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An emerging community in online mathematics teacher professional development: an interactional perspective

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Abstract
Online collaborative and content-focused professional development (PD) is becoming an increasingly important setting for supporting mathematics teachers' professional learning. The purpose of this study was to better understand the process by which a community emerges in such a PD setting by examining how the cohesiveness of 21 mathematics teachers' social network evolves and associated shifts in the quality of mathematics teachers' mathematical discourse. We employed social network analysis (SNA) to examine the evolving cohesiveness of mathematics teachers' social network and coding procedures to examine teachers' mathematical discourse. A key finding was the documentation of an emergent divide between participation in the core and periphery during initial weeks of the PD and then a reduced divide and emergence of a social network that resembles a community. We argue that the instructor's pattern of participation that included distributing their interactions across the subgroups while sending a common message regarding expectations for mathematical discourse in the PD may have contributed to the community formation process. We propose the Interaction Assessment Model, which outlines an approach for PD facilitators to use SNA as a feedback mechanism to differentiate facilitation of online collaborative and content-focused PD and build online communities.

Keywords Online professional development · Teacher communities · Social network analysis · Cohesiveness · Interaction assessment model

Introduction

Meeting the calls of research and policy (e.g., see National Council of Teachers of Mathematics (NCTM), 2018; Schoenfeld 2014; Stein et al. 2008; Sztajn et al. 2012) for teachers implementing mathematics instruction that supports students developing conceptual

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mathematics understanding requires teachers to hold these conceptual mathematics understandings themselves (Silverman and Thompson 2008). Research indicates the importance of mathematics teachers having ongoing opportunities to enhance their mathematics content knowledge (Byerley and Thompson 2017), and professional development (PD) is one mechanism used to support teacher mathematics learning (Fennema et al. 1996; McDonald et al. 2013; Sedova et al. 2016). Furthermore, online PD is becoming an increasingly more important and viable context for supporting teachers' mathematics learning (Lantz-Andersson et al. 2018; Macià and García 2016); however, many questions remain regarding how communities emerge in online collaborative and content-focused PD.

Research indicates that not all PD is equally effective (Bannister 2018; Grossman et al. 2001; Kennedy 2016; Munter et al. 2016). Synthesizing research on PD (e.g., see Penuel et al. 2007; Spillane 1999), Desimone (2009) argued that there is consensus on the core design features of effective PD and these features include content focus, collective participation, active learning, coherence, and sustained duration. Recent work has expanded this list to include coaching, modeling effective instruction, opportunities for feedback (Darling-Hammond et al. 2017), and knowledgeable facilitators (Sztajn et al. 2017). While the effectiveness of these core features has been questioned (Kennedy 2016), our PD efforts draw from these features and aim to support teacher learning of mathematics content through collaborative PD because (1) unfortunately, research indicates that teachers' mathematics content knowledge is underdeveloped (Byerly and Thompson 2017) and (2) our work is grounded in the assumption that collaboration is essential for learning (Lave and Wenger 1991).

Teacher communities, which can emphasize the collaborative aspect of PD, have a history of success in supporting teacher learning, and participation in such communities has been linked to increased student achievement (Langer 2000; McLaughlin and Talbert 2001; Ronfeldt et al. 2015). For example, Ronfeldt et al. (2015) found that teachers who were members of communities engaging in high-quality collaboration were linked to increases in their students' achievement. Typical contexts for teachers participating in community include settings that have a physical location such as teacher work groups in local schools (Horn et al. 2016; Little 2003), teacher education courses (Kutaka et al. 2017)), video clubs (van Es 2012; Wallin and Amador 2019), and externally supported projects and initiatives (Bannister 2015; Stein et al. 1999; Wilson et al. 2017).

Despite research documenting the efficacy of teacher communities for supporting instructional change (Ronfeldt et al. 2015), research has also found shortcomings. Specifically, Cobb et al. (2003) note the potential impact of local instructional norms and the challenges they may present for shifting instruction (i.e., following pacing guides, using common assessments). In addition, school-based teacher communities often present challenges due to teachers' schedules and availability (Chappuis et al. 2009) and the lack of continuity in participation due to high teacher turnover rates (Carver-Thomas and Darling-Hammond 2017). Moreover, Desimone and Garet (2015) found that teachers experience different learning gains from participating in the same community; and one factor contributing to this difference is the beliefs, attitudes, and identities teachers bring to the PD (Goldsmith et al. 2014). This indicates a need for approaches that allow PD leaders to differentiate their facilitation of collaborative PD (Desimone and Garet 2015).

As communication and collaboration technology continue to develop, teacher communities mediated by the internet continue to show promise as a viable modality for teachers participating in generative and productive PD (Lantz-Andersson et al. 2018; Macià and García 2016). An important difference between collaboration in face to face and online settings is that online collaboration can be asynchronous. Such collaboration allows for teachers to be more consistently connected to each other and to activities since they are not limited to engagement at a specific time or place (Fletcher et al. 2007). This implies that online environments afford access to sustained participation in PD as teachers move between schools, which is particularly prevalent in urban districts (Desimone and Garet 2015). In addition to these affordances of online collaboration, Li and Qi (2011) found that an online PD focused on instructional design supported teachers developing a better understanding of mathematics content and instructional strategies. Further, Polizzi et al. (2018) documented success of online communities in supporting new teachers' continued development of effective instructional practices as they transitioned from teacher preparation programs to their initial teaching position.

For these reasons, we have spent nearly a decade studying teachers participating in online communities. In previous studies, we have documented the efficacy of our online work with teachers through supporting teachers' development of mathematical practices (Fukawa-Connelly and Silverman 2015) and mathematical knowledge for teaching (Clay et al. 2012). Additionally, we documented the potential for online asynchronous collaboration to support evidence-based feedback practices (Matranga et al. 2018) and collective practices such as placing student thinking at the center of instructional decision making and supporting the development of increased facility with teacher professional noticing (Fukawa-Connelly et al. 2018). Our current work has shifted toward better understanding and characterizing the process by which communities emerge in online collaborative and content-focused PD and to develop frameworks that allow others to understand and implement similar work.

Purpose

The purpose of this study is to better understand the process by which the cohesiveness of mathematics teachers' social network evolves during online collaborative and content-focused PD and begins to resemble a community as well as to document increases in the quality of mathematics teachers' mathematical discourse in relationship to changes in network cohesion. The cohesiveness of a network or network cohesion "describes attributes of the whole network, indicating the presence of strong socializing relationships among network members, and also the likelihood of their having access to the same information or resources" (Haythornthwaite 1996, p. 332) and can be used as a proxy for examining community formation. Social network analysis (SNA)—an approach to examining patterns in interactions derived from graph theory (Wasserman and Faust 1994)—can be employed to measure network cohesion by quantifying the extent to which mathematics teachers interact. Accordingly, this study used SNA and qualitative approaches to investigate participants' interactions during online asynchronous collaborative problem-solving and how the features of their interactions morphed during the PD course. We investigated the following research questions:

- How does the cohesiveness of participants' social network evolve during the PD course?
- How do participants' interactions in the social network influence cohesiveness of the network, including membership in the core and/or periphery of the network?
- How does the content of participants' interactions provide insight into the cohesiveness of the network and associated changes in the quality of their mathematical discourse?

Network cohesion in online teacher PD

This study is situated in research that examines the pattern in and content of teachers' interactions in online PD. Studies that focus on the content of teachers' interactions in online PD consistently show that teachers largely share and compare information and follow norms of agreement in both informal (Macià and Garcià 2016) and formal (Lantz-Andersson et al. 2018) online communities. Research has documented ways to disrupt such norms and promote quality collaboration in online PD through the use of discourse scaffolds (Yücel and Usluel 2016), design of learning environments (Chieu and Herbst 2016), and instructor facilitation methods (Ouyang and Scharber 2017). Nevertheless, additional research is needed to better understand how to support teacher learning through quality interactions in online collaborative and content-focused PD.

To better understand teacher collaboration in online PD by focusing on the pattern of teachers' interactions, research has examined the cohesiveness of online networks. As noted above, cohesiveness or network cohesion describes features of social networks, specifically highly collaborative settings where information and resources are equally accessible to all members of the network (Haythornthwaite 1996). One theme in research taking a network cohesion lens on teacher participation in online PD is the use of arbitrary points in time of the PD to examine changes in network cohesion. For example, Zhang et al. (2017) facilitated two online lesson study cycles and examined network cohesion after each cycle. Similarly, Sing and Khine (2006) assessed network cohesion at the conclusion of a PD. Thus, a gap in the literature exists regarding the process by which the cohesiveness of teachers' social networks evolve in online PD. Filling this gap can lead to a better understanding of the process of teacher learning in collaborative PD, which can inform the development of frameworks for using SNA as a feedback mechanism (Dado and Bodemer 2017; Wise and Cui 2018) to support the emergence of online teacher communities. The current study makes an initial step at filling this gap.

Studies of network cohesion in online teacher PD range from what we have characterized as low-density and low-reciprocal (not cohesive) to cohesively connected. Zhang et al. (2017) studied 83 teachers in a 6-month online PD and documented a low-density and low-reciprocal network as the density was 0.06 (meaning that 6% of the possible connections were made) and the reciprocity was 0.10 (meaning that 10% of participants' connections were reciprocal). On the other hand, Sing and Khine (2006) examined 11 teachers in an eight-week online PD and found that the density of the network was 0.67, which they concluded was a cohesively connected network. Similarly, Ouyang and Scharber (2017) reported that after seven discussions in an online master's course, the 21 participants formed a cohesively connected social network with a density of 0.776 and a reciprocity of 0.698. While the number of participants in a network has a large impact on potential for cohesiveness of the network (Wasserman and Faust 1994), we use results reported by Sing and Khine (2006) and Ouyang and Scharber (2017) as a benchmark for assessing the cohesiveness of participants' social network in this study.

Core-periphery analysis can be used to investigate network cohesion. Studies have documented the presence of core/periphery structures in online teacher PD, while identifying the presence of an intermediate category of participation that is in between the core and periphery. For example, Li and Li (2013) used a continuous core/periphery model to examine 141 teachers in an online community and their emergent degrees of participation, which the authors characterized as the core, semi-core, and periphery. El-Hani and Greca (2013) examined 87 science teachers in an online PD and used "actions" as a metric for participation (e.g., posting messages, downloading files, etc.). Quantifying actions resulted in three emergent subgroups: an “inner circle” with the most active participants, an “intermediate circle” with moderately active participants, and an “outer circle” with less active participants. Our interpretation of the findings reported in these studies is that the intermediate category of participation provides evidence that the network resembles a community because Jan and Vlachopoulos (2019) argued that an indicator of network cohesion that resembles a community is a decrease in the extent of participation when moving outward in the network. Taken together, these studies suggest the potential of examining network cohesion to better understand the process by which a network evolves and begins to resemble a community.

Communities of practice and network cohesion

We draw from Wenger’s (1998) notion of a community of practice to define “community” in this study. Wenger (1998) argued that there are three features of a community of practice, namely participating members’ mutual engagement in practice (collective practices), a joint enterprise (common goals), and a shared repertoire (emergent concepts and artifacts from collaboration). Learning in a community of practice occurs through a process of legitimate peripheral participation (LPP), which involves the evolution of individuals’ participation from peripheral to full participation in the discourse practices that define a community (Lave and Wenger 1991). Barab et al. (2003) emphasized that learning is a social process that involves developing “connections between the learner and other learners with similar goals” (p. 238). This suggests that social learning processes and legitimate peripheral participants’ changing participation in a community of practice has implications for evolution of discourse practices that define a community (Wenger 1998) *and* the cohesiveness of a community’s social network (Wenger et al. 2011). Thus, communities of practice are dynamic and constantly evolving, both in regard to the qualitative features of the community and the relational patterns. Therefore, we argue that focusing on teachers’ interactional patterns in an online PD has high potential to provide insight into the community formation process.

Cohesiveness has been used as a proxy for studying community formation in online environments through focus on structural characteristics of a social network (Haythornthwaite 1996; Jan and Vlachopoulos 2019). Gaggioli et al. (2015) related the presence of a cohesive social network to online learners developing a collective vision and engaging in goal-oriented participation, while Aviv et al. (2003) found that a cohesively connected network was related to participants developing common practices for knowledge-building. In addition, Nistor et al. (2020) argued that online blogging communities with a large intermediate group of participants successfully fostered LPP because they supported newcomers in moving from peripheral to full participation in the community.

These studies show that there can be links between cohesively connected groups of teachers and productive goals, practices, and generative participation structures. SNA includes procedures for studying teachers’ collaboration (Daly 2010) and can index patterns in interactions that indicate cohesiveness. This study draws from an approach that uses SNA to examine network cohesion for the specific purpose of determining the presence of community.

Jan and Vlachopoulos (2019) argued that indicators of network cohesion that resemble a community include a: (1) large proportion of individuals interacting during the time period

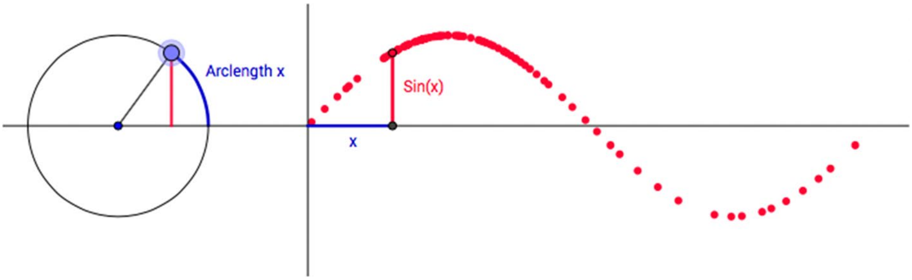

Fig. 1 A mathematics task using an interactive applet

of interest (high-density), (2) group of individuals with strong ties (Granovetter 1973) at the core of the network and a decrease in the extent of and strength in individuals' ties as one moves "outward" in the network; and (3) core-periphery structure that allows for LPP. Further, Zhang et al. (2017) suggest that reciprocal relationships could indicate cohesiveness because multiple interactions between individuals afford negotiation of meaning. The SNA procedures of density, average degree, core-periphery analyses, and reciprocity (discussed below) can quantify these indicators of cohesiveness (Jan and Vlachopoulos 2019). Employing SNA to quantify network cohesion allows us to measure the quality of teachers' collaboration from an interactional perspective. Identifying patterns in how these measures vary over time allows us to surface shifts in teachers' collaboration and characterize the process by which a group of teachers begins to resemble a community. Locating such shifts also allows us to make conjectures regarding what supported these shifts.

Methodology

This section introduces the PD context, the participants, and data sources. We then introduce the SNA procedures used to examine participants' social network, our approach to investigating individuals' interactions, and our process for conducting qualitative analysis of the content of participants' interactions.

PD context

This study is part of a longitudinal mixed-methods research project focusing on mathematics teachers' collaborative learning in online PD. The online PD course is offered by a private university in the Northeast region of the USA. The study was approved by the second author's Institutional Review Board, and all participants in the online PD course consented to participate. The mathematics content goal of the PD is to support teachers reasoning about functions by examining the variation and co-variation of quantitative relationships (e.g., see Carlson et al. 2002). The course mathematics tasks include interactive applets that scaffold participants' focus on quantitative relationships to understand the behavior of functions. An example task (Fig. 1) scaffolds focus on quantities and quantitative relationships from the unit circle to introduce trigonometric functions. The applet affords a dynamic view of the quantitative relationships in the unit circle. One can click and drag a point around the circumference of the unit circle and notice, for example, how the magnitude of the arclength cut out by the horizontal axis and the subtended angle varies (the blue section on the unit circle), how the vertical distance from the point where the subtended angle intersects the circumference of the circle to the horizontal axis varies (the red line segment in the unit circle), and how these two quantities vary simultaneously. The tasks prompt participants to explain their observations with a focus on why the trace of function graphs in the coordinate plane appear the way that they do. These explanations are submitted to the course Discussion Board (DB) and provide foundation for participants' collective mathematical activity. Each week of the course includes similar mathematics tasks that focus on functional relationships and communicating mathematical ideas.

The course has pedagogical goals such as supporting generative feedback practices by scaffolding participants' interactions with online learning technologies and "Noticing and Wondering"—a discourse scaffold for supporting evidence-based reasoning about mathematics and mathematical thinking (Delavan and Matranga 2020; Fukawa-Connelly et al. 2018; Hogan and Alejandre 2010; Shumar 2017). The online asynchronous collaboration model (Silverman and Clay 2010) was applied to structure participants' collective mathematical activity. During a typical week, participants worked privately on a mathematics task, submitted initial thoughts/questions to the course DB, provided feedback on at least two colleagues' mathematics work, revised their initial solution and then collectively reflected on the learning process. Each week followed this work flow except week four, which included an optional DB for participants to discuss the course quiz. Past research on the participants in this PD course indicates that they achieved course goals as we documented the emergence of generative mathematical activity and a pedagogical practice for providing effective feedback (Matranga 2017). Thus, the online PD course presents a rich context for examining community formation in online mathematics teacher PD.

Participants

This study included 21 practicing mathematics teachers who participated in the online PD course, the instructor, and the teaching assistant ($n = 23$). Each of the 21 participants had between one and three years of teaching experience at the secondary level. Participants' geographical locations varied, while 63% of participants were female, 47% were male. Their racial and ethnic backgrounds are unknown.

Data sources

Data for this study were collected from participants collaborating in the online course DB. A third party extracted the data from the DB into a spreadsheet, which included forum name (e.g., week 1 DB), thread name, author of post, content of post, post ID (a unique identifier for each post), and parent post ID (the post ID number a message was in response to). Data preparation included generating a record of the interactions according to who responded to whose post by creating an "author to" column in the spreadsheet and adding the author name that corresponded to the parent post ID. Interactions were directed, meaning that if participant A submitted an initial post to the DB and participant B responded to participant A's post, there was a directed interaction from participant B to participant A. There were ten DBs (one per week) with a total of 1016 directed interactions. Edges were considered binary, meaning that a relationship was considered either established or not. We developed cumulative datasets to provide insight into the evolution of participants' interactional patterns by extracting the author from and author to column from the spreadsheet. The week two dataset, for example, compiled interactions from the week one forum and week two forum. We imported the ten interactional datasets into UCINET 6.0 (Borgatti et al. 2002)—the SNA software used for the analysis. We extracted the forum name, author from, author to, and content columns from the spreadsheet and imported this dataset into Nvivo 11 for the qualitative analysis.

Data analysis

The goal of the SNA was to better understand the process by which participants' interactions evolved during the PD course and began to resemble a community. The analysis began by conducting SNA procedures to examine evolution in the cohesiveness of participants' social network (research question #1) with the purpose of distinguishing stages of evolution, that is time periods during the PD course with similar changes in the structural characteristics—the ways in which a collection of nodes is connected.

The Integrated Methodological Framework (IMF) (Jan and Vlachopoulos 2019) guided our choice of SNA procedures. The IMF identifies SNA procedures effective for examining network cohesion that resembles a community, and these procedures include density, average degree, reciprocity, and core/periphery analysis. Density indicates the extent to which a network is connected by calculating the proportion of connections present to all possible connections. A density of 100% indicates each individual in a network has communicated with everyone in the network. Average degree or average number of interpersonal relationships indicates how many individuals, on average, participants communicate within a network and is calculated by dividing the total number of connections in a network by the number of participants. An average degree of five indicates that on average, participants communicate with five different participants. An average degree of n-1 indicates that each participant in the network interacted with each of his/her colleagues (a density of 100%). Reciprocity indexes the proportion of cases in a network where both participants in a dyad sent a message to one another. A reciprocity of 50% means that in 50% of the pairs of participants, both individuals sent a message to one another.

A core/periphery network structure includes a core group of participants who frequently communicate with one another, a peripheral group who infrequently communicate with one another and on occasion communicate with the core group (Borgatti and Everett 2000). Core/periphery analyses bifurcate a social network into two classes with respect to frequency of participation and with whom that participation is with. The analysis output in UCINET includes core/periphery membership logs and the following densities: core-to-core, core-to-periphery, periphery-to-core, and periphery-to-periphery. These densities indicate the extent to which participants communicate within and across subgroups.

After applying these SNA procedures to the ten interactional data sets, we organized social network metrics into spreadsheets, generated descriptive statistics, examined changes in the social network maps, and created alternative representations of the data. This allowed us to characterize patterns in change in the social network.

This analysis was followed by investigating how individual interactions influenced the structural characteristics of the network (research question #2). We began this analysis by examining ways in which individual participants' membership with the core and peripheral subgroups varied within and across the stages of evolution. This included organizing membership data from the core/periphery analyses into a matrix that had the dimensions "week of PD course" and "participants," where a cell entry of "C" indicated participants' membership with the core. We manipulated the columns of the matrix to uncover patterns in how membership with the core was changing within and across the stages. We also examined individual participants' out degree centrality—a measure of how many times a participant initiates interaction with colleagues—to make meaning out of emergent participation patterns.

We also focused on microlevel processes of how individual participants interacted through existing and/or developed new communication ties both within and across the subgroups of the social network. NetDraw, the network visualization software associated with UCINET, allowed us to manipulate the edges of the network to explore who was initiating interactions in the course and how these interactions impacted the network structure.

Qualitative analyses for this study were conducted with modified grounded theory. Accordingly, we framed analysis of the content of participants' interactions through an existing coding scheme developed in our ongoing research on this PD course. The research team conducted data sessions where the data corpus was analyzed chronologically using constant comparative methods (Strauss and Corbin 1990). Initial coding was done collaboratively in data sessions, allowing for all research team members to develop common understandings and interpretations of the codes. Subsequent coding was done individually by the two authors of the present article, and any questions or disagreements in coding were discussed at whole group data sessions and consensus was reached on the code(s). Four salient codes emerged related to the quality of participants' mathematical discourse and feedback practices, namely visual features, explaining why, covariation to explain why, and challenging practice. The first three codes represent how participants reasoned about functions, while the fourth code represents how participants began to provide feedback to their colleagues that pressed them to develop more sophisticated explanations of functions. Examples of the codes are analyzed and discussed in the final section of the results.

We organized the code counts according to participants' subgroup during each stage of evolution of the network. This allowed us to surface associations between participants' mathematical discourse and their membership with and movement between subgroups (research question #3). The surfaced associations framed our choice of whose sample posts we chose to highlight in the findings section below.

Results

This section provides an overview of the SNA results, characterizes the network's change across two stages of evolution, and shows evidence that the network became cohesively connected by the conclusion of the PD course. We present our analysis of how participants' membership varied between subgroups and the coding results.

Overview of the social network

Table 1 provides an overview of the structural characteristics of participants' network and can be interpreted as follows: at the conclusion of week one, the density was 0.208, which indicates that 20.8% of the possible ways in which participants could reply to their colleagues' posts occurred during week one. By the conclusion of week ten, 75.3% of the ways in which participants could reply to their colleagues' posts occurred. At the conclusion of week one, the average degree was 4.36, meaning that on average, participants sent a message to four or five colleagues during week one. At the conclusion of week one, the reciprocity was 0.438, meaning that of the connections made between pairs of participants, 43.8% included both members of a dyad sending a message to their colleague. Each of the structural characteristics increased throughout the course.

Table 1 Structural characteristics

Week	1	2	3	4	5	6	7	8	9	10
Density	0.208	0.308	0.41	0.417	0.476	0.536	0.648	0.684	0.719	0.753
Avg. degree	4.36	6.78	8.91	9.17	10.47	11.78	14.26	15.04	15.82	16.56
Reciprocity	0.438	0.551	0.615	0.616	0.631	0.686	0.768	0.792	0.802	0.808

Table 2 Core/periphery density model

Week	1	2	3	4	5	6	7	8	9	10
Core-to-Core	0.643	0.833	0.839	0.819	0.833	0.8	0.864	0.879	0.878	0.885
Core-to-Periphery	0.241	0.348	0.517	0.452	0.579	0.591	0.629	0.674	0.715	0.754
Periphery-to-Core	0.107	0.223	0.317	0.302	0.389	0.432	0.614	0.674	0.692	0.738
Periphery-to-Periphery	0.125	0.237	0.276	0.313	0.324	0.364	0.455	0.473	0.489	0.544

Table 2 provides an overview of the network's core-periphery structure and can be interpreted as follows: at the conclusion of week one, the core-to-core density was 0.643. This indicates 64.3% of the ways in which the members of the core could reply to their colleagues' posts occurred during week one. The core-to-periphery density was 0.241, which indicates that 24.1% of the ways in which the members of the core could send a message to the members of the periphery occurred during week one. The core-to-core density increased to 83.3% by week two and then remained relatively stable through week ten, while the core-to-periphery, periphery-to-core, and periphery-to-periphery densities increased throughout the course.

The SNA metrics indicate that the network became cohesively connected and began to resemble a community by the conclusion of the course. The network had a high-density (75.3%), indicating a large proportion of participants interacted during the course. A core-periphery structure with a high-density core (88.5%) and moderate-density periphery (54.4%) was present in the network, which indicates a decrease in density when moving "outwards" in the network. There was a high average number of interpersonal relationships (average degree) (16) and reciprocity (81%). These metrics are consistent with those documented by Sing and Khine (2006) (density of 67%) and Ouyang and Scharber (2017) (density of 78% and reciprocity of 70%), who argued that the networks they studied were cohesively connected.

Two stages of evolution in participants' social network

Looking across the core/periphery density model (Table 2) and the structural characteristics of the network (Table 1) surfaced patterns in the way in which the metrics varied, which was important for understanding how the network evolved. During weeks one through four, we see an increase in each of the structural characteristics (see Table 3); however, those increases were primarily due to the rapid increase in the core-to-core density. For example, the network density increased to 41.7%, while the core-to-core density increased to 81.9%. The structural characteristics of the network continued to increase during weeks five through ten. However, the core-to-core density remained stable, while

Table 3 Changes in SNA metrics

Week	1	2	3	4	5	6	7	8	9	10
Density	0.208	0.1	0.097	0.012	0.059	0.06	0.112	0.036	0.035	0.034
Avg. Degree	4.364	2.419	2.13	0.261	1.304	1.305	2.478	0.782	0.783	0.739
Arc Reciprocity	0.438	0.113	0.064	0.001	0.015	0.055	0.082	0.024	0.01	0.006
Core-to-Core	0.643	0.19	0.006	– 0.02	0.014	– 0.033	0.064	0.015	–0.001	0.007
Core-to-Periphery	0.241	0.107	0.169	– 0.065	0.127	0.012	0.038	0.045	0.041	0.039
Periphery-to-Core	0.107	0.116	0.094	– 0.015	0.087	0.043	0.182	0.06	0.018	0.046
Periphery-to-Periphery	0.125	0.112	0.039	0.037	0.011	0.04	0.091	0.018	0.016	0.055

the core-to-periphery, periphery-to-core, and periphery-to-periphery densities increased toward the core-to-core density. For example, the core-to-core density increased by 1.4% in week five and then decreased by 3.3% in week six, while the other three densities in the model increased each week by larger amounts then the core-to-core density (except in week five where the core-to-core density increased by 1.4% while the periphery-to-periphery density increased by 1.1%).

According to the relationship between how the structural characteristics and the core/periphery density model varied, we defined two stages of the network's evolution and characterized the stages as: Stage one: (a) the structure of the network quickly evolved and became nearly *half way connected* (see Table 1 week four: density = 41.7% and avg. degree = 9.17) and (b) there was an *emergent divide* between participation in the core and periphery (see Table 2 week four: core-to-core density = 81.9% and periphery-to-periphery density = 31.3%). Stage two: (a) the structure of the network became nearly *fully connected* (see Table 1 week ten: density = 75.3% and avg. degree = 16.56) and (b) there was *a reduced divide* between participation in the core and periphery (see Table 2 week ten: core-to-core density = 88.5% and periphery-to-periphery density = 54.4%). The following unpacks and discusses these characterizations.

Stage one

The structural analysis (e.g., density, average degree, reciprocity) of the 425 interactions from stage one showed that the network quickly evolved and became nearly *half way connected*, that is nearly one half of the ways in which participants could reply to their colleagues' posts occurred on the course DB. The proportion of ties present in the network to all possible ties (density) increased each week until reaching 41.7% at the end of stage one. The average outreach of participants' interpersonal relationships (average degree) increased each week until reaching approximately nine at the conclusion of week four, meaning that on average participants sent at least one post to nine of their 22 colleagues (41%) during stage one. The proportion of participants' reciprocal relationships with colleagues (reciprocity) increased each week, reaching 61.6% at the end of week four, meaning that in 61.6% of occasions both participants in a dyad sent a post to one another.

The core/periphery analysis indicated that there was an *emergent divide between the core and periphery* because much of the increases in the network's structural characteristics were due to frequent communication between members of the core (Fig. 2). The proportion of ties present to all possible ties in the core was 81.9% (core-to-core density); the

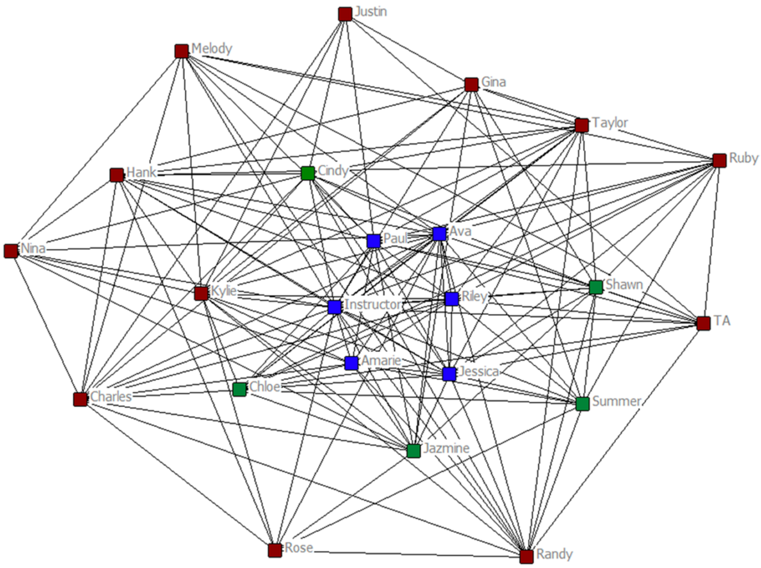

Fig. 2 Stage one core/periphery structure. *Blue = Core; Red = Periphery; Green = changing membership. (Color figure online)

proportion of ties present to all possible ties in the periphery was 31.3% (periphery-to-periphery density). Of the 214 posts that established new connections in the network during stage one, the six members of the core for the entirety of stage one (26% of the class) initiated approximately 40% of these connections (85 posts).

Stage one of the network's evolution shows that participants quickly established connections with their colleagues, which is expected since this was the first time they communicated during the course. The emergent divide in participation indicates that there were qualitatively different ways in which participants engaged in this process of establishing connections. While there are signs of a cohesively connected network because of the high-density core (81.9%) and the core-periphery structure, the emergent divide in the network suggests a community had yet to form in the online PD course.

An emergent divide between subgroups might be an important phase of community development because it might create social context for various levels of engagement and opportunities for LPP, which are important features of a community (Jan and Vlachopoulos 2019; Lave and Wenger 1991). Nevertheless, if the network remained in a "divided state," it could result in the emergence of qualitatively different ways of engaging mathematical and pedagogical activity because of the potential for two separate communities to emerge (Wenger 1998).

Stage two

Analysis of the 591 interactions during stage two showed that the structural characteristics of participants' social network evolved and became nearly *fully connected*, that is nearly three quarters of the way in which participants could reply to their colleagues' posts occurred by the conclusion of the course. The proportion of ties present to all possible ties increased each week (density), reaching 75.3% at the end of stage two. The outreach of participants' interpersonal relationships increased each week (average degree), reaching 16.5

at the end of stage two. The extent to which participants sent a post to a colleague who sent a post back increased each week (reciprocity), reaching 80.1% by the end of the course.

The core/periphery analysis showed that there was a *reduced divide* between the core and peripheral subgroups. The proportion of ties present to all possible ties within the core increased by 5.2%, within the periphery increased by 23.1%, from the core to periphery increased by 30.2%, and from the periphery to core increased by 43.6%. These increases indicate that while the density of connections within the core remained stable, there was an increase in the extent to which members of the core and periphery and between members of the periphery established new connections with one another.

Our interpretation of the change in network cohesion is that there was a shift in the social dynamic of the online course because participants began to establish new communication ties—resulting in the emergence of a more cohesively connected network that resembles a community.

Patterns in individual participant's interactions within and across stages of evolution

We also examined how participants' membership varied between the core and periphery both within and across the two stages of evolution. Table 4 illustrates organized membership data from the core/periphery analysis. The cells with a "C" indicate that UCINET identified the participant as a member of the core for that week. The data show that within each stage there were participants who remained in the core or periphery and participants who moved between these subgroups. For example, in stage one, Paul was in the core each of the four weeks, Gina was in the periphery, and Summer was in the core during weeks one and four while she was in the periphery during weeks two and three. Table 5 shows that there are differences in the average outdegrees of participants who stayed in or moved between a subgroup during a stage. For example, the average outdegree of the core in stage one was 15.2, while the average outdegree of the participants who moved between the core and periphery was 10.6. Outdegree centrality has been shown to be a significant predictor of subgroup membership in online communities (Nistor et al., 2020).

This analysis led to the identification of three participant types: (1) *core participants* remained in the core the entire stage, (2) *intermediate participants* moved between the core and periphery the entire stage, and (3) *peripheral participants* remained in the periphery the entire stage. These varying degrees of participation are consistent with past research (El-Hani and Greca 2013; Li and Li 2013; Nistor et al. 2020) and provide additional evidence of a decrease in the extent of participation when moving "outward," a key component of network cohesion that resembles a community (Jan and Vlachopoulos 2019).

There were three ways in which participants' membership varied between these subgroups across the stages of evolution: (1) *Consistent participation* includes participants who stayed in the core, intermediate, or peripheral subgroup for the entire course, (2) *emerging participation* includes participants who shifted toward the core, and (3) *decreasing participation* includes participants who shifted toward the periphery. In the transition between the network's stages of evolution (Table 7), four of the five intermediate participants appear to settle into roles as members of the core (Cindy, Shawn) or periphery (Chloe, Summer) while Jazmine remained an intermediate participant. Six participants increased their participation from the periphery to the intermediate subgroup. This expanding membership with the intermediate group further illustrates the reduced divide between participation in the core and periphery and evidence of *increasing participation* demonstrates the presence of generative participation structures consistent with LPP (Lave

Table 4 Participant membership with core/periphery*

	Participant	TA	**Gina	Rose	Nina	Justin	Charles	Hank	Summer	Melody	Kylie	Cindy	Amarie	Ava	Instructor	Paul	Riley	Jessica	Shawn	Jazmine	Randy	Ruby	Taylor	Chloe
Week	10									C	C	C	C	C	C	C	C	C	C	C	C		C	
	9						C			C	C	C	C	C	C	C	C	C	C		C	C		
	8										C	C	C	C	C	C	C	C	C	C		C		
	7										C	C	C	C	C	C	C	C	C	C	C	C		
	6										C	C		C	C	C	C	C	C	C		C	C	
	5											C		C	C	C	C	C	C	C			C	
	4								C				C	C	C	C	C	C	C	C				
	3											C	C	C	C	C	C	C	C	C				
	2												C	C	C	C	C	C						C
	1								C				C	C	C	C	C	C						

*C=core and blank=periphery

**All names are pseudonyms

Table 5 Average outdegree

Membership with core/periphery	Stage 1	Stage 2
Remained in core	15.2	19.7
Moved between core and periphery	10.6	16.875
Remained in periphery	7.36	12.75

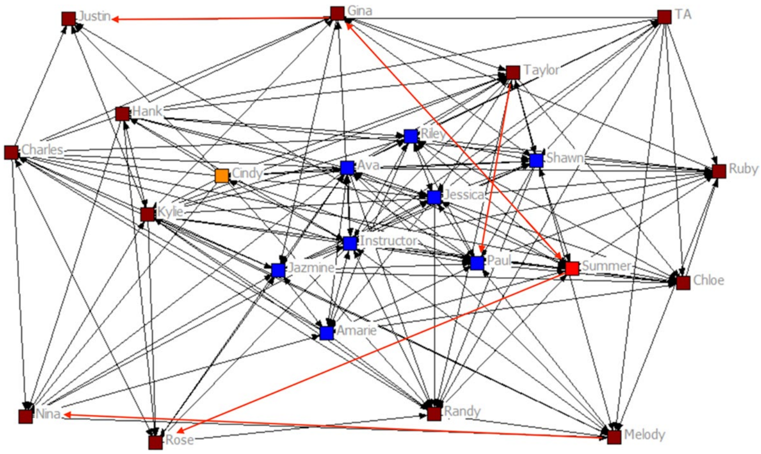

Fig. 3 New connections made by the periphery during week four. *Blue= Core; Maroon=Periphery; Red=Changed from periphery to core; Orange=Changed from core to the periphery; Red edges=New connections during week four. (Color figure online)

& Wenger, 1991) because participants were moving from peripheral toward more central participation.

Patterns in the instructor's interactions during week four

After characterizing the network's evolution and uncovering patterns in participants' interactions within and across the stages, we examined how participants developed new communication ties within and across the core and periphery.[1] We focused specifically on week four because this was the beginning of the network's transition between the stages of evolution.

During week four, the core communicated exclusively through existing communication ties while the periphery reached out to talk to a few colleagues who they had yet to talk to. There were six cases where members of the periphery at beginning of week four (Summer, Gina, Melody and Taylor) sent a message to a colleague who they had yet to talk (connections shown in red in Fig. 3). Taylor (periphery) talked to Paul (core) for the first time while Summer (periphery) sent a post to Gina (periphery) and Rose (core) for

[1] This analysis focused on core and peripheral subgroups because it was during a specific week of the course. The intermediate participants emerged as a result of looking at patterns in participation across multiple weeks. Therefore, at any given moment in the course a participant could be characterized as a member of the core or periphery.

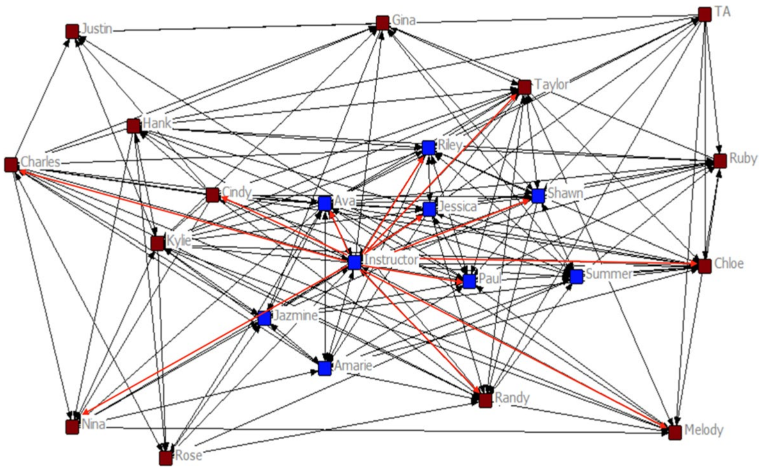

Fig. 4 The instructor's interactions during week four. *Red edges represent posts sent by the instructor. (Color figure online)

the first time. On the other hand, members of the core sent 32 messages to other members of the core who they already talked to, while sending 12 messages to members of the periphery who they already talked to.

The instructor was an exception to this pattern in participation as he distributed his interactions across the core and periphery by sending 23 messages to 12 different participants during week four—10 messages to seven members of the periphery and 13 messages to five members of the core (Fig. 4).

The qualitative analysis uncovered that during week four the instructor sent posts to members of the core and periphery that challenged them to refine their mathematical discourse to include examination of quantitative relationships to provide reasoning for why function graphs behave in certain ways. We coded 10 of the 13 posts the instructor sent to the core and five of the 10 posts the instructor sent to the periphery as *challenging practice*. Consider the following representative sample post the instructor sent to Randy that we coded as such and responded to Randy's rough draft quiz he submitted to the DB:

> While your answers are no doubt correct, I would bet that you would have been able to respond similarly prior to this class. You are taking steps to talking conceptually and about quantities—for example you note "the graph will complete a cycle in a shorter distance" rather than saying for example, "has a shorter period." But I'd urge you to spend time and focus on the quantities, which can explain why that is true.

The instructor quoted a portion of Randy's solution when he said, "you note, 'the graph will complete a cycle in a shorter distance' rather than…" This portion of Randy's explanation quoted by the instructor is an example of Randy focusing on the visual features of the graph to make sense of the function's behavior. The instructor challenged Randy to move beyond a focus on visual features and explain why when he said, "I'd urge you to spend time and focus on the quantities, which can explain why that is true…".

Table 6 Data samples for codes used for the qualitative analysis

Code	Data sample
Visual features	A new observation I was able to make by watching the lengths of the lines on the axis, was that the closer the cities are to the highway as well as closest to the midpoint between the exits, the shorter the lines on the axis were
	sin(θ) has two waves, sin(2θ) has four waves, and sin(3θ) has six waves. All graphs start with a wave above the axis and then alternate dipping below and above the axis
Explaining why	Quantity 'a' multiplies the quantity of sin(x) (which is the vertical length of the sine graph). This vertical length is determined by the ratio of the opposite side and the hypotenuse, and we know that this varies from 0 to 1 to 0 to − 1 to 0. As quantity a varies, now the vertical length will also vary
	As the arc length varies around the circle, the sine function varies from 0 to 1, decreases from 1 to 0, decreases from 0 to −1, and increases from −1 to 0. This is considered one period of the graph
Covariational to explain why	As the arclength x increases from 0 to pi/2, the y value, which represents the vertical length, increases until it reaches the maximum value of 1 at pi/2. Then as x increases from pi/2 to pi the y value decreases as the vertical length decreases. It reaches 0 again at pi since there is no vertical length
	As x increases by 1 rad from 0 to 1, $2x/7$ increases by 2/7 radians from 0 to 2/7, sin($2x/7$) increases by 0.28 rad from 0 to 0.28
Challenging practice	Can you describe why sin(θ) has two waves for one revolution around the circle and why sin(2θ) has four waves for one revolution around the circle?
	I notice that you talk about the shift in direction when the graph begins increasing and decreasing it periods. …and I wonder at which points does the graph shift from a period shorter than 2pi to a period longer than 2pi. Can you explain why it occurs here?

The pattern in the instructor's interactions indicates that he distributed his participation across the core and periphery. The content of the instructor's interactions suggests that he provided support for how to engage mathematical discourse that focuses on quantities and the variation between quantities. Taken together, the pattern in and content of the instructor's interactions may have contributed to the establishment of common expectations for productive mathematical discourse in the class.

Examining the content of interactions in relation to the network's structural characteristics

This section presents results from our coding of the content of participants' interactions. The codes included: *visual features, explaining why, covariation to explain why*, and *challenging practice* (see Table 6 for examples). The first three codes represent an increase in the sophistication of participants' mathematical discourse from focusing on visual features of function graphs to understand its properties, examining underlying quantities to explain why functions behave in certain ways, and examining the variation and covariation of quantities to understand functions. *Visual features* captures discourse that reflects a less sophisticated approach to examining functions because it relies on perceptual and sensorimotor experiences for sense making, which can result in context specific patterns in learners' understandings (i.e., steepness of a linear function indicating rate of change). On the other hand, *explaining why* and *covariation to explain why* capture discourse that reflects a more sophisticated approach because examining

Table 7 Coding results

Code	Visual features		Explaining why		Covariation to explain why		Challenging practice	
Stage	Stage 1	Stage 2	Stage 1	Stage 2	Stage 1	Stage 2	Stage 1	Stage 2
Core	15	9	35	64	5	36	20	29
Intermediate	11	13	17	59	5	39	1	9
Peripheral	28	20	35	43	7	20	0	7
Percentage of interactions	12.7	7.1	20.4	28.1	4	16.1	4.9	7.6

the variation and covariation of quantities to understand the behavior of functions is generalizable to a variety of function families (see Moore and Thompson (2015) for a description of a comparable learning trajectory ranging from static to emergent shape thinking). The *challenging practice*, on the other hand, was an emerging approach to providing feedback, where participants pressed their colleagues to refine their mathematical explanations. Table 7 illustrates the aggregate coding results according to participant type and stage of observation. Table 8 disaggregates these results.

The coding results demonstrate an overall shift in the quality of participants' mathematical discourse throughout the course as there are increases in the presence of each of the latter three codes (explaining why, covariation to explain why, challenging practice) and decreases in the presence of the first code (visual features). The percentage of posts coded as visual features decreased from 12.7% of posts in stage one to 7.1% of posts during stage two. The percentage of posts coded as explaining why, covariation to explain why, or challenging practice increased by approximately 8%, 12%, and 3%, respectively. The coding results also indicate that the core more frequently engaged quality mathematical discourse. For example, the six members of the core *explained why* 35 times in stage one (40% of the occasions of explaining why) and the seven core members explained why 64 times in stage two (39% of the occasions of explaining why), while the 12 peripheral participants *explained why* 35 times in stage one (40% of the occasions of explaining why) and 43 times in stage two (25% of the occasions of explaining why). The following analyzes exemplars of visual features, explaining why, and challenging practice while drawing connections between the structural characteristics of the network and shifts in the quality of participants' mathematical discourse.

Each of the 21 participants' mathematical discourse included making sense of functions by examining *visual features* of functions graphs during stage one (see Table 8). As the course progressed, a large proportion of the participants, including *consistent participants who remained in the periphery* showed decreases in the extent to which they used visual features of graphs to understand functions. There were two exceptions, Nina and Justin. Therefore, we present a post by Nina as a case example of participants' initial mathematical discourse and highlight Nina in particular because she did not show shifts in the quality of her discourse. The following exemplar post by Nina examined the function $f(x) = \sin(x^2)$.

> The first thing I noticed is that the range remains the same, −1 and 1. This makes sense considering that the sin of a function is a ratio. I noticed that there is still the hills and valleys. However, the difference is where we learn the most information.

Table 8 Individual coding results

Subgroup	Stage 1		Stage 2	
Core	Ava	2, 8, 1, 0*	Ava	1, 16, 6, 3
	Paul	5, 6, 2, 1	Paul	3, 11, 5, 16
	Riley	3, 7, 0, 0	Riley	1, 10, 5, 0
	Jessica	3, 13, 1, 0	Jessica	2, 8, 5, 1
	Instructor	0, 0, 0, 18	Instructor	0, 0, 0, 2
	Amarie	2, 1, 1, 1	Cindy	0, 8, 8, 5
			Shawn	2, 11, 7, 2
Intermediate			Randy	3, 12, 4, 1
			Taylor	3, 12, 3, 2
			Charles	0, 11, 6, 0
			Kylie	2, 6, 3, 1
			Melody	1, 5, 5, 1
			Ruby	2, 5, 8, 2
	Jazmine	2, 1, 1, 1	Jazmine	1, 5, 4, 2
	Chloe	2, 1, 0, 0		
	Summer	2, 4, 0, 0		
	Cindy	3, 9, 2, 0		
	Shawn	2, 2, 2, 0		
			Amarie	1, 3, 6, 0
Periphery	Randy	2, 5, 0, 0		
	Taylor	2, 3, 0, 0		
	Charles	2, 2, 1, 0		
	Kylie	2, 3, 0, 0		
	Melody	4, 4, 1,0		
	Ruby	2, 5, 2, 0		
	Hank	2, 7, 2, 0	Hank	0, 10, 8, 5
	Rose	2, 1, 1, 0	Rose	2, 12, 5, 1
			Chloe	2, 10, 2, 0
	Gina	4, 1, 0, 0	Gina	6, 6, 0, 0
			Summer	3, 3, 2, 0
	Nina	4, 2, 0, 0	Nina	4, 1, 0, 0
	Justin	2, 2, 0, 0	Justin	3, 1, 3, 1
	TA	0, 0, 0, 0	TA	0, 0, 0, 0

*Code counts are in the following order: visual features, explaining why, covariation to explain why, challenging practice

> We can look at the hills and valleys almost as oscillations. The oscillations start "slow" and begin to increase in frequency rapidly as *x* approaches infinity.

Nina noticed visual features of the function graph such as "hills and valleys," and how the "oscillations start slow and begin to increase in frequency" as you look from the origin of the Cartesian plane toward positive or negative infinity. These hills and valleys and the increase in the frequency of the oscillations are visual features of the function $\sin(x^2)$ when it is graph on the Cartesian plane. Thus, our interpretation is that Nina's approach to understanding the function was to describe visual features of the function graphed in the Cartesian plane. Examining functions in this way is consistent with what Moore and Thompson (2015) refer to as static shape thinking.

This focus on visual features of function graphs was a defining aspect of all participants' initial mathematical discourse while it was the discourse demonstrated by Nina and Justin throughout the PD course. Nina and Justin's continued engagement in this discourse distinguishes them from other consistent participants in the periphery who demonstrated shifts in the quality of their discourse (e.g., Hank, Rose, Chloe, Gina). Thus, while the cohesiveness of the network evolved and began to resemble a community, a small subgroup of participants increased the extent of their collaboration with colleagues while showing little sign of shifts in the quality of their mathematical discourse.

Participants began to more frequently use underlying quantitative relationships to *explain why* functions behave in certain ways as the course transitioned from stage one to two. This increase was particularly prevalent in the *emerging participation group who moved from the periphery in stage one to intermediate participation in stage two*. Amongst this group, Randy, Taylor, and Charles demonstrated the most pronounced shift toward explaining why when examining functions. Moreover, the instructor challenged Randy, Taylor, Charles and Melody during stage one to move beyond a focus on visual features of functions and begin examining underlying quantities. The following presents Randy's week six post as an exemplar of this feature of participant's mathematical discourse.

> I am going to attempt to explain why the graph looks as it does. x can be any value from negative infinity to infinity. As x varies the $\cos(x)$ varies from 1 to negative 1. As $\cos(x)$ varies between 1 and -1, $e^{\cos(x)}$ varies between e (approximately 2.7918….) and 1/e (approximately 1/2.7918). Some important values would be $e^{\cos(0)} = 1$. This value repeats itself, since $\cos(x)$ is a periodic function, every 2pi radians.

Randy highlighted that he is attempting to explain "why the graph looks as it does" and then examined the function's quantities by describing how these quantities vary ("As x varies, the $\cos(x)$ varies from 1 to negative 1"). Randy further described how the variation of $\cos(x)$ impacts the variation of $e^{\cos(x)}$ ("$e^{\cos(x)}$ varies between e…and 1/e…"). Thus, our interpretation is that Randy focused on varying quantities to understand the function. This approach to examining functions is similar to what Moore and Thompson (2015) refer to as emergent shape thinking. This post is in contrast to an earlier post by Randy that we coded as visual features. Randy explained:

Similar to Nina's post shown above, Randy focused on visual features of the graph in the coordinate plane such as the "solid line" that might appear near the origin when graphing $y = \sin(1/x)$ on a Cartesian coordinate plane. He also used language indicating he was drawing on his perceptual experience (e.g., "it appears," and "I expected to see") to understand the function. Thus, these posts provide evidence that Randy moved beyond a focus on visual features of graphs and began to provide reasons for why functions vary between particular values.

More generally, the exemplar posts illustrate the shift in the quality of participants' mathematical discourse documented in our coding. The shift is illustrated when comparing discourse such as "varies" and descriptions of the relationship between changes in quantities (e.g., "as x varies…the $\cos(x)$ varies…) to discourse such as "hills and valleys" and a "solid line." We suggest that the former is similar to emergent shape thinking, while the latter is consistent with static shape thinking. Moore and Thompson (2015) argued, similarly, that emergent shape thinking is a more sophisticated approach to examining functions than static shape thinking. We observed this shift in the quality of all participants' discourse (accept Nina and Justin). The emerging participants in the intermediate group during stage two (Randy, Taylor, and Charles) in particular were on a trajectory of increased communication with the core, and the quality of their discourse was becoming more sophisticated. Therefore, this association between Randy, Taylor, and Charles' pattern in participation and shifts in their discourse provides additional evidence of LPP in the network.

An emergent feature of the class's discourse included what we characterized as the *challenging practice*, where participants challenged colleagues to shift their focus from visual features of function graphs to underlying reasons for why graphs have certain visual features. This feature of discourse was primarily observed in instructor's posts during stage one. The instructor sent 18 posts that we coded as challenging practice and four of these posts were sent to Paul during week four. We coded 45 posts sent during stage two as challenging practice and 16 of these posts were sent by Paul. Paul was a *consistent participant who remained in the core*. Therefore, we present Paul's post to Nina to illustrate this emerging feature of participants' discourse. The post was in response to Nina's examination of the function $f(x) = \sin(x)$:

> I noticed that you wrote: "This graph appears as it does because of the Unit Circle. Essentially as the values of $\sin(x)$ make their way around the circle, they start again at zero." …and I wonder… if you could elaborate on this concept more. Why do the values start again at zero? Why does the graph have hills and valleys?

Paul quoted a particular aspect of Nina's work when he said, "I noticed that you wrote: 'This graph appears as it does because …'" The quoted portion of Nina's explanation lacks specificity, focus on quantities or the relationship between quantities. In Paul's response, he pushed Nina to refine her explanation with additional detail regarding why $f(x) = \sin(x)$ has this particular pattern of change. Paul also asked a question that pushed Nina to explain why the graph has particular visual features ("hills and valleys"). This example illustrates this emerging feature of participants' discourse and shows Paul, in particular, taking a role in increasing the connectivity of the network through engagement in a generative feedback practice.

The presented data provide a lens into the content of participants' discourse, how it shifted from focusing on visual features to explaining why functions behave in certain ways, and how participants began to support one another in improving the quality of their mathematical discourse by engaging in the challenging practice. While we are not making any claims regarding the association between the pattern in and content of participants' discourse, this analysis provides insight into potential relationships between shifts in interactional patterns and changes in the quality of participants' mathematical discourse.

Discussion

We began this study with the goal of better understanding the process by which the cohesiveness of mathematics teachers' social network evolved and how the quality of discourse shifted in relationship to changes in the network. This study documented the emergence of a cohesively connected network with high-density, high-reciprocity, a core/periphery structure, and a decrease in density when moving outward in the network—key features of a cohesive network that resembles a community (Jan and Vlachopoulos 2019)—as well as evidence of LPP (Lave and Wenger 1991). We also documented increases in the quality of participants' mathematical discourse from a focus on visual features of graphs to understand functions to a focus on explaining why graphs have certain characteristics. Further, past studies on this group of participants documented the emergence of collective discourse practices, goals, and artifacts (Matranga 2017)—three key features of a community (Wenger 1998). Thus, this study provides an empirical example of online mathematics teacher collaboration that resulted in a cohesively connected network where the qualitative features of participants' collaboration also indicated a community was present. Given the benefits of online settings for teacher participation in successful PD and the importance of teachers' participation in community for enhancing student achievement (Ronfeldt et al. 2015), this study is significant because it provides additional evidence that online PD can be a viable context for supporting mathematics teachers' learning through participation in generative, productive and community-oriented PD.

This study builds on previous work that has taken snapshots of how network cohesion changes over time (e.g., see Zhang et al. 2017) by opening up the "black box" of teacher learning in online PD (Little 2003; Goldsmith et al. 2014) and documenting community formation processes. We characterized the network's evolution in two stages: (1) an *emergent divide* between participation in the core and periphery, and (2) a *reduced divide* between participation in the core and periphery and emergence of a more cohesively connected network. Moreover, we identified three subgroups of participants (core, intermediate, and periphery) and three ways in which participants' membership varied between these subgroups (consistent, increasing, decreasing).

One important finding in this study is how the instructor distributed his interactions across the core and periphery during week four and sent a common message regarding what constitutes appropriate mathematical discourse in the class. The concept of humans-with-media defines mathematics learning as a collaborative process that is integrally shaped by the technologies mediating collaboration. Moreover, the affordances and constraints of technologies impact collaboration and emergent learning (Borba and Villarreal 2006). Borba et al. (2018) argued that an affordance of DBs is that they can function as "interactive textbooks" that learners co-construct during collaborative learning and these interactive textbooks, in turn, provide a mechanism for support in the learning process. In an excerpt from an interview of a pre-service mathematics teacher in an online distance education course, Borba et al. (2018) documented a student noting, "of course we have the textbooks, have the handouts of the course, but the [DB] is all we really need [...] it is full of information, solved exercises, videos, links to help with solutions" (p. 281). We argue that there is potential that the instructor increased visibility of and access to course expectations by co-constructing a digital learning environment that includes details (i.e., the posts we coded as challenging practice) regarding what constitutes quality mathematical discourse.

This co-constructive process and perspective of the course DB as an interactive textbook provides one potential explanation for the network's shift from the emergent divide to a more cohesively connected network. Horn et al. (2020) argued that teachers engaging in dialogue contributing to common perspectives of effective math teaching during formal PD meetings can become a "source of homophily" that contributes to increased interactions outside of the meeting. Homophily is a social network concept that indicates individuals with common genders, backgrounds, and/or interests tend to build relationships with one another (Kadushin 2011). This implies that increasing visibility of quality mathematical discourse across the subgroups might have contributed to increased communication between subgroups because of participants' emerging common understanding of how to communicate mathematically. We showed that in the transition from stage one to stage two, participants shifted from the peripheral to intermediate subgroup and these participants (e.g., Randy, Taylor, and Charles) showed increases in the quality of their mathematical discourse along with this shift. This argument is in line with a key finding in Eberle et al. (2014), showing that access to community knowledge was a significant predictor of the extent to which new participants increased their participation in a community. Therefore, there is potential that the pattern in and content of the instructor's facilitation of this online PD impacted the trajectory of participants' collaborative learning process.

The analysis of the content of participants' interactions provided insight into the structural characteristics of the network and teacher learning processes in online settings. Drawing from the aforementioned logic regarding the DB as an interactive textbook (Borba et al. 2018), it might be expected that quality posts are more generative in the core than in the periphery. There is potentially an increased opportunity for members of the core to take up and respond to colleagues' ideas because of the core's increased engagement and increased visibility of their posts. Conversely, quality interactions in the periphery are more likely to go unnoticed because of the decreased engagement by the periphery and lack of visibility of their posts. We showed that Paul challenged Nina to refine her examination of $f(x) = \sin(x)$. Given Nina's position in the periphery of the network, this may have appeared to Nina as an isolated occasion of challenge, whereas participants in the core may have been more likely to observe this interaction and other, similar interactions. Thus, Paul's interactions in the network were potentially generative because he increased the density of the network by frequently interacting with colleagues and potentially increased the visibility of expectations for quality mathematical discourse by challenging colleges to explain why. An implication of this insight is that it might be effective to scaffold online collaborative PD along two dimensions, namely (1) the pattern in participants' interactions and (2) the content of participants' interactions.

Desimone and Garet (2015) found that individual teachers respond differently to the same collaborative PD and proposed differentiating PD for teachers with common needs as a way to address this issue. From an interactional perspective, our study showed that teachers responded differently to our collaborative PD in the online setting. Teachers' varied responses were evidenced by the emergent divide in the network and in the mathematical discourse of participants such as Nina. We propose the *Interaction Assessment Model* as an empirically grounded strategy that allows educators to differentiate facilitation of online PD by scaffolding collaboration in a way that foregrounds the pattern of participants' interactions to increase access to rich mathematical discussions. The model uses SNA as a feedback mechanism (Dado and Bodemer 2017; Wise and Cui 2018) to monitor collaboration. Specifically, the model includes (1) applying core/periphery analyses to monitor the emergence of core and peripheral subgroups in an online PD, (2) using this data to inform how the PD facilitator can distribute interactions across subgroups to enhance the visibility of support, and (3) responding to participants in a way that provides appropriate support to establish what constitutes acceptable mathematical or pedagogical activity. Moreover, it might be useful to explicitly scaffold mathematics teachers in using community-generated artifacts to inform their development of mathematical explanations.

Finally, the implemented analytical method allowed us to unpack the process of teacher collaboration. Looking across the analysis of the networks' structural characteristics and the core/periphery density model provided a better understanding of how participants' collaborative processes shifted throughout the PD. Identifying subgroups and patterns in how participants' membership varied between these subgroups provided a better understanding of how individual participation shifted. Moreover, looking across the SNA results and the coding results allowed us to uncover subgroups of participants such as Randy, Taylor, and Charles who demonstrated generative shifts in the pattern in and content of their participation, providing empirical examples of LPP (Lave and Wenger 1991). While more work is needed to better understand various factors, events, and conditions that support generative shifts in participation (e.g., Randy, Taylor, and Charles), we argue that the implemented SNA method outlines an efficient approach for unpacking teacher learning processes in online PD and focusing follow-up analyses on factors contributing to the observed learning process. If this SNA approach is adopted by mathematics teacher education researchers, there is potential for an organized effort to better understanding the critical question of what mechanisms foster the emergence of generative and productive teacher communities that have positive impacts on student mathematics achievement (Goldsmith et al. 2014; Kennedy 2016).

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