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Nonempty interior of configuration sets via microlocal partition optimization

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Abstract

We prove new results of Mattila–Sjölin type, giving lower bounds on Hausdorff dimensions of thin sets $E \subset \mathbb{R}^d$ ensuring that various k-point configuration sets, generated by elements of E, have nonempty interior. The dimensional thresholds in our previous work (Greenleaf et al., Mathematika 68(1):163–190, 2022) were dictated by associating to a configuration function a family of generalized Radon transforms, and then optimizing L^2-Sobolev estimates for them over all nontrivial bipartite partitions of the k points. In the current work, we extend this by allowing the optimization to be done locally over the configuration's incidence relation, or even microlocally over the conormal bundle of the incidence relation. We use this approach to prove Mattila–Sjölin type results for (i) areas of subtriangles determined by quadrilaterals and pentagons in a set $E \subset \mathbb{R}^2$; (ii) pairs of ratios of distances of 4-tuples in $\mathbb{R}^d$; and (iii) similarity classes of triangles in $\mathbb{R}^d$, as well as to (iv) give a short proof of Palsson and Romero Acosta's result on congruence classes of triangles in $\mathbb{R}^d$.

Keywords Erdos–Falconer configuration problem · Radon transform · Fourier integral operator

Mathematics Subject Classification Primary 28A75 · 28A80 · 58J40; Secondary 52C10

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1 Introduction

This is the third in a series of papers using Fourier integral operator techniques to obtain Mattila–Sjölin type results, by which we mean results showing that certain types of configuration sets have nonempty interior when the underlying sets have sufficiently high Hausdorff dimension. In [9, 10], we showed how to obtain such results for a wide range of configurations, using estimates for generalized Radon transforms, based on analysis of them as linear or multilinear Fourier integral operators. The current paper extends these methods and gives several applications.

A classical result of Steinhaus [32] states that if $E \subset \mathbb{R}^d$, $d \geq 1$, has positive Lebesgue measure, then the difference set $E - E \subset \mathbb{R}^d$ contains a neighborhood of the origin. $E - E$ can be identified with the set of two-point configurations, $x - y$, of points of E modulo the translation group. In the context of the Falconer distance set problem, a theorem of Mattila and Sjölin [24] states that if $E \subset \mathbb{R}^d$, $d \geq 2$, is compact, then the distance set of E, $\Delta(E) =: \{\,|x - y| : x,\ y \in E\} \subset \mathbb{R}$, contains an open interval, i.e., has nonempty interior, if the Hausdorff dimension $\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$. This provided a strengthening of Falconer's original result [5], from $\Delta(E)$ merely having positive Lebesgue measure to having nonempty interior, for the same range of $\dim_{\mathcal{H}}(E)$. This was generalized to distance sets with respect to norms on $\mathbb{R}^d$ having positive curvature unit spheres in Iosevich, Mourgoglou and Taylor [17].

Mattila–Sjölin type results, establishing nonempty interior for sets of configurations in a set E only satisfying a lower bound on $\dim_{\mathcal{H}}(E)$, or results that can be interpreted as such, have been obtained by various authors. These include [1, 3, 8, 17, 19] and, more recently, [21, 26, 29]; see also [4] for a finite field analogue, as well as [26, 27, 31] for analogues in which Hausdorff dimension is replaced by an alternative notion of size, mainly Newhouse thickness.

More general Mattila–Sjölin style theorems were studied by the current authors, for 2-point configurations in [9] and k-point configurations in [10]. In those, as in the present work, the configurations considered are Φ-*configurations*, as defined by Grafakos, Palsson and the first two authors [7], which can be vector-valued, nontranslation-invariant and possibly asymmetric, i.e., among points in sets $E_1, \dots, E_k$ lying in different spaces, e.g., points and circles in $\mathbb{R}^2$. The approach taken was to study the L^2-Sobolev mapping properties of an associated family of generalized Radon transforms, linear in [9] or multilinear in [10]. The main step in showing that the set $\Delta_\Phi(E_1, \dots, E_k)$ has nonempty interior is analysis of the configuration measure $\nu(\mathbf{t})$ (defined below); we show that this measure is absolutely continuous and that its density with respect to Lebesgue measure, $d\mathbf{t}$, is a continuous function of the configuration parameter $\mathbf{t} \in \mathbb{R}^p$ (or other space). This was done in [10] by representing $\nu(\mathbf{t})$ as the pairing of the tensor product of Frostman measures μ_i on some of the E_i with the value of a generalized Radon transform, $\mathcal{R}_{\mathbf{t}}$, acting on the tensor product of Frostman measures on the complementary collection of μ_j-s. Each such partition of the k variables into two groups gives a threshold for $\sum_{j=1}^k \dim_{\mathcal{H}}(E_j)$ ensuring *Int* $(\Delta_\Phi(E_1, \dots, E_k)) \neq \emptyset$; the threshold can potentially be lowered by optimizing over all such partitions. We refer to that approach as *partition optimization*; for a precise statement, see Theorem 2.4 below, which is [10, Thm. 5.2]. (See [6] for a subsequent application of partition optimization.)

The purpose of the current paper is to show that extensions of that approach, performing the partition optimization *locally* or even *microlocally*, allow one to obtain such nonempty interior results for an even wider range of k-point configurations, which fail to satisfy the hypotheses of Theorem 2.4. We do this by considering open covers of the k-fold incidence relation defining the configuration of interest, or more generally allowing microlocal covers of the conormal bundle of the incidence relation by open, conic sets. (This microlocal refinement of the local method is not possible for incidence relations of codimension one, since above each point of the incidence relation is just a line, and an open conic cover of the conormal bundle is equivalent to an open cover of the incidence relation.) On each of these sets, Theorem 2.4 is applicable, but with the Hausdorff dimensional threshold *possibly optimized by different* partitions of the k variables as one ranges over the elements of the cover. Taking the maximum of the thresholds needed, either locally near each point of the incidence relation or microlocally near each point of its conormal bundle, and then optimizing over all covers, yields a threshold which is always less than or equal to that provided by Theorem 2.4; see Theorems 2.5 and 2.6 for the statements of the local and microlocal versions. See Sect. 2 for the background material from [10] and the precise statements and the proofs of the theorems.

We now state some results which can be obtained using this new approach, restricting the discussion to various three-, four- and five-point configurations in $\mathbb{R}^d$.

Areas of triangles generated by vertices of quadrilaterals and pentagons in $\mathbb{R}^2$: In [10, Thm. 1.1] we showed that if $E \subset \mathbb{R}^2$ with $\dim_{\mathcal{H}}(E) > 5/3$, then the set of areas of triangles with vertices in E has nonempty interior in $\mathbb{R}$. For n-tuples of vertices in E, with $n \geq 4$, one can also consider vector-valued configurations consisting of the areas of *some* of the triangles they generate. In [10] we established that the collection of ordered pairs of areas of two of the triangles generated by a quadrilateral $xyzw$ with vertices in E, say $(|xyz|, |xzw|)$, has nonempty interior in $\mathbb{R}^2$ if $\dim_{\mathcal{H}}(E) > 7/4$. Note that there are limits on how far such results can be pushed: since $|xyw| + |yzw| = |xyz| + |xzw|$, the configuration set of all four of these areas would lie in a hyperplane in $\mathbb{R}^4$ and thus would have empty interior. However, using microlocal partition optimization, we are able to obtain (i) a threshold improving upon that in [10, Thm. 1.6]; (ii) a result for triples of areas of triangles generated by a quadrilateral; and (iii) a result for triples of the areas of a fan of triangles generated by a pentagon.

Theorem 1.1 *If $E \subset \mathbb{R}^2$ is compact, then*

(i) *if $\dim_{\mathcal{H}}(E) > 3/2$, then* $\operatorname{Int}\left\{(|xyz|, |xzw|) \in \mathbb{R}^2 : x, y, z, w \in E\right\} \neq \emptyset$;
(ii) *if $\dim_{\mathcal{H}}(E) > 7/4$, then* $\operatorname{Int}\left\{(|xyz|, |xzw|, |xyw|) \in \mathbb{R}^3 : x, y, z, w \in E\right\} \neq \emptyset$; *and*
(iii) *if $\dim_{\mathcal{H}}(E) > 9/5$, then* $\operatorname{Int}\left\{(|xyz|, |xzw|, |xwu|) \in \mathbb{R}^3 : x, y, z, w, u \in E\right\} \neq \emptyset$.

For the proof of Theorem 1.1, see Sect. 3. Also see A. McDonald [25] for related results of Falconer type (i.e., positive Lebesgue measure), formulated in terms of areas of parallelograms generated by pairs of points in E rather than areas of triangles generated by triples.

Pairs of ratios of distances: Another result that can be obtained using microlocal partition optimization concerns ratios of distances of 4-tuples in a set.

Theorem 1.2 *If $E \subset \mathbb{R}^d$, $d \geq 2$, is compact and $\dim_{\mathcal{H}}(E) > (3d+1)/4$, then*

$$\operatorname{Int}\left\{\left(\frac{|x-y|}{|z-w|}, \frac{|x-w|}{|z-y|}\right) \in \mathbb{R}^2 : x, y, z, w \in E, z \neq w,\ z \neq y\right\} \neq \emptyset.$$

In contrast, a modification of this configuration, (4.5), has nonempty interior for a larger range of dimensions and has a more elementary proof. A discussion of these and other results concerning configuration sets defined by ratios of distances, can be found in Sect. 4.

Congruences classes of triangles in $\mathbb{R}^d$, $d \geq 4$: One motivation for developing the microlocal extension of the original partition optimization technique was from trying to understand how a recent result of Palsson and Romero-Acosta [29] related to the FIO framework of [10]. They proved the following:

Theorem 1.3 [29]. *If $E \subset \mathbb{R}^d$, $d \geq 4$, is compact with $\dim_{\mathcal{H}}(E) > (2d+3)/3$, then the set of congruence classes of triangles with vertices in E,*

$$\{ (|x-y|, \; |x-z|, \; |y-z|) : x, y, z \in E \}, \tag{1.1}$$

has nonempty interior in $\mathbb{R}^3$.

In Sect. 5, we present a much shorter proof of this. Both the original partition optimization method from [10] (Thm. 2.4 below) and the local version (Thm. 2.5 below) fail to prove Thm. 1.3, However, it can be proved using, Thm. 2.6. because the optimal partition of the three variables varies with the normal direction to the (codimension 3) incidence relation, even above a single point. However, it can be proved using microlocal partition optimization, Thm. 2.6.

Similarity classes of triangles in $\mathbb{R}^d$, $d \geq 3$. We conclude with a result that has similarities to both Thm. 1.2 and Thm. 1.3. Using Thm. 2.5, in Sect. 6 we prove

Theorem 1.4 *If $E \subset \mathbb{R}^d$, $d \geq 3$, is compact and $\dim_{\mathcal{H}}(E) > (2d+2)/3$, then the set of similarity classes of triangles with vertices in E,*

$$\left\{ \left[\, |x-y| \; : \; |x-z| \; : \; |y-z| \, \right] \in \mathbb{RP}^2 \text{ s.t. } x, y, z \in E \text{ are distinct} \right\} \tag{1.2}$$

has nonempty interior in $\mathbb{RP}^2$.

(Here, $[\, A \; : \; B \; : \; C \,]$ are standard projective coordinates on $\mathbb{RP}^2$.)

Before proving the theorems, in Sect. 2 we recall the framework of Φ-configurations and the method of partition optimization from [9, 10].

2 *k*-point Φ-configuration sets

In order to state microlocal partition optimization, and for the sake of readability, we recall from [9, 10] the framework for studying the Φ-configuration sets of [7] via FIO methods and the original (global) version of partition optimization. Suppose that X^i, $1 \leq i \leq k$, and T, are smooth manifolds of dimensions d_i and p, resp. We sometimes denote $X^1 \times \cdots \times X^k$ by X, and set $d_{\text{tot}} := \dim(X) = \sum_{i=1}^k d_i$.

Definition 2.1 Let $\Phi \in C^\infty(X, T)$. Suppose that $E_i \subset X^i$, $1 \leq i \leq k$, are compact sets. Then the *k-configuration set of the E_i* defined by Φ is

$$\Delta_\Phi (E_1, E_2, \ldots, E_k) := \left\{ \Phi \left(x^1, \ldots, x^k \right) : x^i \in E_i, \; 1 \leq i \leq k \right\} \subset T. \tag{2.1}$$

If $E = E_1 = \cdots = E_k = E$, then we just write $\Delta_\Phi(E)$.

We want to find conditions on the $\dim_{\mathcal{H}}(E_i)$ ensuring that $\Delta_\Phi (E_1, E_2, \ldots, E_k)$ has nonempty interior. To this end, now suppose that $\Phi : X \to T$ is a submersion, so that for each $\mathbf{t} \in T$, $Z_\mathbf{t} := \Phi^{-1}(\mathbf{t})$ is a smooth, codimension p submanifold of X, and these vary smoothly with $\mathbf{t}$. For each $\mathbf{t}$, the measure

$$\lambda_\mathbf{t} := \delta \left(\Phi \left(x^1, \ldots, x^k \right) - \mathbf{t} \right) \tag{2.2}$$

is a smooth density on $Z_\mathbf{t}$; i.e., a smooth multiple of surface measure. In local coordinates $\mathbf{t} = (t_1, \ldots, t_p)$ on T, $\lambda_\mathbf{t}$ can be represented as an oscillatory integral of the form

$$\lambda_\mathbf{t} = \int_{\mathbb{R}^p} e^{i \sum_{l=1}^p (\Phi_l(x^1, \ldots, x^k) - t_l)\tau_l} \, a(\mathbf{t}) 1(\tau) \, d\tau, \tag{2.3}$$

where the $a(\cdot)$ belongs to a partition of unity on T, and $1(\tau)$ comes from the Fourier transform of the delta distribution in $\mathbb{R}^p$. Thus, $\lambda_{\mathbf{t}}$ is a Fourier integral distribution on X; in Hörmander's notation [12, 15, 16],

$$\lambda_{\mathbf{t}} \in I^{(2p-d_{\text{tot}})/4}(X; N^*Z_{\mathbf{t}}),$$

where $N^*Z_{\mathbf{t}} \subset T^*X\backslash 0$ is the conormal bundle of $Z_{\mathbf{t}}$ and the value of the order follows from the amplitude having order zero and the numbers of phase variables and spatial variables being p and d_{tot}, resp., so that the order is $m := 0 + p/2 - d_{\text{tot}}/4$.

We separate the variables $x^1, \ldots, x^k$ into groups on the left and right, associating to Φ a collection of families of generalized Radon transforms indexed by the nontrivial partitions of $\{1, \ldots, k\}$, with each family then depending on the parameter $\mathbf{t} \in T$. Write such a partition as $\sigma = (\sigma_L \mid \sigma_R)$, with $|\sigma_L|,\ |\sigma_R| > 0,\ |\sigma_L| + |\sigma_R| = k$, and let $\mathcal{P}_k$ denote the set of all $2^k - 2$ such partitions. We use i and j to refer to elements of σ_L and σ_R, resp. Define $d_L^\sigma = \sum_{i\in\sigma_L} d^i$ and $d_R^\sigma = \sum_{j\in\sigma_R} d^j$, so that $d_L^\sigma + d_R^\sigma = d_{\text{tot}}$.

For each $\sigma \in \mathcal{P}_k$, $\sigma_L = \left\{i_1, \ldots, i_{|\sigma_L|}\right\}$ and $\sigma_R = \left\{j_1, \ldots, j_{|\sigma_R|}\right\}$, where without loss of generality we may assume that $i_1 < \cdots < i_{|\sigma_L|}$ and $j_1 < \cdots < j_{|\sigma_R|}$. With a slight abuse of notation we still denote the coordinate-partitioned version of x as x,

$$x = (x_L; x_R) := \left(x^{i_1}, \ldots, x^{i_{|\sigma_L|}};\ x^{j_1}, \ldots, x^{j_{|\sigma_R|}}\right).$$

Write the corresponding reordered Cartesian product as

$$X_L \times X_R := \left(X^{i_1} \times \cdots \times X^{i_{|\sigma_L|}}\right) \times \left(X^{j_1} \times \cdots \times X^{j_{|\sigma_R|}}\right);$$

again by abuse of notation, we sometimes still refer to this as X. The dimensions of the two factors are $dim(X_L) = d_L^\sigma$ and $dim(X_R) = d_R^\sigma$, resp. The choice of σ also defines a coordinate-partitioned version of each $Z_{\mathbf{t}}$,

$$Z_{\mathbf{t}}^\sigma := \{(x_L; x_R) :\ \Phi(x) = \mathbf{t}\} \subset X_L \times X_R, \tag{2.4}$$

with spatial projections to the left and right, $\pi_{X_L} : Z_{\mathbf{t}}^\sigma \to X_L$ and $\pi_{X_R} : Z_{\mathbf{t}}^\sigma \to X_R$. The integral geometric double fibration condition for $Z_{\mathbf{t}}^\sigma$ is the requirement that

$$(DF)_\sigma \qquad \pi_L : Z_{\mathbf{t}}^\sigma \to X_L \text{ and } \pi_R : Z_{\mathbf{t}}^\sigma \to X_R \text{ are submersions.} \tag{2.5}$$

(See [12–14].) Note that, for a given σ, a necessary (but not sufficient) condition for $(DF)_\sigma$ to hold is $p \le d_L^\sigma \wedge d_R^\sigma := \min\left(d_L^\sigma, d_R^\sigma\right)$.

If $(DF)_\sigma$ holds, then the generalized Radon transform $\mathcal{R}_{\mathbf{t}}^\sigma$, defined weakly by

$$\mathcal{R}_{\mathbf{t}}^\sigma f(x_L) = \int_{\{x_R:\,\Phi(x_L,x_R)=\mathbf{t}\}} f(x_R),$$

where the integral is with respect to the surface measure induced by $\lambda_{\mathbf{t}}$ on the codimension p submanifold $\{x_R :\ \Phi(x_L, x_R) = \mathbf{t}\} = \left\{x_R : (x_L, x_R) \in Z_{\mathbf{t}}^\sigma\right\} \subset X_R$, which extends from mapping $\mathcal{D}(X_R) \to \mathcal{E}(X_L)$ to

$$\mathcal{R}_{\mathbf{t}}^\sigma : \mathcal{E}'(X_R) \to \mathcal{D}'(X_L).$$

Here, $\mathcal{E},\ \mathcal{D}$ are the standard spaces of C^∞ functions and those of compact support, resp., and $\mathcal{E}',\ \mathcal{D}'$ their dual spaces of distributions. Furthermore,

$$C_{\mathbf{t}}^\sigma := \left(N^*Z_{\mathbf{t}}^\sigma\right)' = \left\{(x_L, \xi_L;\ x_R, \xi_R) :\ (x_L, x_R) \in Z_{\mathbf{t}}^\sigma,\ (\xi_L, -\xi_R) \perp TZ_{\mathbf{t}}^\sigma\right\} \tag{2.6}$$

is contained in $(T^*X_L\backslash 0) \times (T^*X_R\backslash 0)$. Thus, $\mathcal{R}_{\mathbf{t}}^\sigma$ is an FIO, $\mathcal{R}_{\mathbf{t}}^\sigma \in I^m\left(X_L, X_R; C_{\mathbf{t}}^\sigma\right)$, where the order m is determined as in (2) by $m = 0 + p/2 - d_{\text{tot}}/4$ [15, 16]. Given the possible difference in the dimensions of X_L and X_R, due to the clean intersection calculus it is useful to express m as

$$m = m_{\text{eff}}^{\sigma} - \frac{1}{4}\left|d_L^{\sigma} - d_R^{\sigma}\right|,$$

where the *effective* order of $\mathcal{R}_{\mathbf{t}}^{\sigma}$ is defined to be

$$m_{\text{eff}}^{\sigma} := \left(2p - d_{\text{tot}} + \left|d_L^{\sigma} - d_R^{\sigma}\right|\right)/4 = \left(p - \left(d_L^{\sigma} \wedge d_R^{\sigma}\right)\right)/2. \tag{2.7}$$

By standard estimates for FIO [15, 16], if $C_{\mathbf{t}}^{\sigma}$ is a nondegenerate canonical relation, i.e., the cotangent space projections $\pi_L : C_{\mathbf{t}}^{\sigma} \to T^*X_L$ and $\pi_R : C_{\mathbf{t}}^{\sigma} \to T^*X_R$ have differentials of maximal rank, then

$$\mathcal{R}_{\mathbf{t}}^{\sigma} : L_r^2(X_R) \to L_{r-m_{\text{eff}}^{\sigma}}^2(X_L).$$

More generally, if π_L (and thus π_R) drops rank by $\leq q$, then there is a loss of $\leq q/2$ derivatives:

$$\mathcal{R}_{\mathbf{t}}^{\sigma} : L_r^2(X_R) \to L_{r-m_{\text{eff}}^{\sigma}-\frac{q}{2}}^2(X_L). \tag{2.8}$$

It is natural to express the estimates for possibly degenerate FIO in terms of possible losses relative to the optimal estimates. Initially, our basic assumptions is that there is at least one σ such that (i) the double fibration condition (2.5) is satisfied, and (ii) there is a known $\beta^{\sigma} \geq 0$ such that, for all $r \in \mathbb{R}$,

$$\mathcal{R}_{\mathbf{t}}^{\sigma} : L_r^2(X_R) \to L_{r-m_{\text{eff}}^{\sigma}-\beta^{\sigma}}^2(X_L), \tag{2.9}$$

uniformly for $\mathbf{t} \in T$, or at least for $\mathbf{t}$ in some compact set containing any configurations that arise from the E_i of interest.

Remark 2.2 A folk theorem in microlocal analysis is that the estimates for nondegenerate FIO or even those covered by the corank q scenario of (2.8), which are all that we use in the concrete applications in this paper, are stable under small perturbations of the amplitudes and phase functions in C^N norm for N sufficiently large. This is due to the finite number of integrations by parts that are required in the various proofs for FIO and the underlying oscillatory integral operators; see, e.g., [11, Lem. 2.3]. Thus, once one has a single value $\mathbf{t} = \mathbf{t}_0$ of the configuration parameter for which the generalized Radon transform $\mathcal{R}_{\mathbf{t}}^{\sigma}$ is nondegenerate or corank q, one is ensured that there is a neighborhood of $\mathbf{t}_0$ for which this is true and for which (2.8) holds uniformly in $\mathbf{t}$. See the comment at the end of Sect. 3.1.

Now suppose that, for $1 \leq i \leq k$, $E_i \subset X^i$ are compact sets. Our goal is to find conditions on the $\dim_{\mathcal{H}}(E_i)$ ensuring that $\Delta_{\Phi}(E_1, E_2, \dots, E_k)$ has nonempty interior in T. For each i, fix an $s_i < \dim_{\mathcal{H}}(E_i)$ and a Frostman measure μ_i on E_i of finite s_i-energy; translating energy into L^2-based Sobolev space norms, $\mu_i \in L_{(s_i-d_i)/2}^2(X^i)$. (See [22, 23] for further background.) Define measures

$$\mu_L := \mu_{i_1} \times \cdots \times \mu_{i_{|\sigma_L|}} \text{ on } X_L \text{ and } \mu_R := \mu_{j_1} \times \cdots \times \mu_{j_{|\sigma_R|}} \text{ on } X_R,$$

and recall the following result from [10]:

Proposition 2.3 *For $1 \leq j \leq k$, let X^j be a C^{∞} manifold of dimension d_j, and suppose that $u_j \in L_{r_j,comp}^2\left(X^j\right)$, $1 \leq j \leq k$, with each $r_j \leq 0$. Then the tensor product $u_1 \otimes \cdots \otimes u_k$ belongs to $L_{r,comp}^2\left(X^1 \times \cdots \times X^k\right)$, for $r = \sum_{j=1}^k r_j$.*

From this it follows that $\mu_L \in L^2_{r_L}(X_L)$ and $\mu_R \in L^2_{r_R}(X_R)$, where $r_L = \frac{1}{2}\sum_{l=1}^{|\sigma_L|}\left(s_{i_l} - d_{i_l}\right)$ and $r_R = \frac{1}{2}\sum_{l=1}^{|\sigma_R|}\left(s_{j_l} - d_{j_l}\right)$, resp.

As in [10, Eqn. 2.6], for any $\sigma \in \mathcal{P}_k$, the configuration measure can be expressed as

$$\nu(\mathbf{t}) = \left\langle \mathcal{R}^\sigma_{\mathbf{t}}\left(\mu_R\right), \mu_L \right\rangle, \tag{2.10}$$

which representation is justified *ex post facto* for s_i in the admissible range. (See [10, §3.4] for the argument.) Our basic assumption, that the boundedness (2.9) holds for the σ in question, then implies that $\mathcal{R}^\sigma_{\mathbf{t}}\left(\mu_R\right) \in L^2_{r_R - m^\sigma_{\text{eff}} - \beta^\sigma}(X_L)$. Since $\mu_L \in L^2_{r_L}(X_L)$, the pairing in (2.10) is bounded, and yields a continuous function of $\mathbf{t}$ (by continuity of the integral), if

$$r_R - m^\sigma_{\text{eff}} - \beta^\sigma + r_L \geq 0. \tag{2.11}$$

Noting that

$$r_L + r_R = \frac{1}{2}\left[\left(\sum_{i=1}^k s_i\right) - d^{tot}\right],$$

and using (2.7), we see that (2.11) holds iff

$$\begin{aligned}\sum_{i=1}^k s_i &\geq d^{tot} + 2\left(m^\sigma_{\text{eff}} + \beta^\sigma\right) = d^{tot} + p - \min(d_L, d_R) + 2\beta^\sigma \\ &= \max(d_L, d_R) + p + 2\beta^\sigma.\end{aligned}$$

Optimizing over all nontrivial partitions $\sigma \in \mathcal{P}_k$ leads to:

Theorem 2.4 Partition Optimization. [10, Thm. 5.2]

(i) *With the notation and assumptions as above, define*

$$s_\Phi = \min_\sigma \left[\max(d_L, d_R) + p + 2\beta^\sigma\right], \tag{2.12}$$

where the min *is taken over those $\sigma \in \mathcal{P}_k$ for which both the double fibration condition* (2.5) *holds and the uniform boundedness of the generalized Radon transforms $\mathcal{R}^\sigma_{\mathbf{t}}$ with some loss of $\leq \beta^\sigma$ derivatives* (2.9) *hold. Then, if $E_i \subset X^i$, $1 \leq i \leq k$, are compact sets with $\sum_{i=1}^k dim_{\mathcal{H}}(E_i) > s_\Phi$, it follows that* Int $(\Delta_\Phi(E_1, E_2, \ldots, E_k)) \neq \emptyset$.

(ii) *In particular, if $X^1 = \cdots = X^k =: X_0$, with $dim(X_0) = d$, and $E \subset X_0$ is compact, then* Int $(\Delta_\Phi(E)) \neq \emptyset$ *if*

$$dim_{\mathcal{H}}(E) > \frac{1}{k}\left(\min_\sigma \max(d_L, d_R) + p\right), \tag{2.13}$$

where the minimum is taken over all $\sigma \in \mathcal{P}_k$ such that (2.5) *holds and the canonical relation $C^\sigma_{\mathbf{t}}$ is nondegenerate.*

The threshold for $\sum_{i=1}^k \dim_{\mathcal{H}}(E_i)$ in (2.12) can be thought of as the *minimum* over all nontrivial partitions σ of the thresholds determined by the *maximum* microlocal loss (relative to the nondegenerate estimate) over all the points of $C_{\mathbf{t}}{}^\sigma$. On general principle, one can (possibly) lower a *minimum of the maxima* by replacing it with the *maximum of the minima*, and in this setting it is not hard to do this in practice. The goal of this paper is to show that weakening the assumptions in the original partition optimization, by working either locally on $Z_{\mathbf{t}}$ or more generally microlocally on $N^*Z_{\mathbf{t}}$, can allow one to lower the needed threshold on $\dim_{\mathcal{H}}(E)$, or even to obtain a positive result when an application of the original version of partition optimization, Thm. 2.4, would be vacuous.

In particular, in the context of Thm. 2.4 (ii) it is not necessary that *any* of the canonical relations ${C_{\mathbf{t}}}^{\sigma}$ be nondegenerate. Rather, working locally on $Z_{\mathbf{t}}$, it is sufficient that, for every $x \in Z_{\mathbf{t}}$ there is some neighborhood U of x in $Z_{\mathbf{t}}$ and *some* $\sigma \in \mathcal{P}_k$ such that ${C_{\mathbf{t}}}^{\sigma}$ is nondegenerate over U. Even more generally, working microlocally, it suffices that for every point $(x, \xi) \in N^*Z_{\mathbf{t}}$, there exists some $\sigma \in \mathcal{P}_k$ and a conic neighborhood $\mathcal{U}$ of (x, ξ) in $N^*Z_{\mathbf{t}}$ such that ${C_{\mathbf{t}}}^{\sigma}$ is nondegenerate on $\mathcal{U}$ (or rather the image $\mathcal{U}^{\sigma}$ of $\mathcal{U}$ under the σ-separation of the variables to the left and right). Since a partition of unity subordinate to an open cover of $Z_{\mathbf{t}}$ is a special, ξ-independent case of a microlocal partition of unity subordinate to a microlocal cover of $N^*Z_{\mathbf{t}}$, the local version of the new approach is a special case of the microlocal one. However, for clarity we state them separately:

Theorem 2.5 Local Partition Optimization. *Suppose that there is a $\beta \geq 0$ such that, for every point $x \in Z_{\mathbf{t}}$ there exists a neighborhood U and a partition $\sigma \in \mathcal{P}_k$ for which the generalized Radon transform $\mathcal{R}_{\mathbf{t}}^{\sigma}$, localized to U, satisfies both* (2.5) *and* (2.9) *with a loss of at most β derivatives, uniformly in $\mathbf{t}$.*

Then, for $E \subset \mathbb{R}^d$ compact, if

$$dim_{\mathcal{H}}(E) > \frac{1}{k}\left(\max(d_L, d_R) + p + 2\beta\right), \tag{2.14}$$

then Int $(\Delta_{\Phi}(E)) \neq \emptyset$.

Theorem 2.6 Microlocal Partition Optimization. *Suppose there exists a $\beta \geq 0$ such that, for every $(x, \xi) \in N^*Z_{\mathbf{t}}$ there exist a conic neighborhood $\mathcal{U}$ and partition $\sigma \in \mathcal{P}_k$ for which the generalized Radon transform $\mathcal{R}_{\mathbf{t}}^{\sigma}$, microlocalized to $\mathcal{U}$, satisfies both* (2.5) *and* (2.9) *with a loss of at most β derivatives, uniformly in $\mathbf{t}$. Then, for $E \subset \mathbb{R}^d$ compact,* Int $(\Delta_{\Phi}(E)) \neq \emptyset$ *if*

$$dim_{\mathcal{H}}(E) > \frac{1}{k}\left(\max(d_L, d_R) + p + 2\beta\right). \tag{2.15}$$

Since spatial partitions of unity are special cases of microlocal ones, the local theorem will follow immediately from the microlocal one, which in turn is proven by a straightforward refinement of the proof in [10]. We start by forming a standard pseudodifferential partition of unity, $\sum Q_l(x, D) = I$, on X subordinate to the open cover $\{\mathcal{U}_l\}$ of $N^*Z_{\mathbf{t}}$, supplemented by a $\mathcal{U}_0$ disjoint from $N^*Z_{\mathbf{t}}$ which completes the $\mathcal{U}_l$ to be a cover of $T^*X \setminus 0$. Each $Q_l \in \Psi^0_{cl}(X)$, and together their principal symbols, $q_l(x, \xi)$, form a partition of unity on $T^*X \setminus 0$. (For Thm. 2.5, the q_l are independent of ξ.) One can assume that this sum has at most $1 + |\mathcal{P}_k|$ terms. We let σ^l denote a partition such that $\mathcal{R}_{\mathbf{t}}^{\sigma}$ satisfies (2.9) with a loss of $\leq \beta$ derivatives on the conic support of Q_l. The surface measure $\lambda_{\mathbf{t}}$ from (2.2) on $Z_{\mathbf{t}}$ then decomposes as

$$\lambda_{\mathbf{t}} = \sum_l Q_l \lambda_{\mathbf{t}},$$

leading to a similar decomposition of the generalized Radon transforms. Hence, the identity (2.10) for the configuration measure can be replaced by

$$\nu(\mathbf{t}) = \sum_l \left\langle \mathcal{R}_{\mathbf{t}}^{\sigma^l}(\mu_R), \mu_L \right\rangle. \tag{2.16}$$

By the analysis above, if $\dim_{\mathcal{H}}(E)$ is greater than the threshold in (2.15), each of the terms in (2.16) are continuous in $\mathbf{t}$, finishing the proof.

Remark 2.7 We recall, for the proof of Thms. 1.2 below, that β can be taken to be $r/2$ if the projection π_L from each $\mathcal{U}_l$ drops rank by at most r (see [15, 16]).

Remark 2.8 The conormal bundle of $Z_{\mathbf{t}}$ is

$$N^*Z_{\mathbf{t}} = \left\{\left(x, D\Phi(\vec{x})^*(\tau)\right) : x \in Z_{\mathbf{t}},\ \tau \in \mathbb{R}^p \setminus \mathbf{0}\right\}.$$

However, for the calculations needed to verify the microlocal condition in Thm. 2.6 in each particular application, it is convenient to reorganize $N^*Z_{\mathbf{t}}$ by grouping each pair $(x^i, \xi^i) \in T^*X^i$ together, and we define

$$\widetilde{N^*Z_{\mathbf{t}}} = \left\{\left(x^1, \xi^1; x^2, \xi^2; \ldots; x^k, \xi^k\right) : (x^1, \ldots, x^k; \xi^1, \ldots, \xi^k) \in N^*Z_{\mathbf{t}}\right\},$$

and let π_i denote the natural projection onto the i-th factor, T^*X^i.

3 Areas of triangles

We now turn to results that require a microlocal approach, starting with the proofs of the various parts of Thm. 1.1 concerning areas of triangles generated by quadruples and quintuples of points in a planar set.

3.1 Pairs of areas of triangle in quadrilaterals

For part (i), let $\Phi : (\mathbb{R}^2)^4 \to \mathbb{R}^2$ be

$$\begin{aligned}\Phi(x, y, z, w) &= \left(\det[y - x, z - x], \det[z - x, w - x]\right)\\ &= \left((y - x) \cdot (z - x)^\perp, (z - x) \cdot (w - x)^\perp\right), \end{aligned} \tag{3.1}$$

where $\perp$ denotes rotation by $+\pi/2$, which is of course antisymmetric. All of the entries in $D\Phi$ are $\perp$ of simpler expressions, and so in place of $D\Phi$ we work with

$$D\Phi^\perp := \begin{bmatrix} y - z & z - x & w - y & 0 \\ z - w & 0 & vw - x & x - z \end{bmatrix},$$

and we denote the modified conormal bundle computed with $D\Phi^\perp$ by $\widetilde{N^*Z_{\mathbf{t}}}^\perp$.

If $\dim_{\mathcal{H}}(E) > 5/3$, and μ is a Frostman measure for $s > 5/3$, then since the set of degenerate triangles, $\{(x, y, z) \in \mathbb{R}^6 : \det[y - x, z - x] = 0\}$, is an algebraic hypersurface, its Hausdorff dimension equals 5. Hence, $W_1 := \{(x, y, z, w) : \det[y - x, z - x] = 0\}$ has $\otimes^4\mu$–measure 0 in $\mathbb{R}^8$, and without loss of generality we can assume that 4-tuples we consider lie in $\mathbb{R}^8 \setminus W_1$; see [10, Sec. 4.1] for related reasoning. Thus, without loss of generality, we can assume that for each $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$, $Z_{\mathbf{t}} = \Phi^{-1}(\mathbf{t})$ can be parametrized by $x, y, w \in \mathbb{R}^2$, with $z = x + \tilde{z}(x, y, w, \mathbf{t}) \in \mathbb{R}^2$ then being the unique solution of

$$(x - y)^\perp \cdot (z - x) = t_1, \quad (w - x)^\perp \cdot (z - x) = t_2.$$

One can check that $|D\tilde{z}/Dy| \neq 0$ and $|D\tilde{z}/Dw| \neq 0$.

Using the above one computes

$$\begin{aligned}\widetilde{N^*Z_{\mathbf{t}}}^\perp = \Big\{&\big(x, \tau_1(y - x - \tilde{z}) + \tau_2(x - w - \tilde{z});\ y, \tau_1\tilde{z}; x + \tilde{z}, \tau_1(w - y) + \tau_2(w - x);\\ &w, -\tau_2\tilde{z}\big) :\ x, y, z \in \mathbb{R}^2,\ \tau \in \mathbb{R}^2 \setminus \mathbf{0}\Big\}. \end{aligned} \tag{3.2}$$

From this we see that $D(x,\xi)/D(x,\tau)$ is always nonsingular.

If $\tau_1\neq 0$, then $D(y,\eta)/D(y,w)$ is nonsingular, since $|D\tilde z/Dw|\neq 0$. Thus, ordering the variables x, y, z, w, in order $1,2,3,4$, partitioning them by $\sigma=(12|34)$ yields $C_{\mathbf t}{}^{\sigma}$ which is a local canonical graph on $\mathcal U_1=\{\tau_1\neq 0\}$.

On the other hand, if $\tau_2\neq 0$ then $D(w,\omega)/D(w,y)$ is nonsingular, since $|D\tilde z/Dy|\neq 0$, so that using $\sigma=(14|23)$ gives $C_{\mathbf t}{}^{\sigma}$ which is a local canonical graph on $\mathcal U_2=\{\tau_2\neq 0\}$.

Together, $\mathcal U_1$ and $\mathcal U_2$ cover $\widetilde{N^*Z_{\mathbf t}}^{\perp}$, and $d_L=d_R=4$ for both partitions. Picking any particular $\mathbf t_0\in\Delta_\Phi(E)$ for which the above analysis applies, it also holds for $\mathbf t$ close to $\mathbf t_0$, and the estimates for resulting nondegenerate generalized Radon transforms $\mathcal R_{\mathbf t}^{\sigma}$ are locally uniform in $\mathbf t$; cf. Remark 2.2. Thus, Thm. 2.6 applies with $\beta=0$. It follows that if $\dim_{\mathcal H}(E)>\frac14(4+2+0)=3/2$, then $\Delta_\Phi(E)$ has nonempty interior in $\mathbb R^2$.

3.2 Triples of areas of triangles in quadrilaterals

To prove Thm. 1.1(ii) we modify the considerations of the previous section as follows. Let $\Phi:(\mathbb R^2)^4\to\mathbb R^3$ be

$$\begin{aligned}\Phi(x,y,z,w)&=\big(\det[y-x,z-x],\det[z-x,w-x],\det[y-x,w-x]\big)\\&=\Big((y-x)\cdot(z-x)^{\perp},(z-x)\cdot(w-x)^{\perp},(w-x)\cdot(y-x)^{\perp}\Big).\end{aligned}\tag{3.3}$$

As before, in place of $D\Phi$ we work with

$$D\Phi^{\perp}:=\begin{bmatrix} y-z & z-x & x-y & 0\\ z-w & 0 & w-x & x-z\\ w-y & x-w & 0 & y-x\end{bmatrix},$$

and denote the modified conormal bundle computed with $D\Phi^\perp$ by $\widetilde{N^*Z_{\mathbf t}}^{\perp}$.

For $\mathbf t=(t_1,t_2,t_3)\in\mathbb R^3$, $Z_{\mathbf t}=\Phi^{-1}(\mathbf t)$ is determined by

$$(y-x)^{\perp}\cdot(z-x)=-t_1,\ (w-x)^{\perp}\cdot(z-x)=t_2,\ (y-x)^{\perp}\cdot(w-x)=t_3.$$

Solving the last equation first, we can solve for w with one degree of freedom:

$$w=x+t_3\frac{(y-x)^{\perp}}{|y-x|}+s(y-x)=:x+\tilde w(x,y,s;t_3),\ s\in\mathbb R,$$

so that $w-x=\tilde w$. Then, as in the previous section, without loss of generality we can assume that $\det[y-x,z-x]\neq 0$ and so one can solve uniquely for z, incorporating the dependence of w on s:

$$z=x+\tilde z(x,y,s;\mathbf t)\implies z-x=\tilde z.$$

Note that $\partial_s\tilde w=y-x$ and, as in the previous section, $|D\tilde z/Dy|\neq 0$,

We can parametrize the conormal bundle as

$$\begin{aligned}\widetilde{N^*Z_{\mathbf t}}^{\perp}=\Big\{&\big(x,\tau_1(y-x-\tilde z)+\tau_2(\tilde z-\tilde w)+\tau_3(\tilde w-y);\\&y,\tau_1\tilde z-\tau_3\tilde w;\ x+\tilde z,\tau_1(x-y)+\tau_2\tilde w;\\&x+\tilde w,-\tau_2\tilde z+\tau_3(y-x)\big):\ x,y\in\mathbb R^2,s\in\mathbb R,\tau\in\mathbb R^3\setminus\mathbf 0\Big\}.\end{aligned}\tag{3.4}$$

Note that the differential of (x,ξ) with respect to x and any two of the three τ_j is nonsingular. (Here we can assume that $y-x$, $z-w$ and $w-y$ are in general position, i.e., any two are linearly independent, which excludes a variety $W_2 \subset \mathbb{R}^8$ of dimension 5.) This leaves y, s and the remaining τ_j variable to use for another one of the three remaining projections.

Since $\partial_s \tilde{w} = y - x$, one sees that $D(y,\eta)/D(y,s,\tau_1)$ is nonsingular if $\tau_3 \neq 0$, while $D(y,\eta)/D(y,s,\tau_3)$ is nonsingular if $\tau_1 \neq 0$. Hence, $C_{\mathbf{t}}{}^{(12|34)}$ is a local canonical graph on $\mathcal{U}_1 = \{\tau_1 \neq 0 \text{ or } \tau_3 \neq 0\}$.

Combining $\partial_s \tilde{w} = y - x$ with $|D\tilde{z}/Dy| \neq 0$, one sees that $D(z,\zeta)/D(y,s,\tau_1)$ is nonsingular if $\tau_2 \neq 0$, so that $C_{\mathbf{t}}{}^{(13|24)}$ is a local canonical graph on $\mathcal{U}_2 = \{\tau_2 \neq 0\}$.

Since $\mathcal{U}_1, \mathcal{U}_2$ form an open cover of $\widetilde{N^*Z_{\mathbf{t}}}^{\perp}$, and $d_L = d_R = 4$, $\beta = 0$ for all of those partitions, we can apply Thm. 2.6, obtaining that if $\dim_{\mathcal{H}}(E) > \frac{1}{4}(4+3+0) = 7/4$, then $\Delta_\Phi(E)$ has nonempty interior in $\mathbb{R}^3$.

3.3 Triples of areas of triangles in pentagons

For the proof of Thm. 1.1 (iii) we modify the setup for parts (i) and (ii) as follows. Define $\Phi : (\mathbb{R}^2)^5 \to \mathbb{R}^3$, recording the areas of the three adjacent triangles pinned at x, by

$$\begin{aligned}\Phi(x,y,z,w,u) &= \big(\det[y-x, z-x], \det[z-x, w-x], \det[w-x, u-x]\big)\\ &= \Big((y-x)\cdot(z-x)^{\perp}, (z-x)\cdot(w-x)^{\perp}, (w-x)\cdot(u-x)^{\perp}\Big).\end{aligned} \tag{3.5}$$

As before, in place of $D\Phi$ we work with

$$D\Phi^{\perp} := \begin{bmatrix} y-z & z-x & x-y & 0 & 0 \\ z-w & 0 & w-x & x-z & 0 \\ w-u & 0 & 0 & u-x & x-w \end{bmatrix},$$

and denote the modified conormal bundle computed with $D\Phi^{\perp}$ by $\widetilde{N^*Z_{\mathbf{t}}}^{\perp}$:

$$\begin{aligned}\widetilde{N^*Z_{\mathbf{t}}}^{\perp} = \Big\{&\big(x, \tau_1(y-z) + \tau_2(z-w) + \tau_3(w-u);\ y, \tau_1(z-x);\\ &z, \tau_1(x-y) + \tau_2(w-x);\ w, \tau_2(x-z) + \tau_3(u-x);\\ &u, \tau_3(x-w)\big) : (x,y,z,w,u) \in Z_{\mathbf{t}},\ \tau \in \mathbb{R}^3 \setminus \mathbf{0}\Big\}.\end{aligned} \tag{3.6}$$

The linear coordinates τ_1, τ_2, τ_3 on the fibers are intrinsically defined (given that Φ has been fixed), independent of what coordinates we pick on the 7-dimensional base $Z_{\mathbf{t}}$. We claim that on the open conic sets $\mathcal{U}_j = \{\tau_j \neq 0\} \subset \widetilde{N^*Z_{\mathbf{t}}}^{\perp}$, $j = 1, 2, 3$, which form a microlocal cover, the partitions $\sigma = (14|235)$, $(13|245)$, $(13|245)$, resp., give canonical relations $C_{\mathbf{t}}{}^{\sigma}$ which are nondegenerate. (Note that the partitions used on $\mathcal{U}_2$ and $\mathcal{U}_3$ are the same, but we have to treat $\mathcal{U}_2$ and $\mathcal{U}_3$ separately.) Thus, Thm. 2.6 implies that $\Delta_\Phi(E)$ has nonempty interior for for $E \subset \mathbb{R}^2$ with $\dim_{\mathcal{H}}(E) > \frac{1}{5}(\max(4,6)+3+0) = \frac{9}{5}$, proving Thm. 1.1(iii).

To prove the claim above, we use two different coordinate parametrizations of $Z_{\mathbf{t}}$; the first is useful for establishing the claim on $\mathcal{U}_1$ and $\mathcal{U}_2$, and the second for $\mathcal{U}_3$. For the first, we parametrize $Z_{\mathbf{t}}$ by $x, y, w \in \mathbb{R}^2$ and $s \in \mathbb{R}$ by

(i) solving the 2×2 system for z,

$$(y-x)^{\perp} \cdot (z-x) = -t_1,\ (w-x)^{\perp} \cdot (z-x) = t_2,$$

obtaining, for (x, y, w) in general position (in the complement of a hypersurface), a unique solution $z = x + \tilde{z}(x, y, w, \mathbf{t})$; and

(ii) for $w \neq x$, solving $(w-x)^\perp \cdot (u-x) = -t_3$ for u with one degree of freedom,

$$u - x = -t_3 \frac{(w-x)^\perp}{|w-x|} + s \cdot (w-x) =: \tilde{u}(x, w, s; t_3),\ s \in \mathbb{R}.$$

Note that

$$\left| \frac{D(y-x-\tilde{z})}{Dy} \right| \neq 0, \quad \left| \frac{D(w-x-\tilde{z})}{Dw} \right| \neq 0; \tag{3.7}$$

the first follows since the differential maps $(y-x)\cdot\partial_y \to (y-x)\cdot\partial_y$ and $(y-x)^\perp\cdot\partial_y \to c_{y,w,\mathbf{t}}\,(y-x)^\perp\cdot\partial_y + \dots$, and the second is similar. We also have $\partial_s \tilde{u} = w - x \neq 0$.

Adapting (3.6) to this parametrization of $Z_\mathbf{t}$, the conormal bundle of $Z_\mathbf{t}$ is parametrized

$$\begin{aligned}\widetilde{N^*Z_\mathbf{t}}^\perp = \Big\{ \big(& x, \tau_1(y-x-\tilde{z}) + \tau_2(x-w+\tilde{z}) + \tau_3(w-x+\tilde{u});\ y, \tau_1\tilde{z};\\ & x+\tilde{z}, \tau_1(x-y) + \tau_2(w-x);\ w, -\tau_2\tilde{z} + \tau_3\tilde{u};\ x+\tilde{u}, \tau_3(x-w)\big)\\ & : x, y, w \in \mathbb{R}^2, s \in \mathbb{R}, \tau \in \mathbb{R}^3 \setminus \mathbf{0}\Big\}.\end{aligned} \tag{3.8}$$

On $\{\tau_1 \neq 0\}$, we can use the τ_1 term in the expression for ξ in (3.8) together with (3.7) to obtain $|D(x,\xi)/D(x,y)| \neq 0$, while $D(w,\omega)/D(w,\tau_2,\tau_3)| \neq 0$ since $\tilde{z},\ \tilde{u}$ are generically linearly independent. Hence, $\pi_1 \times \pi_4 : C_\mathbf{t}^{(14|235)} \to T^*\mathbb{R}^4$ is a submersion.

On $\{\tau_2 \neq 0\}$, $|D(x,\xi)/D(x,y)| \neq 0$ using (3.7) with the τ_2 term in the expression for ξ in (3.8), while $|D(z,\zeta)/D(y,\tau_1,\tau_2)| \neq 0$ from (3.7) and the generic linear independence of $x-y,\ w-x$. Hence, $\pi_1 \times \pi_3 : C_\mathbf{t}^{(13|245)} \to T^*\mathbb{R}^4$ is a submersion.

To deal with $\mathcal{U}_3 = \{\tau_3 \neq 0\}$, we change the parametrization to $x, z, u \in \mathbb{R}^2$ and $s' \in \mathbb{R}$ by

(i) solving the 2×2 system for w,

$$(z-x)^\perp \cdot (w-x) = -t_2,\ (u-x)^\perp \cdot (w-x) = t_3,$$

obtaining, for (x, z, w) in general position a unique solution $w = x + \tilde{w}(x, z, u; \mathbf{t})$, with $|D(u-x-\tilde{w})/Du| \neq 0$; and

(ii) for $z \neq x$, solving $(z-x)^\perp \cdot (y-x) = t_1$ for y with one degree of freedom,

$$y - x = t_1 \frac{(z-x)^\perp}{|z-x|} + s' \cdot (z-x) =: \tilde{y}(x, z, s'; \mathbf{t}),\ s' \in \mathbb{R}.$$

With respect to these coordinates, the analogue of (3.8) is

$$\begin{aligned}\widetilde{N^*Z_\mathbf{t}}^\perp = \Big\{ \big(& x, \tau_1(x-z+\tilde{y}) + \tau_2(z-x-\tilde{w}) + \tau_3(x-u+\tilde{w});\ x+\tilde{y}, \tau_1(z-x);\\ & z, -\tau_1\tilde{y} + \tau_2\tilde{w};\ x+\tilde{w}, \tau_2(x-z) + \tau_3(u-x);\ u, -\tau_3\tilde{w}\big)\\ & : x, z, u \in \mathbb{R}^2, s' \in \mathbb{R}, \tau \in \mathbb{R}^3 \setminus \mathbf{0}\Big\}.\end{aligned} \tag{3.9}$$

Arguing as above, one sees that, using the τ_3 term in ξ, for $\tau_3 \neq 0$ we have $|D(x,\xi)/D(x,u)| \neq 0$, and $|D(z,\zeta)/D(z,\tau_1,\tau_2)| \neq 0$, which shows that $\pi_1 \times \pi_3$ is a submersion. Hence, $C_\mathbf{t}^{(13|245)}$ is nondegenerate on $\mathcal{U}_3$, and this finishes the proof of Thm. 1.1(iii).

4 Pairs of ratios of distances

We now discuss and prove Thm. 1.2: if $E \subset \mathbb{R}^d$ with $\dim_{\mathcal{H}}(E) > (3d+1)/4$, then

$$\text{Int}\left\{\left(\frac{|x-y|}{|z-w|}, \frac{|x-w|}{|z-y|}\right) \in \mathbb{R}^2 : x, y, z, w \in E,\ z \neq w,\ z \neq y\right\} \neq \emptyset. \tag{4.1}$$

See the left 4-tuple in Fig. 1 below.

To put this in context, we discuss sets of ratios of distances more generally, recalling related previous results and then proving some variations. It follows immediately from the result of Mattila and Sjölin [24] that if $E \subset \mathbb{R}^d$, $\dim_{\mathcal{H}}(E) > (d+1)/2$, then

$$\text{Int}\left\{\frac{|x-y|}{|z-w|} : x, y, z, w \in E,\ z \neq w\right\} \neq \emptyset, \tag{4.2}$$

since both the numerators and denominators range over sets containing an interval.

Later, under the higher dimensional threshold $\dim_{\mathcal{H}}(E) > (d+2)/2$, Peres and Schlag [30, p. 248] proved a stronger, *pinned* version of [24]: there exists an $x \in E$ such that the pinned distance set,

$$\Delta^x(E) := \{\, |x-z| : z \in E \,\}$$

contains an interval, i.e., has nonempty interior. Using such a pin point x, and then fixing any $y \in E,\ y \neq x$, it follows immediately that for $\dim_{\mathcal{H}}(E) > (d+2)/2$,

$$(\exists\, x \in E)\ (\forall\, y \in E,\ y \neq x)\ \text{Int}\left\{\frac{|x-z|}{|x-y|} : z \in E\right\} \neq \emptyset. \tag{4.3}$$

By weakening the statement in (4.3), one can lower the threshold:

Theorem 4.1 *If $E \subset \mathbb{R}^d$, $d \geq 2$, is compact and $\dim_{\mathcal{H}}(E) > (d+1)/2$, then*

$$(\exists\, x \in E)\ \text{s.t.}\ \text{Int}\left\{\frac{|x-z|}{|x-y|} : y, z \in E,\ y \neq x\right\} \neq \emptyset. \tag{4.4}$$

Remark 4.2 This improves upon our result [10, Thm. 1.4], which established (4.4) for $\dim_{\mathcal{H}}(E) > (2d+1)/3$ using the original partition optimization.

Remark 4.3 The elementary proof we give, combining known results, follows closely an argument in a recent preprint of Borges, Iosevich and Ou [2, Sec. 2].

To prove Theorem 4.1, first recall another result of Peres and Schlag [30, Cor. 8.4] (see also [18, 20]): if $\dim_{\mathcal{H}}(E) > (d+1)/2$, there exists $x \in E$ such that the pinned distance set has positive Lebesgue measure, $|\Delta^x(E)|_1 > 0$. Apply this twice for one such pin point, x, with variables z and y, resp., in the definition of $\Delta^x(E)$. For each of these, apply log (a bi-Lipschitz map on compact subsets of $\mathbb{R}_+$) to obtain

$$|\{\log|x-z| : z \in E,\ z \neq x\}|_1 > 0, \quad |\{\log|x-y| : y \in E,\ y \neq x\}|_1 > 0.$$

Now invoking the classical theorem of Steinhaus [32] on difference sets, one has that

$$\mathit{Int}\ \{\log|x-z| - \log|x-y| : y, z \in E,\ \ y \neq x,\ z \neq x\} \neq \emptyset.$$

Applying the exponential map (a diffeomorphism $\mathbb{R} \to \mathbb{R}_+$) preserves nonempty interior, yielding (4.4) and finishing the proof of Theorem 4.1.

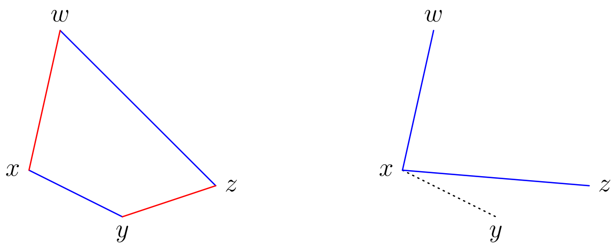

Fig. 1 Two pairs of ratios of distances of points in a 4-tuple. Figure on left corresponds to (4.1) from Thm. 1.2, with ratio of the lower left and upper right side lengths forming the first coordinate of $\Phi(x, y, z, w)$ and the ratio of the upper left and lower right side lengths being the second coordinate. Figure on right illustrates (4.5) for the same 4 points, with the two coordinates of $\Phi(x, y, z, w)$ being the ratios of the lengths of the solid segments to the dotted one

The results above are motivation to study similar ones for other configuration sets constructed from ratios of distances. For example, one such is the following: If $E \subset \mathbb{R}^d$, $d \geq 2$, is compact with $\dim_{\mathcal{H}}(E) > (d+1)/2$, then

$$(\exists\, x \in E)\ (\forall\, y \in E,\ y \neq x)\ \mathrm{Int}\left\{\left(\frac{|x-z|}{|x-y|}, \frac{|x-w|}{|x-y|}\right) \in \mathbb{R}^2 : z, w \in E\right\} \neq \emptyset. \tag{4.5}$$

(See the right 4-tuple in Fig. 1 below.) However, this follows immediately from (4.3) since, with x and y fixed, the set in (4.5) equals the Cartesian product of the set in (4.3) with itself and hence has nonempty interior.

However, the set that is the subject of Thm. 1.2, i.e., the one in (4.1), does not seem to be amenable to such an approach, and we prove it using microlocal partition optimization, Thm. 2.6,with $k = 4$, $p = 2$, which results in requiring the higher dimensional threshold, $\dim_{\mathcal{H}}(E) > (3d+1)/4$.

To start the proof, the configuration function $\Phi : \left(\mathbb{R}^d\right)^4 \to \mathbb{R}^2_+$,

$$\Phi(x, y, z, w) := \left(\frac{|x-y|}{|z-w|}, \frac{|x-w|}{|z-y|}\right), \tag{4.6}$$

defines, for each $\mathbf{t} = (t_1, t_2) \in (\mathbb{R}_+)^2$, an incidence relation $Z_{\mathbf{t}} = \Phi^{-1}(\mathbf{t})$. Defining $F = (F_1, F_2)$, with

$$F_1 = |x-y| - t_1|z-w|, \quad F_2 = |x-w| - t_2|z-y|,$$

so that $Z_{\mathbf{t}} = F^{-1}(\mathbf{0})$, a calculation shows that $Z_{\mathbf{t}}$ can be parametrized by

$$\begin{aligned} Z_{\mathbf{t}} = \Big\{ &(x, y, z, w) = (x, x - t_1 r\omega^1, z, z - r\omega^2) \in \mathbb{R}^{4d} \\ &: x, z \in \mathbb{R}^d,\ x \neq z,\ r > 0,\ \left(\omega^1, \omega^2\right) \in S_{x,z,r} \Big\}, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} S_{x,z,r} = \big\{ &(\omega^1, \omega^2) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \\ &: 2(z-x)\cdot(t_1 t_2^2 2\omega^1 - \omega^2) = (1-t_2)^2|x-z|^2 + (1 - t_1^2 t_2^2)r^2 \big\}. \end{aligned}$$

One further calculates

$$DF = \begin{bmatrix} \frac{x-y}{|x-y|} & -\frac{x-y}{|x-y|} & -t_1\frac{z-w}{|z-w|} & t_1\frac{z-w}{|z-w|} \\ \frac{x-w}{|x-w|} & t_2\frac{z-y}{|z-y|} & -t_2\frac{z-y}{|z-y|} & -\frac{x-w}{|x-w|} \end{bmatrix}$$

$$= \begin{bmatrix} \omega^1 & -\omega^1 & -t_1\omega^2 & t_1\omega^2 \\ \frac{x-z+r\omega^2}{|x-z+r\omega^2|} & t_2\frac{z-x+t_1r\omega^1}{|z-x+t_1r\omega^1|} & -t_2\frac{z-x+t_1r\omega^1}{|z-x+t_1r\omega^1|} & -\frac{x-z+r\omega^2}{|x-z+r\omega^2|} \end{bmatrix},$$

where the second representation is DF evaluated at points of $Z_{\mathbf{t}}$ in terms of the parametrization (4.7). As in Remark 2.8, we let

$$\widetilde{N^*Z_{\mathbf{t}}} = \left\{ \left(x, \tau^t D_x F;\ y, \tau^t D_y F;\ z, \tau^t D_z F;\ w, \tau^t D_w F\right) : (x, y, z, w) \in Z_{\mathbf{t}},\ \tau \in \mathbb{R}^2 \setminus 0 \right\}$$

be the conormal bundle of the incidence relation, with the (x, ξ), (y, η), (z, ζ), (w, ν) variables separated.

Denote the projections from $\widetilde{N^*Z_{\mathbf{t}}}$ onto its four $T^*\mathbb{R}^d$ factors by $\pi_x, \pi_y, \pi_z, \pi_w$, respectively, each of which is a function of x, z, r, ω, τ with values in $T^*\mathbb{R}^d$. Corresponding to the choice of partition $\sigma = (13\,|\,24)$, by the above one has

$$(\pi_x \times \pi_z)(x, z, r, \omega, \tau) = \left(x, \tau_1\omega^1 + \tau_2\frac{x-z+r\omega^2}{|x-z+r\omega^2|};\ z, -\tau_2 t_1\omega^2 - \tau_2 t_2\frac{z-x+t_1r\omega^1}{|z-x+t_1r\omega^1|} \right).$$

From this one can calculate that, for $\tau_1 \neq 0$ and $x - z + t_1r\omega^1 \neq 0$,

$$\text{Rank}\ \frac{D(x, \xi, z, \zeta)}{D(x, z, \omega^1, \tau)} = 3d + 1,$$

and similarly, for $\tau_2 \neq 0$ and $x - z + r\omega^2 \neq 0$,

$$\text{Rank}\ \frac{D(x, \xi, z, \zeta)}{D(x, z, \omega^2, \tau)} = 3d + 1.$$

Note that $W := \{x - z + t_1r\omega^1 = 0\} \cup \{x - z + r\omega^2 = 0\}$ is a variety of dimension $3d$. Since our dimensional threshold will be $\dim_{\mathcal{H}}(E) > (3d+1)/4$, and thus $\dim_{\mathcal{H}}(E \times E \times E \times E) > 3d + 1 > 3d$, this exceptional set is irrelevant for the analysis. See, e.g., [10, Sec. 4] for several instances of this type of reasoning. Thus, if over $Z_{\mathbf{t}} \cap \left(\mathbb{R}^{4d} \setminus W\right)$ we let $\mathcal{U}_j = \{\tau_j \neq 0\} \subset \widetilde{N^*Z_{\mathbf{t}}}$, $j = 1, 2$, then $\{\mathcal{U}_1, \mathcal{U}_2\}$ forms an open cover of $C_{\mathbf{t}}^{\sigma}$ on which $D(\pi_x \times \pi_z)$ drops rank by $\leq d - 1$. Hence, by the FIO estimates (cf. Remark 2.7), the associated $\mathcal{R}_{\mathbf{t}}^{\sigma}$ loses at most $\beta^{\sigma} = (d-1)/2$ derivatives. Applying Theorem 2.6 with codimension $p = 2$ and loss $\beta = (d-1)/2$ then yields that $\Delta_{\Phi}(E)$, the set in (4.1), has nonempty interior if $\dim_{\mathcal{H}}(E) > \frac{1}{4}(2d + 2 + (d-1)) = (3d+1)/4$, finishing the proof of Thm. 1.2.

5 Congruence classes of triangles in $\mathbb{R}^d$

We now show how the result of Palsson and Romero-Acosta [29] follows easily from the microlocal approach taken here. Let $\Phi : \left(\mathbb{R}^d\right)^3 \to \mathbb{R}^3$,

$$\Phi(x, y, z) = (\,|x-y|,\ |x-z|,\ |y-z|\,),$$

so that $\Delta_{\Phi}(E)$ is the set of vectors of side lengths of triangles generated by the points in E and thus, modulo permutations, the set of congruence classes of triangles in E. Letting $\widehat{x} = x/|x|$, one computes

$$D\Phi = \begin{bmatrix} \widehat{x-y} & -\widehat{x-y} & 0 \\ \widehat{x-z} & 0 & -\widehat{x-z} \\ 0 & \widehat{y-z} & -\widehat{y-z} \end{bmatrix}.$$

Furthermore, for $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3_+$, we can parametrize $Z_{\mathbf{t}}$ as follows: We first take $x \in \mathbb{R}^d$ to be arbitrary, and then $y = x - t_1\omega$, with $\omega \in \mathbb{S}^{d-1}$ arbitrary. If one writes $z = x - t_2\tilde{\omega}$ for some $\tilde{\omega} \in \mathbb{S}^{d-1}$, then one computes that $|y - z| = t_3$ iff $\tilde{\omega} \cdot \omega = \frac{t_3 - (t_1^2 + t_2^2)}{2t_1t_2}$. For $\mathbf{t}$ in the complement of a lower dimensional variety,

$$S_{\mathbf{t},\omega} := \left\{ \tilde{\omega} \in \mathbb{S}^{d-1} : \tilde{\omega} \cdot \omega = \frac{t_3 - (t_1^2 + t_2^2)}{2t_1t_2} \right\}$$

is a smooth $(d-2)$-surface in $\mathbb{S}^{d-1}$ (possibly empty), and

$$Z_{\mathbf{t}} = \left\{ (x, x - t_1\omega, x - t_2\tilde{\omega}) : x \in \mathbb{R}^d, \omega \in \mathbb{S}^{d-1}, \tilde{\omega} \in S_{\mathbf{t},\omega} \right\}.$$

Applying $D\Phi^*$ to $\tau \in \mathbb{R}^3$ at these points, we obtain

$$\begin{aligned}\widetilde{N^*Z_{\mathbf{t}}} = \Big\{ \big(x,\ \tau_1\omega + \tau_2\tilde{\omega};\ & x - t_1\omega,\ -(\tau_1 - (t_1/t_3)\tau_3)\,\omega + (t_2/t_3)\tau_3\tilde{\omega}; \\ & x - t_2\tilde{\omega},\ (t_1/t_3)\tau_3\omega - (\tau_2 + (t_2/t_3)\tau_3)\,\tilde{\omega} \big) \\ & : x \in \mathbb{R}^d,\ \omega \in \mathbb{S}^{d-1},\ \tilde{\omega} \in S_{\mathbf{t},\omega},\ \tau \in \mathbb{R}^3 \setminus \mathbf{0} \Big\}. \end{aligned} \tag{5.1}$$

Let $i_\omega : T_\omega\mathbb{S}^{d-1} \hookrightarrow T_\omega\mathbb{R}^d$ and $\tilde{i}_{\tilde{\omega}} : T_{\tilde{\omega}}S_{\mathbf{t},\omega} \hookrightarrow T_{\tilde{\omega}}\mathbb{R}^d$ be the inclusions of the tangent spaces, and note that for generic $\mathbf{t}$,

$$span \left\{ T_\omega\mathbb{S}^{d-1},\ \omega \right\} = T_\omega\mathbb{R}^d \text{ and } span \left\{ T_{\tilde{\omega}}S_{\mathbf{t},\omega},\ \omega,\ \tilde{\omega} \right\} = T_{\tilde{\omega}}\mathbb{R}^d. \tag{5.2}$$

Denoting the projections into the (x, ξ), (y, η) and (z, ζ) variables by π_j, $j = 1, 2, 3$, resp., we calculate their Jacobians with respect to $(x, \omega, \tilde{\omega}, \tau_1, \tau_2, \tau_3)$; to avoid clutter, we indicate unneeded terms by $*$:

$$D\pi_1 = \begin{bmatrix} I_d & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau_1 i_\omega & \tau_2\tilde{i}_{\tilde{\omega}} & \omega & \tilde{\omega} & 0 \end{bmatrix},$$

$$D\pi_2 = \begin{bmatrix} I_d & * & * & * & * & * \\ 0 & -(\tau_1 - (t_1/t_3)\tau_3)\,i_\omega & (t_2/t_3)\tau_3\tilde{i}_{\tilde{\omega}} & -\omega & 0 & (1/t_3)(t_2\tilde{\omega} + t_1\omega) \end{bmatrix},$$

and

$$D\pi_3 = \begin{bmatrix} I_d & * & * & * & * & * \\ 0 & (t_1/t_3)\tau_3 i_\omega & -(\tau_2 + (t_2/t_3)\tau_3)\,\tilde{i}_{\tilde{\omega}} & 0 & -\tilde{\omega} & (1/t_3)(t_1\omega - t_2/\tilde{\omega}) \end{bmatrix}.$$

Examining the column spaces of these and using (5.2), one sees that $D\pi_1$ is surjective except on the line

$$L_1 := \{\tau_1 = \tau_2 = 0\};$$

$D\pi_2$ is surjective except on

$$L_2 := \{\tau_1 + (t_1/t_3)\tau_3 = \tau_3 = 0\} = \{\tau_1 = \tau_3 = 0\};$$

and $D\pi_3$ is surjective except on

$$L_3 := \{\tau_3 = \tau_2 + (t_2/t_3)\tau_3 = 0\} = \{\tau_2 = \tau_3 = 0\}.$$

Furthermore, for each j, the image $\pi_j(L_j)$ lies in the $\mathbf{0}$-section of $T^*\mathbb{R}^d$, which causes problems with the standard theory of FIO. Since this holds for every partition, the original partition optimization of Thm. 2.4 is inapplicable. Furthermore, since the lines of degeneracy exist above all points of $Z_{\mathbf{t}}$, the merely local version, Thm. 2.5, also does not suffice, so that one needs the full strength of the microlocal version.

Setting $\mathcal{U}_j = \{\tau \in \mathbb{R}^3 \setminus L_j\} \subset \widetilde{N^*Z_{\mathbf{t}}}$, it follows that $\{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3\}$ is a conic open cover of $\widetilde{N^*Z_{\mathbf{t}}}$ on which the partitions $\sigma = (1|23),\ (2|13),\ (3|12)$, resp., result in canonical relations $C_{\mathbf{t}}{}^{\sigma} \subset \left(T^*\mathbb{R}^d \backslash \mathbf{0}\right) \times \left(T^*\mathbb{R}^{2d} \backslash \mathbf{0}\right)$ which are nondegenerate, so that Thm. 2.6 applies with $k = 3,\ p = 3, \max(d_L, d_R) = 2d$ and $\beta = 0$. Hence, for $E \subset \mathbb{R}^d$ with

$$\dim_{\mathcal{H}}(E) > \frac{1}{3}(2d + 3 + 0) = (2d + 3)/3,$$

$\Delta_{\Phi}(E)$ has nonempty interior in $\mathbb{R}^3$, reproving the main result of [29].

As a final comment, we remark that in their recent preprint [28], Palsson and Romero-Acosta have extended the results of [29] to $(k-1)$-simplices in $\mathbb{R}^d$, $k \geq 4$, for some thresholds depending on d and k. Calculations along the lines of those used above indicate that some of the conditions required to apply Thm. 2.6 for these higher values of k appear to fail. It would be interesting to see whether further microlocal analysis of the problem can be used to obtain results for these higher dimensional simplices.

6 Similarity classes of triangles in $\mathbb{R}^d$

We conclude with the proof of Thm. 1.4. This is a variation on the result of the previous section, with congruence of triangles replaced by similarity and the resulting threshold lowered by 1/3; however, it can also be viewed as concerning ratios of distances, in the spirit of Sect. 4. Just as (1.1) encodes the congruence classes of triangles with vertices in E (up to permutations of x, y, z, which do not affect the nonempty interior statement), (1.2) encodes the similarity classes of triangles with vertices in E. To make this more explicit, recall that the projective plane $\mathbb{RP}^2$ is covered by three coordinate charts,

$$\mathcal{V}_\alpha := \left\{ [\, A : B : C\,] \text{ s.t. } A, B, C \in \mathbb{R},\ \alpha \neq 0 \right\}, \quad \alpha = A,\ B \text{ or } C.$$

In particular, $\mathcal{V}_A = \left\{ [\, 1 : u : v\,] \text{ s.t. } (u, v) \in \mathbb{R}^2 \right\}$, and to prove Thm. 1.4 it suffices to show that the configuration function

$$\Psi(x, y, z) := \left(\frac{|x - z|}{|x - y|}, \frac{|y - z|}{|x - y|} \right)$$

satisfies Int $(\Delta_\Psi(E)) \neq \emptyset$ for $\dim_{\mathcal{H}}(E) > (2d + 2)/3$.

For $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2_+$, we parametrize $Z_{\mathbf{t}}$ as follows: For $x \in \mathbb{R}^d$, write $y = x + r\omega$ with $r > 0,\ \omega \in \mathbb{S}^{d-1}$, so that $|x - y| = r$. In order for $(x, y, z) \in Z_{\mathbf{t}}$, z must be of the form $z = x + t_1 r\tilde{\omega}$ for some $\tilde{\omega} \in \mathbb{S}^{d-1}$, ensuring that $|x - z| = t_1 r$; the further constraint that $|y - z| = t_2|x - y|$ then becomes

$$\omega \cdot \tilde{\omega} = (2t_1)^{-1}\left(1 + t_1^2 - t_2^2\right) =: a(\mathbf{t}).$$

Thus $\tilde{\omega} = a(\mathbf{t})\omega + b(\mathbf{t})\omega'$ for some $\omega' \in \mathbb{S}^{d-1} \cap \omega^\perp$, where $b(\mathbf{t}) := \left(1 - a^2(\mathbf{t})\right)^{\frac{1}{2}}$; without loss of generality we restrict ourselves to $\mathbf{t}$ such that $|a(\mathbf{t})| < 1$. We can thus parametrize the incidence relation as

$$Z_{\mathbf{t}} = \left\{ \left(x,\ x + r\omega,\ x + t_1 a(\mathbf{t}) r\omega + t_1 b(\mathbf{t}) r\omega'\right) : x \in \mathbb{R}^d,\ \omega \in \mathbb{S}^{d-1},\ \omega' \in \mathbb{S}^{d-1} \cap \omega^\perp,\ r > 0 \right\}. \tag{6.1}$$

Writing $\Psi = (\Psi_1, \Psi_2)$, one has

$$\widetilde{N^*Z_{\mathbf{t}}} = \Big\{ \big(x, \tau_1 d_x\Psi_1 + \tau_2 d_x\Psi_2;\ x + r\omega, \tau_1 d_y\Psi_1 + \tau_2 d_y\Psi_2;\\ x + t_1 a(\mathbf{t}) r\omega + t_1 b(\mathbf{t}) r\omega', \tau_1 d_z\Psi_1 + \tau_2 d_z\Psi_2\big)\\ : x \in \mathbb{R}^d,\ \omega \in \mathbb{S}^{d-1},\ \omega' \in \mathbb{S}^{d-1} \cap \omega^\perp,\ r > 0,\ \tau \in \mathbb{R}^2 \setminus 0 \Big\}.$$

We show that the projection $\pi_1 : \widetilde{N^*Z_{\mathbf{t}}} \to T^*\mathbb{R}^d$ onto the first factor is a submersion, so that Thm. 2.4 applies with $\sigma = (1\,|\,23)$ and no loss ($\beta = 0$), yielding the threshold $\dim_{\mathcal{H}}(E) > (1/3)(\max(d, 2d) + 2 + 0) = (2d + 2)/3$, as claimed. Strictly speaking we are not using Thm. 2.6, but will in fact verify the nondegeneracy of π_1 separately on two open conic sets, $\{\tau_1 \neq 0\}$ and $\{(t_1 - a(\mathbf{t}))\tau_1 - t_2\tau_2 \neq 0\}$, so that the microlocal approach is implicitly being used.

Now, one computes

$$d_x\Psi_1 = |x - y|^{-1}|x - z|^{-1}(x - z) - |x - y|^{-3}|x - z|(x - y),\\ d_x\Psi_2 = -|x - y|^{-3}|y - z|(x - y).$$

Evaluating these at a point of $Z_{\mathbf{t}}$ in terms of the variables in (6.1) using

$$x - y = -r\omega, \quad x - z = -t_1 a(\mathbf{t}) r\omega - t_1 r \big(a(\mathbf{t})\omega + b(\mathbf{t})\omega'\big),\\ |x - y| = r, \qquad |x - z| = t_1 r, \quad |y - z| = t_2 r,$$

we find that

$$d_x(\Psi_1, \Psi_2)\big|_{Z_{\mathbf{t}}} = \big(r^{-1}\big((t_1 - a(\mathbf{t}))\,\omega - b(\mathbf{t})\omega'\big),\ -t_2 r^{-1}\omega\big).$$

Thus,

$$(x, \xi) = \pi_1(x, \omega, \omega', r, \tau) = \big(x, \tau_1 r^{-1}\big((t_1 - a(\mathbf{t}))\,\omega - b(\mathbf{t})\omega'\big) - \tau_2 t_2 r^{-1}\omega\big).$$

Since the spatial component of π_1 is the identity in x, to show that the canonical relation $C_{\mathbf{t}}^{(1|23)}$ is nondegenerate, we need to show that rank $D\xi/D(\omega, \omega', r, \tau) = d$ everywhere. We have

$$D\xi/D\omega = r^{-1}((t_1 - a(\mathbf{t}))\tau_1 - t_2\tau_2) i_\omega, \quad D\xi/D\tau_2 = -t_2 r^{-1}\omega,$$

where $i_\omega : T_\omega\mathbb{S}^{d-1} \hookrightarrow T_\omega\mathbb{R}^d$ is the inclusion of the tangent space of the sphere at ω. Thus, $D\xi/D(\omega, \tau_2)$ is surjective on the set of τ such that $(t_1 - a(\mathbf{t}))\tau_1 - t_2\tau_2 \neq 0$.

On the other hand,

$$D\xi/D\omega' = -r^{-1}\tau_1 b(\mathbf{t}) j_{\omega'}, \qquad D\xi/D\tau_1 = r^{-1}\big((t_1 - a(\mathbf{t}))\,\omega - b(\mathbf{t})\omega'\big),\\ \text{and} \quad D\xi/D\tau_2 = -t_2 r^{-1}\omega,$$

where $j_{\omega'} : T_{\omega'}\big(\mathbb{S}^{d-1} \cap \omega^\perp\big) \hookrightarrow T_{\omega'}\mathbb{R}^d$ is the inclusion of the $(d-2)$-dimensional tangent space. As long as $\tau_1 \neq 0$, this space and the two vectors that follow it span $\mathbb{R}^d$ (note that $b(\mathbf{t}) \neq 0$) and hence $D\xi/D(\omega', \tau_1, \tau_2)$ is surjective. Since $\mathcal{U}_1 = \{\tau_1 \neq 0\}$ and $\mathcal{U}_2 = \{(t_1 - a(\mathbf{t}))\tau_1 - t_2\tau_2 \neq 0\}$ form an open cover of $\widetilde{N^*Z_{\mathbf{t}}}$, we are done.

As in the comment at the end of the previous section concerning congruences, it would be interesting to investigate whether the microlocal approach can be applied to obtain results for similarities of $(k-1)$-simplices in $\mathbb{R}^d$.

Data availability There is no data associated with this research.

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