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Classification of Levi-spherical Schubert varieties

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Abstract

A Schubert variety in the complete flag manifold GL_n/B is *Levi-spherical* if the action of a Borel subgroup in a Levi subgroup of a standard parabolic has an open dense orbit. We give a combinatorial classification of these Schubert varieties. This establishes a conjecture of the latter two authors, and a new formulation in terms of standard Coxeter elements. Our proof uses and contributes to the theory of *key polynomials* (type A Demazure module characters).

Mathematics Subject Classification 14M27 · 05E10 · 05E14

1 Introduction

The question of which Schubert varieties in GL_n/B are singular was first combinatorially characterized by J. Wolper [34] after a geometric characterization by K. Ryan [29]. V. Lakshmibai–B. Sandhya [22] gave an alternative combinatorial characterization in terms of permutation pattern avoidance. These results are at the foundation of subsequent work on the singular structure of Schubert varieties; see the book [7], the surveys [1, 35], and the references therein. In this paper we also classify a different "global" geometric property of Schubert varieties, namely, *sphericality* with respect

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to a Levi subgroup of GL_n. However, in contrast, sphericality is *not* a singularity property.

The study of spherical varieties has garnered significant interest; see, e.g., N. Perrin's survey [28]. For example, the notion of being a spherical variety subsumes that of toric varieties, and moreover, Luna-Vust theory gives a description of all birational models of a spherical variety via colored fans (generalizing the concept of fans in toric geometry). Spherical varieties have many nice features. For example, projective spherical varieties are Mori dream spaces.

It is an unsolved problem to classify all spherical actions on products of flag varieties. For the case of Levi subgroups this is solved; see work of P. Littelmann [24], P. Magyar-J. Weyman-A. Zelevinsky [25, 26], J. Stembridge [31, 32], R. Avdeev–A. Petukov [5, 6]. The results of this paper are complementary (in type A) to these earlier results.

This is a sequel to [18] which gave a geometrically motivated, conjectural, combinatorial classification of Schubert varieties that are spherical for the action of a Levi subgroup. While the paper examined the situation in general type, of particular focus was the GL_n/B case. It is in this situation that one finds direct connections to well-studied elements of algebraic combinatorics. Algebraic combinatorics has at its core the theory of symmetric polynomials and Schur polynomials. Modern aspects of the field concern themselves with asymmetric polynomial families such as the *key polynomials* both in their role as characters of Demazure modules but also for their combinatorial features. The aforementioned Levi-sphericality conjecture motivates the consideration of key polynomials for their split-symmetry and suggests the study of when they are multiplicity-free in the split-Schur basis. A strategy was suggested for proving the conjecture from these considerations. This paper completes this strategy.

The main new idea of this paper is a simpler formulation of the conjecture in terms of *standard Coxeter elements*. While the original conjecture of [18] was founded on a geometric heuristic, our new formulation is compatible with the Demazure operators used to define the key polynomials. Therefore, it is this new version that we actually prove. Separately, we establish the equivalence of the two conjectures in type A, thus proving the original version as well.

In proving our main result, we observe that the set of weights appearing as exponents in a key polynomial associated to a standard Coxeter element decompose into posets isomorphic to intervals in the Bruhat order of a Young subgroup. Extensive computations suggest that this remains true of Demazure characters in general type and we hope to explore this surprising poset structure in future work.

Since the results of this work were first announced, there have been a number of follow-up works. Assuming Theorem 1.3, C. Gaetz [14] proves a pattern avoidance criterion for maximally spherical Schubert varieties, thus proving a conjecture from [18]. Now, in *ibid.*, the conjecture was stated in general type. However, in [16] we gave a counterexample to that general conjecture for SO_8/B. On the other hand, [16, Conjecture 4.1] presents, with supporting evidence, a different conjecture to replace it—indeed one that generalizes our new formulation (Theorem 1.3) below. This conjecture has since been simultaneously and independently proved by M. Can–P. Saha [9] and by the authors [17]. The arguments of those papers are shorter but depend on background in algebraic groups. By comparison, the methods here are essentially completely combinatorial, and we believe contribute to the theory of key polynomials. Moreover, this paper provides proofs of both combinatorial classifications in the GL_n case.

1.1 Main result

Let Flags($\mathbb{C}^n$) be the variety of complete flags $\langle 0\rangle \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n$, where F_i is a subspace of dimension i. The group GL_n of invertible $n \times n$ matrices over $\mathbb{C}$ acts transitively on Flags($\mathbb{C}^n$) by change of basis. The *standard flag* is defined by $F_i = \text{span}(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_i)$ where $\vec{e}_i$ is the i-th standard basis vector. The stabilizer of this flag is $B \subset GL_n$, the Borel subgroup of upper triangular invertible matrices. Hence Flags($\mathbb{C}^n$) $\cong GL_n/B$. B acts on GL_n/B with finitely many orbits; these are the *Schubert cells* $X_w^\circ = BwB/B \cong \mathbb{C}^{\ell(w)}$ indexed by $w \in \mathfrak{S}_n$ (viewed as a permutation matrix). Their closures $X_w := \overline{X_w^\circ}$ are the *Schubert varieties*; these are of interest in algebraic geometry and representation theory. A standard reference is [13].

For $I \subseteq J(w)$, let $L_I \subseteq GL_n$ be the Levi subgroup of invertible block diagonal matrices

$$L_I \cong GL_{d_1-d_0} \times GL_{d_2-d_1} \times \cdots \times GL_{d_k-d_{k-1}} \times GL_{d_{k+1}-d_k}.$$

As explained in, e.g., [18, Section 1.2], L_I acts on X_w.

Definition 1.1 X_w is L_I*-spherical* if X_w has an open dense orbit of a Borel subgroup of L_I. If in addition, $I = J(w)$, X_w is *maximally spherical.*

Our main result is a classification of L_I-spherical Schubert varieties using combinatorics. Let $G = GL_n$. Its Weyl group $W \cong \mathfrak{S}_n$ consists of permutations of $[n] := \{1, 2, \ldots, n\}$. Thus W is generated, as a Coxeter group, by the simple transpositions $S = \{s_i = (i\ i+1) : 1 \leq i \leq n-1\}$. The set of *left descents* is

$$J(w) = \{j \in [n-1] : w^{-1}(j) > w^{-1}(j+1)\}.$$

In other words, $j \in J(w)$ if $j+1$ appears to the left of j in w's one-line notation.

Let $\ell(w)$ denote the *Coxeter length* of w. For $w \in \mathfrak{S}_n$,

$$\ell(w) = \#\{1 \leq i < j \leq n : w(i) > w(j)\}$$

counts *inversions* of w.

A *parabolic subgroup* W_I of W is the subgroup generated by a subset $I \subset S$. A *standard Coxeter element* $c \in W_I$ is any product of the elements of I listed in some order. Let $w_0(I)$ be the longest element of W_I.

Definition 1.2 Let $w \in W$ and fix $I \subseteq J(w)$. Then w is I*-spherical* if $w_0(I)w$ is a standard Coxeter element for some parabolic subgroup $W_{I'}$ of W.

The following is our main theorem:

Theorem 1.3 (*cf.* [18, Conjecture 3.2]) *Let $w \in \mathfrak{S}_n$ and $I \subseteq J(w)$. $X_w \subseteq GL_n/B$ is L_I-spherical if and only if w is I-spherical.*

In [18] another combinatorial definition (Definition 7.1) for I-sphericality is used. However, Definition 1.2 is the cornerstone of our argument, and its significance can be traced to Lemma 3.1. We show in Sect. 7 that Definition 1.2 and Definition 7.1 are equivalent in type A, and therefore Theorem 1.3 gives the first (and currently, only) proof of [18, Conjecture 3.2]. In light of our upcoming paper [17], Definition 1.2 is the correct general type definition, with clear connections to boolean permutations [33].

1.2 Strategy of the proof

Using Theorem 4.13 of [18], our main result, Theorem 1.3 is reduced to Theorem 3.8, a character-theoretic statement. We prove the two directions "$\Rightarrow$" and "$\Leftarrow$" of Theorem 3.8 separately. The "$\Rightarrow$" direction requires a careful analysis on the terms involved in $\kappa_{w\lambda}$, which can be compactly organized using a poset structure $\mathcal{P}_{c\lambda,\gamma}$, introduced in Sect. 4, whose main feature is the "Diamond property" (Theorem 4.4). This "Diamond property", proved in Sect. 5, is the crucial technical lemma that helps to establish the "$\Rightarrow$" direction of Theorem 3.8. Sections 2 and 3 contain basic background and setup for the discussion of $\mathcal{P}_{c\lambda,\gamma}$ and the "Diamond property": Sect. 2 introduces some notation and terminology about symmetric groups, Bruhat order, and a certain poset $\mathcal{S}_{I,\gamma}$ that we define; Sect. 3 recalls notions about *key polynomials*, split-symmetry, and multiplicity-freeness from [18] connecting Coxeter combinatorics to the geometry. The "$\Leftarrow$" direction is then proved in Sect. 6 via explicit construction.

Finally, in Sect. 7 we prove Theorem 7.2; in the process, we establish a root-system uniform result (Proposition 7.8) that shows Definition 1.2 and Definition 7.3 from [18] (a generalization of Definition 7.1) are, in some sense, "close" in general type.

2 Bruhat order of Young subgroups and the poset $\mathcal{S}_{I,\gamma}$

The main objective of this section is the introduction of the poset $\mathcal{S}_{I,\gamma}$, which we show is isomorphic to a Young subgroup of $\mathfrak{S}_n$. Our eventual goal will be to study certain subposets of $\mathcal{S}_{I,\gamma}$ that play a role in the analysis of the terms of the key polynomial $\kappa_{w\lambda}$.

The symmetric group $\mathfrak{S}_n$ has the poset structure of *(strong) Bruhat order* $<_{\text{Bruhat}}$. It is convenient for us to use the "upside down" version. That is, the covering relations are $u <_{\text{Bruhat}} us_{ij}$ where $\ell(u) - 1 = \ell(us_{ij})$ and $s_{ij} = (i\ j)$ is a transposition. Hence, under this choice of convention, the longest permutation $w_0 = n\ n-1\ \ldots\ 3\ 2\ 1$ is the unique minimum, and the identity permutation is the unique maximum.

A sequence of non-negative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a *weak composition*. Let Comp_n be the set of all such compositions. Let Par_t be the set of partitions with at most t nonzero parts. A *split-partition* is

$$(\lambda^1, \ldots, \lambda^k) \in \mathsf{Par}_D := \mathsf{Par}_{d_1-d_0} \times \cdots \times \mathsf{Par}_{d_{k+1}-d_k}.$$

Fix $\gamma \in \mathsf{Par}_D$, where $D = [n-1] - I$ (as in Sect. 1), which we will identify (in the obvious way) with an element of Comp_n.

Definition 2.1 $i, j \in [n]$ are *in the same block* (with respect to $D = [n] - I$) if there exists $t \in [0, k]$ such that $d_t + 1 \leq i, j \leq d_{t+1}$.

Let $\delta_t = (t, t-1, \ldots, 3, 2, 1)$. Given γ, pick $\Delta := \Delta_\gamma \in \mathbb{Z}^n_{\geq 0}$ to be any fixed but arbitrary strictly decreasing vector such that:

- In the i-th block (of size $d_i - d_{i-1}$), the components of Δ are of the form $(f_i, f_i, \ldots, f_i) + \delta_{d_i - d_{i-1}}$ where f_i is some positive integer depending on i.
- $\gamma + \Delta$ is a vector with distinct components.

Let $\widehat{\mathfrak{S}}_n$ be permutations on the (distinct) entries of $\gamma + \Delta$. Clearly there is an isomorphism of Bruhat orders between that of $\mathfrak{S}_n$ and $\widehat{\mathfrak{S}}_n$ that sends w_0 to $\Delta + \gamma$. We will therefore mildly abuse notation and use $<_{\text{Bruhat}}$ for either order, as the context will be clear. Let

$$\Omega : (\mathfrak{S}_n, <_{\text{Bruhat}}) \to (\widehat{\mathfrak{S}}_n, <_{\text{Bruhat}})$$

be this poset isomorphism.

Now, let

$$\tilde{\mathcal{S}}_{I,\gamma} = \widehat{\mathfrak{S}}_{d_1 - d_0} \times \widehat{\mathfrak{S}}_{d_2 - d_1} \times \cdots \times \widehat{\mathfrak{S}}_{d_{k+1} - d_k}$$

be the Young subgroup of $\widehat{\mathfrak{S}}_n$, where $\widehat{\mathfrak{S}}_{d_{i+1} - d_i}$ is the permutation group on the labels of $\Delta + \gamma$ in the i-th block. Thus, strong Bruhat order $<_{\text{Bruhat}}$ on $\widehat{\mathfrak{S}}_n$ restricts to $\tilde{\mathcal{S}}_{I,\gamma}$.

Definition 2.2 Given $\tilde{\beta} \in \tilde{\mathcal{S}}_{I,\gamma}$ (thought of as a vector in $\mathbb{Z}^n_{\geq 0}$), let

$$\Phi(w) = \tilde{\beta} - \Delta.$$

Let $\mathcal{S}_{I,\gamma} := \operatorname{Im} \Phi \subset \mathsf{Comp}_n$. For $x, y \in \mathcal{S}_{I,\gamma}$ define $x <_{\text{Bruhat}} y$ if $\Phi^{-1}(x) <_{\text{Bruhat}} \Phi^{-1}(y)$.

Proposition 2.3 $(\mathcal{S}_{I,\gamma}, <_{\text{Bruhat}}) \cong (\tilde{\mathcal{S}}_{I,\gamma}, <_{\text{Bruhat}}) \cong (\mathfrak{S}_{d_1 - d_0} \times \cdots \times \mathfrak{S}_{d_{k+1} - d_k}, <_{\text{Bruhat}})$.

Proof Φ is injective and hence a bijection onto its image. It is a poset map by construction. This proves the first isomorphism. The second isomorphism is induced from Ω. $\square$

Definition 2.4 If $\beta = (\beta_1, \ldots, \beta_n) \in \mathsf{Comp}_n$ and $i < j \in [n-1]$, define $t_{ij} : \mathsf{Comp}_n \to \mathsf{Comp}_n$ by

$$t_{ij}(\ldots, \beta_i, \ldots, \beta_j, \ldots) = (\ldots, \beta_j - (j-i), \ldots, \beta_i + (j-i), \ldots). \tag{1}$$

Also let $t_i := t_{i\ i+1}$.

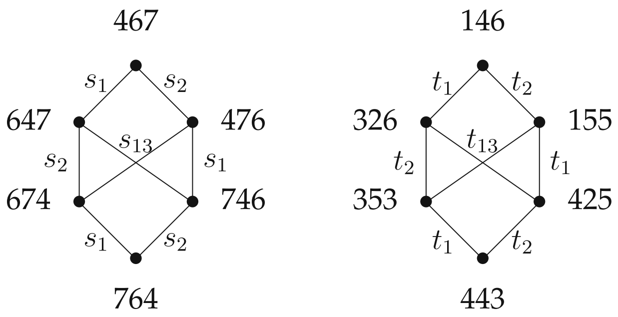


Fig. 1 Example of the poset $\tilde{\mathcal{S}}_{I,\gamma}$ (left) and $\mathcal{S}_{I,\gamma}$ (right)

The next lemma asserts that the role of t_{ij}'s in $\mathcal{S}_{I,\gamma}$ is the same as that of the $s_{ij} = (i\ \ j)$ in $\mathfrak{S}_n$. In particular, the t_i's are analogous to the simple transpositions.

Lemma 2.5 *For $i < j$ in the same block, this diagram commutes:*

$$\begin{array}{ccc} \tilde{\mathcal{S}}_{I,\gamma} & \xrightarrow{\Phi} & \mathcal{S}_{I,\gamma} \\ s_{ij} \downarrow & & \downarrow t_{ij} \\ \tilde{\mathcal{S}}_{I,\gamma} & \xrightarrow{\Phi} & \mathcal{S}_{I,\gamma}. \end{array} \tag{2}$$

Proof Let $\tilde{\beta} \in \tilde{\mathcal{S}}_{I,\gamma}$. By definition of Δ, there is some number f such that $\Delta_k = f - k$ for $i \le k \le j$. We have

$$\begin{aligned} t_{ij}\Phi\tilde{\beta} &= t_{ij}(\ldots, \tilde{\beta}_i - f + i, \ldots, \tilde{\beta}_k - f + k, \ldots, \tilde{\beta}_j - f + j, \ldots) \\ &= (\ldots, \tilde{\beta}_j - f + j - (j - i), \ldots, \tilde{\beta}_k - f + k, \ldots, \tilde{\beta}_i - f + i + (j - i), \ldots) \\ &= (\ldots, \tilde{\beta}_j - f + i, \ldots, \tilde{\beta}_k - f + k, \ldots, \tilde{\beta}_i - f + j, \ldots) \\ &= \Phi(\ldots, \tilde{\beta}_j, \ldots, \tilde{\beta}_k, \ldots, \tilde{\beta}_i, \ldots) = \Phi s_{ij}\tilde{\beta} \end{aligned}$$

as desired. □

Example 2.6 Let $n = 3$, $I = \{1, 2\}$ with a single block, $\gamma = 443$ and $\Delta = 321$. Figure 1 shows the poset $\tilde{\mathcal{S}}_{I,\gamma}$ and $\mathcal{S}_{I,\gamma}$ with the actions of s_{ij}'s and t_{ij}'s respectively.

Remark 2.7 Having formally defined $(\mathcal{S}_{I,\gamma}, <_{\text{Bruhat}})$ above, in the remainder of the paper, one can think of this poset as generated from γ *via* the action of t_{ij}'s, including just the t_i's.

Definition 2.8 For $\beta \in \mathcal{S}_{I,\gamma}$, let $\theta(\beta)$ be the *rank* of β, i.e., there exists a saturated chain

$$\beta = \beta^{(\theta)} \gtrdot_{\text{Bruhat}} \beta^{(\theta-1)} \gtrdot_{\text{Bruhat}} \cdots \gtrdot_{\text{Bruhat}} \beta^{(0)} = \gamma$$

of length $\theta = \theta(\beta)$ from β to the minimum γ in $\mathcal{S}_{I,\gamma}$. Also define the *sign* of β to be $\mathsf{sgn}(\beta) := (-1)^{\theta(\beta)}$.

These facts follow immediately from the usual Bruhat orders and the isomorphism Φ.

Lemma 2.9 *For $\beta \in \mathcal{S}_{I,\gamma}$ and i, j in the same block,*

(i) $\beta_i > \beta_j - (j - i)$ *if and only if* $\beta <_{\text{Bruhat}} t_{ij}\beta$*; in particular,* $\beta_i - i \neq \beta_j - j$ *for* $i \neq j$*;*
(ii) $\mathsf{sgn}(t_{ij}\beta) = -\mathsf{sgn}(\beta)$.

3 Polynomials and sphericality

Below we define key polynomials and highlight a number of their important properties. We recall the relationship between key polynomials, split-symmetry, and multiplicity-freeness that was established in [18]. This allows Theorem 1.3 to be restated as Theorem 3.8; the proof of Theorem 3.8 will then occupy the remainder of this work.

3.1 Key polynomials

Let $\mathsf{Pol} := \mathbb{Z}[x_1, x_2, \ldots, x_n]$ be the polynomial ring in the indeterminates $x_1, x_2, \ldots, x_n$. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathsf{Comp}_n$, the *key polynomial* κ_α is defined as follows. If α is weakly decreasing, then $\kappa_\alpha := \prod_i x_i^{\alpha_i}$. Otherwise, suppose $\alpha_i > \alpha_{i+1}$. Let

$$\pi_i : \mathsf{Pol} \to \mathsf{Pol}, \quad f \mapsto \frac{x_i f(\ldots, x_i, x_{i+1}, \ldots) - x_{i+1} f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}},$$

and

$$\kappa_\alpha = \pi_i(\kappa_{\widehat{\alpha}}) \text{ where } \widehat{\alpha} := (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots).$$

We need facts about the operators π_i; our reference is [23]. The operators π_i satisfy the relations

$$\begin{aligned} \pi_i \pi_j &= \pi_j \pi_i (\text{for } |i - j| > 1) \\ \pi_i \pi_{i+1} \pi_i &= \pi_{i+1} \pi_i \pi_{i+1} \\ \pi_i^2 &= \pi_i. \end{aligned}$$

Recall that the *Demazure product* on $\mathfrak{S}_n$ is defined by

$$w * s_i = \begin{cases} w s_i & \text{if } \ell(w s_i) = \ell(w) + 1 \\ 0 & \text{otherwise.} \end{cases}.$$

This product is associative. Then $R = (s_{i_1}, \cdots, s_{i_\ell})$ is a *Hecke word* of w if $w = s_{i_1} * s_{i_2} * \cdots * s_{i_\ell}$.

For any $w \in \mathfrak{S}_n$ one unambiguously defines

$$\pi_w := \pi_{i_1}\pi_{i_2}\cdots\pi_{i_\ell},$$

where $R = (s_{i_1}, \ldots, s_{i_\ell})$ is any Hecke word of w.

Now suppose $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n)$ is a partition, and $w \in \mathfrak{S}_n$. Define

$$\kappa_{w\lambda} := \kappa_{\lambda_{w^{-1}(1)},\ldots,\lambda_{w^{-1}(n)}}.$$

With this choice of convention, we have

$$\kappa_{w\lambda} = \pi_w \kappa_\lambda. \tag{3}$$

Lemma 3.1 *Suppose $w = w_0(I)c$ where c is a standard Coxeter element and moreover $\ell(w) = \ell(w_0(I)) + \ell(c)$. Then $\kappa_{w\lambda} = \pi_{w_0(I)}\kappa_{c\lambda}$.*

Proof By two applications of (3), and the definition of π_w

$$\kappa_{w\lambda} = \kappa_{w_0(I)c\lambda} = \pi_{w_0(I)c}(\kappa_\lambda) = \pi_{w_0(I)}\pi_c(\kappa_\lambda) = \pi_{w_0(I)}\kappa_{c\lambda}.$$

For any $\alpha \in \mathsf{Comp}_n$, let

$$a_{\alpha_1+n-1,\alpha_2+n-2,\ldots,\alpha_n} := \det(x_j^{\lambda_i+n-i})_{1\leq i,j\leq n}.$$

In particular,

$$\Delta_n := a_{n-1,n-2,\ldots,0} = \prod_{1\leq j<k\leq n}(x_j - x_k)$$

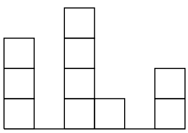
is the Vandermonde determinant. Define a *generalized Schur polynomial* s_α by

$$s_\alpha(x_1,\ldots,x_n) := a_{\alpha_1+n-1,\alpha_2+n-2,\ldots,\alpha_n}/a_{n-1,n-2,\ldots,1,0}. \tag{4}$$

This is well-known, and clear from (4) and the row-swap property of determinants:

Lemma 3.2 *$s_{t_i\alpha}(x_1,\ldots,x_n) = -s_\alpha(x_1,\ldots,x_n)$. Thus, if $\alpha_{i+1} = \alpha_i + 1$ then $s_\alpha(x_1,\ldots,x_n) = 0$.*

A result we need is a characterization of the monomials x^β that appear (with nonzero coefficient) in κ_α. Graphically represent the weak composition α as a *skyline* $D(\alpha)$ of boxes where column i (from the left) is a tower of α_i boxes. For example, if $\alpha = (3, 0, 4, 1, 0, 2)$ then the associated skyline is



Define $\mathsf{Tab}(\alpha)$ to be fillings of $D(\alpha)$ with $\mathbb{N} := \{1, 2, 3, \ldots\}$ such that:

- no label appears twice in a row (*row distinct*); and
- the labels in column i are at most i (*flagged*).

The *weight* of $T \in \mathsf{Tab}(\alpha)$ is the vector $\mathsf{wt}(T) = (c_1, c_2, \ldots)$ where $c_i = \#\{i \in T\}$. The following result is implicit in [2–4] and explicit in [11].

Theorem 3.3 $[x^\beta]\kappa_\alpha \neq 0$ *if and only if there exists* $T \in \mathsf{Tab}(\alpha)$ *with content* β.

Proof We explicate the argument alluded to in [2–4]; we refer to these papers for definitions. This argument differs from the one in [11]. In [4], it is shown that a lattice point β appears in the *Schubitope* associated to $D(\alpha)$ (rotated 90-degrees clockwise) if and only if there exists $T \in \mathsf{Tab}(\alpha)$ with content β. In [12], it is proved that these lattice points correspond exactly to the monomials of κ_α. $\square$

A consequence of Theorem 3.3 that we will use is

Corollary 3.4 *Let* $\alpha, \beta \in \mathsf{Comp}_n$ *and assume* $[x^\beta]\kappa_\alpha > 0$. *Suppose* $i < j$ *and* $\beta_j - \beta_i = t \in \mathbb{Z}_{>0}$. *For* $1 \leq s \leq t$, *let* $\beta' := (\ldots, \beta_i + s, \ldots, \beta_j - s, \ldots)$. *Then* $[x^{\beta'}]\kappa_\alpha > 0$.

Proof By Theorem 3.3 there exists $T \in \mathsf{Tab}(\alpha)$ of content β. By definition, there are β_j distinct rows where T has a label j, and there are β_i distinct rows where T has a label i. Since $\beta_j - \beta_i = t$, there exist s rows where T contains a j but not an i. Define T' by replacing j by i in those s rows. Since $i < j$, we conclude $T' \in \mathsf{Tab}(\beta')$ and hence (by Theorem 3.3), $[\beta']\kappa_\alpha > 0$, as claimed. $\square$

Given α, define the set of *Kohnert diagrams* $\mathsf{Koh}(\alpha)$ iteratively. To start $D(\alpha) \in \mathsf{Koh}(\alpha)$. If $D \in \mathsf{Koh}(\alpha)$, consider the top-most box in any column. Let D' be the result of moving that box left, in the same row, to the rightmost location that is not occupied (if it exists); this operation is a *Kohnert move*. Now include $D' \in \mathsf{Koh}(\alpha)$, as well. We emphasize that $\mathsf{Koh}(\alpha)$ is a finite set (rather than multiset), hence if a diagram D is obtained by two different sequences of Kohnert moves starting from $D(\alpha)$, then D only counts once in $\mathsf{Koh}(\alpha)$.

Given $D \in \mathsf{Koh}(\alpha)$, let

$$\mathsf{Kohwt}(D) = \prod_{i=1}^{n} x_i^{\#\text{boxes of } D \text{ in column } i}.$$

Theorem 3.5 (Kohnert's rule [21]) $\kappa_\alpha = \sum_{D \in \mathsf{Koh}(\alpha)} \mathsf{Kohwt}(D)$.

Define *dominance order* on $\alpha, \beta \in \mathsf{Comp}_n$ such that $|\alpha| := \sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i := |\beta|$ by $\alpha \leq_{\mathsf{dom}} \beta$ if for every $1 \leq t \leq n$ we have $\sum_{i=1}^t \alpha_i \leq \sum_{i=1}^t \beta_i$.

Corollary 3.6 *Let* $\alpha, \beta \in \mathsf{Comp}_n$ *with* $[x^\beta]\kappa_\alpha > 0$. *Then* $\beta \geq_{\mathsf{dom}} \alpha$.

3.2 Split-symmetry

We recall some notions from [18, Section 4]. Suppose

$$d_0 := 0 < d_1 < d_2 < \ldots < d_k < d_{k+1} := n$$

and $D = \{d_1, \ldots, d_k\}$. Let Π_D be the subring of Pol consisting of the polynomials that are separately symmetric in $X_i := \{x_{d_{i-1}+1}, \ldots, x_{d_i}\}$ for $1 \leq i \leq k+1$. If $f \in \Pi_D$, f is *D-split-symmetric*.

The ring Π_D has a basis of *D-Schur polynomials*

$$s_{\lambda^1,\ldots,\lambda^k} := s_{\lambda^1}(X_1)s_{\lambda^2}(X_2)\cdots s_{\lambda^k}(X_k),$$

where

$$(\lambda^1, \ldots, \lambda^k) \in \mathsf{Par}_D := \mathsf{Par}_{d_1-d_0} \times \cdots \times \mathsf{Par}_{d_{k+1}-d_k},$$

and Par_t is the set of partitions with at most t nonzero-parts. See [18, Definition 4.3, Corollary 4.4]. Thus, for any $f \in \Pi_D$ there is a unique expression

$$f = \sum_{(\lambda^1,\ldots,\lambda^k)\in \mathsf{Par}_D} c_{\lambda^1,\ldots,\lambda^k} s_{\lambda^1,\ldots,\lambda^k}.$$

If $c_{\lambda^1,\ldots,\lambda^k} \in \{0, 1\}$ for all $(\lambda^1, \ldots, \lambda^k) \in \mathsf{Par}_D$, f is called *D-multiplicity-free*.

This fact allows us to study Levi-sphericality using key polynomials:

Theorem 3.7 ([18, Theorem 4.13]) *Let $\lambda \in \mathsf{Par}_n$, and $w \in \mathfrak{S}_n$. Suppose $I \subseteq J(w)$ and $D = [n-1] - I$. X_w is L_I-spherical if and only if $\kappa_{w\lambda}$ is D-multiplicity-free for all $\lambda \in \mathsf{Par}_n$.*

In view of Theorem 3.7, the following is equivalent to Theorem 1.3.

Theorem 3.8 *Let $D = [n-1]-I$. w is I-spherical if and only if $\kappa_{w\lambda}$ is D-multiplicity-free for all $\lambda \in \mathsf{Par}_n$.*

Our goal is therefore to prove Theorem 3.8. To do this, we will use the lemma below.

Lemma 3.9 *Let $\beta \in \mathsf{Comp}_n$. Then*

$$\pi_{w_0(I)}(x_1^{\beta_1} \cdots x_n^{\beta_n}) \in \{0, \mathsf{sgn}(\beta) s_{\alpha^1,\ldots,\alpha^k}\},$$

where $(\alpha^1, \ldots, \alpha^k) \in \mathsf{Par}_D$.

Proof First, consider the special case that $w_0(I) = w_0$. By [23, Proposition 1.5.1],

$$\pi_{w_0}(f) = \frac{1}{\Delta_n} x^\rho \sum_{w\in\mathfrak{S}_n} (-1)^{\ell(w)} w(f).$$

Hence by (4), $\pi_{w_0}(x^\beta) = s_\beta$. Rearrange β to be weakly decreasing by application of the operators $t_1, t_2, \ldots$ and swapping two adjacent entries where the left entry is strictly smaller than the other one. This can always be achieved unless during this process one arrives at a composition κ where $\kappa_{i+1} = \kappa_i + 1$. In that case, Lemma 3.2 asserts $s_\beta = 0$. Otherwise we arrive at $\alpha \in \mathsf{Par}_n$ and Lemma 3.2 combined with Definition 2.8 shows $s_\beta = \mathsf{sgn}(\beta)s_\alpha$.

In the general case, $w_0(I)$ is by definition the long element of the Young subgroup $\mathfrak{S}_{d_1-d_0} \times \cdots \times \mathfrak{S}_{d_{k+1}-d_k}$ of $\mathfrak{S}_n$. Hence $w_0(I) = w_0^{(1)} w_0^{(2)} \ldots, w_0^{(k+1)}$ where $w_0^{(i)}$ is the long element of $\mathfrak{S}_{d_i-d_{i-1}} =$ the parabolic subgroup of $\mathfrak{S}_n$ generated by $s_{d_{i-1}+1}, s_{d_{i-1}+2}, \ldots, s_{d_i-1}$. Hence, it follows that

$$\pi_{w_0(I)} = \pi_{w_0^{(1)}} \pi_{w_0^{(2)}} \cdots \pi_{w_0^{(k+1)}}. \tag{5}$$

and the factors commute. Thus, the general case follows from (5) and the special case. □

4 The subposet $\mathcal{P}_{u\lambda,\gamma}$ of $\mathcal{S}_{I,\gamma}$ and the proof of Theorem 3.8 (⇒)

In this section we introduce a subposet $\mathcal{P}_{c\lambda,\gamma}$ of $\mathcal{S}_{I,\gamma}$. This poset is shown, in Sect. 5, to satisfy the "Diamond property" (Theorem 4.4). Assuming this property, we conclude this section with a proof of the "⇒" direction of Theorem 3.8. The central observation is that $\mathcal{P}_{c\lambda,\gamma}$ is poset isomorphic to an interval in the Bruhat order of a Young subgroup. This permits us to reduce "⇒" to basics about the Möbius function of Bruhat order [10].

Lemma 4.1 *$\mathcal{S}_{I,\gamma}$ (as a set) contains all $\beta \in \mathsf{Comp}_n$ such that $\pi_{w_0(I)} x^\beta = \pm s_\gamma$.*

Proof Suppose $\beta \in \mathsf{Comp}_n$ satisfies $\pi_{w_0(I)} x^\beta = \pm s_\gamma (\neq 0)$. As in the proof of Lemma 3.9 by successive applying the operators $t_1, t_2, \ldots$ $(i \in I)$ to β, we either arrive at some $\gamma' \in \mathsf{Par}_D$ or a $\kappa \in \mathsf{Comp}_n$ with $\kappa_{i+1} = \kappa_i + 1$ where $i, i+1$ are in the same block. In the latter case we conclude, by (the proof of) Lemma 3.9 that $\pi_{w_0(I)} x^\beta = 0$, a contradiction. Otherwise we find $\pm s_\gamma = s_{\gamma'}$, which can only happen if $\gamma = \gamma'$. Thus, we have found a sequence of t_i's connecting β to γ. The result then follows from Lemma 2.5 and the definition of $\mathcal{S}_{I,\gamma}$. □

We need a subposet of $\mathcal{S}_{I,\gamma}$ attached to the following datum:

- $w = w_0(I)u \in \mathfrak{S}_n$ where $I \subset J(w)$ and $\ell(w) = \ell(w_0(I)) + \ell(u)$.
- $\alpha = u\lambda$ for some $\lambda \in \mathsf{Par}_n$.
- $\gamma \in \mathsf{Par}_D$ where $D = [n] - I = \{d_1 < d_2 < \cdots < d_k\}$.

Definition 4.2 $\mathcal{P}_{\alpha,\gamma}$ is the subposet of $\mathcal{S}_{I,\gamma}$ induced by those $\beta \in \mathcal{S}_{I,\gamma}$ such that $[x^\beta]\kappa_\alpha \neq 0$.

The following lemma is straightforward from Lemma 3.9, the definition of $\mathcal{P}_{\alpha,\gamma}$ and Lemma 4.1.

Lemma 4.3 *With notations as above, the coefficient of s_γ in $\kappa_{w\lambda}$ expanded in the basis of D-Schur polynomials, denoted $[s_\gamma]\kappa_{w\lambda}$, equals $\sum_{\beta\in\mathcal{P}_{\alpha,\gamma}} \mathsf{sgn}(\beta)[x^\beta]\kappa_\alpha$.*

The next result holds for $u = c$, a standard Coxeter element for a parabolic subgroup.

Theorem 4.4 (Diamond property) *Let $\beta \in \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$. Let $i < j$ in the same block and $p < q$ in the same block with $(i, j) \neq (p, q)$. If both $t_{ij}\beta$ and $t_{pq}\beta$ are in $\mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$ and cover β, then there exists $\beta' \in \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$ such that $t_{ij}\beta, t_{pq}\beta < \beta'$.*

We defer the proof of Theorem 4.4 until Sect. 5. We complete this section by using Theorem 4.4 to prove the "⇒" direction of Theorem 1.3.

The following result is immediate from the Diamond Property (Theorem 4.4) and Newman's diamond lemma [27].

Lemma 4.5 *$\mathcal{P}_{c\lambda,\gamma}$ has a unique maximum.*

Lemma 4.6 *Suppose $\beta \in \mathcal{P}_{\alpha,\gamma}$, $\beta_i < \beta_j - (j - i)$ for some $i < j$ in the same block. Then $t_{ij}\beta \in \mathcal{P}_{\alpha,\gamma}$.*

Proof By Lemma 4.1 $\mathcal{S}_{I,\gamma}$ consists of all β such that $\pi_{w_0(I)}x^\beta = \pm s_\gamma$. Let $\beta' := t_{ij}\beta$. Thus, $\beta'_i = \beta_j - (j - i)$, $\beta'_j = \beta_i + (j - i)$, and $\beta'_k = \beta_k$ if $k \neq i, j$. The hypothesis that $\beta_i < \beta_j - (j - i)$ means $\beta_i < \beta'_i$ and $\beta'_j < \beta_j$ and $\beta'_j - \beta'_i = (j - i) \in \mathbb{Z}_{>0}$. Hence by Corollary 3.4, $[x^{\beta'}]\kappa_\alpha > 0$. Therefore, it follows that $\beta' = t_{ij}\beta \in \mathcal{P}_{\alpha,\gamma}$, as desired. □

Lemma 4.7 *Let $\mathfrak{S} := \mathfrak{S}_{d_1-d_0} \times \cdots \times \mathfrak{S}_{d_{k+1}-d_k}$ be a Young subgroup of $\mathfrak{S}_n$. Suppose $[u, v] \subset \mathfrak{S}$ is an interval. Then*

$$\sum_{u\leq w\leq v} (-1)^{\ell(uw)} = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Proof For a (locally) finite poset P let $\mu_P : P \times P \to \mathbb{R}$ be its Möbius function. This is defined recursively by $\mu_P(x, x) = 1$ and $\mu_P(x, z) = -\sum_{x\leq_P z<_P y} \mu_P(x, z)$. When $P = \mathfrak{S} = \mathfrak{S}_n$, the lemma holds since $(-1)^{\ell(uw)}$ is the Möbius function for $\mathfrak{S}_n$ under Bruhat order [10].

For the general case, recall [30, Proposition 3.8.2], which states that if P and Q be locally finite posets, and $P \times Q$ is their direct product, if $(s, t) \leq (s', t')$ in $P \times Q$ then the Möbius functions of $P \times Q$, P, and Q are related by

$$\mu_{P\times Q}((s, t), (s', t')) = \mu_P(s, s')\mu_Q(t, t'). \tag{7}$$

Elements of $\mathfrak{S}$ are uniquely factorizable as $w = p^{(1)}p^{(2)}\cdots p^{(k+1)}$ where $p^{(i)}$ is an element of the parabolic subgroup $\mathfrak{S}_{d_i-d_{i-1}}$ of $\mathfrak{S}_n$ generated by $s_{d_{i-1}+1}, s_{d_{i-1}+2}, \ldots, s_{d_i-1}$. Similarly, let $u = q^{(1)}q^{(2)}\cdots q^{(k+1)}$ be the factorization of $u \in \mathfrak{S}$, and $u \leq_{\text{Bruhat}} w$. By iterating application of (7) k many times,

$$\mu_{\mathfrak{S}}(u,w)=\prod_{i=1}^{k+1}\mu_{\mathfrak{S}_{d_i-d_{i-1}}}(q^{(i)},p^{(i)})=(-1)^{\sum_{i=1}^{k+1}\ell(q^{(i)}p^{(i)})}=(-1)^{\ell(wu)},$$

and the result follows. □

Proposition 4.8 *$(\mathcal{P}_{c\lambda,\gamma},<_{\text{Bruhat}})$ is isomorphic (as posets) to an interval in $(\mathfrak{S}_{d_1-d_0}\times\cdots\times\mathfrak{S}_{d_{k+1}-d_k},<_{\text{Bruhat}})$.*

Assuming the proof of Theorem 4.4 (given in the next section), we are ready to present:

Proof of Proposition 4.8 *and Theorem* 3.8 (⇒): Let

$$\Gamma:(\mathcal{S}_{I,\gamma},<_{\text{Bruhat}})\to(\mathfrak{S}_{d_1-d_0}\times\cdots\times\mathfrak{S}_{d_{k+1}-d_k},<_{\text{Bruhat}})$$

denote the isomorphism of posets from Proposition 2.3.

Let $\beta_{\max}$ be the unique maximum of $\mathcal{P}_{c\lambda,\gamma}\subseteq\mathcal{S}_{I,\gamma}$, guaranteed to exist by Lemma 4.5. The unique minimum is γ. It follows from Lemma 4.6 that

$$\Gamma(\mathcal{P}_{c\lambda,\gamma})=[\Gamma(\gamma),\Gamma(\beta_{\max})]\subseteq(\mathfrak{S}_{d_1-d_0}\times\cdots\times\mathfrak{S}_{d_{k+1}-d_k},<_{\text{Bruhat}}).$$

This is the assertion of Proposition 4.8.

If $\mathsf{sgn}(\beta)$ is the sign associated to β, then this maps to $(-1)^{\ell(w_\beta)}$, which agrees with the Möbius function on $\mathfrak{S}$. Now apply (6) to conclude s_γ appears in the D-split expansion of $\kappa_{w\lambda}=\pi_{w_0(I)}\kappa_{c\lambda}$ (the equality is Lemma 3.1) with coefficient zero or one, completing the proof of Theorem 3.8. □

Example 4.9 Let $w=765432918$ and $\lambda=987654321$. Hence $J(w)=\{1,2,3,4,5,6,8\}$; let $I=\{2,3,4,5,6\}\subseteq J(w)$. Thus $w_0(I)=176543289$ and we can factor $w=w_0(I)c$ where c is the standard Coxeter element $c=234567918=s_8s_1s_2s_3s_4s_5s_6s_7$. Now, $c^{-1}=812345697$ and $w^{-1}=865432197$. Therefore $\alpha=c\lambda=298765413$, whereas $w\lambda=245678913$.

Since $D=[9]-I=\{1,7,8,9\}$, we have that $\kappa_{w\lambda}=\kappa_{245678913}\in\Pi_D$ is separately symmetric in the sets of indeterminates $\{x_1\},\{x_2,x_3,x_4,x_5,x_6,x_7\},\{x_8\},\{x_9\}$.

Since c is a standard Coxeter element, by [18, Theorem 4.13(II)], we have that $\kappa_{c\lambda}$ is $[n-1]$-multiplicity-free. Consider the term $x^{928765422}$ appearing in $\kappa_{c\lambda}$. Now

$$\pi_{w_0(I)}(x^{928765422})=s_{9,\underline{28}7654,2,2}=-s_{9,7\underline{37}654,2,2}=s_{9,76\underline{46}54,2,2}=-s_{9,765554,2,2},$$

where we have underlined the swaps.

The list of monomials x^β of $\kappa_{c\lambda}$ such that $\pi_{w_0(I)}(x^\beta)=\pm s_{9,765554,2,2}$, together with the signs they contribute are:

$$[9,7,6,5,5,5,4,2,2]\ 1,[9,7,4,7,5,5,4,2,2]\ -1,[9,7,6,4,6,5,4,2,2]\ -1,$$
$$[9,5,8,4,6,5,4,2,2]\ 1,[9,7,3,7,6,5,4,2,2]\ 1,[9,5,8,5,5,5,4,2,2]\ -1,$$
$$[9,2,8,7,6,5,4,2,2]\ -1,[9,3,8,7,5,5,4,2,2]\ 1.$$

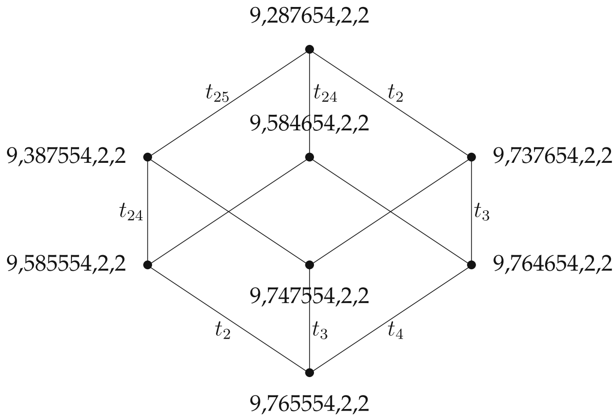


Fig. 2 The poset $\mathcal{P}_{c\lambda,\gamma}$ for $c = 234567918$, $\lambda = 987654321$, $\gamma = 976555422$, $I = \{2, 3, 4, 5, 6\}$ with some edges labeled

These elements form a poset $\mathcal{P}_{c\lambda,\gamma=9,765554,2,2}$ shown in Fig. 2 isomorphic to an interval $[\mathrm{id}, s_2s_3s_4]$ in Bruhat order, consistent with Proposition 4.8.

Indeed the coefficients sum to zero, in agreement with the above discussion about the Möbius function. $\square$

5 Proof of the diamond property (Theorem 4.4)

The initial goal in this section is the proof of Proposition 5.9. This proposition provides a set of linear inequalities on a weak composition $\beta \in \mathcal{P}_{c\lambda,\gamma}$ that characterize when $t_{i,j}\beta$ remains in the poset. This proposition, along with several technical lemmas, is then used to prove the diamond property of the poset $\mathcal{P}_{c\lambda,\gamma}$ (Theorem 4.4).

Throughout this section we fix a decomposition $w = w_0(I)c$ where c is a standard Coxeter element of some parabolic such that $\ell(w) = \ell(w_0(I)) + \ell(c)$, and $\lambda \in \mathsf{Par}_n$.

Lemma 5.1 *Let $w = w_0(I)u \in \mathfrak{S}_n$ with $\ell(w) = \ell(w_0(I)) + \ell(u)$. If $i \in I$, then $(u\lambda)_i \geq (u\lambda)_{i+1}$.*

Proof The length additivity of $w_0(I)$ and u implies $J(u) \cap J(w_0(I)) = J(u) \cap I = \emptyset$. Thus $u^{-1}(i) < u^{-1}(i+1)$, and since λ is a partition, $(u\lambda)_i = \lambda_{u^{-1}(i)} \geq \lambda_{u^{-1}(i+1)} = (u\lambda)_{i+1}$. $\square$

We will use the following notion from [18]:

Definition 5.2 (*Composition patterns*) Let $\mathsf{Comp} := \bigcup_{n=1}^{\infty} \mathsf{Comp}_n$. For $\alpha = (\alpha_1, \ldots, \alpha_\ell), \beta = (\beta_1, \ldots, \beta_k) \in \mathsf{Comp}$, α *contains the composition pattern* β if there exist integers $j_1 < j_2 < \cdots < j_k$ that satisfy:

- $(\alpha_{j_1}, \ldots, \alpha_{j_k})$ is order isomorphic to β ($\alpha_{j_s} \leq \alpha_{j_t}$ if and only if $\beta_s \leq \beta_t$),
- $|\alpha_{j_s} - \alpha_{j_t}| \geq |\beta_s - \beta_t|$.

If α does not contain β, then α *avoids* β.

Lemma 5.3 *$c\lambda$ avoids* 012*,* 1032*,* 0011*,* 0021*,* 1022*.*

Proof Since c is a standard Coxeter element in a parabolic subgroup, $X_c \subseteq GL_n/B$ is a toric variety [20]. Hence, by [18, Theorem 4.13(II)], $\kappa_{c\lambda}$ is $[n-1]$-multiplicity-free for all $\lambda \in \mathsf{Par}_n$. In [19], it is shown that κ_α is $[n-1]$-multiplicity free if and only if α avoids 012, 1032, 0022, 0021, 1022. Thus, since $\kappa_{c\lambda}$ is $[n-1]$-multiplicity-free, $c\lambda$ avoids 012, 1032, 0022, 0021, 1022.

To seek a contradiction, suppose that $c\lambda$ contains the pattern 0011. Let $j_1 < j_2 < j_3 < j_4$ be the integers such that $(c\lambda)_{j_1}, (c\lambda)_{j_2}, (c\lambda)_{j_3}, (c\lambda)_{j_4}$ contains the composition pattern 0011. Let $\tilde{\lambda} \in \mathsf{Par}_n$ be obtained from λ by replacing all part lengths equal to $(c\lambda)_{j_3}$ by $(c\lambda)_{j_3} + 1$. Then $c\tilde{\lambda}$ contains the pattern 0022. We conclude, *via* [19], that $\kappa_{c\tilde{\lambda}}$ is not $[n-1]$-multiplicity-free. By [18, Theorem 4.13(II)], this implies X_c is not a toric variety, a contradiction. Thus $c\lambda$ must also avoid the pattern 0011. □

5.1 The leftmin, rightmax, and center functions

The linear inequalties of Proposition 5.9 will be stated in terms of three functions defined with respect to the fixed weak composition $c\lambda$. We now introduce and prove some basic properties of these functions.

Definition 5.4 Let $\mathsf{leftmin}_\alpha(i) = \min\{\alpha_j : j \le i\}$ and $\mathsf{rightmax}_\alpha(i) = \max\{\alpha_j : j \ge i\}$.

Lemma 5.5 *Let* $1 \le i, j \le n$ *and* $F \in \mathsf{Tab}(c\lambda)$*. Then*

(i) $(\mathsf{wt}(F))_k \ge \mathsf{leftmin}_{c\lambda}(i)$ *for* $1 \le k \le i$*.*
(ii) $(\mathsf{wt}(F))_k \le \mathsf{rightmax}_{c\lambda}(j)$ *for* $j \le k \le n$*.*
(iii) *If* $i < j$ *are in the same block and* $\mathsf{leftmin}_{c\lambda}(i) = (c\lambda)_i$ *and* $\mathsf{rightmax}_{c\lambda}(j) = (c\lambda)_j$*, then* $(\mathsf{wt}(F))_i = (c\lambda)_i$ *and* $(\mathsf{wt}(F))_j = (c\lambda)_j$*.*

Proof (i): By Definition 5.4, for $1 \le k \le i$, $(c\lambda)_k \ge \mathsf{leftmin}_{c\lambda}(i)$. By induction, and the definition of flagged fillings, $F(k,r) = k$ for $1 \le k \le i$ and $1 \le r \le \mathsf{leftmin}_{c\lambda}(i)$. Thus $(\mathsf{wt}(F))_k \ge \mathsf{leftmin}_{c\lambda}(i)$ for $1 \le k \le i$.

(ii): Once again we apply Definition 5.4, concluding $\mathsf{rightmax}_{c\lambda}(k) \le \mathsf{rightmax}_{c\lambda}(j)$ for $j \le k \le n$. By the definition of flagged fillings a value k can only appear once in a fixed row, and only in columns greater than or equal to k. Hence, $(\mathsf{wt}(F))_k \le \mathsf{rightmax}_{c\lambda}(k) \le \mathsf{rightmax}_{c\lambda}(j)$.

(iii): If i, j are in the same block, then Lemma 5.1, applied inductively, implies $(c\lambda)_k \ge (c\lambda)_j$ for $i \le k \le j$. This, combined with $\mathsf{leftmin}_{c\lambda}(i) = (c\lambda)_i$, implies that $\mathsf{leftmin}_{c\lambda}(j) = (c\lambda)_j$. Applying (i) and (ii) to j yields $(\mathsf{wt}(F))_j \ge (c\lambda)_j$ and $(\mathsf{wt}(F))_j \le (c\lambda)_j$. Hence $(\mathsf{wt}(F))_j = (c\lambda)_j$.

Additionally, $(c\lambda)_k \ge (c\lambda)_j$ for $i \le k \le j$ combined with $\mathsf{rightmax}_{c\lambda}(j) = (c\lambda)_j$ gives $\mathsf{rightmax}_{c\lambda}(i) = (c\lambda)_i$. Applying (i) and (ii) to i again yields the desired equality. □

Lemma 5.6 *Let* $i \le j$ *with* $(c\lambda)_k \ge (c\lambda)_{k+1}$ *for* $i \le k < j$*. Let* m *be the maximum value such that* $i \le m \le j$ *and* $(c\lambda)_m \ge \mathsf{leftmin}_{c\lambda}(i)$*. Then*

$$|\{(d,r) \in D(c\lambda) : d \le m\}| = m \text{ for } 1 \le r \le \mathsf{leftmin}_{c\lambda}(i).$$

This implies that for all $F \in \mathsf{Tab}(c\lambda)$,

$$F(d, r) = d \textit{ for } 1 \leq r \leq \mathsf{leftmin}_{c\lambda}(i) \textit{ and } 1 \leq d \leq m.$$

Proof This first claim follows from the definition of $\mathsf{leftmin}_{c\lambda}(i)$. The latter then follows from inductively applying the flagged and row distinct properties of F. □

Definition 5.7 If $i < j$ with $(c\lambda)_k \geq (c\lambda)_{k+1}$ for $i \leq k < j$, $\mathsf{leftmin}_{c\lambda}(i) < (c\lambda)_i$, and $\mathsf{rightmax}_{c\lambda}(j) > (c\lambda)_j$, then we say the pair (i, j) is *interwoven*. For such an (i, j), define

$$\mathsf{center}_{c\lambda}(i, j) = \max\{k : i \leq k \leq j \text{ and } (c\lambda)_k \geq \mathsf{rightmax}_{c\lambda}(j)\}.$$

Notice $\mathsf{center}_{c\lambda}(i, j) \neq -\infty$ since $(c\lambda)_i \geq \mathsf{rightmax}_{c\lambda}(j)$ (otherwise, we have $\mathsf{leftmin}_{c\lambda}(i) < (c\lambda)_i < \mathsf{rightmax}_{c\lambda}(j)$ which says $c\lambda$ contains a 012-pattern, contradicting Lemma 5.3).

Lemma 5.8 *Let* $i < j$ *with* $(c\lambda)_k \geq (c\lambda)_{k+1}$ *for* $i \leq k < j$. *Then*

(i) *If* $\mathsf{leftmin}_{c\lambda}(i) = (c\lambda)_i$ *and* $\mathsf{rightmax}_{c\lambda}(j) > (c\lambda)_j$, *then*

$$|\{(d, r) \in D(c\lambda) : d > i\}| \leq 1 \textit{ for } r > (c\lambda)_i,$$

(ii) *If* $\mathsf{leftmin}_{c\lambda}(i) < (c\lambda)_i$ *and* $\mathsf{rightmax}_{c\lambda}(j) = (c\lambda)_j$, *then*

$$|\{(d, r) \in D(c\lambda) : d \leq j\}| \geq j - 1 \textit{ for } \mathsf{leftmin}_{c\lambda}(i) < r \leq (c\lambda)_j,$$

(iii) *If* $\mathsf{leftmin}_{c\lambda}(i) < (c\lambda)_i$ *and* $\mathsf{rightmax}_{c\lambda}(j) > (c\lambda)_j$, *then*

$$|\{(d, r) \in D(c\lambda) : d \geq \mathsf{center}_{c\lambda}(i, j)\}| = 1 \textit{ for } \mathsf{leftmin}_{c\lambda}(i) < r \leq \mathsf{rightmax}_{c\lambda}(j),$$

and

$$|\{(d, r) \in D(c\lambda) : d \leq \mathsf{center}_{c\lambda}(i, j)\}| = \mathsf{center}_{c\lambda}(i, j) - 1 \textit{ for } \mathsf{leftmin}_{c\lambda}(i) < r \leq \mathsf{rightmax}_{c\lambda}(j).$$

Proof (i): Let $r > (c\lambda)_i$. If $j < d_1 < d_2$, then $c\lambda$ contains the pattern $((c\lambda)_i, (c\lambda)_j, (c\lambda)_{d_1}, (c\lambda)_{d_2})$. Suppose that $(d_1, r), (d_2, r) \in D(c\lambda)$. This implies $(c\lambda)_{d_1}, (c\lambda)_{d_2} \geq (c\lambda)_i$. This, combined with $(c\lambda)_i \geq (c\lambda)_j$, implies $((c\lambda)_i, (c\lambda)_j, (c\lambda)_{d_1}, (c\lambda)_{d_2})$ contains 012, 1032, 0021, 0011, or 1022. This contradicts Lemma 5.3. Thus

$$|\{(d, r) \in D(c\lambda) : d > j\}| \leq 1 \text{ for } r > (c\lambda)_i.$$

Further, since $r > (c\lambda)_i \geq (c\lambda)_k$ for $i \leq k \leq j$,

$$|\{(d, r) \in D(c\lambda) : d > i\}| \leq 1 \text{ for } r > (c\lambda)_i.$$

(ii): Let $\mathsf{leftmin}_{c\lambda}(i) < r \leq (c\lambda)_j$. If $d_1 < d_2 < i$, then $c\lambda$ contains the pattern $((c\lambda)_{d_1}, (c\lambda)_{d_2}, (c\lambda)_i, (c\lambda)_j)$. Suppose that $(d_1, r), (d_2, r) \notin D(c\lambda)$. This implies $(c\lambda)_{d_1}, (c\lambda)_{d_2} \leq (c\lambda)_j$. This, combined with $(c\lambda)_i \geq (c\lambda)_j$, implies $((c\lambda)_{d_1}, (c\lambda)_{d_2}, (c\lambda)_i, (c\lambda)_j)$ contains 012, 1032, 0021, 0011, or 1022. This contradicts Lemma 5.3. Thus

$$|\{(d, r) \in D(c\lambda) : d \leq i\}| \geq i - 1 \text{ for } \mathsf{leftmin}_{c\lambda}(i) < r \leq (c\lambda)_j.$$

Further, since $r \leq (c\lambda)_j \leq (c\lambda)_k$ for $i \leq k \leq j$,

$$|\{(d, r) \in D(c\lambda) : d \leq j\}| \geq j - 1 \text{ for } \mathsf{leftmin}_{c\lambda}(i) < r \leq (c\lambda)_j.$$

(iii): Let x be an integer such that $x < i$ and $(c\lambda)_x = \mathsf{leftmin}_{c\lambda}(i)$, and y be an integer such that $y > j$ and $(c\lambda)_y = \mathsf{rightmax}_{c\lambda}(j)$.

Our claim holds vacuously if $(c\lambda)_x \geq (c\lambda)_y$. Hence, for the rest of the proof we assume $(c\lambda)_x < (c\lambda)_y$. Now $c\lambda$ contains the pattern $((c\lambda)_x, (c\lambda)_i, (c\lambda)_j, (c\lambda)_y)$ and by Lemma 5.3 this pattern avoids 012. This, combined with $(c\lambda)_j < (c\lambda)_y$, implies

$$(c\lambda)_x \geq (c\lambda)_j. \tag{8}$$

It further implies, when combined with $(c\lambda)_x < (c\lambda)_i$, that

$$(c\lambda)_i \geq (c\lambda)_y. \tag{9}$$

Let $(c\lambda)_x < r \leq (c\lambda)_y$. Let $\mathsf{center}_{c\lambda}(i, j) < d_1 < d_2$. Suppose, to obtain a contradiction, that $(d_1, r), (d_2, r) \in D(c\lambda)$. Then

$$(c\lambda)_{d_1}, (c\lambda)_{d_2} > (c\lambda)_x. \tag{10}$$

If $d_1 \leq j$, then the definition of $\mathsf{center}_{c\lambda}(i, j)$ implies $(c\lambda)_{d_1} < (c\lambda)_y$. This implies $c\lambda$ contains the pattern $((c\lambda)_x, (c\lambda)_{d_1}, (c\lambda)_y)$ which is a 012 pattern. This contradicts Lemma 5.3. Otherwise, if $j < d_1 < d_2$, then $c\lambda$ contains the pattern $((c\lambda)_x, (c\lambda)_j, (c\lambda)_{d_1}, (c\lambda)_{d_2})$. By (8) and (10), this pattern contains 012, 1032, 0021, 0011, or 1022. This contradicts Lemma 5.3. Thus

$$|\{(d, r) \in D(c\lambda) : d \geq \mathsf{center}_{c\lambda}(i, j)\}| = 1 \text{ for } \mathsf{leftmin}_{c\lambda}(i) < r \leq \mathsf{rightmax}_{c\lambda}(j). \tag{11}$$

Let $(c\lambda)_x < r \leq (c\lambda)_y$. Let $d_1 < d_2 < \mathsf{center}_{c\lambda}(i, j)$. Suppose, to obtain a contradiction, that $(d_1, r), (d_2, r) \notin D(c\lambda)$. Thus

$$(c\lambda)_{d_1}, (c\lambda)_{d_2} < (c\lambda)_y. \tag{12}$$

If $d_2 \geq i$, then $(c\lambda)_{d_2} \geq (c\lambda)_j$ and the definition of $\mathsf{center}_{c\lambda}(i, j)$ implies $(c\lambda)_{d_2} \geq (c\lambda)_y$. This contradicts (12). Otherwise, if $d_1 < d_2 < i$, then $c\lambda$ contains $((c\lambda)_{d_1}, (c\lambda)_{d_2}, (c\lambda)_i, (c\lambda)_y)$. By (9) and (12), this pattern contains

012, 1032, 0021, 0011, or 1022. This contradicts Lemma 5.3. We conclude $|\{(d,r) \in D(c\lambda) : d \leq \mathsf{center}_{c\lambda}(i,j)\}| \geq \mathsf{center}_{c\lambda}(i,j) - 1$. Since $(c\lambda)_x < r$, we can strengthen this inequality to

$$|\{(d,r) \in D(c\lambda) : d \leq \mathsf{center}_{c\lambda}(i,j)\}| = \mathsf{center}_{c\lambda}(i,j) - 1 \text{ for } \mathsf{leftmin}_{c\lambda}(i) < r \leq \mathsf{rightmax}_{c\lambda}(j).$$

5.2 The linear inequalities governing poset containment

We are now able to state and prove Proposition 5.9. This subsection concludes with the proof of two technical lemmas that will be needed in the proof of Theorem 4.4.

Proposition 5.9 *Let $\beta \in \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$, $i < j$ in the same block, and $\beta_i > \beta_j - (j-i)$. Then $t_{i,j}\beta \in \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$ if and only if*

(1) $\mathsf{leftmin}_{c\lambda}(i) \leq \beta_j - (j-i)$*;*
(2) $\mathsf{rightmax}_{c\lambda}(j) \geq \beta_i + (j-i)$*; and*
(3) *if (i,j) is interwoven, then*

$$\beta_1 + \cdots + \beta_{i-1} + (\beta_j - (j-i)) + \beta_{i+1} + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} \geq (c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)}.$$

Proof ($\Rightarrow$) We prove the contrapositive. That is, we assume that $\mathsf{leftmin}_{c\lambda}(i) > \beta_j - (j-i)$, $\mathsf{rightmax}_{c\lambda}(j) < \beta_i + (j-i)$, or (i,j) is interwoven with $\beta_1 + \cdots + \beta_{i-1} + (\beta_j - (j-i)) + \beta_{i+1} + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} < (c\lambda)_1 + \cdots (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)}$. Let $\tau = t_{i,j}\beta$ and suppose, to seek a contradiction, that $F \in \mathsf{Tab}(c\lambda)$ with $\tau = \mathsf{wt}(F)$.

Case $\mathsf{leftmin}_{c\lambda}(i) > \beta_j - (j-i)$: By the case hypothesis, $\mathsf{leftmin}_{c\lambda}(i) > \tau_i = (\mathsf{wt}(F))_i$. This contradicts Lemma 5.5(i).

Case $\mathsf{rightmax}_{c\lambda}(i) < \beta_j + (j-i)$: By the case hypothesis, $\mathsf{rightmax}_{c\lambda}(i) < \tau_j = (\mathsf{wt}(F))_j$. This contradicts Lemma 5.5(ii).

Case (i,j) is interwoven with $\beta_1 + \cdots + \beta_{i-1} + (\beta_j - (j-i)) + \beta_{i+1} + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} < (c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)}$: The case hypothesis implies that $c\lambda \not\leq_{\mathsf{dom}} \tau$. This contradicts Corollary 3.6.

($\Leftarrow$) Since $[x^\beta]\kappa_{c\lambda} \neq 0$, we know there exists an $F \in \mathsf{Tab}(c\lambda)$ with $\mathsf{wt}(F) = \beta$. There are four cases to consider.

Case $\mathsf{leftmin}_{c\lambda}(i) = (c\lambda)_i$ *and* $\mathsf{rightmax}_{c\lambda}(j) = (c\lambda)_j$: By Lemma 5.5(iii), $\beta_i = (c\lambda)_i$ and $\beta_j = (c\lambda)_j$. Thus

$$(c\lambda)_i = \mathsf{leftmin}_{c\lambda}(i) \leq \beta_j - (j-i) = (c\lambda)_j - (j-i), \tag{13}$$

where the first equality is the case hypothesis, the inequality is the proposition hypothesis. Thus $j > i$ implies that $(c\lambda)_i < (c\lambda)_j$. This is a contradiction of Lemma 5.1, and hence this case cannot occur.

Case $\mathsf{leftmin}_{c\lambda}(i) = (c\lambda)_i$ *and* $\mathsf{rightmax}_{c\lambda}(j) > (c\lambda)_j$: By Lemma 5.6, $F(d, r) = d$ for all $1 \leq d \leq i$ and $r \leq (c\lambda)_i$. Hence, there is an i in every row $r \leq (c\lambda)_i$ of F, and

$$\beta_i \geq (c\lambda)_i. \tag{14}$$

The flagged property of F, combined with Lemma 5.8(i), implies that i and j can not both be in row $r > (c\lambda)_i$ of F. By the definition of F and (14), there are exactly $\beta_i - (c\lambda)_i \geq 0$ such rows containing only i, but not j. By the case and proposition hypotheses,

$$\beta_i - (c\lambda)_i = \beta_i - \mathsf{leftmin}_{c\lambda}(i) \geq \beta_i - (\beta_j - (j - i)) > 0.$$

Setting $v := \beta_i - (\beta_j - (j - i))$ we can choose v rows $r_1, \ldots, r_v > (c\lambda)_i$ in F that contain i and not j.

The filling G is obtained from F by changing the i in rows $r_1, \ldots, r_v$ to a j. By construction, G is row distinct. For $i \leq k \leq j$, the boxes $(k, r_1), \ldots (k, r_v) \notin D(c\lambda)$ since $r_1, \ldots, r_v > (c\lambda)_i \geq (c\lambda)_k$. Hence the flagged property of F implies that the i in these rows of F must appear in a column strictly greater than j. Thus the j in these rows of G appears in a column greater than j, and G is flagged.

Let $\tau = \mathsf{wt}(G)$. Then $\tau_i = \beta_i - v = \beta_i - (\beta_i - (\beta_j - (j - i))) = \beta_j - (j - i)$ and $\tau_j = \beta_j + v = \beta_j + (\beta_i - (\beta_j - (j - i))) = \beta_i + (j - i)$. Otherwise, $\tau_k = \beta_k$ for $r \neq i, j$. Thus $\tau = t_{i,j}\beta$. We conclude that $t_{i,j}\beta$ is an exponent vector of $\kappa_{c\lambda}$.

Case $\mathsf{leftmin}_{c\lambda}(i) < (c\lambda)_i$ *and* $\mathsf{rightmax}_{c\lambda}(j) = (c\lambda)_j$: The row distinct and flagged properties of F, combined with Lemma 5.6 and Lemma 5.8(ii), imply that at least one of i or j are in row r of F for $1 \leq r \leq (c\lambda)_j$. By the case and proposition hypotheses, $\beta_j < \beta_i + (j - i) \leq \mathsf{rightmax}_{c\lambda}(j) = (c\lambda)_j$.

Hence, there are at least $(c\lambda)_j - \beta_j$ rows r, with $1 \leq r \leq (c\lambda)_j$, of F that contain i but not j. Setting $v := \beta_i + (j - i) - \beta_j \leq (c\lambda)_j - \beta_j$, we choose v rows in F, $r_1, \ldots, r_v \leq (c\lambda)_j$, that contain i but not j. By Lemma 5.8(ii) and the flagged property of F, for each $e \in \{r_1, \ldots, r_v\}$ there is exactly one $d_e \leq j$ such that $(d_e, e) \notin D(c\lambda)$. It follows, by the definition of e and the flagged property of F, that the content of row e in the first j columns of F is equal to $\{1, \ldots, j - 1\}$. We use this fact to define the filling G.

The filling G is obtained from F via the following rule. Let $1 \leq e \leq \lambda_1$. Then

(i) $e \notin \{r_1, \ldots, r_v\}$: The e-th row of G equals the e-th row of F.
(ii) $e \in \{r_1, \ldots, r_v\}$: The e-th row of G is defined by filling each of the values in $[j] \setminus \{i\}$ in the minimal column possible. Explicitly, $G(d, e) = d$ for $d < d_e$, $G(d, e) = d - 1$ for $d_e < d \leq i$, $G(d, e) = d$ for $i < r < j$. Then, set $G(j, e) = j$, and for any column greater than j the entries in row e of F and G coincide.

Clearly G is row distinct; for $e \in \{r_1, \ldots, r_v\}$, the content of row e of G is equal to the content of row e of F with the unique i replaced by j. It is equally easy to verify that each of (i)–(ii) leaves the respective column in G satisfying the flagged constraint. Let $\tau = \mathsf{wt}(G)$. Then $\tau_i = \beta_i - v = \beta_i - (\beta_i + (j-i) - \beta_j) = \beta_j - (j-i)$ and $\tau_j = \beta_j + v = \beta_j + (\beta_i + (j-i) - \beta_j) = \beta_i + (j-i)$. Otherwise, $\tau_k = \beta_k$ for $k \neq i, j$. Thus $\tau = t_{i,j}\beta$. We conclude that $t_{i,j}\beta$ is an exponent vector of $\kappa_{c\lambda}$.

Case $\mathsf{leftmin}_{c\lambda}(i) < (c\lambda)_i$ *and* $\mathsf{rightmax}_{c\lambda}(j) > (c\lambda)_j$: Let x be an integer such that $x < i$ and $(c\lambda)_x = \mathsf{leftmin}_{c\lambda}(i)$, and y be an integer such that $y > j$ and $(c\lambda)_y = \mathsf{rightmax}_{c\lambda}(j)$. Suppose, for sake of contradiction, that $(c\lambda)_x \geq (c\lambda)_y$. Then, by Lemma 5.5(i), $\beta_i \geq (c\lambda)_x \geq (c\lambda)_y = \mathsf{rightmax}_{c\lambda}(j)$. Thus, $\beta_i + (j-i) > \mathsf{rightmax}_{c\lambda}(j)$, which contradicts the hypothesis (2). Thus,

$$(c\lambda)_x < (c\lambda)_y. \tag{15}$$

Corollary 3.6 implies $\beta_1 + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} \geq (c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)}$. Now

$$\begin{aligned} &|\left\{(d,r) \in D(c\lambda) : d > \mathsf{center}_{c\lambda}(i,j),\ 1 \leq r \leq (c\lambda)_y, \text{ and } F(d,r) \leq \mathsf{center}_{c\lambda}(i,j)\right\}| \\ &\quad = \beta_1 + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} - ((c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)}). \end{aligned} \tag{16}$$

Then our hypothesis $\beta_1 + \cdots + \beta_{i-1} + (\beta_j - (j-i)) + \beta_{i+1} + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} \geq (c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)}$ is equivalent to $\beta_1 + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} - ((c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)}) \geq \beta_i - (\beta_j - (j-i))$. Applying this to (16) yields

$$\begin{aligned} &|\left\{(d,r) \in D(c\lambda) : d > \mathsf{center}_{c\lambda}(i,j),\ 1 \leq r \leq (c\lambda)_y, \text{ and } F(d,r) \leq \mathsf{center}_{c\lambda}(i,j)\right\}| \\ &\quad \geq \beta_i - (\beta_j - (j-i)). \end{aligned} \tag{17}$$

We can further refine (17). By the definition of $\mathsf{center}_{c\lambda}(i,j)$, (15), and Lemma 5.1, $(c\lambda)_d \geq (c\lambda)_x$ for all $i \leq d \leq \mathsf{center}_{c\lambda}(i,j)$. By Lemma 5.6, $F(d,r) = d$ for all $d \leq \mathsf{center}_{c\lambda}(i,j)$ and $r \leq (c\lambda)_x$. Thus, the row distinct property of F transforms (17) into

$$\begin{aligned} &|\left\{(d,r) \in D(c\lambda) : d > \mathsf{center}_{c\lambda}(i,j),\ (c\lambda)_x < r \leq (c\lambda)_y, \text{ and } F(d,r) \leq \mathsf{center}_{c\lambda}(i,j)\right\}| \\ &\quad \geq \beta_i - (\beta_j - (j-i)). \end{aligned} \tag{18}$$

By Lemma 5.8(iii), the rows $(c\lambda)_x < r \leq (c\lambda)_y$ have $\mathsf{center}_{c\lambda}(i,j)$ boxes in $D(c\lambda)$. By (18), we can pick $v := \beta_i - (\beta_j - (j-i))$ of these rows, where the $\mathsf{center}_{c\lambda}(i,j)$ many boxes of $D(c\lambda)$ are filled using precisely the labels $1, 2, \ldots, \mathsf{center}_{c\lambda}(i,j)$. By Lemma 5.8(iii), for each $e \in \{r_1, \ldots, r_v\}$ there is exactly one $d_e \leq \mathsf{center}_{c\lambda}(i,j)$ such that $(d_e, e) \notin D(c\lambda)$.

The filling G is obtained from F via the following rule. Let $1 \leq e \leq \lambda_1$.

(i) $e \notin \{r_1, \ldots, r_v\}$: The e-th row of G equals the e-th row of F.
(ii) The e-th row of G is defined by filling each of the values in $[\mathsf{center}_{c\lambda}(i,j) - 1] \setminus \{i\}$ in the minimal column possible. Explicitly, $G(d,e) = d$ for $d < d_e$, $G(d,e) = d - 1$ for $d_e < d \leq i$, $G(d,e) = d$ for $i < d \leq \mathsf{center}_{c\lambda}(i,j)$. Then set the value of the unique box in a column greater than $\mathsf{center}_{c\lambda}(i,j)$ to be j.

Clearly G is row distinct; for $e \in \{r_1, \ldots, r_v\}$, the content of row e of G is equal to the content of row e of F with the unique i replaced by j. It is an easy check to verify that each of (i)-(ii) leaves the respective row in G satisfying the flagged constraint. Let $\tau = \mathsf{wt}(G)$. Then $\tau_i = \beta_i - v = \beta_i - (\beta_i - (\beta_j - (j - i))) = \beta_j - (j - i)$, $\tau_j = \beta_j + v = \beta_j + (\beta_i - (\beta_j - (j - i))) = \beta_i + (j - i)$. Otherwise, $\tau_k = \beta_k$ for $k \neq i, j$. Thus $\tau = t_i \beta$. We conclude that $t_i \beta$ is an exponent vector of $\kappa_{c\lambda}$. □

We now prove two lemmas. The first lemma is one of the primary tools used to show that certain inequalities from Proposition 5.9 are satisfied. The second lemma will allow us to simplify one of the cases in the proof of Theorem 4.4.

Lemma 5.10 *Let $i \in [n-1]$ and $\beta \in \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$. Then*

$$\max\{\mathsf{rightmax}_{c\lambda}(i+1) - \mathsf{leftmin}_{c\lambda}(i), 0\} \geq \beta_1 + \cdots + \beta_i - ((c\lambda)_1 + \cdots + (c\lambda)_i).$$

Proof Since $[x^{\beta}]\kappa_{c\lambda} \neq 0$, there exists an $F \in \mathsf{Tab}(c\lambda)$ with $\mathsf{wt}(F) = \beta$. Now,

$$|\{(d, r) \in D(c\lambda) : d > i \text{ and } F(d, r) \leq i\}| = \beta_1 + \cdots + \beta_i - ((c\lambda)_1 + \cdots + (c\lambda)_i). \tag{19}$$

We first prove this lemma for $i \in [n-1]$ with $(c\lambda)_i \geq (c\lambda)_{i+1}$. Let $r \leq \mathsf{leftmin}_{c\lambda}(i)$. Lemma 5.6 implies that $F(d_1, r) = d_1$ for $d_1 \leq i$ and $r \leq \mathsf{leftmin}_{c\lambda}(i)$. Since F is row distinct this implies

$$|\{(d, r) \in D(c\lambda) : d > i \text{ and } F(d, r) \leq i\}| = 0 \text{ for } 1 \leq r \leq \mathsf{leftmin}_{c\lambda}(i). \tag{20}$$

Suppose $\mathsf{leftmin}_{c\lambda}(i) \geq \mathsf{rightmax}_{c\lambda}(i+1)$. Then there exist no $(d, r) \in D(c\lambda)$ such that $d > i$ and $r > \mathsf{leftmin}_{c\lambda}(i)$. This, combined with (19) and (20), implies $\beta_1 + \cdots + \beta_i - ((c\lambda)_1 + \cdots + (c\lambda)_i) = 0$. Thus our result trivially holds.

For the rest of the proof we assume $\mathsf{leftmin}_{c\lambda}(i) < \mathsf{rightmax}_{c\lambda}(i+1)$.

Case $\mathsf{leftmin}_{c\lambda}(i) = (c\lambda)_i$ *and* $\mathsf{rightmax}_{c\lambda}(i+1) = (c\lambda)_{i+1}$: By our assumption $(c\lambda)_i \geq (c\lambda)_{i+1}$ and the case hypothesis, $\mathsf{leftmin}_{c\lambda}(i) = (c\lambda)_i \geq (c\lambda)_{i+1} = \mathsf{rightmax}_{c\lambda}(i+1)$. Thus, since we are assuming $\mathsf{leftmin}_{c\lambda}(i) < \mathsf{rightmax}_{c\lambda}(i+1)$, this case does not occur.

Case $\mathsf{leftmin}_{c\lambda}(i) = (c\lambda)_i$ *and* $\mathsf{rightmax}_{c\lambda}(i+1) > (c\lambda)_{i+1}$: We have that $\mathsf{leftmin}_{c\lambda}(i) = (c\lambda)_i$ paired with (20), and combined with Lemma 5.8(i) implies

$$|\{(d, r) \in D(c\lambda) : d > i \text{ and } F(d, r) \leq i\}| \leq \mathsf{rightmax}_{c\lambda}(i+1) - \mathsf{leftmin}_{c\lambda}(i).$$

Then (19) gives the required inequality.

Case $\mathsf{leftmin}_{c\lambda}(i) < (c\lambda)_i$ *and* $\mathsf{rightmax}_{c\lambda}(i+1) = (c\lambda)_{i+1}$: Lemma 5.8(ii) says

$$|\{(d, r) \in D(c\lambda) : d \leq i+1\}| \geq i \text{ for } \mathsf{leftmin}_{c\lambda}(i) < r \leq (c\lambda)_{i+1},$$

which implies

$$|\{(d,r)\in D(c\lambda): d\le i\}|\ge i-1 \text{ for } \mathsf{leftmin}_{c\lambda}(i)<r\le (c\lambda)_{i+1}. \tag{21}$$

Since $\mathsf{rightmax}_{c\lambda}(i+1)=(c\lambda)_{i+1}$, there exist no $(d,r)\in D(c\lambda)$ such that $d>i$ and $r>(c\lambda)_{i+1}$. This, combined with (20), and the row distinct property of F paired with (21), implies that

$$|\{(d,r)\in D(c\lambda): d>i \text{ and } F(d,r)\le i\}|\le \mathsf{rightmax}_{c\lambda}(i+1)-\mathsf{leftmin}_{c\lambda}(i).$$

Applying (19) concludes the proof in this case.

Case $\mathsf{leftmin}_{c\lambda}(i)<(c\lambda)_k$ *and* $\mathsf{rightmax}_{c\lambda}(i+1)>(c\lambda)_{i+1}$: There exist no $(d,r)\in D(c\lambda)$ such that $d>i$ and $r>\mathsf{rightmax}_{c\lambda}(i+1)$. We apply Lemma 5.8(iii), noting that $\mathsf{center}_{c\lambda}(i,i+1)=i$, and (20) to imply that

$$|\{(d,r)\in D(c\lambda): d>i \text{ and } F(d,r)\le i\}|\le \mathsf{rightmax}_{c\lambda}(i+1)-\mathsf{leftmin}_{c\lambda}(i).$$

Once again we conclude after applying (19).

This completes the proof for i such that $(c\lambda)_i\ge(c\lambda)_{i+1}$. Otherwise, $i\in[n-1]$ with $(c\lambda)_i<(c\lambda)_{i+1}$. If $i=1$ or $i=n-1$ the proof is straightforward. Otherwise, let $x<i<i+1<y$. Then

$$(c\lambda)_{i+1}\ge(c\lambda)_y \tag{22}$$

and $(c\lambda)_x\ge(c\lambda)_i$ by Lemma 5.3 (012-avoidance). If $(c\lambda)_x<(c\lambda)_y$, then $c\lambda$ contains the composition pattern 012, 1032, 0021, 0011, or 1022. This contradicts Lemma 5.3. Thus $(c\lambda)_x\ge(c\lambda)_y$ for all $x<i$. This implies $\mathsf{leftmin}_{c\lambda}(i-1)\ge(c\lambda)_y$ for all $i+1<y$. We conclude $\mathsf{leftmin}_{c\lambda}(i-1)\ge\mathsf{rightmax}_{c\lambda}(i+2)$. By Lemma 5.6 (the second displayed equation, where we have applied it to $i-1$) and the row distinct property of F, this implies

$$\begin{aligned}|\{(d,r)\in D(c\lambda): d>i,\ 1\le r\\ \le \mathsf{rightmax}_{c\lambda}(i+2), \text{ and } F(d,r)\le i-1\big\}|=0.\end{aligned} \tag{23}$$

Then $\mathsf{leftmin}_{c\lambda}(i)=(c\lambda)_i$ by Lemma 5.3 (012-avoidance) and, combined with Lemma 5.6 applied to i, this implies

$$|\{(d,r)\in D(c\lambda): d>i,\ 1\le r\le\mathsf{leftmin}_{c\lambda}(i), \text{ and } F(d,r)\le i\}|=0. \tag{24}$$

Now

$$\begin{aligned}&|\{(d,r)\in D(c\lambda): d>i \text{ and } F(d,r)\le i\}|\\ &\quad\le (c\lambda)_{i+1}-\mathsf{leftmin}_{c\lambda}(i)\\ &\quad= \mathsf{rightmax}_{c\lambda}(i+1)-\mathsf{leftmin}_{c\lambda}(i).\end{aligned}$$

The inequality comes by studying the intervals $[1, \mathsf{leftmin}_{c\lambda}(i)]$, $(\mathsf{leftmin}_{c\lambda}(i), \mathsf{rightmax}_{c\lambda}(i+2)]$, and $(\mathsf{rightmax}_{c\lambda}(i+2), (c\lambda)_{i+1}]$. Respectively, we use (24), and (23) paired with the row distinct property of F, for the first two intervals. For the third interval, we use the fact that there is at most one column, namely $y = i+1$, such that $y > d$ and $(c\lambda)_y > \mathsf{rightmax}_{c\lambda}(i+2)$. The equality follows from (22). □

Lemma 5.11 *Let $i < p < j < q$ be in the same block and $\beta \in \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$. If (i,j) and (p,q) are interwoven, $\beta <_{\text{Bruhat}} t_{i,j}\beta$, and $\beta <_{\text{Bruhat}} t_{p,q}\beta$, then $t_{i,j}\beta \notin \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$ or $t_{p,q}\beta \notin \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$.*

Proof If (i,j) and (p,q) are interwoven, then it is straightforward that $\mathsf{center}_{c\lambda}(i,j) = \mathsf{center}_{c\lambda}(p,q)$. Lemma 5.1 and the definition of $\mathsf{center}_{c\lambda}(i,j)$ implies

$$(c\lambda)_k = \mathsf{rightmax}_{c\lambda}(k) \text{ for } i \le k \le \mathsf{center}_{c\lambda}(p,q),$$

which in turn implies, via Lemma 5.5(ii), that

$$\beta_k - (c\lambda)_k \le 0 \text{ for } i \le k \le \mathsf{center}_{c\lambda}(p,q). \tag{25}$$

In a similar fashion, the definition of $\mathsf{center}_{c\lambda}(i,j)$ and Lemma 5.3 (012-avoidance) implies $(c\lambda)_k = \mathsf{leftmin}_{c\lambda}(k)$ for $\mathsf{center}_{c\lambda}(p,q) < k \le q$. Hence, Lemma 5.5(i) says

$$\beta_k - (c\lambda)_k \ge 0 \text{ for } \mathsf{center}_{c\lambda}(p,q) < k \le q. \tag{26}$$

Suppose that $t_{i,j}\beta, t_{p,q}\beta \in \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$. Let $C := \beta_1 + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} - ((c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)})$. Then,

$$\begin{aligned} \mathsf{rightmax}_{c\lambda}(q) - (c\lambda)_q &= \mathsf{rightmax}_{c\lambda}(q+1) - \mathsf{leftmin}_{c\lambda}(q) \\ &\ge C + (\beta_{\mathsf{center}_{c\lambda}(i,j)+1} + \cdots + \beta_q) \\ &\quad - ((c\lambda)_{\mathsf{center}_{c\lambda}(i,j)+1} + \cdots (c\lambda)_q) \\ &\ge C + (\beta_j - (c\lambda)_j) + (\beta_q - (c\lambda)_q), \end{aligned} \tag{27}$$

where the equality follows from the interweaving assumption combined with Lemma 5.3 (012-avoidance), the first inequality comes from Lemma 5.10 applied to β, and the final inequality follows from (26).

Proposition 5.9(3) says

$$\begin{aligned} &\beta_1 + \cdots + \beta_{i-1} + (\beta_j - (j-i)) + \beta_{i+1} + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} \\ &\quad \ge (c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)}, \end{aligned} \tag{28}$$

$$\begin{aligned} &\beta_1 + \cdots + \beta_{p-1} + (\beta_q - (q-p)) + \beta_{p+1} + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} \\ &\quad \ge (c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)}. \end{aligned} \tag{29}$$

Let $D := (\beta_{i+1}+\cdots+\beta_{\mathsf{center}_{c\lambda}(i,j)})-((c\lambda)_{i+1}+\cdots+(c\lambda)_{\mathsf{center}_{c\lambda}(i,j)})$. Reformulating (28) yields

$$\begin{aligned} 0 &\leq (\beta_1+\cdots+\beta_{i-1})-((c\lambda)_1+\cdots+(c\lambda)_{i-1})+(\beta_j-(j-i)-(c\lambda)_i)+D \\ &\leq (c\lambda)_i-\mathsf{leftmin}_{c\lambda}(i)+(\beta_j-(j-i)-(c\lambda)_i)+D \\ &= (c\lambda)_i-\mathsf{leftmin}_{c\lambda}(i)+((c\lambda)_j+(\beta_j-(c\lambda)_j)-(j-i)-(c\lambda)_i)+D \\ &= ((c\lambda)_j-\mathsf{leftmin}_{c\lambda}(i))+(\beta_j-(c\lambda)_j)-(j-i)+D \\ &\leq (\beta_j-(c\lambda)_j)-(j-i)+D, \end{aligned} \tag{30}$$

where the second inequality is via Lemma 5.10 applied to β (note $(c\lambda)_i = \mathsf{rightmax}_{c\lambda}(i)$ here), and the final inequality follows from Lemma 5.3 (012-avoidance).

Let $E := (\beta_{p+1}+\cdots+\beta_{\mathsf{center}_{c\lambda}(i,j)})-((c\lambda)_{p+1}+\cdots+(c\lambda)_{\mathsf{center}_{c\lambda}(i,j)})$. Reformulating (29),

$$\begin{aligned} 0 &\leq (\beta_1+\cdots+\beta_{p-1})-((c\lambda)_1+\cdots+(c\lambda)_{p-1})+(\beta_q-(q-p)-(c\lambda)_p)+E \\ &\leq (\beta_1+\cdots+\beta_i)-((c\lambda)_1+\cdots+(c\lambda)_i)+(\beta_q-(q-p)-(c\lambda)_p)+E \\ &\leq (\beta_1+\cdots+\beta_i)-((c\lambda)_1+\cdots+(c\lambda)_i)+(\beta_q-(q-p)-(c\lambda)_p) \\ &\leq (\beta_1+\cdots+\beta_i)-((c\lambda)_1+\cdots+(c\lambda)_i)+(\beta_q-(q-p)-\mathsf{rightmax}_{c\lambda}(q)) \\ &= (\beta_1+\cdots+\beta_i)-((c\lambda)_1+\cdots+(c\lambda)_i)+((c\lambda)_q+(\beta_q-(c\lambda)_q)-(q-p)-\mathsf{rightmax}_{c\lambda}(q). \end{aligned} \tag{31}$$

where the second and third inequality are by (25), the fourth inequality is by Lemma 5.3 (012-avoidance).

Adding (30) and (31) we have

$$\begin{aligned} 0 \leq C &+ (\beta_j-(c\lambda)_j)+(\beta_q-(c\lambda)_q)-(j-i) \\ &-(q-p)+((c\lambda)_q-\mathsf{rightmax}_{c\lambda}(q)) \end{aligned} \tag{32}$$

which can be reformulated into

$$\begin{aligned} \mathsf{rightmax}_{c\lambda}(q)-(c\lambda)_q &\leq C+(\beta_j-(c\lambda)_j)+(\beta_q-(c\lambda)_q)-(j-i)-(q-p) \\ &< C+(\beta_j-(c\lambda)_j)+(\beta_q-(c\lambda)_q) \end{aligned} \tag{33}$$

This, combined with (27), gives our desired contradiction. We conclude that $t_{i,j}\beta \notin \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$ or $t_{p,q}\beta \notin \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$. $\square$

5.3 The diamond property

We are now ready for the proof of Theorem 4.4.

Conclusion of the proof of Theorem 4.4: Without loss of generality assume $(i,j) < (p,q)$ in lexicographic order. Both $\tau := t_{i,j}\beta$ and $\phi := t_{p,q}\beta$ cover β, thus

$$\beta_i > \beta_j-(j-i) = \tau_i, \tag{34}$$

$$\beta_p > \beta_q - (q-p) = \phi_p. \tag{35}$$

By Proposition 5.9, we have

$$\beta_i + (j-i) \leq \mathsf{rightmax}_{c\lambda}(j), \tag{36}$$
$$\beta_p + (q-p) \leq \mathsf{rightmax}_{c\lambda}(q), \tag{37}$$
$$\beta_j - (j-i) \geq \mathsf{leftmin}_{c\lambda}(i), \tag{38}$$
$$\beta_q - (q-p) \geq \mathsf{leftmin}_{c\lambda}(p). \tag{39}$$

Moreover, for the same reason, if (i, j) is interwoven, then

$$\begin{aligned}&\beta_1 + \cdots + \beta_{i-1} + \beta_j - (j-i) + \beta_{i+1} + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)}\\ &\geq (c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)}.\end{aligned} \tag{40}$$

If (p, q) is interwoven, then

$$\begin{aligned}&\beta_1 + \cdots + \beta_{p-1} + \beta_q - (q-p) + \beta_{p+1} + \cdots + \beta_{\mathsf{center}_{c\lambda}(p,q)}\\ &\geq (c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(p,q)}.\end{aligned} \tag{41}$$

We now consider five cases depending on the overlap in the values (i, j) and (p, q). In what follows, we will make repeated use of Lemma 2.9(i), which characterizes the covering relation in $(\mathcal{S}_{\gamma,I}, <_{\text{Bruhat}})$.

Case 1.1 (*i and p in the same block, $i = p$, $j < q$*): Suppose, for contradiction, that $\beta_j - (j-i) = \beta_q - (q-p)$. Then, since $i = p$, this equality is equivalent to $\beta_j = \beta_q - (q-j)$. The contradicts Lemma 2.9(i), and hence $\beta_j - (j-i) \neq \beta_q - (q-p)$.

Subcase 1.1.1 $\beta_j - (j-i) > \beta_q - (q-p)$: By the subcase hypothesis, $t_{i,j}\beta <_{\text{Bruhat}} t_{p,q}t_{i,j}\beta$. Then $t_{p,q}t_{i,j}\beta <_{\text{Bruhat}} t_{j,q}t_{p,q}t_{i,j}\beta$ by (34). Combining, we have $t_{i,j}\beta <_{\text{Bruhat}} t_{j,q}t_{p,q}t_{i,j}\beta = t_{p,q}\beta$. This contradicts the hypothesis that $t_{p,q}\beta$ covers β. Hence this subcase cannot occur.

Subcase 1.1.2 $\beta_j - (j-i) < \beta_q - (q-p)$: We will show that $t_{i,j}\phi \in \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$. By the subcase hypothesis, the definition of ϕ, and $i = p$,

$$\phi_i = \phi_p = \beta_q - (q-p) > \beta_j - (j-i) = \phi_j - (j-i). \tag{42}$$

By (35), (36), and $i = p$ we have

$$\begin{aligned}\phi_i + (j-i) &= \beta_q - (q-p) + (j-i) < \beta_p + (j-i)\\ &= \beta_i + (j-i) \leq \mathsf{rightmax}_{c\lambda}(j).\end{aligned} \tag{43}$$

Since $\phi_j = \beta_j$, by (38),

$$\phi_j - (j-i) = \beta_j - (j-i) \geq \mathsf{leftmin}_{c\lambda}(i). \tag{44}$$

Finally, $\phi_r = \beta_r$ for $r \neq p, q$. If (i, j) is interwoven, then (40) and $i = p$ combined with the previous sentence implies

$$\begin{aligned}
&\phi_1 + \cdots + \phi_{i-1} + \phi_j - (j-i) + \phi_{i+1} + \cdots + \phi_{\mathsf{center}_{c\lambda}(i,j)} \\
&\quad = \beta_1 + \cdots + \beta_{i-1} + \beta_j - (j-i) \\
&\qquad + \beta_{i+1} + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} \\
&\quad \geq (c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)}.
\end{aligned} \tag{45}$$

The hypotheses of Proposition 5.9 are satisfied for $t_{i,j}\phi$ by (42), (43), (44), and (45). Hence, $t_{i,j}\phi \in \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$.

By (42), $\phi < t_{i,j}\phi$. By (35), $\beta_i + (j-i) = \beta_p + (j-i) > \beta_q - (q-p) + (j-i) = \beta_q - (q-j)$, and hence $\tau = t_{i,j}\beta <_{\text{Bruhat}} t_{j,q}t_{i,j}\beta = t_{i,j}\phi$.

Case 1.2 (*i and p in the same block, $i < p$, $j = p$*): In this case,

$$\tau_p = \beta_i + (j-i) > \beta_j > \beta_q - (q-p) = \tau_q - (q-p), \tag{46}$$

$$\phi_i = \beta_i > \beta_j - (j-i) > \beta_q - (q-p) - (j-i) = \phi_j - (j-i). \tag{47}$$

Before breaking into subcases we first prove that $\phi <_{\text{Bruhat}} t_{i,j}\phi, t_{p,q}\tau$, and $\tau <_{\text{Bruhat}} t_{i,j}\phi, t_{p,q}\tau$. First, $\phi <_{\text{Bruhat}} t_{i,j}\phi$ and $\tau <_{\text{Bruhat}} t_{p,q}\tau$ follow from (47) and (46). Then, (34) implies

$$\phi_i = \beta_i > \beta_j - (j-i) = \beta_j + (q-j) + (q-i) = \phi_q - (q-i),$$

and hence $\phi <_{\text{Bruhat}} t_{i,q}\phi = t_{p,q}\tau$. Finally, by (35),

$$\tau_i = \beta_j - (j-i) > \beta_q - (q-j) - (j-i) = \tau_q - (q-i),$$

and thus $\tau <_{\text{Bruhat}} t_{i,q}\tau = t_{i,j}\phi$. Hence, in all the following subcases, it remains to show that at least one of $t_{i,j}\phi$ or $t_{p,q}\tau$ are in $\mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$.

Subcase 1.2.1 $\mathsf{leftmin}_{c\lambda}(i) = (c\lambda)_i$ *and* $\mathsf{rightmax}_{c\lambda}(j) = (c\lambda)_j$: By Lemma 5.5(iii), $\beta_i = (c\lambda)_i$ and $\beta_j = (c\lambda)_j$. Thus

$$(c\lambda)_i = \mathsf{leftmin}_{c\lambda}(i) \leq \beta_j - (j-i) = (c\lambda)_j - (j-i),$$

where the first equality is the case hypothesis and the inequality is (38). Now $j > i$ implies that $(c\lambda)_i < (c\lambda)_j$. This contradicts Lemma 5.1, and hence this case cannot occur.

Subcase 1.2.2 $\mathsf{leftmin}_{c\lambda}(i) = (c\lambda)_i$ *and* $\mathsf{rightmax}_{c\lambda}(j) > (c\lambda)_j$: By Lemma 5.1, the subcase hypothesis implies

$$\mathsf{leftmin}_{c\lambda}(k) = (c\lambda)_k \text{ for } i \leq k \leq q. \tag{48}$$

By Lemma 5.5(i) this implies

$$\beta_k \geq (c\lambda)_k \text{ for } i \leq k \leq q. \tag{49}$$

In this subcase, (38) and (39) become

$$\beta_j - (j-i) \geq (c\lambda)_i = (c\lambda)_j + ((c\lambda)_i - (c\lambda)_j), \tag{50}$$
$$\beta_q - (q-p) \geq (c\lambda)_p = (c\lambda)_q + ((c\lambda)_p - (c\lambda)_q). \tag{51}$$

Thus

$$\begin{aligned}
\mathsf{rightmax}_{c\lambda}(q) - (c\lambda)_q &= \mathsf{rightmax}_{c\lambda}(q+1) - \mathsf{leftmin}_{c\lambda}(q) \\
&\geq \beta_1 + \cdots + \beta_q - ((c\lambda)_1 + \cdots + (c\lambda)_q) \\
&= \left(\sum_{t=1}^{i-1} \beta_t - (c\lambda)_t\right) + [\beta_i - (c\lambda)_i] + \left(\sum_{t=i+1, t\neq p}^{q-1} \beta_t - (c\lambda)_t\right) \\
&\quad + [\beta_p - (c\lambda)_p] + [\beta_q - (c\lambda)_q] \\
&\geq [\beta_i - (c\lambda)_i] + [\beta_p - (c\lambda)_p] + [\beta_q - (c\lambda)_q] \\
&\geq [\beta_i - (c\lambda_i)] + [(j-i) + ((c\lambda)_i - (c\lambda)_j)] + [(q-p) + ((c\lambda)_p - (c\lambda)_q)] \\
&= \beta_i + (q-i) - (c\lambda)_q,
\end{aligned} \tag{52}$$

where the first equality follows from the subcase hypotheses and (48), the first inequality from Lemma 5.10 with $\mathsf{rightmax}_{c\lambda}(q+1) - \mathsf{leftmin}_{c\lambda}(q) \geq 0$, the second inequality by Corollary 3.6 and (49), and the third inequality is by (50), (51), and the final equality by $p = j$. Rewriting (52), we arrive at $\mathsf{rightmax}_{c\lambda}(q) \geq \beta_i + (q-i) = \tau_p + (q-p)$. Further, by (39), $\tau_q - (q-p) = \beta_q - (q-p) \geq \mathsf{leftmin}_{c\lambda}(p)$. The hypotheses of Proposition 5.9 are satisfied for $t_{p,q}\tau$ by the preceding two sentences, the subcase hypothesis, and (46). Hence, $t_{p,q}\tau \in \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$ (notice (p,q) cannot be interwoven since $j = p$ and $c\lambda$ is 012-avoiding by Lemma 5.3).

Subcase 1.2.3 $\mathsf{leftmin}_{c\lambda}(i) < (c\lambda)_i$ *and* $\mathsf{rightmax}_{c\lambda}(j) = (c\lambda)_j$: By the subcase hypotheses,

$$\mathsf{rightmax}_{c\lambda}(k) = (c\lambda)_k \text{ for } i \leq k \leq j, \tag{53}$$

and hence by Lemma 5.5(ii)

$$\beta_k \leq (c\lambda)_k \text{ for } i \leq k \leq j. \tag{54}$$

In this subcase, (36) becomes

$$\beta_i + (j-i) \leq (c\lambda)_j = (c\lambda)_i + ((c\lambda)_j - (c\lambda)_i). \tag{55}$$

By Corollary 3.6 applied to $\phi = t_{p,q}\beta$,

$$\beta_1 + \cdots + \beta_{i-1} - ((c\lambda)_1 + \cdots + (c\lambda)_{i-1})$$

$$\geq -((\beta_i + \cdots + \beta_{j-1} + (\beta_q - (q-p)) \\ -((c\lambda)_i + \cdots + (c\lambda)_j)). \tag{56}$$

We conclude

$$\begin{aligned}(c\lambda)_i - \mathsf{leftmin}_{c\lambda}(i) &= \mathsf{rightmax}_{c\lambda}(i) - \mathsf{leftmin}_{c\lambda}(i-1) \\ &\geq \beta_1 + \cdots + \beta_{i-1} - ((c\lambda)_1 + \cdots + (c\lambda)_{i-1}) \\ &\geq -((\beta_i + \cdots + \beta_{j-1} + (\beta_q - (q-p)) - ((c\lambda)_i + \cdots + (c\lambda)_j)) \\ &= -\left(\sum_{t=i+1}^{j-1} \beta_t - (c\lambda)_t\right) - [\beta_i - (c\lambda)_i] - [\beta_q - (q-p) - (c\lambda)_j] \\ &\geq -[\beta_i - (c\lambda)_i] - [\beta_q - (q-p) - (c\lambda)_j] \\ &\geq -[-(j-i) + ((c\lambda)_j - (c\lambda)_i)] - [(\beta_q - (q-p) - (c\lambda)_j] \\ &= (q-i) - (\beta_q - (c\lambda)_i);\end{aligned} \tag{57}$$

the first equality follows by the subcase hypotheses, the first inequality from Lemma 5.10 with $\mathsf{rightmax}_{c\lambda}(i) - \mathsf{leftmin}_{c\lambda}(i-1) \geq 0$, the second inequality by (56), the third inequality by (54), the fourth by (55), and the final equality by $p = j$. Now (57) is equivalent to

$$\mathsf{leftmin}_{c\lambda}(i) \leq \beta_q - (q-i) = \phi_j - (j-i).$$

Further, by (36),

$$\phi_i + (j-i) = \beta_i + (j-i) \geq \mathsf{rightmax}_{c\lambda}(j).$$

The hypotheses of Proposition 5.9 are satisfied for $t_{i,j}\phi$ by the preceding two sentences, the subcase hypothesis, and (47). Hence, $t_{i,j}\phi \in \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$.

Subcase 1.2.4 $\mathsf{leftmin}_{c\lambda}(i) < (c\lambda)_i$, $\mathsf{rightmax}_{c\lambda}(j) > (c\lambda)_j$: In this subcase, $\mathsf{leftmin}_{c\lambda}(j) = (c\lambda)_j$, since $\mathsf{leftmin}_{c\lambda}(j) < (c\lambda)_j$ would imply that $c\lambda$ contains 012. Thus, since (i, j) is interwoven, $\mathsf{center}_{c\lambda}(i, j) < j$ and the definition of $\mathsf{center}_{c\lambda}(i, j)$ and Lemma 5.3 (012-avoidance) implies

$$\mathsf{leftmin}_{c\lambda}(k) = (c\lambda)_k \text{ for } \mathsf{center}_{c\lambda}(i, j) < k \leq q. \tag{58}$$

Corollary 3.6, applied to β and τ, respectively, implies

$$(\beta_1 + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)}) - ((c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)}) \geq 0,$$

and

$$(\beta_1 + \cdots + \beta_{i-1} + \beta_j - (j-i) + \beta_{i+1} + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)}) \\ -((c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)}) \geq 0.$$

The second inequality yields

$$\begin{aligned}&(\beta_1 + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)}) - ((c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)})\\&\quad \geq \beta_i - (\beta_j - (j-i)).\end{aligned} \tag{59}$$

Thus

$$\begin{aligned}\mathsf{rightmax}_{c\lambda}(q) - (c\lambda)_q &= \mathsf{rightmax}_{c\lambda}(q+1) - \mathsf{leftmin}_{c\lambda}(q)\\&\geq \beta_1 + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} - ((c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)})\\&\quad + (\beta_{\mathsf{center}_{c\lambda}(i,j)+1} - (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)+1}) + \cdots + (\beta_{j-1} - (c\lambda)_{j-1})\\&\quad + (\beta_q - (q-p) - (c\lambda)_j) + (\beta_{j+1} - (c\lambda)_{j+1}) + \cdots + (\beta_{q-1} - (c\lambda)_{q-1})\\&\quad + (\beta_j + (q-p) - (c\lambda)_q)\\&\geq \beta_1 + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} - ((c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)})\\&\quad + (\beta_j + (q-p) - (c\lambda)_q)\\&\geq \beta_i - (\beta_j - (j-i)) + (\beta_j + (q-p) - (c\lambda)_q)\\&= \beta_i + (q-i) - (c\lambda)_q,\end{aligned} \tag{60}$$

where the first relation follows from the subcase hypotheses, the second relation from Lemma 5.10 with $\mathsf{rightmax}_{c\lambda}(q+1) - \mathsf{leftmin}_{c\lambda}(q) \geq 0$ applied to τ, the third from (58), Corollary 3.6, and Lemma 5.5(i), the fourth by (59), and the final by $p = j$. Hence, (60) implies $\mathsf{rightmax}_{c\lambda}(q) \geq \beta_i + (q-i) = \tau_p + (q-p)$. By (39), $\mathsf{leftmin}_{c\lambda}(p) \leq \beta_q - (q-p) = \tau_q - (q-p)$. We conclude by Proposition 5.9 applied to $t_{p,q}\tau$ that $t_{p,q}\tau \in \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$ (notice (p,q) cannot be interwoven since $j = p$ and $c\lambda$ is 012-avoiding by Lemma 5.3).

Case 1.3 (*i and p in the same block, $i < p$, $j = q$*): Lemma 2.9(i) implies $\beta_i \neq \beta_p - (p-i)$.

Subcase 1.3.1 $\beta_i > \beta_p - (p-i)$: It is easily checked that $t_{p,q}\beta <_{\text{Bruhat}} t_{i,p}t_{p,q}\beta <_{\text{Bruhat}} t_{p,q}t_{i,p}t_{p,q}\beta = t_{i,j}\beta$. Hence $t_{i,j}\beta$ is not a cover of β and this subcase cannot occur.

Subcase 1.3.2 $\beta_i < \beta_p - (p-i)$: By the subcase hypothesis, the definition of τ, and $j = q$,

$$\begin{aligned}\tau_p = \beta_p &> \beta_i + (p-i) = \beta_i + (j-i) - (q-p)\\&= \tau_j - (q-p) = \tau_q - (q-p).\end{aligned} \tag{61}$$

By (34), (39), and $j = q$ we have

$$\begin{aligned}\tau_q - (q-p) &= \tau_j - (q-p) = \beta_i + (j-i) - (q-p) > \beta_j - (q-p)\\&= \beta_q + (q-p) \geq \mathsf{leftmin}_{c\lambda}(q).\end{aligned} \tag{62}$$

Since $\tau_p = \beta_p$, by (37),

$$\tau_p + (q-p) = \beta_p + (q-p) \leq \mathsf{rightmax}_{c\lambda}(q). \tag{63}$$

Finally, $\tau_r = \beta_r$ for $r \neq i, j$. If (p, q) is interwoven, then (41) and $j = q$ combined with the previous sentence implies

$$\begin{aligned}
&\tau_1 + \cdots + \tau_{p-1} + \tau_q - (q-p) + \tau_{p+1} + \cdots + \tau_{\mathsf{center}_{c\lambda}(p,q)} \\
&\quad = \beta_1 + \cdots + \beta_{i-1} + \beta_j - (j-i) + \beta_{i+1} + \cdots \\
&\qquad + \beta_{p-1} + \beta_i + (j-i) - (q-p) + \beta_{p+1} + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} \\
&\quad = \beta_1 + \cdots + \beta_{p-1} + \beta_j - (q-p) + \beta_{p+1} + \cdots + \beta_{\mathsf{center}_{c\lambda}(i,j)} \\
&\quad \geq (c\lambda)_1 + \cdots + (c\lambda)_{\mathsf{center}_{c\lambda}(i,j)}.
\end{aligned} \tag{64}$$

The hypotheses of Proposition 5.9 are satisfied for $t_{pq}\tau$ by (61), (62), (63), and (64). Hence, $t_{p,q}\tau \in \mathcal{P}_{c\lambda,\boldsymbol{\gamma}}$.

We conclude by (42) that $\tau <_{\text{Bruhat}} t_{p,q}\tau$. By (34), $\phi_i = \beta_i > \beta_j - (j-i) = \beta_q - (q-p) - (p-i) = \phi_p - (p-i)$, and hence $\phi = t_{p,q}\beta <_{\text{Bruhat}} t_{i,p}t_{p,q}\beta = t_{p,q}\tau$.

Case 1.4 ($i < p < j < q$ *are all disjoint*): In this case $\tau, \phi <_{\text{Bruhat}} t_{i,j}t_{p,q}\beta = t_{p,q}t_{i,j}\beta$. By Lemma 5.11, at least one of (i, j) or (p, q) is not interwoven. If (i, j) is not interwoven then it follows from applying Proposition 5.9 to $t_{i,j}\beta \in \mathcal{P}_{c\lambda,\gamma}$ that $t_{p,q}\beta$ satisfies the hypotheses of Proposition 5.9 yielding $t_{i,j}t_{p,q}\beta \in \mathcal{P}_{c\lambda,\gamma}$. Similarly, if (p, q) is not interwoven, Proposition 5.9 implies $t_{p,q}t_{i,j}\beta \in \mathcal{P}_{c\lambda,\gamma}$.

Case 1.5 ($i < j < p < q$ *are all disjoint*): Once again $\tau, \phi <_{\text{Bruhat}} t_{i,j}t_{p,q}\beta = t_{p,q}t_{i,j}\beta$. It is easy to check that $t_{p,q}\beta$ satisfies the hypotheses of Proposition 5.9 yielding $t_{i,j}t_{p,q}\beta \in \mathcal{P}_{c\lambda,\gamma}$. $\square$

6 Proof of Theorem 3.8 ($\Leftarrow$)

Let us restate the "$\Leftarrow$" direction of Theorem 3.8:

Proposition 6.1 *Let $w \in \mathfrak{S}_n$, $I \subset J(w)$ and $D = [n-1] - I$ where w is not I-spherical. There exists $\lambda \in \mathsf{Par}_D$ such that $\kappa_{w\lambda}$ is not D-multiplicity-free.*

Our strategy is to construct such a λ explicitly.

Proof Let $u = w_0(I) \cdot w$. Since w is not I-spherical, by Definition 1.2, u is not a product of distinct generators. By Proposition 7.9, u contains 321 or 3412. We divide our analysis into cases based on the patterns contained in u. For $\mu \in \mathsf{Comp}_n$ write $\mu|_D = (\mu^1, \ldots, \mu^k)$ to denote the splitting of μ into blocks of sizes $d_1 - d_0, \ldots, d_{k+1} - d_k = n - d_k$. Note that $\mu|_D \in \mathsf{Par}_D$ if it is weakly decreasing in each block.

Case 1 (u *contains the pattern* 321): Choose the partition λ whose parts are in $\{2, 1, 0\}$ so that $u\lambda$ contains the values 0, 1, 2 at indices $p' < q < r'$. Choose the pattern 012 so that $r' - p'$ is minimized. Also choose the minimum $p \leq p'$ such that $u\lambda$ contains only 0's at indices $p, \ldots, p'$ and choose the maximum $r \geq r'$ such that $u\lambda$ contains only 2's at indices $r', \ldots, r$. An example of a skyline diagram of $u\lambda$ is shown in Fig. 3.

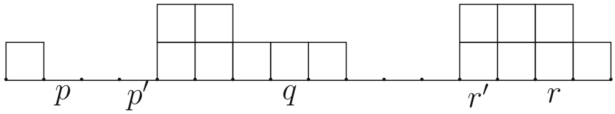


Fig. 3 A skyline diagram for $u\lambda$ that contains 012

Here, $(u\lambda)_{p'} = 0$, $(u\lambda)_q = 1$, $(u\lambda)_{r'} = 2$. In the interval $[p'+1, q]$, $u\lambda$ can take on 1's or 2's, and all the 2's are left of the 1's by minimality of $r' - p'$. Similarly, in the interval $[q, r'-1]$, $u\lambda$ takes on values 1's followed by 0's. Thus, in the interval $[p'+1, r'-1]$, say $u\lambda$ takes on $k_2 \geq 0$ many 2's, then $k_1 \geq 1$ many 1's and then $k_0 \geq 0$ many 0's, and $(u\lambda)_q = 1$.

Since $I \subset J(w)$, $D = [n-1] - I$, $w\lambda$ is weakly increasing in each block so $u\lambda$ is weakly decreasing in each block, i.e., $(u\lambda)|_D \in \mathsf{Par}_D$. The argument that follows only uses this property of D.

Consider the following composition

$$\gamma = (\gamma^1, \ldots, \gamma^k) = (u\lambda + \vec{e}_p - \vec{e}_r)|_D.$$

It is easily checked that if $(u\lambda)_i \geq (u\lambda)_{i+1}$, then $\gamma_i \geq \gamma_{i+1}$ by our choice of p and r. Thus each γ^i is indeed a partition, meaning that $\gamma \in \mathsf{Par}_D$.

Recall the poset $\mathcal{P}_{u\lambda,\gamma}$ (Sect. 4) contains all vectors β such that the monomial x^β appears in the expansion of $\kappa_{u\lambda}$ and $\pi_{w_0(I)} x^\beta = \pm s_\gamma$ (see Lemma 4.1). By Lemma 4.6, $\mathcal{P}_{u\lambda,\gamma}$ is an order ideal in $\mathcal{S}_{I,\gamma}$. Also each element β can be generated from γ via the moves t_{ij}.

Claim 6.2 *$\mathcal{P}_{u\lambda,\gamma}$ has height at most 1. Moreover it has at most $k_1 - 1$ many β such that $\theta(\beta) = 1$.*

Proof of Claim 6.2 Since all part sizes of $u\lambda$ belong in $\{0, 1, 2\}$, it is straightforward from Lemma 2.9(i) that the only t_{ij}'s that increase the rank of β are

$$t_i : (\ldots, 1, 1, \ldots) \mapsto (\ldots, 0, 2, \ldots)$$

for i and $i+1$ in the same block. The number of nonzero values in the composition decreases by one when we apply such a move. Let $\#_{\neq 0}\beta$ be the number of nonzero values in β. By Kohnert's rule (Theorem 3.5), $\#_{\neq 0}\beta \geq \#_{\neq 0}u\lambda$ for $[x^\beta]\kappa_{u\lambda} > 0$. At the same time, $\#_{\neq 0}\gamma = \#_{\neq 0}u\lambda + 1$, meaning that for all $\beta \in \mathcal{P}_{u\lambda,\gamma}$, β can be obtained from γ via at most one such move t_i.

Next, let $\beta = t_i\gamma \in \mathcal{P}_{u\lambda,\gamma}$. Since $\beta \geq_{\mathsf{dom}} u\lambda$, by Corollary 3.6, we necessarily have $p' < i < r'$ so i is one of $r' + k_2 + 1, \ldots, r' + k_2 + k_1 - 1$ such that i and $i+1$ are in the same block. Thus, there are at most $k_1 - 1$ choices for i. $\square$

Claim 6.3 *If $\beta \in \mathcal{P}_{u\lambda,\gamma}$ and $\theta(\beta) = 1$ then $[x^\beta]\kappa_{u\lambda} = 1$.*

Proof of Claim6.3 For each such $\beta = t_i\gamma$, there is exactly one corresponding Kohnert diagram, as we need to move the top box in column r of $u\lambda$ to column $i+1$, and the single box in column i of $u\lambda$ to column p. An example of such Kohnert diagrams corresponding to the example in Fig. 3 is shown in Fig. 4. $\square$

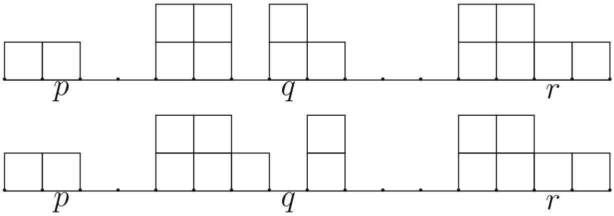


Fig. 4 Kohnert diagrams with weight $x^\beta = x^{t_{ij}\gamma}$ where $\beta \in \mathcal{P}_{u\lambda,\gamma}$

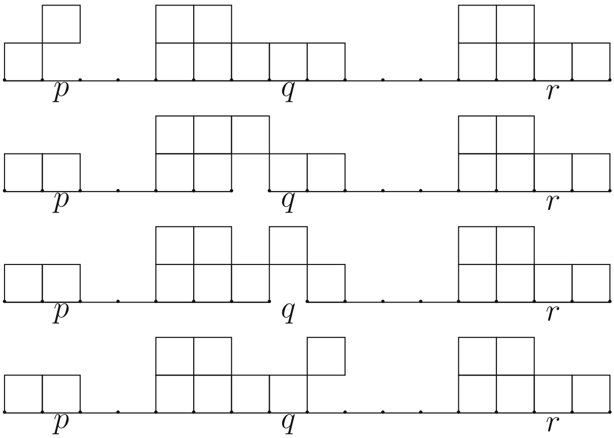


Fig. 5 Kohnert diagrams with weight x^γ

Claim 6.4 $[x^\gamma]\kappa_{u\lambda} = k_1 + 1$.

Proof of Claim 6.4 The $D \in \mathsf{Koh}(u\lambda)$ such that $\mathsf{Kohwt}(D) = \gamma$ are obtained by either

- moving the top box of column r in $u\lambda$ moved to column p; or
- moving the unique box in the column $z \in \{p' + k_2 + 1, \ldots, p' + k_2 + k_1\}$ to column p followed by moving the top box in column r to column z.

These Kohnert diagrams corresponding to the example shown in Fig. 3 are shown in Fig. 5. □

Hence, by Claims 6.2, 6.3, 6.4, and Lemma 4.3,

$$[s_\gamma]\kappa_{w\lambda} = \sum_{\beta \in \mathcal{P}_{u\lambda,\gamma}} \mathsf{sgn}(\beta)[x^\beta]\kappa_{u\lambda} \geq (k_1 + 1) - (k_1 - 1) = 2$$

so $\kappa_{w\lambda}$ is not D-multiplicity-free.

Case 2 (*u avoids the pattern* 321 *but u contains the pattern* 3412): Pick $\lambda \in \mathsf{Par}_n$ to consist of values in $\{3, 2, 1, 0\}$ so that $u\lambda$ contains the values $1, 0, 3, 2$ at indices $p' < q' < r' < z'$ so that $z' - p'$ is minimized. Analogous to Case 1, choose the minimum $p \leq p'$ such that $u\lambda$ contains only 1's in the interval $[p, p']$ and choose the

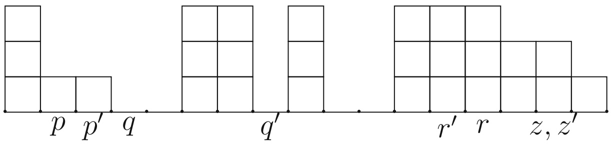


Fig. 6 A skyline diagram for $u\lambda$ that contains 1032 and avoids 012 (possibly $z = z'$)

maximum $z \geq z'$ such that $u\lambda$ contains only 2's on $[z', z]$. Let $q > p$ be the minimum index such that $(u\lambda)_q = 0$ and let $r < z$ be the maximum index such that $(u\lambda)_r = 3$. Since u avoids 321, $u\lambda$ avoids 012, and together with the minimality of $z' - p'$, we see that $(u\lambda)_{p'+1}, \ldots, (u\lambda)_{z'-1}$ can only take on values in $\{0, 3\}$. An example of a skyline diagram of $u\lambda$ is shown in Fig. 6.

Similar to Case 1, let

$$\gamma = (\gamma^1, \ldots, \gamma^k) = (u\lambda + \vec{e}_p + \vec{e}_q - \vec{e}_r - \vec{e}_z)|_D \in \mathsf{Par}_D.$$

Claim 6.5 $\mathcal{P}_{u\lambda,\gamma} = \{\gamma\}$.

Proof of Claim 6.5 By Proposition 2.3, Lemma 2.5, and Lemma 4.6, it suffices to show that there does not exist $i, i+1$ in the same block such that $\beta = t_i\gamma \in \mathcal{P}_{u\lambda,\gamma}$. If such a t_i exists, then $[x^\beta]\kappa_{u\lambda} > 0$ and so $\beta \geq_{\mathsf{dom}} u\lambda$, by Corollary 3.6. Also we must have $p \leq i < z$ since γ and $u\lambda$ only differ in that interval. Let $\beta_{\leq j} := (\beta_1, \ldots, \beta_j)$ and recall that $\#_{\neq 0}\beta$ is the number of nonzero entries in β. By Kohnert's rule, Theorem 3.5, for $\beta \in \mathcal{P}_{u\lambda,\gamma}$, $\#_{\neq 0}\beta_{\leq j} \geq \#_{\neq 0}(u\lambda)_{\leq j}$ for all j. Consider the following cases:

- $p = i < q$, $t_i : \gamma = (\ldots, 2, 1, \ldots) \mapsto (\ldots, 0, 3, \ldots)$, $\#_{\neq 0}\beta_{\leq i} < \#_{\neq 0}(u\lambda)_{\leq i}$;
- $p < i < q$, $t_i : \gamma = (\ldots, 1, 1, \ldots) \mapsto (\ldots, 0, 2, \ldots)$, $\#_{\neq 0}\beta_{\leq i} < \#_{\neq 0}(u\lambda)_{\leq i}$;
- $q \leq i < r$, $t_i : \gamma = (\ldots, 1, 0, \ldots) \mapsto (\ldots, -1, 2, \ldots)$ or $(\ldots, 3, \beta_{i+1}, \ldots) \mapsto (\ldots, \beta_{i+1} - 1, 4, \ldots)$, with impossible part sizes;
- $r \leq i < z$, $t_i : \gamma = (\ldots, 2, 2, \ldots) \mapsto (\ldots, 1, 3, \ldots)$ or $(\ldots, 2, 1, \ldots) \mapsto (\ldots, 0, 3, \ldots)$, where the newly generated part of size 3 cannot be obtained by Kohnert's rule, Theorem 3.5, since $u\lambda$, γ and β only differ on the interval $[p, z]$, that is $\beta \notin \mathcal{P}_{u\lambda,\gamma}$, a contradiction.

As a result, no such t_i exists. □

Claim 6.6 $[x^\gamma]\kappa_{u\lambda} = 2$.

Proof of Claim 6.6 The $D \in \mathsf{Koh}(u\lambda)$ such that $\mathsf{Kohwt}(D) = \gamma$ are obtained from $u\lambda$ by

- moving the top box of column r to column p and moving the top box of column z to column q; or
- moving the top box of column r to column q and moving the top box of column z to column p;

as shown in Fig. 7. □

Therefore, by Claim 6.5 and Claim 6.6, $[s_\lambda]\kappa_{w\lambda} = [x^\gamma]\kappa_{u\lambda} = 2$, as desired. □

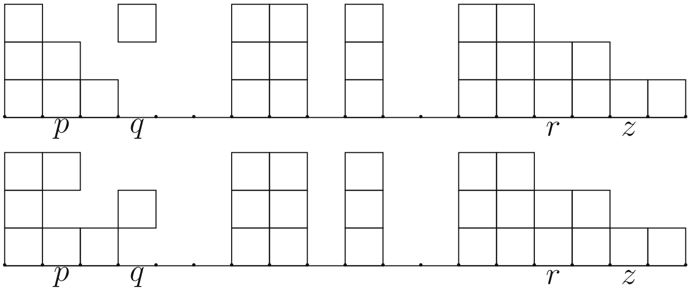


Fig. 7 Kohnert diagrams with weight x^γ

7 Equivalence of definitions

Let's first recall the definition of I-spherical in [18]. Let $\mathrm{Red}(w)$ be the set of *reduced expressions* $w = s_{i_1} \cdots s_{i_{\ell(w)}}$. Let $D := [n-1] - I = \{d_1 < d_2 < \cdots < d_k\}$; $d_0 := 0, d_{k+1} := n$.

Definition 7.1 ([18, Definition 3.1]) Let $w \in \mathfrak{S}_n$ and $I \subseteq J(w)$. Then w is *I-spherical* if $R = s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}} \in \mathrm{Red}(w)$ exists such that

(S.1) s_{d_i} appears at most once in R; and
(S.2) $\#\{m : d_{t-1} < i_m < d_t\} < \binom{d_t - d_{t-1}+1}{2}$ for $1 \le t \le k+1$.

Theorem 7.2 *Definitions 1.2 and 7.1 are equivalent.*

Theorems 7.2 and 1.3 were used in C. Gaetz's [14], which proves [18, Conjecture 3.8]. This gives a pattern avoidance criterion for maximally spherical Schubert varieties [14, Theorem 1.4, Corollary 1.5]. We refer to [18] for further information.

We first derive some results valid for any finite crystallographic root system Φ. Let the positive roots be Φ^+, with simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$. Let W be its finite Weyl group with corresponding simple generators $S = \{s_1, s_2, \ldots, s_r\}$, where we have fixed a bijection of $[r] := \{1, 2, \ldots, r\}$ with the nodes of the Dynkin diagram $\mathcal{G}$. Let $\mathrm{Red}(w)$ be the set of the *reduced expressions* $w = s_{i_1} \cdots s_{i_k}$, where $k = \ell(w)$ is the *Coxeter length* of w. The *left descents* of w are

$$J(w) = \{j \in [r] : \ell(s_j w) < \ell(w)\}.$$

For $I \in 2^{[r]}$, let $\mathcal{G}_I$ be the induced subdiagram of $\mathcal{G}$. Write

$$\mathcal{G}_I = \bigcup_{z=1}^{m} \mathcal{C}^{(z)} \tag{65}$$

as its decomposition into connected components. Let $w_0^{(z)}$ be the longest element of the parabolic subgroup $W_{I^{(z)}}$ generated by $I^{(z)} = \{s_j : j \in \mathcal{C}^{(z)}\}$. The generalization of Definition 7.1 to general type was given as follows:

Definition 7.3 Let $w \in W$ and fix $I \subset J(w)$. Then w is *I-spherical* if there exists $R = s_{i_1} \cdots s_{i_{\ell(w)}} \in \mathrm{Red}(w)$ such that

- $\#\{t \mid i_t = j\} \leq 1$ for all $j \in [r] - I$, and
- $\#\{t \mid i_t \in \mathcal{C}^{(z)}\} \leq \ell(w_0^{(z)}) + \#\mathrm{vertices}(\mathcal{C}^{(z)})$ for $1 \leq z \leq m$.

Such an R is called an *I-witness.*

Definition 1.2 makes sense in the general context as well. However, that notion differs from Definition 7.3 in type D_4 and F_4 (this reduces confidence in the general-type classification conjecture for Levi-spherical Schubert varieties [18, Conjecture 1.9]). We plan to study this further in future work.[1]

We now develop some preliminary results.

Lemma 7.4 *Let $w \in W$ and fix $I \subset J(w)$. Let $R = s_{i_1} \cdots s_{i_{\ell(w)}}$ and $R' = s_{j_1} \cdots s_{j_{\ell(w)}} \in \mathrm{Red}(w)$ be such that each s_t, $t \in [r] - I$, appears at most once in R, and at most once in R'. Then for each $1 \leq z \leq m$,*

$$\#\{t \mid i_t \in \mathcal{C}^{(z)}\} = \#\{t \mid j_t \in \mathcal{C}^{(z)}\}.$$

Proof We may assume without loss of generality that Φ is irreducible. Furthermore, we may assume without loss of generality that each $s_i \in S$ is used in any (equivalently, all) $R'' \in \mathrm{Red}(w)$, since otherwise we work individually on the root systems associated to each irreducible component of $\Delta \setminus \{\alpha_i\}$.

We induct on $m \geq 1$. In the base case $m = 1$, then

$$\#\{t \mid i_t \in \mathcal{C}^{(1)}\} = \ell(w) - (r - \#I)$$

is independent of any choice of R'', so we are done.

For the induction step, consider a fixed $\mathcal{C} \in \{\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(m)}\}$. Fix some $t_0 \in [r] - I$ such that not all of $\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(m)}$ lie in the same connected component of (the Dynkin diagram of) $S \setminus \{t_0\}$. Such t_0 can be chosen because $m \geq 2$ and the Dynkin diagram for W is a tree. Let $J_1, J_2, \ldots, J_p$ be the connected components of $S \setminus \{t_0\}$ and assume $\mathcal{C} \subset J_1$.

Note that generators in different J_i's commute with each other. For the reduced word R, we can regroup it as $w_{J_1} \cdots w_{J_p} s_{t_0} u_{J_1} \cdots u_{J_p}$ where $w_{J_i}, u_{J_i} \in W_{J_i}$, the parabolic subgroup generated by J_i. We can rearrange it as

$$w = (w_{J_2} \cdots w_{J_p})(w_{J_1} s_{t_0} u_{J_1})(u_{J_2} \cdots u_{J_p}).$$

Similarly, for R' we obtain

$$\begin{aligned} w &= w'_{J_1} \cdots w'_{J_p} s_{t_0} u'_{J_1} \cdots u'_{J_p} \\ &= (w'_{J_2} \cdots w'_{J_p})(w'_{J_1} s_{t_0} u'_{J_1})(u'_{J_2} \cdots u'_{J_p}). \end{aligned}$$

[1] As mentioned in the introduction, in later work [16, Section 4] such a counterexample was indeed verified using Demazure character computations.

Since $w_{J_1}s_{t_0}u_{J_1}$ does not contain any simple generators associated to $K = J_2 \cup \cdots \cup J_p$, it is the unique minimal double coset representative of $W_K w W_K$. This implies that $w_{J_1}s_{t_0}u_{J_1} = w'_{J_1}s_{t_0}u'_{J_1}$, where we obtained the same Weyl group element from different reduced decompositions.

Now apply the induction hypothesis by replacing S by $J_1 \cup \{t_0\}$, I by $I \cap J_1 \cup \{t_0\}$, w by the minimal length coset representative of $W_K w W_K$, R (and R') by the subword of R (and R') that equals $w_{J_1}s_{t_0}u_{J_1} = w'_{J_1}s_{t_0}u'_{J_1}$, and leaving $\mathcal{C}$ unchanged. □

For each $\alpha \in \Phi^+$, define its *support* to be

$$\mathrm{Supp}(\alpha) = \{\alpha_i \in \Delta \mid \alpha - \alpha_i \text{ is a nonnegative linear combination of } \Delta\}.$$

Also, for each positive root $\alpha = \sum_{i=1}^r c_i\alpha_i$, written as a nonnegative linear combination of Δ, define its *height* to be $\mathrm{ht}(\alpha) = \sum_{i=1}^r c_i$. The next folklore result is well-known, but we do not know a precise reference with proof. We include one here:

Lemma 7.5 *For each $\alpha \in \Phi^+$, $\mathrm{Supp}(\alpha)$ is a connected subgraph in the Dynkin diagram.*

Proof We use induction on $\mathrm{ht}(\alpha)$. The base case $\mathrm{ht}(\alpha) = 1$, i.e., $\alpha \in \Delta$, is clear.

In the induction step, for each $\alpha \in \Phi^+ \setminus \Delta$, there exists $i \in [r]$ such that $\alpha' := s_i\alpha = \alpha - k\alpha_i \in \Phi^+$ for some positive integer k. We know that $\mathrm{ht}(\alpha') < \mathrm{ht}(\alpha)$ so $\mathrm{Supp}(\alpha')$ is connected by induction hypothesis. At the same time, $\mathrm{Supp}(\alpha) = \mathrm{Supp}(\alpha') \cup \{\alpha_i\}$. If $\alpha_i \in \mathrm{Supp}(\alpha')$, then $\mathrm{Supp}(\alpha) = \mathrm{Supp}(\alpha')$ is connected. Thus, we assume $\alpha_i \notin \mathrm{Supp}(\alpha')$. Let $\langle -, - \rangle$ denote the standard inner product on the ambient vector space containing our root system. We have

$$\alpha = s_i\alpha' = \alpha' - \frac{2\langle \alpha', \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i \neq \alpha'.$$

As $\langle \alpha', \alpha_i \rangle \neq 0$, there exists some $\alpha_j \in \mathrm{Supp}(\alpha')$ such that $\langle \alpha_j, \alpha_i \rangle \neq 0$, meaning that the node j is connected to the node i in the Dynkin diagram. Therefore, $\mathrm{Supp}(\alpha) = \mathrm{Supp}(\alpha') \cup \{\alpha_i\}$ is connected. □

Lemma 7.6 *Suppose that we have an equality of reduced words $s_{i_1}s_{i_2}\cdots s_{i_{k-1}} = s_{i_2}s_{i_3}\cdots s_{i_k}$. Then $\#\{t \mid i_t = j\} \geq 2$ for all j on the path (excluding i_1 and i_k) between i_1 and i_k in $\mathcal{G}$.*

Proof Let $w = s_{i_1}s_{i_2}\cdots s_{i_{k-1}} = s_{i_2}s_{i_3}\cdots s_{i_k}$. As $s_{i_2}\cdots s_{i_k}$ is reduced, α_{i_k} is a right inversion of w, where α_{i_k} is the simple root corresponding to s_{i_k}, i.e., $\alpha_{i_k} \in \Phi^+$ and $w\alpha_{i_k} \in \Phi^-$. Let $-\beta = w\alpha_{i_k}$ so $\beta \in \Phi^+$. We have that

$$\beta = -s_{i_2}\cdots s_{i_k}\alpha_{i_k} = -s_{i_2}\cdots s_{i_{k-1}}(-\alpha_{i_k}) = s_{i_2}\cdots s_{i_{k-1}}\alpha_{i_k}.$$

This means that $s_{i_1}\beta = w\alpha_{i_k} = -\beta$ so $\beta = \alpha_{i_1}$.

Note that since $s_{i_j}\cdots s_{i_k}$ is reduced and has α_{i_k} as its right descent, we know

$$s_{i_j}\cdots s_{i_{k-1}}s_{i_k}\alpha_{i_k} \in \Phi^-,\ s_{i_j}\cdots s_{i_{k-1}}\alpha_{i_k} \in \Phi^+.$$

Consider the sequence of positive roots

$$\alpha_{i_k}, s_{i_{k-1}}\alpha_{i_k}, \ldots, s_{i_2}\cdots s_{i_{k-1}}\alpha_{i_k} = \alpha_{i_1}.$$

By definition, $s_\alpha(x) = x - \frac{2\langle x,\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha$. Hence the symmetric difference

$$\mathrm{Supp}(s_{i_t}\cdots s_{i_{k-1}}\alpha_{i_k}) \,\Delta\, \mathrm{Supp}(s_{i_{t+1}}\cdots s_{i_{k-1}}\alpha_{i_k}) \subseteq \{\alpha_{i_t}\}, \text{ for } t = k-1, \ldots, 2.$$

Recall that for each $\alpha \in \Phi^+$, its support $\mathrm{Supp}(\alpha)$ is connected in the Dynkin diagram (Lemma 7.5). Fix any j on the path between i_1 and i_k in the Dynkin diagram. As a result, there exists some p such that $\alpha_j \in \mathrm{Supp}(s_{i_p}\cdots s_{i_{k-1}}\alpha_{i_k})$. Thus, there must be some s_j among $s_{i_p}, \ldots, s_{i_{k-1}}$ so that a positive multiple of α_j can be added from α_{i_k}, and there must be some s_j among $s_{i_2}, \ldots, s_{i_{p-1}}$ so that a positive multiple of α_j can be subtracted to obtain α_{i_1}. □

We use this textbook result:

Proposition 7.7 (Deletion property [8, Proposition 1.4.7]) *Let $w = s_{i_1}\cdots s_{i_\ell}$ be a reduced word. Then for a left descent s_{i_0} of w, i.e. $\ell(s_{i_0}w) = \ell(w) - 1$, we have another reduced word $w = s_{i_0}s_{i_1}\cdots\widehat{s_{i_j}}\cdots s_{i_\ell}$, where $\widehat{s_{i_j}}$ means the deletion of s_{i_j}.*

The culmination of the above root-system uniform arguments is this next proposition, which says that Definition 7.3 is, in general, "close" to Definition 1.2.

Proposition 7.8 *If $w \in W$ is I-spherical (in the sense of Definition 7.3), then there exists an I-witness R of w of the form $R = R'R''$ where $R' \in \mathrm{Red}(w_0(I))$ and $R'' \in \mathrm{Red}(w_0(I)w)$.*

Proof Let $R^{(0)} = s_{i_1}\cdots s_{i_\ell}$ be an I-witness of w. Pick any $R' = s_{k_1}\cdots s_{k_{\ell'}} \in \mathrm{Red}(w_0(I))$. We gradually modify $R^{(0)}$, so that at each step it remains an I-witness, until it is of the desired form. For each $j = \ell', \ldots, 1$, add s_{k_j} to the start of R. By the deletion property (Proposition 7.7), some $s_{i_{j'}}$ is deleted resulting in $R^{(1)} \in \mathrm{Red}(w)$. By Lemma 7.6, k_j and $i_{j'}$ must be in the same $\mathcal{C}^{(z)}$ since otherwise, some s_i with $i \notin I$ on the path from k_j to $i_{j'}$ in the Dynkin diagram is used at least twice in $R^{(0)}$, contradicting that $R^{(0)}$ is an I-witness. Thus, in $R^{(1)}$, $\#\{t \mid i_t \in \mathcal{C}^{(z)}\}$ remains unchanged for each z. Repeating this, $k_{\ell'}$ many times, we obtain an I-witness $R^{(k_{\ell'})} = R'R''$, as claimed. □

Henceforth, we assume that $W = \mathfrak{S}_n$. Recall that $w \in \mathfrak{S}_n$ *contains the pattern* $u \in \mathfrak{S}_k$ if there exists $i_1 < i_2 < \ldots < i_k$ such that $w(i_1), w(i_2), \ldots, w(i_k)$ is in the same relative order as $u(1), u(2), \ldots, u(k)$. Furthermore w *avoids* u if no such indices exist.

We need the following proposition relating pattern avoidance and standard Coxeter elements. A more general statement for finite Weyl groups can be found in [15].

Proposition 7.9 ([33]) *A permutation $w \in \mathfrak{S}_n$ is a product of distinct generators,* i.e., *a standard Coxeter element in some parabolic subgroup, if and only if w avoids 321 and 3412.*

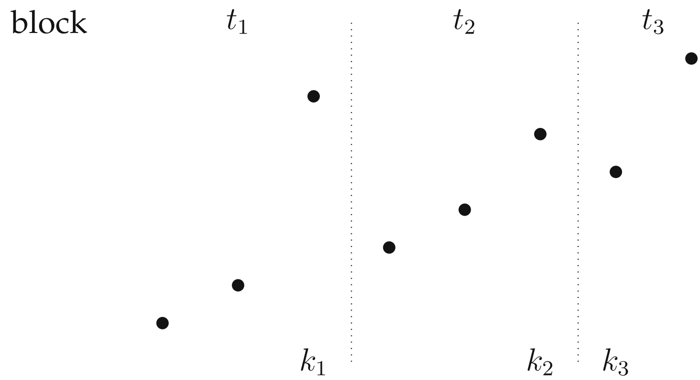


Fig. 8 An example of a permutation u^{-1} containing 321 in the proof of Theorem 7.2

Conclusion of the proof of Theorem 7.2: If $w \in \mathfrak{S}_n$ satisfies Definition 1.2 then it satisfies Definition 7.1 as the length-additive expression $w = w_0(I)c$ provides an I-witness, where c is a product of distinct simple reflections.

Conversely, suppose $w \in \mathfrak{S}_n$ satisfies Definition 7.1. We now show that it satisfies Definition 1.2. Recall $D = [n] - I = \{d_1 < d_2 < \cdots < d_k\}$; $d_0 = 0$, $d_{k+1} = n$. Let

$$A_i = \{d_{i-1} + 1, \ldots, d_i\} \text{ for } i = 1, \ldots, k+1.$$

Assume w is I-spherical with some I-witness. By Proposition 7.8 and Definition 7.1, we can write $w = w_0(I)u$ such that there is a reduced word $R'' = s_{i_1} \cdots s_{i_{\ell(u)}}$ of u such that

- s_{d_i} appears at most once in R''; and
- $\#\{m \mid d_{t-1} < i_m < d_t\} < \binom{d_t - d_{t-1}+1}{2} - \binom{d_t - d_{t-1}}{2} = d_t - d_{t-1}$ for $1 \le t \le k+1$.

By Proposition 7.9, it suffices to show that $u = w_0(I) \cdot w$ avoids 321 and 3412, or equivalently, u^{-1} avoids 321 and 3412. Since Proposition 7.8 implies $\ell(w) = \ell(w_0(I)) + \ell(u)$, $u = w_0(I) \cdot w$ does not have left descents in I. In other words, u^{-1} is increasing on the indices A_i for $1 \le i \le k+1$.

Think about R'' as successive multiplications of u^{-1} on the right by simple transpositions of R'' (read right to left) until one reaches id (for example, if $u^{-1} = 2413$, $R'' = s_1 s_3 s_2$ represents $2\underline{41}3 \to 21\underline{43} \to \underline{21}34 \to 1234$). Since s_{d_i} appears at most once in R'', we know $|\{u^{-1}(1), u^{-1}(2), \ldots, u^{-1}(d_i)\} \backslash [d_i]| \le 1$. Moreover, if this cardinality is 1, s_{d_i} swaps $\max\{u^{-1}(1), \ldots, u^{-1}(d_i)\}$ at index d_i with $\min\{u^{-1}(d_i+1), \ldots, u^{-1}(n)\}$ at index $d_i + 1$.

First suppose u^{-1} contains 3412 at indices $k_1 < k_2 < k_3 < k_4$. Then any reduced expression of u^{-1} contains at least two copies of s_j for $k_2 \le j < k_3$. Since $u^{-1}(k_2) > u^{-1}(k_3)$, k_2 and k_3 lie in different A_i's. This means that there exists some $k_2 \le j < k_3$ with $j \notin I$ such that s_j is used at least twice in R'', a contradiction.

If u^{-1} contains 321 at indices $k_1 < k_2 < k_3$ with $k_i \in A_{t_i}$, then $t_1 < t_2 < t_3$. We concentrate on the block A_{t_2} and will show that simple transpositions in A_{t_2} are used at least $d_{t_2} - d_{t_2-1}$ times in R''. A visualization of u^{-1} is shown in Fig. 8.

Recall that $s_{d_{t_2-1}}$ exchanges the maximum value in indices $A_1 \cup \cdots \cup A_{t_2-1}$ with the minimum value in indices $A_{t_2} \cup \cdots \cup A_{k+1}$. Since $u^{-1}(k_2) > u^{-1}(k_3)$, the value $u^{-1}(k_2)$ is not the minimum among $u^{-1}(A_{t_2} \cup \cdots \cup A_{k+1})$ and thus cannot arrive left of index $d_{t_2-1}+1$ during this $s_{d_{t_2-1}}$ swap. Similarly, since $u^{-1}(k_2) < u^{-1}(k_1)$, the value $u^{-1}(k_2)$ cannot go to the right of index $d_{t_2}-1$. As a result, the value of $u^{-1}(k_2)$ occurs among $u^{-1}(A_{t_2})$ as we are using R'' to transform u^{-1} into id.

In order to put $u^{-1}(k_1), u^{-1}(k_2), u^{-1}(k_3)$ into the correct order, both the values $u^{-1}(k_1)$ and $u^{-1}(k_3)$ must enter A_{t_2} and exchange with $u^{-1}(k_2)$. In particular, all of the simple transpositions s_j, $j = d_{t_2-1}+1, \ldots, d_{t_2}-1$ must be used in order to exchange $u^{-1}(k_1)$ with $u^{-1}(k_3)$. Moreover, certain s_j need to be applied twice: if $u^{-1}(k_1)$ switches with $u^{-1}(k_2)$ at transposition s_j before $u^{-1}(k_2)$ switches with $u^{-1}(k_3)$, then s_j must be used again; and if $u^{-1}(k_3)$ switches with $u^{-1}(k_2)$ first at s_j, then s_j must be used again as well to eventually switch $u^{-1}(k_2)$ and $u^{-1}(k_1)$. Either way, in this case, the total number of times that s_j, $j = d_{t_2-1}+1, \ldots, d_{t_2}-1$, is used is at least $d_{t_2} - d_{t_2-1}$. $\square$

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