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Levi-spherical Schubert varieties

Yibo Gao [a], Reuven Hodges [b,*], Alexander Yong [c]

[a] *Beijing International Center for Mathematical Research, Peking University, Beijing, China*
[b] *Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA*
[c] *Department of Mathematics, U. Illinois at Urbana-Champaign, Urbana, IL 61801, USA*

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We prove a short, root-system uniform, combinatorial classification of Levi-spherical Schubert varieties for any generalized flag variety G/B of finite Lie type. We apply this to the study of multiplicity-free decompositions of a Demazure module into irreducible representations of a Levi subgroup.

1. Introduction

1.1. History and motivation

In his essay [19] on representation theory and invariant theory, R. Howe discusses the significance of multiplicity-free actions as an organizing principle for the subject. Clas-

* Corresponding author.
E-mail addresses: gaoyibo@bicmr.pku.edu.cn (Y. Gao), rmhodges@ku.edu (R. Hodges), ayong@illinois.edu (A. Yong).

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sical invariant theory focuses on actions of a reductive group G on symmetric algebras, which is to say, coordinate rings of vector spaces. Now one also considers G-actions on varieties X and their coordinate rings $\mathbb{C}[X]$. Such an action is multiplicity-free if $\mathbb{C}[X]$ decomposes, as a G-module, into irreducible G-modules each with multiplicity one. An important example is when X is the *base affine space* of a complex, semisimple algebraic group G [3]; in this case the coordinate ring is a multiplicity-free direct sum of the irreducible representations of G. Lustzig's theory of dual canonical bases [27] provides a basis for it. In the early 2000s, understanding this basis was a motivation for S. Fomin and A. Zelevinsky's theory of Cluster algebras [13].

The notion of multiplicity-free actions is closely connected to that of *spherical varieties*. Let G be a connected, complex, reductive algebraic group; we say that a variety X is a G-variety if X is equipped with an action of G that is a morphism of varieties. A spherical variety is a normal G-variety where a Borel subgroup of G has an open, and therefore dense, orbit. An normal, affine G-variety X is spherical if and only if $\mathbb{C}[X]$ decomposes into irreducible G-modules each with multiplicity one [34]. If X is instead a normal, projective G-variety then one can still recover one direction of this implication. That is, if the induced G-action on the homogeneous coordinate ring of X is multiplicity-free, then X is G-spherical [17, Proposition 4.0.1].

Spherical varieties possess numerous nice properties. For instance, projective spherical varieties are Mori Dream Spaces. Moreover, Luna-Vust theory describes all the birational models of a spherical variety in terms of colored fans; these fans generalize the notion of fans used to classify toric varieties (which are themselves spherical varieties). N. Perrin's excellent survey covers additional background on spherical varieties [30].

It is an open problem to classify all spherical actions on products of flag varieties. This is solved in the case of Levi subgroups; we point to the work of P. Littelmann [26], P. Magyar–J. Weyman–A. Zelevinsky [28,29], J. Stembridge [32,33], R. Avdeev–A. Petukhov [1,2]. Connecting back to the representation-theoretic perspective of [19], in [32,33], J. Stembridge relates this classification problem to the multiplicity-freeness of restrictions of *Weyl modules* [14, Lecture 6]. Indeed, the homogeneous coordinate ring of a single flag variety is a multiplicity-free sum of spaces of global sections on the variety with respect to line bundles associated to each dominant integral weight. By the Borel-Weil-Bott theorem, these spaces are isomorphic to the irreducible representations of G. This is the central object of interest in *Standard Monomial Theory* [25] and is closely related to the coordinate ring of base affine space mentioned above. As remarked above a product of flag varieties is G-spherical if its homogeneous coordinate ring is multiplicity-free as an G-module.

This paper solves a related problem. We classify all *Levi-spherical* Schubert varieties in a single flag variety; that is, Schubert varieties that are spherical for the action of a Levi subgroup. Here, the relevant ring is the homogeneous coordinate ring of a Schubert variety and the attendant representation theory is that of *Demazure modules* [12], which are Borel subgroup representations. Critically for this paper, they are also Levi subgroup representations. Multiplicity-freeness in this setting refers to the decomposition of these modules into irreducible Levi subgroup representations. This study was initiated in [18] and the authors solved the problem for the GL_n case in [15]. In [16] we conjectured an answer for all finite rank Lie types; this paper proves that conjecture. During the completion of this article, we learned that M. Can-P. Saha [10] independently proved the conjecture.

1.2. Background

Throughout, let G be a complex, connected, reductive algebraic group and let $B \leq G$ be a choice of Borel subgroup along with a maximal torus T contained in B. The *Weyl group* is $W := N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. The orbits of the homogeneous space G/B under the action of B by left translations are the *Schubert cells* $X_w^\circ, w \in W$. Their Zariski closures

$$X_w := \overline{X_w^\circ}$$

are the *Schubert varieties*. It is relevant below that these varieties are normal [11,31].

A *parabolic subgroup* of G is a closed subgroup $B \subset P \subsetneq G$ such that G/P is a projective variety. Each such P admits a *Levi decomposition*

$$P = L \ltimes R_u(P)$$

where L is a reductive subgroup called a *Levi subgroup* of P and $R_u(P)$ is the unipotent radical. One parabolic subgroup is $P_w := \mathrm{stab}_G(X_w)$. Any of the parabolic subgroups $B \subseteq Q \subseteq P_w$ act on X_w.

Let L_Q be a Levi subgroup of Q. A variety X is *H-spherical* for the action of a complex reductive algebraic group H if it is normal and contains an open, and therefore dense, orbit of a Borel subgroup of H. Our reference for spherical varieties is [30]; toric varieties are examples of spherical varieties.

Definition 1.1 *([18, Definition 1.8]).* Let $B \subseteq Q \subseteq P_w$ be a parabolic subgroup of G. $X_w \subseteq G/B$ is *L_Q-spherical* if has a dense, open orbit of a Borel subgroup of L_Q under left-translations.

1.3. The main result

We give a root-system uniform combinatorial criterion to decide if X_w is L_Q-spherical. Let $\Phi := \Phi(\mathfrak{g}, T)$ be the root system of weights for the adjoint action of T on the Lie algebra $\mathfrak{g}$ of G. It has a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots. Let $\Delta \subset \Phi^+$ be the base of simple roots. The parabolic subgroups $Q = P_I \supset B$ are in bijection with subsets I of Δ; let $L_I := L_Q$. The set of *left descents* of w is

$$\mathcal{D}_L(w) = \{\beta \in \Delta : \ell(s_\beta w) < \ell(w)\},$$

where $\ell(w) = \dim X_w$ is the *Coxeter length* of w. Under the bijection, $P_w = P_{\mathcal{D}_L(w)}$, and $B \subset Q \subseteq P_w = P_{\mathcal{D}_L(w)}$ satisfy $Q = P_I$ for some $I \subseteq \mathcal{D}_L(w)$.

For $I \subset \Delta$, a *parabolic subgroup* $W_I \subseteq W$ is the subgroup generated by $S_I := \{s_\beta : \beta \in I\}$. A *standard Coxeter element* $c \in W_I$ is any product of the elements of S_I listed in some order. Let $w_0(I)$ be the longest element of W_I. The following definition was given in type A in [15, Definition 1.1] and in general type in [16, Section 4]:

Definition 1.2. Let $w \in W$ and $I \subseteq \mathcal{D}_L(w)$ be fixed. Then w is I-spherical if $w_0(I)w$ is a standard Coxeter element for W_J where $J \subseteq \Delta$.

We first note that if $I \subseteq \mathcal{D}_L(w)$, then the left inversion set $\mathcal{I}(w)$, defined in Section 3, contains all the positive roots in the root subsystem generated by I, and thus $w = w_0(I)d$ is a length-additive expression for some $d \in W$, by Proposition 3.1.3 in [4].

Theorem 1.3. *Fix $w \in W$ and $I \subseteq \mathcal{D}_L(w)$. X_w is L_I-spherical if and only if w is I-spherical.*

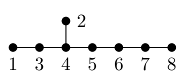
Theorem 1.3 resolves the main conjecture of the authors' earlier work [16, Conjecture 4.1] and completes the main goal set forth in [18]. In [15], Theorem 1.3 was established in the case $G = GL_n$ using essentially algebraic combinatorial methods concerning *Demazure characters* (or in their type A embodiment, the *key polynomials*). In contrast, the geometric arguments of this paper are quite different, significantly shorter, but require more background of the reader in algebraic groups. Theorem 1.3 is a generalization of work of P. Karuppuchamy [24] that characterizes Schubert varieties that are toric, which in our setup means spherical for the action of $L_\emptyset = T$. Using work of R. S. Avdeev–A. V. Petukhov [1], Theorem 1.3 may also be seen as a generalization of some results of P. Magyar–J. Weyman–A. Zelevinsky [28] and J. Stembridge [32,33] on spherical actions on G/B; see [18, Theorem 2.4]. Previously, there was not even a finite algorithm to decide L_I-sphericality of X_w in general.

1.4. Organization

Examples of the main result are given in Section 2. Sections 3 and 4 prove Theorem 1.3. Section 5 offers an application of our main result to the study of Demazure modules [12].

2. Examples of Theorem 1.3

We begin with a few examples illustrating Theorem 1.3.

Example 2.1 *(E_8 cf. [18, Example 1.3]).* The E_8 Dynkin diagram is 

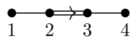
. One associates the simple roots β_i ($1 \leq i \leq 8$) with this labeling and writes $s_i := s_{\beta_i}$. Suppose

$$w = s_2s_3s_4s_2s_3s_4s_5s_4s_2s_3s_1s_4s_5s_6s_7s_6s_8s_7s_6 \in W.$$

Then $\mathcal{D}_L(w) = \{\beta_2, \beta_3, \beta_4, \beta_5, \beta_7, \beta_8\}$. Let $I = \mathcal{D}_L(w)$. Here

$$w_0(I) = s_3s_2s_4s_3s_2s_4s_5s_4s_3s_2s_4s_5 \cdot s_7s_8s_7 \quad \text{and} \quad w_0(I)w = s_1s_6s_7s_8.$$

Since $w = w_0(I)c$ where $c = s_1s_6s_7s_8$ is a standard Coxeter element, Theorem 1.3 asserts that X_w is L_I-spherical in the complete flag variety for E_8.

Example 2.2 *(F_4 cf. [18, Example 1.5]).* The F_4 diagram is 

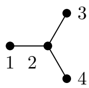
. First suppose

$$w = s_4s_3s_4s_2s_3s_4s_2s_3s_2s_1s_2s_3s_4 \ (I = \mathcal{D}_L(w) = \{\beta_2, \beta_3, \beta_4\}).$$

Then $w_0(I) = s_2s_3s_2s_3s_4s_3s_2s_3s_4$ and $w_0(I)w = s_1s_2s_3s_4$ is standard Coxeter. Hence X_w is L_I-spherical. On the other hand if

$$w' = s_2s_1s_4s_3s_2s_1s_3s_2s_4s_3s_2s_1 \ (I = \mathcal{D}_L(w') = \{\beta_2, \beta_4\}),$$

then $w_0(I) = s_2s_4$ and $w_0(I)w = s_1s_3s_2s_1s_3s_2s_4s_3s_2s_1$ is not standard Coxeter and X_w is not L_I-spherical.

Example 2.3 *(D_4).* The D_4 diagram is 

. Let

$$w = s_3s_2s_3s_4s_2s_1s_2 \ (I = \mathcal{D}_L(w) = \{\beta_2, \beta_3\}).$$

Thus $w_0(I) = s_2s_3s_2$ and $w_0(I)w = s_4s_2s_1s_2$ is not standard Coxeter. Hence X_w is not L_I-spherical. The interested reader can check w is I-spherical in the (different) sense of [18, Definition 1.2]. Therefore, this w provides a counterexample to [18, Conjecture 1.9] in type D_4. This counterexample was also (implicitly) verified in [16] using a different method, namely Demazure character computations, the topic of Section 5.

3. An equivariant isomorphism

The primary goal of this section is to construct a torus equivariant isomorphism from a specified affine subspace of the open cell of a Schubert variety to the open cell of a distinguished Schubert subvariety. In what follows, we assume standard facts from the theory of algebraic groups. References we draw upon are [20,6,25].

Let $w \in W$. Let n_w be a coset representative of w in $N_G(T)$. By definition of $N_G(T)$ being the normalizer of T in G, $t \mapsto n_w t n_w^{-1}$ defines an automorphism $\gamma_w : T \to T$.

Lemma 3.1. *The automorphism γ_w does not depend on our choice of coset representative n_w.*

Proof. Suppose that m_w is another coset representative of w. Then $m_w = n_w s$ for some $s \in T$. Hence $m_w t m_w^{-1} = n_w s t s^{-1} n_w^{-1} = n_w t s s^{-1} n_w^{-1} = n_w t n_w^{-1}$. □

In light of Lemma 3.1, henceforth for $w \in W$ we will also let w denote a coset representative of w in $N_G(T)$. Let X be a T-variety with action denoted by $\cdot$. For each $w \in W$ we define an action $\cdot_w$ on X by $t \cdot_w x = \gamma_w(t) \cdot x$ for all $x \in X$ and $t \in T$.

Lemma 3.2. *For all $w \in W$, the T-variety X has an open, dense T-orbit for the action $\cdot$ if and only if it has an open, dense T-orbit for the action $\cdot_w$. Indeed, the set of T-orbits in X for these two actions is identical.*

Proof. Let $\mathcal{O}$ be a T-orbit in X for the $\cdot$ action. Let $x, y \in \mathcal{O}$ and $t \in T$ be such that $t \cdot x = y$. As γ_w is an automorphism, there exists a $t' \in T$ such that $\gamma_w(t') = t$. Then

$$t' \cdot_w x = \gamma_w(t') \cdot x = t \cdot x = y.$$

Thus $\mathcal{O}$ is contained in the T-orbit $\mathcal{O}'$ of x for the action $\cdot_w$. The reverse containment is true by definition of $\cdot_w$. The lemma follows. □

For the remainder, we fix $\cdot$ to be the restriction to T of the action of G on G/B by left multiplication. The *left inversion set* of $w \in W$ is

$$\mathcal{I}(w) := \Phi^+ \cap w(\Phi^-) = \{\alpha \in \Phi^+ | w^{-1}(\alpha) \in \Phi^-\}.$$

Recall two standard facts regarding left inversion sets [21, Chapter 1]. For $w \in W$,

$$|\mathcal{I}(w)| = \ell(w) = \dim_{\mathbb{C}} X_w, \tag{1}$$

and

$$\mathcal{I}(w_0(I)) = \Phi^+(I), \tag{2}$$

where $\Phi(I) = \Phi(\mathfrak{l}_I, T)$ is the root system for the adjoint action of T on $\mathfrak{l}_I = \mathrm{Lie}(L_I)$.

We say that an algebraic group H is *directly spanned* by its closed subgroups $H_1, \ldots, H_n$, in the given order, if the product morphism

$$H_1 \times \cdots \times H_n \to H$$

is bijective. For $w \in W$, define $U_w := U \cap wU^- w^{-1}$, where U consists of the unipotent elements of B and similarly, U^- is the unipotent part of $B^- := w_0 B w_0$. This is a subgroup of U that is closed and normalized by T. Hence, by [6, §14.4], U_w is directly spanned, in any order, by the *root subgroups* U_α, $\alpha \in \Phi^+$, contained in U_w. Since by [22, Part II, 1.4(5)],

$$wU_\alpha w^{-1} = U_{w(\alpha)}, \tag{3}$$

these are the U_α such that $\alpha \in \Phi^+ \cap w(\Phi^-) = \mathcal{I}(w)$. Thus

$$U_w = \prod_{\alpha \in \mathcal{I}(w)} U_\alpha, \tag{4}$$

where the products U_α may be taken in any order.

Lemma 3.3. *For a coset $wB \in G/B$, we have*

$$X_w^\circ := BwB = U_w wB = \prod_{\alpha \in \mathcal{I}(w)} U_\alpha \; wB. \tag{5}$$

Moreover, X_w° is isomorphic to the affine space $\mathbb{A}^{\ell(w)}$ (as varieties).

Proof. It is a well-known fact that U_w is isomorphic to X_w° (as varieties) under the map $u \mapsto uwB$, $u \in U_w$ [6, §14.12]. The final equality in (5) is (4). By [6, Remark in §14.4], U_w is isomorphic, as a variety, to $\mathbb{A}^{\ell(w)}$. $\square$

We say that $w = uv \in W$ is *length additive* if $\ell(uv) = \ell(u) + \ell(v)$. Under this hypothesis, by [5, Ch. VI, §1, Cor. 2 of Prop. 17] one has

$$\mathcal{I}(uv) = \mathcal{I}(u) \sqcup u(\mathcal{I}(v)).$$

Therefore, in particular, if we assume $w_0(I)d \in W$ is *length additive*, then

$$\mathcal{I}(w_0(I)d) = \mathcal{I}(w_0(I)) \sqcup w_0(I)(\mathcal{I}(d)). \tag{6}$$

Define

$$V_d := w_0(I)U_d w_0(I)^{-1} = w_0(I)U_d w_0(I).$$

Lemma 3.4. *V_d is a closed subgroup of $U_{w_0(I)d}$ that is normalized by T.*

Proof. Since U_d is a closed subgroup normalized by T, so is V_d. Indeed, V_d is a subgroup of $U_{w_0(I)d}$ since

$$V_d = w_0(I) \prod_{\alpha \in \mathcal{I}(d)} U_\alpha w_0(I) = \prod_{\alpha \in w_0(I)(\mathcal{I}(d))} U_\alpha \leq U_{w_0(I)d}, \tag{7}$$

where the first equality is (4), the second is (3), and the subgroup claim is (4) and (6). $\square$

Lemma 3.5. *$U_{w_0(I)d}$ is directly spanned by $U_{w_0(I)}$ and V_d:*

$$U_{w_0(I)d} = U_{w_0(I)} V_d = V_d U_{w_0(I)}. \tag{8}$$

Proof. This follows from (4), (6), and (7) combined. □

Define

$$\tilde{O} := V_d w_0(I) dB \subseteq G/B.$$

Lemma 3.6. *$\tilde{O}$ is T-stable for the action $\cdot$.*

Proof. The claim follows since

$$V_d w_0(I) dB = (tV_d t^{-1}) t w_0(I) dB \subseteq V_d w_0(I) dB,$$

where the final step follows from the fact that V_d is normalized by T and that $w_0(I)dB$ is a T-fixed point in G/B. □

The following is the main point of this section:

Proposition 3.7. *If $w_0(I)d \in W$ is length additive then*

$$X^\circ_{w_0(I)d} = U_{w_0(I)d}\, w_0(I) dB.$$

Hence $\tilde{O} \subset X^\circ_{w_0(I)d}$. Moreover, $\tilde{O}$ with the T-action $\cdot$ is T-equivariantly isomorphic to X°_d with the T-action $\cdot_{w_0(I)}$.

Proof. By (5), $X^\circ_{w_0(I)d} = U_{w_0(I)d}\, w_0(I)dB$. Combining this with Lemma 3.4, one concludes that $\tilde{O} \subseteq X^\circ_{w_0(I)d}$. Define a map

$$\begin{aligned} \phi : \tilde{O} &\longrightarrow X^\circ_d \\ aB &\longmapsto w_0(I) aB. \end{aligned}$$

Now,

$$\phi(\tilde{O}) = w_0(I) V_d\, w_0(I) dB = U_d\, dB = X^\circ_d,$$

where the second equality is by the definition of V_d, and the final equality is Lemma 3.3. Thus ϕ is well-defined and surjective.

As ϕ is simply left multiplication by $w_0(I)$ it is injective. Further, by Lemma 3.3 X°_d is isomorphic as a variety to $\mathbb{A}^{\ell(d)}$, and thus is smooth, and in particular normal. Hence, by Zariski's main lemma, ϕ is an isomorphism of varieties.

To see that ϕ is T-equivariant for the specified actions, let $t \in T$ and $aB \in \tilde{O}$. Then

$$\phi(t \cdot aB) = w_0(I) t aB = w_0(I) t w_0(I) w_0(I) aB = \gamma_{w_0(I)}(t) \cdot \phi(aB) = t \cdot_{w_0(I)} \phi(aB). \quad \square$$

4. Proof of the main result

We need a lemma examining the L_I-action on $\tilde{O}$. This lemma is then used in conjunction with Proposition 3.7 to prove our main result.

Let $B_{L_I} = L_I \cap B$ and let U_{L_I} be the unipotent radical of B_{L_I}. Then B_{L_I} is a Borel subgroup in L_I [6, §14.17] with $U_{L_I} = B_{L_I} \cap U$ and $B_{L_I} = T \ltimes U_{L_I}$. Since L_I is the subgroup of G generated by T and $\{U_\alpha \mid \alpha \in \Phi(I)\}$ [25, §3.2.2], it is straightforward to show that

$$U_{L_I} = \prod_{\alpha \in \Phi^+(I)} U_\alpha,$$

where the product is taken in any order [6, §14.4].

Lemma 4.1. *Let $w = w_0(I)d \in W$ be length additive. Let $x \in X^\circ_{w_0(I)d} \setminus \tilde{O}$ and $y, z \in \tilde{O}$.*

(i) *$uy \notin \tilde{O}$ for all $u \in U_{L_I}$ with $u \neq e$.*
(ii) *$tx \notin \tilde{O}$ for all $t \in T$.*
(iii) *There exists $b \in B_{L_I}$ such that $by = z$ if and only if there exists $t \in T$ such that $ty = z$.*

Proof. (i) We have

$$U_{L_I} = \prod_{\alpha \in \Phi^+(I)} U_\alpha = U_{w_0(I)},$$

where the final equality is (4). Thus $u \in U_{w_0(I)}$.

Since $y \in \tilde{O}$, we have that $y = vw_0(I)dB$ for some $v \in V_d$. By Lemma 3.5, $uv \in U_{w_0(I)d} \setminus V_d$ for $u \neq e$. Thus $uvw_0(I)dB \in X^\circ_{w_0(I)d} \setminus \tilde{O}$ by Lemma 3.3.
(ii) This follows immediately from the fact that $\tilde{O}$ is T-stable.
(iii) The Borel $B_{L_I} = T \ltimes U_{L_I}$, and thus for all $b \in B_{L_I}$ we may express $b = tu$ for unique $t \in T, u \in U_{L_I}$. If $u \neq e$, then $uy \notin \tilde{O}$ by (i) and so $by = tuy \notin \tilde{O}$ by (ii). Hence, if $by = z$, then $u = e$ and $b = t \in T$. The converse direction is immediate since $T \subseteq B_{L_I}$. □

We now have the necessary ingredients to complete the proof of our main result.

Proof of Theorem 1.3: (⇐) Let w be I-spherical. Then $w = w_0(I)c$ is length additive and c is a standard Coxeter element. Our goal is to exhibit a $x \in \tilde{O}$ such that $\dim(B_{L_I} \cdot x) = \dim X^\circ_{w_0(I)c}$.

The Schubert variety X_c is a toric variety [24]; it contains an open, dense T-orbit $\mathcal{O}$ for the T-action $\cdot$. Since X°_c is an open, dense subset of X_c, $\mathcal{O} \cap X^\circ_c$ is open and dense in X°_c; since X°_c is T-stable we have that $\mathcal{O} \cap X^\circ_c$ is a T-orbit in X°_c for the T-action $\cdot$. Lemma 3.2 implies that $\mathcal{O} \cap X^\circ_c$ is an open, dense T-orbit for the T-action $\cdot_{w_0(I)}$.

By Proposition 3.7, there is a T-equivariant isomorphism $\phi : \tilde{O} \to X_c^\circ$. Let

$$\Theta = \phi^{-1}(\mathcal{O} \cap X_c^\circ);$$

this is an open, dense T-orbit in $\tilde{O}$ for the T-action $\bullet$. Let $x \in \Theta$. By Lemma 4.1(iii), the isotropy subgroup $(B_{L_I})_x$ is equal to the isotropy subgroup T_x. By [8, Proposition 1.11], for any variety X equipped with the action of an algebraic group H, the orbit $H \cdot x$, $x \in X$, is a subvariety of X of dimension $\dim H - \dim H_x$,

$$\dim(H \cdot x) = \dim H - \dim H_x. \tag{9}$$

The above combine to imply that

$$\dim(B_{L_I})_x = \dim T_x = \dim T - \dim(T \bullet x) = \dim T - \dim \Theta = \dim T - \ell(c). \tag{10}$$

We conclude, applying (9) and (10), that

$$\begin{aligned} \dim(B_{L_I} \cdot x) &= \dim B_{L_I} - \dim(B_{L_I})_x \\ &= \ell(w_0(I)) + \dim T - (\dim T - \ell(c)) \\ &= \ell(w_0(I)) + \ell(c) \\ &= \ell(w_0(I)c), \end{aligned}$$

and thus there exists an dense B_{L_I}-orbit in $X_{w_0(I)c}$. Indeed, this dense orbit must also be open in its closure by [6, Proposition 1.8]. Hence, $X_{w_0(I)c}$ is L_I-spherical.
($\Rightarrow$) Suppose w is not I-spherical. Then $w = w_0(I)d$ where d is not a standard Coxeter element. Moreover, by the hypothesis that $I \subseteq \mathcal{D}_L(w)$, this factorization is length additive.

The Schubert variety X_d is not a toric variety for the $\bullet$ action of T [24]. If X_d° contained an open, dense T-orbit, then X_d would be a toric variety for $\bullet$. Thus X_d° is not a toric variety for $\bullet$. In general, a normal G-variety is spherical if and only if it has finitely many B-orbits (see [30, Theorem 2.1.2]). If $G = T$ then $B = T$ and hence there are infinitely many T-orbits in X_d° for T-action $\bullet$; and for the T-action $\bullet_{w_0(I)}$ by Lemma 3.2.

By Proposition 3.7, $\tilde{O}$ is T-equivariantly isomorphic as an affine variety to X_d°. Thus, there are infinitely many T-orbits in $\tilde{O}$ for T-action $\bullet$. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two such orbits, and $x_1 \in \mathcal{O}_1, x_2 \in \mathcal{O}_2$. The fact that x_1 and x_2 reside in different orbits implies that there does not exist a $t \in T$ such that $tx_1 = x_2$. Thus Lemma 4.1(iii) implies $B_{L_I} \bullet x_1 \cap B_{L_I} \bullet x_2 = \emptyset$. As these were an arbitrary pair among the infinite T-orbits, there must be infinitely many B_{L_I} orbits in $X_{w_0(I)d}^\circ$ and hence in $X_{w_0(I)d}$. We conclude that $X_{w_0(I)d}$ is not L_I-spherical by the same result [30, Theorem 2.1.2] mentioned above. □

5. Application to Demazure modules

As an application of these results we give a sufficient condition for a Demazure module to be a multiplicity-free L_I-module; equivalently, a sufficient condition for a Demazure character to be multiplicity-free with respect to the basis of irreducible L_I-characters.

Let $\mathfrak{X}(T)$ denote the lattice of weights of T; our fixed Borel subgroup B determines a subset of dominant integral weights $\mathfrak{X}(T)^+ \subset \mathfrak{X}(T)$. The finite-dimensional irreducible G-representations are indexed by $\lambda \in \mathfrak{X}(T)^+$. Denoting the associated representation by V_λ, there is a class of B-submodules of V_λ, first introduced by Demazure [12], that are indexed by $w \in W$. If v_λ is a nonzero highest weight vector, then the *Demazure module* V_λ^w is the minimal B-submodule of V_λ containing wv_λ.

There is a geometric construction of these Demazure modules. For $\lambda \in \mathfrak{X}(T)^+$, let $\mathfrak{L}_\lambda$ be the associated line bundle on G/B. For $w \in W$, we write $\mathfrak{L}_\lambda|_{X_w}$ for the restriction of $\mathfrak{L}_\lambda$ to the Schubert subvariety $X_w \subseteq G/B$. Then the Demazure module V_λ^w is isomorphic to the dual of the space of global sections of $\mathfrak{L}_\lambda|_{X_w}$, that is

$$V_\lambda^w \cong H^0(X_w, \mathfrak{L}_\lambda|_{X_w})^*.$$

This geometric perspective highlights the fact that V_λ^w is not just a B-module, but is in fact also a L_I-module via the action induced on $H^0(X_w, \mathfrak{L}_\lambda|_{X_w})$ by the left multiplication action of L_I on X_w.

As L_I is a reductive group over characteristic zero, any L_I-module decomposes into a direct sum of irreducible L_I-modules. Let $\mathfrak{X}_{L_I}(T)^+$ be the set of dominant integral weights with respect to the choice of maximal torus and Borel subgroup $T \subseteq B_I \subseteq L_I$. For $\mu \in \mathfrak{X}_{L_I}(T)^+$, let $V_{L_I,\mu}$ be the associated irreducible L_I-module. If M is a L_I-module and

$$M = \bigoplus_{\mu \in \mathfrak{X}_{L_I}(T)^+} V_{L_I,\mu}^{\oplus m_{L_I,\mu}}$$

is the decomposition into irreducible L_I-modules, then we say that M is a *multiplicity-free L_I-module* if $m_{L_I,\mu} \in \{0,1\}$. Similarly, if $\operatorname{char}(M)$ is the formal T-character of M and

$$\operatorname{char}(M) = \sum_{\mu \in \mathfrak{X}_{L_I}(T)^+} m_{L_I,\mu} \operatorname{char}(V_{L_I,\mu}),$$

then we say that $\operatorname{char}(M)$ is *I-multiplicity-free* if $m_{L_I,\mu} \in \{0,1\}$.

The following argument was given for type A in [18, Theorem 4.13(II)]. We include the general type argument (which is essentially the same) for sake of completeness:

Theorem 5.1. *Let $w \in W$ with $I \subseteq D_L(w)$. Then X_w is L_I-spherical if and only if for all $\lambda \in \mathfrak{X}(T)^+$, the Demazure module V_λ^w is multiplicity-free L_I-module.*

Proof. Let X be a quasi-projective, normal variety with the action of a complex, connected, reductive algebraic group G. Then X is G-spherical if and only if the G-module $H^0(X, \mathfrak{L})$ is a multiplicity free G-module for all G-linearized line bundles $\mathfrak{L}$ [30, Theorem 2.1.2].

All Schubert varieties $X_w \subseteq G/B$ are normal, quasi-projective varieties [23]. Further, as L_I is reductive and we are in characteristic zero, V_λ^w is a multiplicity-free L_I-module if and only if its dual space $(V_\lambda^w)^* = H^0(X_w, \mathfrak{L}_\lambda|_{X_w})$ is a multiplicity-free L_I-module [20, §31.6]. The above combines to imply our desired result once we show that the L_I-linearized line bundles on X_w are precisely of the form $\mathfrak{L}_\lambda|_{X_w}$ for $\lambda \in \mathfrak{X}(T)^+$.

The line bundles, with non-trivial spaces of global sections, on G/B are precisely $\mathfrak{L}_\lambda$, for $\lambda \in \mathfrak{X}(T)^+$; these line bundles are all G-linearized [7, §1.4]. Every line bundle on X_w is the restriction of a line bundle on G/B [7, Proposition 2.2.8]. We are done since the restriction $\mathfrak{L}_\lambda|_{X_w}$ of the G-linearized line bundle $\mathfrak{L}_\lambda$, for $\lambda \in \mathfrak{X}(T)^+$, is L_I-linearized. □

Corollary 5.2. *Let $w \in W$ be I-spherical for $I \subseteq D_L(w)$. For all $\lambda \in \mathfrak{X}(T)^+$, the Demazure module V_λ^w is a multiplicity-free L_I-module.*

Proof. By Theorem 1.3, if w is I-spherical then X_w is L_I-spherical. Therefore, by Theorem 5.1, V_λ^w is a multiplicity-free L_I-module for $\lambda \in \mathfrak{X}(T)^+$. □

Corollary 5.3. *Let $w \in W$ be I-spherical for $I \subseteq D_L(w)$. For all $\lambda \in \mathfrak{X}(T)^+$, the Demazure character* $\mathrm{char}(V_\lambda^w)$ *is I-multiplicity-free.*

These two corollaries appear non-trivial from a combinatorial perspective, even for a *specific choice* of dominant weight λ with fixed $w \in W$. The Demazure character can be recursively computed using Demazure operators. There is also a combinatorial rule for the character in terms of crystal bases (in instantiations such as the *Littelmann path model* or the *alcove walk model*); see, e.g., the textbook [9]. However, an argument based on these methods eludes in general type, although we have an argument in type A [15].

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