehat{\mathbf{w}}_{i+2\lfloor \frac{N}{2} \rfloor - 2} & & \end{array} \quad (30)$$

An observation on (28) gives us that, for $|\widehat{y}_{2-i+N-n,m}^{\mathbf{w}}|$ the related terms $\widehat{\mathbf{z}}_k \widehat{\mathbf{w}}_l \omega^{lm}$ are arranged as in (30). Motivated by such a structure, we next use the (multi-window) measurements $\{|\widehat{y}_{2-i+N-n,m}^{(s)}| : s = 1, 2, 3, 4\}$ to do the PR for $\mathbf{z}$, which is stated below as our second main theorem.

Theorem 3 Assume that the STFT separation parameter L satisfies $\lceil N/L \rceil \geq 3$. Suppose that the four windows $\mathbf{w}^{(s)} \in \mathbb{C}^N$, $s = 1, \dots, 4$ are $(\lceil \frac{N}{2} \rceil + 1)$ -bandlimited such that they satisfy (27) with $i \in \{0, \dots, N-1\}$, $\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor}^{(1)} \neq 0$, and let $m_1, m_2, m_3 \in \{0, 1, \dots, \lceil N/L \rceil - 1\}$ be three distinct numbers. If the matrix

$$\mathcal{A}_0 := \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{21}^{(1)} & a_{22}^{(1)} \\ a_{11}^{(2)} & a_{12}^{(2)} & a_{21}^{(2)} & a_{22}^{(2)} \\ a_{11}^{(3)} & a_{12}^{(3)} & a_{21}^{(3)} & a_{22}^{(3)} \\ a_{11}^{(4)} & a_{12}^{(4)} & a_{21}^{(4)} & a_{22}^{(4)} \end{pmatrix} \quad (31)$$

is invertible, where

$$\begin{cases} a_{11}^{(s)} = |\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor - 1}^{(s)}|^2, & a_{12}^{(s)} = \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor - 1}^{(s)} \overline{\widehat{\mathbf{w}}_{i-1+N}^{(s)}}, \\ a_{21}^{(s)} = \widehat{\mathbf{w}}_{i-1+N}^{(s)} \overline{\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor - 1}^{(s)}}, & a_{22}^{(s)} = |\widehat{\mathbf{w}}_{i-1+N}^{(s)}|^2, \end{cases} \quad (32)$$

then any generic analytic signal $\mathbf{z} \in \mathbb{C}^N$ can be determined (up to a global sign) by its $(3\lfloor \frac{N}{2} \rfloor + 1)$ number of STFT measurements

$$\left\{ |\widehat{\mathbf{y}}_{1-i+N,0}^{\mathbf{w}^{(1)}}|, |\widehat{\mathbf{y}}_{1-i+N,0}^{\mathbf{w}^{(2)}}|, |\widehat{\mathbf{y}}_{1-i+N,0}^{\mathbf{w}^{(3)}}|, |\widehat{\mathbf{y}}_{1-i+N,0}^{\mathbf{w}^{(4)}}|, |\widehat{\mathbf{y}}_{k-(i+\lfloor \frac{N}{2} \rfloor - 1), m_j}^{\mathbf{w}^{(1)}}| : k = 1, \dots, \lfloor \frac{N}{2} \rfloor - 1, \right. \\ \left. j = 1, 2, 3 \right\}. \quad (33)$$

Proof Consider the equation system w.r.t $(\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor})$:

$$|\widehat{\mathbf{y}}_{1-i+N,0}^{\mathbf{w}^{(s)}}| = \frac{1}{N} |\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor - 1}^{(s)} + \widehat{\mathbf{z}}_0 \widehat{\mathbf{w}}_{i-1+N}^{(s)}|, \quad s = 1, 2, 3, 4. \quad (34)$$

Note that (34) is equivalent to

$$\left(\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor - 1}^{(s)} + \widehat{\mathbf{z}}_0 \widehat{\mathbf{w}}_{i-1+N}^{(s)} \right) \overline{\left(\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor - 1}^{(s)} + \widehat{\mathbf{z}}_0 \widehat{\mathbf{w}}_{i-1+N}^{(s)} \right)} = N^2 |\widehat{\mathbf{y}}_{1-i+N,0}^{\mathbf{w}^{(s)}}|^2, \\ s = 1, 2, 3, 4. \quad (35)$$

Through the direct calculation, (35) is equivalent to

$$\mathcal{A}_0 \begin{pmatrix} |\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor}|^2 \\ \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \widehat{\mathbf{z}}_0 \\ \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \widehat{\mathbf{z}}_0 \\ \widehat{\mathbf{z}}_0^2 \end{pmatrix} = \begin{pmatrix} N^2 |\widehat{\mathbf{y}}_{1-i+N,0}^{\mathbf{w}^{(1)}}|^2 \\ N^2 |\widehat{\mathbf{y}}_{1-i+N,0}^{\mathbf{w}^{(2)}}|^2 \\ N^2 |\widehat{\mathbf{y}}_{1-i+N,0}^{\mathbf{w}^{(3)}}|^2 \\ N^2 |\widehat{\mathbf{y}}_{1-i+N,0}^{\mathbf{w}^{(4)}}|^2 \end{pmatrix}. \quad (36)$$

Since $\mathcal{A}_0$ is invertible and $\widehat{\mathbf{z}}_0 \in \mathbb{R}$, $(\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor})$ can be determined up to a sign by the four measurements in (34). We denote such a recovery result by $\epsilon(\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor})$ with $\epsilon \in \{1, -1\}$.

We first consider the equation system w.r.t $\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor - 1}$:

$$|\widehat{y}_{N-i, m_j}^{w(1)}| = \frac{1}{N} |\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor - 1} \widehat{\mathbf{w}}_{i + \lfloor \frac{N}{2} \rfloor - 1}^{(1)} \omega^{(i + \lfloor \frac{N}{2} \rfloor - 1)m_j} + \epsilon \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \widehat{\mathbf{w}}_{i + \lfloor \frac{N}{2} \rfloor}^{(1)} \omega^{(i + \lfloor \frac{N}{2} \rfloor)m_j}|, j = 1, 2, 3. \quad (37)$$

Note that (37) is equivalent to

$$\frac{N |\widehat{y}_{N-i, m_j}^{w(1)}|}{|\widehat{\mathbf{w}}_{i + \lfloor \frac{N}{2} \rfloor - 1}^{(1)} \omega^{(i + \lfloor \frac{N}{2} \rfloor - 1)m_j}|} = |\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor - 1} + v_{j, \lfloor \frac{N}{2} \rfloor - 1}|, j = 1, 2, 3, \quad (38)$$

where $v_{j, \lfloor \frac{N}{2} \rfloor - 1} := \frac{\epsilon \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \widehat{\mathbf{w}}_{i + \lfloor \frac{N}{2} \rfloor}^{(1)} \omega^{(i + \lfloor \frac{N}{2} \rfloor)m_j}}{\widehat{\mathbf{w}}_{i + \lfloor \frac{N}{2} \rfloor - 1}^{(1)} \omega^{(i + \lfloor \frac{N}{2} \rfloor - 1)m_j}}$. For the generic analytic signal $\mathbf{z} \in \mathbb{C}^N$, we have $\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \neq 0$. Therefore, for $j = 1, 2, 3$ we have

$$\frac{v_{1, \lfloor \frac{N}{2} \rfloor - 1} - v_{2, \lfloor \frac{N}{2} \rfloor - 1}}{v_{1, \lfloor \frac{N}{2} \rfloor - 1} - v_{3, \lfloor \frac{N}{2} \rfloor - 1}} = \frac{\omega^{m_1} - \omega^{m_2}}{\omega^{m_1} - \omega^{m_3}}. \quad (39)$$

By (39) and Lemma 3, we have $\Im\left(\frac{v_{1, \lfloor \frac{N}{2} \rfloor - 1} - v_{2, \lfloor \frac{N}{2} \rfloor - 1}}{v_{1, \lfloor \frac{N}{2} \rfloor - 1} - v_{3, \lfloor \frac{N}{2} \rfloor - 1}}\right) \neq 0$. Now it follows from

Lemma 1 that there exists a unique solution to the equation system (38) w.r.t $\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor - 1}$. Clearly, $\epsilon \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor - 1}$ is a solution. Then it is the unique solution. Suppose that for any $k \in \{\lfloor \frac{N}{2} \rfloor, \lfloor \frac{N}{2} \rfloor - 1, \dots, k_0, 0\}$ where $k_0 \in \{\lfloor \frac{N}{2} \rfloor, \lfloor \frac{N}{2} \rfloor - 1, \dots, 2\}$, the component $\widehat{\mathbf{z}}_k$ has been determined by the measurements

$$\{|\widehat{y}_{1-i+N, 0}^{w(s)}|, |\widehat{y}_{\ell - (i + \lfloor \frac{N}{2} \rfloor - 1), m_j}^{w(1)}| : \ell = \lfloor \frac{N}{2} \rfloor - 1, \dots, k_0, s = 1, 2, 3, 4, j = 1, 2, 3\}.$$

We next recover $\widehat{\mathbf{z}}_{k_0 - 1}$. Consider the equation system w.r.t $\widehat{\mathbf{z}}_{k_0 - 1}$:

$$\begin{aligned} |\widehat{y}_{k_0 - 1 - (i + \lfloor \frac{N}{2} \rfloor - 1), m_j}^{w(1)}| &= \frac{1}{N} |\widehat{\mathbf{z}}_{k_0 - 1} \widehat{\mathbf{w}}_{i + \lfloor \frac{N}{2} \rfloor - 1}^{(1)} \omega^{(i + \lfloor \frac{N}{2} \rfloor - 1)m_j} \\ &\quad + \sum_{l=1}^{\lfloor \frac{N}{2} \rfloor - k_0 + 1} \epsilon \widehat{\mathbf{z}}_{k_0 - 1 + l} \widehat{\mathbf{w}}_{i + \lfloor \frac{N}{2} \rfloor - 1 + l}^{(1)} \omega^{(i + \lfloor \frac{N}{2} \rfloor - 1 + l)m_j}|, j = 1, 2, 3. \end{aligned} \quad (40)$$

Note that (40) is equivalent to

$$\frac{N |\widehat{y}_{k_0 - 1 - (i + \lfloor \frac{N}{2} \rfloor - 1), m_j}^{w(1)}|}{|\widehat{\mathbf{w}}_{i + \lfloor \frac{N}{2} \rfloor - 1}^{(1)} \omega^{(i + \lfloor \frac{N}{2} \rfloor - 1)m_j}|} = |\widehat{\mathbf{z}}_{k_0 - 1} + v_{j, k_0 - 1}|, j = 1, 2, 3, \quad (41)$$

where

$$v_{j,k_0-1} := \frac{\epsilon \widehat{\mathbf{z}}_{k_0} \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor}^{(1)} \omega^{(i+\lfloor \frac{N}{2} \rfloor)m_j} + \sum_{l=2}^{\lfloor \frac{N}{2} \rfloor - k_0 + 1} \epsilon \widehat{\mathbf{z}}_{k_0-1+l} \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1+l}^{(1)} \omega^{(i+\lfloor \frac{N}{2} \rfloor-1+l)m_j}}{\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \omega^{(i+\lfloor \frac{N}{2} \rfloor-1)m_j}}. \quad (42)$$

Define

$$\begin{aligned} f(\widehat{\mathbf{z}}_{k_0}) &:= \frac{v_{1,k_0-1} - v_{2,k_0-1}}{v_{1,k_0-1} - v_{3,k_0-1}} \\ &= \frac{a\widehat{\mathbf{z}}_{k_0} + b}{c\widehat{\mathbf{z}}_{k_0} + d}, \end{aligned} \quad (43)$$

where

$$\begin{cases} a = \epsilon \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor}^{(1)} (\omega^{m_1} - \omega^{m_2}), \\ b = \sum_{l=2}^{\lfloor \frac{N}{2} \rfloor - k_0 + 1} \epsilon \widehat{\mathbf{z}}_{k_0-1+l} \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1+l}^{(1)} (\omega^{lm_1} - \omega^{lm_2}), \\ c = \epsilon \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor}^{(1)} (\omega^{m_1} - \omega^{m_3}), \\ d = \sum_{l=2}^{\lfloor \frac{N}{2} \rfloor - k_0 + 1} \epsilon \widehat{\mathbf{z}}_{k_0-1+l} \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1+l}^{(1)} (\omega^{lm_1} - \omega^{lm_3}). \end{cases} \quad (44)$$

Since $\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor}^{(1)} \neq 0$, $ac \neq 0$. For the generic analytic signal $\mathbf{z}$, we have $ad - bc \neq 0$. That is, $f(\widehat{\mathbf{z}}_{k_0})$ meets the requirements in Lemma 2. Then $\Im[f(\widehat{\mathbf{z}}_{k_0})] \neq 0$. By Lemma 1, $\epsilon \widehat{\mathbf{z}}_{k_0-1}$ can be determined. This completes the proof. $\square$

Remark 6 The condition $\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor}^{(1)} \neq 0$ in Theorem 3 is also important since if otherwise, then the equation system w.r.t $\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor-1}$:

$$\frac{N|\widehat{\mathbf{y}}_{N-i,m_j}^{\mathbf{w}(1)}|}{|\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \omega^{(i+\lfloor \frac{N}{2} \rfloor-1)m_j}|} = \left| \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor-1} + \frac{\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor} \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor}^{(1)} \omega^{(i+\lfloor \frac{N}{2} \rfloor)m_j}}{\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \omega^{(i+\lfloor \frac{N}{2} \rfloor-1)m_j}} \right|, j = 1, 2, 3$$

degenerates to

$$\frac{N|\widehat{\mathbf{y}}_{N-i,m_j}^{\mathbf{w}(1)}|}{|\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \omega^{(i+\lfloor \frac{N}{2} \rfloor-1)m_j}|} = \left| \widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor-1} \right|, j = 1, 2, 3.$$

Clearly, the above system is underdetermined and $\widehat{\mathbf{z}}_{\lfloor \frac{N}{2} \rfloor-1}$ cannot be recovered exactly.

Remark 7 The overall requirements for the four windows $\mathbf{w}^{(s)} \in \mathbb{C}^N$, $s = 1, \dots, 4$ are as follows: (1) $\mathbf{w}^{(s)} \in \mathbb{C}^N$, $s = 1, \dots, 4$ are $(\lceil \frac{N}{2} \rceil + 1)$ -bandlimited. (2) Associated with the four matrices, the matrix

$$\mathcal{A}_0 = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{21}^{(1)} & a_{22}^{(1)} \\ a_{11}^{(2)} & a_{12}^{(2)} & a_{21}^{(2)} & a_{22}^{(2)} \\ a_{11}^{(3)} & a_{12}^{(3)} & a_{21}^{(3)} & a_{22}^{(3)} \\ a_{11}^{(4)} & a_{12}^{(4)} & a_{21}^{(4)} & a_{22}^{(4)} \end{pmatrix} \quad (45)$$

is invertible. Uniformly choose the first and fourth columns of $\mathcal{A}_0$ from $\mathbb{R}^4$, and the second and third columns from $\mathbb{C}^4$. Since invertible matrices are dense in $\mathbb{C}^{4 \times 4}$ (or in $\mathbb{R}^{4 \times 4}$), $\mathcal{A}_0$ is invertible with probability one.

The following provides a design for the windows in Theorem 3.

Example 4 Choose a $(\lceil \frac{N}{2} \rceil + 1)$ -bandlimited window $\widehat{\mathbf{w}}^{(1)}$ such that (27) holds with $i \in \{0, \dots, N-1\}$. Consequently, $\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor}^{(1)} \neq 0$. Now choose the other three $(\lceil \frac{N}{2} \rceil + 1)$ -bandlimited windows $\widehat{\mathbf{w}}^{(s)}$, $s = 2, 3, 4$ such that $\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(2)} = (\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)})^2$, $\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(3)} = (\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)})^3$, $\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(4)} = (\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)})^4$, $\widehat{\mathbf{w}}_{i-1+N}^{(2)} = (\widehat{\mathbf{w}}_{i-1+N}^{(1)})^2$, $\widehat{\mathbf{w}}_{i-1+N}^{(3)} = (\widehat{\mathbf{w}}_{i-1+N}^{(1)})^3$ and $\widehat{\mathbf{w}}_{i-1+N}^{(4)} = (\widehat{\mathbf{w}}_{i-1+N}^{(1)})^4$. Additionally, it is required that $\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \neq \widehat{\mathbf{w}}_{i-1+N}^{(1)}$, $\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \widehat{\mathbf{w}}_{i-1+N}^{(1)} \notin \mathbb{R}$, $|\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)}| \neq |\widehat{\mathbf{w}}_{i-1+N}^{(1)}|$ and $\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \widehat{\mathbf{w}}_{i-1+N}^{(1)} \neq 0$. Then $\mathcal{A}_0$ in (31) can be expressed as

$$\mathcal{A}_0 = \begin{pmatrix} |\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)}|^2 \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \overline{\widehat{\mathbf{w}}_{i-1+N}^{(1)}} & \overline{\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)}} \widehat{\mathbf{w}}_{i-1+N}^{(1)} & |\widehat{\mathbf{w}}_{i-1+N}^{(1)}|^2 \\ |\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)}|^4 (\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \overline{\widehat{\mathbf{w}}_{i-1+N}^{(1)}})^2 & (\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \overline{\widehat{\mathbf{w}}_{i-1+N}^{(1)}})^2 |\widehat{\mathbf{w}}_{i-1+N}^{(1)}|^4 \\ |\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)}|^6 (\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \overline{\widehat{\mathbf{w}}_{i-1+N}^{(1)}})^3 & (\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \overline{\widehat{\mathbf{w}}_{i-1+N}^{(1)}})^3 |\widehat{\mathbf{w}}_{i-1+N}^{(1)}|^6 \\ |\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)}|^8 (\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \overline{\widehat{\mathbf{w}}_{i-1+N}^{(1)}})^4 & (\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \overline{\widehat{\mathbf{w}}_{i-1+N}^{(1)}})^4 |\widehat{\mathbf{w}}_{i-1+N}^{(1)}|^8 \end{pmatrix}. \quad (46)$$

Clearly,

$$\mathcal{A}_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ |\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)}|^2 \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \overline{\widehat{\mathbf{w}}_{i-1+N}^{(1)}} & \overline{\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)}} \widehat{\mathbf{w}}_{i-1+N}^{(1)} & |\widehat{\mathbf{w}}_{i-1+N}^{(1)}|^2 \\ |\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)}|^4 (\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \overline{\widehat{\mathbf{w}}_{i-1+N}^{(1)}})^2 & (\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \overline{\widehat{\mathbf{w}}_{i-1+N}^{(1)}})^2 |\widehat{\mathbf{w}}_{i-1+N}^{(1)}|^4 \\ |\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)}|^6 (\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \overline{\widehat{\mathbf{w}}_{i-1+N}^{(1)}})^3 & (\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor-1}^{(1)} \overline{\widehat{\mathbf{w}}_{i-1+N}^{(1)}})^3 |\widehat{\mathbf{w}}_{i-1+N}^{(1)}|^6 \end{pmatrix}$$

$$\times \begin{pmatrix} |\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor - 1}^{(1)}|^2 & 0 & 0 & 0 \\ 0 & \widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor - 1}^{(1)} \overline{\widehat{\mathbf{w}}_{i-1+N}^{(1)}} & 0 & 0 \\ 0 & 0 & \overline{\widehat{\mathbf{w}}_{i+\lfloor \frac{N}{2} \rfloor - 1}^{(1)}} \widehat{\mathbf{w}}_{i-1+N}^{(1)} & 0 \\ 0 & 0 & 0 & |\widehat{\mathbf{w}}_{i-1+N}^{(1)}|^2 \end{pmatrix}. \quad (47)$$

Then $\mathcal{A}_0$ is invertible, and the four windows $\mathbf{w}^{(s)}$, $s = 1, 2, 3, 4$ meet the requirements in Theorem 3. As an example for $(N, B, i) = (48, 25, 25)$, the graphs of $\mathbf{w}^{(s)}$, $s = 1, 2, 3, 4$ and their DFTs are plotted in Fig. 2.

3.4 The Third Main Result: The Analytic Window Case

The main purpose of this subsection is to show that if N is even and all the windows are analytic, then fewer measurements than Theorem 2 and 3 are required for the recovery.

Lemma 5 Suppose that N is even, and $\mathbf{z}, \tilde{\mathbf{z}} \in \mathbb{C}^N$ are both generic analytic signals with DFTs $\widehat{\mathbf{z}} = (\widehat{z}_0, \widehat{z}_1, \dots, \widehat{z}_{\frac{N}{2}}, 0, \dots, 0)$ and $\widehat{\tilde{\mathbf{z}}} = (\widehat{\tilde{z}}_0, \widehat{\tilde{z}}_1, \dots, \widehat{\tilde{z}}_{\frac{N}{2}}, 0, \dots, 0)$. Assume that the STFT separation parameter $0 < L < N$ satisfies $\lceil N/L \rceil \geq 3$, and $\mathbf{w}^{(1)}$ is an analytic window such that $\widehat{\mathbf{w}}_1^{(1)} \widehat{\mathbf{w}}_0^{(1)} \neq 0$. Let $m_1, m_2, m_3 \in \{0, 1, \dots, \lceil N/L \rceil - 1\}$ be three distinct parameters. If $|\widehat{\tilde{\mathbf{z}}}_{\frac{N}{2}}| \neq |\widehat{\mathbf{z}}_{\frac{N}{2}}|$ and $\tilde{\mathbf{z}}$ has the same STFT (associated with the window $\mathbf{w}^{(1)}$) magnitudes as $\mathbf{z}$ at $(\frac{N}{2} - 1, m_j)$, $j = 1, 2, 3$, then

$$\frac{\widehat{\tilde{\mathbf{z}}}_{\frac{N}{2}}^2}{|\widehat{\tilde{\mathbf{z}}}_{\frac{N}{2}-1}|^2} = \frac{(\widehat{\mathbf{w}}_0^{(1)})^2}{|\widehat{\mathbf{w}}_1^{(1)}|^2}. \quad (48)$$

Proof By Proposition 2, both $\widehat{\mathbf{z}}_{\frac{N}{2}}$ and $\widehat{\tilde{\mathbf{z}}}_{\frac{N}{2}}$ are real-valued. Suppose that $\widehat{\tilde{\mathbf{z}}}_{\frac{N}{2}} = \lambda_{\mathbf{z}, \tilde{\mathbf{z}}} \widehat{\mathbf{z}}_{\frac{N}{2}}$ such that $\pm 1 \neq \lambda_{\mathbf{z}, \tilde{\mathbf{z}}} \in \mathbb{R}$. Since the STFT magnitudes of $\mathbf{z}$ at $(\frac{N}{2} - 1, m_j)$, $j = 1, 2, 3$ are identical to those of $\tilde{\mathbf{z}}$, we have that

$$\begin{aligned} \frac{1}{N} |\widehat{\mathbf{z}}_{\frac{N}{2}-1} \widehat{\mathbf{w}}_0^{(1)} + \widehat{\mathbf{z}}_{\frac{N}{2}} \widehat{\mathbf{w}}_1^{(1)} \omega^{m_j}| &= |\widehat{\mathbf{y}}_{\frac{N}{2}-1, m_j}^{\mathbf{w}^{(1)}}| \\ &= \frac{1}{N} |\widehat{\tilde{\mathbf{z}}}_{\frac{N}{2}-1} \widehat{\mathbf{w}}_0^{(1)} + \widehat{\tilde{\mathbf{z}}}_{\frac{N}{2}} \widehat{\mathbf{w}}_1^{(1)} \omega^{m_j}|, \quad j = 1, 2, 3. \end{aligned} \quad (49)$$

Since $\widehat{\tilde{\mathbf{z}}}_{\frac{N}{2}} = \lambda_{\mathbf{z}, \tilde{\mathbf{z}}} \widehat{\mathbf{z}}_{\frac{N}{2}}$,

$$|\widehat{\mathbf{z}}_{\frac{N}{2}-1} \widehat{\mathbf{w}}_0^{(1)} + \widehat{\mathbf{z}}_{\frac{N}{2}} \widehat{\mathbf{w}}_1^{(1)} \omega^{m_j}|^2 = |\widehat{\tilde{\mathbf{z}}}_{\frac{N}{2}-1} \widehat{\mathbf{w}}_0^{(1)} + \lambda_{\mathbf{z}, \tilde{\mathbf{z}}} \widehat{\tilde{\mathbf{z}}}_{\frac{N}{2}} \widehat{\mathbf{w}}_1^{(1)} \omega^{m_j}|^2. \quad (50)$$

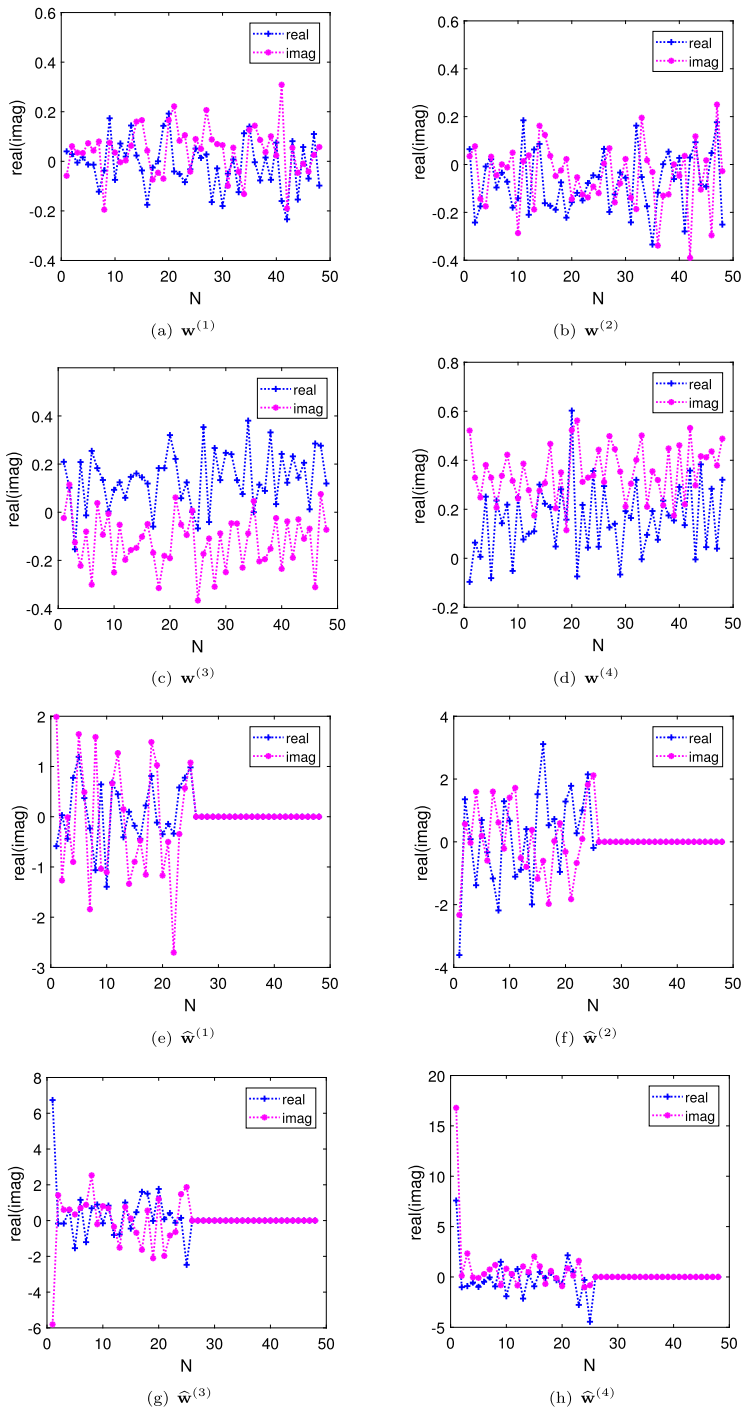


Fig. 2 (a–d) Real and imaginary parts of the four windows $\mathbf{w}^{(s)}$; (e–h) Real and imaginary parts of $\hat{\mathbf{w}}^{(s)}$, $s = 1, 2, 3, 4$

Using Proposition 2 again, $\widehat{\mathbf{w}}_0^{(1)}$ is real-valued. Then (50) is equivalent to

$$\begin{aligned} & (\lambda_{\mathbf{z}, \tilde{\mathbf{z}}}^2 - 1) \widehat{\mathbf{z}}_{\frac{N}{2}}^2 |\widehat{\mathbf{w}}_1^{(1)}|^2 + (|\widehat{\mathbf{z}}_{\frac{N}{2}-1}|^2 - |\widehat{\mathbf{z}}_{\frac{N}{2}-1}|^2) (\widehat{\mathbf{w}}_0^{(1)})^2 \\ & + 2\Re\{\omega^{mj} \widehat{\mathbf{w}}_1^{(1)} \widehat{\mathbf{w}}_0^{(1)} \widehat{\mathbf{z}}_{\frac{N}{2}} (\lambda_{\mathbf{z}, \tilde{\mathbf{z}}} \widehat{\mathbf{z}}_{\frac{N}{2}-1} - \widehat{\mathbf{z}}_{\frac{N}{2}-1})\} = 0. \end{aligned} \quad (51)$$

Multiplying by ω^{mj} on both sides of (51) leads to

$$\begin{aligned} & [(\lambda_{\mathbf{z}, \tilde{\mathbf{z}}}^2 - 1) \widehat{\mathbf{z}}_{\frac{N}{2}}^2 |\widehat{\mathbf{w}}_1^{(1)}|^2 + (|\widehat{\mathbf{z}}_{\frac{N}{2}-1}|^2 - |\widehat{\mathbf{z}}_{\frac{N}{2}-1}|^2) (\widehat{\mathbf{w}}_0^{(1)})^2] \omega^{mj} \\ & + (\lambda_{\mathbf{z}, \tilde{\mathbf{z}}} \widehat{\mathbf{z}}_{\frac{N}{2}-1} - \widehat{\mathbf{z}}_{\frac{N}{2}-1}) \widehat{\mathbf{w}}_1^{(1)} \widehat{\mathbf{w}}_0^{(1)} \widehat{\mathbf{z}}_{\frac{N}{2}} \omega^{2mj} + (\lambda_{\mathbf{z}, \tilde{\mathbf{z}}} \widehat{\mathbf{z}}_{\frac{N}{2}-1} - \widehat{\mathbf{z}}_{\frac{N}{2}-1}) \widehat{\mathbf{w}}_0^{(1)} \widehat{\mathbf{w}}_1^{(1)} \widehat{\mathbf{z}}_{\frac{N}{2}} = 0. \end{aligned} \quad (52)$$

Consider the following equation w.r.t x :

$$\begin{aligned} & [(\lambda_{\mathbf{z}, \tilde{\mathbf{z}}}^2 - 1) \widehat{\mathbf{z}}_{\frac{N}{2}}^2 |\widehat{\mathbf{w}}_1^{(1)}|^2 + (|\widehat{\mathbf{z}}_{\frac{N}{2}-1}|^2 - |\widehat{\mathbf{z}}_{\frac{N}{2}-1}|^2) (\widehat{\mathbf{w}}_0^{(1)})^2] x \\ & + (\lambda_{\mathbf{z}, \tilde{\mathbf{z}}} \widehat{\mathbf{z}}_{\frac{N}{2}-1} - \widehat{\mathbf{z}}_{\frac{N}{2}-1}) \widehat{\mathbf{w}}_1^{(1)} \widehat{\mathbf{w}}_0^{(1)} \widehat{\mathbf{z}}_{\frac{N}{2}} x^2 + (\lambda_{\mathbf{z}, \tilde{\mathbf{z}}} \widehat{\mathbf{z}}_{\frac{N}{2}-1} - \widehat{\mathbf{z}}_{\frac{N}{2}-1}) \widehat{\mathbf{w}}_0^{(1)} \widehat{\mathbf{w}}_1^{(1)} \widehat{\mathbf{z}}_{\frac{N}{2}} = 0. \end{aligned} \quad (53)$$

If the polynomial on the left-hand side of (53) is a nonzero polynomial, then there are at most two solutions to the above equation. By (52), ω^{mj} , $j = 1, 2, 3$ are the three distinct solutions to (53). Therefore, all the coefficients in (53) are zero. Then

$$(\lambda_{\mathbf{z}, \tilde{\mathbf{z}}} \widehat{\mathbf{z}}_{\frac{N}{2}-1} - \widehat{\mathbf{z}}_{\frac{N}{2}-1}) \widehat{\mathbf{w}}_0^{(1)} \widehat{\mathbf{w}}_1^{(1)} \widehat{\mathbf{z}}_{\frac{N}{2}} = 0 \quad (54)$$

and

$$(\lambda_{\mathbf{z}, \tilde{\mathbf{z}}}^2 - 1) \widehat{\mathbf{z}}_{\frac{N}{2}}^2 |\widehat{\mathbf{w}}_1^{(1)}|^2 + (|\widehat{\mathbf{z}}_{\frac{N}{2}-1}|^2 - |\widehat{\mathbf{z}}_{\frac{N}{2}-1}|^2) (\widehat{\mathbf{w}}_0^{(1)})^2 = 0. \quad (55)$$

Since $\mathbf{z}$ and $\tilde{\mathbf{z}}$ are generic analytic signals, we get that $\widehat{\mathbf{z}}_{\frac{N}{2}}$, $\widehat{\mathbf{z}}_{\frac{N}{2}-1}$, $\widehat{\mathbf{z}}_{\frac{N}{2}-1}$, $\lambda_{\mathbf{z}, \tilde{\mathbf{z}}}$ are nonzero. From (54), we have $\widehat{\mathbf{z}}_{\frac{N}{2}-1} = \frac{1}{\lambda_{\mathbf{z}, \tilde{\mathbf{z}}}} \widehat{\mathbf{z}}_{\frac{N}{2}-1}$. Combining this with (55), we have that

$$(\lambda_{\mathbf{z}, \tilde{\mathbf{z}}}^2 - 1) \widehat{\mathbf{z}}_{\frac{N}{2}}^2 |\widehat{\mathbf{w}}_1^{(1)}|^2 + (\frac{1}{\lambda_{\mathbf{z}, \tilde{\mathbf{z}}}^2} - 1) |\widehat{\mathbf{z}}_{\frac{N}{2}-1}|^2 (\widehat{\mathbf{w}}_0^{(1)})^2 = 0, \quad (56)$$

which implies that

$$\widehat{\mathbf{z}}_{\frac{N}{2}}^2 = -\frac{(\frac{1}{\lambda_{\mathbf{z}, \tilde{\mathbf{z}}}^2} - 1) |\widehat{\mathbf{z}}_{\frac{N}{2}-1}|^2 (\widehat{\mathbf{w}}_0^{(1)})^2}{(\lambda_{\mathbf{z}, \tilde{\mathbf{z}}}^2 - 1) |\widehat{\mathbf{w}}_1^{(1)}|^2} = \frac{1}{\lambda_{\mathbf{z}, \tilde{\mathbf{z}}}^2} \frac{|\widehat{\mathbf{z}}_{\frac{N}{2}-1}|^2 (\widehat{\mathbf{w}}_0^{(1)})^2}{|\widehat{\mathbf{w}}_1^{(1)}|^2}. \quad (57)$$

It follows from $\lambda_{\mathbf{z}, \hat{\mathbf{z}}} = \frac{\hat{\mathbf{z}}_{\frac{N}{2}}}{\hat{\mathbf{z}}_{\frac{N}{2}}}$ and (57) that

$$\frac{\hat{\mathbf{z}}_{\frac{N}{2}}^2}{|\hat{\mathbf{z}}_{\frac{N}{2}-1}|^2} = \frac{(\hat{\mathbf{w}}_0^{(1)})^2}{|\hat{\mathbf{w}}_1^{(1)}|^2}, \quad (58)$$

which completes the proof. $\square$

Now we are ready to prove our third main result.

Theorem 4 Assume that N is even and the STFT separation parameter L satisfies $\lceil N/L \rceil \geq 3$. Let $m_1, m_2, m_3 \in \{0, 1, \dots, \lceil N/L \rceil - 1\}$ be distinct. If the two windows $\mathbf{w}^{(1)}$ and $\mathbf{w}^{(2)}$ are analytic such that $\hat{\mathbf{w}}_1^{(1)}\hat{\mathbf{w}}_0^{(1)} \neq 0$, $\hat{\mathbf{w}}_{\frac{N}{2}}^{(1)}\hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} \neq 0$ and $\hat{\mathbf{w}}_0^{(1)}\hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - \hat{\mathbf{w}}_0^{(2)}\hat{\mathbf{w}}_{\frac{N}{2}}^{(1)} \neq 0$, then any generic analytic signal $\mathbf{z} \in \mathbb{C}^N$ can be determined (up to a sign) by its $(\frac{3N}{2} - 1)$ number of STFT magnitudes

$$\{|\hat{\mathbf{y}}_{\frac{N}{2},0}^{\mathbf{w}^{(1)}}|, |\hat{\mathbf{y}}_{\frac{N}{2},0}^{\mathbf{w}^{(2)}}|, |\hat{\mathbf{y}}_{k,m_j}^{\mathbf{w}^{(1)}}| : k = 1, \dots, \frac{N}{2} - 1, j = 1, 2, 3\}. \quad (59)$$

Proof Since $\mathbf{z}$, $\mathbf{w}^{(1)}$ and $\mathbf{w}^{(2)}$ are all analytic, it follows from Proposition 2 (i) that the six numbers $\hat{\mathbf{z}}_0$, $\hat{\mathbf{z}}_{\frac{N}{2}}$, $\hat{\mathbf{w}}_0^{(1)}$, $\hat{\mathbf{w}}_0^{(2)}$, $\hat{\mathbf{w}}_{\frac{N}{2}}^{(1)}$ and $\hat{\mathbf{w}}_{\frac{N}{2}}^{(2)}$ are all real-valued.

Step 1: The determination of $(\hat{\mathbf{z}}_0, \hat{\mathbf{z}}_{\frac{N}{2}}, \hat{\mathbf{z}}_{\frac{N}{2}-1})$.

In this step, we prove that $(\hat{\mathbf{z}}_0, \hat{\mathbf{z}}_{\frac{N}{2}}, \hat{\mathbf{z}}_{\frac{N}{2}-1})$ can be determined, up to a sign, by the five measurements $\{|\hat{\mathbf{y}}_{\frac{N}{2},0}^{\mathbf{w}^{(1)}}|, |\hat{\mathbf{y}}_{\frac{N}{2},0}^{\mathbf{w}^{(2)}}|, |\hat{\mathbf{y}}_{\frac{N}{2}-1,m_j}^{\mathbf{w}^{(1)}}| : j = 1, 2, 3\}$. Consider the equation system w.r.t the variable $(\hat{\mathbf{z}}_0, \hat{\mathbf{z}}_{\frac{N}{2}}) \in \mathbb{R}^2$:

$$\begin{cases} |\hat{\mathbf{y}}_{\frac{N}{2},0}^{\mathbf{w}^{(1)}}| = \frac{1}{N}|\hat{\mathbf{z}}_{\frac{N}{2}}\hat{\mathbf{w}}_0^{(1)} + \hat{\mathbf{z}}_0\hat{\mathbf{w}}_{\frac{N}{2}}^{(1)}|, \\ |\hat{\mathbf{y}}_{\frac{N}{2},0}^{\mathbf{w}^{(2)}}| = \frac{1}{N}|\hat{\mathbf{z}}_{\frac{N}{2}}\hat{\mathbf{w}}_0^{(2)} + \hat{\mathbf{z}}_0\hat{\mathbf{w}}_{\frac{N}{2}}^{(2)}|. \end{cases} \quad (60)$$

It follows from $\hat{\mathbf{w}}_0^{(1)}\hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - \hat{\mathbf{w}}_0^{(2)}\hat{\mathbf{w}}_{\frac{N}{2}}^{(1)} \neq 0$ that the solutions (up to a global sign ϵ) to (60) are

$$(\hat{\mathbf{z}}_0, \hat{\mathbf{z}}_{\frac{N}{2}}) = \left(\frac{N(-|\hat{\mathbf{y}}_{\frac{N}{2},0}^{\mathbf{w}^{(1)}}|\hat{\mathbf{w}}_0^{(2)} + |\hat{\mathbf{y}}_{\frac{N}{2},0}^{\mathbf{w}^{(2)}}|\hat{\mathbf{w}}_0^{(1)})}{\hat{\mathbf{w}}_0^{(1)}\hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - \hat{\mathbf{w}}_0^{(2)}\hat{\mathbf{w}}_{\frac{N}{2}}^{(1)}}, \frac{N(|\hat{\mathbf{y}}_{\frac{N}{2},0}^{\mathbf{w}^{(1)}}|\hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - |\hat{\mathbf{y}}_{\frac{N}{2},0}^{\mathbf{w}^{(2)}}|\hat{\mathbf{w}}_{\frac{N}{2}}^{(1)})}{\hat{\mathbf{w}}_0^{(1)}\hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - \hat{\mathbf{w}}_0^{(2)}\hat{\mathbf{w}}_{\frac{N}{2}}^{(1)}} \right) \quad (61)$$

and

$$(\hat{\mathbf{z}}_0, \hat{\mathbf{z}}_{\frac{N}{2}}) = \left(\frac{N(-|\hat{\mathbf{y}}_{\frac{N}{2},0}^{\mathbf{w}^{(1)}}|\hat{\mathbf{w}}_0^{(2)} - |\hat{\mathbf{y}}_{\frac{N}{2},0}^{\mathbf{w}^{(2)}}|\hat{\mathbf{w}}_0^{(1)})}{\hat{\mathbf{w}}_0^{(1)}\hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - \hat{\mathbf{w}}_0^{(2)}\hat{\mathbf{w}}_{\frac{N}{2}}^{(1)}}, \frac{N(|\hat{\mathbf{y}}_{\frac{N}{2},0}^{\mathbf{w}^{(1)}}|\hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} + |\hat{\mathbf{y}}_{\frac{N}{2},0}^{\mathbf{w}^{(2)}}|\hat{\mathbf{w}}_{\frac{N}{2}}^{(1)})}{\hat{\mathbf{w}}_0^{(1)}\hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - \hat{\mathbf{w}}_0^{(2)}\hat{\mathbf{w}}_{\frac{N}{2}}^{(1)}} \right). \quad (62)$$

For any $\hat{\mathbf{z}}_{\frac{N}{2}}$ given through (61) or (62), the following equations w.r.t $\hat{\mathbf{z}}_{\frac{N}{2}-1}$:

$$|\hat{y}_{\frac{N}{2}-1, m_j}^{\mathbf{w}(1)}| = \frac{1}{N} |\hat{\mathbf{z}}_{\frac{N}{2}-1} \hat{\mathbf{w}}_0^{(1)} + \hat{\mathbf{z}}_{\frac{N}{2}} \hat{\mathbf{w}}_1^{(1)} \omega^{m_j}|, j = 1, 2, 3 \quad (63)$$

have a unique solution if and only if the three circles w.r.t the variable $\hat{\mathbf{z}}_{\frac{N}{2}-1}$:

$$\frac{N |\hat{y}_{\frac{N}{2}-1, m_j}^{\mathbf{w}(1)}|}{|\hat{\mathbf{w}}_0^{(1)}|} = |\hat{\mathbf{z}}_{\frac{N}{2}-1} + \frac{\hat{\mathbf{z}}_{\frac{N}{2}} \hat{\mathbf{w}}_1^{(1)} \omega^{m_j}}{\hat{\mathbf{w}}_0^{(1)}}|, j = 1, 2, 3 \quad (64)$$

have only one intersection point. We next prove that for the two choices of $\hat{\mathbf{z}}_{\frac{N}{2}}$ given by (61) and (62):

$$\hat{\mathbf{z}}_{\frac{N}{2}} = \frac{N(|\hat{y}_{\frac{N}{2}, 0}^{\mathbf{w}(1)}| \hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - |\hat{y}_{\frac{N}{2}, 0}^{\mathbf{w}(2)}| \hat{\mathbf{w}}_{\frac{N}{2}}^{(1)})}{\hat{\mathbf{w}}_0^{(1)} \hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - \hat{\mathbf{w}}_0^{(2)} \hat{\mathbf{w}}_{\frac{N}{2}}^{(1)}} \text{ and } \hat{\mathbf{z}}_{\frac{N}{2}} = \frac{N(|\hat{y}_{\frac{N}{2}, 0}^{\mathbf{w}(1)}| \hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} + |\hat{y}_{\frac{N}{2}, 0}^{\mathbf{w}(2)}| \hat{\mathbf{w}}_{\frac{N}{2}}^{(1)})}{\hat{\mathbf{w}}_0^{(1)} \hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - \hat{\mathbf{w}}_0^{(2)} \hat{\mathbf{w}}_{\frac{N}{2}}^{(1)}}, \quad (65)$$

there is only one choice such that the corresponding three circles in (64) have only one intersection point. By Lemma 5, we just need to prove the two aspects: (1) the two numbers in (65) do not have the same absolute values; (2) Lemma 5 (48) does not hold.

If (1) does not hold then

$$|\hat{y}_{\frac{N}{2}, 0}^{\mathbf{w}(1)}| \hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} = 0 \text{ or } |\hat{y}_{\frac{N}{2}, 0}^{\mathbf{w}(2)}| \hat{\mathbf{w}}_{\frac{N}{2}}^{(1)} = 0. \quad (66)$$

For the generic analytic signal $\mathbf{z}$, it follows from (60) that $|\hat{y}_{\frac{N}{2}, 0}^{\mathbf{w}(1)}| \neq 0$ and $|\hat{y}_{\frac{N}{2}, 0}^{\mathbf{w}(2)}| \neq 0$. This combining with $\hat{\mathbf{w}}_{\frac{N}{2}}^{(1)} \hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} \neq 0$ leads to that (66) does not hold. Therefore, (1) hold.

Next we prove (2). Without loss of generality, denote

$$\hat{\mathbf{z}}_{\frac{N}{2}} = \frac{N(|\hat{y}_{\frac{N}{2}, 0}^{\mathbf{w}(1)}| \hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - |\hat{y}_{\frac{N}{2}, 0}^{\mathbf{w}(2)}| \hat{\mathbf{w}}_{\frac{N}{2}}^{(1)})}{\hat{\mathbf{w}}_0^{(1)} \hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - \hat{\mathbf{w}}_0^{(2)} \hat{\mathbf{w}}_{\frac{N}{2}}^{(1)}}, \hat{\mathbf{z}}_{\frac{N}{2}} = \frac{N(|\hat{y}_{\frac{N}{2}, 0}^{\mathbf{w}(1)}| \hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} + |\hat{y}_{\frac{N}{2}, 0}^{\mathbf{w}(2)}| \hat{\mathbf{w}}_{\frac{N}{2}}^{(1)})}{\hat{\mathbf{w}}_0^{(1)} \hat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - \hat{\mathbf{w}}_0^{(2)} \hat{\mathbf{w}}_{\frac{N}{2}}^{(1)}}. \quad (67)$$

It follows from (1) that $|\hat{\mathbf{z}}_{\frac{N}{2}}| \neq |\hat{\mathbf{z}}_{\frac{N}{2}}|$. By (61) and (62), (48) is equivalent to

$$\begin{aligned} & A_1 \hat{\mathbf{z}}_{\frac{N}{2}}^4 + A_2 \hat{\mathbf{z}}_{\frac{N}{2}}^2 \hat{\mathbf{z}}_0^2 + A_3 \hat{\mathbf{z}}_{\frac{N}{2}}^3 \hat{\mathbf{z}}_0 + A_4 \hat{\mathbf{z}}_{\frac{N}{2}} \hat{\mathbf{z}}_0^3 - A_5 \hat{\mathbf{z}}_{\frac{N}{2}}^2 |\hat{\mathbf{z}}_{\frac{N}{2}-1}|^2 \\ & - A_6 \hat{\mathbf{z}}_0^2 |\hat{\mathbf{z}}_{\frac{N}{2}-1}|^2 - A_7 \hat{\mathbf{z}}_{\frac{N}{2}} \hat{\mathbf{z}}_0 |\hat{\mathbf{z}}_{\frac{N}{2}-1}|^2 + C^2 K^4 |\hat{\mathbf{z}}_{\frac{N}{2}-1}|^4 = 0, \end{aligned} \quad (68)$$

where all the coefficients A_i depend only on $\widehat{\mathbf{w}}_0^{(1)}, \widehat{\mathbf{w}}_0^{(2)}, \widehat{\mathbf{w}}_{\frac{N}{2}}^{(1)}$ and $\widehat{\mathbf{w}}_{\frac{N}{2}}^{(2)}$, and $K = \widehat{\mathbf{w}}_0^{(1)} \widehat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - \widehat{\mathbf{w}}_{\frac{N}{2}}^{(1)} \widehat{\mathbf{w}}_0^{(2)}$, $C = \frac{(\widehat{\mathbf{w}}_0^{(1)})^2}{|\widehat{\mathbf{w}}_1^{(1)}|^2}$. Clearly, $K \neq 0$ and $C \neq 0$. Define a polynomial as follows

$$H(x_0, x_1, x_2, x_3) = A_1 x_3^4 + A_2 x_3^2 x_0^2 + A_3 x_3^3 x_0 + A_4 x_3 x_0^3 - A_5 x_3^2 x_1 x_2 - A_6 x_0^2 x_1 x_2 - A_7 x_3 x_0 x_1 x_2 + C^2 K^4 x_1^2 x_2^2. \quad (69)$$

Since $K \neq 0$ and $C \neq 0$, H is a nonzero polynomial. Replacing $\widehat{\mathbf{z}}_k$ by $\sum_{n=0}^{N-1} \mathbf{z}_n e^{-2\pi i k n / N}$, then it follows from (69) that there exists a polynomial $\tilde{H}(\mathbf{z}_0, \dots, \mathbf{z}_{N-1})$ such that $\tilde{H}(\mathbf{z}_0, \dots, \mathbf{z}_{N-1}) = H(\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_{\frac{N}{2}-1}, \overline{\widehat{\mathbf{z}}_{\frac{N}{2}-1}}, \widehat{\mathbf{z}}_{\frac{N}{2}})$. Since H is a nonzero polynomial, $\tilde{H}$ is also a nonzero polynomial. Moreover, as those of H the coefficients of $\tilde{H}$ depend only on $\widehat{\mathbf{w}}_0^{(1)}, \widehat{\mathbf{w}}_0^{(2)}, \widehat{\mathbf{w}}_{\frac{N}{2}}^{(1)}$ and $\widehat{\mathbf{w}}_{\frac{N}{2}}^{(2)}$. Now it follows from (68) that $\tilde{H}(\mathbf{z}_0, \dots, \mathbf{z}_{N-1}) = H(\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_{\frac{N}{2}-1}, \overline{\widehat{\mathbf{z}}_{\frac{N}{2}-1}}, \widehat{\mathbf{z}}_{\frac{N}{2}}) = 0$. But for the generic signal $\mathbf{z}$, we have $\tilde{H}(\mathbf{z}_0, \dots, \mathbf{z}_{N-1}) \neq 0$. This is a contradiction.

Summarizing what addressed above, (1) and (2) hold. Consequently, only one of two choices of $\widehat{\mathbf{z}}_{\frac{N}{2}}$ in (65) is feasible. Combining (61) or (62), $(\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_{\frac{N}{2}})$ can be determined up to a sign. With $\epsilon(\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_{\frac{N}{2}})$ at hand, (63) is equivalent to

$$\frac{N |\widehat{\mathbf{y}}_{\frac{N}{2}-1, m_j}^{\mathbf{w}(1)}|}{|\widehat{\mathbf{w}}_0^{(1)}|} = |\widehat{\mathbf{z}}_{\frac{N}{2}-1} + v_{j, \frac{N}{2}-1}|, j = 1, 2, 3 \quad (70)$$

where $v_{j, \frac{N}{2}-1} = \frac{\epsilon \widehat{\mathbf{z}}_{\frac{N}{2}} \widehat{\mathbf{w}}_1^{(1)} \omega^{m_j}}{\widehat{\mathbf{w}}_0^{(1)}}$. For the generic analytic signal $\mathbf{z} \in \mathbb{C}^N$, we have $\widehat{\mathbf{z}}_{\frac{N}{2}} \neq 0$. Therefore,

$$\begin{aligned} & \frac{v_{1, \frac{N}{2}-1} - v_{2, \frac{N}{2}-1}}{v_{1, \frac{N}{2}-1} - v_{3, \frac{N}{2}-1}} \\ &= \frac{\omega^{m_1} - \omega^{m_2}}{\omega^{m_1} - \omega^{m_3}}. \end{aligned}$$

By Lemma 3, we have

$$\Im \left(\frac{v_{1, \frac{N}{2}-1} - v_{2, \frac{N}{2}-1}}{v_{1, \frac{N}{2}-1} - v_{3, \frac{N}{2}-1}} \right) \neq 0.$$

Now it follows from Lemma 1 that there exists a unique solution to the equation system (70) w.r.t $\widehat{\mathbf{z}}_{\frac{N}{2}-1}$. Clearly, $\epsilon \widehat{\mathbf{z}}_{\frac{N}{2}-1}$ is a solution to (70). Then $\epsilon \widehat{\mathbf{z}}_{\frac{N}{2}-1}$ is the unique solution. Summarizing what has been addressed above, from the five measurements $\{|\widehat{\mathbf{y}}_{\frac{N}{2}, 0}^{\mathbf{w}(1)}|, |\widehat{\mathbf{y}}_{\frac{N}{2}, 0}^{\mathbf{w}(2)}|, |\widehat{\mathbf{y}}_{\frac{N}{2}-1, m_j}^{\mathbf{w}(1)}| : j = 1, 2, 3\}$ the vector $\epsilon(\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_{\frac{N}{2}}, \widehat{\mathbf{z}}_{\frac{N}{2}-1})$ with $\epsilon \in \{1, -1\}$ can be obtained.

Step 2: The determination of other components $\widehat{\mathbf{z}}_1, \dots, \widehat{\mathbf{z}}_{\frac{N}{2}-2}$.

Having $\epsilon(\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_{\frac{N}{2}}, \widehat{\mathbf{z}}_{\frac{N}{2}-1})$ at hand, through the similar procedures in the proof of Theorem 3, other components $\widehat{\mathbf{z}}_1, \dots, \widehat{\mathbf{z}}_{\frac{N}{2}-2}$ can be determined (up to the sign ϵ) by the $(\frac{3N}{2} - 6)$ measurements

$$\left\{ |\widehat{y}_{k,m_j}^{\mathbf{w}(1)}| = \frac{1}{N} \left| \sum_{l=0}^{N-1} \widehat{\mathbf{z}}_{k+l} \widehat{\mathbf{w}}_l^{(1)} \omega^{lm_j} \right| : k = 1, \dots, \frac{N}{2} - 2, j = 1, 2, 3 \right\}. \quad (71)$$

This completes the proof. $\square$

Remark 8 Overall, the two windows $\mathbf{w}^{(1)}$ and $\mathbf{w}^{(2)}$ in Theorem 4 are analytic such that $\widehat{\mathbf{w}}_1^{(1)} \widehat{\mathbf{w}}_0^{(1)} \neq 0$, $\widehat{\mathbf{w}}_{\frac{N}{2}}^{(1)} \widehat{\mathbf{w}}_{\frac{N}{2}}^{(2)} \neq 0$ and $\widehat{\mathbf{w}}_0^{(1)} \widehat{\mathbf{w}}_{\frac{N}{2}}^{(2)} - \widehat{\mathbf{w}}_0^{(2)} \widehat{\mathbf{w}}_{\frac{N}{2}}^{(1)} \neq 0$. Through the similar analysis in Remark 7, any two generic analytic windows satisfy these requirements.

4 Conclusion

This paper concerns the phase retrieval of analytic signals in $\mathbb{C}^N$ by using very few STFT measurements. There are three main contributions. (1) If the windows are B -bandlimited with $2 \leq B \leq \lceil \frac{N}{2} \rceil$, our main results state that a generic analytic signal can be determined up to a sign by $(3\lfloor \frac{N}{2} \rfloor + 1)$ measurements. (2) Suppose that $\mathbf{w}_l \in \mathbb{C}^N$, $l = 1, \dots, 4$ are the structured B -bandlimited windows for STFT where $B = \lceil \frac{N}{2} \rceil + 1$. Then for a generic analytic signal $\mathbf{z} \in \mathbb{C}^N$, it can be recovered (up to a sign) by its $(3\lfloor \frac{N}{2} \rfloor + 1)$ number of STFT measurements. (3) If N is even and the windows are also analytic, then a generic analytic signal can be determined (up to a sign) by its $(\frac{3N}{2} - 1)$ STFT measurements.

Phase derivative is a feature that characterizes the changing rate of phase. It is widely applied in optics such as phase unwrapping (cf. [47]) and the approximation to the effect of a changing pulse frequency (cf. [24]). On the other hand, it follows from Remark 1 (2) that for real-valued signals their phase distributions are defined via their analytic signals and phase derivatives. Therefore, our results have the potential merit for phase unwrapping and the approximation to the effect of a changing pulse frequency.

Funding Youfa Li is partially supported by the Natural Science Foundation of China (No: 61961003) and the Guangxi Science and Technology base and Talent Project (No. AD22080047). Deguang Han is partially supported by the NSF grant DMS-2105038.

Data Availability Data sharing is not applicable to this article, as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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