String C-group representations of transitive groups: a case study with degree 11

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Abstract

In this paper we give a non-computer-assisted proof of the following result: if G is an even transitive group of degree 11 and has a string C-group representation with rank $r \in \{4, 5\}$ then $G \cong \mathrm{PSL}_2(11)$. Moreover this string C-group is the group of automorphisms of the rank 4 polytope known as the 11-cell.

The insights gained from this case study include techniques and observations concerning permutation representation graphs of string C-groups. The foundational lemmas yield a natural and intuitive understanding of these groups. These and similar approaches can be replicated and are applicable to the study of other transitive groups.

Keywords: Abstract regular polytopes, string C-groups, symmetric groups, alternating groups, permutation groups.

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1   Introduction

It is well-known that abstract regular polytopes are in one to one correspondence with string C-groups [16]. In this day and age, and given the right circumstances in terms of access to sufficiently powerful computing technology, it is possible to create, by computer, classifications of abstract regular polytopes for any given rank and "small-enough" group. In contrast, this paper presents a detailed exposition of a variety of computer-free methods by which one may approach such a problem. The example on which we focus is even permutation groups of degree 11, and in this way we obtain a classification of such abstract regular polytopes for rank 4 or 5. This provides an illustrative demonstration of a methodology for classifying string C-groups, and it establishes a pathway for tackling unsolved open-problems such as the classification of high-rank string C-groups for alternating groups of arbitrary degree.

The "Aveiro theorem" states that the maximal rank of an abstract regular polytope with alternating group of degree n as its automorphism group is $\lfloor \frac{n-1}{2} \rfloor$ when $n \geq 12$ [2]. For the alternating groups of degrees 5, 9, 10 and 11, the maximal ranks are 3, 4, 5 and 6, respectively. The remaining alternating groups are not the automorphism groups of any such polytope. In [8] it was proved that there exists an abstract regular polytope for each rank $r \in \{3, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ when $n \geq 12$. The set of all possible ranks of abstract regular polytopes for alternating groups with a degree different from 11 is either empty or an interval, as shown in Table 1.

Group	Set of ranks
A_5	{3}
A_6	$\emptyset$
A_7	$\emptyset$
A_8	$\emptyset$
A_9	{3,4}
A_{10}	{3,4,5}
A_{11}	{3,6}
$A_n, n \geq 12$	$\{3, \ldots, \lfloor (n-1)/2 \rfloor\}$

Table 1: The set of possible ranks of abstract regular polytopes for A_n for each $n \geq 5$.

The alternating group A_5 is the first alternating group that is the group of automorphisms of a regular polytope, namely there are, up to duality, exactly two abstract regular polytopes for A_5, the hemi-icosahedron and the hemi-great dodecahedron. In his doctoral thesis Conder proved that all but finitely many alternating groups are the automorphism group of a regular map of type $\{3, m\}$ with $m > 6$ (this result can also be found in [3, 4]). As regular maps for alternating groups are precisely abstract regular polyhedra [5, Corollary 4.2], this means that the number 3 belongs to each set of ranks of Table 1, the exceptions being $n = 3$, 4, 6, 7 or 8. The lists of all abstract regular polytopes for alternating groups up to degree 9 are available in [14]. In [10] the authors give permutation representation graphs of all abstract regular polytopes for A_9 and A_{10} having ranks $r \in \{4, 5\}$, and some examples of rank 6 abstract regular polytopes for the group A_{11}. In [9], their compu-

tations revealed the non-existence of abstract regular polytopes of ranks 4 and 5 for A_{11}. In 2018, Meynaert in his master's thesis [17] gave a complete classification of the representations of A_{11} as a string group generated by an independent set of involutions with rank 4 or 5. Meynaert used permutation representation graphs in his classification approach, but he did not explore the potential of fracture graphs in his work.

During a problem session in the 2022 Edition of the Symmetries in Graphs, Maps, and Polytopes Workshop, A_{11} was again highlighted as an interesting case of study because it is the unique known example of a group whose set of ranks is not an interval.

The 11-cell is a rank 4 polytope discovered by Coxeter and Grünbaum in the 80's, and is the only known abstract regular polytope with rank $r \in \{4, 5\}$ having an even transitive group of degree 11 as its automorphism group, namely the $PSL_2(11)$. The group $PSL_2(11)$ is the unique transitive even group of degree 11 which is the automorphism group of an abstract regular polytope having rank 4 or 5. Moreover the only rank 4 polytope for $PSL_2(11)$ is the 11-cell, which is self-dual, and there is no abstract regular polytopes of rank 5 for $PSL_2(11)$.

Our approach to show this result uses the concept of fracture graphs, as first introduced in [1], which provides a method for tackling the problem, dividing it into three distinct cases: absence of a fracture graph, presence of a split, and the existence of a 2-fracture graph. This method gives a way to determine string group generated by involutions representations of a transitive group [1, 11], such as the illustrative example of even groups of degree 11. A string group generated by involutions may not be a string C-group, then it is necessary to test whether the intersection property is satisfied. For groups of degree 11, this evaluation is straightforward using computer-based methods. However, in contrast to a simple "yes" or "no" outcome, our approach provides a more profound understanding by elucidating the reasons behind the failure.

In the first four sections we give the tools that will be used in this classification but that can also be used in a more general setting.

- Section 2: String C-groups.

- Section 3: Permutation representation graphs.

- Section 4: Fracture graphs.

- Section 5: Conditions leading to the failure of the intersection property.

In the following remaining sections, we show how the tools described above can be used on our example, in which we assume that $\mathcal{G}$ is any permutation representation graph of an even transitive string C-group of degree 11. We start by dividing into the cases where $\mathcal{G}$ has a fracture graph (with a split or without a split), and after we give a classification that shows what we have claimed above.

- Section 6: When $\mathcal{G}$ has a fracture graph with a split.

- Section 7: When $\mathcal{G}$ has a 2-fracture graph.

- Section 8: A classification of even transitive string C-groups of degree 11.

Our results rely on the atlas of finite groups and on classifications of regular polyhedra for $PSL_2(11)$ which are available and well known among the researchers working on abstract polytopes and maps.

2 String C-groups

A group G is the automorphism group of an abstract regular polytope of rank r if and only if it has a *string C-group representation* $\Gamma = (G, \{\rho_0, \ldots, \rho_{r-1}\})$ such that:

(1) $G = \langle \rho_0, \ldots, \rho_{r-1} \rangle$;

(2) $\{\rho_0, \ldots, \rho_{r-1}\}$ is an ordered set of involutions;

(3) $\forall i, j \in \{0, \ldots, r-1\}$, $|i - j| > 1 \Rightarrow (\rho_i \rho_j)^2 = 1$ (commuting property);

(4) $\forall J, K \subseteq \{0, \ldots, r-1\}, \langle \rho_j \mid j \in J \rangle \cap \langle \rho_k \mid k \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle$.

The sequence $\{p_1, \ldots, p_{r-1}\}$ where p_i is the order of $\rho_{i-1}\rho_i$ is the *(Schläfli) type* of Γ. A representation $\Gamma = (G, \{\rho_0, \ldots, \rho_{r-1}\})$ that satisfies (1), (2) and (3) is called a *string group generated by involutions* or, for short, a sggi. The *dual* of a sggi is obtained by reversing the sequence of generators.

Let us consider the following notation.

$$\Gamma_{i_1, \ldots, i_k} := (G_{i_1, \ldots, i_k}, \{\rho_j : j \notin \{i_1, \ldots, i_k\}\});$$
$$\Gamma_{\{i_1, \ldots, i_k\}} := (G_{\{i_1, \ldots, i_k\}}, \{\rho_j : j \in \{i_1, \ldots, i_k\}\});$$
$$\Gamma_{<i} := (G_{<i}, \{\rho_0, \ldots, \rho_{i-1}\}) \qquad (i \neq 0);$$
$$\Gamma_{>i} := (G_{>i}, \{\rho_{i+1}, \ldots, \rho_{r-1}\}) \qquad (i \neq r-1).$$

The *maximal parabolic subgroups* of Γ are the subgroups G_i with $i \in \{0, \ldots, r-1\}$.

The following result shows that when Γ_0 and Γ_{r-1} are string C-groups, the intersection property for Γ is verified by checking only one condition.

Proposition 2.1 ([16, Proposition 2E16]). *Let* $\Gamma = (G, S)$ *be a sggi with* $S := \{\rho_0, \ldots, \rho_{r-1}\}$. *Suppose that* Γ_0 *and* Γ_{r-1} *are string C-groups. If* $G_0 \cap G_{r-1} \cong G_{0, r-1}$, *then* Γ *is a string C-group.*

2.1 Sesqui-extensions

The term sesqui-extension was first introduced in [10]. Let us recall its meaning. Let $\Gamma = (G, \{\rho_0, \ldots, \rho_{r-1}\})$ be a sggi, and let τ be an involution in a supergroup of G such that $\tau \notin G$ and τ centralizes G. For a fixed k, we define the sggi $\Gamma^* = (G^*, \{\rho_i \tau^{\eta_i} \mid i \in \{0, \ldots, r-1\})$ where $\eta_i = 1$ if $i = k$ and 0 otherwise, the *sesqui-extension* of Γ with respect to ρ_k and τ.

Proposition 2.2 ([10, Proposition 3.3]). *If* Γ *is a string C-group, and* Γ^* *is a sesqui-extension of* Γ *with respect to the first generator, then* Γ^* *is a string C-group.*

Lemma 2.3 ([9, Lemma 5.4]). *Let* $\Gamma = (G, \{\rho_0, \ldots, \rho_{r-1}\})$ *be a sggi. If* $\Gamma^* = (G^*, \{\rho_i \tau^{\eta_i} \mid i \in \{0, \ldots, r-1\})$ *where* $\eta_i = 1$ *if* $i = k$ *and* 0 *otherwise, then the following hold:*

(a) $G^* \cong G$ *or* $G^* = G \times \langle \tau \rangle \cong G \times C_2$.

(b) *If the identity element of* G *can be written as a product of generators involving* ρ_k *an odd number of times, then* $G^* = G \times \langle \tau \rangle$.

(c) *If G is a finite permutation group, τ and ρ_k are odd permutations, and all other ρ_i are even permutations, then $G^* \cong G$.*

(d) *Whenever $\tau \notin G^*$, Γ is a string C-group if and only if Γ^* is a string C-group.*

3   Permutation representation graphs

Suppose that G is a permutation group of degree n and let $\Gamma = (G, \{\rho_0, \ldots, \rho_{r-1}\})$ be a sggi. The *permutation representation graph* $\mathcal{G}$ of Γ is an r-edge-labelled multigraph with n vertices and with an i-edge $\{a, b\}$ whenever $a\rho_i = b$ with $a \neq b$. The *dual* of a permutation representation graph is obtained by reverting the labels of the edges according to the correspondence $(0, \ldots, r - 1) \mapsto (r - 1, \ldots, 0)$. Let $\mathcal{G}_{i_1,\ldots,i_k}$ (resp. $\mathcal{G}_{\{i_1,\ldots,i_k\}}$) denote the permutation representation graph of $\Gamma_{i_1,\ldots,i_k}$ (resp. $\Gamma_{\{i_1,\ldots,i_k\}}$). Notice that when ρ_i is a k-transposition (a product of k disjoint transpositions), $\mathcal{G}_{\{i\}}$ is a matching with k edges.

If $a\rho_i = a\rho_j = b$ with $a \neq b$ and $i \neq j$ then we say that the graph has a *double $\{i, j\}$-edge*. Similarly, triple edges with labels i, j and k are called *triple $\{i, j, k\}$-edges*. These multiple edges are represented as follows (respectively).

$$\textcircled{a}\overset{\{i,j\}}{=\!=\!=}\textcircled{b} \qquad \textcircled{a}\overset{\{i,j,k\}}{\equiv\!\equiv}\textcircled{b}$$

A square with alternating labels in the set $\{i, j\}$ is called an $\{i, j\}$-*square*.

A consequence of the commuting property is that, if i and j are nonconsecutive the connected components of $\mathcal{G}_{\{i,j\}}$ with more then two vertices are $\{i, j\}$-squares. We also have the following lemma which is a direct consequence of the commuting property.

Lemma 3.1. *If j is the label of an edge of $\mathcal{G}$ connecting a vertex of $\mathrm{Fix}(\rho_i)$ and a vertex of its complement $\overline{\mathrm{Fix}(\rho_i)}$ then $j \in \{i - 1, i + 1\}$.*

Another consequence of the commuting property is that ρ_0 centralizes $G_{0,1}$, for that reason we may state the following results about the connected components of $\mathcal{G}_{0,1}$. The dual of the following lemmas also can be applied to $\mathcal{G}_{r-1,r-2}$.

Lemma 3.2. *Let U and V be distinct $G_{0,1}$-orbits.*

(a) *If $x\rho_0 = y$ with $x, y \in U$ and $x \neq y$, then $|U|$ is even.*

(b) *If $x\rho_0 = y$ with $x \in U$ and $y \in V$, then the permutation representation subgraph of $\mathcal{G}_{0,1}$ induced by U is a copy of the one induced by V.*

Proof. This is a consequence of the commuting property of Γ. ☐

Lemma 3.3. *If G_0 is transitive and ρ_0 is an even permutation then one of the following situations occurs.*

(a) *$G_{0,1}$ has at least one orbit of even size.*

(b) *$G_{0,1}$ has at least four odd orbits.*

Proof. Suppose that all $G_{0,1}$-orbits are odd. Then, by Lemma 3.2, ρ_0 cannot swap a pair of vertices in the same $G_{0,1}$-orbit. Then ρ_0 swaps vertices in different $G_{0,1}$-orbits pair-wisely. Let O_1 and O_2 be $G_{0,1}$-orbits such that $O_1\rho_0 = O_2$. If ρ_0 fixes the remaining points, then ρ_0 is a product of $|O_1|(=|O_2|)$ transpositions, hence ρ_0 is odd, a contradiction. Thus there exists another pair of (odd) orbits O_3 and O_4 such that $O_3\rho_0 = O_4$, as wanted. $\square$

Lemma 3.4. *If the permutation representation subgraphs induced by each of the $G_{0,1}$-orbits are all different, then ρ_0 acts non-trivially only on $G_{0,1}$-orbits of even size (fixing the odd orbits pointwisely).*

Proof. This is an immediate consequence of Lemma 3.2 (b). $\square$

4   Fracture graphs

Suppose that all maximal parabolic subgroups of Γ are intransitive. A *fracture graph* of $\mathcal{G}$ is a subgraph of $\mathcal{G}$ having n vertices and, for each $i \in \{0, \ldots, r-1\}$, one i-edge chosen among the i-edges between vertices in different G_i-orbits [11]. A fracture graph of $\mathcal{G}$ thus has exactly r edges.

In general a sggi has multiple fracture graphs. Indeed only S_n has a string C-group representation, corresponding to the simplex, having a uniquely determined fracture graph. An i-edge that belongs to every fracture graph of $\mathcal{G}$ is called an *i-split* of Γ [1]. A split is a bridge of $\mathcal{G}$, therefore it satisties the following property.

Proposition 4.1. *Any path (not containing an i-edge) from an i-split to an edge with label l, where $l \neq i$, contains all labels between l and i.*

Proof. This is a consequence of Proposition 5.18 of [2]. $\square$

Lemma 4.2. *Let $\Gamma := (G, \{\rho_0, \ldots, \rho_{n-1}\})$ be a sggi with a permutation representation graph $\mathcal{G}$ having a fracture graph. If ρ_i is a 2-transposition and $\mathcal{G}$ has a double $\{i, j\}$-edge, for some $i, j \in \{0, \ldots, n-1\}$, then $\mathcal{G}$ has an i-split.*

Proof. This is an immediate consequence of a definition of a split. $\square$

Suppose that $\mathcal{G}$ admits a fracture graph. If in addition $\mathcal{G}$ has no splits then, for every $i \in \{0, \ldots, r-1\}$, there are at least two i-edges between vertices in different G_i-orbits. In this case $\mathcal{G}$ admits a *2-fracture graph*, that is a subgraph of $\mathcal{G}$ with n vertices and with exactly two i-edges between vertices in different G_i-orbits, for each $i \in \{0, \ldots, r-1\}$ [2]. A 2-fracture graph of $\mathcal{G}$ thus has exactly $2r$ edges.

5   Conditions leading to the failure of the intersection property

In this section, we give sufficient conditions for the intersection property to fail. Relying on these, we now prove that all sggi's given in the Appendix A are not string C-groups. These sggi's are a result of the case-by-case analysis.

Proposition 5.1 ([1, Proposition 6.1]).

(i) *Let G be a primitive permutation group containing a 3-cycle. Then G is the alternating or symmetric group.*

(ii) *Let G be an intransitive permutation group containing a 3-cycle α. Let X be the orbit of one of the points of α, and H the group induced on X by G. If $A_X \leq H$, then $A_X \leq G$.*

Lemma 5.2. *Let $\Gamma = (G, \{\rho_0, \rho_1, \rho_2, \rho_3\})$ be an even sggi. Suppose that*

- *D is a $G_{0,3}$-orbit with at least four points,*

- *X is the G_0-orbit containing D and*

- *Y is the G_3-orbit containing D.*

*If the following two conditions hold then Γ is not a string C-group.*

(a) *G_0 is primitive on X and there exists a permutation $\alpha \in G_0$ such that α is a 3-cycle on X, fixing the complement $\bar{X}$ point-wisely;*

(b) *G_3 is primitive on Y and there exists a permutation $\beta \in G_3$ such that β is a 3-cycle on Y, fixing the complement $\bar{Y}$ point-wisely.*

Proof. Suppose that the conditions (a) and (b) are satisfied. By Proposition 5.1 we conclude that G_0 contains all even permutations on X, that is, $A_X \leq G_0$. In particular $A_D \leq G_0$. Similarly we get that $A_D \leq G_3$. Hence $A_D \leq G_0 \cap G_3$ and $|D| \geq 4$. But $G_{0,3}$ is a dihedral group, thus $G_0 \cap G_3 \neq G_{0,3}$. □

Using Lemma 5.2, the failure of the intersection property of the permutation representations given in the Appendix A can be proven for most cases. For the remaining rank 4 cases, the proof of the following proposition also gives an alternative approach, relying heavily on the fact that $G_{0,3}$ is a dihedral group with an intransitive action. This allows us to find a permutation in $G_0 \cap G_3$ that does not belong to that dihedral group.

Proposition 5.3. *The sggi's of the* Appendix A *are independent generating sets for A_{11} but they are not string C-groups.*

Proof. In all cases of the Appendix A, we have that G is a transitive permutation group of prime degree 11, hence primitive. Suppose that $\Gamma = (G, \{\rho_0, \rho_1, \rho_2, \rho_3\})$ is the sggi corresponding to the graph (A1) with the following numeration of the vertices.

$$①\overset{1}{-}②\overset{0}{-}③\overset{1}{-}④\overset{\{0,2\}}{=}⑤\overset{1}{-}⑥\overset{2}{-}⑦\overset{3}{-}⑧\overset{2}{-}⑨\overset{\{1,3\}}{=}⑩\overset{2}{-}⑪$$
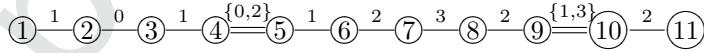
Consider the permutations $a \in G_0$ and $b \in G_3$ defined as follows.

$$a := \rho_1\rho_2\rho_3 := (1,2)(3,5,8,10,7,6,4)(9,11)$$
$$b := \rho_0\rho_1\rho_2 = (1,2,5,3)(4,7,6)(8,9,11,10)$$

Since the generators of G are even and b^4 is a 3-cycle, by Proposition 5.1, we conclude that $G \cong A_{11}$. Now consider the sets:

$$D := \{3, \ldots, 7\}, \ X = \{3, \ldots, 11\} \text{ and } Y := \{1, \ldots, 7\}.$$

Consider the action of G_0 on X. As $|X| = 9$ and a has a 7-cycle on its cyclic decomposition permuting elements of X, G_0 is primitive on X. As $|Y| = 7$, G_3 is primitive

on Y. Notice that a^7 and its conjugate by ρ_1 are both 2-transpositions whose product is a 3-cycle, that is $\alpha := a^7(a^7)^{\rho_1} = (9, 10, 11)$ is a 3-cycle, satisfying condition (a) of Lemma 5.2. Finally the permutation $\beta := b^4$ satisfies the (b) of Lemma 5.2, thus Γ is not a string C-group.

The remaining sggi of the Appendix A can be dealt in the same way, with few exceptions. Namely for graphs (B14), (B15), (C1), (D1), (D2) and (F3) a different argument should be applied. In these cases $I := G_0 \cap G_3$ acts on each orbit as a dihedral group but it happens that I is a bigger group. Let $\Gamma = (G, \{\rho_0, \rho_1, \rho_2, \rho_3\})$ be the sggi corresponding to the graph (B14) of the Appendix A with the following numeration of the vertices.

$$\textcircled{1} \overset{\{0,2\}}{=\!=\!=} \textcircled{2} \overset{1}{-\!-} \textcircled{3} \overset{0}{-\!-} \textcircled{4} \overset{1}{-\!-} \textcircled{5} \overset{2}{-\!-} \textcircled{6} \overset{3}{-\!-} \textcircled{7} \overset{2}{-\!-} \textcircled{8} \overset{\{1,3\}}{=\!=\!=} \textcircled{9} \overset{2}{-\!-} \textcircled{10} \overset{1}{-\!-} \textcircled{11}$$
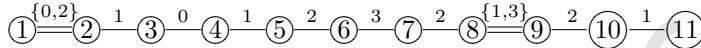
By similar arguments as before we conclude that G_0 acts as a symmetric group on the set $\{4, \ldots, 11\}$. In particular, as G is even, either $(7, 8)(9, 10) \in G_0$ or $(7, 8)(9, 10)(1, 2, 3) \in G_0$. Since $((7, 8)(9, 10)(1, 2, 3))^3 = (7, 8)(9, 10)$ in either case we have that $(7, 8)(9, 10) \in G_0$. In addition $(\rho_2(\rho_1\rho_0)^2)^3 = (7, 8)(9, 10) \in G_3$. This implies that $(7, 8)(9, 10) \in G_0 \cap G_3$. Moreover, $\rho_2(7, 8)(9, 10) = (1, 2)(5, 6) \in G_0 \cap G_3$, as well as its conjugate by ρ_1. But then $S_3 \times D_{10} \leq G_0 \cap G_3$, and therefore Γ is not a string C-group. Similar arguments can be used when Γ has one of the permutation representation graphs (B15), (C1), (D1), (D2) or (F3).

For the sggis of rank 5 in the Appendix A, it can be shown that Γ_0 is not a string C-group, consequently Γ is not a string C-group. $\square$

6 When $\mathcal{G}$ has a fracture graph with a split

Let G be an even transitive group of degree 11 and rank $r \in \{3, 4, 5\}$. Notice that $r \neq 3$ otherwise $\mathcal{G}_i$ has nine edges with precisely two labels demanding that one of the permutations is odd. In addition assume that G_j is intransitive for every $j \in \{0, \ldots, r - 1\}$. Suppose that $\mathcal{G}$ has a split $\{a, b\}$ with label i. Then G_i has exactly two orbits O_1 and O_2. Let $a \in O_1$ and $b \in O_2$. For $j \neq i$, $\rho_j = \alpha_j \beta_j$ where α_j acts on O_1 and β_j acts on O_2, and $\rho_i = \alpha_i \beta_i(a, b)$ where α_i acts on O_1 and β_i acts on O_2. Let $J_A := \{j \in \{0, \ldots, r - 1\} \setminus \{i\} \mid \alpha_j \neq 1_G\}$ and $J_B := \{j \in \{0, \ldots, r - 1\} \setminus \{i\} \mid \beta_j \neq 1_G\}$. We then have $A = \langle \alpha_j \mid j \in J_A \rangle$ and $B = \langle \beta_j \mid j \in J_B \rangle$. If one of the groups is trivial then the corresponding set of indices is empty.

In what follows m_i and k_i denote, respectively, the number of blocks and the size of a block for an imprimitive action on O_i.

Proposition 6.1 ([2, Proposition 5.1])**.** *If B is primitive, then the set J_B is an interval. The same result holds for A.*

We start by considering that the $\mathcal{G}_i$ components have at least two vertices. Later we deal with the other case, where one of the components is trivial.

### 6.1 Case: $\mathcal{G}_i$ has two nontrivial components.

Proposition 6.2. *A and B cannot both be imprimitive.*

Proof. Suppose that $A \leq S_{k_1} \wr S_{m_1}$ and $B \leq S_{k_2} \wr S_{m_2}$ with k_1, k_2, m_1, $m_2 \geq 2$. Then $k_1 m_1 + k_2 m_2 = 11$. But then either $k_1 m_1$ or $k_2 m_2$ is odd. Hence $k_1 m_1 + k_2 m_2 \geq 3 \times 3 + 2 \times 2 = 13$, a contradiction. $\square$

Let us assume without loss of generality that $|O_1| < |O_2|$. In the next three propositions we consider all possibilities for the sizes of O_1, leading to the conclusion that B must be primitive. Note that this is also true when $|O_1| = 4$, as in this case $|O_2|$ equals 7, a prime number. So in what follows we need only to consider $|O_1| = 2$, $|O_1| = 3$ and $|O_1| = 5$.

**Proposition 6.3.** *If $|O_1| = 2$ then B is primitive.*

Proof. Suppose that $B \leq S_{k_2} \wr S_{m_2}$ with $k_2 = m_2 = 3$. As the i-split does not belong to a square, $J_A \subseteq \{i-1, i+1\}$. Suppose first that J_B is not an interval. Both ρ_{i-1} and ρ_{i+1} must act nontrivially on O_2. Then $\mathcal{G}$ must contain the following graph.
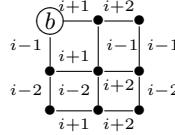
Either ρ_{i-2} or ρ_{i+2} fixes O_1, thus G contains a 3-transposition, a contradiction. Thus J_B must be an interval. Suppose, up to duality, that all labels in J_B are greater than i. If $i \neq 0$ then, as $0 \notin J_B$, ρ_0 is odd, a contradiction. Thus $i = 0$ and, by Proposition 4.1 we necessarily have $J_A = \{1\}$, thus the permutation graph has the following subgraph.
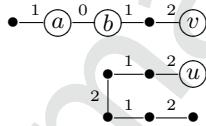
Moreover, without loss of generality, the permutation representation graph of G has an edge $\{u, v\}$ which either has label 1 or 3. If it is 1 then ρ_1 is a 5-transposition, a contradiction. If the edge $\{u, v\}$ has label 3 then G_0 has one of the following permutation representation subgraphs.
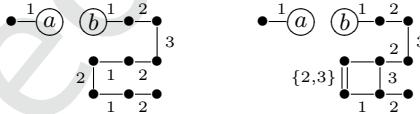

In any case ρ_3 is odd, a contradiction. With this we conclude that if $|O_1| = 2$, then B is primitive, as required. $\square$

Proposition 6.4. *If $|O_1| = 3$ then B is primitive.*

Proof. Suppose that B is an imprimitive permutation group (of degree 8). Let us assume without loss of generality that $J_A = \{i+1, i+2\}$. Then $\{i+1, i+2\} \subseteq J_B$ otherwise ρ_{i+1} and ρ_{i+2} are odd. Suppose that J_B is not an interval, then $i \in \{1, 2\}$. If $i = 1$, then $J_A = \{2, 3\}$ and, as $\langle \rho_0 \rangle$ is an intransitive normal subgroup of B, ρ_0 determines a block system for the group B, with four blocks of size two. As G_0 is intransitive, ρ_0 is the unique permutation acting non-trivially within the blocks. This forces ρ_2 and ρ_3 to be odd permutations. Thus $i = 2$ and $J_A = \{3, 4\}$. Notice that $\langle \rho_0, \rho_1 \rangle$ cannot be transitive on O_2, otherwise ρ_0 is odd. Thus, the orbits of $\langle \rho_0, \rho_1 \rangle$ determine a block system and necessarily have size greater than 2. Then $k_2 = 4$ and $m_2 = 2$. Then ρ_3 must be the unique permutation swapping the two blocks of size four. But then as ρ_3 commutes with ρ_1, ρ_3 is odd, a contradiction.

Thus J_B is an interval. Therefore $i = 0$ and $m_2 = 4$, otherwise ρ_1 is odd. We find the following possibilities for G_0 with ρ_2 and ρ_3 being even. For $r = 5$:
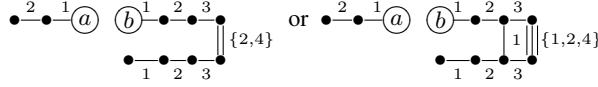

For $r = 4$:
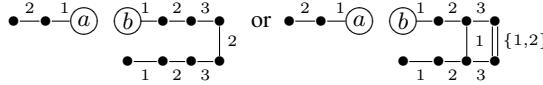
When $r = 5$, G_4 is transitive, ρ_4 is odd and ρ_1 is odd. When $r = 4$, ρ_1 is odd. In any case we have a contradiction. $\qquad\square$

**Proposition 6.5.** *If* $|O_1| = 5$ *then* B *is primitive.*

Proof. As $|O_1|$ is prime, A is primitive. By Proposition 6.1 we may assume that all labels in J_A are greater than i. Now assume that B is embedded into $S_{k_2} \wr S_{m_2}$ with $k_2, m_2 > 1$ and $k_2 m_2 = 6$.

Suppose first that $m_2 = 2$ and $k_2 = 3$. The i-split $\{a, b\}$ does not belong to a square, hence the permutation swapping the blocks is either ρ_{i-1} or ρ_{i+1}. If it is ρ_{i-1} then, as ρ_{i-1} fixes O_1, we have that ρ_{i-1} is an odd permutation, a contradiction. Thus ρ_{i+1} is the unique permutation swapping the blocks. Thus ρ_{i+1} acts as an odd permutation in both orbits, that is, both α_{i+1} and β_{i+1} are odd. Then A cannot be the even group D_{10}, hence $i + 3 \in J_A$. By Proposition 4.1 α_{i+3} must be odd, hence $i + 3 \in J_B$ and β_{i+3} is odd. But ρ_{i+3} commutes with ρ_{i+1} (the permutation swapping the blocks), which forces β_{i+3} to be an even permutation, a contradiction.

Now suppose that $m_2 = 3$ and $k_2 = 2$. If J_B is not an interval then $B \cong C_2 \times S_3$, particularly B is also embedded into $S_3 \wr C_2$. We have just concluded that this case leads to a contradiction. Thus J_B is an interval. If the labels of J_B are smaller than i then $r = 5$ and $i = 2$. Then there is only one possibility for the permutation representation graph of G_2 on the orbit O_2, which is as follows.
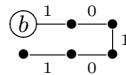
Then ρ_1 is odd, a contradiction. If all labels in J_B are greater than i, then $i = 0$. Then there are three possible permutation representation graphs of G_0 on the orbit O_2.
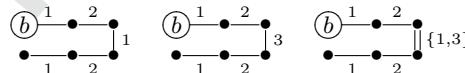
As any path in O_1 containing two 3-edges has at least six vertices, the possibility on the left can be excluded. In the other cases we get that either ρ_0 or ρ_3 are odd, a contradiction. $\qquad\square$

Now let us consider the case when B is primitive. By Proposition 6.1, J_B is an interval. Let us assume, without loss of generality, that any label in J_B is greater than i.

Proposition 6.6. $|O_1| \neq 2$.

Proof. Suppose that $|O_1| = 2$. If $i \neq 0$ then $i - 1 \in J_A$ and $i - 1 \notin J_B$. This implies that ρ_{i-1} is odd, a contradiction. Hence $i = 0$ and $J_A = \{1\}$. Let us now prove that ρ_1 is a 4-transposition. Suppose that ρ_1 is a 2-transposition. Then the orbit of $G_{>1}$ containing $b\rho_1$ has 8 vertices. As ρ_0 must act non-trivially on this orbit and ρ_0 centralizes $G_{>1}$, ρ_0 is a 5-transposition, a contradiction. Indeed, $G_{>1}$ needs to have more than two orbits on O_2.

Let us now prove that ρ_3 is a 2-transposition. Suppose that ρ_3 is a 4-transposition. As the shortest path from b including the first 3-edge must have four vertices, by Proposition 4.1, this implies that the size of $|O_2| \geq 4 + 3 \times 2 = 10$ a contradiction. Let us now consider the cases $r = 4$ and $r = 5$ separately.

$\underline{r = 5}$: By Proposition 4.1 ρ_4 must be a 2-transposition. Suppose that $\mathcal{G}$ has a $\{2, 4\}$-square. As observed before, $G_{>1}$ has more than two orbits on O_2. Hence a path from b to the $\{2, 4\}$-square must contain two 1-edges. This gives the following possibility for $\mathcal{G}_0$.
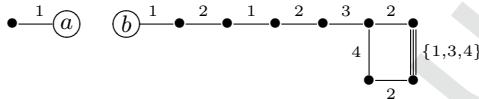
But then G_4 is transitive, a contradiction. We get the same contradiction if we admit that the graph has $\{2, 4\}$-edges. Hence, ρ_2 is a 2-transposition.

As ρ_1 is a 4-transposition and ρ_4 is a 2-transposition, $\mathcal{G}$ cannot contain a $\{1, 4\}$-square. Suppose that $\mathcal{G}$ contains a $\{1, 4\}$-square, then as ρ_4 is a 2-transposition, there exist a 3-edge incident to exactly one vertex of the $\{1, 4\}$-square. Then $\mathcal{G}$ has a $\{1, 3\}$-square sharing an 1-edge with the $\{1, 4\}$-square. Moreover, the vertex b, of the split, cannot be a vertex of that $\{1, 3\}$-square. This implies that the graph has least five 1-edges, a contradiciton.

Thus an edge adjacent to a 4-edge must have label 3. Thus $\mathcal{G}$ contains the following graph.

This implies that ρ_1 is a 3-transposition, a contradiction.

$\underline{r = 4}$: As ρ_3 is a 2-transposition, for connectedness, ρ_2 must be a 4-transposition, that means that ρ_2 is fixed-point-free in $O_2 \setminus \{b\}$. As ρ_0 must be odd in O_2, $\mathcal{G}$ must have a $\{0, 2\}$-edge. If it is adjacent to a 3-edge we get the following possibility with G_3 being transitive, a contradiction.
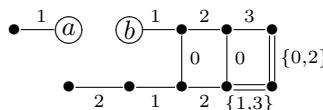

Thus the $\{0, 2\}$-edge is not adjacent to a 3-edge. If $\mathcal{G}$ contains a $\{0, 2\}$-square, then we can determine the components of $G_{1,3}$ in $O_2 \setminus \{b\}$: a $\{0, 2\}$-edge, a $\{0, 2\}$-square and a 2-edge. Thus, the 3-edges cannot connect the components above. Hence, G_3 is transitive, a contradiction. By the same reason $\mathcal{G}$ has exactly one $\{0, 2\}$-edge. Thus ρ_0 is a 2-transposition. Note also that if $\mathcal{G}$ contains a $\{1, 3\}$-square then the $\{0, 2\}$-edge must share at least one vertex with that square, otherwise, since ρ_1 is a 4-transposition, the graph is disconnected. As ρ_3 is a 2-transposition, both vertices of the $\{0, 2\}$-edge belong to the $\{1, 3\}$-square. But in this case G_3 is transitive, a contradiction. Thus $\mathcal{G}$ does not contain a $\{1, 3\}$-square. This gives only the four possibilities corresponding to graphs (A1) to (A4) in the Appendix A, which are not string C-groups, a contradiction.

□

Proposition 6.7. $|O_1| \neq 3$.

Proof. Suppose that $|O_1| = 3$. In this case J_A and J_B are both intervals. If $i \neq 0$ then $A = G_{<i}$ and $B = G_{>i}$. Moreover $G_{<i}$ is an even string C-group of degree 3, but a string C-group of degree 3 is isomorphic to S_3, a contradiction.

Thus $i = 0$ and $J_A = \{1, 2\}$. Moreover the O_1 component of $\mathcal{G}_0$ is a path. Let $O_1 = \{1, 2, 3\}$ and $\{a, b\} = \{3, 4\}$ as is the following figure.

Suppose first that $G_{0,1}$ has exactly four orbits: $\{1, 2\}, \{3\}, \{4\}$ and $\{5, \dots, 11\}$. In this case $\Gamma_{0,1}$ is a sesqui-extension with respect to ρ_2 of string C-group Λ acting transitively on $11 - 4 = 7$ points. Moreover, as $(1, 2) \notin G_{0,1}$, by Lemma 2.3, Λ is a string group representation of a group isomorphic to $G_{0,1}$ and Λ must have an index 2 subgroup (which is the even subgroup of Λ of the elements that can be written with an even number of ρ_2's). Then there is only one possibility which is $G_{0,1} \cong S_7$. In particular $r = 5$.

By Proposition 4.1, note that ρ_3 and ρ_4 must be 2-transpositions. But then for connectedness of $\mathcal{G}$, ρ_2 must be a 4-transposition. Then $\mathcal{G}$ has an $\{2, 4\}$-square. This gives the possibilities (B1), (B2) and (B3) of the Appendix A which are not string C-groups by Proposition 5.3.

Now suppose that $G_{0,1}$ has orbits: $\{1, 2\}, \{3\}, \{4\}, \{5, 6\}$ and $\{7, \dots, 11\}$. By Proposition 4.1, $r = 4$ and $\mathcal{G}$ must have the following spanning subgraph, where $G_{0,1} \cong D_{20}$.

Now $\mathcal{G}$ must have precisely one more 1-edge and at least one more 0-edge. By the commuting property ρ_0 must fix the vertices $\{7, 8, 9, 10, 11\}$, so there are only two possibilities for the other 0-edge, $\{1, 2\}$ or $\{5, 6\}$. In addition there are also two possibilities for the other 1-edge, $\{8, 9\}$ or $\{10, 11\}$. Hence $\mathcal{G}$ is one of the graphs (B4), (B5), (B6) or (B7) of the Appendix A.

Now suppose that the orbit of the vertex 5, in $G_{0,1}$ has more than two points (and less than 7) then ρ_3 does not fix the vertex 6. If both ρ_1 and ρ_3 act non-trivially on the vertex 6 then there is either a $\{1, 3\}$-square or a double $\{1, 3\}$-edge. In total this yields six possibilities for $\mathcal{G}$, the graphs (B8)-(B13) of the Appendix A.

If ρ_1 fixes the vertex 6, then we get the graphs (B14) and (B15) of the Appendix A.

Thus, in every case, Γ is not a string C-group by Proposition 5.3, a contradiction.

$\square$

Proposition 6.8. $|O_1| \neq 4$.

Proof. Suppose $|O_1| = 4$. Now suppose that J_A is not an interval. In this case A is imprimitive with two blocks of size 2. As only edges with labels $i \pm 1$ are incident to a, and remembering that J_B is an interval with labels greater than i, there are only two possibilities either $J_A = \{0, 2\}$ or $J_A = \{0, 2, 3\}$. In any case $i = 1$ and the permutation representation of G_1 restricted to O_1 is one of the following graphs.
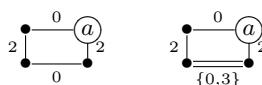
Recalling that, by Proposition 6.2, J_B is an interval, we may exclude the second possibility, for otherwise G_0 is transitive. Thus we have to consider the case $J_A = \{0, 2\}$ with A having the permutation representation graph on the left.

Now, if J_A is an interval then $i = 0$ and there are only the following three possibilities for the permutation representation of G_0 in the orbit O_1.
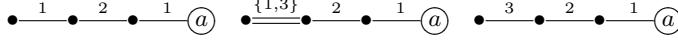
Hence, we have to consider the three cases above (when $i = 0$) plus the $\{0, 2\}$-square when $i = 1$. Let us prove that in any case $r = 4$. Suppose that $r > 4$. Then $4 \in J_B$ and the labelling set of any path from b to a 4-edge must contain the set $\{i + 1, \ldots, 3\}$. As $4 \notin J_A$ and $|O_2| = 7$, $\mathcal{G}$ must contain one of the following two paths with 7 vertices of O_2 having at least two 4-edges. In the first $i = 1$ and the second $i = 0$.


In the first case ρ_3 is odd. In the second case ρ_0 fixes O_2 pointwisely, hence it must act non-trivially on O_1. Thus we must have $J_A = \{1, 2\}$, but then ρ_1 is odd, a contradiction. Thus $r = 4$.

We now deal separately with the cases $J_A = \{0, 2\}$, $J_A = \{1, 2\}$ and $J_A = \{1, 2, 3\}$.

If $J_A = \{0, 2\}$ then $\langle \rho_2, \rho_3 \rangle$ must be transitive in O_2, which has size 7. This implies that both ρ_2 and ρ_3 are odd, a contradiction.

If $J_A = \{1, 2\}$ then ρ_3 fixes O_1, and the commuting property forces a path starting at the vertex b and containing two 3-edges, to have exactly 6 vertices. Having in mind that G is even, we find the sggi which is the permutation graph (C1) of the Appendix A and the following two permutation representation graphs.
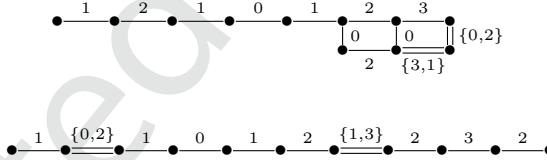
If $\mathcal{G}$ is the first graph G_3 is transitive, a contradiction. If $\mathcal{G}$ is the second graph above then it has a split with label 3 and G_3 has an orbit of size two, contradicting Proposition 6.6.

If $J_A = \{1, 2, 3\}$ then ρ_3 will swap exactly one pair of vertices of O_2. Indeed since $i = 0$ and $|O_2| = 7$, ρ_3 cannot swap three pairs of vertices of O_2. Consider the minimal path, in $\mathcal{G}$, starting at b and containing the 3-edge of O_2. This path must have 4 or 6 vertices, thanks to the commuting property. If it has 4 vertices we get the sggi (C2), (C3) or (C4) of the Appendix A. If it has 6 vertices then we get either the sggi (C5) or (C6) of the Appendix A, or the following graph which may be dismissed by Proposition 6.6 since it has a 3-split with one orbit of size two.
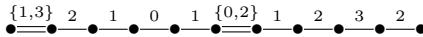
The sggis of the Appendix A do not satisfy the intersection property and so, in all of the remaining cases, Γ is not a string C-group by Proposition 5.3, a contradiction.

$\square$

Proposition 6.9. $|O_1| \neq 5$.

Proof. Since $|O_1| = 5$, then A is primitive and thus J_A is an interval. Suppose $i \neq 0$. Then since J_B only has labels greater than i, all the labels of J_A are all smaller than i. Therefore $A = G_{<i}$ and $B = G_{>i}$. This implies that $r = 5$ and the groups A and B must be dihedral. But D_{12} is odd, a contradiction. Consequently $i = 0$. Now if the rank is 5 then the permutation graph of Γ contains the following path of size 10.

$$\bullet \overset{4}{\rule{1.2em}{0.4pt}} \bullet \overset{3}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{1}{\rule{1.2em}{0.4pt}} \bullet \overset{0}{\rule{1.2em}{0.4pt}} \bullet \overset{1}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{3}{\rule{1.2em}{0.4pt}} \bullet \overset{4}{\rule{1.2em}{0.4pt}} \bullet$$
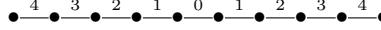
This graph cannot be a subgraph of $\mathcal{G}$, as this forces ρ_0 to be odd. Thus $r \leq 4$. As G, is even $r = 4$.

Suppose first that ρ_3 fixes O_1 pointwisely. Then we get the following permutation representation graphs of $\mathcal{G}_0$.

$$\bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{1}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{1}{\rule{1.2em}{0.4pt}} \textcircled{a} \qquad \textcircled{b} \overset{1}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{3}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \overset{\{1,3\}}{=\!=} \bullet$$

or

$$\bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{1}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{1}{\rule{1.2em}{0.4pt}} \textcircled{a} \qquad \textcircled{b} \overset{1}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{\{1,3\}}{=\!=} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{3}{\rule{1.2em}{0.4pt}} \bullet$$
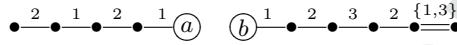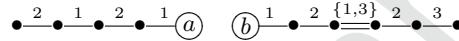
In both cases 3 is the label of a split and G_3 has one orbit of size 3 or 1, respectively, and so by Proposition 6.7 we may exclude the first of these graphs. From the second graph we get the graphs (D1) and (D2) of the Appendix A. From the second graph we get the graphs (D1) and (D2) of the Appendix A.

Now consider that ρ_3 has a non-trivial action in O_1. In this case Γ_0 has one of the following permutation representation graphs, giving graphs (D3) and (D4) of the Appendix A.

$$\bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{\{1,3\}}{=\!=} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{1}{\rule{1.2em}{0.4pt}} \textcircled{a} \qquad \textcircled{b} \overset{1}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{3}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{1}{\rule{1.2em}{0.4pt}} \bullet$$

$$\bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{3}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{1}{\rule{1.2em}{0.4pt}} \textcircled{a} \qquad \textcircled{b} \overset{1}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{1}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{\{1,3\}}{=\!=} \bullet$$

$$\bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{\{1,3\}}{=\!=} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{1}{\rule{1.2em}{0.4pt}} \textcircled{a} \qquad \textcircled{b} \overset{1}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{1}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{3}{\rule{1.2em}{0.4pt}} \bullet$$

$$\bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{3}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{1}{\rule{1.2em}{0.4pt}} \textcircled{a} \qquad \textcircled{b} \overset{1}{\rule{1.2em}{0.4pt}} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{\{1,3\}}{=\!=} \bullet \overset{2}{\rule{1.2em}{0.4pt}} \bullet \overset{1}{\rule{1.2em}{0.4pt}} \bullet$$

As before, there is a 3-split for the first, second and fourth graphs, where G_3 has either an orbit of size two or three. By Propositions 6.6 and 6.7 all the possibilities for $\mathcal{G}$ are given in the Appendix A. In any case we have a contradiction with the intersection property by Proposition 5.3.

$\square$

The case where the connected components of $\mathcal{G}_i$ are nontrivial is now completed and the conclusion is the following.

Proposition 6.10. *Let* Γ *be an even string C-group of degree* 11 *and rank* $r \in \{3, 4, 5\}$. *Suppose that* Γ *has a fracture graph. If* i *is the label of a split, then* G_i *has one trivial orbit.*

Proof. This is a consequence of Propositions 6.6, 6.7, 6.8 and 6.9. $\square$

6.2   Case: $\mathcal{G}_i$ has a trivial component.

Proposition 6.11. *If* $|O_1| = 1$ *then* B *is primitive and* $i \in \{0, r - 1\}$.

Proof. Suppose that B is embedded into $S_{k_2} \wr S_{m_2}$ with $k_2 m_2 = 10$, $k_2, m_2 > 1$. If $m_2 = 2$ then the permutation swapping the blocks is a 5-transposition, a contradiction. Hence $k_2 = 2$ and $m_2 = 5$. First suppose that $i \notin \{0, r - 1\}$. In that case $B = G_i = G_{<i} \times G_{>i}$. If $G_{>i}$ is transitive on O_2, as ρ_0 centralizes $G_{>i}$, ρ_0 is fixed-point-free on O_2. Then ρ_0 is a 5-transposition, a contradiction. Thus $G_{>i}$ is intransitive and, by the same argument, $G_{<i}$ is intransitive. If either $G_{<i}$ or $G_{>i}$ is a cyclic group, then we have the same contradiction as before. Hence both groups have two orbits of size 5. As neither $G_{<i}$ nor $G_{>i}$ is cyclic, we have that $i \notin \{1, r - 2\}$. Consider the blocks of size 5 corresponding to the $G_{<i}$-orbits. Then as ρ_{i+1} centralizes $G_{<i}$, it cannot fix the blocks which have odd size, hence ρ_{i+1} swaps the blocks. Similarly ρ_{i+2} swaps the blocks. Hence G_{i+1} and G_{i+2} are transitive, a contradiction. Consequently $i \in \{0, r - 1\}$.

Without loss of generality lets assume that $i = 0$. Note that a 0-split $\{a, b\}$ does not belong to a square, therefore $G_{>1}$ fixes b and $G_{>2}$ fixes $b\rho_1$. Hence we get the following possibilities for the graph representing the block action of G_0.
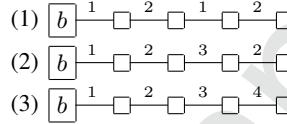
But ρ_0 must have a non-trivial action on the orbit of size 10. If ρ_0 permutes two vertices in a block, then it permutes another pair of vertices in an adjacent block, which forces ρ_0 to be odd. Thus ρ_0 swaps a pair of vertices in different blocks. This is only possible when the block action is as in (1) or (2), corresponding to the following permutation graphs of G_0 for O_2.

By the commuting property it is impossible to place an odd number of 0-edges into either of the above diagrams. Thus B is primitive. By Proposition 6.1 J_B must be an interval, hence $i \in \{0, r - 1\}$. $\square$

Proposition 6.12. *Let Γ be an even string C-group of degree 11 and rank $r \leq 5$. If Γ has a fracture graph then either $\mathcal{G}_0$ or $\mathcal{G}_{r-1}$ has a 2-fracture graph.*

Proof. Suppose that neither $\mathcal{G}_0$ nor $\mathcal{G}_{r-1}$ has a 2-fracture graph. This is only possible if both $i = 0$ and $i = r - 1$ are labels of splits. Suppose that $r = 4$ in this case $G_{0,3} \cong D_{18}$. Then G_0 and G_3 are even transitive groups of degree 10 containing D_{18}. Hence, both G_0 and G_3 contain a 9-cycle, therefore they are primitive. This is only possible if $G_0 \cong A_{10}$ and $G_3 \cong A_{10}$ [6]. But then $G_{0,3} \cong A_9$ not D_{18}, contradicting the intersection property. Thus $r = 5$. Let $\{c, d\}$ be the 4-split.

ρ_0 and ρ_4 are 2-transpositions: Suppose that ρ_0 is a 4-transposition. As ρ_0 and ρ_4 commute, $\{c, d\} \subseteq \text{Fix}(\rho_0)$. By Lemma 3.1 an edge connecting a vertex of $\overline{\text{Fix}(\rho_0)}$ with a vertex of $\text{Fix}(\rho_0)$ must have label 1. As $\mathcal{G}_4$ has a pendant 3-edge, the vertices of this edge must belong to $\text{Fix}(\rho_0)$. Hence, we have identified the three fixed points of ρ_0. But then as $\mathcal{G}$ is connnected there is a 1-edge from a vertex of $\text{Fix}(\rho_0)$ and a vertex of $\overline{\text{Fix}(\rho_0)}$. But then by the commuting property the 4-split belongs to a $\{1, 4\}$-square, a contradiction. Therefore ρ_0 is a 2-transposition and, by duality, ρ_4 is also 2-transposition.

ρ_2 is a 2-transposition: If ρ_2 is a 4-transposition then, as the three permutations ρ_0, ρ_2 and ρ_4 commute pairwisely, the vertices of the 0-split and the vertices of the 4-split must belong to $\mathrm{Fix}(\rho_2)$. Thus $\{a, b, c, d\} \subseteq \mathrm{Fix}(\rho_2)$, a contradiction.

ρ_1 and ρ_3 are 2-transpositions: Suppose that ρ_1 is a 4-transposition. By Lemma 3.1 an edge connecting a vertex of $\overline{\mathrm{Fix}(\rho_1)}$ with a vertex of $\mathrm{Fix}(\rho_1)$ must have either label 0 or 2. Then, as the 4-split $\{c, d\}$ does not belong to a square, $\{c, d\} \subseteq \mathrm{Fix}(\rho_1)$. As $\mathcal{G}_4$ has a pendant 3-edge, the vertices of this edge must also belong to $\mathrm{Fix}(\rho_1)$. However with this, there are no possibilities for a 0-split with a trivial orbit, a contradiction. Hence ρ_1 is a 2-transposition and, by duality, ρ_3 is also a 2-transposition.

As 0 and 4 are labels of splits, $\mathcal{G}$ has neither $\{0, k\}$-squares ($k \neq 0$) nor $\{4, k\}$-squares ($k \neq 4$). Consequently $\mathcal{G}$ also does not have $\{0, 4\}$-edges. Thus the 0-edges and the 4-edges have no vertices in common. But then there exists a 2-edge meeting either a 0-edge or a 4-edge. This implies that $\mathcal{G}$ either has a $\{0, 2\}$-edge or a $\{2, 4\}$-edge. Up to duality we may assume that $\mathcal{G}$ has a $\{0, 2\}$-edge. Then $\mathcal{G}_{1,3}$ is the following graph.
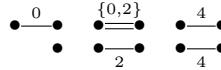
Now as $\mathcal{G}_{1,3}$ has six connected components and since $\mathcal{G}$ has exactly two 1-edges and two 3-edges, then $\mathcal{G}$ is disconnected, a contradiction.

This proves that $\mathcal{G}$ cannot have both a 0-split and an $(r-1)$-split. Consequently, by the Propositions 6.6 to 6.9, it may be assumed up to duality that $\mathcal{G}_0$ has a 2-fracture graph. $\qquad \square$

Let us consider separately the cases $r = 4$ and $r = 5$. Assume, up to duality, the $\mathcal{G}_0$ has a 2-fracture graph.

Lemma 6.13. *If $r = 4$ then $\mathcal{G}$ has exactly*

(a) *one $\{1, 3\}$-square;*

(b) *four 1-edges;*

(c) *one double $\{1, 3\}$-edge, if ρ_3 is a 4-transposition;*

(d) *four 2-edges;*

Proof. (a) Suppose that $\mathcal{G}_0$ does not have a $\{1, 3\}$-square. Then, $\mathcal{G}_0$ cannot have $\{1, 2\}$-squares either, for otherwise, any edge incident to one of the vertices of the $\{1, 2\}$-square must belong to a $\{1, 3\}$-square, a contradiction. Similarly $\mathcal{G}_0$ cannot have other squares nor double $\{1, 2\}$-edges nor double $\{2, 3\}$-edges. Thus two incident edges of $\mathcal{G}_0$ must have consecutive labels and the only admissible double edges of $\mathcal{G}_0$ have label-set $\{1, 3\}$. We have that ρ_2 is a 4-transposition, otherwise we would have $\{1, 3\}$-squares. Let us prove that ρ_1 is also a 4-transposition. Suppose that ρ_1 is a 2-transposition. As $\mathcal{G}$ is connected, $\mathcal{G}_0$ has at least nine edges, hence ρ_3 is a 4-transposition. Then there exists a 1-edge meeting a 3-edge, which is only possible if we have a double $\{1, 3\}$-edge, contradicting Lemma 4.2. Thus ρ_1 is a 4-transposition and, by similar arguments, we may also conclude that ρ_3 is a 4-transposition. Since there are no $\{1, 2\}$-double edges nor $\{2, 3\}$-double edges, there are three double $\{1, 3\}$-edges. Hence 1 and 3 are labels of splits, a contradiction. Hence, $\mathcal{G}$ contains a $\{1, 3\}$-square.

Let us prove uniqueness. Suppose that there are two $\{1,3\}$-squares. Recall that the 0-split of $\mathcal{G}$ is adjacent to a pendant 1-edge of $\mathcal{G}_0$. The existence of two $\{1,3\}$-squares and a pendant 1-edge, implies that $\mathcal{G}_0$ has at least five 1-edges, which is clearly is not possible.

(b) As $\mathcal{G}_0$ has a pendant 1-edge and $\mathcal{G}$ has a $\{1,3\}$-square, we conclude that $\mathcal{G}$ has exactly four 1-edges.

(c) Suppose that ρ_3 is a 4-transposition. Then, as ρ_1 is also a 4-transposition, $\mathcal{G}_0$ has at least $8 + 8 - 10 = 6$ vertices that belong to both a 3-edge and a 1-edge. Since ρ_1 commutes with ρ_3, and that there can only be one $\{1,3\}$-square, then there must exist exactly one $\{1,3\}$-double edge.

(d) From (a)-(c) we may conclude that the orbits of $\mathcal{G}_{0,2}$ acting on O_2 are one of the following
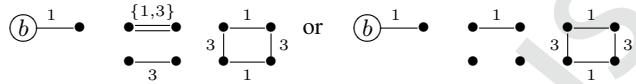
Then, there must be at least three 2-edges connecting the orbits. Hence, ρ_2 is a 4-transposition.

□

Proposition 6.14. *Let $r = 4$. The non-trivial connected components of $\mathcal{G}_{0,1}$ are either as in* (1) *or as in* (2).
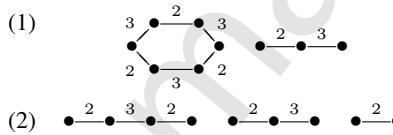
Proof. The group $\tilde{G} = \langle \beta_0, \rho_1, \rho_2, \rho_3 \rangle$ is a transitive group on 10 points and $\tilde{G}_0 = G_0$ is transitive. Hence Lemmas 3.2 and 3.4 can be used to restrict the sizes of the orbits of $G_{0,1}$.

Let s denote the size of the largest connected component of $\mathcal{G}_{0,1}$. Let us consider separately all the possibilities for s. Recall that the 0-split $\{a, b\}$ of $\mathcal{G}$ must be adjacent to a 1-edge, to be precise, $\mathcal{G}_0$ has a pendant 1-edge. Furthermore $\mathcal{G}_{0,1}$ has a pendant 2-edge and fixes the points a and b. Hence $s \notin \{10, 11\}$. Moreover, as ρ_0 is an even permutation, $s \notin \{8, 9\}$. In addition $s > 2$ for otherwise ρ_2 and ρ_3 would commute, a contradiction.

$\underline{s = 7}$: In this case the non-trivial components of $\mathcal{G}_{0,1}$ are an alternating path with the sequence of labels $(2, 3, 2, 3, 2, 3)$ and a double $\{2,3\}$-edge. Since ρ_3 is a 4-transposition, by Lemma 6.13 (a) and (c), $\mathcal{G}$ has a $\{1,3\}$-square and a double $\{1,3\}$-edge. Moreover the $\{1,3\}$-square must have the double $\{2,3\}$-edge, which implies that $\mathcal{G}$ either has a split with label 3 or G_3 is transitive, a contradiction.

$\underline{s = 6}$: The largest orbit is either a path or a hexagon. Suppose first it is a path. Then $\mathcal{G}_{0,1}$ has three isolated vertices and the following non-trivial components.
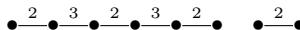
Then the 1-edges of the unique $\{1,3\}$-square of $\mathcal{G}$ are between vertices of the path. But then $\mathcal{G}$ must have at least another three 1-edges to connect the remaining components (besides the 0-split), a contradiction.

Now suppose that the largest component is a hexagon. Then, as we need a pendant edge labelled 2, the non-trivial components of $\mathcal{G}_{0,1}$ are as in (1).

$\underline{s = 5}$: In this case we have the following possibilities for the non-trivial components of $\mathcal{G}_{0,1}$.

(a)   (b)

(c)
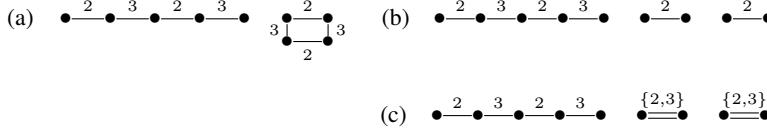
In (a) there is only one even component of size $4 \not\equiv 2 \mod 4$, a contradiction. In (b), similar to the case when $s = 6$, the existence of a $\{1,3\}$-square and the connectness of $\mathcal{G}$ forces the existence of at least five 1-edges, a contradiction. In (c) the 1-edges connecting these components must belong to at least two $\{1,3\}$-squares, a contradiction.

$\underline{s = 4}$: The largest component of $\mathcal{G}_{0,1}$ must be either a square or a path. Assume first it is a square. If $\mathcal{G}_{0,1}$ has another component of size 4, then the action ρ_0 is odd, a contradiction. Now, as $\mathcal{G}$ has four 2-edges, $\mathcal{G}_{0,1}$ has exactly three non-trivial components. Hence the possibilities are as follows.

(a)   (b)   (c)
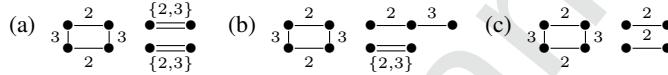
In (a) and (b) the 1-edges between these components, will induce more than one $\{1,3\}$-square, a contradiction. In (c) the existence of a $\{1,3\}$-square implies that $\mathcal{G}$ is disconnected, a contradiction.

Now suppose that the largest component is a path. Since, by Lemma 6.13 (d), $\mathcal{G}$ has four 2-edges, the path must have the sequence of labels $(2, 3, 2)$. Thus $\mathcal{G}$ has only two 3-edges, the ones belonging to the $\{1,3\}$-square (that must exist by Lemma 6.13 (a)). If there is another path with four vertices, then the action of ρ_0 is odd, a contradiction. Moreover, if there is a $\{2,3\}$-edge, then by Lemma 4.2 either $\mathcal{G}$ has a 3-split or G_3 is transitive, a contradiction. Then there is only one possibility corresponding to the graph (2) of this proposition.

$\underline{s = 3}$: Lastly, if the largest orbit has three points, and since ρ_2 is a 4-transposition, we have one of the following possibilities for the non-trivial components of $\mathcal{G}_{0,1}$.

(a)   (b)

In both cases, by Lemma 4.2 either there is a 3-split or G_3 is transitive, a contradiction.

Hence, the only possibilities are the ones stated in this proposition.   □

In what follows we analyse the situation when Γ has rank five.

Lemma 6.15. *Let $r = 5$. The permutation representation graph $\mathcal{G}$ has*

(a) *exactly two 4-edges, no double edges with label 4 neither $\{3,4\}$-squares;*

(b) *exactly two 3-edges and no double edges having with label 3;*

(c) *a $\{2,4\}$-square and four 2-edges.*

Proof. (a) If there are four 4-edges a minimal path in $\mathcal{G}$ starting in the vertex a and containing the four 4-edges must have size at least $4 + 8 = 12$ by Lemma 4.1. This gives a contradiction. By Lemma 4.2, $\mathcal{G}$ does not have double $\{i, 4\}$-edges ($i \neq 4$). If there is a $\{3, 4\}$-square, then ρ_3 and ρ_4 would commute, a contradiction.

(b) Suppose that ρ_3 is a 4-transposition. Consider first that ρ_1 is a 2-transposition. As ρ_1 commutes with ρ_3 and $\mathcal{G}_0$ has a pendant 1-edge, then $\mathcal{G}$ has a $\{1, 3\}$-double edge, but then by Lemma 4.2 $\mathcal{G}$ has a 1-split, a contradiction. Hence, ρ_1 is a 4-transposition. As ρ_1 commutes with ρ_3 and they move at most 10 points, then there are at least $6 = 8 + 8 - 10$ vertices moved by both ρ_1 and ρ_3. If there are no $\{1, 3\}$-squares, there are three $\{1, 3\}$-double edges, but then $\mathcal{G}$ has a 1-split and a 3-split, a contradiction. This shows that $\mathcal{G}$ has a $\{1, 3\}$-square. As $\mathcal{G}_0$ has a pendant 1-edge, there also exists exactly one $\{1, 3\}$-double edge. This determines the graph $\mathcal{G}_{0,2,4}$ which has exactly five components: a $\{1, 3\}$-square, a $\{1, 3\}$-double edge, a 1-edge, a 3-edge and a single vertex. Now by (a) the two 4-edges must connect vertices in different components of $\mathcal{G}_{0,2,4}$. This gives the following possibilities for the non-trivial components of $\mathcal{G}_{0,2}$.
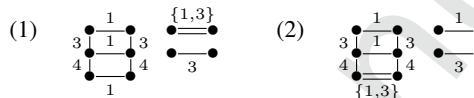

In (1) $\mathcal{G}_0$ does not have a pendant 1-edge, and in (2) there exists a 1-split. In both cases we get a contradiction. This shows that ρ_3 is a 2-transposition. By Lemma 4.2, the label 3 cannot be one of the labels of a double edge.

(c) Suppose that there is no $\{2, 4\}$-square. Let us first deal with the case where $\mathcal{G}$ has a $\{1, 4\}$-square. Since the ρ_3 and ρ_4 are 2-transpositions and cannot commute with each other, then the non-trivial component of $\mathcal{G}_0$ is one of the following two graphs.
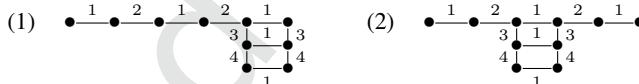
In both cases ρ_1 is odd, a contradiction.

Now consider that $\mathcal{G}$ does not have $\{i, 4\}$-squares for $i \in \{1, 2, 3\}$. The $\{1, 3\}$-squares are also forbidden otherwise there must exist a 4-edge incident to a vertex of this square (recall that ρ_3 is a 2-transposition and ρ_3 cannot commute with ρ_4), but then $\mathcal{G}$ has an $\{1, 4\}$-square, a contradiction. By a similar argument $\mathcal{G}$ does not have $\{2, 3\}$-squares. Keep in mind that by (a) and (b), $\mathcal{G}$ does not have double edges containing the labels 3 or 4. Hence $\mathcal{G}_0$ restricted to O_2 must be one of the following graphs.


But in these cases neither $\mathcal{G}_0$ has a pendant 1-edge, nor is ρ_2 an even permutation, a contradiction. This proves the existence of a $\{2, 4\}$-square.

Now suppose that ρ_2 is a 2-transposition. Since ρ_1 cannot commute with ρ_2, there must exist a 1-edge incident to a vertex of the $\{2, 4\}$-square but then there exists another $\{1, 4\}$-square, making ρ_4 is a 4-transposition, contradicting (a).

Consequently ρ_2 is a 4-transposition.

$\square$

Proposition 6.16. *Let $r = 5$. The non-trivial components of $\mathcal{G}_{0,1}$ are either as in* (a), (b) *or* (c).
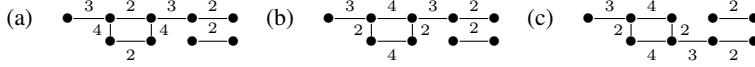
Proof. The group $\tilde{G} = \langle \beta_0, \rho_1, \rho_2, \rho_3, \rho_4 \rangle$ is a transitive group on 10 points and $\tilde{G}_0 = G_0$ is transitive. Hence Lemmas 3.2 and 3.4 can be used to restrict the sizes of the orbits of $G_{0,1}$. Let s be the size of the largest orbit of $\mathcal{G}_{0,1}$. Recall that a and b (the vertices of the 0-split) are isolated vertices of $\mathcal{G}_{0,1}$. Hence $s \leq 9$.

$\underline{s = 4}$: In this case the $\{2,4\}$-square (that exists by Lemma 6.15 (c)) determines a maximal component of $\mathcal{G}_{0,1}$. As, by Lemma 6.15 (a), ρ_4 is a 2-transposition and the $\{2,4\}$-square must be connected to the rest of the permutation graph, this can only happen via a 3-edge. But then $s > 4$, a contradiction.

$\underline{s = 5}$: By what we have proved in the previous case, we have the following possibilities for the largest orbit of size 5.
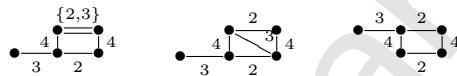

In the first case, $\mathcal{G}$ has a $\{2,3\}$-edge, contradicting Lemma 6.15 (b). In the second graph G_4 is transitive, a contradiction. Hence, we may assume the largest orbit of size 5 is given by the third graph. Since we cannot have an orbit with a $\{2,3\}$-double edge (by Lemma 6.15 (b)) and we need a pendant edge with label 2, then the non-trivial components of $\mathcal{G}_{0,1}$ are as follows.
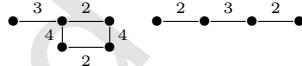

By similar arguments to the ones given by Lemmas 3.3 and 3.4, we get that $\mathcal{G}$ has an odd number of 0-edges, contradiction.

$\underline{s = 6}$: In this case the largest component must contain the $\{2,4\}$-square. If the 3-edges are both incident to the $\{2,4\}$-square then we get the following possibilities for that component of size 6 (keep in mind that ρ_2 is a 4-transposition).
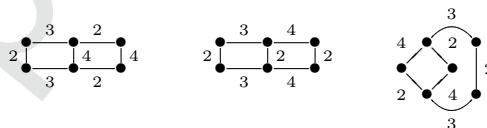

But then G_4 is transitive or, by Lemma 6.15 (b), ρ_2 and ρ_3 commute. In any case we get a contradiction.

Now suppose that there exists only one 3-edge incident to $\{2,4\}$-square. Having in mind that $\mathcal{G}_0$ does not have fracture graph and does not have $\{2,3\}$-edges (by Lemma 6.15), then the largest component of $\mathcal{G}_{0,1}$ is as follows.
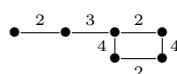
But now, as by Lemma 6.15 (b) $\mathcal{G}$ does not have double $\{2,3\}$-edges and G_4 is not transitive, the non-trivial orbits of $\mathcal{G}_{0,1}$ are as follows.
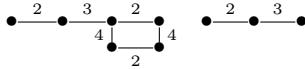
If ρ_0 acts non-trivially on these orbits, then it can only happen on the largest orbit. However, this implies that $\mathcal{G}$ has a double $\{0,3\}$-edge, a contradiction by Lemma 6.15 (b). Hence, ρ_0 is odd, a contradiction.

$\underline{s = 7}$: The largest orbit with size 7 can be either the ones presented in the statement of this proposition or the following.
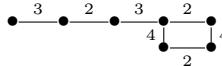

In any case the other non-trivial component of $\mathcal{G}_{0,1}$ is a single 2-edge. In the case above it is not possible to connect the two components of $\mathcal{G}_{0,1}$ with 1-edges (recall that the graph has exactly two 4-edges and two 3-edges by Lemma 6.15).

$\underline{s \in \{8,9\}}$: In this case ρ_0 is odd.

The only possibilities for $\mathcal{G}_{0,1}$ are the ones stated in the proposition. $\qquad\square$

Proposition 6.17. *Let G be an even group of degree 11 and rank $r \leq 5$. If Γ has a fracture graph then either Γ has a 2-fracture graph or $\mathcal{G}$ is as follows.*
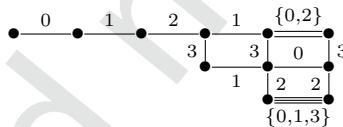
Proof. If Γ does have a split then either 0 or $r-1$ is a label of a split but not both by Propositions 6.10 and 6.12. Suppose without loss of generality that $\mathcal{G}_0$ has a 2-fracture graph. By Propositions 6.13 and 6.14 the only possibilities when the rank of Γ is equal to 4 are the sggi's (E1)-(E10) of the Appendix A which are not string C-groups, or Γ has the permutation representation graph given in the statement of this theorem.

If $r = 5$, then by Propositions 6.15 and 6.16 we find the possibilities (E11), (E12) and (E13) of the Appendix A which again are not string C-groups, by Proposition 5.3, a contradiction. $\qquad\square$

Later we will see that the permutation representation graph given in the above proposition corresponds to the 11-cell.

7   When $\mathcal{G}$ has a 2-fracture graph

In this section we assume that $\Gamma = (G, \{\rho_0, \ldots, \rho_r\})$ with $r \in \{4, 5\}$, is a string C-group representation for an even transitive group of degree 11 having a permutation representation graph $\mathcal{G}$ that admits a 2-fracture graph.

In Proposition 4.9 of [1] the authors give a classification of the string C-groups of degree n admitting a 2-fracture of rank at least $(n-1)/2$. If Γ has rank 5, then a 2-fracture graph has exactly 10 edges and 11 vertices. Then if it is connected it is a tree, otherwise

there is a 2-fracture graph having exactly two components, one being a tree and the other one having an alternating square [2, Proposition 4.12]. Having this in mind it is possible to find all possibilities for $\mathcal{G}$. This was precisely the idea behind the classification given in Proposition 4.9 of [1]. A consequence of this is the following.

Proposition 7.1. *If $\mathcal{G}$ has a 2-fracture graph, then $r \neq 5$.*

We now consider the case $r = 4$.

Lemma 7.2. *Suppose that ρ_i is a 4-transposition. If ρ_j swaps a pair of vertices of $\mathrm{Fix}(\rho_i)$ and $|j - i| \neq 1$ then $\mathcal{G}$ has an $\{i, j\}$-edge and an $\{i, j\}$-square. In particular, ρ_j is a 4-transposition.*

Proof. In this case $|\mathrm{Fix}(\rho_i)| = 3$. As ρ_j is even, by Lemma 3.1, ρ_j must swap an odd number of pairs of vertices in $\overline{\mathrm{Fix}(\rho_i)}$. The rest follows from the fact that $\mathcal{G}$ has a 2-fracture graph. □

Proposition 7.3. *If $\mathcal{G}$ has a 2-fracture graph, then $r \neq 4$.*

Proof. The connectedness of $\mathcal{G}$ implies that the generating set of G contains at least one permutation which is a 4-transposition, suppose first it is ρ_0. Let $\mathrm{Fix}(\rho_0) = \{A, B, C\}$. In this case, ρ_1 is the only generator which may interchange a vertex of $\mathrm{Fix}(\rho_0)$ with a vertex of $\overline{\mathrm{Fix}(\rho_0)}$. Hence up to a relabelling of the points that are fixed by ρ_0, we have three cases: $\{A, B\}$ is a 3-edge (Case 1); $\{A, B\}$ is a 2-edge (Case 2); the vertices A, B and C are fixed by ρ_2 and ρ_3 (Case 3). Let us deal with each case separately.

Case 1: By Lemma 7.2 $\mathcal{G}$ has $\{0, 3\}$-square, a $\{0, 3\}$-edge and another 0-edge that we denote by $\{D, F\}$. Thus there exists a 2-fracture graph of $\mathcal{G}$ containing the $\{0, 3\}$-square. The $\{0, 3\}$-edge must (for transitivity) be connected to other vertices of the graph. It cannot be at distance one from the $\{0, 3\}$-square for otherwise the edges of the $\{0, 3\}$-square would not belong to a 2-fracture graph. If it is at distance one from the other 0-edge $\{D, F\}$ then it must be via a 2-edge. Then we get $\{0, 2\}$-square (with an extra 3-edge) and a $\{0, 3\}$-square that cannot be at distance one from each other. Thus both squares must be at distance one (via a 1-edge) from one of the vertices of the set $\{A, B, C\}$ as shown in the following graph, on the left, where the dashed line can only be crossed by 1-edges.
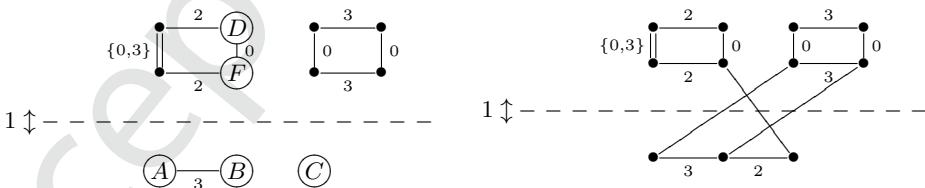
Now C cannot be at distance one from $\{0, 3\}$-square thus, for connectedness we get the graph on the right and no more edges can be added for otherwise $\mathcal{G}$ does not have a 2-fracture graph. Hence ρ_1 and ρ_2 are odd, a contradiction. Hence the double $\{0, 3\}$-edge cannot be part of a $\{0, 2\}$-square and only A and B are at distance one from the $\{0, 3\}$-edge.

This forces the existence of 1-edges from the $\{0, 3\}$-edge to vertices of $\{A, B, C\}$. That is only possible if $\mathcal{G}$ has a $\{1, 3\}$-square containing the edge $\{A, B\}$. Now connectedness

implies $\mathcal{G}$ contains the following graph where the edges that cross the dashed lines have label 1.
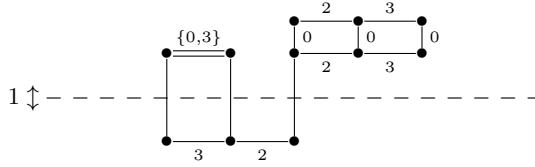
As G is even, there must be another single 1-edge, forming a $\{1,3\}$-edge, but then G_3 is transitive, giving a contradiction with the fact that Γ has a fracture graph.

<u>Case 2</u>: Let $\{A, B\}$ be a 2-edge of $\mathcal{G}$. By Lemma 7.2 $\mathcal{G}$ has a $\{0,2\}$-square and a $\{0,2\}$-edge. Moreover the $\{0,2\}$-square belongs to a 2-fracture graph. As Case 1 gives a contradiction we may assume that A, B, $C \in \text{Fix}(\rho_3)$. As $\mathcal{G}$ has a 2-fracture graph, $\mathcal{G}$ must have exactly two 3-edges. This gives the following two possibilities for $\mathcal{G}_1$.
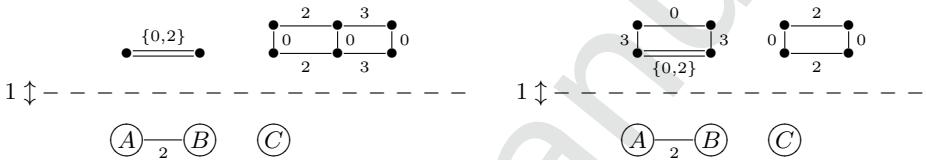
Transitivity implies that $\mathcal{G}$ has at least four 1-edges (three of those connecting the graph). In addition the 1-edges connecting different components cannot be adjacent to the 3-edges. Then, in the case on the left, evenness implies the existence of a $\{1,3\}$-edge, making G_3 transitive. In the case on the right, the $\{0,3\}$-square cannot be connected to the rest of the graph. Each case leads to a contradiction.

<u>Case 3</u>: In this case A, B and C are vertices of degree one in $\mathcal{G}$. Thus $\mathcal{G}$ has at least three 1-edges (crossing the dotted line in the graph below). As ρ_3 commutes with ρ_1, the 3-edges cannot intersect the 1-edges which cross the dotted line, thus ρ_3 is a 2-transposition. Hence we get the following subgraph of $\mathcal{G}$.
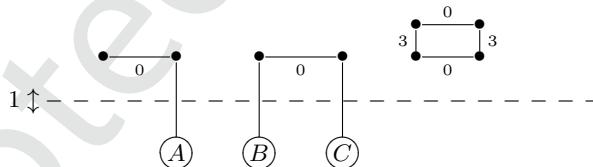
But the existence of a fourth 1-edge, implies that $\mathcal{G}$ has a $\{1,3\}$-edge, which by Lemma 4.2 and the fact the $\mathcal{G}$ has a 2-fracture graph leads to a contradiction.

Hence, the above cases contradict that ρ_0 is a 4-transpositions. Thus ρ_0 is a 2-transposition, and by duality ρ_3 is also a 2-transposition. If ρ_2 is a 2-transposition, then, for transitivity, ρ_1 must be a 4-transposition. In this case $\mathcal{G}$ has exactly ten edges, thus it is a tree. Then $|\text{Fix}(\rho_1)| = 3$ and only ρ_0 and ρ_2 can swap vertices between $\text{Fix}(\rho_1)$ and $\overline{\text{Fix}(\rho_1)}$. But in order to avoid squares and double edges, ρ_3 fixes $\overline{\text{Fix}(\rho_1)}$ pointwisely, thus ρ_3 cannot be a 2-transposition, a contradiction. Hence ρ_1 and ρ_2 are 4-transpositions. Therefore ρ_1 and ρ_3 must both act non-trivially on at least one point $(8+4-11=1)$. Thus by Lemma 4.2 $\mathcal{G}$ has a $\{1,3\}$-square. Similarly $\mathcal{G}$ has a $\{0,2\}$-square. Let d be the distance between these two squares. We have that $d \in \{2, 4\}$. If $d = 2$ we get the first nine graphs of (F1) to (F9) of the Appendix A. If $d = 4$ we get only one possibility corresponding to graph (F10) of the Appendix A. In any case Γ is not a string C-group by Proposition 5.3, a contradiction. $\square$

8   A classification of even transitive string C-groups of degree 11

The only transitive even groups of degree 11 that we need to consider are $\mathrm{PSL}_2(11)$, M_{11} and A_{11}, as those are the unique ones that can be generated by involutions.

Proposition 8.1. *If $r \in \{4,5\}$ and $G \cong \mathrm{PSL}_2(11)$ then Γ is the abstract regular 4-polytope known as the 11-cell and has the permutation representation graph given in* Theorem 6.17.

Proof. The proper even transitive subgroups of $\mathrm{PSL}_2(11)$ are not generated by involutions [6]. Thus if $G \cong \mathrm{PSL}_2(11)$ then $\mathcal{G}$ has a fracture graph. Moreover, by Propositions 7.1 and 7.3, $\mathcal{G}$ has a split.

The 11-cell is a well known 4-polytope whose automorphism group is $\mathrm{PSL}_2(11)$ [7]. Consequently, the graph given in Proposition 6.17 must be the permutation representation graph of the 11-cell. $\square$

From Propositions 6.17, 7.1, 7.3 and 8.1 we have the following.

Corollary 8.2. *If G is an even transitive string C-group of degree 11 with a fracture graph of rank $r \in \{3,4,5\}$ then one of the following two situations occurs:*

- *Γ is the abstract regular 4-polytope known as the 11-cell;*

- *Γ has rank 3 and has a 2-fracture graph.*

The classification of abstract regular polyhedra (reflexible maps) for $\mathrm{PSL}_2(11)$ can be found in various atlases that are available online, either related to polytopes or to maps, particularly in [12] and [14].

Lemma 8.3. *There are, up to duality, three abstract regular polyhedra for $PSL_2(11)$.*

Let us now determine the faithful transitive permutation representations graphs of the abstract regular polyhedra for $PSL_2(11)$.

Proposition 8.4. *There are, up to duality, four abstract regular polytopes for $\mathrm{PSL}_2(11)$ with rank $r \in \{3,4\}$. Their (five) faithful transitive permutation representation graphs are, up to duality, given in the following table.*

Type	Permutation representation graphs	
$\{5,5\}$		(1)
$\{5,6\}$		(2)
		(3)
$\{6,6\}$		(4)
$\{3,5,3\}$		(5)

Proof. As $\mathrm{PSL}_2(11)$ is simple and has exactly two distinct conjugacy classes of subgroups of index 11 (which are isomorphic to A_5), there are exactly two faithful transitive permutation representation of $\mathrm{PSL}_2(11)$ on 11 points. The graphs (1) and (4) are the graphs for the regular polyhedra of types $\{5, 5\}$ and $\{6, 6\}$ respectively [12]. The other permutation representation graphs of these polyhedra are obtained by interchanging the labels 0 and 2, that is, the dual graph of the ones presented.

The regular polyhedron of type $\{5, 6\}$ has the permutation representations graphs (2) and (3). As only (2) can be found in [12], we obtained the graph (3) using the Todd-Coxeter Algorithm [13].

The graph (5) is the permutation representation graph of the unique rank 4 string C-group by Proposition 6.17 and Corollary 8.2. Similarly to what happens with the polyhedron of type $\{5, 5\}$ and $\{6, 6\}$, the other faithful permutation representation is the dual of (5). □

At this point, it remains to consider the cases where G is either A_{11} or M_{11}. From Corollary 8.2, G_i must be transitive for some i. Let us consider the case where G_i is $\mathrm{PSL}_2(11)$.

Lemma 8.5. *Let* $r \in \{4, 5\}$. *If* G *is any transitive even group of degree* 11 *of rank* r, *then* G_i *cannot be* $\mathrm{PSL}_2(11)$.

Proof. By the commuting property $G_i = G_{<i} \times G_{>i}$ for $i \in \{1, \ldots, r - 2\}$. Thus only G_0 or G_{r-1} can be simple groups. For a contradiction, and up to duality, let us assume that $G_0 \cong \mathrm{PSL}_2(11)$. Then we find ten possibilites for $\mathcal{G}_0$ corresponding to the five permutation representation graphs of Proposition 8.4 and their duals. We now use Lemmas 3.3 and 3.4 to reduce the possibilities due to the sizes of the orbits of $G_{0,1}$. In the following table we list the possibilities for the sizes of the connected components of $\mathcal{G}_{0,1}$. For each of the graphs of Proposition 8.4 we need also to consider the duals, for this reason each cases gives two possibilities.

Graph	Sizes of the $\mathcal{G}_{0,1}$-components	Dual of Graph	Sizes of the $\mathcal{G}_{0,1}$-components
(1)	$1, 5, 5$	(1)	$1, 5, 5$
(2)	$2, 3, 6$	(2)	$1, 5, 5$
(3)	$2, 3, 6$	(3)	$1, 5, 5$
(4)	$2, 3, 6$	(4)	$2, 3, 6$
(5)	$1, 10$	(5)	$5, 6$

The cases where the sizes of the orbits are $1, 5, 5$ are excluded by Lemma 3.3. The case where the orbits have sizes $1, 10$ or $5, 6$ are excluded by Lemma 3.4 and by the impossibility of having an even ρ_0. The remaining cases are those where the G_1-orbits have sizes $2, 3, 6$. In these cases, by Lemma 3.4, ρ_0 must be fixed-point-free on the orbits of even size. The graph (2) and (3) give the permutation representation graphs (A) and (B) below, respectively. Taking the graph (4), we get, up to duality, the permutation representation graph (C).
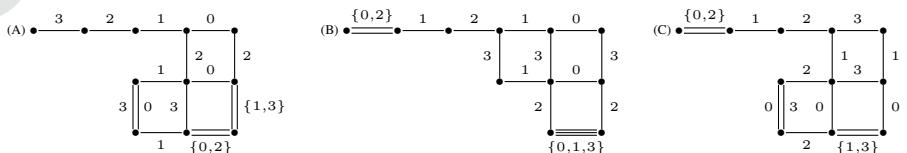
In any case the $\rho_0 = (\rho_3\rho_2)^3 \in G_0$, contradicting the intersection property. □

Corollary 8.6. *There are no abstract regular polytopes of rank* 4 *and* 5 *for* M_{11}.

The non-existance of abstract regular polyhedra for M_{11} is a consequence of the following result.

Lemma 8.7 ([15]). *None of the groups* $G \in \{M_{11}, M_{22}, M_{23}, M^cL\}$ *has a generating set of three involutions two of which commute.*

To summarize, we have proved the following result.

Theorem 8.8. *There is exactly one abstract regular polytope of rank* $r \in \{4, 5\}$ *for an even transitive group of degree* 11*, namely the* 11*-cell, which is self-dual and of rank* 4*.*

Proof. This is a consequence of Propositions 8.2, 8.5 and 8.6. □

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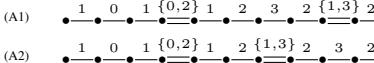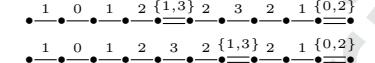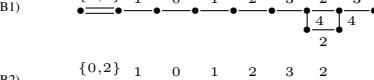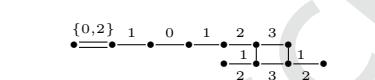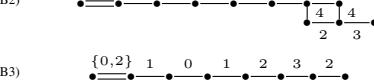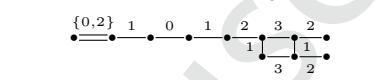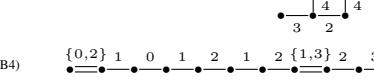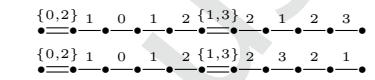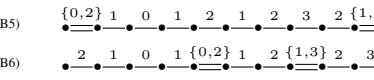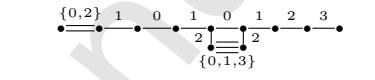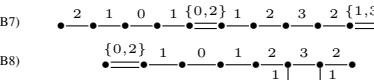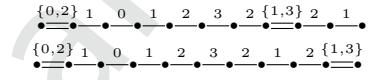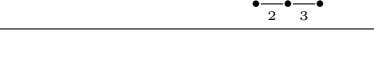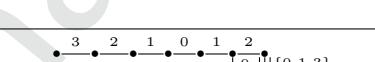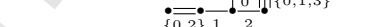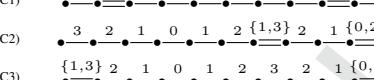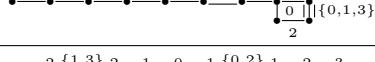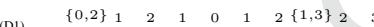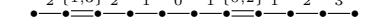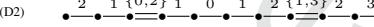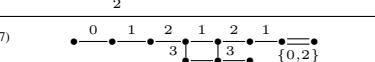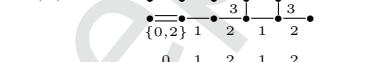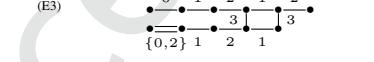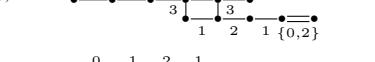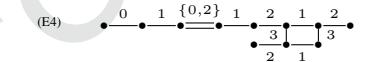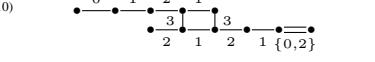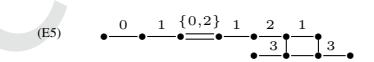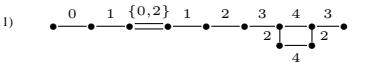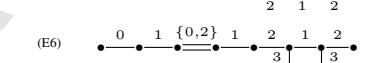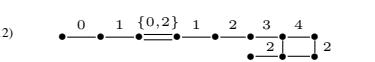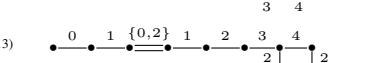
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A Appendix: Table of sggi's for A_{11}

(Continued.)

(F1)   2   1
       0   0   3   3
     1   2   1   2   1   2

(F6)   0   3
       2   2   1   1
     1   0   1   2   3   2

(F2)   2   1   2
       0   0   3   3
     1   2   1   2   1

(F7)   2   1
       0   0   3   3
     2   1   2   1   2   1

(F3)   2   3
       0   0   1   1
     1   2   1   2   3   2

(F8)   2   1   2
       0   0   3   3
     2   1   2   1   2   1

(F4)   1   2   1   2
       0   0   3   3
     2   1   2   1

(F9)   2   3
       0   0   1   1
     2   1   2   1   3   2   1

(F5)   1   2   3
       0   0   1   1
     2   1   2   3   2

(F10) 2   1
       0   3   3
     2   1   2   1   2   1