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A computer-free classification of orientably-regular maps on surfaces of genus $p + 1$ for prime p*

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Abstract

A classification of orientably-regular maps of genus $p + 1$ for primes $p > 13$ was published in 2010 by M. Conder, J. Širáň and T. Tucker. This involved a computer-free proof for primes $p > 83$, while for $17 \leq p \leq 83$ it followed from computations undertaken in 2006 by M. Conder. Classification of such maps for $p \leq 7$ was already available from much earlier work by others on such maps of small genus dating back to the 1930s, but without explicit proofs in a number of cases.

In this paper we give a computer-free classification of orientably-regular maps of genus $p + 1$ for all primes $p \leq 83$, complementing the 2010 work by M. Conder, J. Širáň and T. Tucker for all larger primes p.

Keywords: Regular map, classification of orientably regular maps.

Math. Subj. Class.: 57M60, 05E18, 20B25, 20F05, 57M15

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1 Introduction

An *orientably-regular map* is a 2-cell embedding of a finite, connected graph or multigraph on a compact orientable surface, such that the group of all orientation-preserving automorphisms of the embedding acts transitively and hence regularly on the arcs of the graph. Classical examples on a sphere are the five Platonic maps, which all admit also orientation-reversing automorphisms (reflections) and hence are reflexible, and fully regular. On a torus, one can take as an example the famous Heawood trivalent map consisting of seven hexagons; this is an orientably-regular map admitting no reflection, and hence is 'chiral'). Orientably-regular maps may be viewed as generalisations of these maps to those on orientable surfaces of arbitrary genus.

Interest in orientably-regular maps goes back more than a century. For this and further historical background we refer to the survey [25], and continue to describe one of the central problems in this field of research, namely the classification of orientably-regular maps on an orientable surface of given genus g.

For $g = 0$ (the spherical case), all orientably-regular maps are reflexible, and besides the equatorial maps (embedding cycles) and their duals (the 'polar' maps), the only non-trivial examples are the five Platonic maps. For $g = 1$ (the Euclidean or toroidal case), the classification has been known for about a century, and was described in detail by Coxeter and Moser in [13], for example. In this case there are infinitely many orientably-regular maps of each of the three possible types (namely $(3, 6)$, $(4, 4)$ and $(6, 3)$), and for each such type, infinitely many of these toroidal maps are reflexible and infinitely many are chiral.

For genus $g > 1$ (the hyperbolic case), however, the problem is much more open. Below we briefly summarise the history of the classification of orientably-regular maps of given genus g for small values of $g \geq 2$. For genus 2, a first attempt appears to be that of Erréra [14] a hundred years ago, carried further by Brahana [4] who left out one case, and the analysis was eventually completed by Threlfall [24]. A classification for genus 3 had to wait for almost three decades until the appearance of a paper by Sherk [22]. About a decade later, classifications for genera 4, 5 and 6 were published by Garbe [15], followed another two decades later by a classification for genus 7 by Bergau and Garbe [3].

Here it is important to note that the proofs given in [4], in combination with what was done in [24] for $g = 2$ and in [22] for $g = 3$, are explicit and complete, based on a mixture of geometric and group-theoretic considerations. This, however, is not the case for the next four genera 4 to 7. Although Tables II, III and IV of [15] contain complete lists of all orientably-regular maps of genus 4, 5 and 6, only a schematic description of what needs to be done to identify suitable normal subgroups of appropriate triangle groups to obtain orientably-regular maps of those genera is given, on page 42 in Section 3 of [15], along with a hint about using the Reidemeister method (with reference to [20]) in the process. Not a single worked example demonstrating the method in practice is presented there, although the paper does contains a classification of all orientably-regular maps with at most 6 faces. A similar comment applies to the paper [3], which includes (among other things) a complete table of orientably-regular maps of genus 7, but again with no worked example, although this time the paper includes a little more information about looking for suitable normal subgroups of triangle groups by checking all possibilities for certain permutation representations.

The above summary represents the state of the classification of orientably-regular maps of given genus g (for $g > 1$) in the late 1980s, with complete tables for genera $g \leq 7$ available, but with checkable published proofs only for genus 2 and 3. An outcome worth mentioning was the non-existence of chiral examples for every genus between 2 and 6.

A new era for the problem coincided with the dawn of the new millennium, with Conder and Dobcsányi [8] producing a computer-assisted classification of all orientably-regular and reflexible maps on orientable surfaces of genus 2 to 15, confirming the previously obtained lists for genus 2 to 7. Over the next 12 years this was gradually taken further by Conder, thanks to continual improvement of both hardware and algorithms, first in [11] for genus up to 101 and then in [6] for genus up to 301. (These also have non-orientable counterparts up to non-orientable genus 30, 202 and 602, respectively.) The current record goes to orientable genus 1501 and non-orientable genus 1502 [9].

Following the appearance of the first computer-assisted classifications for larger genera, a quest began for classification of orientably-regular maps and non-orientable regular maps on infinite families of surfaces. In 2005, Belolipetsky and Jones [2] obtained a classification of orientably-regular maps M of genus $p+1$ with $|\mathrm{Aut}^+(M)| \geq \lambda p$, where $\lambda > 6$ and p is any prime such that $p \geq c_\lambda$ for some constant c_λ. (For example, for $\lambda = 8$ one may take $p \geq 17$.) In the same year, Breda, Nedela and Širáň [5] derived a classification of regular maps of non-orientable genus $p+2$ where p is prime. While the work for [2] was based on the theory of Riemann surfaces, the approach of [5] was purely group-theoretic. Methods of [5] were then substantially extended by Conder, Širáň and Tucker [12], to obtain a classification of orientably-regular maps M of genus $p+1$ for any prime $p \geq 17$, and hence for any prime p (thanks to earlier computer-assisted findings).

The proof given in [12] splits into two parts, depending on whether the order of the group $\mathrm{Aut}^+(M)$ is relatively prime to p or divisible by p. The first part follows from a far more general theorem given in [12], proved without the need for computations (but guided by the results in [11] and further computational experiments), and resulting in a classification of all orientably-regular maps M of genus g with $|\mathrm{Aut}^+(M)|$ relatively prime to $g-1$. The second part involves a computer-free proof in [12] for $p \geq 89$, with the remainder relying on the earlier computations [11] for primes in the range $17 \leq p \leq 83$. Here it is important to note that the theoretical arguments are very accessible, involving little beyond the Schur-Zassenhaus theorem, Reidemeister-Schreier theory, Schur's theorems on the transfer and centre-by-finite groups, and the known classification of 'almost Sylow-cyclic' groups.

The aim of this article is to develop a few more tools which together with [12] give a very accessible, purely group-theoretic and computer-free proof of a complete classification of orientably-regular maps of genus $p+1$ for an arbitrary given prime p.

By our summary this amounts to supplying computer-free arguments for classification of orientably-regular maps M with $|\mathrm{Aut}^+(M)|$ divisible by p for primes p in the range $17 \leq p \leq 83$ as considered seperately in [12], but our approach will be more general and valid for all primes $p \geq 5$. A task like this has recently been accomplished also by Izquierdo, Jones and Reyes-Carocca [17], in a computer-free classification of orientably-regular *hypermaps* of genus $p+1$ with the orientation-preserving automorphism group having order divisible by p, for primes $p \geq 5$ (but with no counterpart for the case where the group order is relatively prime to p). Their arguments, however, are based on the theory of Riemann surfaces and go beyond elementary group theory, whereas our approach emphasises easy accessibility.

For completeness we include also a classification for the remaining primes 3 and 2, the first one because of unavailability of a proof in the literature, and the second one to give more streamlined arguments compared to those in the original resource [22].

The structure of this paper is as follows. Sections 2 and 3 contain preliminary material together with a review of the approach of [12] and arithmetic consequences of Euler's formula. In sections 4 and 5 we deal with orientably-regular maps M of genus $p+1$ with the order of $G = \mathrm{Aut}^+(M)$ divisible by p for $p \geq 5$, when G is soluble and insoluble, respectively. Finally, sections 6 and 7 handle the remaining cases $p = 2$ and $p = 3$.

2 Preliminaries

Recall that an *orientably-regular map* M is a 2-cell embedding of a finite, connected (but not necessarily simple) graph on a compact orientable surface, such that the group $\mathrm{Aut}^+(M)$ of all the orientation-preserving automorphisms of M acts regularly on the set of all arcs of M. The regular action of $\mathrm{Aut}^+(M)$ on arcs provides a close connection between the map and the group itself, and this connection and the numerous relationships of such maps with group theory, hyperbolic geometry and Riemann surfaces have been explored in great detail in [18], so here we just recall a few facts.

By regularity, all vertices of an orientably-regular map M have the same valency, say k, and all faces of M are bounded by closed walks of the same length, say ℓ. We then say that M is of *type* (k, ℓ). For our purposes the latter notation appears to be more appropriate than the traditional $\{\ell, k\}$ (or Schläfli notation $\{\ell \,|\, k\}$), listing the face length first and using curly brackets.

Next, let $a = (v, w)$ be an arc of M emanating from a vertex v. Then regularity of $\mathrm{Aut}^+(M)$ on arcs implies also that there exist $x, y \in \mathrm{Aut}^+(M)$ such that the x-image a^x of a is the locally clockwise next arc to a emanating from v, and the y-image a^y of a is the locally clockwise next arc to a on the boundary of the face f containing both a and a^x, and then a^{xy} is the reverse (w, v) of the arc a. Since a non-identity orientation-preserving automorphism of a map cannot fix any arc, we see that x and y are automorphisms of M acting like rotations about the vertex v and the centre of a face f, respectively, with the sense of rotation being consistent with a chosen orientation of the carrier surface, and then xy then acts like a 'half-turn' about the centre of the edge $e = \{v, w\}$ incident with both v and f. In particular, x, y and xy have orders k, ℓ and 2, respectively.

The subgroup generated by x and y contains elements that act locally around each arc incident with v or f in the analogous way to x and y, and by connectedness of the underlying graph of M it follows that this subgroup acts transitively on arcs and hence is equal to the entire group $G = \mathrm{Aut}^+(M)$. In particular, x and y generate G, which therefore admits a presentation of the form $G = \langle x, y \,|\, x^k, y^\ell, (xy)^2, \ldots \rangle$, and hence is a smooth quotient of the ordinary $(k, \ell, 2)$-*triangle group* $\Delta^+(k, \ell, 2) = \langle X, Y \,|\, X^k, Y^\ell, (XY)^2 \rangle$.

Groups having such a presentation will be called $(k, \ell, 2)$-presented, or $(k, \ell, 2)$-groups. (They are also said to be $(k, \ell, 2)$-generated.) The map M can then be identified with this presentation by letting the set of arcs, edges, vertices and faces be, respectively, the set of right cosets of the subgroups $\langle 1 \rangle$, $\langle xy \rangle$, $\langle x \rangle$ and $\langle y \rangle$, with incidence between arcs, edges, vertices and faces given by non-empty intersection of the corresponding cosets, and with G acting as the group of all orientation-preserving automorphisms by right multiplication. All other relators satisfied by x and y in G trace out closed walks in the map M.

In this way one may associate with any $(k, \ell, 2)$-presentation $\langle x, y \,|\, x^k, y^\ell, (xy)^2, \ldots \rangle$ for a group G a uniquely determined orientably-regular map, denoted by $\mathrm{Map}\langle x, y \rangle$, or by $\mathrm{Map}(G)$ if we think of G in terms of a given $(k, \ell, 2)$-presentation for it, rather than as an abstract group. Another way of saying this is that orientably-regular maps of type (k, ℓ)

are in a 1-to-1 correspondence with torsion-free normal subgroups of the ordinary $(k,\ell,2)$-triangle group $\Delta^+(k,\ell,2)$. For many other algebraic connections and their topological counterparts we refer the reader to [18].

In general, two maps are isomorphic if there is an incidence-preserving bijection between their respective vertex, edge and face sets. Here, in the context of orientable regularity, map isomorphism reduces to the existence of a particular group isomorphism, namely as follows: if $G = \langle\, x, y \,|\, x^k, y^\ell, (xy)^2, \dots \rangle$ and $H = \langle\, r, s \,|\, r^k, s^\ell, (rs)^2, \dots \rangle$ are $(k,\ell,2)$-presented groups, then the maps $\mathrm{Map}\langle x,y\rangle$ and $\mathrm{Map}\langle r,s\rangle$ are isomorphic if and only if there is a group isomorphism from G to H taking (x,y) to (r,s).

An orientably-regular map $M = \mathrm{Map}\langle x,y\rangle$ may sometimes also admit an orientation-reversing automorphism. If this is the case then the map is called *reflexible*; otherwise it is called *chiral*. As in the case of map isomorphism, reflexibility can be expressed in the language of groups and is equivalent to the existence of an involutory group automorphism of $G = \langle x,y\rangle$ inverting both x and y. Equivalently, $M = \mathrm{Map}\langle x,y\rangle$ is reflexible if and only if G has an automorphism inverting x and fixing xy, or inverting y and fixing xy. (These three automorphisms are obviously different when $k > 2$ and $\ell > 2$.)

Similarly, the *dual* of $M = \mathrm{Map}\langle x,y\rangle$ is the map $M^* = \mathrm{Map}\langle y,x\rangle$, and so M is self-dual if and only if G has an automorphism interchanging x and y. It then follows easily that reflexibility is preserved by map duality.

We now look at implications of Euler's formula for an orientably-regular map $M = \mathrm{Map}(G)$ of type (k,ℓ) when $G = \langle\, x, y \,|\, x^k, y^\ell, (xy)^2, \dots \rangle$. First, M has $|G|/|\langle x\rangle| = |G|/k$ vertices, $|G|/|\langle y\rangle| = |G|/\ell$ faces, and of course $|G|/2$ edges, and so if χ and g are the Euler chraracteristic and genus of the carrier surface of M, then by Euler's formula,

$$|G|\left(\frac{1}{k} - \frac{1}{2} + \frac{1}{\ell}\right) = \chi = 2 - 2g,$$

which for $g > 1$ is equivalent to

$$|G| = 2(g-1)\mu(k,\ell) \quad \text{where} \quad \mu(k,\ell) = \left(\frac{1}{2} - \frac{1}{k} - \frac{1}{\ell}\right)^{-1} = \frac{2k\ell}{k\ell - 2k - 2\ell}\,. \tag{2.1}$$

In particular, in the hyperbolic case $\mu(k,\ell) > 0$ and so $1/k + 1/\ell < 1/2$. It is well known (and easy to see) that the maximum possible value of $1/k + 1/\ell$ in this case is $1/3 + 1/7 = 10/21$, and hence $\mu(k,\ell) \le 42$, which implies the *Hurwitz bound* $|G| \le 84(g-1)$, with equality occurring if and only if the type (k,ℓ) is $(3,7)$ or $(7,3)$.

The theory outlined above tells us that for $g \ge 2$, classification of orientably-regular maps on a surface of a genus g is equivalent to classification of the finite $(k,\ell,2)$-presented groups G satisfying the condition $|G| = 2(g-1)\mu(k,\ell)$. By the Hurwitz bound, the order of any such group G is bounded, and it follows that up to isomorphism the number of orientably-regular maps on a surface of genus g is finite for every $g \ge 2$. This fact makes the classification only a little easier, but in what follows we will simplify matters further by assuming that the hyperbolic type (k,ℓ) satisfies $k \le \ell$, since the types satisfying the reverse of the latter inequality can be handled by duality.

3 First observations about orientably-regular maps M of genus $p+1$ with p dividing $|\mathrm{Aut}^+(M)|$

We begin this section by giving more details about the structure and principal results of [12]. One major theorem [12, Theorem 8.4] implies a classification of all the orientably-regular maps M of given genus $g > 2$ with the property that $|\mathrm{Aut}^+(M)|$ is relatively prime to $g-1$. For any such map M, in the expression $2(g-1)\mu(k,\ell)$ for $|G|$ given in (2.1) in the previous section, the term $g-1$ is completely absorbed by the denominator of $\mu(k,\ell)$. In particular, if $g-1=p$ for an odd prime p, this gives all of the orientably-regular maps M of genus $p+1$ such that p does not divide $|\mathrm{Aut}^+(M)|$. Again, from the point of view of the aim of this paper, we emphasise that the proof of this theorem in [12] is computer-free.

To complete our classification of *all* orientably-regular maps M of genus $p+1$ for odd primes p, it therefore remains to consider the situation when p divides $|\mathrm{Aut}^+(M)|$. This was done for $p \geq 17$ by the fairly general Theorem 3.1 of [12], but the proof of the latter theorem in [12] relies on computation for p in the range $17 \leq p \leq 83$. To eliminate the need for computer-based arguments here, we present a modified version of Theorem 3.1 of [12] and indicate below how a computer-free proof this modified version can be extracted from the original arguments of [12].

Theorem 3.1. *Let M be an orientably-regular map of type (k,ℓ) with $k \leq \ell$, and of genus $p+1$ for some odd prime $p \geq 5$ dividing neither k nor ℓ but dividing the order of $G = \mathrm{Aut}^+(M)$. If C_p is a normal subgroup of G, then only the following three cases can occur:*

(a) *M has type $(8,8)$ and $G \cong C_p \rtimes C_8$, of order $8p$, with $p \equiv 1 \bmod 8$,*

(b) *M has type $(5,10)$ and $G \cong C_p \rtimes C_{10}$, of order $10p$, with $p \equiv 1 \bmod 10$,*

(c) *M has type $(6,6)$ and $G \cong C_p \rtimes (C_6 \times C_2)$, of order $12p$, with $p \equiv 1 \bmod 6$.*

Moreover, up to isomorphism in cases (a) and (c), there is a unique chiral pair of such maps, while in case (b) there are two such chiral pairs.

The differences between the statement above and the original Theorem 3.1 of [12] are as follows. First, the original statement of Theorem 3.1 in [12] assumes that $p > 13$ and p divides $|G|$, and its proof in [12] begins by assuming that $p > 83$, which, together with the assumption that p divides $|G|$, is shown to imply that (i) p divides no entry of the type of the map, and (ii) G contains a normal subgroup of order p. Using (i) and (ii), the conclusions (a) to (c) are subsequently derived in [12] for $p > 83$, and the proof in [12] finishes with a remark that the validity of the conclusions can be verified also for primes p in the range $13 < p \leq 83$ by the computations described in [11]). That part of the proof, however, is valid *without any restriction* on the size of the odd prime p, when (i) and (ii) are included as *extra assumptions*, as we have done in the revised statement. We have also replaced the assumption $p > 13$ by $p \geq 5$, for the same reason. A further modification of the statement could be made to cover *all* primes p, but such a change is rather pointless, because for $p \in \{2,3\}$ there is no orientably-regular map M of genus $p+1$ such that $|\mathrm{Aut}^+(M)|$ is divisible by p and assumptions (i) and (ii) are both satisfied.

Now let M be an orientably-regular map of type (k,ℓ) with $k \leq \ell$ and of genus $p+1$, where p is a prime dividing $|\mathrm{Aut}^+(M)|$. Then (k,ℓ) is hyperbolic and so $1/k+1/\ell < 1/2$. Also besides the obvious divisibility of $|\mathrm{Aut}^+(M)|$ by $\mathrm{lcm}(k,\ell,2)$, invoking (2.1) tells us that the number $|\mathrm{Aut}^+(M)|/(2p) = \mu(k,\ell)$ is an integer.

Let us consider what the last condition says for $k \in \{3,4,5\}$:

$$\text{if } k = 3, \text{ then } \mu(3, \ell) = \frac{6\ell}{\ell - 6} = 6 + \frac{36}{\ell - 6}, \text{ so } \ell \in \{7, 8, 9, 10, 12, 15, 18, 24, 42\};$$
$$\text{if } k = 4, \text{ then } \mu(4, \ell) = \frac{8\ell}{2\ell - 8} = 4 + \frac{16}{\ell - 4}, \text{ so } \ell \in \{5, 6, 8, 12, 20\};$$
$$\text{if } k = 5, \text{ then } \mu(5, \ell) = \frac{10\ell}{3\ell - 10} = 3 + \frac{\ell + 30}{3\ell - 10}, \text{ so } \ell \in \{5, 10, 20\}.$$

Next, suppose $6 \le k \le \ell$. Note that $\mu(k, \ell) = 2k\ell/(k\ell - 2k - 2\ell) > 2$ for every hyperbolic pair (k, ℓ), and so for $\mu(k, \ell)$ to be a positive integer we must have $\mu(k, \ell) \ge 3$, which is equivalent to $2k\ell \ge 3k\ell - 6k - 6\ell$ and hence to $(k - 6)(\ell - 6) \le 36$; and also $\mu(k, \ell) = 3$ if and only if $(k - 6)(\ell - 6) = 36$. On the other hand, observe that if $6 \le k \le \ell$ then $(k - 3)(\ell - 3) \ge 9$ and hence $4k\ell - 12k - 12\ell \ge 0$ which gives $\mu(k, \ell) = 2k\ell/(k\ell - 2k - 2\ell) \le 6$; and also $\mu(k, \ell) = 6$ if and only if $(k - 6)(\ell - 6) = 36$. Thus $\mu(k, \ell) \in \{3, 4, 5, 6\}$, and similarly we find that $\mu(k, \ell) = 4$ if and only if $(k - 4)(\ell - 4) = 16$, and $\mu(k, \ell) = 5$ if and only if $(3k - 10)(3\ell - 10) = 100$. These give just eight more possibilities for (k, ℓ), namely $(7, 42)$, $(8, 24)$, $(9, 18)$, $(10, 15)$ and $(12, 12)$ with $\mu(k, \ell) = 3$, plus $(6, 12)$ and $(8, 8)$ with $\mu(k, \ell) = 4$, plus $(6, 6)$ with $\mu(k, \ell) = 6$.

Thus we have a total of exactly $17 + 8 = 25$ pairs (k, ℓ) with $k \le \ell$ such that $\mu(k, \ell) \in \mathbb{Z}$.

Now these include the eight pairs $(3, 15)$, $(3, 24)$, $(3, 42)$, $(4, 20)$, $(6, 12)$, $(8, 24)$, $(9, 18)$ and $(12, 12)$, with $|G| = 2\mu(k, \ell)p = 20p$, $16p$, $14p$, $10p$, $8p$, $6p$, $6p$ and $6p$, respectively, but as $p \ge 5$, none of the listed group orders is divisible by the first entry k of the corresponding pair (k, ℓ), contradicting the fact that one generator of G has order k. Hence we can eliminate all of them from consideration. Also we can eliminate two further pairs, namely $(7, 42)$ and $(5, 20)$: in the former case, $|G| = 2\mu(k, \ell)p = 6p$ and is divisible by $\ell = 42$, and so $p = 7$, but then $G \cong C_{42}$ which is clearly not $(7, 42, 2)$-generated, while in the latter case $|G| = 2\mu(k, \ell)p = 8p$ and is divisible by $\ell = 20$, so $p = 5$, and then $|G| = 40$ and so G has a normal Sylow 5-subgroup P, but then the quotient G/P has order 8 and so is clearly not $(1, 4, 2)$-generated.

This leaves us with only 15 hyperbolic pairs (k, ℓ) with $k \le \ell$ such that $\mu = \mu(k, \ell)$ is an integer, as given in Table 1. We will split our consideration of the hypothetical $(k, \ell, 2)$-groups for these pairs by taking solubility (or otherwise) of the groups into account.

k	3	3	3	3	3	3	4	4	4	4	5	5	6	8	10
ℓ	7	8	9	10	12	18	5	6	8	12	5	10	6	8	15
μ	42	24	18	15	12	9	20	12	8	6	10	5	6	4	3

Table 1: The remaining 15 pairs (k, ℓ) with $k \le \ell$ and $\mu(k, \ell) \in \mathbb{Z}$ for $p \ge 5$.

4 Maps with a soluble automorphism group

The following proposition will be our basic tool for handling soluble $(k, \ell, 2)$-groups G of order $2\mu(k, \ell)p$ for the pairs (k, ℓ) listed in Table 1, and we state it in a form applicable to hypermaps as well. Using number-theoretic notation, for any prime s and any positive integer m, we let $\nu_s(m)$ be the largest non-negative integer e such that s^e divides m.

Proposition 4.1. *Let $p \geq 5$ be a prime, and let $n > 1$ be a positive integer such that all prime divisors of n are less than p, and let G be a soluble group of order np generated by two non-trivial elements of orders k and ℓ, both coprime to p, and let $m = \mathrm{lcm}(k, \ell)$. Further, suppose that if s is any prime, then $\nu_s(n) \leq \nu_s(m) + 1$ if s divides m (that is, if $\nu_s(m) > 0$), while $\nu_s(n) \leq 2$ otherwise. Then G contains a normal subgroup of order p.*

Proof. First, if n is a prime r smaller than p, then the claim follows from Sylow theory. (Note here that for an arbitrary prime $r \geq 2$ dividing $p-1$, the semi-direct products $C_p \rtimes C_r$ are examples of groups as in the statement of the proposition, generated by two elements of order r.) So from now on we may assume that n is composite. We will proceed by induction on n. Note that p does not divide n, and so $\nu_p(n) = 0$.

As G is soluble, G contains an elementary abelian normal subgroup, say $N \cong C_r^j$ for some prime r and some integer $j \geq 1$. If $r = p$ then $|N| = p$ (since $\nu_p(n) = 0$), and there is nothing left to prove, so r is a prime a divisor of n, and hence $r < p$, and indeed $r + 1 < p$ since p and r are primes with $p \geq 5$.

Next, let u and v be two generators for G of orders k and ℓ. As p divides $|G/N|$, neither of u and v can be contained in N (for otherwise G/N would be cyclic, of order a multiple of p, implying that one of the generators would have to have order a multiple of p, contrary to our assumptions).

So let $k' > 1$ and $\ell' > 1$ be the orders of Nu and Nv in G/N, and let $m' = \mathrm{lcm}(k', \ell')$. Since N is an elementary abelian r-group, it follows that either $k' = k$ or $k' = k/r$, and, similarly, $\ell' = \ell$ or $\ell' = \ell/r$. Also let $n' = n/r^j$, so that $|G/N| = n'p$, and $\nu_r(n') = \nu_r(n) - j \leq \nu_r(n) - 1$. Also let $e = \nu_r(m)$, and recall the hypethsis that $\nu_r(n) \leq e + 1$ if $e \geq 1$, while $\nu_r(n) \leq 2$ if $e = 0$.

Letting $e' = \nu_r(m')$ it is obvious that either $e' = e$, or $e - 1$ if $e > 0$, and so we have

$$\nu_r(n) - 1 \ \leq \ 2 - 1 = 1 \quad \text{if } e = e' = 0, \text{ and}$$

$$\nu_r(n) - 1 \ \leq \ e + 1 - 1 = 1 \quad \text{if } e = 1 \text{ and } e' = 0, \text{ while}$$

$$\nu_r(n) - 1 \ \leq \ e + 1 - 1 = e \ \leq \ e' + 1 \quad \text{if } e \ \geq \ e' > 0.$$

Hence in all cases $\nu_r(n') = \nu_r(n) - j \leq \nu_r(n) - 1 \leq e' + 1$ if $e' > 0$, while otherwise $\nu_r(n') = \nu_r(n) - j \leq \nu_r(n) - 1 = 1 \leq 2$ (when $e' = 0$).

This shows that the 'largest exponent condition' for the prime r is satisfied in G/N. Also this condition is clearly satisfied for any prime $s \neq r$ automatically.

The above observations show that the (soluble) group G/N of order $n'p$, generated by a pair of elements of orders $k', \ell' > 1$ as above, satisfies all the required assumptions. By the induction hypothesis, the group G/N contains a normal subgroup isomorphic to C_p, and it follows that G has a normal subgroup K containing N such that $K/N \cong C_p$. Since N and C_p have relatively prime orders, the Schur-Zassenhaus theorem gives $K \cong N \rtimes C_p$.

Moreover, we can show that $|N| = r$ or r^2. Let L be a a Sylow r-subgroup L of G. Then L contains the normal r-subgroup N of G, which has exponent r and order r^j, while L itself has order $r^{\nu_r(n)}$. By the assumption on $e = \nu_r(m)$, we know that $|L|$ divides r^{e+1} if $e \geq 1$, or divides r^2 if $e = 0$. If $e = 0$ then $|N| \leq |L| \leq r^2$. On the other hand, if $e > 0$ then since m divides both k and ℓ, which are the orders of two elements of G, the Sylow subgroup L contains a cyclic subgroup of order r^e, and so the exponent of L/N is a multiple of r^{e-1}, which gives $|N| \leq |L|/(|L/N|) \leq r^{e+1}/r^{e-1} = r^2$. Accordingly, in both cases $|N| = r$ or r^2, so $j = 1$ or 2, and $K \cong C_r \rtimes C_p$ or $C_r^2 \rtimes C_p$.

We now examine the semi-direct product K, which is determined by some group homomorphism from C_p into $\mathrm{Aut}(C_r^j)$. For this homomorphism to be non-trivial, we need an element of order p in $\mathrm{Aut}(C_r^j)$. But there is no such element, because $\mathrm{Aut}(C_r) \cong C_{r-1}$, which has order $r-1 < p$, while $\mathrm{Aut}(C_r^2) \cong \mathrm{GL}(2,r)$, the order of which is $(r^2-1)(r^2-r) = r(r-1)^2(r+1)$ and hence is not divisible by p (because $r+1 < p$).

We conclude that the subgroup C_p of K acts trivially on N, and then since $p \nmid |N|$, it follows that $K \cong N \times C_p$, and so the subgroup C_p is characteristic in K and therefore normal in G. This completes the proof by induction. □

Proposition 4.1 enables us to establish the existence of a normal subgroup of order p in the vast majority of cases of (k,ℓ)-groups covered by Table 1.

Proposition 4.2. *Let G be a soluble $(k,\ell,2)$-group of order $2\mu(k,\ell)p$ for some odd prime p, and for a pair (k,ℓ) and corresponding value of $\mu(k,\ell)$ as given in* Table 1. *If $p \geq 7$, or if $p = 5$ but $5 \nmid \ell$, then G contains a normal subgroup isomorphic to C_p.*

Proof. First, suppose $(k,\ell) = (3,7)$. By considering the commutator subgroup G' of G and the induced presentation of the abelianisation G/G' for a $(3,7,2)$-group G one finds that $G' = G$, implying that in any such case G is insoluble. An inspection shows that for all the remaining 9 types $(k,\ell) \neq (3,7)$ with $5 \nmid \ell$ listed in Table 1, the conditions of Proposition 4.1 are satisfied for the given prime p, and the rest follows easily. □

To complete this section, we deal with the five cases listed in Table 1 but not covered by Proposition 4.2, namely the pairs $(3,10)$, $(4,5)$, $(5,5)$, $(5,10)$ and $(10,15)$, all with $p = 5$. We do this by using a trivial but useful consequence of Sylow theory coupled with the fact that if a Sylow p-subgroup is normal in the commutator subgroup G' of G, then it is also characteristic in G' and therefore normal in G as well:

(F) If $|G| = tp^j$ or if $|G'| = tp^j$ for some prime p and some positive integer $t < p$, then the group G has a unique (and hence normal) Sylow p-subgroup.

The situation for those five remaining cases is as follows.

Proposition 4.3. *For $p = 5$, there is:*

(a) *a unique $(3,10,2)$-group of order $2\cdot\mu(3,10)\cdot p = 150$, isomorphic to $(C_5\times C_5)\rtimes S_3$, with presentation $\langle\, x, y \mid x^3, y^{10}, (xy)^2, (x^{-1}y^2)^3 \,\rangle$,*

(b) *no $(4,5,2)$-group of order $2\cdot\mu(4,5)\cdot p = 200$,*

(c) *no $(5,5,2)$-group of order $2\cdot\mu(5,5)\cdot p = 100$,*

(d) *a unique $(5,10,2)$-group of order $2\cdot\mu(5,10)\cdot p = 50$, isomorphic to $(C_5\times C_5)\rtimes C_2$, with presentation $\langle\, x, y \mid x^5, y^{10}, (xy)^2, [x, y^2] \,\rangle$, and*

(e) *a unique $(10,15,2)$-group of order $2\cdot\mu(10,15)\cdot p = 30$, isomorphic to $C_{15}\rtimes C_2$, with presentation $\langle\, x, y \mid x^{10}, y^{15}, (xy)^2, xyx^{-1}y^4 \,\rangle$.*

Proof. We will consider the five pairs (k,ℓ) one by one.

Case (a). Let $G = \langle\, x, y \mid x^3, y^{10}, (xy)^2, \dots \rangle$ be a $(3,10,2)$-group, with order $2\cdot 15\cdot 5 = 150$. In what follows, we will work with an equivalent presentation of G obtainable by taking $a = xy = y^{-1}x^{-1}$ and $b = x$, namely $G = \langle\, a, b \mid a^2, b^3, (ab)^{10}, \dots \rangle$. As the order of no non-abelian simple group divides 150, such a group is soluble, and then since the abelianisation G/G' can only be C_2, it follows that $|G'| = 75$. Moreover, by the observation (F) above, the group G has a normal Sylow 5-subgroup N, with G/N being $(3,2,2)$-generated and therefore isomorphic to S_3. The subgroup N has order 25 and hence is abelian, but N is not cyclic, for otherwise $K = \langle (ab)^2 \rangle$ would be a normal subgroup in G, with G/K having order 30, and yet $G/K \cong S_3$ by the same argument as in the previous sentence, so $N \cong C_5^2$. Next, by the Schur-Zassenhaus theorem, $G \cong N \rtimes S_3$. Letting $u = (ab)^2$ and $v = (ba)^2$, we see that conjugation by a interchanges a with b, and conjugation by b takes v to u, and also the two elements u and v generate N (for otherwise u and v would generate the same cyclic normal subgroup J of order 5 with quotient $G/J \cong S_3$). Next, using $b^2 = b^{-1}$ we have $1 = [u,v] = [(ab)^2,(ba)^2] = b^{-1}ab^{-1}aab^{-1}ab^{-1}ababbaba = (b^{-1}aba)^3$, and with the help of this we also obtain $b^{-1}ub = b^{-1}abab^{-1} = ab^{-1}abab^{-1}a = ab^{-1}ab^{-1}b^{-1}ab^{-1}a = v^{-1}u^{-1}$. Hence the action of S_3 by conjugation on N is completely and uniquely determined by the extra relator $(b^{-1}aba)^3$, which is conjugate to $(x^{-1}y^2)^3$ in the original notation. This proves uniqueness of our group of order 150, and its given $(3,10,2)$-presentation.

Case (b). Here $\mu(k,\ell) = \mu(4,5) = 20$ and $p = 5$, and $G = \langle\, x, y \,|\, x^4, y^5, (xy)^2, \dots \rangle$ has order 200. As in the previous case, by (F) the group G would have a normal Sylow 5-subgroup N, but then G/N would be $(4,1,2)$-generated, which is impossible.

Case (c). Here $\mu(k,\ell) = \mu(5,5) = 10$ and $p = 5$, and $G = \langle\, x, y \,|\, x^5, y^5, (xy)^2, \dots \rangle$ has order 100, and by Sylow theory would have a normal Sylow 5-subgroup N, but then x and y would lie in N and so could not generate G.

Case (d). Let $G = \langle\, x, y \,|\, x^5, y^{10}, (xy)^2, \dots \rangle$ be a $(5,10,2)$-group for $p = 5$, with order 50. Then G has a Sylow 5-subgroup N, and clearly $G = N \rtimes \langle xy \rangle \cong N \rtimes C_2$. Also N not cyclic (for otherwise G would have a normal subgroup K of order 5 such that G/K is a $(1,2,2)$-generated group of order 10), and so $N \cong C_5^2$. Next, N contains both x and y^2, but $\langle x \rangle \neq \langle y^2 \rangle$ by the immediately previous argument, so $\langle x, y^2 \rangle = N$, and $[x, y^2] = 1$. The latter relation now implies that $(xy)^{-1}x(xy) = y^{-1}xy = y^{-2}yxy = y^{-2}x^{-1} = (xy^2)^{-1}$, and that $(xy)^{-1}y^2(xy) = y^{-1}x^{-1}y^2xy = y^{-1}y^2y = y^2$, and as these completely and uniquely determine the action of xy by conjugation on N, it gives the stated presentation for G.

Case (e). Let $G = \langle\, x, y \,|\, x^{10}, y^{15}, (xy)^2, \dots \rangle$ be a $(15,10,2)$-group for $p = 5$, with order 30. Then $H = \langle y \rangle$ is a normal subgroup of G of index 2, and so conjugation by the involution xy gives $(xy)y(xy)^{-1} = y^j$ for some square root j of 1 in $\mathbb{Z}_{15}$, namely $j \in \{1, 4, 11, 14\}$. Then also $xyx^{-1} = y^j$, and so $1 = (yx)^2 = yxyx^{-1}x^2 = y^{1+j}x^2$, so that $y^{1+j} = x^{-2}$ which has order 5, and so $1 + j \equiv 3, 6, 9$ or $12 \bmod 15$. The intersection of the two congruences for j leaves only $j \equiv 11 \equiv -4 \bmod 15$, giving G as a semi-direct product $\langle y \rangle \rtimes \langle xy \rangle \cong C_{15} \rtimes C_2$, determined by the extra relator $xyx^{-1}y^4$. □

The orientably-regular maps M from the first, fourth and fifth cases of Proposition 4.3 are all reflexible: it is easy to check that in each case the assignment $(x, y) \mapsto (x^{-1}, y^{-1})$ extends to an automorphism of $\mathrm{Aut}^+(M)$, preserving the given group presentation.

The remaining $(k,\ell,2)$-groups G of order $2\mu(k,\ell)p$, which are insoluble, are handled separately in the next section.

5 Maps with an insoluble automorphism group, and summary of the classification for $p \geq 5$

Again in this section we suppose that M is an orientably-regular map with type (k, ℓ) as given in Table 1, and with genus $p + 1$ for some prime $p \geq 5$, such that the order of $G = \mathrm{Aut}^+(M)$ is divisible by p, but now also such that G is insoluble. Moreover, if $p > 84$ then because the Hurwitz bound gives $|G| \leq 84p$, we know that the group G would have a normal Sylow p-subgroup P, but then the insoluble quotient G/P would have to be A_5 and so $|G| = 60p$, which is impossible since $\mu(k, \ell) \neq 30$ for all the cases in Table 1. Hence we may suppose that $5 \leq p \leq 83$.

Next, $|G| = 2\mu p$ must be a multiple of the order of some non-abelian simple group H, with $|H| \leq |G| \leq 84 \cdot 83 = 6972$. There are exactly thirteen such simple groups, namely A_n for $n \in \{5, 6, 7\}$, $\mathrm{PSL}(2, q)$ for $q \in \{7, 8, 11, 13, 16, 17, 19, 23\}$, $\mathrm{PSL}(3, 3)$ and $\mathrm{PSU}(3, 3)$, with orders 60, 360, 2520, 168, 504, 660, 1092, 4080, 2448, 3420, 6072, 5616 and 6048 respectively. (This may be checked at numerous online references; for an article reference pre-dating the Classification of Finite Simple Groups see [16].)

Comparison of these orders with the entries in Table 1 reveals that there are just eleven possibilities for $(k, \ell, \mu, p, |G|)$, and a unique H for each one, as in Table 2.

We consider these eleven cases in detail, showing that only three of them (cases (b), (d) and (h)) are realisable.

Case	k	ℓ	μ	p	$\|G\| = 2\mu p$	H
(a)	3	7	42	5	$84 \cdot 5 = 420$	A_5
(b)	3	7	42	13	$84 \cdot 13 = 1092$	$\mathrm{PSL}(2, 13)$
(c)	3	8	24	5	$48 \cdot 5 = 240$	A_5
(d)	3	8	24	7	$48 \cdot 7 = 336$	$\mathrm{PSL}(2, 7)$
(e)	3	9	18	5	$36 \cdot 5 = 180$	A_5
(f)	3	12	12	5	$24 \cdot 5 = 120$	A_5
(g)	3	12	12	7	$24 \cdot 13 = 168$	$\mathrm{PSL}(2, 7)$
(h)	4	6	12	5	$24 \cdot 5 = 120$	A_5
(i)	4	6	12	7	$24 \cdot 7 = 168$	$\mathrm{PSL}(2, 7)$
(j)	4	12	6	5	$12 \cdot 5 = 60$	A_5
(k)	6	6	6	5	$12 \cdot 5 = 60$	A_5

Table 2: The eleven possibilities for $(k, \ell, \mu, p, |G|)$ and simple H.

In case (a), we have $|G| = 420 = 60 \cdot 7$ and $H = A_5$, so the group G must have A_5 and C_7 as composition factors, with one of them being the quotient of G by the other, but neither of those groups is a quotient of the $(3, 7, 2)$-triangle group, so this case can be eliminated.

In contrast, for case (b) the group G is $\mathrm{PSL}(2, 13)$, which is known to be a quotient of the $(3, 7, 2)$-triangle group in three different ways, as a consequence of a theorem by Macbeath [19, Theorem 8]. Details were given in [23], but also explicit matrix representations for these three ways can be extracted from [10, 21] to show that the corresponding group presentations can be obtained by adding the relator $[x, y]^6$, $[x, y^3]^3$ or $[x, y]^7$ to the presentation $\langle\, x, y \mid x^3 = y^7 = (xy)^2 = 1 \,\rangle$ for the $(3, 7, 2)$-triangle group; see also [7].

In case (c), the group G of order $240 = 60 \cdot 4$ cannot have A_5 as a quotient, because A_5 has no element of order 8 and is not a quotient of either of the $(3, 2, 2)$- and $(3, 4, 2)$-

triangle groups (which have orders 6 and 24), and so G must have the cyclic group C_2 as a quotient, and hence a normal subgroup K of index 2. This subgroup K has order 120 and so must be isomorphic to one of $A_5 \times C_2$, $\mathrm{SL}(2,5)$ or $S_5 \cong \mathrm{PGL}(2,5)$. But if x and y are generators for G satisfying $x^3 = y^8 = (xy)^2 = 1$, then K is generated by the two elements y and xyx^{-1} $(= (xy)x(xy)^{-1})$ of order 3, so K cannot have C_2 as a quotient, and hence $K \cong \mathrm{SL}(2,5)$. But now this implies that K contains a unique involution, which must be equal to y^4, and if $L = \langle y^4 \rangle$ then L is characteristic in K and hence normal in G, and the quotient G/L has order 240 but is a quotient of the $(3,4,2)$-triangle group of order 24, a contradiction.

In case (d), the group G must have $\mathrm{PSL}(2,7)$ and C_2 as composition factors, but cannot have $\mathrm{PSL}(2,7)$ as a quotient by the same argument as in the first sentence of the previous case, and so $G \cong \mathrm{PGL}(2,7)$. Using the known enumeration of all (k,ℓ)-groups isomorphic to $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$ for some prime power q, completed in [21] and independently in [1] and later revisited in [10], we find that $\mathrm{PGL}(2,7)$ is a quotient of the $(3,8,2)$-triangle group in two different ways, with presentations obtainable by adding the relator $[x,y]^4$ or $[x,y^3]^2$ to the presentation $\langle\, x, y \mid x^3 = y^8 = (xy)^2 = 1 \,\rangle$ for the $(3,8,2)$-triangle group.

In case (e), the group G must have A_5 and C_3 as composition factors, but cannot have A_5 as a quotient, because A_5 has no element of order 9 and is not a quotient of the $(3,3,2)$-triangle group (which has order 12). Thus G has a normal subgroup K isomorphic to A_5 with $G/K \cong C_3$, but $G \ncong A_5 \times C_3$. Now $\mathrm{Aut}(K) \cong \mathrm{Aut}(A_5) \cong S_5$ with all elements of order 3 being inner automorphisms, and so conjugation by y (which has order 9) induces the same automorphism of K as conjugation by some element w of order 3 in K. This implies that yw^{-1} centralises K, and as it lies outside K, it follows that $Z(G)$ is non-trivial, and must have order 3, so $G \cong K \times C_3 \cong A_5 \times C_3$, a contradiction.

In cases (f) and (h), the group G has order 120 but cannot have A_5 as a quotient because A_5 is not a quotient of the $(3,12,2)$ or the $(4,6,2)$-triangle group, and so $G \cong S_5$, But S_5 has no element of order 12, and so this eliminates case (f). On the other hand, S_5 is a $(4,6,2)$-group, being generated by $x = (1,2,3,4)$ and $y = (1,2)(3,4,5)$ with $xy = (2,4)(3,5)$, and this choice of generators is unique up to conjugacy in $\mathrm{Aut}(S_5) = S_5$.

Finally, cases (g), (i), (j) and (k) are easy to eliminate, as in each case $G = H$, which has no element of order ℓ. (Both $\mathrm{PSL}(2,7)$ and A_5 have no element of order 12 or 6.)

Summing up our findings gives the following.

Proposition 5.1. *Let G be an insoluble $(k,\ell,2)$-group of order $2 \cdot \mu(k,l) \cdot p$, where the type (k,ℓ) is given in* Table 1*, and p is a prime ≥ 5. Then $p = 5$ and $G = S_5$, expressible as a $(4,6,2)$-group uniquely, or $p = 7$ and $G = \mathrm{PGL}(2,7)$, expressible as a $(3,8,2)$-group in two ways, or $p = 13$ and $G = \mathrm{PSL}(2,13)$, expressible as a $(3,7,2)$-group in three ways.* □

All the orientably-regular maps identified above are reflexible, because in each case the $(k,\ell,2)$-presented group $G = \langle x, y \rangle$ admits an automorphism taking (x,y) to (x^{-1}, y^{-1}). This can be verified directly for the map in (h) using conjugation by the permutation $(1,2)(3,4)$, while for all the remaining maps it follows from the fact that relators of the form x^r, y^s, $(xy)^t$ and $[x^r, y^s]^t$ are preserved by such inversion. (For example, $[x^r, y^s]^t$ is taken to $[x^{-r}, y^{-s}]^t = (x^r y^s x^{-r} y^{-s})^t$, which is conjugate to $(x^{-r} y^{-s} x^r y^s)^t = [x^r, y^s]^t$.)

Theorem 3.1 and Propositions 4.2 to 5.1 together give the following.

Theorem 5.2. *Up to isomorphism and duality, every orientably-regular map M of genus $p+1$ where p is prime, $p \geq 5$, and p divides the order of $G = \mathrm{Aut}^+(M)$, occurs as one of the maps in the list below:*

(a) *one chiral pair of maps of type $(8,8)$ with $G \cong C_p \rtimes C_8$ of order $8p$, for every prime $p \equiv 1 \bmod 8$, with $p \geq 17$;*

(b) *two chiral pairs of maps of type $(5,10)$ with $G \cong C_p \rtimes C_{10}$ of order $10p$, for every prime $p \equiv 1 \bmod 10$, with $p \geq 11$;*

(c) *a unique chiral pair of maps of type $(6,6)$ with $G \cong C_p \rtimes (C_6 \times C_2)$ of order $12p$, for every prime $p \equiv 1 \bmod 6$, with $p \geq 7$;*

(d) *three reflexible maps of type $(3,7)$ with $G \cong \mathrm{PSL}(2,13)$ of order $84p$ for $p = 13$;*

(e) *two reflexible maps of type $(3,8)$ with $G \cong \mathrm{PGL}(2,7)$ of order $48p$ for $p = 7$;*

(f) *a unique reflexible map of each of the types $(3,10)$, $(4,6)$, $(5,10)$ and $(10,15)$, with groups G isomorphic to $(C_5 \times C_5) \rtimes S_3$ of order $30p$, S_5 of order $24p$, $(C_5 \times C_5) \rtimes C_2$ of order $10p$, and $C_{15} \rtimes C_2$ of order $6p$, respectively, and all for $p = 5$.*

To complete our computer-free proof of the classification of orientably-regular maps on surfaces of genus $p+1$ where p is prime, it remains to deal with the cases $p = 2$ (genus 3) and $p = 3$ (genus 4), which we do in the next (and final) two sections.

6 Orientably-regular maps of genus 3

It is easy to compile a table of possible types (k,ℓ) with $k \leq \ell$ for an orientably-regular map M of genus 3 (for $p = 2$), together with the corresponding integer value of $2\mu(k,\ell)$ and the order $4\mu(k,\ell)$ of $G = \mathrm{Aut}^+(M)$, which must be divisible by k and ℓ – see Table 3.

k	3	3	3	3	3	3	3	3	4	4	4	4	4	5	5	5	6	6	7	8	12
ℓ	7	8	9	10	12	14	18	30	5	6	8	12	20	5	6	10	6	9	14	8	12
2μ	84	48	36	30	24	21	18	15	40	24	16	12	10	20	15	10	12	9	7	8	6
$\lvert G \rvert$	168	96	72	60	48	42	36	30	80	48	32	24	20	40	30	20	24	18	14	16	12

Table 3: Potential hyperbolic pairs (k,ℓ) with $k \leq \ell$ for $p = 2$.

The values in Table 3 will eventually give 12 group presentations and hence 12 orientably-regular maps of genus 3, up to isomorphism and duality. We will obtain groups of orders 168, 96, 48, 32, 24, 16, 14 and 12, among which each of the orders 48, 32, 24, and 116 will give a pair of non-isomorphic maps. We show this in the following two subsections.

6.1 Genus 3: non-existence

Here we exclude pairs (k,ℓ) from Table 3 for which no suitable $(k,\ell,2)$-group exists.

• *Groups with a cyclic subgroup of index* 1 *or* 2. If G is a abelian group of generated by two elements, then $\mathrm{ord}(ab)$ divides $\mathrm{lcm}(\mathrm{ord}(a),\mathrm{ord}(b))$ for any pair generating pair (a,b), and so ℓ divides $\mathrm{lcm}(k,2)$. Hence there is no $(3,30,2)$-generated group of order 30, and no $(4,20,2)$-generated group of order 20. Similarly, if G is a group generated by two elements a and b of orders r and s where r is odd and $|G| = 2s$, then b generates a normal subgroup N of index 2, but then N contains both a (as it has odd order) and b, a contradiction.

Hence there is no $(3, 18, 2)$-generated group of order 36, and no $(5, 10, 2)$-generated group of order 20. Also there is no $(6, 9, 2)$-generated group of order 18, for in that case $N = \langle y \rangle$ would have index 2 in G, so $x^2 = y^{\pm 3}$ (to have order 3), and also $x^{-1}yx = y^j$ where $j = \pm 1$, but then $y^{\mp 3} = x^{-2} = x^{-2}(xy)^2 = x^{-1}yxy = y^{j+1}$ and so $j = 2$ or 5, a contradiction.

• *Groups of order* $2rs$ *where* r *and* s *are distinct odd primes, and* x *has order* r *while* y *has order* s *or* $2s$. If r and s are small enough that we can assume that G is soluble, then consider a minimal normal subgroup N of G. This must be elementary abelian, and hence cyclic of (prime) order 2, r or s. But then respectively G/N is $(r, s, 1)$-generated, or $(1, s, 2)$- or $(1, 2s, 2)$-generated, or $(r, 1, 2)$ or $(r, 2, 2)$-generated, all of which are impossible. Hence there is no $(3, 14, 2)$-generated group of order 42, and no $(5, 6, 2)$-generated group of order 30. Also for later use, the same argument shows there is no $(3, 10, 2)$-generated group of order 30, and we note that the only $(3, 5, 2)$-generated group is A_5, of order 60.

• *A* $(5, 5, 2)$*-group of order* 40. Any such group G would have a normal Sylow 5-subgroup N (of index 8), but then G/N would be $(1, 1, 2)$-generated, which is impossible.

• *A* $(4, 5, 2)$*-group of order* 80. Any such group G would be soluble, and its commutator subgroup G' would have index 2 and order 40, with a unique Sylow 5-subgroup N that is normal in G, but then G/N would be $(4, 1, 2)$-generated, which is impossible.

• *A* $(3, 9, 2)$*-group of order* 72. Any such group G would be soluble, and its abelianisation G/G' would have order 3. so G' would have order 24, and contain the involution xy. Also x lies in a conjugate of the cyclic Sylow subgroup of order 9 generated by y, so $h^{-1}xh = y^{\pm 3}$ for some $h \in G$, and therefore G' contains $h^{-1}xhx^{-1} = y^{\pm 3}x^{-1} = y^{\pm 3}yy^{-1}x^{-1} = y^{1\pm 3}xy$. Hence G' contains $y^{1\pm 3} = y^4$ or y^7, both of which have order 9, a contradiction.

• *A* $(3, 10, 2)$*-group of order* 60. Any such group G is soluble (because A_5 has no element of order 10), and then a minimal normal subgroup N of G has order 2, 3, 4 or 5. The last three cases are impossible since G/N cannot be $(1, 10, 2)$-, $(3, 5, 1)$- or $(3, 2, 2)$-generated, and so $|N| = 2$. Then similarly, because G/N cannot be $(3, 5, 1)$- or $(3, 10, 1)$-generated, we find that G/N is $(3, 5, 2)$- or $(3, 10, 2)$-generated, but there is no such group of order 30 (as observed at the end of the second bullet point above).

The above arguments exclude 11 of the 21 cases in Table 3, leaving only the 10 pairs $(3, 7)$, $(3, 8)$, $(3, 12)$, $(4, 6)$, $(4, 8)$, $(4, 12)$, $(6, 6)$, $(7, 14)$, $(8, 8)$ and $(12, 12)$.

6.2 Genus 3: existence

Here we examine the remaining 10 pairs (k, ℓ) from Table 3, which will give the total of 12 orientably-regular maps of genus 3 and type (k, ℓ) with $k \leq \ell$. In the course of the analysis we also derive presentations for the corresponding $(k, \ell, 2)$-groups.

• *Pairs* $(k, \ell) = (12, 12)$ *and* $(7, 14)$. For these pairs, $|G| = \ell$ and hence G is cyclic, with $xy = y^{\ell/2}$ as the unique involution, and $x = y^{\ell/2-1}$. This gives unique $(k, \ell, 2)$-presentations $\langle\, x, y \mid x^{12}, y^{12}, (xy)^2, x^5y^{-1} \,\rangle$ for C_{12} and $\langle\, x, y \mid x^7, y^{14}, (xy)^2, x^{-1}y^6 \,\rangle$ for C_{14}.

• *Pairs* $(k, \ell) = (4, 12)$ *and* $(8, 8)$. For these pairs, $|G| = 2\ell$ and hence $N = \langle y \rangle$ is a cyclic subgroup of index 2, and $x^2 \in N$ but $x \notin N$, with $x^2 = y^\varepsilon$ for some ε such that y^ε has order $k/2$. Now conjugation by the involution xy gives $(xy)y(y^{-1}x^{-1}) = xyxx^{-2} = y^{-1}x^{-2} = y^{-1-\varepsilon}$, and so $G \cong \langle y \rangle \rtimes \langle xy \rangle \cong C_\ell \rtimes C_2$. This gives a unique

$(k, \ell, 2)$-presentation in the first case, namely $\langle\, x, y \mid x^4, y^{12}, (xy)^2, x^2y^6 \,\rangle$ for $C_{12} \rtimes C_2$, and two $(k, \ell, 2)$-presentations in the second case, namely $\langle\, x, y \mid x^8, y^8, (xy)^2, x^2y^2 \,\rangle$ and $\langle\, x, y \mid x^8, y^8, (xy)^2, x^2y^{-2} \,\rangle$ for $C_8 \rtimes C_2$.

• *The pair* $(k, \ell) = (6, 6)$. For this pair, the number of distinct products x^iy^j in G is equal to $6 \times 6/|\langle x\rangle \cap \langle y\rangle|$ and as this cannot exceed $|G| = 24$, and G is not cyclic, we find that $x^i = y^j$ for some $i, j \in \{2, 3, 4\}$, and so either $x^2 = y^{\pm 2}$ or $x^3 = y^3$. In the first case the intersection $\langle x^2\rangle = \langle x\rangle \cap \langle y\rangle = \langle y^2\rangle$ would be normal in G and give a $(2, 2, 1)$- or $(2, 2, 2)$-generated quotient of order 8, which is impossible, so $N = \langle x^3\rangle = \langle y^3\rangle$ is normal in G, with G/N a $(3, 3, 2)$-group of order 12 and hence isomorphic to A_4. Also by centrality of $x^3 = y^3$ we find $(x^2y^2)^2 = (x^{-1+3}y^{-1+3})^2 = (x^{-1}y^{-1})^2 = (yx)^{-2} = 1$, so $G \cong \langle x^2, y^2\rangle \times \langle x^3\rangle \cong A_4 \times C_2$, and $\langle\, x, y \mid x^6, y^6, (xy)^2, x^3y^3 \,\rangle$ is a unique $(6, 6, 2)$-presentation for G.

• *The pair* $(k, \ell) = (4, 8)$. Here we consider the permutation representation of the group G (of order 32) on right cosets of $H = \langle y\rangle \cong C_8$. Clearly $H \neq Hx$, and also if $Hx^2 = H$, then $x^2 \in H$ and so $x^2 = y^4$, but then this would be a central involution in G and give a $(2, 4, 2)$-generated quotient of order 8, a contradiction. Hence the four cosets H, Hx, Hx^2 and Hx^{-1} are distinct. On the other hand, right multiplication by y fixes H and takes Hx to $Hxy = Hy^{-1}x^{-1} = Hx^{-1}$, and then since y has even order it must induce an involution, swapping Hx with Hx^{-1} and hence fixing Hx^2. It follows that $xy^{-1}x = x(xyx)x = x^2yx^{-2} \in H$, so $xy^{-1}x = x^2yx^{-2} = y^j$ for some $j \in \{\pm 1, \pm 3\}$, noting that y has order 8, and then squaring gives $x^2y^2x^{-2} = y^{2j}$. Also $(Hx)y^2 = Hx$, so $xy^2x^{-1} \in H$ and hence $xy^2x^{-1} = y^{\pm 2}$ and then conjugating again gives $y^{2j} = x^2y^2x^{-2} = y^2$, which implies that $j \in \{1, -3\}$. Next, conjugation by the involution xy on $\langle x^2, y\rangle \cong \langle y\rangle \rtimes \langle x^2\rangle$ is given by $(xy)y(xy) = xyx^{-1} = xyxx^{-2} = y^{-1}x^2$ and $(xy)x^2(xy) = (y^{-1}x^{-1})x^2(xy) = y^{-1}x^2y = y^{-1}(x^2yx^{-2})x^2 = y^{-1+j}x^2$. This gives $G \cong (\langle y\rangle \rtimes \langle x^2\rangle) \rtimes \langle xy\rangle$, where for $j = 1$ the leftmost semi-direct product is a direct product. Hence we have a unique $(4, 8, 2)$-presentation for each of two groups of order 32, namely $\langle\, x, y \mid x^4, y^8, (xy)^2, xy^{-1}xy^{-1} \,\rangle$ for a semi-direct product $(C_8 \times C_2) \rtimes C_2$ when $j = 1$, and $\langle\, x, y \mid x^4, y^8, (xy)^2, xy^{-1}xy^3 \,\rangle$ for a semi-direct product $(C_8 \rtimes C_2) \rtimes C_2$ when $j = -3$.

• *The pair* $(k, \ell) = (4, 6)$. For this pair, the group G of order 48 is soluble, and G/G' is a non-trivial quotient of the Klein four-group $V_4 \cong C_2^2$, so $|G'| = 12$ or 24, and G/G' has exponent 2, and hence G' contains both x^2 and y^2. Next, a Sylow 3-subgroup P of G cannot be normal in G, for otherwise G/P would be a $(4, 2, 2)$-generated group, and hence also P cannot be normal in G'.

Now suppose $|G'| = 24$. Then $G \cong \mathrm{SL}(2, 3)$, $A_4 \times C_2$ or S_4, because it is known (as a consequence of Sylow theory) that these are the only groups of order 24 not having a normal subgroup of order 3. In the first two cases, G' contains a unique involution, which has to be x^2 and which generates a subgroup N of order 2 that is characteristic in G' and hence normal in G, but with $(2, 3, 2)$-generated quotient G/N of order 12, a contradiction. Hence $G' \cong S_4$. In particular, $G'' \cong A_4$, but then G/G'' has order 4 and hence is abelian, contradicting the fact that G/G' has order 2.

Thus $|G'| = 12$, and $G' \cong A_4$ (the only group of order 12 without a normal subgroup of order 3), and then also $G' = \langle x^2, y^2\rangle$, because A_4 is generated by any pair (a, b) of its elements with $\mathrm{o}(a) = 2$ and $\mathrm{o}(b) = 3$. Clearly the three involutions in $G' \cong A_4$ are x^2 and its conjugates under $y^{\pm 2}$, and these generate a subgroup isomorphic to V_4 that is characteristic in G' and hence normal in G, so $y^{-1}x^2y$ must equal one of them. But if $y^{-1}x^2y = x^2$ then x^2 commutes with y and hence with y^2, while if $y^{-1}x^2y = x^2 =$

$y^{-2}x^2y^2$ then the same thing happens, and so $y^{-1}x^2y = y^2x^2y^{-2}$, which is equivalent to $[x^2, y^3] = 1$.

It follows that the subgroup $K = \langle x^2, y\rangle$ is the direct product $\langle x^2, y^2\rangle \times \langle y^3\rangle \cong A_4 \times C_2$, of order 24 and normal of index 2 in G. In particular, y^3 generates $Z(K)$ and hence is centralised by x (and xy), so $[x, y^3] = 1$, which implies the relation $[x^2, y^3] = 1$ found in the previous sentence. Finally, $x^{-1}y^2x = x^{-1}y^{-1}y^3x = y^3x^{-1}y^{-1}x = y^3yxx = y^{-2}x^2$, while $y^{-1}x^2y = y^2x^2y^{-2}$ as before, and these relations determine conjugation of $G' = \langle x^2, y^2\rangle$ (and hence also conjugation of $\langle G', y^3\rangle = K$) by x and y.

The result is just one possibility for G, namely $(A_4 \times C_2) \rtimes C_2$, with unique $(4, 6, 2)$-presentation $\langle\, x, y \mid x^4, y^6, (xy)^2, [x, y^3]\,\rangle$.

• *The pair* $(k, \ell) = (3, 12)$. Here we consider the permutation representation of the group G (of order 48) on the four right cosets of $H = \langle y\rangle \cong C_{12}$. Clearly multiplication by x induces a 3-cycle (H, Hx, Hx^{-1}), fixing the fourth coset, while multiplication by y fixes H and takes Hx to $Hxy = Hy^{-1}x^{-1} = Hx^{-1}$, but then by transitivity of $\langle x, y\rangle = G$ it cannot fix the fourth coset, and so it must induce the 3-cycle $(Hx, Hx^{-1}, Hx^{-1}y)$. In particular, $Hxy^3 = Hx$ and so $xy^3x^{-1} \in H$, and then because $\langle y\rangle \cong C_{12}$ has no automorphism of order 3, we deduce that $xy^3x^{-1} = y^3$, and therefore $N = \langle y^3\rangle$ is a central cyclic subgroup of order 4. Finally, the quotient G/N of order 12 must be $(3, 3, 2)$-generated and hence is isomorphic to A_4, which makes G a central (non-split) extension of C_4 by A_4, with unique $(3, 12, 2)$-presentation $\langle\, x, y \mid x^3, y^{12}, (xy)^2, [x, y^3]\,\rangle$.

• *The pair* $(k, \ell) = (3, 8)$. In this case the group G (of order 96) is soluble, and $G/G' \cong C_2$ with $x, y^2 \in G'$. In fact, $G' = \langle x, y^2\rangle$, because $y^{-1}x^{-1}y = xyy \in \langle x, y^2\rangle$ and hence the latter subgroup is normal, with index 2. Moreover, the fact that $xy^2 = y^{-1}x^{-1}y$ implies that xy^2 has order 3, and so G' is $(3, 3, 4)$-generated, by $u = x^{-1}$ and $v = xy^2$. Next, let $a = uv = y^2$ and $b = vu = xy^2x^{-1}$. Then clearly $x^{-1}bx = a$ and $yay^{-1} = a$, while $x^{-1}ax = x^{-1}y^2x = xxyyx = xy^{-1}x^{-1}x^{-1}y^{-1} = xy^{-1}xy^{-1} = xy^{-1}xyy^{-2} = xy^{-1}y^{-1}x^{-1}y^{-2} = xy^{-2}x^{-1}y^{-2} = b^{-1}a^{-1}$ and then also $yby^{-1} = yxy^2x^{-1}y^{-1} = x^{-1}y^{-1}y^2yx = x^{-1}y^2x = x^{-1}ax = b^{-1}a^{-1}$, and so $K = \langle a, b\rangle$ is normalised by x and y^{-1}, and hence is normal in G. The quotient G/K is $(3, 2, 2)$-generated and therefore isomorphic to S_3, and so $G'' = K = \langle a, b\rangle = \langle y^2, xy^2x^{-1}\rangle$, which must have order 16 and be generated by two elements of order 4.

Now let N be a minimal normal subgroup of G contained in G''. Then N is an elementary abelian 2-group, but cannot have order 2 because there is no $(3, 8, 2)$-generated group of order $|G|/2 = 48$ (as it would have to give an orientably-regular map on the sphere), and it cannot have order 8 because there is no $(3, s, 2)$-generated group of order $|G|/8 = 12$ with $s \in \{2, 4\}$. Thus $|N| = 4$, with G/N isomorphic to the $(3, 4, 2)$-generated group S_4. Also N contains $y^4 = a^2$ and its two conjugates $xa^2x^{-1} = b^2$ and $x^{-1}a^2x = (ab)^{-2}$, which must be the three involutions in $N \cong V_4$, and so they commute with each other, with trivial product. Hence $(ab)^2 = a^2b^2$ and so $ba = ab$, making K itself abelian, and isomorphic to $C_4 \times C_4$. In particular, $1 = [a, b] = [y^2, xy^2x^{-1}] = y^{-2}xy^{-2}x^{-1}y^2xy^2x^{-1} = y^{-2}xy^{-2}xxyyxy^2xx = y^{-2}xy^{-2}xy^{-1}x^{-1}yxyyxx = y^{-2}xy^{-2}xy^{-1}x^{-2}x^{-1}y^{-1}x = y^{-2}xy^{-2}xy^{-2}x = (y^{-2}x)^3$, and so $(x^{-1}y^2)^3 = 1$. Conversely, this relation implies that $[a, b] = 1$ and hence gives a unique $(3, 8, 2)$-presentation $G = \langle x, y \mid x^3, y^8, (xy)^2, (x^{-1}y^2)^3\rangle$ for $G \cong ((C_4 \times C_4) \rtimes C_3) \rtimes C_2$.

• *The pair* $(k, \ell) = (3, 7)$. In this case G is the unique (smallest) Hurwitz group $\mathrm{PSL}(2, 7)$, which is known to have a unique $(3, 7, 2)$-presentation, namely $\langle x, y \mid x^3, y^7, (xy)^2, [x, y]^4\rangle$; see [19] (or [7]).

6.3 Genus 3: summary

In summary, we have a classification of all orientably-regular maps of genus 3, in a somewhat different form but equivalent to earlier findings by [22], and confirmed by computations in [6, 8, 11]. Note also that all of these maps are reflexible, as in each case the $(k, \ell, 2)$-presentation for G is preserved under inversion of the generators x and y.

Theorem 6.1. *Up to isomorphism and duality, an orientably-regular map M has genus 3 if and only if $G = \mathrm{Aut}^+(M)$ is one of the twelve $(k, \ell, 2)$-presented groups with presentation of the form $G = \langle x, y \mid x^k, y^\ell, (xy)^2, \dots \rangle$ as given in* Table 4*. All of these maps are reflexible.*

Type	$\lvert G\rvert$	Additional relators	Group structure
$(12, 12)$	12	x^5y^{-1}	C_{12}
$(7, 14)$	14	$x^{-1}y^6$	C_{14}
$(4, 12)$	24	x^2y^6	$C_{12} \rtimes C_2$
$(8, 8)$	16	x^2y^2	$C_8 \times C_2$
$(8, 8)$	16	x^2y^{-2}	$C_8 \rtimes C_2$
$(6, 6)$	24	x^3y^3	$A_4 \times C_2$
$(4, 8)$	32	$(xy^{-1})^2$	$(C_8 \times C_2) \rtimes C_2$
$(4, 8)$	32	$xy^{-1}xy^3$	$(C_8 \rtimes C_2) \rtimes C_2$
$(4, 6)$	48	$[x, y^3]$	$(A_4 \times C_2) \rtimes C_2$
$(3, 12)$	48	$[x, y^3]$	$C_4 \cdot A_4$
$(3, 8)$	96	$(x^{-1}y^2)^3$	$((C_4 \times C_4) \rtimes C_3) \rtimes C_2$
$(4, 7)$	168	$[x, y]^4$	$\mathrm{PSL}(2, 7)$

Table 4: Orientably-regular maps of genus 3 (up to duality).

7 Orientably-regular maps of genus 4

For a classification of orientably-regular maps M of genus 4 (for $p = 3$) we need only consider the situation when the order of the group $G = \mathrm{Aut}^+(M)$ is divisible by 3, for the remainder are furnished by Theorem 8.4 of [12].

More specifically, Theorem 8.4 of [12] tells us that up to duality there are exactly four orientably-regular maps M of genus 4 with $\lvert\mathrm{Aut}^+(M)\rvert$ not divisible by 3. These consist of three maps of types $(4, 10)$, $(4, 16)$ and $(10, 10)$ with group G having order 40, 32 and 20 (respectively), all of which arise from case (A2) of Theorem 8.4, and one map of type $(16, 16)$ with group G having order 16, arising from case (A0) of Theorem 8.4.

Again we compile a table of possible types (k, ℓ) with $k \le \ell$ for an orientably-regular map M, this time with the order $\lvert G\rvert = \lvert\mathrm{Aut}^+(M)\rvert$ divisible by 3, k and ℓ – see Table 5.

k	3	3	3	3	3	3	3	3	3	4	4	4	4	5	5	6	6	8	9
ℓ	7	8	9	10	12	15	18	24	42	5	6	8	12	5	10	6	12	8	18
$\mu(k, \ell)$	42	24	18	15	12	10	9	8	7	20	12	8	6	10	5	6	4	4	3
$\lvert G\rvert$	252	144	108	90	72	60	54	48	42	120	72	48	36	60	30	36	24	24	18

Table 5: Potential hyperbolic pairs (k, ℓ) with $k \le \ell$ for $p = 3$.

The values in Table 5 will give eight orientably-regular maps of genus 4, with types $(3, 12)$, $(4, 5)$, $(4, 6)$, $(5, 5)$, $(6, 6)$, $(6, 6)$, $(6, 12)$ and $(9, 18)$. Together with the four maps

M of types $(4, 10)$, $(4, 16)$, $(10, 10)$ and $(16, 16)$ mentioned earlier with $|\mathrm{Aut}^+(M)|$ not divisible by 3, these will give the complete family of all twelve orientably-regular maps of genus 4, up to isomorphism and duality. Again we split the analysis into two subsections, the first dealing with pairs (k, ℓ) from Table 5 for which no suitable $(k, \ell, 2)$-generated group exists.

7.1 Genus 4 and group order divisible by 3: non-existence

• *Groups with a cyclic subgroup of index* 1 *or* 2. These can be handled as for genus 3, and result in elimination of the cases with $(k, \ell) = (3, 24)$ or $(3, 42)$ from Table 5.

• *A* $(3, 18, 2)$*-group* G *of order* 54. In this case, we see that in the natural action of G on the three right cosets of $H = \langle y\rangle$, the generator y fixes H and induces the transposition (Hx, Hx^{-1}), since $Hxy = Hy^{-1}x^{-1} = Hx^{-1}$. Hence the image of this permutation representation of G is isomorphic to S_3, and so its kernel of order 9 is $N = \langle y^2\rangle$. Now N is centralised by y, so conjugation by x and xy induce the same automorphism of N, of order dividing $\gcd(3, 2) = 1$. Hence x centralises y^2, but this implies that $x^{-1}y^{-2}$ has order 9, and yet $x^{-1}y^{-2} = yxy^{-1}$, which has order 3, a contradiction.

• *An* $(8, 8, 2)$*-group* G *of order* 24. Here the formula $|\langle x\rangle\langle y\rangle| = |\langle x\rangle|\,|\langle y\rangle|/|\langle x\rangle \cap \langle y\rangle|$ gives $24 = |G| \geq |\langle x\rangle\langle y\rangle| = 64/|\langle x\rangle \cap \langle y\rangle|$ and so $|\langle x\rangle \cap \langle y\rangle| = 8$ or 4, but the former is impossible (as G is not cyclic), and so is the latter since it would imply that $K = \langle x\rangle \cap \langle y\rangle = \langle x^2\rangle = \langle y^2\rangle$ is central in G, with $(2, 2, 2)$-generated quotient G/K of order 6.

• *A* $(5, 10, 2)$*-group* G *of order* 30. In this case, if N is a minimal normal subgroup of G, then $|N| \neq 2$ or 5, as G/N cannot be a $(5, 5, 1)$-generated group of order 15 or a $(1, 2, 2)$-generated group of order 6, so $|N| = 3$. But then any element of order 5 (such as x) centralises N, so G has a cyclic subgroup of order 15 and index 2, and hence a normal subgroup of order 5 after all, a contradiction.

• *A* $(4, 12, 2)$*-group* G *of order* 36. In this case, in the natural action of G on the three right cosets of $H = \langle y\rangle$, the generator x of order 4 induces a transposition, and so $x^2 \in H$, but then $(xy)^{-1}y(xy) = y^{-1}x^{-1}yy^{-1}x^{-1} = y^{-1}x^{-2} \in H$ and therefore $G = \langle y, xy\rangle = \langle y\rangle \rtimes \langle xy\rangle$, which has order 24, a contradiction.

• *A* $(4, 8, 2)$*-group* G *of order* 48. Here the number of Sylow 2-subgroups of G (with order 16) is 1 or 3, but is not 1 since C_3 cannot be a quotient of the $(4, 8, 2)$-generated group G, similarly, is not 3 because neither C_3 nor S_3 can be a quotient of G.

• *A* $(3, 15, 2)$*-group* G *of order* 60. In this case $H = \langle y\rangle$ has index 4 and so the element y^3 (of order 5) must induce the identity permutation on right cosets of H, and therefore $Hxy^3 = Hx$, which implies that x normalises $N = \langle y^3\rangle \cong C_5$. Moreover, since x has order 3, it must centralise y^3. It follows that N is a central subgroup of $\langle x, y\rangle = G$, of order 5, with the $(3, 3, 2)$-generated quotient G/N of order 12 isomorphic to A_4. By the Schur-Zassenhaus theorem, $G \cong C_5 \rtimes A_4$ and hence $G \cong C_5 \times A_4$ (because $N \cong C_5$ is central) and hence. But now the pre-image V in G of the subgroup V_4 of A_4 is normal in G, which is impossible because G/V cannot be $(3, 15, 1)$-generated.

• *A* $(3, 10, 2)$*-group* G *of order* 90. Here G is soluble, and its abelianisation G/G' must be C_2, so $|G'| = 45$. But then by Sylow theory, G' has a normal Sylow 3-subgroup N of order 9, which is characteristic in G' and hence normal in G, which is impossible since G/N cannot be $(1, 10, 2)$-generated.

• *A* $(3,9,2)$*-group* G *of order* 108. In this case we observe that G is soluble, and that there is no $(3,9,2)$-generated group of order 54 or 36, by Euler's formula and the fact that there is no orientably-regular map of genus 2 with type $\{3,9\}$. Hence every non-trivial quotient of G is isomorphic to either the $(3,3,1)$-generated group C_3, or the $(3,3,2)$-generated group A_4, or G itself, and so G has a unique minimal normal subgroup N of order 9, isomorphic to C_3^2, and with quotient $G/N \cong A_4$. Now $G/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N) \cong \mathrm{Aut}(C_3^2) \cong \mathrm{GL}(2,3)$, and this has no subgroup H isomorphic to A_4 (because A_4 has no subgroup of index 2 and yet cannot be isomorphic to a subgroup $\mathrm{SL}(2,3)$ as it contains more than one involution). It follows that $G/C_G(N) \cong C_3$ and hence $C_G(N) \cong N \times K \cong C_3^2 \times V_4$ for some subgroup $K \cong V_4$, by the Schur-Zassenhaus theorem. But then because K is characteristic in $C_G(N)$ and hence normal in G, we find that G has a quotient G/K of order 27, a contradiction.

• *A* $(3,8,2)$*-group* G *of order* 144. In a similar way to the previous case, such a group G is soluble, and there is no $(3,8,2)$-generated group of order 24 or 72 (by Euler's formula), and it follows that every non-trivial quotient of G is isomorphic to C_2, or S_3, or S_4, or the unique $(3,8,2)$-generated group H of order 48 (associated with the orientably-regular map of genus 2 with type $\{3,8\}$), or G itself, Thus G has a unique minimal normal subgroup N, of order 3. Moreover, from the classification of orientably-regular maps of genus 2 we know that $G/N \cong H$ has $(3,8,2)$-presentation $\langle\, u,v \mid u^3, v^8, (uv)^2, [u,v^4] \,\rangle$, and hence the normal subgroup N of order 3 is generated by $z = [x,y^4]$. Then since $\mathrm{Aut}(N) \cong \mathrm{Aut}(C_3) \cong C_2$, we find that N is centralised by x (of order 3) and y^2, and in particular, $[z,y^4] = 1$ and therefore zy^4 has order $3 \cdot 2 = 6$. But also $x^{-1}y^4x = x^{-1}y^{-4}x = [x,y^4]y^4 = zy^4$, and so zy^4 is conjugate to y^4, which has order 2, a contradiction.

• *A* $(3,7,2)$*-group* G *of order* 252. Every $(3,7,2)$-generated group is perfect and therefore insoluble, but every group of order 252 is soluble, as its order is not divisible by 60 or 168.

7.2 Genus 4 and group order divisible by 3: existence

In this subsection we examine the remaining seven pairs (k,ℓ) from Table 5, which will give the eight orientably-regular maps of genus 4 and type (k,ℓ) with $k \le \ell$ and with $|G| = |\mathrm{Aut}^+(M)|$ divisible by 3, and we give presentations for the corresponding groups.

• *The pair* $(k,\ell) = (9,18)$. In this case, a $(9,18,2)$-group G of order 18 is cyclic, with unique $(9,18,2)$-presentation $\langle\, x,y \mid x^9, y^{18}, (xy)^2, x^{-1}y^8 \,\rangle$.

• *The pair* $(k,\ell) = (6,12)$. Here the subgroup $H = \langle y\rangle$ is normal in G of index 2 in G, so $(xy)y(xy)^{-1} = xyx^{-1} = y^j$ for some square root j of 1 mod 12, namely ±1 or ±5. Also $x \notin H$ but $x^2 \in H$, so $x^2 = y^{\pm4}$ (because it has order 3), and now it follows that $y^{\mp4} = x^{-2} = (yxy)(yy^{-1}x^{-1}) = y(xy)y(xy)^{-1} = yy^j = y^{j+1}$, implying that $j+1 \in \{\mp4\}$ and therefore $j = -5$, and $x^2 = y^4$. This gives $G = \langle y\rangle \rtimes \langle xy\rangle \cong C_{12} \rtimes C_2$, with unique $(6,12,2)$-presentation $\langle\, x,y \mid x^6, y^{12}, (xy)^2, x^{-2}y^4 \,\rangle$.

• *The pair* $(k,\ell) = (6,6)$. In this case $\langle x\rangle \cap \langle y\rangle$ is trivial, for otherwise it would be a central subgroup N order 2 or 3 and then G/N would be a $(3,3,2)$-generated group of order 18 or a $(2,2,2)$-generated group of order 12, both of which are impossible. It follows that $G = \langle x\rangle\langle y\rangle$, and so in the action of G by right multiplication on the right cosets of $H = \langle y\rangle$, the generator x induces a 6-cycle. On the other hand, right multiplication by y fixes H, and takes Hx to Hx^{-1} as usual, since $Hxy = Hy^{-1}x^{-1} = Hx^{-1}$.

Now if also $Hx^{-1}y = Hx$, then $x^{-1}yx^{-1} \in H$ and so $x^{-1}y^2x = x^{-1}yx^{-1}y^{-1} \in H$, which implies that $\langle y^2\rangle$ is normal in G, and indeed $x^{-1}y^2x = y^{2j}$ where $j = \pm1$. Hence

$K = \langle y^2, x\rangle$ is a subgroup of order 18 and index 2, isomorphic to the direct product $C_3 \times C_6$ when $j = 1$ or a semi-direct product $C_3 \rtimes C_6$ when $j = -1$, and then $G \cong K \rtimes C_2$, with conjugation of K by the involution xy given by $(xy)^{-1}x(xy) = y^{-1}xy = y^{-2}x^{-1}$ and $(xy)^{-1}y^2(xy) = y^{-1}x^{-1}y^2xy = y^{-1}y^{2j}y = y^{2j}$. Moreover, when $j = -1$ the relation $x^{-1}y^2x = y^{-2}$ implies $1 = x^{-1}y^2xy^2 = x^{-1}y(yxy)y = x^{-1}yx^{-1}y$ (and vice versa), and hence is equivalent to $(x^{-1}y)^2 = 1$. Accordingly, we obtain two possibilities for G, namely $(C_3 \times C_6) \rtimes C_2$ with unique $(6, 6, 2)$-presentation $\langle\, x, y \mid x^6, y^6, (xy)^2, [x, y^2]\,\rangle$, and $(C_3 \rtimes C_6) \rtimes C_2$ with unique $(6, 6, 2)$-presentation $\langle\, x, y \mid x^6, y^6, (xy)^2, (x^{-1}y)^2\,\rangle$.

On the other hand, suppose $Hx^{-1}y \neq Hx$. Then since y has order 6 it follows that Hx and Hx^{-1} lie in a 3-cycle, so $Hxy^3 = Hx$, which gives $xy^3x^{-1} \in H$ and so $xy^3x^{-1} = y^3$, and then $Hx^iy^3 = Hx^i$ for every i, so y must induce a single 3-cycle on right cosets of H and hence fix Hx^2 or Hx^{-2} (as well as H). It follows that $x^2yx^{-2} \in H$ or $x^{-2}yx^2 \in H$, and so $x^2yx^{-2} = y^{\pm1}$ or $x^{-2}yx^2 = y^{\pm1}$, and in both cases this implies that x^2 centralises y, because x^2 has order 3. Thus $[x^2, y] = 1$, which is dual to one of the two sub-cases above.

• *The pair* $(k, \ell) = (5, 5)$. First, suppose G is soluble, and let N be a minimal normal subgroup of G. Then $|N| \neq 5$ or 3 or 4, for otherwise G/N would be $(1, 1, 2)$-generated, or a $(5, 5, 2)$-generated group of order 20 with a normal Sylow 5-subgroup K/N such that $(G/N)/(K/N)$ is $(1, 1, 2)$-generated, or a $(5, 5, 1)$ generated group of order 15. Thus $|N| = 2$, and G/N is a $(5, 5, 2)$-generated group of order 30, but this is also impossible, for otherwise G/N would have a normal Sylow 3-subgroup K/N such that $(G/N)/(K/N)$ is a $(5, 5, 2)$-generated group of order 10 (again with a $(1, 1, 2)$-generated quotient).

Thus G is insoluble, and so $G \cong A_5$. Finally, it is well known (and quite easy to verify) that A_5 can be $(5, 5, 2)$-generated in just one way up to conjugacy in $\mathrm{Aut}(A_5) \cong S_5$, such as by $x = (12345)$ and $y = (12435)$, and then because $xy^{-1} = (2, 4, 3)$ for this x and y (with xy^{-1} and y^2 giving the more standard $(3, 5, 2)$-presentation for A_5), it follows that $\langle\, x, y \mid x^5, y^5, (xy)^2, (xy^{-1})^3\,\rangle$ is a unique $(5, 5, 2)$-presentation for G.

• *The pair* $(k, \ell) = (3, 12)$. In this case, again let $H = \langle y\rangle$, and consider the action of G by right multiplication on the six cosets of H. As y has order 12, it must induce a permutation of order dividing 4 or 6. Now if y^6 fixes Hx, then $xy^6x^{-1} \in H$ and so $xy^6x^{-1} = y^6$, and then $K = \langle y^6\rangle$ is central in G with $(3, 6, 2)$-generated quotient G/K of order 36. Such a quotient would be the group of orientation-preserving automorphisms of a toroidal map of type $(3, 6)$. By the theory of such maps (as in [13]), however, this is impossible because $36 = |G/K|$ is not equal to $6(a^2 - ab + b^2)$ for any integers a and b. Hence y^4 fixes Hx, which gives $xy^4x^{-1} \in H$ and so $xy^4x^{-1} = y^{\pm4}$, indeed $xy^4x^{-1} = y^4$ since x has order 3. Thus $N = \langle y^4\rangle \cong C_3$ is central in G, with $(3, 4, 2)$-generated quotient G/N isomorphic to S_4, and it follows that G is isomorphic to a central extension $C_3 \cdot S_4$ of C_3 by S_4, with unique $(3, 12, 2)$-presentation $\langle\, x, y \mid x^3, y^{12}, (xy)^2, [x, y^4]\,\rangle$.

• *The pair* $(k, \ell) = (4, 6)$. Here we start by noting that potential non-smooth quotients of G and hence of $\Delta^+(4, 6, 2)$ are abelian of order 1, 2 or 4, dihedral of order 6, 8 or 12, or isomorphic to the $(4, 3, 2)$-generated group S_4. Next, G (of order 72) is soluble, and so a minimal normal subgroup N has order 2, 3, 4, 8 or 9. But $|N| \neq 2$, for otherwise G/N would be a $(4, 6, 2)$-generated group of order 36, violating Euler's theorem, and $|N| \neq 4$ or 8, for otherwise G/N would be a non-smooth quotient of of $\Delta^+(4, 6, 2)$ of order 9 or 18. Hence $|N| = 3$ or 9, and $N \cong C_3$ or C_3^2.

Now if $N \cong C_3$ then the quotient G/N of order 24 cannot be $(4, 2, 2)$-generated, so must be $(4, 6, 2)$-generated, and by Euler's formula is isomorphic to the orientation-preserving group of automorphisms of the regular map of genus 2 with type $\{4,6\}$, having presentation $\langle\, u, v \,|\, u^4, v^6, (uv)^2, (uv^{-1})^2 \,\rangle$. Thus $N = \langle z \rangle$ where $z = (xy^{-1})^2$, and by normality of N it follows that every element of G either centralises or inverts z by conjugation. In particular, both xy^{-1} and y^2 must centralise z, and therefore so does $(xy^{-1})^{-1}x^2 = yx$, and it follows that zyx has order $3 \cdot 2 = 6$. But on the other hand, $zyx = (xy^{-1})^2yx = xy^{-1}x^2 = xy^{-1}x^{-2} = x(y^{-1}x^{-1})x^{-1} = x(xy)x^{-1}$, so zyx has order 2, a contradiction.

Hence $N \cong C_3^2$, and G/N is isomorphic to the $(4,2,2)$-generated group D_4, and then by the Schur-Zassenhaus theorem, $G \cong C_3^2 \rtimes D_4$. Next, N contains y^2 and its conjugate $x^{-1}y^2x$, and $x^{-1}y^2x \neq y^{\pm 2}$ for otherwise $L = \langle y^2 \rangle$ would be normal in G with $(4,2,2)$-generated quotient of order 36, and it follows that $N = \langle y^2, x^{-1}y^2x \rangle$. Also because D_4 has abelianisation C_2^2, the abelianisation G/G' of G is the $(2,2,2)$-generated group C_2^2, and therefore $G' = \langle N, x^2 \rangle$, of order 18. Moreover, x^2 cannot centralise y^2, for otherwise x^2 centralises $x^{-1}y^2x$ and hence also G', but then $G' = \langle y^2, x^{-1}y^2x \rangle \times \langle x^2 \rangle \cong N \times C_2$, which implies that $\langle x^2 \rangle$ is characteristic in G' and hence normal in G, with $(2,6,2)$- or $(2,3,2)$-generated quotient of order 36, which is impossible. Hence conjugation by x induces an automorphism of N of order 4. Then since $\mathrm{Aut}(N) \cong \mathrm{Aut}(C_3^2) \cong \mathrm{GL}(2,3)$, conjugation by x^2 behaves like the only involution in $\mathrm{SL}(2,3)$, namely $-I_2$, and hence inverts every element of N. In particular, $x^{-2}y^2x^2 = y^{-2}$ and so $(x^2y^2)^2 = 1$, which implies this in reverse. Finally, conjugation of N by G is given by the 4-cycles $(y^2, x^{-1}y^2, y^{-2}, x^{-1}y^{-2}x)$ induced by x on $\{y^{\pm 2}, x^{-1}y^{\pm 2}x\}$ and the assignment $(y^2, x^{-1}y^2x) \mapsto (y^2, x^{-1}y^{-2}x)$ induced by y, noting that $y^{-1}(x^{-1}y^2x)y = xyy^2y^{-1}x^{-1} = xy^2x^{-1} = x^{-1}y^{-2}x$, and we have a unique $(4,6,2)$-presentation $\langle\, x, y \,|\, x^4, y^6, (xy)^2, (x^2y^2)^2 \,\rangle$ for G.

• *The pair* $(k,\ell) = (4,5)$. In this case, first suppose that the group G (of order 120) is soluble, and let N be a minimal normal subgroup of G. Then $|N| \neq 5$ for otherwise G/N would be $(4,1,2)$-generated, and $|N| \neq 3$ for otherwise G/N would be a $(4,5,2)$-generated group of order 40 and hence give rise to an orientably-regular map of genus 2 with type $\{4,5\}$, but no such map exists. Hence $|N|$ divides 8, and as x has order 4 it follows that $N \cong C_2$ or C_2^2, but then since G/N has order 60 or 30 it cannot be $(2,5,2)$-generated, so it must be $(4,5,2)$-generated, which is impossible by Euler's formula when $|G/N| = 60$, and because there is no element order 4 in G/N when $|G/N| = 30$.

Thus G is insoluble, and hence isomorphic to one of $\mathrm{SL}(2,5)$, $A_5 \times C_2$ or S_5. The first two groups contain a central subgroup K of order 2, with quotient $G/K \cong A_5$, but this is neither $(4,5,2)$-generated nor $(2,5,2)$-generated, and so $G \cong S_5$. Finally, it is well known (and easy to verify) that S_5 can be $(4,5,2)$-generated in just one way up to conjugacy in $\mathrm{Aut}(S_5) \cong S_5$, such as by $x = (1234)$ and $y = (15432)$, with $xy = (45)$ and $[x,y] = (145)$ and with $\langle y, x^{-1}yx \rangle = A_5$, it follows that $\langle\, x, y \,|\, x^4, y^5, (xy)^2, [x,y]^3 \,\rangle$ is a unique $(4,5,2)$-presentation for G.

7.3 Genus 4: summary

The analysis in the previous two subsections contributes to a complete classification of the orientably-regular maps of genus $p+1$ for $p = 3$, up to isomorphism and duality. These arise from eight kinds of map M for which the order of the group $G = \mathrm{Aut}^+(M)$ is a multiple of 3 as determined above, plus four more for which $|G|$ is not divisible by 3, coming from the classification in [12]. The result is equivalent to the classification given with just an indication of the method of proof in [15], and is confirmed by computations described in [6, 8, 11]. Note also that all of these maps are reflexible, as in each case the $(k, \ell, 2)$-presentation for G is preserved under inversion of the generators x and y.

Theorem 7.1. *Up to isomorphism and duality, an orientably-regular map M has genus 4 if and only if $G = \mathrm{Aut}^{(}M)$ is one of the twelve $(k, \ell, 2)$-presented groups with presentation of the form $G = \langle x, y \mid x^k, y^\ell, (xy)^2, \ldots \rangle$ as given in* Table 6. *All of these maps are reflexible.*

Type	$\lvert G\rvert$	Additional relators	Group structure
$(9, 18)$	18	$x^{-1}y^8$	C_{18}
$(6, 12)$	24	$x^{-2}y^4$	$C_{12} \rtimes C_2$
$(6, 6)$	36	$[x, y^2]$	$(C_3 \times C_6) \rtimes C_2$
$(6, 6)$	36	$(x^{-1}y)^2$	$(C_3 \rtimes C_6) \rtimes C_2$
$(5, 5)$	60	$(xy^{-1})^3$	A_5
$(3, 12)$	72	$[x, y^4]$	$C_3 \cdot S_4$
$(4, 6)$	72	$(x^2y^2)^2$	$(C_3 \times C_3) \rtimes D_4$
$(4, 5)$	120	$[x, y]^3$	S_5
$(16, 16)$	16	$x^{-1}y^7$	C_{16}
$(10, 10)$	20	x^2y^2	$C_{10} \times C_2$
$(4, 16)$	32	x^2y^8	$C_{16} \rtimes C_2$
$(4, 10)$	40	$(xy^{-1})^2$	$(C_2 \times C_{10}) \rtimes C_2$

Table 6: Orientably-regular maps of genus 4 (up to duality).

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