Journal of Functional Analysis 285 (2023) 110151

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Full Length Article

Determination of compactly supported functions in shift-invariant space by single-angle Radon samples ☆

Youfa Li [a,*], Shengli Fan [b], Deguang Han [c]

[a] *School of Mathematics and Information Science, Center for Applied Mathematics of Guangxi, Guangxi University, Nanning, Guangxi, 530004, PR China*
[b] *CREOL College of Optics & Photonics University of Central Florida, Orlando, FL 32816, United States of America*
[c] *Department of Mathematics, University of Central Florida, Orlando, FL 32816, United States of America*

ARTICLE INFO

Article history:
Received 2 September 2022
Accepted 20 August 2023
Available online 29 August 2023
Communicated by Camil Muscalu

MSC:
primary 42C40, 65T60, 94A20

Keywords:
Single-angle Radon transform
Shift-invariant spaces
Direction vector
Sampling set

ABSTRACT

While traditionally the computerized tomography of a function $f \in L^2(\mathbb{R}^2)$ depends on the samples of its Radon transform at multiple angles, the real-time imaging sometimes requires the reconstruction of f by the samples of its Radon transform $\mathcal{R}_{\boldsymbol{p}}f$ at a single angle θ, where $\boldsymbol{p} = (\cos\theta, \sin\theta)$ is the direction vector. This naturally leads to the question of identifying those functions that can be determined by their Radon samples at a single angle θ. The shift-invariant space $V(\varphi, \mathbb{Z}^2)$ generated by φ is a type of function space that has been widely considered in many fields including wavelet analysis and signal processing. In this paper we examine the single-angle reconstruction problem for compactly supported functions $f \in V(\varphi, \mathbb{Z}^2)$. The central issue for the problem is to identify the eligible $\boldsymbol{p}$ and sampling set $X_{\boldsymbol{p}} \subseteq \mathbb{R}$ such that f can be determined by its single-angle Radon (w.r.t. $\boldsymbol{p}$) samples at $X_{\boldsymbol{p}}$. For the general generator φ, we address the eligible $\boldsymbol{p}$ for the two cases: (1) φ being nonvanishing ($\int_{\mathbb{R}^2}\varphi(\boldsymbol{x})d\boldsymbol{x} \neq 0$) and (2) being vanishing ($\int_{\mathbb{R}^2}\varphi(\boldsymbol{x})d\boldsymbol{x} = 0$). We prove that eligi-

☆ Youfa Li is partially supported by Natural Science Foundation of China (No: 61961003). Deguang Han is partially supported by the NSF grant DMS-2105038.

* Corresponding author.
E-mail addresses: youfalee@hotmail.com (Y. Li), shengli.fan@knights.ucf.edu (S. Fan), Deguang.Han@ucf.edu (D. Han).

https://doi.org/10.1016/j.jfa.2023.110151

ble $X_{\boldsymbol{p}}$ exists for general φ. In particular, $X_{\boldsymbol{p}}$ can be explicitly constructed if $\varphi \in C^1(\mathbb{R}^2)$. Positive definite functions form an important class of functions that have been widely applied in scattered data interpolation. For the case that φ is positive definite, the corresponding single-angle problem in SIS $V(\varphi, \mathbb{Z}^2)$ is addressed such that $X_{\boldsymbol{p}}$ can be constructed easily. Besides using the samples of the single-angle Radon transform, another common feature for our recovery results is that the number of the required samples is minimum.

1. Introduction

1.1. CT and Radon transform

We start with the X-ray computerized tomography (CT) on $\mathbb{R}^2$.

Its core mathematics includes the Radon transform and its inversion. For a function $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ its Radon transform at $t \in \mathbb{R}$, w.r.t. a direction vector $\boldsymbol{p} = (\cos\theta, \sin\theta)$, is defined as the integral of f along the line $(x, y) = t\boldsymbol{p} + s(-\sin\theta, \cos\theta)$ on $\mathbb{R}^2$:

$$\mathcal{R}_{\boldsymbol{p}} f(t) := \int_{-\infty}^{\infty} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta) ds. \tag{1.1}$$

If $f \in L^1(\mathbb{R}^2)$ then we can prove that $\mathcal{R}_{\boldsymbol{p}} f \in L^1(\mathbb{R})$:

$$\begin{aligned} \int_{-\infty}^{\infty} |\mathcal{R}_{\boldsymbol{p}} f(t)| dt &= \int_{-\infty}^{\infty} \Big| \int_{-\infty}^{\infty} f((t,s)A) ds \Big| dt \\ &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f((t,s)A)| ds dt \\ &= \|f\|_{L^1(\mathbb{R}^2)} \end{aligned} \tag{1.2}$$

where $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. If $f \in L^2(\mathbb{R}^2)$ is compactly supported, then by the Cauchy-Schwarz inequality one can check that

$$\mathcal{R}_{\boldsymbol{p}} f \in L^2(\mathbb{R}). \tag{1.3}$$

The Fourier transforms of $\mathcal{R}_{\boldsymbol{p}} f$ and f are correlated via

$$\widehat{\mathcal{R}_{\boldsymbol{p}} f}(\xi) = \widehat{f}(\boldsymbol{p}^T \xi), \quad \xi \in \mathbb{R}, \tag{1.4}$$

where $\widehat{g}(\gamma) := \int_{\mathbb{R}^d} g(\mathbf{x})e^{-\mathrm{i}\mathbf{x}\cdot\gamma}d\mathbf{x}$ is the Fourier transform of any function $g \in L^p(\mathbb{R}^d)$. It follows from (1.4) that $\widehat{\mathcal{R}_{\boldsymbol{p}}f}$ is essentially obtained by taking the cross-section of $\widehat{f}$ on the subspace (slice) $\{\boldsymbol{p}^T\xi : \xi \in \mathbb{R}\}$.

The central problem of CT is to use the Radon transform to reconstruct the source function f. The most classical reconstruction approach is the filtered backprojection (FBP) (cf. [31,32]). It states that if f is bandlimited then it can be reconstructed via (cf. S. Helgason [19]):

$$f(x,y) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{\infty}\widehat{\mathcal{R}_{\boldsymbol{p}}f}(\xi)e^{\mathrm{i}\xi(x\cos\theta+y\sin\theta)}\xi d\xi d\theta, \tag{1.5}$$

where $\boldsymbol{p} = (\cos\theta, \sin\theta)$.

1.2. Traditional reconstruction approaches conducted by Radon transform at multiple angles and our single angle-based problem

Theoretically, the reconstruction of f via (1.5) requires the cross-sections $\widehat{\mathcal{R}_{\boldsymbol{p}}f}(\xi) = \widehat{f}(\boldsymbol{p}^T\xi)$ for all angles $\theta \in [0, 2\pi)$. In practice, however, what one can observe are the samples of a limited number of cross-sections. Therefore, the essential problem of CT is to reconstruct f by the samples of finitely many cross-sections. Based on (1.5), many reconstruction algorithms have been designed (cf. [9,10,24]). Some recent alternatives to FBP have been introduced (e.g. [29,43]). Unlike FBP, they are conducted by the samples of Radon transforms. For example, based on the Chebyshev orthogonal polynomial system, Xu [43] established the approach to CT. McCann and Unser [29] established a spline-based reconstruction.

Note that the samples required for the above approaches are derived from Radon transforms at multiple angles, and naturally we confront the following problem:

Q : Can a function be exactly reconstructed by its Randon (transform) samples at a single angle (SA)? (1.6)

Such a single-angle problem is essentially the injectivity problem of Radon transform (cf. S. Helgason [19]). Due to (1.5), we do not anticipate the injectivity can be achieved for any function in $L^2(\mathbb{R}^2)$. Instead it follows from [19] that it can be achieved in some subspaces of $L^2(\mathbb{R}^2)$. While there are some results on such an injectivity problem (e.g. [7,12,19,21,23]), the related sampling problem in (1.6) remains less explored. In what follows we briefly explain why such a sampling problem is significant from the real-time imaging perspective.

1.3. SACT is required for real-time imaging

Optical imaging has been widely used in observing biological objects, such as blood cells (thin objects) and bones (thick objects). The thin objects are commonly imaged directly by refractive-index distributions, which is achieved by holographic tomography (HT) ([26]). However, for imaging thick objects, CT is usually employed.

CT commonly requires samples (measurements) of the light fields penetrating through the object from different angles (views). To do so, the object needs to be rotated by a rotation motor ([44]) or the illumination needs to be scanned by a beam steering device, which not only causes instability for the imaging system, but makes the system bulky ([3,22]). More importantly, limited by the time of recording fields, rotating objects or scanning illuminations become not suitable for real-time imaging, especially for observing fast dynamic events ([22]). Therefore this naturally leads to the following imaging problem:

$$\text{Under what condition can CT be achieved by the samples of Radon transform at SA?} \tag{1.7}$$

Most recently, R. Horisaki, K. Fujii, and J. Tanida [22] established a SA method for HT by inserting a diffuser. Note that the samples used in [22] are required to contain the diffraction information while the Radon samples for CT commonly do not contain (cf. [30, section 1]). Here the diffraction of light waves at an aperture is computed by the Fresnel integral

$$U(x,y) = \frac{1}{\mathtt{i}\lambda z} \int_{\mathbb{R}^2} U(x_0,y_0) e^{\mathtt{i}\frac{\kappa}{2z}[(x-x_0)^2+(y-y_0)^2]} dx_0 dy_0,$$

where $U(x_0, y_0)$ is the transmission field, $U(x, y)$ is the field on the view plane, z is the distance between the aperture and the view plane, and λ and κ are the wavelength and wave number, respectively. Therefore, the SA method in [22] is not applicable for CT. To the best of our knowledge, the theoretical study of sampling problem (1.7) (or (1.6)) has not been fully explored yet in optics.

1.4. The SACT problem in shift-invariant space (SIS)

The shift-invariant space (SIS) is a type of function space that is widely applied in approximation theory, wavelet analysis and signal processing (e.g. [1,2,4,6,11,18,37,38, 40]). Throughout this paper, the SIS is denoted by

$$V(\varphi, \mathbb{Z}^2) = \Big\{ \sum_{\boldsymbol{k}\in\mathbb{Z}^2} c_{\boldsymbol{k}} \varphi(\cdot - \boldsymbol{k}) : \sum_{\boldsymbol{k}\in\mathbb{Z}^2} |c_{\boldsymbol{k}}|^2 < \infty \Big\}, \tag{1.8}$$

where $\varphi \in L^2(\mathbb{R}^2)$ is referred to as the generator.

Note 1.1. SIS-based multiple-angle CT models. There are many multiple-angle CT approaches (e.g. [8,33–35]) modeling the continuous-domain representations of biomedical images as the functions in SISs. The generators for these SISs are compactly supported functions including box splines ([8]), Kaiser-Bessel window functions ([34]) and refinable functions ([33,35]).

Our purpose in this paper is to examine the SACT problem (1.6) in the SIS setting:

$$\textbf{Q}: \text{How can a compactly supported function } f \in V(\varphi, \mathbb{Z}^2) \text{ be exactly reconstructed by its Randon (transform) samples at a single angle (SA)?} \tag{1.9}$$

1.5. Assumption on the support of source function, and definition of positive definite function

Before introducing our main contributions, some denotations are necessary. Throughout this paper, suppose that the generator $\varphi \in L^2(\mathbb{R}^2)$ is compactly supported such that

$$\operatorname{supp}(\varphi) \subseteq [N_1, M_1] \times [N_2, M_2], \tag{1.10}$$

and the shift system $\{\varphi(\cdot - \boldsymbol{k}) : \boldsymbol{k} \in \mathbb{Z}^2\}$ is linearly independent in $L^2(\mathbb{R}^2)$. Moreover, the arbitrary source function $f \in V(\varphi, \mathbb{Z}^2)$ is compactly supported such that

$$\operatorname{supp}(f) \subseteq [a_1, b_1] \times [a_2, b_2]. \tag{1.11}$$

By (1.10) and (1.11), there exists a finite sequence $\{c_{\boldsymbol{k}_l}, l = 1, \ldots, \#E\} \subseteq \mathbb{C}$ such that f can be expressed as

$$f = \sum_{l=1}^{\#E} c_{\boldsymbol{k}_l} \varphi(\cdot - \boldsymbol{k}_l), \tag{1.12}$$

where $E := \{\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{\#E}\} = \big\{\big[\lceil a_1 - M_1\rceil, \lfloor b_1 - N_1\rfloor\big] \times \big[\lceil a_2 - M_2\rceil, \lfloor b_2 - N_2\rfloor\big]\big\} \cap \mathbb{Z}^2$, $\#E$ is the cardinality of E, and $\lceil x\rceil$ ($\lfloor x\rfloor$) is the smallest (largest) integer that is not smaller (larger) than $x \in \mathbb{R}$, respectively. In what follows we explain that the assumption in (1.11) is reasonable.

Remark 1.1. Throughout this paper, as in (1.11) we assume that the function to be reconstructed is compactly supported and its support is contained in a known rectangle. Such an assumption is reasonable for CT (e.g. [43]) since from the optical perspective, the function to be reconstructed in CT is the difference between the refractive index distribution of the object and that of the surrounding medium (cf. [30]), and consequently it is generally compactly supported. Moreover, the support of the function is known when the boundary of the object is clear.

In what follows we recall the definition of positive definite functions which have been extensively applied to scattered data interpolation, approximation theory and harmonic analysis (e.g. [14,20,25,42]).

Definition 1.1. We say that a function $\phi : \mathbb{R}^d \longrightarrow \mathbb{C}$ is positive semi-definite if for all $N \in \mathbb{N}$, all sets $X = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\} \subseteq \mathbb{R}^d$, and all vectors $\mathbf{0} \neq (\alpha_1, \dots, \alpha_N)^T \in \mathbb{C}^N$, the quadratic form

$$\begin{aligned}&\sum_{j=1}^N \sum_{k=1}^N \alpha_j \overline{\alpha}_k \phi(\mathbf{x}_j - \mathbf{x}_k)\\ &= (\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_N) \begin{pmatrix} \phi(\mathbf{0}) & \phi(\mathbf{x}_1 - \mathbf{x}_2) & \cdots & \phi(\mathbf{x}_1 - \mathbf{x}_N) \\ \phi(\mathbf{x}_2 - \mathbf{x}_1) & \phi(\mathbf{0}) & \cdots & \phi(\mathbf{x}_2 - \mathbf{x}_N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\mathbf{x}_N - \mathbf{x}_1) & \phi(\mathbf{x}_N - \mathbf{x}_2) & \cdots & \phi(\mathbf{0}) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}\\ &\geq 0.\end{aligned} \tag{1.13}$$

Furthermore, the function ϕ is positive definite if the above quadratic form is positive for all $\mathbf{0} \neq (\alpha_1, \dots, \alpha_N)^T$. We will recall more properties of positive definite functions in subsection 5.3.

1.6. Main contributions and their common features

Our central task is summarized as follows.

> We prove the existences of the vectors $\boldsymbol{p}$ and the corresponding sampling set $X_{\boldsymbol{p}}$ such that the compactly supported source function $f \in V(\varphi, \mathbb{Z}^2)$ can be determined by the samples of $\mathcal{R}_{\boldsymbol{p}} f$ at $X_{\boldsymbol{p}}$. Moreover, the designing problem of $\boldsymbol{p}$ and $X_{\boldsymbol{p}}$ is also addressed. These problems are investigated from the perspective: (1) The nonvanishing case: $\widehat{\varphi}(\mathbf{0}) = \int_{\mathbb{R}^2} \varphi(\mathbf{x}) d\mathbf{x} \neq 0$; (2) The vanishing case: $\widehat{\varphi}(\mathbf{0}) = \int_{\mathbb{R}^2} \varphi(\mathbf{x}) d\mathbf{x} = 0$. In particular, we address the case that φ is positive definite such that $\boldsymbol{p}$ and $X_{\boldsymbol{p}}$ can be constructed easily.

There are five main results in this paper. They will be established in subsections 4.3, 4.4, 5.2, 5.4 and 5.5. From the perspective of the properties satisfied by the generator φ, these main results are organized briefly as follows.

• **The nonvanishing case** ($\widehat{\varphi}(\mathbf{0}) = \int_{\mathbb{R}^2} \varphi(\mathbf{x}) d\mathbf{x} \neq 0$). The set Λ of eligible direction vectors is constructed for the SACT of any $f \in V(\varphi, \mathbb{Z}^2)$ satisfying (1.11). It is proved that for any $\boldsymbol{p} \in \Lambda$, there exists a sampling set $X_{\boldsymbol{p}} \subseteq \mathbb{R}$ (having the cardinality $\# E$) such that f can be determined uniquely by its SA Radon samples at $X_{\boldsymbol{p}}$, where the set E is correlated with f via (1.12). Additionally, if $\varphi \in C^1(\mathbb{R}^2)$ then $X_{\boldsymbol{p}}$ is constructed explicitly.

• **The vanishing case** ($\widehat{\varphi}(\mathbf{0}) = \int_{\mathbb{R}^2} \varphi(\boldsymbol{x})d\boldsymbol{x} = 0$). As in the nonvanishing case, the set Ω of eligible direction vectors is constructed for the SACT of any $f \in V(\varphi, \mathbb{Z}^2)$ satisfying (1.11). The set Ω is different from the above Λ in the nonvanishing case. For any $\boldsymbol{p} \in \Omega$, the existence of the eligible sampling set $X_{\boldsymbol{p}} \subseteq \mathbb{R}$ (also having the cardinality $\#E$) is proved such that f can be determined uniquely by its SA Radon samples at $X_{\boldsymbol{p}}$. Additionally, for the case that $\varphi \in C^1(\mathbb{R}^2)$ the sampling set $X_{\boldsymbol{p}}$ is constructed explicitly.

• **The positive definite generator case**. Suppose that φ is positive definite. Eligible direction vector sets are constructed for the nonvanishing and vanishing cases, respectively. For any eligible direction vector $\boldsymbol{p}$, the source function $f \in V(\varphi, \mathbb{Z}^2)$ satisfying (1.11) can be determined uniquely by its SA samples at $\{\boldsymbol{p}\boldsymbol{k}_1, \ldots, \boldsymbol{p}\boldsymbol{k}_{\#E}\}$, where $\{\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{\#E}\} = E$ is correlated with f via (1.12).

Remark 1.2. There are two common features of the above three main contributions. (1) The samples for CT are derived from the SA Radon transform but not from multiple-angle Radon transforms. (2) Note that $\#E$ Radon samples are used to determined f. Recall again that $\{\varphi(\cdot - \boldsymbol{k}) : \boldsymbol{k} \in \mathbb{Z}^2\}$ is linearly independent, and by (1.12), $f = \sum_{l=1}^{\#E} c_{\boldsymbol{k}_l}\varphi(\cdot - \boldsymbol{k}_l)$. Then f is determined uniquely by the $\#E$ coefficients: $c_{\boldsymbol{k}_1}, \ldots, c_{\boldsymbol{k}_{\#E}}$. Therefore we only use the **minimum** number of samples in our SA-based reconstruction.

1.7. Outline of the paper

In Theorem 3.1 a sufficient and necessary condition is established on the pair $(\varphi, \boldsymbol{p})$ such that, an arbitrary compactly supported source function $f \in V(\varphi, \mathbb{Z}^2)$ satisfying (1.11) can be determined uniquely by its SA Radon transform $\mathcal{R}_{\boldsymbol{p}}f$. With the help of Paley-Wiener theorem, it will be explained in subsection 3.2 that such a determination problem is absolutely nontrivial. Based on Theorem 3.1 we will address the SACT sampling problem (1.9) in section 4 and section 5.

Section 4 concerns on the problem (1.9) for compactly supported functions in $V(\varphi, \mathbb{Z}^2)$ where φ is a general generator. Theorem 4.1 establishes a sufficient and necessary condition on $(\varphi, \boldsymbol{p}, X_{\boldsymbol{p}})$ such that the SACT sampling (1.9) can be achieved by the SA Radon samples at $X_{\boldsymbol{p}}$. For the general generator φ case, a natural problem is the existence of $\boldsymbol{p}$ and $X_{\boldsymbol{p}}$. The answer to this problem will be addressed in Theorem 4.3 for the nonvanishing ($\widehat{\varphi}(\mathbf{0}) \neq 0$) case and in Theorem 4.5 for the vanishing ($\widehat{\varphi}(\mathbf{0}) = 0$) case, where a set of eligible direction vectors Λ (respectively, Ω) is provided in Theorem 4.3 (respectively, Theorem 4.5) such that for any $\boldsymbol{p} \in \Lambda$ (or $\boldsymbol{p} \in \Omega$) there exists a sampling set $X_{\boldsymbol{p}}$, and consequently f can be determined uniquely by its SA Radon samples at $X_{\boldsymbol{p}}$. In particular, an explicit construction of a sampling set $X_{\boldsymbol{p}}$ was presented in Theorem 4.4 and Proposition 4.6 for the case when $\varphi \in C^1(\mathbb{R}^2)$.

The purpose of Section 5 is to address the condition on $(\varphi, \boldsymbol{p})$ such that the compactly supported $f \in V(\varphi, \mathbb{Z}^2)$ satisfying (1.11) can be determined uniquely by its SA Radon samples at $\{\boldsymbol{p}\boldsymbol{k}_1, \ldots, \boldsymbol{p}\boldsymbol{k}_{\#E}\}$, where $\{\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{\#E}\} = E$. Such a condition is established in Theorem 5.1. Based on Theorem 5.1, we address the case that φ is positive definite in Theorems 5.4 and 5.6. In particular, Theorem 5.4 applies to the nonvanishing case while Theorem 5.6 applies to the vanishing case. A numerical example is provided in Example 5.1 to check the recovery result.

2. Preliminary

2.1. On the support of $\mathcal{R}_{\mathbf{p}}f$

For a function $f \in L^1(\mathbb{R}^2)$ and a direction vector $\boldsymbol{p} = (\cos\theta, \sin\theta)$, motivated by [16,18] we next address the relationship between $\mathcal{R}_{\boldsymbol{p}}f$ and f in the spatial domain. Denote the singular value decomposition (SVD) of $\boldsymbol{p}$ by $\boldsymbol{p} = \Sigma V^T$, where V is a 2×2 real-valued unitary matrix and $\Sigma = (1, 0)$. Now it follows from [16,18] that

$$\mathcal{R}_{\boldsymbol{p}}f = \Sigma(V^T f), \tag{2.1}$$

where $V^T f(\mathbf{x}) = f((V^T)^{-1}\mathbf{x})$ with $\mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2$, and for any g on $\mathbb{R}^2$ the function Σg on $\mathbb{R}$ is defined by

$$\Sigma g(x_1) = \int_{\mathbb{R}} g(x_1, x_2) dx_2. \tag{2.2}$$

The following remark is derived from [16, section 1].

Remark 2.1. If $f \in L^2(\mathbb{R}^2)$ is compactly supported then its Radon transform $\mathcal{R}_{\boldsymbol{p}}f$ can be expressed as $\Sigma(V^T f)$ in (2.1).

It has been stated in (1.3) that if $f \in L^2(\mathbb{R}^2)$ is compactly supported then $\mathcal{R}_{\boldsymbol{p}}f \in L^2(\mathbb{R})$. We include its proof together with support information in the following lemma.

Lemma 2.1. *Suppose that $f \in L^2(\mathbb{R}^2)$ with $supp(f) \subseteq [a_1, b_1] \times [a_2, b_2]$. Then*

$$supp(\mathcal{R}_{\mathbf{p}}f) \subseteq [-\sqrt{2}\max\{|b_i|, |a_i| : i = 1, 2\}, \sqrt{2}\max\{|b_i|, |a_i| : i = 1, 2\}], \tag{2.3}$$

and $\mathcal{R}_{\mathbf{p}}f \in L^2(\mathbb{R})$.

Proof. Let V be the real unitary matrix from the SVD of $\boldsymbol{p}$ such that $\boldsymbol{p} = \Sigma V^T$ with $\Sigma = (1, 0)$. Denote $V^T f$ by g. Then for any $x_1 \in \mathbb{R}$, we have

$$\begin{aligned}\mathcal{R}_{\boldsymbol{p}}f(x_1) &= \Sigma g(x_1)\\ &= \int_{\mathbb{R}} g(x_1,x_2)dx_2\\ &= \int_{\mathbb{R}} [V^T f](x_1,x_2)dx_2\\ &= \int_{\mathbb{R}} f(V(x_1,x_2)^T)dx_2, \ (2.4\text{A})\end{aligned} \tag{2.4}$$

where the first and second equalities are derived from Remark 2.1 and (2.2), respectively. It follows from $\text{supp}(f) \subseteq [a_1,b_1]\times[a_2,b_2]$ that for any $\boldsymbol{x} \in \text{supp}(f)$, we have

$$\|\boldsymbol{x}\|_2 \le \sqrt{2}\max\{|b_i|,|a_i| : i=1,2\}. \tag{2.5}$$

It follows from (2.4) and the fact that V is a unitary matrix, we have $|x_1| \le \sqrt{2}\max\{|b_i|,|a_i| : i=1,2\}$. Then (2.3) holds.

Define $G(x_1,x_2) := f(V(x_1,x_2)^T)$. By (2.5) and V being a real unitary matrix, we have

$$|x_2| \le \sqrt{2}\max\{|b_i|,|a_i| : i=1,2\} \tag{2.6}$$

for any $(x_1,x_2)^T \in \text{supp}(G)$. Moreover,

$$\begin{aligned}\|\mathcal{R}_{\boldsymbol{p}}f\|^2_{L^2(\mathbb{R})} &= \|\Sigma(V^T f)\|^2_{L^2(\mathbb{R})}\\ &= \int_{\mathbb{R}} \Big|\int_{\mathbb{R}} [V^T f](x_1,x_2)dx_2\Big|^2 dx_1\\ &\le \sqrt{2}\max\{|b_i|,|a_i| : i=1,2\}\int_{\mathbb{R}}\int_{\mathbb{R}} |[V^T f](x_1,x_2)|^2 dx_2 dx_1\\ &= \sqrt{2}\max\{|b_i|,|a_i| : i=1,2\}\|f\|^2_{L^2(\mathbb{R}^2)}\\ &< \infty,\end{aligned} \tag{2.7}$$

where the first inequality is derived from (2.6) and the Cauchy-Schwarz inequality. This completes the proof. □

2.2. (Quasi) shift-invariant space

For a generator $\varphi \in L^2(\mathbb{R}^2)$, as in (1.8) its associated shift-invariant space (SIS) $V(\varphi,\mathbb{Z}^2)$ is defined to be

$$V(\varphi,\mathbb{Z}^2) := \Big\{\sum_{\boldsymbol{k}\in\mathbb{Z}^2} c_{\boldsymbol{k}}\varphi(\cdot-\boldsymbol{k}) : \{c_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2)\Big\}, \tag{2.8}$$

where $\ell^2(\mathbb{Z}^2)$ is the space of square summable sequences such that any $\{c_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2)$ satisfies $\|\{c_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2}\|_{\ell^2(\mathbb{Z}^2)} = (\sum_{\boldsymbol{k}\in\mathbb{Z}^2}|c_{\boldsymbol{k}}|^2)^{1/2} < \infty$. As mentioned in section 1, throughout the paper the system $\{\varphi(\cdot - \boldsymbol{k}) : \boldsymbol{k} \in \mathbb{Z}^2\}$ is required to be linearly independent in $L^2(\mathbb{R}^2)$. A sufficient condition for the linear independence is that $\{\varphi(\cdot - \boldsymbol{k}) : \boldsymbol{k} \in \mathbb{Z}^2\}$ satisfies the so called Riesz basis condition, namely, there exist constants $0 < C_1 \leq C_2 < \infty$ such that for any $\{c_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2)$ there holds

$$C_1\|\{c_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2}\|^2_{\ell^2(\mathbb{Z}^2)} \leq \Big\| \sum_{\boldsymbol{k}\in\mathbb{Z}^2} c_{\boldsymbol{k}}\varphi(\cdot - \boldsymbol{k})\Big\|^2_{L^2(\mathbb{R}^2)} \leq C_2\|\{c_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2}\|^2_{\ell^2(\mathbb{Z}^2)}. \tag{2.9}$$

For a generator $\varphi \in L^2(\mathbb{R}^2)$ and the shift set $\mathcal{X} = \{x_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2} \subseteq \mathbb{R}^2$, its associated quasi shift-invariant space (QSIS) is defined as

$$V(\varphi, \mathcal{X}) := \Big\{ \sum_{\boldsymbol{k}\in\mathbb{Z}^2} c_{\boldsymbol{k}}\varphi(\cdot - x_{\boldsymbol{k}}) : \{c_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2)\Big\}. \tag{2.10}$$

If $\mathcal{X} = \mathbb{Z}^2$ then $V(\varphi, \mathcal{X})$ degenerates to a SIS. As implied in [14], the recovery for the functions in $V(\varphi, \mathcal{X})$ ($\mathcal{X} \neq \mathbb{Z}^2$) is much more complicated than that for the SIS. For such a recovery, by [14, section 3.1(A1)] it is required that φ is positive definite.

2.3. Sobolev smoothness of a function

For any $\varsigma \in \mathbb{R}$, the Sobolev space $H^\varsigma(\mathbb{R}^d)$ (cf. [15,27,28]) is defined as

$$H^\varsigma(\mathbb{R}^d) := \Big\{ f : \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2(1 + \|\xi\|_2^2)^\varsigma d\xi < \infty \Big\}. \tag{2.11}$$

Clearly, if $\varsigma \geq 0$ then $H^\varsigma(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$. The deduced norm is defined by

$$\|f\|_{H^\varsigma(\mathbb{R}^d)} := \frac{1}{(2\pi)^{d/2}} \Big(\int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2(1 + \|\xi\|_2^2)^\varsigma d\xi \Big)^{1/2}, \quad \forall f \in H^\varsigma(\mathbb{R}^d).$$

The Sobolev smoothness of f is defined as $\nu_2(f) := \sup\{\varsigma : f \in H^\varsigma(\mathbb{R}^d)\}$. The following lemma is derived from [17, Lemma 2.4]. It states that for a compactly supported $f \in L^2(\mathbb{R}^2)$, the Sobolev smoothness of $\mathcal{R}_{\mathbf{p}}f$ is not smaller than $\nu_2(f)$.

Lemma 2.2. *Suppose that $f \in H^\varsigma(\mathbb{R}^2), \varsigma \geq 0$ is compactly supported. Then $\nu_2(\mathcal{R}_{\mathbf{p}}f) \geq \nu_2(f)$ for any direction vector* $\mathbf{p}$.

With the help of Lemma 2.2 we next address the continuity of the Radon transform.

Proposition 2.3. *Suppose that $f \in H^\varsigma(\mathbb{R}^d)$ such that $\varsigma > d/2$. Then we have*

(1) f is continuous.

Suppose that $g \in H^s(\mathbb{R}^2)$ with $s > 1/2$ is compactly supported. Then we have

(2) the Radon transform $\mathcal{R}_{\mathbf{p}}g$ is continuous for any direction vector $\mathbf{p}$.

Proof. The first part of the proposition is the standard result on Sobolev space (cf. [27, section 1.1], [37, Chapter 9.1]). For any compactly supported $g \in H^s(\mathbb{R}^2)$ with $s > 1/2$, by Lemma 2.2 the Sobolev smoothness $\nu_2(\mathcal{R}_{\boldsymbol{p}} g) \ge \nu_2(g) > 1/2$. By the first part of the present proposition, $\mathcal{R}_{\boldsymbol{p}} g$ is continuous. The proof is concluded. □

Remark 2.2. For $f \in H^{\varsigma}(\mathbb{R}^d)$ with $\varsigma > d/2$, by Proposition 2.3 (1) we have $f \in C(\mathbb{R}^d)$. But it does not necessarily imply that $f \in C^1(\mathbb{R}^d)$. For example, define $f(x_1, x_2) = [\chi_{(0,1]} \star \chi_{(0,1]}](x_1)[\chi_{(0,1]} \star \chi_{(0,1]}](x_2)$, where $\star$ is the convolution and $\chi_{(0,1]}$ is the characteristic function on the interval $(0, 1]$. By direct calculation we have

$$[\chi_{(0,1]} \star \chi_{(0,1]}](x_j) = \begin{cases} x_j, & 0 < x_j \le 1, \\ 2 - x_j, & 1 < x_j \le 2, \\ 0, & \text{else.} \end{cases} \tag{2.12}$$

On the other hand, one can check that

$$\widehat{\chi_{(0,1]}}(\xi_j) = e^{-\mathrm{i}\xi_j/2} \frac{\sin \xi_j/2}{\xi_j/2}. \tag{2.13}$$

Therefore,

$$\widehat{f}(\xi_1, \xi_2) = e^{-\mathrm{i}\xi_1}[\frac{\sin \xi_1/2}{\xi_1/2}]^2 e^{-\mathrm{i}\xi_2}[\frac{\sin \xi_2/2}{\xi_2/2}]^2.$$

From this, one can check that $f \in H^{\varsigma}(\mathbb{R}^2)$ for any $\varsigma < 3/2$. But it is clear from (2.12) that $f \notin C^1(\mathbb{R}^2)$.

Remark 2.3. The purpose here is to state that there exist functions which are discontinuous but their Radon transforms are continuous. For example, define $f(x_1, x_2) = \chi_{(0,1]}(x_1)\chi_{(0,1]}(x_2)$. It is clear that f is discontinuous. It follows from (2.13) that $\widehat{f}(\xi_1, \xi_2) = e^{-\mathrm{i}\xi_1/2} \frac{\sin \xi_1/2}{\xi_1/2} e^{-\mathrm{i}\xi_2/2} \frac{\sin \xi_2/2}{\xi_2/2}$. Now for any $\boldsymbol{p} = (\cos\theta, \sin\theta)$ such that $\cos\theta \sin\theta \ne 0$, we have

$$\widehat{\mathcal{R}_{\boldsymbol{p}} f}(\xi) = \widehat{f}(\xi\cos\theta, \xi\sin\theta) = e^{-\mathrm{i}\frac{\xi\cos\theta}{2}} \frac{\sin\frac{\xi\cos\theta}{2}}{\frac{\xi\cos\theta}{2}} e^{-\mathrm{i}\frac{\xi\sin\theta}{2}} \frac{\sin\frac{\xi\sin\theta}{2}}{\frac{\xi\sin\theta}{2}}.$$

For $|\xi| > 1$, $|\widehat{\mathcal{R}_{\boldsymbol{p}} f}(\xi)| \le \frac{|\frac{2}{\cos\theta}||\frac{2}{\sin\theta}|}{\xi^2}$. From this and the continuity of $\widehat{\mathcal{R}_{\boldsymbol{p}} f}$, one can prove that the Sobolev smoothness $\nu_2(\mathcal{R}_{\boldsymbol{p}} f) > 1/2$. By Proposition 2.3 (1), $\mathcal{R}_{\boldsymbol{p}} f$ is continuous.

3. A necessary and sufficient condition for the SA Radon transform-based determination

The following establishes a necessary and sufficient condition on the pair $(\varphi, \boldsymbol{p})$ such that any compactly supported function $f \in V(\varphi, \mathbb{Z}^2)$ can be determined by its SA Radon transform $\mathcal{R}_{\boldsymbol{p}}f$. Although such a determination depends on $\mathcal{R}_{\boldsymbol{p}}f$ and does not use its samples directly, it will be helpful for answering the SACT sampling problem (1.9). As previously, the vectors in $\mathbb{R}^d$ are considered as column vectors, while the direction vector $\boldsymbol{p}$ is a row vector.

3.1. Determination result

The following is the main result of the present section.

Theorem 3.1. *Suppose that $\varphi \in L^2(\mathbb{R}^2)$ such that supp$(\varphi) \subseteq [N_1, M_1] \times [N_2, M_2]$. Moreover, $\{\varphi(\cdot - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^2\}$ is linearly independent in $L^2(\mathbb{R}^2)$. Then any $f \in V(\varphi, \mathbb{Z}^2)$ such that supp$(f) \subseteq [a_1, b_1] \times [a_2, b_2]$ can be determined uniquely by its SA Radon transform $\mathcal{R}_{\mathbf{p}}f$ if and only if $\{\mathcal{R}_{\mathbf{p}}\varphi(\cdot - \mathbf{pk}) : \mathbf{k} \in E\}$ is linearly independent in $L^2(\mathbb{R})$, where $E = \big\{\big[\lceil a_1 - M_1\rceil, \lfloor b_1 - N_1\rfloor\big] \times \big[\lceil a_2 - M_2\rceil, \lfloor b_2 - N_2\rfloor\big]\big\} \cap \mathbb{Z}^2$.*

Proof. We first prove that $\mathcal{R}_{\boldsymbol{p}}(\varphi(\cdot - \boldsymbol{k})) = \mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{pk})$ for any $\boldsymbol{k} \in \mathbb{Z}^2$. Actually, the Fourier transform of $\varphi(\cdot - \boldsymbol{k})$ at $\boldsymbol{x} \in \mathbb{R}^2$ is $e^{-\mathbf{i}\boldsymbol{k}\cdot\boldsymbol{x}}\widehat{\varphi}(\boldsymbol{x})$. Then by the Radon transform representation (1.4) in the Fourier domain, the Fourier transform of the Radon transform $\mathcal{R}_{\boldsymbol{p}}(\varphi(\cdot - \boldsymbol{k}))$ at $\xi \in \mathbb{R}$ is $e^{-\mathbf{i}\boldsymbol{k}\cdot\boldsymbol{p}^T\xi}\widehat{\varphi}(\boldsymbol{p}^T\xi)$. Clearly,

$$e^{-\mathbf{i}\boldsymbol{k}\cdot\boldsymbol{p}^T\xi}\widehat{\varphi}(\boldsymbol{p}^T\xi) = e^{-\mathbf{i}\boldsymbol{pk}\xi}\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}(\xi). \tag{3.1}$$

Stated another way,

$$\mathcal{R}_{\boldsymbol{p}}(\varphi(\cdot - \boldsymbol{k})) = \mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{pk}). \tag{3.2}$$

Since $\varphi \in L^2(\mathbb{R}^2)$ is compactly supported, then it follows from Lemma 2.1 that $\mathcal{R}_{\boldsymbol{p}}(\varphi) \in L^2(\mathbb{R})$. Consequently, $\mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{pk}) \in L^2(\mathbb{R})$ for any $\boldsymbol{k} \in E$.

For convenient narration, denote E by $\{\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{\#E}\}$. It follows from $\{\varphi(\cdot - \boldsymbol{k}) : \boldsymbol{k} \in \mathbb{Z}^2\}$ being linearly independent, supp$(\varphi) \subseteq [N_1, M_1] \times [N_2, M_2]$ and supp$(f) \subseteq [a_1, b_1] \times [a_2, b_2]$ that there exists uniquely a finite sequence $\{c_{\boldsymbol{k}_1}, \ldots, c_{\boldsymbol{k}_{\#E}}\} \subseteq \mathbb{C}$ such that

$$f = \sum_{l=1}^{\#E} c_{\boldsymbol{k}_l}\varphi(\cdot - \boldsymbol{k}_l). \tag{3.3}$$

Now by (3.3) and (3.2), we have

$$\mathcal{R}_{\boldsymbol{p}}f = \sum_{l=1}^{\#E} c_{\boldsymbol{k}_l}\mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{pk}_l). \tag{3.4}$$

($\Longleftarrow$): Since $\{\mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{pk}_l) : l = 1, \ldots, \#E\}$ is linearly independent in $L^2(\mathbb{R})$, then $\{c_{\boldsymbol{k}_l} : l = 1, \ldots, \#E\}$ can be determined uniquely by $\mathcal{R}_{\boldsymbol{p}}f$. Note that $\{\varphi(\cdot - \boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^2}$

is linearly independent. Then with the sequence $\{c_{\boldsymbol{k}_l} : l = 1, \dots, \#E\}$ at hand, $f = \sum_{l=1}^{\#E} c_{\boldsymbol{k}_l} \varphi(\cdot - \boldsymbol{k}_l)$ can be determined uniquely.

($\Longrightarrow$): If $\{\mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_l) : l = 1, \dots, \#E\}$ is linearly dependent, then there exists a nonzero sequence $\{\widehat{c}_{\boldsymbol{k}_l} : l = 1, \dots, \#E\}$ such that $\| \sum_{l=1}^{\#E} \widehat{c}_{\boldsymbol{k}_l} \mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_l)\|_{L^2(\mathbb{R})} = 0$. Recall that $\{\varphi(\cdot - \boldsymbol{k}) : \boldsymbol{k} \in \mathbb{Z}^2\}$ is linearly independent. Then $\tilde{f} := \sum_{l=1}^{\#E} \widehat{c}_{\boldsymbol{k}_l} \varphi(\cdot - \boldsymbol{k}_l) \not\equiv 0$ but $\mathcal{R}_{\boldsymbol{p}}\tilde{f} \equiv 0$. Now $\tilde{f}$ is not distinguishable from $g \equiv 0 \in V(\varphi, \mathbb{Z}^2)$ since their Radon transforms (w.r.t. the direction vector $\boldsymbol{p}$) are both zero. This leads to a contradiction. □

Remark 3.1. (1) The sampling problem is not considered in Theorem 3.1. Therefore $\mathcal{R}_{\boldsymbol{p}}\varphi$ is not required to be continuous therein. (2) If the set $\{\boldsymbol{p}\boldsymbol{k} : \boldsymbol{k} \in E\}$ is not contained in $\mathbb{Z}$, then it follows from (3.4) that $\mathcal{R}_{\boldsymbol{p}}f$ sits in the quasi-SIS (QSIS) generated by $\mathcal{R}_{\boldsymbol{p}}\varphi$. As addressed in section 2.2, the recovery problem in QSIS is absolutely not the trivial generalization of that in SIS.

The following subsection states that the SA Radon-based determination problem in Theorem 3.1 is absolutely not trivial.

3.2. A nontrivial problem: what pair $(\varphi, \mathbf{p})$ ensures the system $\{\mathcal{R}_{\mathbf{p}}\varphi(\cdot - \mathbf{p}\mathbf{k}) : \mathbf{k} \in E\}$ being linearly independent

Note that in Theorem 3.1 the system $\{\mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}) : \boldsymbol{k} \in E\}$ is required to be linearly independent in $L^2(\mathbb{R})$. Our purpose of this subsection is to explain that such a requirement is absolutely not trivial. The following lemma is necessary for our discussion. It is derived from [42, Lemma 6.7].

Lemma 3.2. *Suppose that $\mathbf{x}_k \in \mathbb{R}^d, k = 1, \dots, N$ are pairwise distinct. Then the set $\{e^{i\mathbf{x}_k \cdot \xi}\}_{k=1}^N$ is linearly independent on any interval $I \subseteq \mathbb{R}^d$, namely, for any vector $(\alpha_1, \dots, \alpha_N) \in \mathbb{C}^N$ if $\sum_{k=1}^N \alpha_k e^{i\mathbf{x}_k \cdot \xi} \equiv 0$ then $(\alpha_1, \dots, \alpha_N) = \boldsymbol{0}$.*

In what follows, we establish the equivalent characterizations for the linear independence of $\{\mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}) : \boldsymbol{k} \in E\}$.

Proposition 3.3. *Let the compactly supported φ and $E = \{\mathbf{k}_1, \dots, \mathbf{k}_{\#E}\} \subseteq \mathbb{Z}^2$ be as in Theorem 3.1. Then the following statements are equivalent:*

(1) The system $\{\mathcal{R}_{\mathbf{p}}\varphi(\cdot - \mathbf{p}\mathbf{k}_j) : j = 1, \dots, \#E\}$ is linearly independent in $L^2(\mathbb{R})$.

(2) For any vector $\boldsymbol{0} \neq (c_1, \dots, c_{\#E})^T \in \mathbb{C}^{\#E}$ it holds that

$$\int_{\mathbb{R}} |\sum_{j=1}^{\#E} c_j e^{-i\mathbf{p}\mathbf{k}_j \xi}|^2 |\widehat{\varphi}(\mathbf{p}^T\xi)|^2 d\xi > 0. \tag{3.5}$$

(3) $\widehat{\varphi}(\mathbf{p}^T \cdot) \not\equiv 0$, and if $\#E > 1$ then for any $j \neq n \in \{1, \dots, \#E\}$ we have $\mathbf{p}\mathbf{k}_j \neq \mathbf{p}\mathbf{k}_n$.

Proof. By (3.1) we can check that the Fourier transform of $\sum_{j=1}^{\#E} c_j \mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{pk}_j)$ is $\sum_{j=1}^{\#E} c_j e^{-\mathrm{i}\boldsymbol{pk}_j\xi}\widehat{\varphi}(\boldsymbol{p}^T\xi)$. From this we have (1) $\Longleftrightarrow$ (2). If $\widehat{\varphi}(\boldsymbol{p}\cdot) \equiv 0$ then the integral in (3.5) is zero. On the other hand, if $\#E > 1$ and $\boldsymbol{pk}_{i_1} = \boldsymbol{pk}_{i_2}$ for some $i_1, i_2 \in \{1, 2, \ldots, \#E\}$ then the integral is zero when choosing $0 \neq c_{i_1} = -c_{i_2}$ and $c_j = 0$ for $j \neq i_1, i_2$. Then (2) $\Longrightarrow$ (3). Next we prove that (3) $\Longrightarrow$ (2). Actually, since φ is compactly supported, it follows from Lemma 2.1 that $\mathcal{R}_{\boldsymbol{p}}\varphi$ is also compactly supported. Then $0 \not\equiv \widehat{\mathcal{R}_{\boldsymbol{p}}\varphi} = \widehat{\varphi}(\boldsymbol{p}^T\cdot) \in C^\infty(\mathbb{R})$, and consequently there exists an interval denoted by $[\xi_0 - \delta_0, \xi_0 + \delta_0]$ such that for any $\xi \in [\xi_0 - \delta_0, \xi_0 + \delta_0]$ we have $|\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}(\xi)| > 0$. Additionally, it follows from Lemma 3.2 that $\{e^{-\mathrm{i}\boldsymbol{pk}_l} : l = 1, \ldots, \#E\}$ is linearly independent on $[\xi_0 - \delta_0, \xi_0 + \delta_0]$. Then

$$\int_{\mathbb{R}} |\sum_{j=1}^{\#E} c_j e^{-\mathrm{i}\boldsymbol{pk}_j\xi}|^2 |\widehat{\varphi}(\boldsymbol{p}^T\xi)|^2 d\xi \geq \int_{\xi_0-\delta_0}^{\xi_0+\delta_0} |\sum_{j=1}^{\#E} c_j e^{-\mathrm{i}\boldsymbol{pk}_j\xi}|^2 |\widehat{\varphi}(\boldsymbol{p}^T\xi)|^2 d\xi > 0. \tag{3.6}$$

Consequently, (3.5) holds. This completes the proof. $\square$

The following is a counterexample such that the condition in Proposition 3.3 is not satisfied. Therefore, the problem of the linear independence of $\{\mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{pk}) : \boldsymbol{k} \in E\}$ is not trivial.

Example 3.1. The generator φ is defined such that

$$\widehat{\varphi}(\xi_1, \xi_2) = \sin(\xi_1 - \xi_2)\widehat{g}(\xi_1, \xi_2), \tag{3.7}$$

where $0 \not\equiv g \in L^2(\mathbb{R}^2)$ is compactly supported. One can check that $\varphi(x_1, x_2) = \frac{1}{2\mathrm{i}} g(x_1 + 1, x_2 - 1) - \frac{1}{2\mathrm{i}} g(x_1 - 1, x_2 + 1)$ and is compactly supported as well. Clearly, $\widehat{\varphi}(\boldsymbol{p}^T\cdot) \equiv 0$ if choosing $\boldsymbol{p} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

Analysis with the help of Paley-Wiener theorem. From the perspective of zero distribution, $\widehat{\varphi}$ in Example 3.1 has zeros along the line $\xi_1 - \xi_2 = 0$ on $\mathbb{R}^2 = \{(\xi_1, \xi_2)^T : \xi_1, \xi_2 \in \mathbb{R}\}$. This implies that $\widehat{\varphi}$ has non-isolated zeros on $\mathbb{R}^2$. For better understanding this issue, in what follows we explain it from the perspective of zero distribution of entire functions. The classical Paley-Wiener theorem (cf. [36]) states that a function $g \in L^2(\mathbb{R}^d)$ is the Fourier transform of a square integrable function with compact support if and only if it is the boundary value on $\mathbb{R}^d$ of an entire function on $\mathbb{C}^d$ of exponential type. Now for the compactly supported generator $\varphi \in L^2(\mathbb{R}^d)$, by the Paley-Wiener theorem we conclude that its Fourier transform $\widehat{\varphi}$ is the boundary value on $\mathbb{R}^d$ of an entire function on $\mathbb{C}^d$. It is well-known that for $d \geq 2$ an entire function on $\mathbb{C}^d$ may have non-isolated zeros (cf. [13]). Therefore, it is no wonder that there exists a pair $(\varphi, \boldsymbol{p})$ such that $\widehat{\varphi}(\boldsymbol{p}^T\xi) = 0$ for any $\xi \in \mathbb{R}$. Correspondingly, the system $\{\mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{pk}_j) : j = 1, \ldots, \#E\}$ in Proposition 3.3 is linearly dependent.

4. SA-Radon samples based reconstruction for compactly supported functions in SIS

This section concerns on the SACT sampling problem (1.9) for compactly supported functions in the SIS generated by a compactly supported generator φ. The main results will be organized in Theorems 4.3, 4.4 and 4.5. For the better readability, we quickly sketch the structure of this section. A necessary and sufficient condition on $(\varphi, \boldsymbol{p})$ and the sampling set $X \subseteq \mathbb{R}$ will be established in Theorem 4.1, such that a compactly supported function $f \in V(\varphi, \mathbb{Z}^2)$ can be determined uniquely by its SA Radon samples at X. Based on Theorem 4.1, our two main results are organized in Theorems 4.3, 4.4, and Theorem 4.5 and Proposition 4.6. Theorems 4.3 and 4.4 hold for the nonvanishing case ($\widehat{\varphi}(\mathbf{0}) \neq 0$) while Theorem 4.5 and Proposition 4.6 hold for the vanishing case ($\widehat{\varphi}(\mathbf{0}) = 0$).

4.1. A sufficient and necessary condition on the pair $(\varphi, \mathbf{p})$ and the sampling set X such that the SACT sampling (1.9) can be achieved

As previously, any $\mathbf{x} \in \mathbb{R}^2$ is considered as a column vector while the direction vector $\boldsymbol{p}$ is a row vector.

Theorem 4.1. *Suppose that $\varphi \in L^2(\mathbb{R}^2)$ such that supp$(\varphi) \subseteq [N_1, M_1] \times [N_2, M_2]$ and $\{\varphi(\cdot - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^2\}$ is linearly independent, and $\mathbf{p} = (\cos\theta, \sin\theta)$ is a direction vector such that $\mathcal{R}_{\mathbf{p}}\varphi$ is continuous. Moreover, $f \in V(\varphi, \mathbb{Z}^2)$ is an arbitrary source function such that supp$(f) \subseteq [a_1, b_1] \times [a_2, b_2]$. Let $E = \big\{\big[\lceil a_1 - M_1\rceil, \lfloor b_1 - N_1\rfloor\big] \times \big[\lceil a_2 - M_2\rceil, \lfloor b_2 - N_2\rfloor\big]\big\} \cap \mathbb{Z}^2$ and denote it by $\{\mathbf{k}_1, \dots, \mathbf{k}_{\#E}\}$. Then f can be determined uniquely by its SA Radon (w.r.t. $\mathbf{p}$) samples at $X = \{x_1, \dots, x_{\#E}\} \subseteq \mathbb{R}$ if and only if the $\#E \times \#E$ matrix*

$$A_{\varphi,\mathbf{p},X} := \begin{pmatrix} \mathcal{R}_{\mathbf{p}}\varphi(x_1 - \mathbf{p}\mathbf{k}_1) & \mathcal{R}_{\mathbf{p}}\varphi(x_1 - \mathbf{p}\mathbf{k}_2) & \cdots & \mathcal{R}_{\mathbf{p}}\varphi(x_1 - \mathbf{p}\mathbf{k}_{\#E}) \\ \mathcal{R}_{\mathbf{p}}\varphi(x_2 - \mathbf{p}\mathbf{k}_1) & \mathcal{R}_{\mathbf{p}}\varphi(x_2 - \mathbf{p}\mathbf{k}_2) & \cdots & \mathcal{R}_{\mathbf{p}}\varphi(x_2 - \mathbf{p}\mathbf{k}_{\#E}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{R}_{\mathbf{p}}\varphi(x_{\#E} - \mathbf{p}\mathbf{k}_1) & \mathcal{R}_{\mathbf{p}}\varphi(x_{\#E} - \mathbf{p}\mathbf{k}_2) & \cdots & \mathcal{R}_{\mathbf{p}}\varphi(x_{\#E} - \mathbf{p}\mathbf{k}_{\#E}) \end{pmatrix} \tag{4.1}$$

is invertible.

Proof. ($\Longleftarrow$) We first prove that if $A_{\varphi,\boldsymbol{p},X}$ is invertible then $\{\mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_n) : n = 1, \dots, \#E\}$ is linearly independent in $L^2(\mathbb{R})$. Otherwise, there exists a nonzero vector $(\widehat{d}_1, \dots, \widehat{d}_{\#E})^T \in \mathbb{C}^{\#E}$ such that

$$\|\sum_{n=1}^{\#E} \widehat{d}_n \mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_n)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |\sum_{n=1}^{\#E} \widehat{d}_n \mathcal{R}_{\boldsymbol{p}}\varphi(x - \boldsymbol{p}\boldsymbol{k}_n)|^2 dx = 0. \tag{4.2}$$

It follows from (4.2) and the continuity of $\mathcal{R}_{\boldsymbol{p}}\varphi$ that for any $x_l \in X$ we have $\sum_{n=1}^{\#E} \widehat{d}_n \mathcal{R}_{\boldsymbol{p}}\varphi(x_l - \boldsymbol{p}\boldsymbol{k}_n) = 0$, which implies that the matrix $A_{\varphi,\boldsymbol{p},X}$ is singular. This is a contradiction. Next we prove that $\mathcal{R}_{\boldsymbol{p}}f$ can be determined by its samples at X if $A_{\varphi,\boldsymbol{p},X}$ is invertible.

As in (3.3) and (3.4), there exists uniquely $(c_{\boldsymbol{k}_1}, \ldots, c_{\boldsymbol{k}_{\#E}})^T \in \mathbb{C}^{\#E}$ such that

$$f = \sum_{n=1}^{\#E} c_{\boldsymbol{k}_n} \varphi(\cdot - \boldsymbol{k}_n) \tag{4.3}$$

and consequently,

$$\mathcal{R}_{\boldsymbol{p}}f = \sum_{n=1}^{\#E} c_{\boldsymbol{k}_n} \mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_n). \tag{4.4}$$

Now it follows from (4.4) that

$$A_{\varphi,\boldsymbol{p},X}(c_{\boldsymbol{k}_1}, \ldots, c_{\boldsymbol{k}_{\#E}})^T = (\mathcal{R}_{\boldsymbol{p}}f(x_1), \ldots, \mathcal{R}_{\boldsymbol{p}}f(x_{\#E}))^T. \tag{4.5}$$

Since $A_{\varphi,\boldsymbol{p},X}$ is invertible then $(c_{\boldsymbol{k}_1}, \ldots, c_{\boldsymbol{k}_{\#E}})^T$ can be determined uniquely by the SA Radon samples $\mathcal{R}_{\boldsymbol{p}}f(x_1), \ldots, \mathcal{R}_{\boldsymbol{p}}f(x_{\#E})$. Since $\{\mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_n) : n = 1, \ldots, \#E\}$ is linearly independent, $\mathcal{R}_{\boldsymbol{p}}f$ represented via (4.4) can be determined from the vector $(c_{\boldsymbol{k}_1}, \ldots, c_{\boldsymbol{k}_{\#E}})^T$. Now by Theorem 3.1, $f = \sum_{n=1}^{\#E} c_{\boldsymbol{k}_n} \varphi(\cdot - \boldsymbol{k}_n)$ can be determined uniquely.

($\Longrightarrow$) If $A_{\varphi,\boldsymbol{p},X}$ is not invertible then $(c_{\boldsymbol{k}_1}, \ldots, c_{\boldsymbol{k}_{\#E}})^T$ can not be determined uniquely by (4.5). Recall again that $\{\varphi(\cdot - \boldsymbol{k}) : \boldsymbol{k} \in E\}$ is linearly independent, then f in (4.3) can not be determined uniquely. $\square$

Remark 4.1. For the sampling problem in the SIS $V(\varphi, \mathbb{Z}^2)$, it is required that φ is continuous (cf. Aldroubi and Gröchenig [1]). Therefore, if φ is discontinuous then the sampling in $V(\varphi, \mathbb{Z}^2)$ is not well-defined. On the other hand, it follows from Remark 2.3 that even though φ is discontinuous, the Radon transform $\mathcal{R}_{\boldsymbol{p}}\varphi$ may be continuous. From this perspective, when φ is discontinuous Theorem 4.1 may provide an alternative sampling-based recovery for compactly supported functions in $V(\varphi, \mathbb{Z}^2)$.

4.2. Direction vector set and null set

The concepts of direction vector set and null set will be necessary for SACT sampling.

Definition 4.1. (1) Suppose that $\mathcal{S} \subseteq \mathbb{R}^2$ such that $\mathcal{S}\backslash\{\mathbf{0}\}$ is not empty. Define its direction vector set as

$$\mathrm{dv}_{\mathcal{S}} = \{(\cos\theta, \sin\theta) : \text{all } \mathbf{0} \neq \mathbf{x} = \|\mathbf{x}\|_2(\cos\theta, \sin\theta)^T \in \mathcal{S}\}. \tag{4.6}$$

The direction vector sets of the empty set $\emptyset$ and $\{\mathbf{0}\}$ are both simply defined as $\emptyset$.

(2) For $\mathcal{S} \subseteq \mathbb{R}^2$ such that $\mathcal{S}\backslash\{\mathbf{0}\}$ is not empty, its null set $\mathcal{N}_{\mathcal{S}}$ is defined as

$$\{\mathbf{0} \neq \boldsymbol{y} \in \mathbb{R}^2 : \text{there exists } \mathbf{0} \neq \mathbf{x} \in \mathcal{S} \text{ such that } \mathbf{x}^T \boldsymbol{y} = 0\}. \tag{4.7}$$

The null sets of $\emptyset$ and $\{\mathbf{0}\}$ are both simply defined as $\emptyset$. Correspondingly, if $\mathcal{S}\backslash\{\mathbf{0}\}$ is not empty then the direction vector set $\mathrm{dv}_{\mathcal{N}_{\mathcal{S}}}$ is defined via (4.6).

Remark 4.2. (1) For $\mathbf{x}_0 \in \mathbb{R}^2$ and its open disc

$$\mathring{\mathcal{D}}(\mathbf{x}_0, \delta) := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{x}_0\|_2 < \delta\}, \tag{4.8}$$

if $\mathbf{0} \in \mathring{\mathcal{D}}(\mathbf{x}_0, \delta)$ then $\mathrm{dv}_{\mathring{\mathcal{D}}(\mathbf{x}_0,\delta)}$ is the unit circle $\{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi)\}$. (2) Suppose that $\mathcal{S} \subseteq \mathbb{R}^2$ is finite such that $\mathcal{S}\backslash\{\mathbf{0}\}$ is not empty. The null set $\mathcal{N}_{\mathcal{S}}$ of $\mathcal{S}$ is defined via (4.7). Then its cardinality $\#\mathrm{dv}_{\mathcal{N}_{\mathcal{S}}} < \infty$.

Proof. Item (1) is obvious. We just need to prove item (2). Denote $\mathcal{S}\backslash\{\mathbf{0}\}$ by $\{\mathbf{x}_1, \ldots, \mathbf{x}_L\}$. For any $\mathbf{0} \neq \mathbf{x}_l = (\mathbf{x}_{l,1}, \mathbf{x}_{l,2})^T \in \mathcal{S}$, suppose that $\mathbf{0} \neq \boldsymbol{y} = \|\boldsymbol{y}\|_2(\cos\theta_{\boldsymbol{y}}, \sin\theta_{\boldsymbol{y}})^T$ such that $\mathbf{x}_l^T \boldsymbol{y} = 0$. Without loss of generality, let $\mathbf{x}_{l,2} \neq 0$. Then $\tan\theta_{\boldsymbol{y}} = -\frac{\mathbf{x}_{l,1}}{\mathbf{x}_{l,2}}$. By $\mathcal{S}$ being finite, the proof can be completed. □

The following direction vector set is related to a function.

Definition 4.2. Suppose that $0 \not\equiv g : \mathbb{R}^2 \to \mathbb{C}$ is continuous. For $\mathbf{x}_0 \in \mathbb{R}^2$ such that $g(\mathbf{x}_0) \neq 0$, let $\delta^g_{\mathbf{x}_0,\max} > 0$ be the maximum value in $(0, \infty]$ such that for any $\mathbf{x} \in \mathring{\mathcal{D}}(\mathbf{x}_0, \delta^g_{\mathbf{x}_0,\max})$ we have $g(\mathbf{x}) \neq 0$, where $\mathring{\mathcal{D}}(\mathbf{x}_0, \delta^g_{\mathbf{x}_0,\max})$ is defined via (4.8). Following Definition 4.1 (4.6), the set of direction vectors $\mathrm{dv}_{\mathring{\mathcal{D}}(\mathbf{x}_0,\delta^g_{\mathbf{x}_0,\max})}$ of $\mathring{\mathcal{D}}(\mathbf{x}_0, \delta^g_{\mathbf{x}_0,\max})$ is defined as

$$\{(\cos\theta, \sin\theta) : \mathbf{0} \neq \mathbf{x} = \|\mathbf{x}\|_2(\cos\theta, \sin\theta)^T \in \mathring{\mathcal{D}}(\mathbf{x}_0, \delta^g_{\mathbf{x}_0,\max})\}. \tag{4.9}$$

Definition 4.3. Suppose that $0 \not\equiv \varphi \in L^2(\mathbb{R}^2)$ is compactly supported and vanishing (i.e. $\widehat{\varphi}(\mathbf{0}) = 0$). Denote the nonzero set of $\widehat{\varphi}$ by $\mathcal{G}_{\widehat{\varphi}}$ such that $\widehat{\varphi}(\mathbf{x}) \neq 0$ for any $\mathbf{x} \in \mathcal{G}_{\widehat{\varphi}}$. Define

$$\mathrm{DV}_{\widehat{\varphi}} = \bigcup_{\mathbf{x} \in \mathcal{G}_{\widehat{\varphi}}} \mathrm{dv}_{\mathring{\mathcal{D}}(\mathbf{x},\delta^{\widehat{\varphi}}_{\mathbf{x},\max})}, \tag{4.10}$$

where $\mathrm{dv}_{\mathring{\mathcal{D}}(\mathbf{x},\delta^{\widehat{\varphi}}_{\mathbf{x},\max})}$ is defined via Definition 4.2. Correspondingly, the angle set of $\mathrm{DV}_{\widehat{\varphi}}$ is defined as

$$\arg_{\mathrm{DV}_{\widehat{\varphi}}} = \{\theta \in [0, 2\pi) : (\cos\theta, \sin\theta) \in \mathrm{DV}_{\widehat{\varphi}}\}. \tag{4.11}$$

Proposition 4.2. *Let φ and $\arg_{DV_{\widehat{\varphi}}}$ be as in Definition 4.3. Then (1) the Lebesgue measure $\mu(\arg_{DV_{\widehat{\varphi}}})$ of $\arg_{DV_{\widehat{\varphi}}}$ on $\mathbb{R}$ is positive; (2) $\widehat{\mathcal{R}_{\mathbf{p}}\varphi} = \widehat{\varphi}(\mathbf{p}^T \cdot) \neq 0$ for $\mathbf{p} = (\cos\theta, \sin\theta)$ with any $\theta \in \arg_{DV_{\widehat{\varphi}}}$.*

Proof. We first prove item (1). Since $0 \not\equiv \varphi \in L^2(\mathbb{R}^2)$ is compactly supported, $0 \not\equiv \widehat{\varphi} \in C^\infty(\mathbb{R}^2)$. Then the nonzero set $\mathcal{G}_{\widehat{\varphi}}$ of $\widehat{\varphi}$ is not empty. Choose any $\mathbf{x} \in \mathcal{G}_{\widehat{\varphi}}$ and consider $\mathrm{dv}_{\mathring{\mathcal{D}}(\mathbf{x},\delta^{\widehat{\varphi}}_{\mathbf{x},\max})}$. As in (4.11), define the angle set of $\mathrm{dv}_{\mathring{\mathcal{D}}(\mathbf{x},\delta^{\widehat{\varphi}}_{\mathbf{x},\max})}$ as $\arg_{\mathrm{dv}_{\mathring{\mathcal{D}}(\mathbf{x},\delta^{\widehat{\varphi}}_{\mathbf{x},\max})}} = \{\theta \in [0, 2\pi) : (\cos\theta, \sin\theta) \in \mathrm{dv}_{\mathring{\mathcal{D}}(\mathbf{x},\delta^{\widehat{\varphi}}_{\mathbf{x},\max})}\}$. Since $\delta^{\widehat{\varphi}}_{\mathbf{x},\max} > 0$, the Lebesgue measure $\mu(\arg_{\mathrm{dv}_{\mathring{\mathcal{D}}(\mathbf{x},\delta^{\widehat{\varphi}}_{\mathbf{x},\max})}}) > 0$. Therefore, $\mu(\arg_{\mathrm{DV}_{\widehat{\varphi}}}) > 0$.

Next we prove item (2). For any $\theta \in \arg_{\mathrm{DV}_{\widehat{\varphi}}}$, by (4.11) the corresponding direction vector $\boldsymbol{p} = (\cos\theta, \sin\theta) \in \mathrm{DV}_{\widehat{\varphi}}$. Now by (4.10) there exists $\mathbf{x} \in \mathcal{G}_{\widehat{\varphi}}$ such that $\widehat{\varphi}(\mathbf{x}) \neq 0$ and $\boldsymbol{p} \in \mathrm{dv}_{\mathring{\mathcal{D}}(\mathbf{x},\delta^{\widehat{\varphi}}_{\mathbf{x},\max})}$. By the definition of $\mathring{\mathcal{D}}(\mathbf{x}, \delta^{\widehat{\varphi}}_{\mathbf{x},\max})$ in Definition 4.2, there exists $\gamma > 0$ such that $\widehat{\varphi}(\gamma \boldsymbol{p}^T) \neq 0$. Therefore, $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}(\gamma) = \widehat{\varphi}(\gamma \boldsymbol{p}^T) \neq 0$. By Lemma 2.1, $\mathcal{R}_{\boldsymbol{p}}\varphi$ is compactly supported and consequently $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi} \in C^\infty(\mathbb{R})$. Now by the continuity of $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}$ one can prove that $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi} \not\equiv 0$. This completes the proof. $\square$

4.3. The first main result: SACT sampling for compactly supported functions in a SIS generated by a non-vanishing generator φ

The following is the first main theorem in this section.

Theorem 4.3. *Suppose that $\varphi \in L^2(\mathbb{R}^2)$ is compactly supported such that supp$(\varphi) \subseteq [N_1, M_1] \times [N_2, M_2]$, $\{\varphi(\cdot - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^2\}$ is linearly independent and*

(i) the Sobolev smoothness $\nu_2(\varphi) > 1/2$,

(ii) $\widehat{\varphi}(\boldsymbol{0}) = \int_{\mathbb{R}^2} \varphi(\mathbf{x})d\mathbf{x} \neq 0$ (non-vanishing property).

As previously, suppose that $f \in V(\varphi, \mathbb{Z}^2)$ is an arbitrary source function such that supp$(f) \subseteq [a_1, b_1] \times [a_2, b_2]$. Correspondingly, define two sets

$$E = \big\{\big[\lceil a_1 - M_1\rceil, \lfloor b_1 - N_1\rfloor\big] \times \big[\lceil a_2 - M_2\rceil, \lfloor b_2 - N_2\rfloor\big]\big\} \cap \mathbb{Z}^2$$

and

$$E^+ = \begin{cases} \emptyset, & \#E = 1, \\ \{\mathbf{x} - \mathbf{y} : \mathbf{x} \neq \mathbf{y} \in E\}, & \#E > 1. \end{cases} \tag{4.12}$$

Then for any $\mathbf{p} \in \{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi)\} \setminus dv_{\mathcal{N}_{E^+}}$, there exists a sampling set $X_{\mathbf{p}} \subseteq \mathbb{R}$ having the cardinality $\#X_{\mathbf{p}} = \#E$ such that f can be determined by its SA Radon (w.r.t. $\mathbf{p}$) samples at $X_{\mathbf{p}}$.

Proof. Denote $E = \{\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{\#E}\}$. We first prove $\{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi)\} \setminus \mathrm{dv}_{\mathcal{N}_{E^+}}$ is not empty. It is sufficient to prove that $\#\mathrm{dv}_{\mathcal{N}_{E^+}} < \infty$. If $\#E = 1$ then $E^+ = \emptyset$ and by Definition 4.1 (1) we have $\mathrm{dv}_{\mathcal{N}_{E^+}} = \emptyset$ and $\#\mathrm{dv}_{\mathcal{N}_{E^+}} = 0$. If $\#E > 1$ then $\#E^+ = \#E(\#E - 1) < \infty$. By Proposition 4.2 (2) we have $\#\mathrm{dv}_{\mathcal{N}_{E^+}} < \infty$.

Since supp$(\varphi) \subseteq [N_1, M_1] \times [N_2, M_2]$ and supp$(f) \subseteq [a_1, b_1] \times [a_2, b_2]$, as in (4.3) we denote $f = \sum_{l=1}^{\#E} c_{\boldsymbol{k}_l}\varphi(\cdot - \boldsymbol{k}_l)$ for $(c_{\boldsymbol{k}_1}, \ldots, c_{\boldsymbol{k}_{\#E}}) \in \mathbb{C}^{\#E}$. Consequently, by (4.4) we have

$$\mathcal{R}_{\boldsymbol{p}} f = \sum_{l=1}^{\#E} c_{\boldsymbol{k}_l} \mathcal{R}_{\boldsymbol{p}} \varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_l). \tag{4.13}$$

We first prove that for any $\boldsymbol{p} \in \{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi)\} \setminus \mathrm{dv}_{\mathcal{N}_{E^+}}$, the system $\{\mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_l) : l = 1, \dots, \#E\}$ is linearly independent. For the equivalence of the linear independence established in Proposition 3.3 for the above system, we just need to prove that Proposition 3.3 (3) is satisfied any $\boldsymbol{p} \in \{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi)\} \setminus \mathrm{dv}_{\mathcal{N}_{E^+}}$. Clearly, $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}(0) = \widehat{\varphi}(\boldsymbol{0}) \neq 0$ for any $\boldsymbol{p}$. Then

$$\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi} = \widehat{\varphi}(\boldsymbol{p}^T \cdot) \not\equiv 0. \tag{4.14}$$

On the other hand, if $E^+ = \emptyset$ then $\mathrm{dv}_{\mathcal{N}_{E^+}} = \emptyset$. This combining (4.14) implies that item (3) of Proposition 3.3 is naturally satisfied for any $\boldsymbol{p} \in \{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi)\}$. If $E^+ \neq \emptyset$ then it follows from the definition of $\mathcal{N}_{E^+}$ in Definition 4.1 (2) that for any $\boldsymbol{p} \notin \mathrm{dv}_{\mathcal{N}_{E^+}}$ we have $\boldsymbol{p}\boldsymbol{k}_l \neq \boldsymbol{p}\boldsymbol{k}_n$ for any $l \neq n \in \{1, \dots, \#E\}$. That is, for the case that $E^+ \neq \emptyset$ item (3) of Proposition 3.3 is also satisfied. Then it follows from Proposition 3.3 that $\{\mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_l) : l = 1, \dots, \#E\}$ is linearly independent.

By the above independence there exist constants $0 < C_{1,\boldsymbol{p}} \leq C_{2,\boldsymbol{p}} < \infty$ such that

$$C_{1,\boldsymbol{p}} \sum_{l=1}^{\#E} |d_{\boldsymbol{k}_l}|^2 \leq \int_{\mathbb{R}} |\sum_{l=1}^{\#E} d_{\boldsymbol{k}_l} \mathcal{R}_{\boldsymbol{p}}\varphi(x - \boldsymbol{p}\boldsymbol{k}_l)|^2 dx \leq C_{2,\boldsymbol{p}} \sum_{l=1}^{\#E} |d_{\boldsymbol{k}_l}|^2 \tag{4.15}$$

for any $(d_{\boldsymbol{k}_1}, \dots, d_{\#E})^T \in \mathbb{R}^{\#E}$. On the other hand, it follows from Proposition 2.1 (2.3) that $\mathrm{supp}(\mathcal{R}_{\boldsymbol{p}}\varphi) \subseteq [-L_\varphi, L_\varphi]$, where $L_\varphi = \sqrt{2}\max\{|N_i|, |M_i| : i = 1, 2\}$. Denote $a_{\boldsymbol{p},1} = \min\{\boldsymbol{p}\boldsymbol{k}_l : l = 1, \dots, \#E\}$ and $a_{\boldsymbol{p},2} = \max\{\boldsymbol{p}\boldsymbol{k}_l : l = 1, \dots, \#E\}$. One can check that

$$\mathrm{supp}\big(\sum_{l=1}^{\#E} d_{\boldsymbol{k}_l} \mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_l)\big) \subseteq [L_{\boldsymbol{p},1}, L_{\boldsymbol{p},2}],$$

where $L_{\boldsymbol{p},1} = -L_\varphi + a_{\boldsymbol{p},1}$ and $L_{\boldsymbol{p},2} = L_\varphi + a_{\boldsymbol{p},2}$. Then (4.15) is equivalent to

$$C_{1,\boldsymbol{p}} \sum_{l=1}^{\#E} |d_{\boldsymbol{k}_l}|^2 \leq \int_{L_{\boldsymbol{p},1}}^{L_{\boldsymbol{p},2}} |\sum_{l=1}^{\#E} d_{\boldsymbol{k}_l} \mathcal{R}_{\boldsymbol{p}}\varphi(x - \boldsymbol{p}\boldsymbol{k}_l)|^2 dx \leq C_{2,\boldsymbol{p}} \sum_{l=1}^{\#E} |d_{\boldsymbol{k}_l}|^2. \tag{4.16}$$

The rest of the proof is to find a sampling set $X_{\boldsymbol{p}} \subseteq \mathbb{R}$ with the cardinality $\#X_{\boldsymbol{p}} = \#E$ such that f can be determined by its SA Radon (w.r.t. $\boldsymbol{p}$) samples at $X_{\boldsymbol{p}}$. Since $\nu_2(\varphi) > 1/2$, by Proposition 2.3 (2) we have that $\mathcal{R}_{\boldsymbol{p}}\varphi$ is continuous. Consequently, all $\mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_l), l = 1, \dots, \#E$ are uniformly continuous on the interval $[L_{\boldsymbol{p},1}, L_{\boldsymbol{p},2}]$. Then there exists $\delta_{\boldsymbol{p}} \leq (L_{\boldsymbol{p},2} - L_{\boldsymbol{p},1})$ such that for any $l \in \{1, \dots, \#E\}$ and any $x', x'' \in [L_{\boldsymbol{p},1}, L_{\boldsymbol{p},2}]$ satisfying $|x' - x''| < \delta_{\boldsymbol{p}}$ we have

$$|\mathcal{R}_{\boldsymbol{p}}\varphi(x'-\boldsymbol{p}\boldsymbol{k}_l)-\mathcal{R}_{\boldsymbol{p}}\varphi(x''-\boldsymbol{p}\boldsymbol{k}_l)|\leq\sqrt{\frac{C_{1,\boldsymbol{p}}}{3\#E(L_{\boldsymbol{p},2}-L_{\boldsymbol{p},1})}}. \tag{4.17}$$

Now let $K_{\boldsymbol{p}}=\lceil\frac{L_{\boldsymbol{p},2}-L_{\boldsymbol{p},1}}{\delta_{\boldsymbol{p}}}\rceil$. Construct

$$Y_{\boldsymbol{p}}=\{x_k=L_{\boldsymbol{p},1}+\frac{L_{\boldsymbol{p},2}-L_{\boldsymbol{p},1}}{K_{\boldsymbol{p}}}(k-1):k=1,\dots,K_{\boldsymbol{p}}+1\}$$

such that

$$|x_k-x_j|\leq\delta_{\boldsymbol{p}} \tag{4.18}$$

for any x_k,x_j. Define an approximation to $\mathcal{R}_{\boldsymbol{p}}(\cdot-\boldsymbol{p}\boldsymbol{k}_l)$ as $h_l(x)=\sum_{k=1}^{K_{\boldsymbol{p}}}\mathcal{R}_{\boldsymbol{p}}\varphi(x_k-\boldsymbol{p}\boldsymbol{k}_l)\chi_{[x_k,x_{k+1})}(x)$. Then one can check that

$$\begin{aligned}&\int_{L_{\boldsymbol{p},1}}^{L_{\boldsymbol{p},2}}|\sum_{l=1}^{\#E}d_{\boldsymbol{k}_l}(\mathcal{R}_{\boldsymbol{p}}\varphi(x-\boldsymbol{p}\boldsymbol{k}_l)-h_l(x))|^2dx\\&\leq\sum_{j=1}^{\#E}|d_{\boldsymbol{k}_j}|^2\int_{L_{\boldsymbol{p},1}}^{L_{\boldsymbol{p},2}}\sum_{l=1}^{\#E}|\mathcal{R}_{\boldsymbol{p}}\varphi(x-\boldsymbol{p}\boldsymbol{k}_l)-h_l(x)|^2dx\ (4.19\text{A})\\&=\sum_{j=1}^{\#E}|d_{\boldsymbol{k}_j}|^2\sum_{n=1}^{K_p}\int_{x_n}^{x_{n+1}}\sum_{l=1}^{\#E}|\mathcal{R}_{\boldsymbol{p}}\varphi(x-\boldsymbol{p}\boldsymbol{k}_l)-h_l(x)|^2dx\end{aligned} \tag{4.19}$$

where (4.19A) is derived from the Cauchy-Schwart inequality. We continue to estimate (4.19) as follows,

$$\begin{aligned}&\int_{L_{\boldsymbol{p},1}}^{L_{\boldsymbol{p},2}}|\sum_{l=1}^{\#E}d_{\boldsymbol{k}_l}(\mathcal{R}_{\boldsymbol{p}}\varphi(x-\boldsymbol{p}\boldsymbol{k}_l)-h_l(x))|^2dx\\&\leq\sum_{j=1}^{\#E}|d_{\boldsymbol{k}_j}|^2\sum_{n=1}^{K_{\boldsymbol{p}}}\int_{x_n}^{x_{n+1}}\sum_{l=1}^{\#E}|\mathcal{R}_{\boldsymbol{p}}\varphi(x-\boldsymbol{p}\boldsymbol{k}_l)-h_l(x)|^2dx\\&=\sum_{j=1}^{\#E}|d_{\boldsymbol{k}_j}|^2\sum_{n=1}^{K_{\boldsymbol{p}}}\int_{x_n}^{x_{n+1}}\sum_{l=1}^{\#E}|\mathcal{R}_{\boldsymbol{p}}\varphi(x-\boldsymbol{p}\boldsymbol{k}_l)-\mathcal{R}_{\boldsymbol{p}}\varphi(x_n-\boldsymbol{p}\boldsymbol{k})|^2dx\ (4.20\text{A})\\&\leq\sum_{j=1}^{\#E}|d_{\boldsymbol{k}_j}|^2\#E\frac{C_{1,\boldsymbol{p}}}{3\#E(L_{\boldsymbol{p},2}-L_{\boldsymbol{p},1})}K_{\boldsymbol{p}}\delta_{\boldsymbol{p}}\ (4.20\text{B})\\&\leq\frac{C_{1,\boldsymbol{p}}}{3}\sum_{j=1}^{\#E}|d_{\boldsymbol{k}_j}|^2,\ (4.20\text{C})\end{aligned} \tag{4.20}$$

where (4.20A) is from the definition of $h_l(x)$, (4.20B) is from (4.17) and (4.18), and (4.20C) is from $K_{\boldsymbol{p}}\delta_{\boldsymbol{p}} \le L_{\boldsymbol{p},2} - L_{\boldsymbol{p},1}$. Then

$$\begin{aligned}
\Big(\int_{L_{\boldsymbol{p},1}}^{L_{\boldsymbol{p},2}} \Big|\sum_{l=1}^{\#E} d_{\boldsymbol{k}_l} h_l(x)\Big|^2 dx\Big)^{1/2} &\ge -\Big(\int_{L_{\boldsymbol{p},1}}^{L_{\boldsymbol{p},2}} \Big|\sum_{l=1}^{\#E} d_{\boldsymbol{k}_l} (\mathcal{R}_{\boldsymbol{p}}\varphi(x-\boldsymbol{p}\boldsymbol{k}_l) - h_l(x))\Big|^2 dx\Big)^{1/2} \quad (4.21\text{A}) \\
&+\Big(\int_{L_{\boldsymbol{p},1}}^{L_{\boldsymbol{p},2}} \Big|\sum_{l=1}^{\#E} d_{\boldsymbol{k}_l} \mathcal{R}_{\boldsymbol{p}}\varphi(x-\boldsymbol{p}\boldsymbol{k}_l)\Big|^2 dx\Big)^{1/2} \\
&\ge (1-\sqrt{1/3})\sqrt{C_{1,\boldsymbol{p}}}\Big(\sum_{l=1}^{\#E} |d_{\boldsymbol{k}_l}|^2\Big)^{1/2}, \quad (4.21\text{B})
\end{aligned} \tag{4.21}$$

where (4.21A) is from the triangle inequality, and (4.21B) is from (4.15) and (4.20). Then for any $(d_{\boldsymbol{k}_1}, \dots, d_{\boldsymbol{k}_{\#E}}) \neq \mathbf{0}$ we have

$$\begin{aligned}
0 < C_{1,\boldsymbol{p}}(1-\sqrt{1/3})^2 \sum_{l=1}^{\#E} |d_{\boldsymbol{k}_l}|^2 &\le \int_{L_{\boldsymbol{p},1}}^{L_{\boldsymbol{p},2}} \Big|\sum_{l=1}^{\#E} d_{\boldsymbol{k}_l} h_l(x)\Big|^2 dx \\
&= \sum_{j=1}^{K_{\boldsymbol{p}}} \int_{x_j}^{x_{j+1}} \Big|\sum_{l=1}^{\#E} d_{\boldsymbol{k}_l} h_l(x)\Big|^2 dx \\
&= \sum_{j=1}^{K_{\boldsymbol{p}}} \Big|\sum_{l=1}^{\#E} d_{\boldsymbol{k}_l} \mathcal{R}_{\boldsymbol{p}}\varphi(x_j - p\boldsymbol{k}_l)\Big|^2.
\end{aligned} \tag{4.22}$$

By (4.22), we conclude that there exists $X_{\boldsymbol{p}} := \{x_{j_1}, \dots, x_{j_{\#E}}\} \subseteq Y_{\boldsymbol{p}}$ such that the corresponding $\#E \times \#E$ matrix

$$A_{\varphi,\boldsymbol{p},X_{\boldsymbol{p}}} = \begin{pmatrix}
\mathcal{R}_{\boldsymbol{p}}\varphi(x_{j_1} - \boldsymbol{p}\boldsymbol{k}_1) & \mathcal{R}_{\boldsymbol{p}}\varphi(x_{j_1} - \boldsymbol{p}\boldsymbol{k}_2) & \cdots & \mathcal{R}_{\boldsymbol{p}}\varphi(x_{j_1} - \boldsymbol{p}\boldsymbol{k}_{\#E}) \\
\mathcal{R}_{\boldsymbol{p}}\varphi(x_{j_2} - \boldsymbol{p}\boldsymbol{k}_1) & \mathcal{R}_{\boldsymbol{p}}\varphi(x_{j_2} - \boldsymbol{p}\boldsymbol{k}_2) & \cdots & \mathcal{R}_{\boldsymbol{p}}\varphi(x_{j_2} - \boldsymbol{p}\boldsymbol{k}_{\#E}) \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{R}_{\boldsymbol{p}}\varphi(x_{j_{\#E}} - \boldsymbol{p}\boldsymbol{k}_1) & \mathcal{R}_{\boldsymbol{p}}\varphi(x_{j_{\#E}} - \boldsymbol{p}\boldsymbol{k}_2) & \cdots & \mathcal{R}_{\boldsymbol{p}}\varphi(x_{j_{\#E}} - \boldsymbol{p}\boldsymbol{k}_{\#E})
\end{pmatrix} \tag{4.23}$$

is invertible. Now by Theorem 4.1, the source function f can be determined uniquely by its Radon (w.r.t. $\boldsymbol{p}$) samples at $X_{\boldsymbol{p}}$. Specifically, the vector $(c_{\boldsymbol{p}\boldsymbol{k}_1}, \dots, c_{\boldsymbol{p}\boldsymbol{k}_{\#E}})^T$ can be determined by

$$(c_{\boldsymbol{p}\boldsymbol{k}_1}, \dots, c_{\boldsymbol{p}\boldsymbol{k}_{\#E}})^T = A^{-1}_{\varphi,\boldsymbol{p},X_{\boldsymbol{p}}}(\mathcal{R}_{\boldsymbol{p}}f(x_{j_1}), \dots, \mathcal{R}_{\boldsymbol{p}}f(x_{j_{\#E}}))^T. \tag{4.24}$$

This completes the proof. □

In what follows we explain why the condition $\nu_2(\varphi) > 1/2$ in Theorem 4.3 is required.

Remark 4.3. Since $\nu_2(\varphi) > 1/2$, by Proposition 2.3 (2) we conclude that $\mathcal{R}_{\boldsymbol{p}}\varphi$ is continuous. If such a condition is not satisfied, then $\mathcal{R}_{\boldsymbol{p}}\varphi$ may be discontinuous for some $\boldsymbol{p}$. As in Remark 2.3, let $\varphi(x_1, x_2) = \chi_{(0,1]}(x_1)\chi_{(0,1]}(x_2)$. Through the direct calculation we have $\widehat{\varphi}(\xi_1, \xi_2) = \frac{1-e^{-\mathrm{i}\xi_1}}{\mathrm{i}\xi_1}\frac{1-e^{-\mathrm{i}\xi_2}}{\mathrm{i}\xi_2}$. By the Sobolev smoothness definition in subsection 2.3 one can check that $\nu_2(\varphi) = 1/2$. If $\boldsymbol{p} = (1, 0)$ or $(0, 1)$ then $\mathcal{R}_{\boldsymbol{p}}\varphi = \chi_{(0,1]}$ which is discontinuous. As a result, there may not exist $\delta_{\boldsymbol{p}}$ such that (4.17) holds.

Remark 4.4. Define the $\#E \times \#E$ Gram matrix

$$G_{\varphi,\boldsymbol{p}} = \Big(\langle \mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_j), \mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_n)\rangle\Big)_{j,n=1}^{\#E}, \tag{4.25}$$

where the inner product $\langle \mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_j), \mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_n)\rangle = \int_{\mathbb{R}} \mathcal{R}_{\boldsymbol{p}}\varphi(x - \boldsymbol{p}\boldsymbol{k}_j)\overline{\mathcal{R}_{\boldsymbol{p}}\varphi}(x - \boldsymbol{p}\boldsymbol{k}_n)dx$. Then (4.15) or (4.16) is equivalent to

$$C_{1,\boldsymbol{p}}\|(d_{\boldsymbol{k}_1}, \ldots, d_{\boldsymbol{k}_{\#E}})\|_2^2 \le (d_{\boldsymbol{k}_1}, \ldots, d_{\boldsymbol{k}_{\#E}})G_{\varphi,\boldsymbol{p}}(d_{\boldsymbol{k}_1}, \ldots, d_{\boldsymbol{k}_{\#E}})^* \le C_{2,\boldsymbol{p}}\|(d_{\boldsymbol{k}_1}, \ldots, d_{\boldsymbol{k}_{\#E}})\|_2^2, \tag{4.26}$$

where D^* is the conjugate and transpose of a matrix D. Note that $G_{\varphi,\boldsymbol{p}}$ is a Hermitian matrix. Then (4.26) implies that $G_{\varphi,\boldsymbol{p}}$ is a positive definite matrix, and consequently $0 < C_{1,\boldsymbol{p}} \le \lambda_{\min}(G_{\varphi,\boldsymbol{p}})$ and $\lambda_{\max}(G_{\varphi,\boldsymbol{p}}) \le C_{2,\boldsymbol{p}} < \infty$, where $\lambda_{\max}(G_{\varphi,\boldsymbol{p}}) > 0$ and $\lambda_{\min}(G_{\varphi,\boldsymbol{p}}) > 0$ are the maximum and minimum eigenvalues of $G_{\varphi,\boldsymbol{p}}$, respectively. Particularly, in (4.17) one can choose

$$C_{1,\boldsymbol{p}} = \lambda_{\min}(G_{\varphi,\boldsymbol{p}}). \tag{4.27}$$

The following states that if $\varphi \in C^1(\mathbb{R}^2)$ then $\delta_{\boldsymbol{p}}$ in the proof of Theorem 4.3 can be chosen explicitly. Consequently, the SA Radon sampling point set $X_{\boldsymbol{p}}$ in Theorem 4.3 can be constructed explicitly.

Theorem 4.4. *Let the compactly supported generator $\varphi \in C^1(\mathbb{R}^2)$ such that $\widehat{\varphi}(\boldsymbol{0}) \neq 0$ and the source function $f \in V(\varphi, \mathbb{Z}^2)$. As in Theorem 4.3 suppose that supp$(\varphi) \subseteq [N_1, M_1] \times [N_2, M_2]$ and supp$(f) \subseteq [a_1, b_1] \times [a_2, b_2]$. Define two sets*

$$E = \Big\{\big[\lceil a_1 - M_1\rceil, \lfloor b_1 - N_1\rfloor\big] \times \big[\lceil a_2 - M_2\rceil, \lfloor b_2 - N_2\rfloor\big]\Big\} \cap \mathbb{Z}^2$$

and

$$E^+ = \begin{cases} \emptyset, & \#E = 1, \\ \{\mathbf{x} - \mathbf{y} : \mathbf{x} \neq \mathbf{y} \in E\}, & \#E > 1. \end{cases} \tag{4.28}$$

Choose a direction vector $\mathbf{p} \in \{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi)\} \setminus dv_{\mathcal{N}_{E^+}}$*, and correspondingly denote*

$$\begin{cases} L_{\mathbf{p},1} = -\sqrt{2}\max\{|N_i|, |M_i| : i = 1,2\} + \min\{\mathbf{pk} : \mathbf{k} \in E\}, \\ L_{\mathbf{p},2} = \sqrt{2}\max\{|N_i|, |M_i| : i = 1,2\} + \max\{\mathbf{pk} : \mathbf{k} \in E\}, \\ \delta_{\mathbf{p}} = \sqrt{\frac{\lambda_{\min}(G_{\varphi,\mathbf{p}})}{3\#E(L_{\mathbf{p},2} - L_{\mathbf{p},1})}} \Big/ \big(2(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty) \max\{|N_i|, |M_i| : i = 1,2\}\big), \\ K_{\mathbf{p}} = \lceil \frac{L_{p,2} - L_{p,1}}{\delta_{\mathbf{p}}} \rceil, \end{cases} \tag{4.29}$$

where $\lambda_{\min}(G_{\varphi,\mathbf{p}})$ *is the minimum eigenvalue of the Gram matrix* $G_{\varphi,\mathbf{p}}$ *defined in* (4.25)*,* $\varphi_1(x_1, x_2)$ *and* $\varphi_2(x_1, x_2)$ *are the partial derivatives of* $\varphi(x_1, x_2)$ *w.r.t. the variables* x_1 *and* x_2*, respectively such that*

$$\|\varphi_1\|_\infty = \max_{(x_1,x_2)\in[N_1,M_1]\times[N_2,M_2]} |\varphi_1(x_1, x_2)|, \|\varphi_2\|_\infty = \max_{(x_1,x_2)\in[N_1,M_1]\times[N_2,M_2]} |\varphi_2(x_1, x_2)|.$$

Explicitly construct

$$Y_{\mathbf{p}} = \{x_k = L_{\mathbf{p},1} + \frac{L_{\mathbf{p},2} - L_{\mathbf{p},1}}{K_{\mathbf{p}}}(k-1) : k = 1, \dots, K_{\mathbf{p}} + 1\}. \tag{4.30}$$

Then there exists $X_{\mathbf{p}} = \{x_{i_1}, \dots, x_{i_{\#E}}\} \subseteq Y_{\mathbf{p}}$ *such that the matrix* $A_{\varphi,\mathbf{p},X_{\mathbf{p}}}$ *in* (4.23) *is invertible and consequently,* f *can be determined uniquely by its SA Radon samples at* $X_{\mathbf{p}}$*.*

Proof. By Remark 4.4 (4.27), $C_{1,\boldsymbol{p}}$ in (4.26) can be chosen as $\lambda_{\min}(G_{\varphi,\boldsymbol{p}})$. If (4.17) holds with $C_{1,\boldsymbol{p}}$ replaced by $\lambda_{\min}(G_{\varphi,\boldsymbol{p}})$, then by the similar procedures ((4.19)-(4.22)) in the proof of Theorem 4.3 one can prove that there exists $X_{\boldsymbol{p}} = \{x_{i_1}, \dots, x_{i_{\#E}}\} \subseteq Y_{\boldsymbol{p}}$ such that $A_{\varphi,\boldsymbol{p},X_{\boldsymbol{p}}}$ in (4.23) is invertible. Consequently, f can be determined by (4.24). Therefore, we just need to prove that (4.17) holds.

The SVD of $\boldsymbol{p} = (\cos\theta, \sin\theta)$ is ΣV^T such that $V = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and $\Sigma = (1, 0)$. Since $\varphi \in C^1(\mathbb{R}^2)$ is compactly supported, we have

$$\begin{aligned} &|\mathcal{R}_{\boldsymbol{p}}\varphi(x' - \boldsymbol{pk}_l) - \mathcal{R}_{\boldsymbol{p}}\varphi(x'' - \boldsymbol{pk}_l)| \\ &= \Big| \int_{\mathbb{R}} \varphi\big((x' - \boldsymbol{pk}_l)\cos\theta + x_2\sin\theta, (x' - \boldsymbol{pk}_l)\sin\theta - x_2\cos\theta\big) \\ &\qquad - \varphi((x'' - \boldsymbol{pk}_l)\cos\theta + x_2\sin\theta, (x'' - \boldsymbol{pk}_l)\sin\theta - x_2\cos\theta)dx_2 \Big| \quad (4.31\text{A}) \\ &= \Big| \int_{-\max\{|N_i|,|M_i|:i=1,2\}}^{\max\{|N_i|,|M_i|:i=1,2\}} \varphi\big((x' - \boldsymbol{pk}_l)\cos\theta + x_2\sin\theta, (x' - \boldsymbol{pk}_l)\sin\theta - x_2\cos\theta\big) \end{aligned}$$

$$-\varphi\big((x''-\boldsymbol{p}\boldsymbol{k}_l)\cos\theta + x_2\sin\theta, (x''-\boldsymbol{p}\boldsymbol{k}_l)\sin\theta - x_2\cos\theta\big)dx_2\Big| \quad (4.31\text{B})$$

$$\le |x'-x''| \int_{-\max\{|N_i|,|M_i|:i=1,2\}}^{\max\{|N_i|,|M_i|:i=1,2\}} (\|\varphi_1\|_\infty + \|\varphi_2\|_\infty)dx_2 \quad (4.31\text{C})$$

$$= 2|x'-x''|(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty)\max\{|N_i|,|M_i| : i=1,2\}, \tag{4.31}$$

where the (4.31A) and (4.31B) are derived from (2.4A) and (2.6), respectively, and (4.31C) is from the differential mean value theorem. It is required that $|x'-x''| \le \delta_{\boldsymbol{p}}$. Then it follows from (4.31) that

$$|\mathcal{R}_{\boldsymbol{p}}\varphi\big(x'-\boldsymbol{p}\boldsymbol{k}_l\big) - \mathcal{R}_{\boldsymbol{p}}\varphi\big(x''-\boldsymbol{p}\boldsymbol{k}_l\big)| \le 2\delta_{\boldsymbol{p}}(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty)\max\{|N_i|,|M_i| : i=1,2\}. \tag{4.32}$$

Now by (4.32) we can choose

$$\delta_{\mathbf{p}} = \sqrt{\frac{\lambda_{\min}(G_{\varphi,\mathbf{p}})}{3\#E(L_{\mathbf{p},2}-L_{\mathbf{p},1})}}\Big/\big(2(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty)\max\{|N_i|,|M_i| : i=1,2\}\big)$$

such that (4.17) holds with $C_{1,\boldsymbol{p}}$ replaced by $\lambda_{\min}(G_{\varphi,\boldsymbol{p}})$. The proof is completed. □

4.4. The second main result: SACT sampling for compactly supported functions in a SIS generated by a vanishing generator φ

In this subsection suppose that the generator φ is vanishing, namely, $\widehat{\varphi}(\mathbf{0}) = \int_{\mathbb{R}^2}\varphi(\mathbf{x})d\mathbf{x} = 0$.

Theorem 4.5. *Suppose that $\varphi \in L^2(\mathbb{R}^2)$ is compactly supported such that supp$(\varphi) \subseteq [N_1, M_1] \times [N_2, M_2]$, the system $\{\varphi(\cdot - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^2\}$ is linearly independent, and*

(i) the Sobolev smoothness $\nu_2(\varphi) > 1/2$,

(ii) $\widehat{\varphi}(\boldsymbol{0}) = \int_{\mathbb{R}^2}\varphi(\mathbf{x})d\mathbf{x} = 0$ (vanishing property).

Moreover, as previously suppose that $f \in V(\varphi, \mathbb{Z}^2)$ is an arbitrary source function such that supp$(f) \subseteq [a_1, b_1] \times [a_2, b_2]$. Define

$$E = \big\{\big[\lceil a_1 - M_1\rceil, \lfloor b_1 - N_1\rfloor\big] \times \big[\lceil a_2 - M_2\rceil, \lfloor b_2 - N_2\rfloor\big]\big\} \cap \mathbb{Z}^2$$

and

$$E^+ = \begin{cases} \emptyset, & \#E = 1, \\ \{\mathbf{x}-\mathbf{y} : \mathbf{x} \ne \mathbf{y} \in E\}, & \#E > 1. \end{cases} \tag{4.33}$$

Then for any direction vector $\mathbf{p} \in DV_{\widehat{\varphi}} \setminus dv_{\mathcal{N}_{E^+}}$, there exists a sampling set $X_{\mathbf{p}} \subseteq \mathbb{R}$ having the cardinality $\#X_{\mathbf{p}} = \#E$ such that f can be determined uniquely by its SA

Radon (w.r.t. $\mathbf{p}$*) samples at* $X_{\mathbf{p}}$*, where* $DV_{\widehat{\varphi}}$ *and* $dv_{\mathcal{N}_{E+}}$ *are defined via Definitions 4.3 and 4.1.*

Proof. Denote E by $\{\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{\#E}\}$. It has been proved in the proof of Theorem 4.3 that $\#\mathrm{dv}_{\mathcal{N}_{E+}} < \infty$. Now by Proposition 4.2 (1) one can prove that $\mathrm{DV}_{\widehat{\varphi}} \setminus \mathrm{dv}_{\mathcal{N}_{E+}}$ is not empty. It follows from Proposition 4.2 (2) that $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi} \not\equiv 0$ for any direction vector $\boldsymbol{p} \in \mathrm{DV}_{\widehat{\varphi}}$. Moreover, as in the proof of Theorem 4.3 one can prove that for any direction vector $\boldsymbol{p} \in \mathrm{DV}_{\widehat{\varphi}} \setminus \mathrm{dv}_{\mathcal{N}_{E+}}$ the system $\{\mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_l) : l = 1, \ldots, \#E\}$ is linearly independent. Through the similar procedures of the proof of Theorem 4.3, one can prove there exists a sampling set $X_{\boldsymbol{p}}$ such that $\#X_{\boldsymbol{p}} = \#E$ and f can be determined uniquely by its SA Radon samples at $X_{\boldsymbol{p}}$. □

If $\varphi \in C^1(\mathbb{R}^2)$, by the similar proof of Theorem 4.4 one can prove the following result.

Proposition 4.6. *If* $\varphi \in C^1(\mathbb{R}^2)$*, then for* $\mathbf{p} \in DV_{\widehat{\varphi}} \setminus dv_{\mathcal{N}_{E+}}$ *the sampling point set* $X_{\mathbf{p}}$ *in Theorem 4.5 can be constructed explicitly through the similar procedures in Theorem 4.4* (4.29) *and* (4.30)*.*

5. Pairs of $(\varphi, \boldsymbol{p})$ such that the corresponding SACT can be achieved by the sampling set $\{\boldsymbol{p}\boldsymbol{k}_1, \ldots, \boldsymbol{p}\boldsymbol{k}_{\#E}\}$

5.1. Motivation

If the Sobolev smoothness $\nu_2(\varphi) > 1/2$, it has been proved in Theorems 4.3 and 4.5 that there exists a sampling set $X_{\boldsymbol{p}}$ such that the source function $f \in V(\varphi, \mathbb{Z}^2)$ can be determined uniquely by its SA samples at $X_{\boldsymbol{p}}$. Moreover, if $\varphi \in C^1(\mathbb{R}^2)$ then it is stated in Theorem 4.4 and Proposition 4.6 that $X_{\boldsymbol{p}}$ can be constructed explicitly. On the other hand, however, it follows from Remark 2.2 that $\nu_2(\varphi) > 1/2$ does not necessarily imply that $\varphi \in C^1(\mathbb{R}^2)$. Then a natural problem is, without the C^1 condition how can one explicitly construct the sampling set $X_{\boldsymbol{p}}$. Before introducing our scheme, let us recall (4.13) as

$$\mathcal{R}_{\boldsymbol{p}} f = \sum_{l=1}^{\#E} c_{\boldsymbol{k}_l} \mathcal{R}_{\boldsymbol{p}}\varphi(\cdot - \boldsymbol{p}\boldsymbol{k}_l). \tag{5.1}$$

Note that $X_{\boldsymbol{p}}$ in Theorem 4.4 and Proposition 4.6 is not necessarily $\{\boldsymbol{p}\boldsymbol{k}_1, \ldots, \boldsymbol{p}\boldsymbol{k}_{\#E}\}$. Naturally, one asks:

Under what condition on the pair $(\varphi, \boldsymbol{p})$, can f be determined uniquely by its SA samples at $\{\boldsymbol{p}\boldsymbol{k}_1, \ldots, \boldsymbol{p}\boldsymbol{k}_{\#E}\}$?

If such a determination can be achieved then compared with those in Theorem 4.4 and Proposition 4.6, it is more efficient to conduct SACT of f since we do not require to consider the sampling set.

We quickly describe the structure of this section. In subsection 5.2 we give a condition on the pair $(\varphi, \boldsymbol{p})$ such that the above determination can be achieved. We also address the determination in subsections 5.4 and 5.5 for the case that φ being positive definite.

5.2. The third main result: a condition on $(\varphi, \mathbf{p})$ such that $\{\mathbf{pk}_1, \ldots, \mathbf{pk}_{\#E}\}$ is eligible for the SACT

From the perspective of the signs of the real and imaginary parts of $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}$, a condition is given in the following theorem such that the sampling set $X_{\boldsymbol{p}} = \{\boldsymbol{pk}_1, \ldots, \boldsymbol{pk}_{\#E}\}$ is eligible for the SACT. Incidentally, for $0 \neq y \in \mathbb{R}$ its sign $\mathrm{sgn}(y)$ takes 1 and -1 for $y > 0$ and $y < 0$, respectively. For a function $0 \not\equiv g : \mathbb{R} \longrightarrow \mathbb{R}$ we say that its sign function $\mathrm{sgn}(g(x))$ is **unchanged** if $g(x) \geq 0$ for any $x \in \mathbb{R}$ (or $g(x) \leq 0$ for any $x \in \mathbb{R}$).

Theorem 5.1. *As previously, suppose that the generator $\varphi \in L^2(\mathbb{R}^2)$ satisfies supp(φ) $\subseteq [N_1, M_1] \times [N_2, M_2]$, and $f \in V(\varphi, \mathbb{Z}^2)$ is an arbitrary source function such that supp$(f) \subseteq [a_1, b_1] \times [a_2, b_2]$. Additionally, suppose that $\mathbf{p} = (\cos\theta, \sin\theta)$ is a direction vector such that $\mathcal{R}_{\mathbf{p}}\varphi$ is continuous. Define $E = \{\mathbf{k}_1, \ldots, \mathbf{k}_{\#E}\} = \big\{\big[\lceil a_1 - M_1\rceil, \lfloor b_1 - N_1\rfloor\big] \times \big[\lceil a_2 - M_2\rceil, \lfloor b_2 - N_2\rfloor\big]\big\} \cap \mathbb{Z}^2$. If $\widehat{\mathcal{R}_{\mathbf{p}}\varphi} = \widehat{\mathcal{R}_{\mathbf{p}}\varphi}_{\Re} + \boldsymbol{i}\widehat{\mathcal{R}_{\mathbf{p}}\varphi}_{\Im}$ satisfies the following item (i) or (ii), and E satisfies item (iii):*

(i) the real part $\widehat{\mathcal{R}_{\mathbf{p}}\varphi}_{\Re} \not\equiv 0$ and its sign function sgn$(\widehat{\mathcal{R}_{\mathbf{p}}\varphi}_{\Re})$ is unchanged;

(ii) the imaginary part $\widehat{\mathcal{R}_{\mathbf{p}}\varphi}_{\Im} \not\equiv 0$ and its sign function sgn$(\widehat{\mathcal{R}_{\mathbf{p}}\varphi}_{\Im})$ is unchanged;

(iii) if $\#E > 1$ then $\mathbf{pk}_l \neq \mathbf{pk}_n$ for any $l \neq n$;

then the matrix $A_{\varphi,\mathbf{p},X_{\mathbf{p}}}$ defined via (4.1) is invertible where $X_{\mathbf{p}} = \{\mathbf{pk}_1, \ldots, \mathbf{pk}_{\#E}\}$; Consequently, f can be determined uniquely by its SA Radon samples at $X_{\mathbf{p}}$.

Proof. The requirement for the continuity of $\mathcal{R}_{\boldsymbol{p}}\varphi$ in Theorem 4.1 is satisfied here. If the corresponding matrix $A_{\varphi,\boldsymbol{p},X_{\boldsymbol{p}}}$ defined via (4.1) is invertible, then it follows from Theorem 4.1 that f can be determined uniquely by its SA Radon samples at $X_{\boldsymbol{p}} = \{\boldsymbol{pk}_1, \ldots, \boldsymbol{pk}_{\#E}\}$. We next prove that $A_{\varphi,\boldsymbol{p},X_{\boldsymbol{p}}}$ is invertible. For any nonzero vector $(\alpha_1, \ldots, \alpha_{\#E})^T \in \mathbb{C}^{\#E}$ we have

$$\begin{aligned}\sum_{j=1}^{\#E}\sum_{n=1}^{\#E} \alpha_j \overline{\alpha}_n \mathcal{R}_{\boldsymbol{p}}\varphi(\boldsymbol{pk}_j - \boldsymbol{pk}_n) &= \frac{1}{2\pi}\sum_{j=1}^{\#E}\sum_{n=1}^{\#E} \alpha_j \overline{\alpha}_n \int_{\mathbb{R}} \widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}(\xi) e^{\mathtt{i}(\boldsymbol{pk}_j - \boldsymbol{pk}_n)\xi} d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}(\xi) |\sum_{j=1}^{\#E} \alpha_j e^{\mathtt{i}\boldsymbol{pk}_j\xi}|^2 d\xi \quad (5.2\mathrm{A})\end{aligned} \tag{5.2}$$

where (5.2A) is derived from the quadratic form $\sum_{j=1}^{\#E}\sum_{n=1}^{\#E} \alpha_j e^{\mathtt{i}\boldsymbol{pk}_j\xi}\overline{\alpha}_n e^{-\mathtt{i}\boldsymbol{pk}_n\xi} = |\sum_{j=1}^{\#E} \alpha_j e^{\mathtt{i}\boldsymbol{pk}_j\xi}|^2$. By $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}(\xi) = \widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}_{\Re}(\xi) + \mathtt{i}\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}_{\Im}(\xi)$, (5.2) can be further expressed as

$$\sum_{j=1}^{\#E}\sum_{n=1}^{\#E}\alpha_j\overline{\alpha}_n\mathcal{R}_{\boldsymbol{p}}\varphi(\boldsymbol{pk}_j-\boldsymbol{pk}_n) = \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}_{\Re}(\xi)|\sum_{j=1}^{\#E}\alpha_j e^{\mathtt{i}\boldsymbol{pk}_j\xi}|^2 d\xi + \frac{\mathtt{i}}{2\pi}\int_{\mathbb{R}}\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}_{\Im}(\xi)|\sum_{j=1}^{\#E}\alpha_j e^{\mathtt{i}\boldsymbol{pk}_j\xi}|^2 d\xi. \tag{5.3}$$

Since $\varphi\in L^2(\mathbb{R}^2)$ is compactly supported, it follows from Lemma 2.1 that $\mathcal{R}_{\boldsymbol{p}}\varphi$ is compactly supported as well and belongs to $L^2(\mathbb{R})$. Then $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}\in C^\infty(\mathbb{R})$. Item (i) or (ii) implies that $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}\not\equiv 0$. Without loss of generality it is assumed that $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}_{\Re}\not\equiv 0$ and $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}_{\Re}\geq 0$. By the continuity of $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}_{\Re}$ there is $\delta>0$ and $\zeta\in\mathbb{R}$ such that

$$\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}_{\Re}(\xi)>0 \tag{5.4}$$

for any $\xi\in[\zeta-\delta,\zeta+\delta]$. We next prove that $\{e^{\mathtt{i}\boldsymbol{pk}_n\xi}\}_{n=1}^{\#E}$ is linearly independent on $[\zeta-\delta,\zeta+\delta]$. If $\#E=1$ then the linear independence is clear. If $\#E>1$ then it follows from item (iii) that $\boldsymbol{pk}_n\neq\boldsymbol{pk}_j$ for any $n\neq j$. By Lemma 3.2 we have that $\{e^{\mathtt{i}\boldsymbol{pk}_n\xi}\}_{n=1}^{\#E}$ is linearly independent on $[\zeta-\delta,\zeta+\delta]$. Consequently, there exists $\xi_0\in[\zeta-\delta,\zeta+\delta]$ such that for the above nonzero vector $(\alpha_1,\ldots,\alpha_{\#E})^T\in\mathbb{C}^{\#E}$ we have $\sum_{n=1}^{\#E}\alpha_n e^{\mathtt{i}\boldsymbol{pk}_n\xi_0}\neq 0$. By the continuity of the functions in $\{e^{\mathtt{i}\boldsymbol{pk}_n\xi}\}_{n=1}^{\#E}$ we conclude that

$$\int_{\zeta-\delta}^{\zeta+\delta}|\sum_{n=1}^{\#E}\alpha_n e^{\mathtt{i}\boldsymbol{pk}_n\xi}|^2 d\xi>0. \tag{5.5}$$

Now it follows from (5.4) and (5.5) that

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}_{\Re}(\xi)|\sum_{n=1}^{\#E}\alpha_n e^{\mathtt{i}\boldsymbol{pk}_n\xi}|^2 d\xi &\geq \frac{1}{2\pi}\int_{\zeta-\delta}^{\zeta+\delta}\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}_{\Re}(\xi)|\sum_{n=1}^{\#E}\alpha_k e^{\mathtt{i}\boldsymbol{pk}_n\xi}|^2 d\xi\\ &\geq \frac{1}{2\pi}\min_{\xi\in[\zeta-\delta,\zeta+\delta]}\{\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}_{\Re}(\xi)\}\int_{\zeta-\delta}^{\zeta+\delta}|\sum_{n=1}^{\#E}\alpha_n e^{\mathtt{i}\boldsymbol{pk}_n\xi}|^2 d\xi\\ &>0.\end{aligned} \tag{5.6}$$

This combining with (5.3) leads to that

$$\sum_{j=1}^{\#E}\sum_{n=1}^{\#E}\alpha_j\overline{\alpha}_n\mathcal{R}_{\boldsymbol{p}}\varphi(\boldsymbol{pk}_j-\boldsymbol{pk}_n)>0. \tag{5.7}$$

Recall that

$$\sum_{j=1}^{\#E}\sum_{n=1}^{\#E}\alpha_j\overline{\alpha}_n\varphi(\boldsymbol{pk}_j-\boldsymbol{pk}_n)=(\bar{\alpha}_1,\dots,\bar{\alpha}_{\#E})A_{\varphi,\mathbf{p},X_{\mathbf{p}}}(\alpha_1,\dots,\alpha_{\#E})^T \tag{5.8}$$

and $(\alpha_1,\dots,\alpha_{\#E})^T\in\mathbb{C}^{\#E}$ is an arbitrary nonzero vector. Now it follows from (5.7) and (5.8) that $A_{\varphi,\mathbf{p},X_{\mathbf{p}}}$ is invertible. By Theorem 4.1, f can be determined uniquely by its SA Radon samples at $\{\boldsymbol{pk}_1,\dots,\boldsymbol{pk}_{\#E}\}$. □

5.3. Preliminary on positive (semi-)definite function

The positive semi-definite function has been defined in Definition 1.1. The celebrated result on positive semi-definite functions is their characterization in terms of Fourier transform, which was established by Bochner [5]. It is as follows.

Lemma 5.2. *A continuous function $\phi:\mathbb{R}^d\longrightarrow\mathbb{C}$ is positive semi-definite if and only if it is the Fourier transform of a finite nonnegative Borel measure μ on $\mathbb{R}^d$ such that $\phi(\mathbf{x})=\int_{\mathbb{R}^d}e^{-i\mathbf{x}\cdot\xi}d\mu(\xi)$.*

Based on Lemma 5.2, Wendland [42, Theorem 6.11] established the following tool for checking the positive definite property, which will be used in Theorems 5.4 and 5.6 for SACT sampling.

Lemma 5.3. *Suppose that $\phi\in L^1(\mathbb{R}^d)$ is continuous. Then ϕ is positive definite if and only if ϕ is bounded and its Fourier transform $\widehat{\phi}$ is nonnegative and nonvanishing. Here $\widehat{\phi}$ being nonvanishing means that $\int_{\mathbb{R}^d}\widehat{\phi}(\xi)d\xi=(2\pi)^{d/2}\phi(\boldsymbol{0})\neq 0$.*

The following remark concerns on the determination of functions by the positive definite property.

Remark 5.1. If $\phi:\mathbb{R}^d\longrightarrow\mathbb{C}$ is positive definite and continuous, then the system $\{\phi(\cdot-\mathbf{x}_k)\}_{k=1}^N$ is linearly independent for any set $X=\{\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_N\}\subseteq\mathbb{R}^d$. Moreover, any function $f=\sum_{k=1}^N c_k\phi(\cdot-\mathbf{x}_k)$ can be determined uniquely by its samples at X.

Proof. If $\{\phi(\cdot-\mathbf{x}_k)\}_{k=1}^N$ is linearly dependent then there exists a nonzero vector $(\alpha_1,\dots,\alpha_N)\in\mathbb{C}^N$ such that $\sum_{k=1}^N\alpha_k\phi(\cdot-\mathbf{x}_k)\equiv 0$. Particularly, for any $\mathbf{x}_j\in\{\mathbf{x}_1,\dots,\mathbf{x}_N\}$ we have $\sum_{k=1}^N\alpha_k\phi(\mathbf{x}_j-\mathbf{x}_k)=0$. Then the quadratic form

$$\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_j\overline{\alpha}_k\phi(\mathbf{x}_j-\mathbf{x}_k)=0.$$

This contradicts with the positive definite property

$$\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_j\overline{\alpha}_k\phi(\mathbf{x}_j-\mathbf{x}_k)>0. \tag{5.9}$$

Therefore, $\{\phi(\cdot - \mathbf{x}_k)\}_{k=1}^N$ is linearly independent. Additionally,

$$\begin{pmatrix} \phi(\mathbf{0}) & \phi(\mathbf{x}_1 - \mathbf{x}_2) & \cdots & \phi(\mathbf{x}_1 - \mathbf{x}_N) \\ \phi(\mathbf{x}_2 - \mathbf{x}_1) & \phi(\mathbf{0}) & \cdots & \phi(\mathbf{x}_2 - \mathbf{x}_N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\mathbf{x}_N - \mathbf{x}_1) & \phi(\mathbf{x}_N - \mathbf{x}_2) & \cdots & \phi(\mathbf{0}) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} f(\mathbf{x}_1) \\ f(\mathbf{x}_2) \\ \vdots \\ f(\mathbf{x}_N) \end{pmatrix}. \tag{5.10}$$

Since ϕ is positive definite, it follows from (5.9) that the above matrix is invertible. Then the coefficient vector $(c_1, c_2, \ldots, c_N)^T$ can be determined by the samples $f(\mathbf{x}_1), f(\mathbf{x}_2), \ldots, f(\mathbf{x}_N)$. Recall that $\{\phi(\cdot - \mathbf{x}_k)\}_{k=1}^N$ is linearly independent. With $(c_1, \ldots, c_N)^T$ at hand, f can be determined uniquely. □

5.4. The fourth main result: pairs of $(\varphi, \mathbf{p})$ such that $\{\mathbf{pk}_1, \ldots, \mathbf{pk}_{\#E}\}$ is eligible for SACT sampling, where φ is positive definite and nonvanishing

The following is the main result in this subsection. It applies to the case that φ is positive definite and nonvanishing ($\widehat{\varphi}(\mathbf{0}) \neq 0$).

Theorem 5.4. *Suppose that $\varphi \in C(\mathbb{R}^2)$ is compactly supported and positive definite such that its Sobolev smoothness $\nu_2(\varphi) > 1/2$, $\widehat{\varphi}(\mathbf{0}) > 0$ and supp$(\varphi) \subseteq [N_1, M_1] \times [N_2, M_2]$. Moreover, the arbitrary source function $f \in V(\varphi, \mathbb{Z}^2)$ is compactly supported such that supp$(f) \subseteq [a_1, b_1] \times [a_2, b_2]$. As previously, define $E = \{\mathbf{k}_1, \ldots, \mathbf{k}_{\#E}\} = \big\{ \big[\lceil a_1 - M_1 \rceil, \lfloor b_1 - N_1 \rfloor\big] \times \big[\lceil a_2 - M_2 \rceil, \lfloor b_2 - N_2 \rfloor\big] \big\} \cap \mathbb{Z}^2$, and correspondingly*

$$E^+ = \begin{cases} \emptyset, & \#E = 1, \\ \{\mathbf{x} - \mathbf{y} : \mathbf{x} \neq \mathbf{y} \in E\}, & \#E > 1. \end{cases} \tag{5.11}$$

Then f can be determined uniquely by its SA Radon (w.r.t. $\mathbf{p}$) samples at $\{\mathbf{pk}_1, \ldots, \mathbf{pk}_{\#E}\}$, where $\mathbf{p}$ is an arbitrary direction vector from $\{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi)\} \setminus dv_{\mathcal{N}_{E^+}}$ with $\mathcal{N}_{E^+}$ defined in Definition 4.1.

Proof. Recall that it has been proved in the proof of Theorem 4.3 that $\{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi)\} \setminus \mathrm{dv}_{\mathcal{N}_{E^+}}$ is not empty. Next we prove the following three items.

(1) If $\#E > 1$ then for any $l \neq n \in \{1, \ldots, \#E\}$ and any direction vector $\boldsymbol{p} \in \{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi)\} \setminus \mathrm{dv}_{\mathcal{N}_{E^+}}$, we have $\boldsymbol{pk}_l \neq \boldsymbol{pk}_n$.

(2) For any direction vector $\boldsymbol{p}$, the Radon transform $\mathcal{R}_{\boldsymbol{p}}\varphi$ is continuous.

(3) Suppose that $\boldsymbol{p}$ is any fixed direction vector. Then we have $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi} \not\equiv 0$ and $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi} \geq 0$. Clearly, if the above three items are satisfied then the requirements in Theorem 5.1 are satisfied for any direction vector $\boldsymbol{p} \in \{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi)\} \setminus \mathrm{dv}_{\mathcal{N}_{E^+}}$. Consequently, it follows from Theorem 5.1 that f can be determined uniquely by its SA Radon (w.r.t. $\boldsymbol{p}$) samples at $\{\boldsymbol{pk}_1, \ldots, \boldsymbol{pk}_{\#E}\}$.

We first prove (1). One can check that, for any $l \neq n \in \{1, 2, \ldots, \#E\}$ it holds that $\boldsymbol{p}\boldsymbol{k}_l \neq \boldsymbol{p}\boldsymbol{k}_n$ if and only if $\boldsymbol{p} \notin \mathcal{N}_{E^+}$. Then for any $\boldsymbol{p} \in \{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi)\} \setminus \mathrm{dv}_{\mathcal{N}_{E^+}}$, item (1) holds.

Next we prove item (2). Recall that $\nu_2(\varphi) > 1/2$. Then it follows from Proposition 2.3 (2) that $\mathcal{R}_{\boldsymbol{p}}\varphi$ is continuous.

Finally, we need to prove item (3). Since $\varphi \in C(\mathbb{R}^2)$ is positive definite, by Lemma 5.3 we have $\widehat{\varphi} \geq 0$. Now for any direction vector $\boldsymbol{p} = (\cos\theta, \sin\theta)$ we have that $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi} = \widehat{\varphi}(\boldsymbol{p}^T\cdot) \geq 0$. Additionally, $\varphi \in C(\mathbb{R}^2)$ is compactly supported then $\widehat{\varphi} \in C^\infty(\mathbb{R}^2)$. This together with $\widehat{\varphi}(\mathbf{0}) = \int_{\mathbb{R}^2} \varphi(\boldsymbol{x})d\boldsymbol{x} > 0$ leads to that exists a closed disc $\mathcal{D}(\mathbf{0}, \delta) = \{\xi \in \mathbb{R}^2 : \|\xi\|_2 \leq \delta\}$ such that for any $\xi \in \mathcal{D}(\mathbf{0}, \delta)$ we have $\widehat{\varphi}(\xi) > 0$. For any $\gamma \in \mathbb{R}$ such that $|\gamma| \leq \delta$ we have $\gamma\boldsymbol{p}^T \in U(\mathbf{0}, \delta)$ and consequently $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}(\gamma) = \widehat{\varphi}(\gamma\boldsymbol{p}^T) > 0$. That is, $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi} \not\equiv 0$.

Now by Theorem 5.1 the source function f can be determined uniquely by its SA Radon samples at $\{\boldsymbol{p}\boldsymbol{k}_1, \ldots, \boldsymbol{p}\boldsymbol{k}_{\#E}\}$. The proof is completed. □

Remark 5.2. As addressed in item (3) of the proof of Theorem 5.4, the nonvanishing property $\widehat{\varphi}(\mathbf{0}) = \int_{\mathbb{R}^2} \varphi(\boldsymbol{x})d\boldsymbol{x} > 0$ guarantees that $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi} \not\equiv 0$ for any direction vector $\boldsymbol{p}$. Such a property brings great flexibility of $\boldsymbol{p}$ for the SACT sampling. Next we introduce a class of box-splines which are positive and nonvanishing.

The mth cardinal B-spline B_m is defined by $B_m := \overbrace{\chi_{(0,1]} \star \ldots \star \chi_{(0,1]}}^{m \text{ copies}}$ (cf. [39,41]), where $m \in \mathbb{N}$, as in Remark 2.2 $\chi_{(0,1]}$ is the characteristic function of $(0, 1]$ and $\star$ is the convolution. Through the simple calculation (cf. [6]) we have $\mathrm{supp}(B_m) = (0, m]$, and

$$\widehat{B_m}(\xi) = e^{-\mathrm{i}m\xi/2}[\frac{\sin\xi/2}{\xi/2}]^m. \tag{5.12}$$

Remark 5.3. For $s < m - 1/2$, one can check that $\int_{\mathbb{R}} |\widehat{B_m}(\xi)|^2(1+\xi^2)^s d\xi < \infty$. Then the Sobolev smoothness $\nu_2(B_m) = m - 1/2$. By Proposition 2.3, B_m is continuous for $m \geq 2$.

Proposition 5.5. *Through the tensor product we define the box-spline $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $\varphi(x_1, x_2) = \prod_{k=1}^2 B_{2n_k}(x_k + n_k)$, $n_k \in \mathbb{N}$. Then φ is compactly supported, continuous and positive definite such that $\widehat{\varphi}(\mathbf{0}) = \int_{\mathbb{R}^2} \varphi(\mathbf{x})d\mathbf{x} > 0$. Moreover, the Sobolev smoothness $\nu_2(\varphi) \geq 1$. Consequently, it satisfies the requirement in Theorem 5.4.*

Proof. By Remark 5.3, both B_{2n_1} and B_{2n_2} are continuous. So are $B_{2n_1}(\cdot + n_1)$ and $B_{2n_2}(\cdot + n_2)$. Then their tensor product φ is also continuous. Through the direct calculation one can check that $\widehat{\varphi}(\xi_1, \xi_2) = \prod_{k=1}^2 \left[\frac{\sin\xi_k/2}{\xi_k/2}\right]^{2n_k}$, and it follows from $\mathrm{supp}(B_{2n_i}) = (0, 2n_i]$ that $\mathrm{supp}(\varphi) = (-n_1, n_1] \times (-n_2, n_2]$. Clearly, $\widehat{\varphi} \geq 0$ and $\widehat{\varphi}(\mathbf{0}) = \int_{\mathbb{R}^2} \varphi(\boldsymbol{x})d\boldsymbol{x} = 1$. As in (5.2A), for any nonzero $(\alpha_1, \ldots, \alpha_N) \in \mathbb{C}^N$ and any set $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subseteq \mathbb{R}^2$ one can check that

$$\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_j\overline{\alpha}_k\varphi(\mathbf{x}_j-\mathbf{x}_k) \quad = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2}\widehat{\varphi}(\xi)|\sum_{k=1}^{N}\alpha_k e^{\mathbf{i}\mathbf{x}_k\cdot\xi}|^2 d\xi. \tag{5.13}$$

From this and $\widehat{\varphi}\geq 0$ we have that φ is positive semi-definite. By Lemma 3.2, the set of continuous functions $\{e^{\mathbf{i}\mathbf{x}_k\cdot\xi}: k=1,\dots,N\}$ are linearly independent. Since $\varphi\in C(\mathbb{R}^2)$ is compactly supported then $\widehat{\varphi}\in C^\infty(\mathbb{R}^2)$. Now combining the continuities of $\widehat{\varphi}$ and $\{e^{\mathbf{i}\mathbf{x}_k\cdot\xi}: k=1,\dots,N\}$, the above linear independence and $\widehat{\varphi}(\mathbf{0})>0$, through the similar procedures in (5.5) and (5.6) one can prove the integral in (5.13) is positive. Consequently, φ is positive definite.

In what follows, we prove that $\nu_2(\varphi)\geq 1$. First, we have

$$\begin{aligned}\int\limits_{\mathbb{R}^2}|\widehat{\varphi}(\xi_1,\xi_2)|^2(1+\xi_1^2+\xi_2^2)d\xi_1 d\xi_2 \quad &\leq \int\limits_{|\xi_1|\leq 1,\xi_2\in\mathbb{R}}|\widehat{\varphi}(\xi_1,\xi_2)|^2(1+\xi_1^2+\xi_2^2)d\xi_1 d\xi_2\\ &+\int\limits_{|\xi_2|\leq 1,\xi_1\in\mathbb{R}}|\widehat{\varphi}(\xi_1,\xi_2)|^2(1+\xi_1^2+\xi_2^2)d\xi_1 d\xi_2\\ &+\int\limits_{|\xi_1|>1,\xi_2>1}|\widehat{\varphi}(\xi_1,\xi_2)|^2(1+\xi_1^2+\xi_2^2)d\xi_1 d\xi_2\\ &:= I_1+I_2+I_3.\end{aligned} \tag{5.14}$$

We first estimate I_1 as follows,

$$\begin{aligned} I_1 \quad &= \int\limits_{|\xi_1|\leq 1,\xi_2\in\mathbb{R}}|\widehat{\varphi}(\xi_1,\xi_2)|^2(1+\xi_1^2+\xi_2^2)d\xi_1 d\xi_2\\ &= \int\limits_{|\xi_1|\leq 1,|\xi_2|\leq\sqrt{2}}|\widehat{\varphi}(\xi_1,\xi_2)|^2(1+\xi_1^2+\xi_2^2)d\xi_1 d\xi_2\\ &+\int\limits_{|\xi_1|\leq 1,|\xi_2|>\sqrt{2}}|\widehat{\varphi}(\xi_1,\xi_2)|^2(1+\xi_1^2+\xi_2^2)d\xi_1 d\xi_2\\ &:= I_{11}+I_{12}.\end{aligned} \tag{5.15}$$

Recall that $\widehat{\varphi}\in C^\infty(\mathbb{R}^2)$. By the continuity of $\widehat{\varphi}$ we have $I_{11}<\infty$. Moreover,

$$\begin{aligned} I_{12} \quad &= \int\limits_{|\xi_1|\leq 1,|\xi_2|>\sqrt{2}}|\widehat{\varphi}(\xi_1,\xi_2)|^2(1+\xi_1^2+\xi_2^2)d\xi_1 d\xi_2\\ &\leq 2\int\limits_{|\xi_1|\leq 1}\left[\frac{\sin\xi_1/2}{\xi_1/2}\right]^{2n_1}d\xi_1\int\limits_{|\xi_2|>1}\left[\frac{\sin\xi_2/2}{\xi_2/2}\right]^{4n_2}\xi_2^2 d\xi_2 \quad (5.16\text{A})\\ &<\infty,\end{aligned} \tag{5.16}$$

where (5.16A) is from $1+\xi_1^2 \le \xi_2^2$. Consequently, $I_1 = I_{11} + I_{12} < \infty$. Similarly, one can prove that $I_2 < \infty$. Additionally,

$$\begin{aligned} I_3 &\le 2 \int\limits_{|\xi_1|>1,\xi_2>1} |\widehat{\varphi}(\xi_1,\xi_2)|^2(\xi_1^2+\xi_2^2)d\xi_1 d\xi_2 \\ &\le 2\times 2^{2(n_1+n_2)}\Big[\int\limits_{|\xi_1|>1,\xi_2>1} \frac{1}{\xi_1^{4n_1-2}\xi_2^{4n_1}}d\xi_1 d\xi_2 + \int\limits_{|\xi_1|>1,\xi_2>1} \frac{1}{\xi_2^{4n_1-2}\xi_1^{4n_1}}d\xi_1 d\xi_2\Big] \\ &< \infty. \end{aligned} \tag{5.17}$$

Now by (5.14), we have $\int_{\mathbb{R}^2} |\widehat{\varphi}(\xi_1,\xi_2)|^2(1+\xi_1^2+\xi_2^2)d\xi_1 d\xi_2 < \infty$, and consequently, $\nu_2(\varphi) \ge 1$. This completes the proof. □

Example 5.1. As mentioned previously, the SISs generated from box splines are used in [8] to model the continuous-domain representations of biomedical images. Motivated by this, we check the single-angle Radon samples-based recovery result in Theorem 5.4 for the function in a SIS generated from a positive definite box spline. Let $\phi_B(x_1,x_2) = B_2(x_1-1)B_2(x_2-1)$ where $B_2 = \chi_{(0,1]} \star \chi_{(0,1]}$ is the cardinal B-spline of order 2. By Proposition 5.5, ϕ_B is positive definite. By (5.12) we have $\widehat{B_2}(\xi) = e^{-\mathrm{i}\xi}(\frac{\sin\xi/2}{\xi/2})^2$. From this, $\widehat{\phi_B}(\mathbf{0}) = 1$. That is, ϕ_B is nonvanishing. For any fixed direction vector $\boldsymbol{p} = [\cos\theta, \sin\theta]$, its Radon transform $\mathcal{R}_{\boldsymbol{p}}\phi_B$ can be calculated directly from (2.4). Without loss of generality, we choose $\boldsymbol{p} = [\cos\theta,\sin\theta]$ such that $0<\theta<\pi/2$ and $\tan\theta \ge 2$. From (2.4) we calculate that

$$\mathcal{R}_{\boldsymbol{p}}\phi_B(x) = \begin{cases} \frac{(\tan\theta - \frac{x}{\cos\theta})\left[(\frac{x}{\cos\theta}-\tan\theta-\frac{3}{2})^2+\frac{3}{4}\right]+1}{6\cos\theta\tan^2\theta}, & x\in(\sin\theta,\sin\theta+\cos\theta) \\ \frac{3(\frac{x}{\cos\theta}-\tan\theta)^2+(\frac{x}{\cos\theta}-\tan\theta)^3+3\tan\theta-\frac{3x}{\cos\theta}+1}{6\cos\theta\tan^2\theta}, & x\in(\sin\theta-\cos\theta,\sin\theta] \\ \frac{\tan\theta-\frac{x}{\cos\theta}}{\cos\theta\tan^2\theta}, & x\in[\cos\theta,\sin\theta-\cos\theta] \\ \frac{\frac{x}{\cos\theta}(\frac{2x^2}{\cos^2\theta}-\frac{6x}{\cos\theta}+3)+6\tan\theta-\frac{3x}{\cos\theta}-2}{6\cos\theta\tan^2\theta}, & x\in[0,\cos\theta) \\ \mathcal{R}_{\boldsymbol{p}}\phi_B(-x), & x\in(-\cos\theta,0] \\ \mathcal{R}_{\boldsymbol{p}}\phi_B(-x), & x\in[-\sin\theta+\cos\theta,-\cos\theta] \\ \mathcal{R}_{\boldsymbol{p}}\phi_B(-x), & x\in[-\sin\theta,-\sin\theta+\cos\theta) \\ \mathcal{R}_{\boldsymbol{p}}\phi_B(-x), & x\in(-\sin\theta-\cos\theta,-\sin\theta) \\ 0, & \text{else}. \end{cases} \tag{5.18}$$

Without bias, we choose a source function

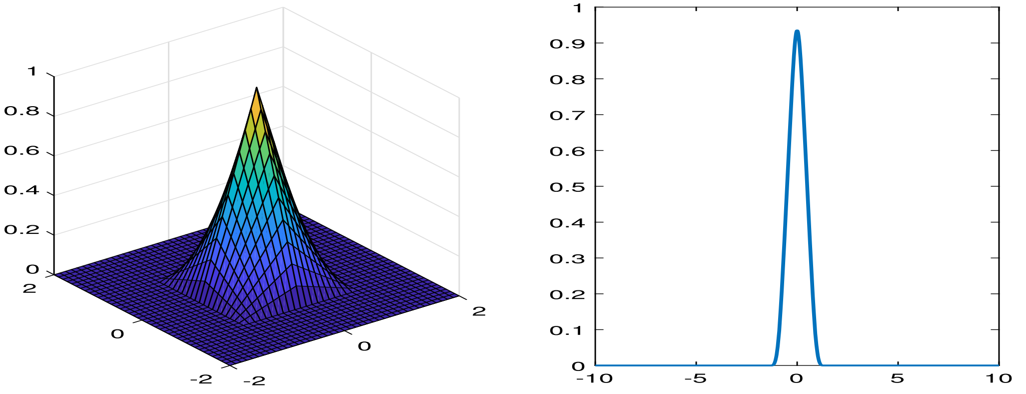

Fig. 5.1. Left: the plot of ϕ_B. Right: the plot of $\mathcal{R}_{\boldsymbol{p}}\phi_B(x)$ with $\boldsymbol{p} = [\cos(1.2208), \sin(1.2208)]$.

$$f = \sum_{\boldsymbol{k}=(i,j)\in\{0,1,2,3,4\}^2} c_{\boldsymbol{k}}\phi_B(\cdot - \boldsymbol{k}) \in V(\phi_B), \tag{5.19}$$

where the coefficient matrix

$$C = \left(c_{(i,j)}\right)_{i,j=0}^{4} = \begin{pmatrix} 0.5377 & -1.3077 & -1.3499 & -0.2050 & 0.6715 \\ 1.8339 & -0.4336 & 3.0349 & -0.1241 & -1.2075 \\ -2.2588 & 0.3426 & 0.7254 & 1.4897 & 0.7172 \\ 0.8622 & 3.5784 & -0.0631 & 1.4090 & 1.6302 \\ 0.3188 & 2.7694 & 0.7147 & 1.4172 & 0.4889 \end{pmatrix}. \tag{5.20}$$

By (5.1) we have

$$\mathcal{R}_{\boldsymbol{p}}f = \sum_{l=1}^{25} c_{\boldsymbol{k}_l}\mathcal{R}_{\boldsymbol{p}}\phi_B(\cdot - \boldsymbol{pk}_l) \tag{5.21}$$

where $\{\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{25}\} = \{0, 1, 2, 3, 4\}^2$ is arranged in the lexicographical order. By Theorem 5.4, f can be determined exactly by its single-angle Radon samples $\{\mathcal{R}_{\boldsymbol{p}}f(\boldsymbol{pk}_l) : l = 1, \ldots, 25\}$ if and only if

$$\boldsymbol{pk}_l \neq \boldsymbol{pk}_n \text{ for any } l \neq n. \tag{5.22}$$

Without bias, we choose $\theta = 1.2208$ such that (5.22) holds. The function f (or the sequence $\{c_{\boldsymbol{k}_l}\}_{l=1}^{25}$) can be determined by the equation system (4.5) with $X = \{\boldsymbol{pk}_1, \ldots, \boldsymbol{pk}_{25}\}$. We found that the recovery error is

$$\text{error} = \frac{\|\{c_{\boldsymbol{k}_l} - \widehat{c}_{\boldsymbol{k}_l}\}_{l=1}^{25}\|_2}{\|\{c_{\boldsymbol{k}_l}\}_{l=1}^{25}\|_2} = 3.1206e - 13 \tag{5.23}$$

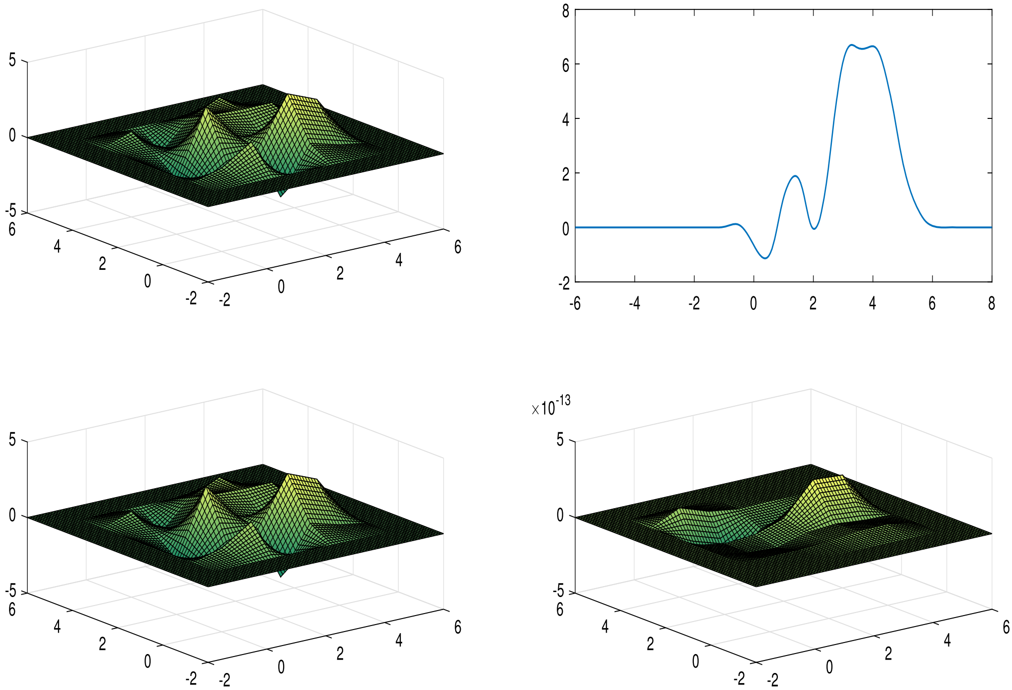

Fig. 5.2. Top left: the plot of f; Top right: the plot of $\mathcal{R}_{\mathbf{p}}f$; Bottom left: the plot of recovery version $\tilde{f}$ of f; Bottom right: the plot of $f - \tilde{f}$.

where $\{\widehat{c}_{\mathbf{k}_l}\}_{l=1}^{25}$ is the recovery version of $\{c_{\mathbf{k}_l}\}_{l=1}^{25}$. The graphs of ϕ_B and $\mathcal{R}_{\mathbf{p}}\phi_B$ are plotted in Fig. 5.1, and the graphs f, $\mathcal{R}_{\mathbf{p}}f$, the recovery version $\tilde{f}$ and $f - \tilde{f}$ are plotted in Fig. 5.2. From Fig. 5.2, f can be recovered by its single-angle Radon samples.

5.5. The fifth main result: pairs of $(\varphi, \mathbf{p})$ such that $\{\mathbf{pk}_1, \ldots, \mathbf{pk}_{\#E}\}$ is eligible for SACT sampling, where φ is positive definite and vanishing

It follows from Lemma 5.3 that for a continuous positive definite function φ, its Fourier transform $\widehat{\varphi}$ is necessarily nonvanishing, namely, $\varphi(\mathbf{0}) \neq 0$. But φ itself is not necessarily nonvanishing, namely, $\widehat{\varphi}(\mathbf{0}) \neq 0$ does not necessarily hold. That is, there exist positive definite and vanishing functions. We next provide an example to explain this. It is the motivation for this subsection.

5.5.1. A motivation example

Example 5.2. Let

$$\phi_1(x_1) = B_2(x_1+1) = \begin{cases} x_1 + 1, & -1 < x_1 \leq 0, \\ 1 - x_1, & 0 < x_1 < 1, \\ 0, & |x_1| \geq 1. \end{cases} \tag{5.24}$$

By (5.12) we have $\widehat{\phi_1}(\xi_1) = (\frac{\sin \xi_1/2}{\xi_1/2})^2$. Define φ_1 via

$$\widehat{\varphi_1}(\xi_1) = (\frac{e^{\mathtt{i}\xi_1/2} - e^{-\mathtt{i}\xi_1/2}}{2\mathtt{i}})^2 \widehat{\phi}_1(\xi_1/2) = \sin^2(\xi_1/2)(\frac{\sin(\xi_1/4)}{\xi_1/4})^2 \geq 0.$$

Additionally, in the time-domain $\varphi_1(x_1) = -\frac{1}{2}\phi_1(2x_1+1) + \phi_1(2x_1) - \frac{1}{2}\phi_1(2x_1-1)$. It is straightforward to check that φ_1 is continuous and bounded, and $\varphi_1(0) = 1$. By Lemma 5.3, φ_1 is positive definite. But it is clear that $\widehat{\varphi_1}(0) = 0$. Now through the tensor product we define

$$\varphi(x_1, x_2) = \varphi_1(x_1)\varphi_2(x_2). \tag{5.25}$$

Using Lemma 5.3 again, one can check that φ is also positive definite. But $\widehat{\varphi}(\mathbf{0}) = 0$. That is, φ is vanishing.

As summarized in Remark 5.2 the nonvanishing property is key in Theorem 5.4 for providing great flexibility for the choice of direction vector $\boldsymbol{p}$. On the other hand, Example 5.2 confirms the existence of positive definite but vanishing functions, and such functions do not reach the requirement of Theorem 5.4. As such, for the vanishing case we need to address what direction vector $\boldsymbol{p}$ is eligible for the SACT sampling.

5.5.2. The SACT sampling result when φ is positive definite and vanishing

Now it is ready to establish the fifth main result in the following Theorem 5.6. On the generator, the difference between the Theorem 5.6 and Theorem 5.4 is that the generator φ here is vanishing here, namely, $\widehat{\varphi}(\mathbf{0}) = 0$ while that in Theorem 5.4 is nonvanishing. The following definition will be necessary for Theorem 5.6.

Definition 5.1. Let $\varphi : \mathbb{R}^2 \to \mathbb{C}$ be positive definite and compactly supported. For $\mathbf{x}_0 \in \mathbb{R}^2$ such that $\widehat{\varphi}(\mathbf{x}_0) > 0$, as in Definition 4.2, $\delta^{\widehat{\varphi}}_{\mathbf{x}_0,\max} \in (0, \infty]$ is supposed to be the maximum value such that $\widehat{\varphi}(\boldsymbol{x}) > 0$ for any $\boldsymbol{x} \in \mathring{\mathcal{D}}(\mathbf{x}_0, \delta^{\widehat{\varphi}}_{\mathbf{x}_0,\max})$. Denote the nonzero set of $\widehat{\varphi}$ by $\mathcal{G}_{\widehat{\varphi}}$ such that $\widehat{\varphi}(\boldsymbol{x}) > 0$ for any $\boldsymbol{x} \in \mathcal{G}_{\widehat{\varphi}}$. As in Definition 4.3, define

$$\mathrm{DV}_{\widehat{\varphi}} = \bigcup_{\boldsymbol{x} \in \mathcal{G}_{\widehat{\varphi}}} \mathrm{dv}_{\mathring{\mathcal{D}}(\mathbf{x}, \delta^{\widehat{\varphi}}_{\mathbf{x},\max})}, \tag{5.26}$$

where $\mathrm{dv}_{\mathring{\mathcal{D}}(\mathbf{x}, \delta^{\widehat{\varphi}}_{\mathbf{x},\max})}$ is defined via Definition 4.2.

Theorem 5.6. *Suppose that $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{C}$ is compactly supported, continuous, positive and vanishing such that supp$(\varphi) \subseteq [N_1, M_1] \times [N_2, M_2]$, its Sobolev smoothness $\nu_2(\varphi) > 1/2$ and $\widehat{\varphi}(\boldsymbol{0}) = 0$. Moreover, $f \in V(\varphi, \mathbb{Z}^2)$ is an arbitrary source function such that supp$(f) \subseteq [a_1, b_1] \times [a_2, b_2]$. As previously, define $E = \{\mathbf{k}_1, \dots, \mathbf{k}_{\#E}\} = \{[\lceil a_1 - M_1\rceil, \lfloor b_1 - N_1\rfloor] \times [\lceil a_2 - M_2\rceil, \lfloor b_2 - N_2\rfloor]\} \cap \mathbb{Z}^2$, and*

$$E^+ = \begin{cases} \emptyset, & \#E = 1, \\ \{\mathbf{x} - \mathbf{y} : \mathbf{x} \neq \mathbf{y} \in E\}, & \#E > 1. \end{cases} \tag{5.27}$$

Then f can be determined uniquely by its SA Radon (w.r.t. $\mathbf{p}$) samples at $\{\mathbf{pk}_1, \ldots, \mathbf{pk}_{\#E}\}$, where $\mathbf{p}$ is an arbitrary direction vector from $DV_{\widehat{\varphi}} \setminus dv_{\mathcal{N}_{E^+}}$, with $dv_{\mathcal{N}_{E^+}}$ defined in Definition 4.1.

Proof. It has been proved in the proof of Theorem 4.5 that $\mathrm{DV}_{\widehat{\varphi}} \setminus \mathrm{dv}_{\mathcal{N}_{E^+}}$ is not empty. Since the only difference between the generator φ here and that in Theorem 5.4 is the vanishing property $\widehat{\varphi}(\mathbf{0}) = 0$, we simplify the proof and focus on something related to the difference. Firstly, since $\nu_2(\varphi) > 1/2$ then it follows from Proposition 2.3 (2) that $\mathcal{R}_{\boldsymbol{p}}\varphi$ is continuous for any $\boldsymbol{p}$. Secondly, for the case that $\#E > 1$ as in the proof of Theorem 5.4 one can check that for any direction vector $\boldsymbol{p} \in \{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi)\} \setminus \mathrm{dv}_{\mathcal{N}_{E^+}}$, we have $\boldsymbol{pk}_l \neq \boldsymbol{pk}_n$ for any $l \neq n \in \{1, \ldots, \#E\}$. Then item (iii) of Theorem 5.1 holds. Now we focus on the proof that $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi} \geq 0$ and $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi} \not\equiv 0$ for any $\boldsymbol{p} \in \mathrm{DV}_{\widehat{\varphi}}$. By $\widehat{\varphi} \geq 0$ we have $\widehat{\mathcal{R}_{\boldsymbol{p}}\varphi}(\xi) = \widehat{\varphi}(\boldsymbol{p}^T\xi) \geq 0$ for any $\boldsymbol{p} \in \mathrm{DV}_{\widehat{\varphi}}$. Since $\boldsymbol{p} \in \mathrm{DV}_{\widehat{\varphi}}$, it follows from Proposition 4.2 (2) that $\mathcal{R}_{\boldsymbol{p}}\varphi \not\equiv 0$. Then item (i) of Theorem 5.1 holds. Now by Theorem 5.1, the proof is completed. □

Data availability

No data was used for the research described in the article.

Acknowledgments

The authors thank both the editor and anonymous reviewer for their comments which helped improve the quality of this paper.

References

[1] A. Aldroubi, K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, SIAM Rev. 43 (2001) 585–620.

[2] A. Aldroubi, Q. Sun, W.S. Tang, Convolution, average sampling, and a Calderon resolution of the identity for shift-invariant spaces, J. Fourier Anal. Appl. 11 (2005) 215–244.

[3] N. Antipa, G. Kuo, R. Heckel, B. Mildenhall, E. Bostan, R. Ng, L. Waller, DiffuserCam: lensless single-exposure 3D imaging, Optica 5 (2018) 1–9.

[4] A. Bhandari, A.I. Zayed, Shift-invariant and sampling spaces associated with the fractional Fourier transform domain, IEEE Trans. Signal Process. 60 (2012) 1627–1637.

[5] S. Bochner, Monotone Funktionen, Stieltjes Integrale und harmonische Analyse, Math. Ann. 108 (1933) 378–410.

[6] C.K. Chui, An Introduction to Wavelets, Academic Press, 1992.

[7] T. Derikvand, R.A. Kamyabi-Gol, M. Janfada, The Radon transform on function spaces related to homogenous spaces, arXiv:1702.06432, 2017.

[8] A. Entezari, M. Nilchian, M. Unser, A box spline calculus for the discretization of computed tomography reconstruction problems, IEEE Trans. Med. Imaging 31 (2012) 1532–1541.

[9] S. Fan, S. S-Dryden, J. Zhao, S. Gausmann, A. Schülzgen, G. Li, B. Saleh, Optical fiber refractive index profiling by iterative optical diffraction tomography, J. Lightwave Technol. 36 (2018) 5754–5763.

[10] S. Fan, S. S-Dryden, G. Li, B. Saleh, Reconstructing complex refractive-index of multiply-scattering media by use of iterative optical diffraction tomography, Opt. Express 28 (2020) 6846–6858.
[11] R.S. Feris, V. Krueger, R.M.C. Junior, A wavelet subspace method for real-time face tracking, Real-Time Imaging 10 (2004) 339–350.
[12] S. Flynn, Injectivity of the Heisenberg X-ray transform, J. Funct. Anal. 280 (2021) 108886.
[13] P.M. Gauthier, Lectures on Several Complex Variables, Springer, Cham, 2014.
[14] K. Hamm, J. Ledford, On the structure and interpolation properties of quasi shift-invariant spaces, J. Funct. Anal. 274 (2018) 1959–1992.
[15] B. Han, Z. Shen, Dual wavelet frames and Riesz bases in Sobolev spaces, Constr. Approx. 29 (2009) 369–406.
[16] B. Han, The projection method for multidimensional framelet and wavelet analysis, Math. Model. Nat. Phenom. 7 (2012) 32–59.
[17] B. Han, Projectable multidimensional refinable functions and biorthogonal wavelets, Appl. Comput. Harmon. Anal. 13 (2002) 89–102.
[18] B. Han, Framelets and Wavelets: Algorithms, Analysis, and Applications, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, Cham, 2017, xxxiii +724 pp.
[19] S. Helgason, Integral Geometry and Radon Transform, Springer, New York, 2011.
[20] A. Hinrichs, J. Vybíral, On positive positive-definite functions and Bochner's Theorem, J. Complex. 27 (2011) 264–272.
[21] A. Homan, H. Zhou, Injectivity and stability for a generic class of generalized Radon transforms, J. Geom. Anal. 27 (2017) 1515–1529.
[22] R. Horisaki, K. Fujii, J. Tanida, Diffusion-based single-shot diffraction tomography, Opt. Lett. 44 (2019) 1964–1967.
[23] J. Ilmavirta, On Radon transforms on tori, J. Fourier Anal. Appl. 21 (2015) 370–382.
[24] A.C. Kak, M. Slaney, Principles of Computerized Tomographic Imaging, SIAM, Philadelphia, PA, 2001.
[25] N. Karimi, S. Kazem, D. Ahmadian, H. Adibi, L.V. Ballestra, On a generalized Gaussian radial basis function: analysis and applications, Eng. Anal. Bound. Elem. 112 (2020) 46–57.
[26] M.K. Kim, Principles and techniques of digital holographic microscopy, SPIE Rev. 1 (2010) 1–51.
[27] Y. Li, D. Han, S. Yang, G. Huang, Nonuniform sampling and approximation in Sobolev space from the perturbation of framelet system, Sci. China Math. 64 (2021) 351–372.
[28] Y. Li, S. Yang, D. Yuan, Bessel multiwavelet sequences and dual multiframelets in Sobolev spaces, Adv. Comput. Math. 38 (2013) 491–529.
[29] M.T. McCann, M. Unser, High-quality parallel-ray X-ray CT back projection using optimized interpolation, IEEE Trans. Image Process. 26 (2017) 4639–4647.
[30] P. Müller, M. Schürmann, J. Guck, The theory of diffraction tomography, arXiv:1507.00466, 2015.
[31] F. Natterer, The Mathematics of Computerized Tomography, Classics Appl. Math., vol. 32, SIAM, Philadelphia, PA, 2001.
[32] F. Natterer, F. Wübbeling, Mathematical Methods in Image Reconstruction, SIAM, Philadelphia, PA, 2001.
[33] K. Niinimäki, S. Siltanen, V. Kolehmainen, Bayesian multiresolution method for local tomography in dental x-ray imaging, Phys. Med. Biol. 52 (2007) 6663–6678.
[34] M. Nilchian, J.P. Ward, C. Vonesch, M. Unser, Optimized Kaiser-Bessel window functions for computed tomography, IEEE Trans. Image Process. 24 (2015) 3826–3833.
[35] M. Rantala, S. Vänskä, S. Järvenpää, M. Kalke, M. Lassas, J. Moberg, S. Siltanen, Wavelet-based reconstruction for limited-angle X-ray tomography, IEEE Trans. Med. Imaging 25 (2006) 210–217.
[36] E.M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ. Press, Princeton, 1971.
[37] M. Stéphane, A Wavelet Tour of Signal Processing, Elsevier Inc., 2009.
[38] Q. Sun, Local reconstruction for sampling in shift-invariant spaces, Adv. Comput. Math. 32 (2010) 335–352.
[39] W. Sun, Local sampling theorems for spaces generated by splines with arbitrary knots, Math. Comput. 78 (2009) 225–239.
[40] W. Sun, X. Zhou, Characterization of local sampling sequences for spline subspaces, Adv. Comput. Math. 30 (2009) 153–175.
[41] W. Sun, Local and global phaseless sampling in real spline spaces, Math. Comput. 90 (2020) 335–352.
[42] H. Wendland, Scattered Data Approximation, vol. 17, Cambridge University Press, 2004.
[43] Y. Xu, A new approach to the reconstruction of images from Radon projections, Adv. Appl. Math. 36 (2006) 388–420.

[44] A.D. Yablon, Multi-wavelength optical fiber refractive index profiling by spatially resolved Fourier transform spectroscopy, J. Lightwave Technol. 28 (2010) 360–364.