



Proper elements of Coxeter groups

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Abstract

We extend the notion of *proper elements* to all finite Coxeter groups. For all infinite families of finite Coxeter groups we prove that the probability a random element is *proper* goes to zero in the limit. This proves a conjecture of the third author and Alexander Yong regarding the proportion of Schubert varieties that are Levi spherical for all infinite families of Weyl groups. We also enumerate the proper elements in the exceptional Coxeter groups.

Keywords Schubert varieties · Spherical varieties · Proper elements · Chernoff bounds

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1 Introduction

The study of proper elements is motivated by the study of reductive group actions on Schubert varieties. Let G be a complex, connected reductive group, with a fixed maximal torus T contained in a fixed Borel subgroup B . The *Weyl group* of G is defined to be $W = N(T)/T$; it is a finite Coxeter group of rank n , where n is the semisimple rank of G . The *flag variety* G/B is an object of central importance in algebraic geometry and representation theory. The B -orbits in G/B are indexed by $w \in W$, and the Zariski-closure of the B -orbit indexed by w is the *Schubert variety* X_w .

The standard Levi subgroups of G are families of reductive subgroups that act on Schubert varieties in G/B . For each $I \subseteq [n] := \{1, \dots, n\}$, there is an associated standard parabolic subgroup $P_I \supseteq B$. Each P_I decomposes as a semidirect product

$$P_I = L_I \ltimes U_I,$$

where L_I is a reductive group called a standard Levi subgroup and U_I is the unipotent radical of P_I .

The group G acts on G/B by left multiplication. If $J(w) \subseteq [n]$ is the left descent set of w (see Definition 1.2), then $\text{stab}_G(X_w) = P_{J(w)}$ [1, Lemma 8.2.3]. For any $I \subseteq J(w)$, $L_I < P_I \leq P_{J(w)}$ and hence L_I is a reductive group that acts on X_w by left multiplication.

A normal variety X is a *spherical variety* for the action of a reductive group R if a Borel subgroup of R has an open dense orbit in X . In [11], the third author and Alexander Yong initiated a study of when a Schubert variety in G/B is *Levi spherical*; that is, when it is a spherical variety under the left multiplication action of a standard Levi subgroup of G .

In this work we define proper elements of a Coxeter group and show that if X_w is L_I -spherical, then w is proper. We then analyze the limiting behavior of properness which yields a proof of [11, Conjecture 3.7] for all infinite families of Weyl groups.

Theorem 1.1 *Let G be a simple group with Weyl group W of type A_n , B_n , C_n , or D_n . Let w be sampled uniformly at random from W . Then as $n \rightarrow \infty$,*

$$\Pr[X_w \subseteq G/B \text{ is } L_{J(w)}\text{-spherical}] \longrightarrow 0.$$

1.1 Proper elements in finite Coxeter groups

An $n \times n$ matrix M is a *Coxeter matrix* if it is a symmetric matrix with entries in $\{1, 2, \dots, \infty\}$ such that $M_{ij} = 1$ if and only if $i = j$. The *Coxeter group* associated to M is the group

$$W = \langle s_1, \dots, s_n : (s_i s_j)^{M_{ij}} = e \text{ for all } M_{ij} \neq \infty \rangle.$$

The matrix M is visualized by the *Coxeter diagram* $\mathcal{G}$, a graph whose nodes are labeled by $[n]$, and nodes i and j are connected by an edge labeled by M_{ij} if $M_{ij} \geq 3$. By convention, the label on edges with $M_{ij} = 3$ is omitted. See Fig. 1 below for examples.

Let $S = \{s_1, \dots, s_n\}$. The pair (W, S) is called a *Coxeter system*. If W is a finite group, then (W, S) is *finite*; (W, S) is *irreducible* if $\mathcal{G}$ is connected. The finite Coxeter groups were classified in [7]. The irreducible, finite Coxeter groups consist of four infinite families A_n, B_n, D_n , and $I_2(n)$ as well as six exceptional groups: E_6, E_7, E_8, F_4, H_3 , and H_4 . The *Coxeter length*, $\ell(w)$, of $w \in W$ is equal to the minimal number of elements of S required to express w . There is a unique element of maximal length in W denoted by $w_0(W)$.

We index the nodes in the Coxeter diagram $\mathcal{G}$ by $[n]$. For $I \subseteq [n]$, let $\mathcal{G}_I$ be the induced subdiagram of $\mathcal{G}$. There is a decomposition of $\mathcal{G}_I$ into m connected components

$$\mathcal{G}_I = \bigcup_{z=1}^m \mathcal{C}^{(z)}, \quad (1)$$

where each $\mathcal{C}^{(z)}$ is a Coxeter diagram with associated Coxeter group $W^{(z)}$. Let W_I be the parabolic subgroup of W generated by $S_I := \{s_i : i \in I\}$. Then $\mathcal{G}_I$ is the Coxeter diagram of W_I , and

$$\ell(w_0(W_I)) = \sum_{z=1}^m \ell(w_0(W^{(z)})). \quad (2)$$

Definition 1.2 For an element $w \in W$, the set of *left descents* is

$$J(w) := \{j \in [n] : \ell(s_j w) < \ell(w)\}.$$

The number of left descents will be denoted by $d(w) := |J(w)|$. For a nonnegative integer $x \leq n$ define $\max w_0(W, x) := \max \{\ell(w_0(W_I)) : I \subseteq [n] \text{ and } |I| = x\}$.

Definition 1.3 An element $w \in W$ is *proper* if $\ell(w) \leq n + \max w_0(W, d(w))$.

Example 1.4 Let W be the B_3 Coxeter group with $w = s_3 s_2 s_3 s_1 s_2 s_3 s_1 \in W$. Then $J(w) = \{2, 3\}$ and $d(w) = 2$. For $I = \{2, 3\} \subseteq [3]$, W_I is the B_2 Coxeter group with $\ell(w_0(W_I)) = 4$. This I achieves the maximum possible value for $\ell(w_0(W_I))$ over all $I \subseteq [3]$ with $|I| = d(w) = 2$. Hence, $\max w_0(W, d(w)) = 4$. We conclude

$$\ell(w) = 7 \leq 3 + 4 = n + \max w_0(W, d(w)),$$

and so w is proper.

Example 1.5 Let W be the A_4 Coxeter group with $w = s_2 s_3 s_4 s_1 s_2 s_3 s_1 s_2 \in W$. Then $J(w) = \{2, 3\}$, $d(w) = 2$, and $\max w_0(W, d(w)) = 3$ by Proposition 2.1 below. Hence,

$$\ell(w) = 8 \not\leq 4 + 3 = n + \max w_0(W, d(w)),$$

and so w is not proper.

We analyze the limiting behavior of this property.

Theorem 1.6 *Let W be a Coxeter group of type A_n , B_n , D_n , or $I_2(n)$. Let w be sampled uniformly at random from W . Then as $n \rightarrow \infty$,*

$$\Pr[w \text{ is proper}] \longrightarrow 0.$$

In Proposition 2.2, it is shown that if $X_w \subseteq G/B$ is L_I -spherical for $I \subseteq J(w)$, then w is proper. Hence, Theorem 1.6 implies Theorem 1.1.

Section 7 enumerates the proper elements for W an exceptional finite Coxeter group. In Theorem 6.4, we give a non-trivial lower bound on the number of proper elements in Coxeter groups of type A_n , B_n , and D_n .

1.2 Classifying Levi-spherical Schubert varieties

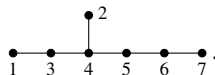
A type-independent classification of Levi-spherical Schubert varieties was conjectured in [11, Conjecture 1.9] by the third author and Yong. This proposed classification is further motivated by its connection to the theory of *Demazure characters*, or *key polynomials*, and the study of their “split-symmetry” [11, Section 4.1].

The proposed classification of Levi-spherical Schubert varieties in [11] is in terms of *spherical elements*. A *reduced expression* of $w \in W$ is a word $s_{i_1} \dots s_{i_{\ell(w)}} = w$. Denote the set of reduced words of w by $\text{Red}(w) := \text{Red}_{(W,S)}(w)$.

Definition 1.7 (*I-spherical elements*) Let $w \in W$ and fix $I \subseteq J(w)$. Then w is *I-spherical* if there exists $s_{i_1} \dots s_{i_{\ell(w)}} \in \text{Red}(w)$ such that:

- (S.1) $\#\{t : i_t = j\} \leq 1$ for all $j \in [n] - I$, and
- (S.2) $\#\{t : i_t \in \mathcal{C}^{(z)}\} \leq \ell(w_0(W^{(z)})) + \#\text{vertices}(\mathcal{C}^{(z)})$ for $1 \leq z \leq m$.

Example 1.8 Let W be the E_7 Coxeter group. The E_7 Coxeter diagram is



Let $w = s_4 s_2 s_3 s_4 s_3 s_2 s_5 s_7 s_4 s_3 s_2 s_4 s_1 s_5 s_3 \in W$. Then $J(w) = \{2, 3, 4, 5, 7\}$. If $I = J(w)$ then

$$\mathcal{C}^{(1)} = \begin{array}{c} & 2 \\ & \swarrow \downarrow \searrow \\ 3 & 4 & 5 \end{array} \quad \text{and} \quad \mathcal{C}^{(2)} = \begin{array}{c} 7 \end{array}.$$

Here $W^{(1)}$ is the D_4 Coxeter group with longest element

$$w_0(W^{(1)}) = s_3 s_2 s_4 s_3 s_2 s_4 s_5 s_4 s_3 s_2 s_4 s_5$$

and $\ell(w_0(W^{(1)})) = 12$. On the other hand, $W_{I^{(2)}}$ is the A_1 Coxeter group with the longest element $w_0(W^{(2)}) = s_7$ and $\ell(w_0(W^{(2)})) = 1$. The reduced word given to define w satisfies (S.1) since s_1 appears once and s_6 appears zero times. This reduced word also satisfies (S.2) since s_2, s_3, s_4 , and s_5 appear 13 times, which is less than $12 + 4 = 16$ and s_7 appears 1 time, which is less than $1 + 1 = 2$. Thus w is I -spherical.

Proposition 2.3 below implies that since w is I -spherical it is also proper. This is easily verified to be the case. We have $\ell(w) = 15$. And this is less than $n + \max w_0(W, d(w)) = 7 + \max w_0(W, 5) = 27$, where the final equality follows from Proposition 2.1 below.

Example 1.9 We also give an example of an element that is not I -spherical. Let W be the B_3 Coxeter group. The Coxeter diagram is $\bullet_1 - \bullet_2 \overset{4}{-} \bullet_3$. The element $w = s_3 s_2 s_3 s_1 s_2 s_3 \in W$ has $J(w) = \{3\}$. Let $I = J(w)$. This w has only two reduced words, $s_3 s_2 s_3 s_1 s_2 s_3$ and $s_3 s_2 s_1 s_3 s_2 s_3$. Each of these reduced words fails (S.1); in both, s_2 appears 2 times which is greater than 1. Thus w is not I -spherical.

Conjecture 1.10 ([11]) *Let $I \subseteq J(w)$. X_w is L_I -spherical if and only if w is I -spherical.*

The third author, joint with Gao and Yong, proved Conjecture 1.10 in type A_n [10]. This was then used by Gaetz in [9] to prove [11, Conjecture 3.8], giving a pattern avoidance criterion for a Schubert variety to be Levi-spherical in type A_n . The pattern avoidance criterion, in combination with the Marcus–Tardos theorem, implies Theorem 1.1 in type A_n . It is an open question if a pattern avoidance criterion exists for a Schubert variety to be Levi-Spherical in types B_n , C_n , and D_n .

Theorem 1.11 *Let W be a Coxeter group of type A_n , B_n , D_n , or $I_2(n)$. Let w be sampled uniformly at random from W . Then as $n \rightarrow \infty$,*

$$\Pr[w \text{ is } J(w)\text{-spherical}] \rightarrow 0.$$

Proposition 2.3 below shows that $w \in W$ is I -spherical implies w is proper. Hence Theorem 1.11 follows from Theorem 1.6.

1.3 Previous results on proper permutations

Proper permutations were first introduced in [4]. We highlight that our definition differs slightly from the original definition. The original definition of properness was motivated by the study of Levi-spherical Schubert varieties in GL_n/B . To study Levi-spherical Schubert varieties in G/B , for G a simple Lie group, requires a definition of properness that corresponds to Levi-spherical Schubert varieties in SL_n/B (SL_n being a simple Lie group). This introduces a difference of 1 on the right-hand side of

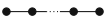
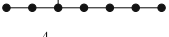
Type	Diagram	Length of longest element	Number of elements
A_n		$\binom{n+1}{2}$	$(n+1)!$
B_n, C_n		n^2	$2^n n!$
D_n		$n^2 - n$	$2^{n-1} n!$
E_6		36	51,840
E_7		63	2,903,040
E_8		120	696,729,600
F_4		24	1,152
H_3		15	120
H_4		60	14,400
$I_2(n)$		n	$2n$

Fig. 1 The finite Coxeter groups [12]

Definition 1.3 as compared to [4, Definition 1]; this is due to the fact that the Levi subgroups in SL_n have dimension one less than the corresponding Levi subgroups in GL_n . This updated definition is considerably more natural in the general type setting.

The upper bound achieved for the number of proper permutations in [4] uses Chebyshev's inequality. In this paper, we apply Chernoff bounds to achieve much tighter bounds, exponentially better than those in [4]. We apply these techniques not only in type A_n , but also in types B_n and D_n .

We now describe the layout of this paper. In Sect. 2 explicit formulas for $\max w_0(W, x)$ are given for each type. Next, it is shown that for an element w in a Weyl group W , X_w being L_I -spherical implies that w is proper. And for W a finite Coxeter group with $w \in W$, w is I -spherical implies that w is proper. Hence Theorem 1.6 implies Theorems 1.1 and 1.11. In Sect. 3 we derive concentration bounds that will be used to bound the number of proper elements in Coxeter groups of type A_n , B_n , and D_n . Section 4 recalls several well-known combinatorial models for the Coxeter groups of type A_n , B_n , and D_n . Theorem 1.6 is proved in Sect. 5 by giving asymptotic bounds on the number of proper elements. In Sect. 6 we give nontrivial lower bounds for the number of proper elements in Coxeter groups of type A_n , B_n , and D_n . We conclude with a table, presented in Sect. 7, enumerating the number of proper elements in each of the exceptional finite Coxeter groups.

2 Preparation

2.1 An analysis of $\max w_0(W, x)$

We begin with a study of $\max w_0(W, x)$ in Coxeter groups of each type, in preparation for probabilistic analysis in later sections.

Proposition 2.1 *Let (W, S) be a finite Coxeter system.*

- (i) *If W is of type A_n , then $\max w_0(W, x) = \binom{x+1}{2}$.*
- (ii) *If W is of type B_n or C_n , then $\max w_0(W, x) = x^2$.*
- (iii) *If W is of type D_n , then*

$$\max w_0(W, x) := \begin{cases} x^2 - x & x > 3, \\ \binom{x+1}{2} & x \leq 3. \end{cases}$$

- (iv) *If W is of type E_6, E_7 , or E_8 , then*

$$\max w_0(W, x) := \begin{cases} 120 & x = 8, \\ 63 & x = 7, \\ 36 & x = 6, \\ 20 & x = 5, \\ 12 & x = 4, \\ 6 & x = 3, \\ 3 & x = 2, \\ 1 & x = 1, \\ 0 & x = 0. \end{cases}$$

- (v) *If W is of type F_4 , then*

$$\max w_0(W, x) := \begin{cases} 24 & x = 4, \\ 9 & x = 3, \\ 4 & x = 2, \\ 1 & x = 1, \\ 0 & x = 0. \end{cases}$$

- (vi) *If W is of type H_3 or H_4 , then*

$$\max w_0(W, x) := \begin{cases} 60 & x = 4, \\ 15 & x = 3, \\ 5 & x = 2, \\ 1 & x = 1, \\ 0 & x = 0. \end{cases}$$

- (vii) *If W is of type $I_2(n)$, then*

$$\max w_0(W, x) := \begin{cases} n & x = 2, \\ x & x < 2. \end{cases}$$

Proof Figure 1 contains the Coxeter diagrams and the length of the longest element for each of the finite Coxeter groups in this proof. Let $I \subset [n]$ with $|I| = x$.

(i) When W is of type A_n , then $\mathcal{G}_I$ decomposes into connected components $C^{(1)}, \dots, C^{(z)}$ of type $A_{k_1}, \dots, A_{k_z}$ respectively, with $k_1, \dots, k_z > 0, z \geq 1$, and $k_1 + \dots + k_z = x$. By (2),

$$\begin{aligned} \ell(w_0(W_I)) &= \ell(w_0(A_{k_1})) + \dots + \ell(w_0(A_{k_z})) \\ &= \binom{k_1+1}{2} + \dots + \binom{k_z+1}{2} \\ &= \frac{k_1^2 + \dots + k_z^2 + k_1 + \dots + k_z}{2} \\ &\leq \frac{(k_1 + \dots + k_z + 1)(k_1 + \dots + k_z)}{2} \\ &= \binom{k_1 + \dots + k_z + 1}{2} = \binom{x+1}{2}. \end{aligned}$$

This upper bound is realized when $z = 1$ and $k_1 = x$. Thus $\max w_0(W, x) = \binom{x+1}{2}$.

(ii) When W is of type B_n or C_n , then $\mathcal{G}_I$ decomposes into connected components of type $A_{k_1}, \dots, A_{k_z}$ and B_m , with $k_1, \dots, k_z > 0, z \geq 0, m \geq 0$ and $k_1 + \dots + k_z + m = x$. Note that when $m = 0$ (respectively, $z = 0$) we take this to mean that there is no connected component of $\mathcal{G}_I$ of type B_s (respectively, of type A_s) for any natural number s . Thus, (2) implies

$$\begin{aligned} \ell(w_0(W_I)) &= \ell(w_0(A_{k_1})) + \dots + \ell(w_0(A_{k_z})) + \ell(w_0(B_m)) \\ &= \binom{k_1+1}{2} + \dots + \binom{k_z+1}{2} + m^2 \\ &\leq \binom{k_1 + \dots + k_z + 1}{2} + m^2 \\ &= \binom{x-m+1}{2} + m^2 \\ &= \frac{3}{2}m^2 - \frac{2x+1}{2}m + \frac{x^2+x}{2} =: f_1(m). \end{aligned}$$

Then m is an integer value in the closed interval $[0, x]$. The function f_1 is convex on the closed interval $[0, x]$ and hence f_1 achieves its maximum at one of the endpoints. For $x \geq 0$, $f_1(0) = \binom{x+1}{2} \leq x^2 = f_1(x)$. This upper bound is realized when $z = 0$, $m = x$. It follows that $\max w_0(W, x) = x^2$.

(iii) When W is of type D_n , then $\mathcal{G}_I$ decomposes into connected components of type $A_{k_1}, \dots, A_{k_z}$ and D_m , with $k_1, \dots, k_z > 0, z \geq 0, m \geq 0$ with $m \neq 1, 2, 3$, and $k_1 + \dots + k_z + m = x$. As in (ii), if $m = 0$ (respectively, $z = 0$), then we take this to mean there are no connected components in $\mathcal{G}_I$ of type D_s (respectively, of type A_s)

for any natural number s . Now, (2) implies

$$\begin{aligned}
 \ell(w_0(W_I)) &= \ell(w_0(A_{k_1})) + \cdots + \ell(w_0(A_{k_z})) + \ell(w_0(D_m)) \\
 &= \binom{k_1+1}{2} + \cdots + \binom{k_z+1}{2} + m^2 - m \\
 &\leq \binom{k_1 + \cdots + k_z + 1}{2} + m^2 - m \\
 &= \binom{x-m+1}{2} + m^2 - m \\
 &= \frac{3}{2}m^2 - \frac{2x+3}{2}m + \frac{x^2+x}{2} =: f_2(m).
 \end{aligned}$$

Then m is an integer value in the interval $[0, x]$. The function f_2 is convex on the closed interval $[0, x]$ and hence f_2 achieves its maximum at one of the endpoints. If $x > 3$, then $f_2(0) = \binom{x+1}{2} \leq x^2 - x = f_2(x)$. The upper bound is realized when $z = 0$ and $m = x$ for $x > 3$. Hence $\max w_0(W, x) = x^2 - x$ for $x > 3$.

If $x \leq 3$, then $f_2(0) = \binom{x+1}{2} \geq x^2 - x = f_2(x)$. This upper bound is realized when $z = 1$, $m = 0$, and $k_1 = x$. Hence $\max w_0(W, x) = \binom{x+1}{2}$ for $x \leq 3$.

(iv)–(vii) Each of these cases can be trivially checked via the enumeration of all induced subdiagrams of a fixed size. $\square$

2.2 Spherical implies proper

We show that sphericity, both in the geometric and Coxeter sense, implies properness. This allows the proofs of Theorems 1.1, 1.11 to be reduced to Theorem 1.6. The following is a generalization of [4, Proposition 3.1].

Proposition 2.2 *Let G be a rank r simple group with Weyl group W . If $X_w \subseteq G/B$ is L_I -spherical for $I \subseteq J(w)$, then w is proper.*

Proof If X_w is L_I -spherical, then X_w is $L_{J(w)}$ -spherical [11, Proposition 2.13]. By definition, X_w is $L_{J(w)}$ -spherical implies that there is a Borel subgroup $K \subset L_{J(w)}$ with an open dense orbit $\mathcal{O}$ in X_w . For $x \in \mathcal{O}$, let K_x be the isotropy group of x . By [5, Proposition 1.11], $\mathcal{O} = K \cdot x$ is a smooth, closed subvariety of X_w of dimension $\dim(K) - \dim(K_x)$. Thus

$$\dim(X_w) = \dim(\mathcal{O}) = \dim(K) - \dim(K_x) \leq \dim(K), \quad (3)$$

where the first equality follows since $\mathcal{O}$ is dense in X_w . All Borel subgroups of a connected algebraic group are conjugate [3, Section 11.1]. Hence

$$\dim(K) = \dim(B_{J(w)}) = \dim(T) + \dim(U_{J(w)}), \quad (4)$$

where the final equality follows from [3, Section 11.1]. Finally, we use the fact that $\dim(U_{J(w)})$ equals the number of positive roots in the root system of $L_{J(w)}$, which

is in turn equal to $\ell(w_0(W_{J(w)}))$ [12, Section 1.7]. Combining this with (3) and (4) yields

$$\begin{aligned}\ell(w) &= \dim(X_w) \\ &\leq \dim(T) + \dim(U_{J(w)}) = r + \ell(w_0(W_{J(w)})) \leq r + \max w_0(W, d(w)).\end{aligned}$$

We conclude that w is proper. $\square$

Proposition 2.3 *Let W be a finite Coxeter group with $w \in W$ and $I \subseteq J(w)$. If w is I -spherical, then w is proper.*

Proof Let $\mathcal{C}^{(1)}, \dots, \mathcal{C}^{(m)}$ be the connected components of $\mathcal{G}_{J(w)}$, and $W^{(z)}$ the parabolic subgroup of W with Coxeter diagram $\mathcal{C}^{(z)}$. By [11, Proposition 2.12], if w is I -spherical, then w is $J(w)$ -spherical. Thus there must be an $R \in \text{Red}(w)$ satisfying (S.1) and (S.2). This implies, via (S.1), that at most $n - d(w)$ factors of R must be of the form s_j for $j \notin J(w)$. Which implies at least $\ell(w) - (n - d(w))$ factors of R are of the form s_j for $j \in J(w)$. Hence, by (S.2) and (2),

$$\begin{aligned}\ell(w) - (n - d(w)) &\leq \sum_{z=1}^m (\ell(w_0(W^{(z)})) + \# \text{ vertices}(\mathcal{C}^{(z)})) \\ &= d(w) + \sum_{z=1}^m \ell(w_0(W^{(z)})) \\ &= d(w) + \ell(w_0(W_{J(w)})) \\ &\leq d(w) + \max w_0(W, d(w)).\end{aligned}$$

It follows that w is proper. $\square$

We are now able to prove both Theorems 1.1 and 1.11, assuming Theorem 1.6.

Proof of Theorem 1.1 If X_w is $L_{J(w)}$ -spherical, then w is proper by Proposition 2.2. Hence

$$\Pr[X_w \subseteq G/B \text{ is } L_{J(w)}\text{-spherical}] \leq \Pr[w \text{ is proper}] \quad (5)$$

when w is sampled from W uniformly at random. Theorem 1.6 implies that $\Pr[w \text{ is proper}] \rightarrow 0$ as $n \rightarrow \infty$. Thus our desired result follows by (5) and the squeeze theorem. $\square$

Proof of Theorem 1.11 This follows by an identical argument after applying Proposition 2.3. $\square$

3 Concentration bounds

3.1 Concentration bounds

In this section we compute some concentration bounds that will be useful for bounding the number of proper elements of a Coxeter group in Sect. 5. Lemmas 3.2 and 3.3 will be used to bound the number of elements with left descents deviating far from the average. Lemmas 3.5 and 3.6 will be used to bound the number of elements with length deviating far from the average. Finally, Lemma 3.7 will be used to bound the number of elements satisfying a certain inequality involving length and left descents. See [14, Chapter 3] for some of the terminology and basic properties of conditional expectations and martingales that are used in this section.

Theorem 3.1 (Chernoff Bound; from [8, Theorem 1.1]) *Let $X_1, \dots, X_n$ be (mutually) independently distributed random variables in the range $[0, 1]$, and let $X := \sum_i X_i$. Then for any $\Delta \in (0, 1)$,*

$$\Pr[X < (1 - \Delta)\mathbb{E}[X]] \leq e^{-\Delta^2\mathbb{E}[X]/2}.$$

Lemma 3.2 *Let $X_1, \dots, X_m$ be mutually independently distributed random variables in the range $\{0, 1\}$. Let $X := \sum_i X_i$. Then for any $\delta > 0$, we have both*

$$\Pr[X - \mathbb{E}[X] > \delta m] \leq e^{-\delta^2 m/2} \quad \text{and} \quad \Pr[\mathbb{E}[X] - X > \delta m] \leq e^{-\delta^2 m/2}.$$

Proof If $\mathbb{E}[X] = 0$, then $X = 0$, in which case $\Pr[\mathbb{E}[X] - X > \delta m] = \Pr[\delta m < 0] = 0$. If $\delta m \geq \mathbb{E}[X]$, then $\Pr[\mathbb{E}[X] - X > \delta m] \leq \Pr[\mathbb{E}[X] - X > \mathbb{E}[X]] = \Pr[X < 0] = 0$. In the case when $\delta m < \mathbb{E}[X]$,

$$\begin{aligned} \Pr[\mathbb{E}[X] - X > \delta m] &= \Pr[X < (1 - \delta m/\mathbb{E}[X])\mathbb{E}[X]] \\ &\leq \exp\left(-\left(\frac{\delta m}{\mathbb{E}[X]}\right)^2 \cdot \frac{\mathbb{E}[X]}{2}\right) \quad \text{by Theorem 3.1} \\ &= \exp\left(\frac{-\delta^2 m}{2} \cdot \frac{m}{\mathbb{E}[X]}\right) \\ &\leq e^{-\delta^2 m/2} \end{aligned}$$

where the last inequality due to the fact that $m \geq \mathbb{E}[X]$. To show the former bound, set $X'_i := 1 - X_i$ and $X' := \sum_i X'_i$. Then $X - \mathbb{E}[X] = \mathbb{E}[X'] - X'$, and a similar analysis to the one above can be performed on X' . $\square$

Lemma 3.3 *Let $\{I_\alpha\}_{\alpha \in \Gamma}$ be a set of identically distributed random variables in the range $\{0, 1\}$, and let $\Gamma_1, \dots, \Gamma_k \subseteq \Gamma$ be such that:*

- (i) $\Gamma_i \cap \Gamma_j = \emptyset$, for all $i \neq j$;
- (ii) $\sum_i |\Gamma_i| = n - 1$;

- (iii) $\lfloor (n-1)/k \rfloor \leq |\Gamma_i| \leq \lfloor n/k \rfloor$;
 (iv) for each $i \in [k]$, the elements of $\{I_a\}_{a \in \Gamma_i}$ are mutually independent.

Let $S_i = \sum_{a \in \Gamma_i} I_a$ and $S = \sum_i S_i$. Then for any $\delta > 0$, we have both

$$\Pr[S - \mathbb{E}[S] > \delta n] \leq k e^{-\delta^2(n-2)/2k} \quad \text{and} \quad \Pr[\mathbb{E}[S] - S > \delta n] \leq k e^{-\delta^2(n-2)/2k}.$$

Proof By Lemma 3.2 we have

$$\Pr[S_i - \mathbb{E}[S_i] > \delta |\Gamma_i|] \leq e^{-\delta^2 |\Gamma_i|/2} \quad (6)$$

for each $i \in [k]$. Thus,

$$\begin{aligned} \Pr[S - \mathbb{E}[S] \geq \delta n] &= \Pr\left[\sum_i (S_i - \mathbb{E}[S_i]) \geq \delta n\right] \\ &\leq \Pr\left[\bigcup_i \{S_i - \mathbb{E}[S_i] \geq \delta n/k\}\right] \\ &\leq \sum_i \Pr[S_i - \mathbb{E}[S_i] \geq \delta n/k] \quad \text{by the union bound} \\ &\leq \sum_i \Pr[S_i - \mathbb{E}[S_i] \geq \delta |\Gamma_i|] \quad \text{since } |\Gamma_i| \leq \lfloor n/k \rfloor \leq n/k \\ &\leq \sum_i e^{-\delta^2 |\Gamma_i|/2} \quad \text{by (6)} \\ &\leq k e^{-\delta^2(n-2)/2k} \quad \text{since } (n-2)/k \leq \lfloor (n-1)/k \rfloor \leq |\Gamma_i|. \end{aligned}$$

Similarly, $\Pr[\mathbb{E}[S] - S \geq \delta n] \leq k e^{-\delta^2(n-2)/2k}$ since the bound from Lemma 3.2 is two-tailed. $\square$

Theorem 3.4 (Azuma–Hoeffding Inequality; [8, Theorem 5.8]) *Let $X_0, X_1, \dots$ be a martingale and let $b_1, b_2, \dots$ be a sequence of non-negative constants such that $|X_i - X_{i-1}| \leq b_i$ for each $i \geq 1$. Then,*

$$\Pr[X_n > X_0 + t] \leq \exp\left(-\frac{t^2}{2\beta_n}\right) \quad \text{and} \quad \Pr[X_n < X_0 - t] \leq \exp\left(-\frac{t^2}{2\beta_n}\right)$$

where $\beta_n = \sum_{i=1}^n b_i^2$.

We use the following elementary lemma; we omit the proof.

Lemma 3.5 *Let $0 \leq x_0, \dots, x_m \leq 1$ be such that $x_0 + \dots + x_m = 1$ and $x_k = x_{m-k}$ for $k = 0, \dots, m$. Then we have $\sum_{k=0}^m k x_k = m/2$.*

Lemma 3.6 *Suppose $Q_0, Q_1, Q_2, \dots$ is a sequence of random variables such that*

- (i) $Q_0 = 0$,
- (ii) $\Pr[Q_{t+1} \mid Q_t, \dots, Q_0] = \Pr[Q_{t+1} \mid Q_t]$,
- (iii) $\Pr[Q_{t+1} = Q_t + a \mid Q_t] = \Pr[Q_{t+1} = Q_t + (c_t - a) \mid Q_t]$ for all $a = 0, \dots, c_t$, and
- (iv) $\Pr[Q_{t+1} = Q_t + a \mid Q_t] = 0$ for all $a \notin \{0, \dots, c_t\}$,

where $c_0, c_1, c_2, \dots$ is a sequence of non-negative integers. Let $\gamma_n = \sum_{i=0}^{n-1} c_i^2$. Then for all $\varepsilon > 0$,

$$\Pr[Q_n - \mathbb{E}[Q_n] \geq \varepsilon] \leq e^{-2\varepsilon^2/\gamma_n} \quad \text{and} \quad \Pr[\mathbb{E}[Q_n] - Q_n \geq \varepsilon] \leq e^{-2\varepsilon^2/\gamma_n}.$$

Proof Firstly,

$$\sum_{a=0}^{c_t} a \cdot \Pr[Q_{t+1} = Q_t + a \mid Q_t] = c_t/2$$

by Lemma 3.5. From this,

$$\mathbb{E}[Q_{t+1} \mid Q_t] = Q_t + \sum_{a=0}^{c_t} a \cdot \Pr[Q_{t+1} = Q_t + a \mid Q_t] = Q_t + c_t/2.$$

Thus, by the law of total expectation,

$$\mathbb{E}[Q_{t+1}] = \mathbb{E}[\mathbb{E}[Q_{t+1} \mid Q_t]] = \mathbb{E}[Q_t] + c_t/2.$$

Letting $Z_t := Q_t - \mathbb{E}[Q_t]$ with $Z_0 = Q_0$, then $(Z_t)_{t \geq 0}$ is a martingale.

Also, $\mathbb{E}[Z_t] = 0$ for each t , using (i) for $t = 0$. Since $Q_t - Q_{t-1} \in \{0, \dots, c_{t-1}\}$ and $\mathbb{E}[Q_t] - \mathbb{E}[Q_{t-1}] = c_{t-1}/2$,

$$|Z_t - Z_{t-1}| = |(Q_t - Q_{t-1}) + (\mathbb{E}[Q_{t-1}] - \mathbb{E}[Q_t])| \leq c_{t-1}/2.$$

Applying Theorem 3.4 to $(Z_t)_{t \geq 0}$ with the bounded differences $b_t := c_{t-1}/2$ for $t \geq 1$ gives that $\Pr[Z_n \geq \varepsilon]$ and $\Pr[Z_n \leq -\varepsilon]$ are both bounded above by $e^{-2\varepsilon^2/\gamma_n}$. By the definition of Z_n , we have that $\Pr[Q_n - \mathbb{E}[Q_n] \geq \varepsilon] = \Pr[Z_n \geq \varepsilon]$ and $\Pr[\mathbb{E}[Q_n] - Q_n \geq \varepsilon] = \Pr[Z_n \leq -\varepsilon]$. $\square$

Lemma 3.7 Let $\{X_i\}_{i \in \Gamma}$ be a set of random variables. Let $\{f_i\}_{i \in \Gamma}$ be increasing functions over the non-negative reals. Let (Γ^+, Γ^-) be a partition of Γ . For any set $\{\varepsilon_i\}_{i \in \Gamma}$ of non-negative reals and any real number r that satisfy

- (i) $\varepsilon_a \leq \mathbb{E}[X_a]$ for all $a \in \Gamma^+$,
- (ii) $\varepsilon_b \geq -\mathbb{E}[X_b]$ for all $b \in \Gamma^-$, and
- (iii) $\sum_{a \in \Gamma^+} f_a(\mathbb{E}[X_a] - \varepsilon_a) - \sum_{b \in \Gamma^-} f_b(\mathbb{E}[X_b] + \varepsilon_b) > r$,

we have

$$\begin{aligned} \Pr \left[\sum_{a \in \Gamma^+} f_a(X_a) - \sum_{b \in \Gamma^-} f_b(X_b) \leq r \right] \\ \leq \sum_{a \in \Gamma^+} \Pr[\mathbb{E}[X_a] - X_a \geq \varepsilon_a] + \sum_{b \in \Gamma^-} \Pr[X_b - \mathbb{E}[X_b] \geq \varepsilon_b]. \end{aligned}$$

Proof We have

$$\begin{aligned} \Pr \left[\sum_{a \in \Gamma^+} f_a(X_a) - \sum_{b \in \Gamma^-} f_b(X_b) > r \right] \\ \geq \Pr \left[\left(\bigcap_{a \in \Gamma^+} \{X_a > \mathbb{E}[X_a] - \varepsilon_a\} \right) \cap \left(\bigcap_{b \in \Gamma^-} \{X_b < \mathbb{E}[X_b] + \varepsilon_b\} \right) \right] \\ = \Pr \left[\left(\bigcap_{a \in \Gamma^+} \{\mathbb{E}[X_a] - X_a < \varepsilon_a\} \right) \cap \left(\bigcap_{b \in \Gamma^-} \{X_b - \mathbb{E}[X_b] < \varepsilon_b\} \right) \right] \\ = 1 - \Pr \left[\left(\bigcup_{a \in \Gamma^+} \{\mathbb{E}[X_a] - X_a \geq \varepsilon_a\} \right) \cup \left(\bigcup_{b \in \Gamma^-} \{X_b - \mathbb{E}[X_b] \geq \varepsilon_b\} \right) \right] \\ \geq 1 - \sum_{a \in \Gamma^+} \Pr[\mathbb{E}[X_a] - X_a \geq \varepsilon_a] - \sum_{b \in \Gamma^-} \Pr[X_b - \mathbb{E}[X_b] \geq \varepsilon_b]. \end{aligned}$$

The first step uses the fact that $\{f_i\}_{i \in \Gamma}$ is a set of increasing functions; the last step uses the union bound. Taking the probability complement of this yields our desired result. $\square$

4 The infinite families

The finite, irreducible Coxeter groups of types A_{n-1} , B_n , and D_n have combinatorial interpretations in terms of permutations. We follow the notation and repeatedly use results from [2] in this section. The type A_{n-1} Coxeter group is isomorphic to the symmetric group S_n under the map that sends each $s_i \in S$ to the simple transposition $(i, i+1)$. Thus the order of A_{n-1} is $n!$. For any bijection w on a subset $S \subset \mathbb{Z}$, let the number of inversions be

$$\text{inv}(w) := \# \{(i, j) : w(i) > w(j), 1 \leq i < j \leq n\}.$$

The length of an element of type A_{n-1} is equal to the number of inversions of w (thinking of w as an element of S_n); that is, $\ell_A(w) = \text{inv}(w)$.

The type B_n Coxeter group is isomorphic to the group of *signed* permutations, S_n^B , which is the collection of bijections w on the set $[\pm n] = \{-n, \dots, -1, 1, \dots, n\}$ with the property that $w(-i) = -w(i)$ for every $i \in [\pm n]$, under the binary operation of function composition. The order of this group is $2^n n!$. The length of $w \in S_n^B$ can be

expressed as

$$\ell_B(w) = \text{inv}_B(w) := \text{inv}(w) + \text{nsp}(w) + \text{neg}(w)$$

where

$$\begin{aligned} \text{nsp}(w) &:= \# \{(i, j) : w(i) + w(j) < 0, 1 \leq i < j \leq n\}, \\ \text{neg}(w) &:= \# \{i : w(i) < 0, 1 \leq i \leq n\}. \end{aligned}$$

The type D_n Coxeter group is isomorphic to the subgroup $S_n^D \subset S_n^B$ containing all w such that $\text{neg}(w)$ even. The order of this group is $2^{n-1}n!$. The length of $w \in S_n^D$ can be expressed as

$$\ell_D(w) = \text{inv}_D(w) := \text{inv}_B(w) - \text{neg}(w) = \text{inv}(w) + \text{nsp}(w).$$

For any bijection w on a subset $S \subset \mathbb{Z}$, let

$$\text{des}(w) := \# \{i : w^{-1}(i) > w^{-1}(i+1), 1 \leq i < n\}.$$

The number of left descents (Definition 1.2) of an element w in a Coxeter group of type A_{n-1} , B_n , and D_n can be computed, respectively, as

- (i) $d_A(w) = \text{des}(w)$,
- (ii) $d_B(w) = \text{des}_B(w) := \text{des}(w) + \mathbb{1}[0 > w^{-1}(1)]$,
- (iii) $d_D(w) = \text{des}_D(w) := \text{des}(w) + \mathbb{1}[w^{-1}(-2) > w^{-1}(1)]$,

where $\mathbb{1}[P]$ is the *Iverson bracket* defined as

$$\mathbb{1}[P] = \begin{cases} 1 & P \text{ is true;} \\ 0 & \text{otherwise.} \end{cases}$$

In the analysis that follows, the *asymptotic substitution* $f(n) = O(g(n))$ means $f(n) \leq c \cdot g(n)$ for all $n \geq N$ where N and c are some absolute positive constants with respect to n . We write $f(n) = \Omega(g(n))$ to mean $g(n) = O(f(n))$. When either $O(\cdot)$ or $\Omega(\cdot)$ appears as part of an inequality, for all possible asymptotic substitutions on the left-hand side, there must exist asymptotic substitutions on the right-hand side such that the inequality holds. The general technique will be to generate a uniformly at random Coxeter element and bound its Coxeter related statistics with high probability.

The *one-line notation* of a permutation $w \in S_n$ is the string $w(1) \cdots w(n)$. For an element w in either S_n^B or S_n^D , its one-line notation is

$$(w(-n), \dots, w(-1), w(+1), \dots, w(+n)),$$

and has indices in $[\pm n]$.

5 Limiting behavior for the proportion of proper elements

In [4], it was shown that the proportion of proper *permutations* is $O(n^{-1})$ and hence is asymptotically zero. In this section, we show that every group of type A_{n-1} has an $e^{-\Omega(n)}$ proportion of proper elements. We then give similar asymptotic results for types B_n and D_n .

5.1 Type A_{n-1}

We begin with type A_{n-1} Coxeter groups.

Proposition 5.1 *The number of proper elements in the type A_{n-1} Coxeter group is at most $n! \cdot e^{-\Omega(n)}$.*

Proof. For $1 \leq i < j \leq n$, let $X_{i,j}^{(n)}: S_n \rightarrow \{0, 1\}$ be the function that maps $w \in S_n$ to 1 if $w^{-1}(i) > w^{-1}(j)$ and 0 otherwise. Then

$$\text{inv}(w) = \sum_{i=1}^n \sum_{j=i+1}^n X_{i,j}^{(n)}(w^{-1}) \quad \text{and} \quad \text{des}(w) = \sum_{i=1}^{n-1} X_{i,i+1}^{(n)}(w)$$

for $w \in S_n$. Further, $\mathbb{E}_w[X_{i,j}^{(n)}(w)] = 1/2$ via symmetry. Thus $\mathbb{E}_w[\text{inv}(w)] = \binom{n}{2}/2$ and $\mathbb{E}_w[\text{des}(w)] = (n-1)/2$ by the linearity of expectation. We also have that the random variables $X_{i,j}^{(n)}$ and $X_{i',j'}^{(n)}$ are independent when $\{i, j\} \cap \{i', j'\} = \emptyset$.

Let $\Gamma = \{(i, j) : 1 \leq i < j \leq n\}$ with $\Gamma_1 = \{(i, i+1) : 1 \leq i < n, i \equiv 1 \pmod{2}\}$ and $\Gamma_2 = \{(i, i+1) : 1 \leq i < n, i \equiv 0 \pmod{2}\}$. Notice Γ_1 and Γ_2 are disjoint subsets of Γ , and that both $\{X_a^n\}_{a \in \Gamma_1}$ and $\{X_b^n\}_{b \in \Gamma_2}$ are sets of independent random variables. Further, $|\Gamma_1| + |\Gamma_2| = n-1$ with $|\Gamma_1| = \lceil (n-1)/2 \rceil$ and $|\Gamma_2| = \lfloor (n-1)/2 \rfloor$. Also,

$$\sum_{a \in \Gamma_1} X_a^{(n)}(w) + \sum_{b \in \Gamma_2} X_b^{(n)}(w) = \sum_{i=1}^{n-1} X_{i,i+1}^{(n)}(w) = \text{des}(w)$$

for $w \in S_n$. Thus Lemma 3.3 with $\delta = \varepsilon/n$ gives us

$$\Pr[\text{des}(w) - \mathbb{E}[\text{des}(w)] \geq \varepsilon] \leq 2 \exp\left(-\frac{\varepsilon^2(n-2)}{4n^2}\right) \quad (7)$$

for any $\varepsilon > 0$ when w is sampled uniformly at random from S_n .

Next, let $\pi_0, \pi_1, \pi_2, \dots$ be a sequence of random permutations, evolving such that given $\pi_{t-1} \in S_{t-1}$, we create π_t by inserting t at a (uniformly) random index in π_{t-1} , such that $\Pr[\pi_t(i) = t \mid \pi_{t-1}] = 1/t$ for each $i \in [t]$. Let $Q_t := \text{inv}(\pi_t)$. From this, we see that $\Pr[Q_{t+1} = Q_t + a \mid Q_t] = 1/(t+1)$ for $a = 0, \dots, t$. Since the distribution is uniform, it is symmetric about $a = t/2$. Then setting $c_t = t$, Lemma 3.6 gives

$$\Pr[\mathbb{E}[Q_n] - Q_n \geq \varepsilon] \leq \exp\left(-\frac{12\varepsilon^2}{n(n-1)(2n-1)}\right) \quad (8)$$

for any $\varepsilon > 0$.

Notice that $\text{inv}(w_1)$ and $\text{inv}(w_2^{-1})$ are identically distributed when w_1 and w_2 are independently sampled uniformly at random from S_n since

$$\text{inv}(w) = \ell_A(w) = \ell_A(w^{-1}) = \text{inv}(w^{-1}).$$

If π_t is the evolved permutation after t insertions, then π_{t-1} is completely determined by π_t . Further, there is only one insertion of t that evolves π_{t-1} into π_t . The conclusion, via induction, is that the probability of seeing π_t after t insertions is $1/t!$. The above implies that Q_n has an identical distribution to $\text{inv}(w^{-1})$, and hence $\text{inv}(w)$, when w is sampled uniformly at random from S_n . This, combined with (8), implies

$$\Pr_w[\mathbb{E}[\text{inv}(w)] - \text{inv}(w) \geq \varepsilon] \leq \exp\left(-\frac{12\varepsilon^2}{n(n-1)(2n-1)}\right) \quad (9)$$

for any $\varepsilon > 0$.

Finally, take f_1 and f_2 to be functions over the reals with $f_1(x) = x$ and $f_2(x) = (x+1)x/2$. Also, let $r = n-1$, $\varepsilon_1 = n^2/16$, and $\varepsilon_2 = n/16$. Using Lemma 3.7, for sufficiently large n , with (7) and (9) gives

$$\begin{aligned} \#\{w \in S_n : w \text{ is proper}\} &= |S_n| \cdot \Pr_{w \leftarrow S_n}[w \text{ is proper}] \\ &= n! \cdot \Pr_{w \leftarrow S_n}\left[\text{inv}(w) \leq (n-1) + \binom{\text{des}(w)+1}{2}\right] \\ &\leq n! \cdot \left[2 \exp\left(-\frac{n-2}{256}\right) + \exp\left(-\frac{3n^3}{64(n-1)(2n-1)}\right)\right] \\ &\leq n! \cdot e^{-\Omega(n)}. \end{aligned} \quad \square$$

Corollary 5.2 *The proportion of proper elements in the Coxeter group of type A_{n-1} vanishes as n goes to infinity.*

Proof We know the type A_{n-1} Coxeter group has group structure S_n . Also, $|S_n| = n!$. Thus, the claim follows from Proposition 5.1 since the proportion of proper elements is

$$\Pr_{w \leftarrow S_n}[w \text{ is proper}] \leq e^{-\Omega(n)}$$

which tends to zero as n goes to infinity. $\square$

5.2 Types B_n and D_n

Next, we give a bound involving type A_{n-1} left descents, type B_n elements, and type D_n inversions. We do this to consolidate our bounds in types B_n and D_n . Types B_n and D_n have asymptotically the same number of elements, so their analysis is similar.

In providing the following, we upper bound a large set and then later show that this set is a superset of both the set of type B_n and type D_n proper elements.

Proposition 5.3 *We have that*

$$\#\{w \in S_n^B : \text{inv}_D(w) \leq n + (\text{des}(w) + 1)^2\} \leq 2^n n! \cdot e^{-\Omega(n)}.$$

Proof This proof is similar to the proof of Proposition 5.1. For $1 \leq i < j \leq n$, let $X_{i,j}^{(n)} : S_n^B \rightarrow \{0, 1\}$ be the function that maps $w \in S_n^B$ to 1 if $w^{-1}(i) > w^{-1}(j)$ and 0 otherwise. Similarly, let $Y_{i,j}^{(n)} : S_n^B \rightarrow \{0, 1\}$ be the function that maps $w \in S_n^B$ to 1 if $w^{-1}(-i) > w^{-1}(j)$ and 0 otherwise.

Then

$$\text{inv}_D(w) = \sum_{i=1}^n \sum_{j=i+1}^n X_{i,j}^{(n)}(w) + Y_{i,j}^{(n)}(w)$$

and

$$\text{des}(w) = \sum_{i=1}^{n-1} X_{i,i+1}^{(n)}(w),$$

for $w \in S_n^B$ since by [2, p. 253, (8.18)]

$$\begin{aligned} \text{inv}_D(w) &= \text{inv}_B(w) - \text{neg}(w) \\ &= \text{inv}(w) + \#\{1 \leq i < j \leq n : w(-i) > w(j)\} \\ &= \text{inv}_D(w^{-1}). \end{aligned}$$

For $i, j \in [n]$ with $i < j$, let $\iota_{i,j} : S_n^B \rightarrow S_n^B$ be the bijective involution that swaps the values at indices i and j in the one-line notation of $w \in S_n^B$. Then both $X_{i,j}^{(n)}(w) = 1 - X_{i,j}^{(n)}(\iota_{i,j}(w))$ and $Y_{i,j}^{(n)}(w) = 1 - Y_{i,j}^{(n)}(\iota_{i,j}(w))$. Hence $\mathbb{E}X_{i,j}^{(n)} = \mathbb{E}Y_{i,j}^{(n)} = 1/2$ by symmetry. Thus, $\mathbb{E}[\text{inv}_D(w)] = \binom{n}{2}$ and $\mathbb{E}[\text{des}(w)] = (n-1)/2$ by the linearity of expectation. Further, $X_{i,j}^{(n)}$ and $X_{i',j'}^{(n)}$ are independent if $\{i, j\} \cap \{i', j'\} = \emptyset$.

Let $\Gamma = \{(i, j) : 1 \leq i < j \leq n\}$ with $\Gamma_1 = \{(i, i+1) : 1 \leq i < n, i \equiv 1 \pmod{2}\}$ and $\Gamma_2 = \{(i, i+1) : 1 \leq i < n, i \equiv 0 \pmod{2}\}$. Notice Γ_1 and Γ_2 are disjoint subsets of Γ , and that both $\{X_a^n\}_{a \in \Gamma_1}$ and $\{X_b^n\}_{b \in \Gamma_2}$ are sets of independent random variables. Further, $|\Gamma_1| + |\Gamma_2| = n-1$ with $|\Gamma_1| = \lceil (n-1)/2 \rceil$ and $|\Gamma_2| = \lfloor (n-1)/2 \rfloor$. Since for $w \in S_n^B$, $\sum_{a \in \Gamma_1} X_a^{(n)}(w) + \sum_{b \in \Gamma_2} X_b^{(n)}(w) = \sum_{i=1}^{n-1} X_{i,i+1}^{(n)}(w) = \text{des}(w)$, applying Lemma 3.3 with $\delta = \varepsilon/n$ gives us

$$\Pr[\text{des}(w) - \mathbb{E}[\text{des}(w)] \geq \varepsilon] \leq 2 \exp\left(-\frac{\varepsilon^2(n-2)}{4n^2}\right) \quad (10)$$

for any $\varepsilon > 0$ where w is sampled uniformly from S_n^B .

Next, let $\pi_0, \pi_1, \pi_2, \dots$ be a sequence of random *signed* permutations, evolving such that given $\pi_{t-1} \in S_{t-1}^B$, we create π_t by inserting t at a (uniformly) random index in π_{t-1} , such that $\Pr[\pi_t(i) = t \mid \pi_{t-1}] = 1/2t$ for each $i \in [\pm t]$. Then $-t$ is inserted at the unique index that makes π_t a signed permutation. Let $Q_t := \text{inv}_D(\pi_t)$.

If t is placed in a positive index k , then $\pi_t(i) < \pi_t(k)$ and $\pi_t(-i) < \pi_t(k)$ for all $1 \leq i < k$, and $\pi_t(k) > \pi_t(j)$ and $\pi_t(-k) < \pi_t(j)$ for all j such that $k < j \leq t$. This leads to a total increase of $t - k$ to Q_{t-1} . If instead $-t$ is placed at a positive index k , then $\pi_t(i) > \pi_t(k)$ and $\pi_t(-i) > \pi_t(k)$ for all $1 \leq i < k$, and $\pi_t(k) < \pi_t(j)$ and $\pi_t(-k) > \pi_t(j)$ for all j such that $k < j \leq t$. This leads to a total increase of $2(k-1) + (t-k) = t+k-2$ to Q_{t-1} .

The two possibilities are equally likely, as are their indices. For $k_+, k_- \in [t]$, $t - k_+ = t + k_- - 2$ only if $t - k_+ = t + k_- - 2 = t - 1$. Thus

$$\Pr[Q_t = Q_{t-1} + a \mid Q_{t-1}] = \begin{cases} 1/t & \text{if } a = t - 1, \\ 1/2t & \text{if } a \in (\{0, \dots, 2(t-1)\} \setminus \{t-1\}), \\ 0 & \text{otherwise} \end{cases}$$

for each t . Thus,

$$\Pr[Q_{t+1} = Q_t + a \mid Q_t] = \begin{cases} \frac{1}{t+1} & \text{if } a = t, \\ \frac{1}{2(t+1)} & \text{if } a \in (\{0, \dots, 2t\} \setminus \{t\}), \\ 0 & \text{otherwise} \end{cases}$$

and so $\Pr[Q_{t+1} = Q_t + a \mid Q_t] = \Pr[Q_{t+1} = Q_t + 2t - a \mid Q_t]$. Then setting $c_t = 2t$, Lemma 3.6 gives

$$\Pr[\mathbb{E}[Q_n] - Q_n \geq \varepsilon] \leq \exp\left(-\frac{3\varepsilon^2}{n(n-1)(2n-1)}\right) \quad (11)$$

for any $\varepsilon > 0$.

Notice that $\text{inv}_D(w_1)$ and $\text{inv}_D(w_2^{-1})$ are identically distributed when w_1 and w_2 are independently sampled uniformly at random from S_n^B since $\text{inv}_D(w^{-1}) = \text{inv}_D(w)$ for each $w \in S_n^B$. If π_t is the evolved signed permutation after t insertions, then π_{t-1} is completely determined by π_t . Further, there is only one insertion of t that evolves π_{t-1} into π_t . The conclusion, via induction, is that the probability of seeing π_t after t insertions is $1/(2^t t!)$. The above implies that Q_n has an identical distribution to $\text{inv}_D(w^{-1})$, and hence $\text{inv}_D(w)$, when w is sampled uniformly at random from S_n^B . This, combined with (11), implies that for w sampled uniformly from S_n^B

$$\Pr[\mathbb{E}[\text{inv}_D(w)] - \text{inv}_D(w) \geq \varepsilon] \leq \exp\left(-\frac{3\varepsilon^2}{n(n-1)(2n-1)}\right) \quad (12)$$

for any $\varepsilon > 0$.

Finally, take f_1 and f_2 to be functions over the reals with $f_1(x) = x$ and $f_2(x) = (x+1)^2$. Also, let $r = n$, $\varepsilon_1 = n^2/16$, and $\varepsilon_2 = n/16$. Using Lemma 3.7, for

sufficiently large n , with (10) and (12) gives

$$\begin{aligned}
 & \# \{w \in S_n^B : \text{inv}_D(w) \leq n + (\text{des}(w) + 1)^2\} \\
 &= |S_n^B| \cdot \Pr_{w \leftarrow S_n^B} [\text{inv}_D(w) \leq n + (\text{des}(w) + 1)^2] \\
 &\leq 2^n n! \cdot \left[2 \exp\left(-\frac{n-2}{256}\right) + \exp\left(-\frac{3n^3}{256(n-1)(2n-1)}\right) \right] \\
 &\leq 2^n n! e^{-\Omega(n)}. \quad \square
 \end{aligned}$$

Proposition 5.4 *The number of proper elements in the Coxeter group of type B_n is at most $2^n n! e^{-\Omega(n)}$.*

Proof We have that

$$\begin{aligned}
 \# \{w \in S_n^B : w \text{ is proper}\} &= \# \{w \in S_n^B : \text{inv}_B(w) \leq n + (\text{des}_B(w))^2\} \\
 &\leq \# \{w \in S_n^B : \text{inv}_B(w) \leq n + (\text{des}(w) + 1)^2\} \\
 &\leq \# \{w \in S_n^B : \text{inv}_D(w) \leq n + (\text{des}(w) + 1)^2\} \\
 &\leq 2^n n! e^{-\Omega(n)},
 \end{aligned}$$

where the first step follows by Proposition 2.1 (ii), the second by $\text{des}(w) + 1 \geq \text{des}_B(w)$, the third by $\text{inv}_D(w) \leq \text{inv}_B(w)$, and the last by Proposition 5.3. $\square$

Corollary 5.5 *The proportion of proper elements in the Coxeter group of type B_n vanishes as n goes to infinity.*

Proof Since $|S_n^B| = 2^n n!$, it follows from Proposition 5.4 that

$$\Pr_{w \leftarrow S_n^B} [w \text{ is proper}] \leq e^{-\Omega(n)},$$

which tends to zero as n goes to infinity. $\square$

Proposition 5.6 *The number of proper elements in the Coxeter group of type D_n is at most $2^n n! e^{-\Omega(n)}$.*

Proof We have that

$$\begin{aligned}
 \# \{w \in S_n^D : w \text{ is proper}\} &= \# \{w \in S_n^D : \text{inv}_D(w) \leq n + \max w_0(S_n^D, \text{des}_D(w))\} \\
 &\leq \# \{w \in S_n^D : \text{inv}_D(w) \leq n + (\text{des}_D(w))^2\} \\
 &\leq \# \{w \in S_n^D : \text{inv}_D(w) \leq n + (\text{des}(w) + 1)^2\} \\
 &\leq \# \{w \in S_n^B : \text{inv}_D(w) \leq n + (\text{des}(w) + 1)^2\} \\
 &\leq 2^n n! e^{-\Omega(n)},
 \end{aligned}$$

where the first step follows by the definition of proper, the second by Proposition 2.1 (iii), the third by $\text{des}(w) + 1 \geq \text{des}_D(w)$, the fourth by $S_n^D \subset S_n^B$, and the last by Proposition 5.3. $\square$

Corollary 5.7 *The proportion of proper elements in the Coxeter group of type D_n vanishes as n goes to infinity.*

Proof Since $|S_n^D| = 2^{n-1}n!$, it follows from Proposition 5.6 that

$$\Pr_{w \leftarrow S_n^D} [w \text{ is proper}] \leq e^{-\Omega(n)},$$

which tends to zero as n goes to infinity. $\square$

5.3 Type I

The type $I_2(n)$ Coxeter group is presented with generators $S = \{s_1, s_2\}$ which satisfy $s_1^2 = s_2^2 = (s_1 s_2)^n = e$.

Proposition 5.8 *Let W_n be a Coxeter group of type $I_2(n)$ and let w be drawn from W_n uniformly at random. Then $\Pr[w \text{ is proper}] \rightarrow 0$ as $n \rightarrow \infty$.*

Proof We know that W_n has order $2n$. Further, for $w \in W_n$,

$$d(w) = \begin{cases} 2 & \text{if } w = w_0, \\ 0 & \text{if } w = e, \\ 1 & \text{otherwise.} \end{cases}$$

- If $w = w_0$, then $\max w_0(W_n, d(w)) = \max w_0(W_n, 2) = n$. Thus, w is proper if and only if $\ell(w) \leq 2 + n$, which is true for all n since $\ell(w_0) = n$.
- If $w = e$, then $\max w_0(W_n, d(w)) = \max w_0(W_n, 0) = 0$. Thus w is proper if and only if $\ell(w) \leq 2$, which is true since $\ell(e) = 0$.
- If $w \neq e, w_0$, then $\max w_0(W_n, d(w)) = \max w_0(W_n, 1) = 1$. Thus, w is proper if and only if $\ell(w) \leq 3$. There are at most a constant number of $w \in W_n$ with length at most 3: $w = s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1$, and $s_2 s_1 s_2$.

Thus the number of proper elements in W_n is constant with respect to n . $\square$

5.4 Proof of Theorem 1.6

Theorem 1.6 now follows immediately from Corollaries 5.2, 5.5, and 5.7 and Proposition 5.8.

6 Lower bounds

6.1 Types A_{n-1} , B_n , and D_n

We have given upper bounds on the number of proper elements in each of the infinite families of finite irreducible Coxeter groups. Next, we give a non-trivial lower bound on the number of proper elements for families A_{n-1} , B_n , and D_n . In order to show

these lower bounds, we construct a subset of S_n such that all elements of the set are proper. Further, we can use this subset to create subsets of S_n^B and S_n^D that contain only proper elements. This will give us a lower bound on the number of elements in each of the groups A_{n-1} , B_n , and D_n .

The next lemma gives mappings that, when applied to each element of a set of proper elements of S_n , will result in a set that only contains proper elements of S_n^B and a set that only contains proper elements of S_n^D .

Lemma 6.1 *Let $\varphi_B: S_n \rightarrow S_n^B$ and $\varphi_D: S_n \rightarrow S_n^D$ be the homomorphisms $w \mapsto w'$ where $w'(i) = w(i)$ and $w'(-i) = -w(i)$ for each $i \in [n]$. For n large enough and for all $w \in S_n$, if w is proper then $\varphi_B(w)$ and $\varphi_D(w)$ are proper.*

Proof We have

$$\ell_B(\varphi_B(w)) = \ell_D(\varphi_D(w)) = \ell_A(w) \quad (13)$$

since $w(i) > 0$ for each $i \in [n]$, by the definitions of inv , inv_B , and inv_D . Further, $(\varphi_B(w))^{-1} = \varphi_B(w^{-1})$ and $(\varphi_D(w))^{-1} = \varphi_D(w^{-1})$ imply that

$$d_B(\varphi_B(w)) = d_D(\varphi_D(w)) = d_A(w). \quad (14)$$

If $w \in S_n$ is proper, then $\ell_A(w) \leq (n-1) + \binom{d_A(w)+1}{2} \leq n + m_A(w)$ by Proposition 2.1. Thus, since

$$\max w_0(A_{n-1}, d_A(w)) \leq \min \{ \max w_0(B_n, d_B(\varphi_B(w))), \max w_0(D_n, d_D(\varphi_D(w))) \}$$

for large enough n , (13), (14), and Proposition 2.1 imply our result. $\square$

Next, we give a construction of a subset of elements of S_n that, with the optimized parameters, are all proper. To construct an element of S_n , we first imagine we have a row of n empty cells. We group the cells into a contiguous *chunks* of q cells. There might be some $r < q$ cells leftover on the rightmost side of the row. Let us focus on one chunk and denote this chunk α . We will place the numbers $1, \dots, q$ into chunk α such that there are many descents. More specifically, we group the numbers $1, \dots, q$ into s contiguous (when the numbers are arranged in increasing order) *runs* of size b . There might be $d < b$ remaining larger numbers. We will focus on one run of $1, \dots, q$ and call this run β . We place the elements of run β into the empty cells of chunk α such that when looking at the cells from left to right, the elements of β are in reverse (decreasing) order. We do this for each run β and place the excess elements of β in the remaining empty cells arbitrarily. Then we add $\alpha \cdot q$ to each cell entry of chunk α so that globally we do not have any repeated numbers in the cells. We repeat this process for all chunks. The remaining empty cells are filled in increasing order with the remaining elements of $[n]$ that have not been used.

The following is a formalization of the above intuition.

Construction 6.2 Let $q, s \in \mathbb{Z}_{\geq 0}$ be such that $n = aq + r$ for some unique $a \in \mathbb{Z}$, $0 \leq r < q$, and $q = sb + d$ for some $b \in \mathbb{Z}$, $0 \leq d < s$. Let $\pi_0, \dots, \pi_{a-1} \in S_q$ with the property that

$$\pi_j^{-1}(st+1) > \pi_j^{-1}(st+2) > \dots > \pi_j^{-1}(st+s)$$

for each $t = 0, \dots, b-1$ and $j = 0, \dots, a-1$. Now set $w \in S_n$ to be such that for $i = \alpha q + \beta \in [n]$ with $\alpha \in \mathbb{Z}$ and $\beta \in [q]$,

$$w(i) = \alpha q + \begin{cases} \pi_\alpha(\beta) & \text{if } \alpha < a, \\ \beta & \text{otherwise.} \end{cases}$$

Let P_n be the set of these constructed permutations on $[n]$.

We note that the above construction results in proper elements of S_n only when the chunk sizes and runs are optimized correctly. We give some examples of this construction.

Example 6.3 Some examples in one-line notation are listed below.

1. $w = 2143\ 6587 \in P_n$ with $n = 8$, $q = 4$, $s = 2$ and $a = 2$, $r = 0$, $b = 2$, $d = 0$. In this example, $\pi_0 = \pi_1 = 2143$.
2. $w = 2431\ 65879 \in P_n$ with $n = 9$, $q = 4$, $s = 2$ and $a = 2$, $r = 1$, $b = 2$, $d = 0$. In this example, $\pi_0 = 2431$ and $\pi_1 = 2143$.
3. $w = 453261\ 789 \in P_n$ with $n = 9$, $q = 6$, $s = 4$ and $a = 1$, $r = 3$, $b = 1$, $d = 2$. In this example, $\pi_0 = 453,261$.

By way of Construction 6.2, we force the number of descents to be somewhat large within each chunk (and thus globally), and we force the inversions to occur only within each chunk which makes the number of inversions somewhat small. In the following theorem, we show that there always exist chunk and run sizes that result in P_n containing only proper elements of S_n . Further, we give lower bounds for the sizes of P_n and the corresponding constructed sets for S_n^B and S_n^D .

Theorem 6.4 *The number of proper elements of S_n , S_n^D , and S_n^B is at least $(cn)^n$ for some absolute constant $c > 0$.*

Proof We will show there exist q, s, a , and b from Construction 6.2 such that each element of P_n is proper. We can assume n is sufficiently large, since for smaller n , we can increase c such that the statement holds. Let $a = 10$, $s = 5$, $q = \lfloor n/a \rfloor$, $b = \lfloor q/s \rfloor$ such that $q, b > 0$ for sufficiently large n . Fix $w \in P_n$. Since inversions of w only take place within each of the a chunks, we have $\ell_A(w) \leq a \binom{q}{2} \leq aq^2/2 \leq 10 \cdot (n/10)^2/2 = n^2/20$. Also, in each chunk of w , there are at least s runs of size b placed in descending order (and not necessarily consecutively); this means $d_A(w) \geq asb$. Thus $\binom{d_A(w)+1}{2} \geq (d_A(w))^2/2 \geq (asb)^2/2 \geq (50b)^2/2 \geq (n-60)^2/2$. This means that for n sufficiently large, $\ell_A(w) \leq n + \binom{d_A(w)+1}{2}$ and hence w is proper.

Type	Number of proper elements
E_6	10690
E_7	159368
E_8	4854344
F_4	297
H_3	48
H_4	594

Fig. 2 The number of proper elements for the finite families were calculated using the aid of a computer [6]

Further, we have that $|P_n| \geq \frac{(q!)^a}{(s!)^{ab}}$ by the constraints of Construction 6.2. For $\varepsilon > 0$ and sufficiently large n ,

$$\begin{aligned} \frac{(q!)^a}{(s!)^{ab}} &\geq \frac{(\lfloor n/a \rfloor!)^a}{(s!)^{n/s}} \\ &\geq \left(\frac{\lfloor n/a \rfloor^{\lfloor n/a \rfloor}}{e^{\lfloor n/a \rfloor - 1}} \right)^a \cdot \left(\frac{e^s - 1}{s^{s+1}} \right)^{n/s} \end{aligned} \quad (15)$$

$$\begin{aligned} &\geq \left(\frac{n}{(a + \varepsilon) \cdot e} \right)^{n-a} \cdot \left(\frac{e^s - 1}{s^{s+1}} \right)^{n/s} \cdot e^{-a} \\ &= (\gamma_n \cdot n)^n, \end{aligned} \quad (16)$$

where

$$\gamma_n = ((a + \varepsilon) \cdot e^{1/s} \cdot s^{1+1/s})^{-1} \cdot v_n \quad \text{and} \quad v_n = \left(\frac{a + \varepsilon}{n} \right)^{a/n}.$$

Step (15) uses the bound

$$n^n / e^{n-1} \leq n! \leq n^{n+1} / e^{n-1}$$

from [13, Section 1.2.5, Exercise 24], and step (16) uses the bound $\lfloor n/a \rfloor \geq n/a - 1 \geq \frac{n}{a+\varepsilon}$ for $n \geq a \cdot (1 + a/\varepsilon)$. Because $\log v_n \geq -\frac{a}{(a+\varepsilon) \cdot e}$ by elementary calculus, there is some absolute constant $c > 0$ such that $\gamma_n \geq c$.

This is a non-trivial lower bound for the number of proper elements of S_n . The rest of the proof follows from Lemma 6.1. $\square$

7 The exceptional groups

7.1 Types E_6, E_7, E_8, F_4, H_3 , and H_4

The Coxeter groups of types E_6, E_7, E_8, F_4, H_3 , and H_4 are finite families. Figure 2 lists the number of proper elements for each of these groups. Using a computer we computed all of their proper elements [6]. The algorithm we used is described below.

- (1) *Generating the Coxeter group elements by length.* Let $W = \langle s_1, \dots, s_n \rangle$ be a finite Coxeter group and let $l \in \{0, \dots, \ell(w_0(W))\}$. At step t of the algorithm, we maintain a set $Z_t \subseteq W$ with $Z_0 = \{e\}$. Then $Z_{t+1} := \{s_i w : w \in Z_t, s_i \in [n] \setminus J(w)\}$. We terminate at the first T such that $Z_{T+1} = \emptyset$. Then for $Z := \bigcup_{t=0}^T Z_t$, we collect the elements $Y := \{w \in Z : \ell(w) = l\}$.
- (2) *Storing the Coxeter group elements by length.* The elements of Y are written to a named file (e.g. `data/E8/23.txt`, where $l = 23$ in this example). Each line of this file is a reduced word of W with length l , represented by the ordered indices of the generators. Then we compress the directory for each Coxeter group name.
- (3) *Computing the number of proper elements by length.* For each Coxeter group, we (in parallel) extract each of its files and compute the the number of proper elements. Then we take the sum of these results.

The computation for E_8 took the longest: it took approximately three weeks to run steps (1) and (2). Step (3) took approximately 75 min to run. We used a 3.0GHz 5th Gen Intel Core i7-5500U processor with 1TB of disk space and 16GB of RAM. Steps (1) and (3) heavily utilize SageMath and Coxeter3 libraries for most Coxeter group operations.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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