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Realizations of the higher rank toroids

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Abstract

We determine and fully describe all pure realizations of the cubic toroids, that is, the regular $(n+1)$-polytopes of type $\{4, 3^{n-2}, 4\}_{\mathbf{b}}$, where $\mathbf{b} = (b, 0^{n-1}), (b, b, 0^{n-2})$ or (b^n), $n \geqslant 2$ and $b \geqslant 2$.

Keywords: Abstract regular polytopes, regular toroids, realizations.

Math. Subj. Class.: 51M20, 52B15

1 Introduction

In [13, 14] we explicitly described all the pure realizations of the various toroidal maps of types $\{3, 6\}_{(b,0)}$ and $\{3, 6\}_{(b,b)}$ (and their duals), and then $\{4, 4\}_{(b,0)}$ and $\{4, 4\}_{(b,b)}$. The constructive part of these papers involves looking at a twisting operation applied to a Coxeter group K, which is in turn the direct product of a certain number of copies of the dihedral group $\mathbb{I}_b$ of order $2b$. This is illustrated in Figure 1, which is a reworking of [14, Figure 2]. A beautiful feature of the construction is that hidden in a picture of the toroid itself we find a parametrization of all pure realizations, along with their dimensions.

To prove that our list of realizations was complete we used the character-like relations for pure realizations found in [9] and [15]. Unfortunately, these relations needed the corrections in [8]. In fact, because of our Proposition 3.3, we can detour around this hazard. This simple, but very useful, fact seems not to be explicitly mentioned in the literature.

In Section 4, we generalize the results of [14] to include all cubical toroids of rank $n+1 \geqslant 4$. The core result is Theorem 4.16, where we find that the pure realizations of the cubical $(n+1)$-toroid $\mathcal{H}_{(b,0,\ldots,0)}$ are indexed by integer parameters $0 \leqslant l_1 \leqslant l_2 \leqslant \ldots \leqslant l_n \leqslant b/2$ taken modulo b. For each pure realization the corresponding group representation is explicitly described. The dimensions of the realizations are given in Theorem 5.4. In

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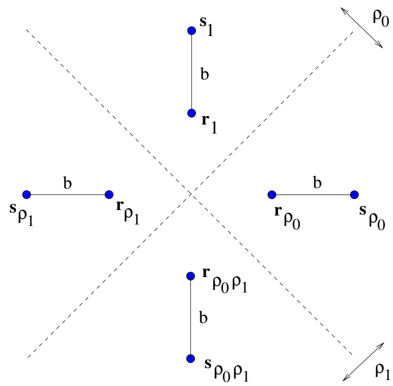


Figure 1: The Coxeter group K for toroids of type $\{4,4\}$. (In $\overline{B}_2$, $\rho_0\rho_1 = \rho_1\rho_0$.)

fact, most pure realizations will have dimension $2^n \cdot n!$. Our two main theorems have a particularly nice visual description set out in Corollary 5.6.

The pure realizations for the cubical toroids $\mathcal{H}_{(c^k,0^{n-k})}$, where $k = 2$ or n, are exhibited in Theorems 6.1 and 6.2.

First, however, let us set up the machinery we need to construct the cubical toroids of higher rank. Proposition 2.5(b), which led the author astray, may be unfamiliar to the reader.

2 Cubes and cubical toroids

The n-cube $\mathbf{P}$ is likely the most familar regular convex polytope in Euclidean n-space $\mathbb{E}^n$ [1, Section 7.2]. To begin our discussion, let us describe $\mathbb{E}^n$ in the customary way as the vector space of real n-tuples $\mathbf{x} = (x_1, \ldots, x_n)$ equipped with the standard inner product. As vertices of $\mathbf{P}$ we may take the 2^n sign change vectors

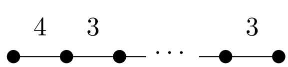
$$(e_1, \ldots, e_n) \in \{\pm 1\}^n. \tag{2.1}$$

The symmetry group B_n of the cube now consists of all permutations and sign changes of the n coordinates, so that the group has order $2^n \cdot n!$.

Recall that $B_n \simeq S_n \ltimes C_2^n$ is a semidirect product; each element $\gamma \in B_n$ factors uniquely as $\gamma = \alpha\mu$, where $\alpha \in S_n$ is a permutation of $\{1, \ldots, n\}$ (labelling the n coordinates); and $\mu = (\mu_1, \ldots, \mu_n) \in C_2^n = \{\pm 1\}^n$ is a sign change vector, just as in equation (2.1). (One can more precisely view α as an $n \times n$ permutation matrix and μ as a diagonal matrix.) The rule $\mu\alpha = \alpha\mu^\alpha$, with $\mu^\alpha := \alpha^{-1}\mu\alpha$, makes easy work of calculations in B_n. The action of B_n on $\mathbb{E}^n$ is described by

$$\mathbf{x}^\alpha = (x_{1\alpha^{-1}}, x_{2\alpha^{-1}}, \ldots, x_{n\alpha^{-1}}) \ \text{ and } \ \mathbf{x}^\mu = (\mu_1 x_1, \ldots, \mu_n x_n)\,.$$

From a more abstract point of view, $B_n = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is (isomorphic to) the Coxeter group having the diagram



(2.2)

For $1 \leqslant j \leqslant n-1$, the generator ρ_j can be identified with the transposition $(j, j+1)$; and ρ_0 with the sign change in the first coordinate, that is, reflection in the first coordinate hyperplane (orthogonal to $(1, 0, \dots, 0)$). Thus reflection in the j-th coordinate hyperplane, $1 \leqslant j \leqslant n$, is

$$\sigma_j := \rho_0^{\rho_1 \cdots \rho_{j-1}} . \tag{2.3}$$

The product of these n special reflections, in any order, is the central element $\zeta\colon \mathbf{x} \mapsto -\mathbf{x}$. Note that $\sigma_1 = \rho_0$. It is easy to check that

$$\zeta = (\rho_{n-1} \cdots \rho_1 \rho_0)^n = (\kappa \rho_0)^n \ , \tag{2.4}$$

where for future use we let $\kappa := \rho_{n-1} \cdots \rho_1$. Thus κ can be identified with the n-cycle $(1, 2, \dots, n)$.

Note that the subgroup $\langle \rho_1, \dots, \rho_{n-1} \rangle$ of B_n is isomorphic to the symmetric group S_n (and to the Coxeter group A_{n-1}). Up to similarity there is clearly a unique non-zero point $\mathbf{p} = (1, 1, \dots, 1)$ fixed by S_n. We may regard $\mathbf{p}$ as a *base vertex*; its orbit under B_n is the full vertex set of $\mathbf{P}$, as described in (2.1). The *base edge* of $\mathbf{P}$ has vertices $\mathbf{p}$ and $\mathbf{p}^{\rho_0} = (-1, 1, \dots, 1)$, so that the edge length is 2. In fact, for any j, the base j-face is the j-cube whose vertex set is the orbit of $\mathbf{p}$ under the subgroup $\langle \rho_0, \dots, \dots, \rho_{j-1} \rangle$.

Just as we can view B_n either as a concrete group of signed permutation matrices or as an abstract Coxeter group, we can likewise reimagine the regular convex polytope $\mathbf{P}$ in combinatorial terms. To do so, just take the abstract n-cube $\mathcal{P} = \{4, 3^{n-2}\}$ to be the face lattice of $\mathbf{P}$.

We will soon encounter other abstract regular polytopes, but there is little need to outline here the general theory (covered fully in [12]). Let us only recall that an abstract n-polytope $\mathcal{Q}$ is a ranked, partially ordered set (with quite natural properties, of course). Its elements of rank j, where $-1 \leqslant j \leqslant n$, are called j-faces; these correspond to faces of dimension j in the convex case. But $\mathcal{Q}$ can be very general; it need not be finite nor a lattice (qua poset).

We say that the n-polytope $\mathcal{Q}$ is (abstractly) *regular* if its automorphism group $\Gamma(\mathcal{Q})$ is transitive on *flags* (maximal chains) in $\mathcal{Q}$. It follows [12, Chapter 2] that $\Gamma(\mathcal{Q})$ is a *string C-group*. This means that

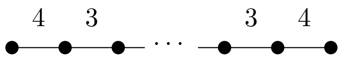
$$\Gamma(\mathcal{Q}) = \langle \gamma_0, \dots, \gamma_{n-1} \rangle$$

is generated in the manner of a Coxeter group with string diagram on n nodes; and furthermore, these generators satisfy a natural 'intersection condition' [12, Proposition 2B10]. Conversely, from any such group Γ we can reconstruct $\mathcal{Q}$ up to isomorphism as a coset geometry over the group [12, Theorem 2E11]. Now the j-faces are all right cosets of $\Gamma_j = \langle \gamma_0, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_{n-1} \rangle$, taking Γ_{-1} and Γ_n to be two distinct copies of Γ. For $j \leqslant k$, two cosets of Γ_j and Γ_k are incident just when their intersection is non-empty.

We see all this for the abstract cube above. More generally, the classical convex regular polytopes are isomorphic to those abstract regular polytopes for which the symmetry group actually is a finite Coxeter group (with string diagram). But usually, the generators of $\Gamma(\mathcal{Q})$ satisfy 'extra' relations which prevent the group from being a Coxeter group; see, for example, equation (2.8) for the toroids appearing below.

Let us now examine a familiar infinite abstract regular polytope. The $(n+1)$-polytope $\mathcal{H} = \{4, 3^{n-2}, 4\}$ is realized faithfully by the face-to-face tessellation $\mathbf{H}$ of $\mathbb{E}^n$ by identical n-cubes, which we may take to be copies of $\mathbf{P}$ from before. The automorphism group $\Gamma(\mathcal{H})$

is the infinite Coxeter group $\widetilde{C_n} = \langle \rho_0, \ldots, \rho_{n-1}, \rho_n \rangle$ with diagram



The symmetry group of the *base facet* for $\mathcal{H}$ is just the subgroup $B_n = \langle \rho_0, \ldots, \rho_{n-1} \rangle$. The final generator $\rho_n \colon (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1}, 2 - x_n)$ is reflection in the hyperplane $x_n = 1$. We have

$$\Gamma(\mathcal{H}) \simeq \widetilde{C_n} \simeq \langle \tau_1, \ldots, \tau_n \rangle \rtimes B_n, \tag{2.5}$$

where $T = \langle \tau_1, \ldots, \tau_n \rangle \simeq \mathbb{Z}^n$ is the *translation subgroup*. Here τ_j is translation through 2 units parallel to the x_j-axis. In terms of the ρ_j's we have

$$\tau_1 = \rho_0 \rho_1 \cdots \rho_{n-1} \rho_n \rho_{n-1} \cdots \rho_1 = \rho_0 \cdot \rho_n^{\rho_{n-1} \cdots \rho_1}, \tag{2.6}$$

and

$$\tau_j = \tau_1^{\rho_1 \cdots \rho_{j-1}}, \; 1 \leqslant j \leqslant n.$$

Notice that ρ_n also fixes the base vertex $\mathbf{p} = (1, 1, \ldots, 1)$ for the cube $\mathbf{P}$. It is because of this that $\Gamma(\mathcal{H})$ takes copies of $\mathbf{P}$ and effectively assembles them face-to-face, so as to produce the tessellation $\mathbf{H}$. Furthermore, the stabilizer of $\mathbf{p}$ in $\Gamma(\mathcal{H})$ is the *vertex-figure subgroup* $\Gamma_0 = \langle \rho_1, \ldots, \rho_n \rangle$; and we have a dual semidirect factorization

$$\Gamma(\mathcal{H}) \simeq \langle \tau_1, \ldots, \tau_n \rangle \rtimes \Gamma_0. \tag{2.7}$$

The important geometric conclusion is that the translation subgroup acts in a sharply transitive way on the vertex set of $\mathbf{H}$. (Another consequence is that both Γ_0 and B_n $(= \Gamma_n)$ are isomorphic to the *point group* for the crystallographic group $\widetilde{C_n}$.)

We now move to the toroidal polytopes central to this paper, following [12, Section 6D]. In general, a *regular* $(n+1)$*-toroid* $\mathcal{Q}_\Lambda$ is the quotient of a regular tessellation $\mathcal{Q}$ of $\mathbb{E}^n$ by a (non-trivial) normal subgroup Λ of translations in $\Gamma(\mathcal{Q})$. We specialize right away to the cubical tessellation $\mathcal{H}$, with its faithful realization $\mathbf{H}$. (Here we anticipate Section 3. Put simply, $\mathbf{H}$, with its natural facial structure, is isomorphic to the partially ordered set $\mathcal{H}$.)

The *regular cubical toroid* $\mathcal{H}_\Lambda$ can be constructed inside a familiar topological space, namely the quotient of $\mathbb{E}^n$ by a non-trivial, normal subgroup Λ of translation symmetries. For example, if $\tau_1^b \in \Lambda$, for some integer $b \geqslant 1$, then $\tau_j^b \in \Lambda$ for all j. If these generate Λ, then we may think of $\mathcal{H}_\Lambda$ as a $b \times \cdots \times b$ block of n-cubes, still of edge length 2 and all packed into a larger n-cube whose outermost facets (supported by hyperplanes $x_j = -b, x_j = b$) are identified in pairs.

It suits us in what follows to focus on the groups and so work on a more abstract level. First of all, to generate Λ as a *normal* subgroup of $\widetilde{C_n}$ we actually need only one translation, say $\tau = \tau_1^{b_1} \cdots \tau_n^{b_n}$. After applying symmetries in B_n, we can further assume that $\mathbf{b} = (b_1, \ldots, b_n)$ takes one of the forms described in Theorem 2.1. (See the discussion preceding Proposition 5.10 below.) We use $(b^k, 0^{n-k})$ as shorthand for the vector with b repeated k times and 0 repeated $n - k$ times.

Theorem 2.1 ([12, Theorems 6D1, 6D4]). *For $n \geqslant 2$ the regular cubical $(n+1)$-toroids can be described as follows. For $b \geqslant 2$ and $k = 1, 2$ or n, let $\mathbf{b} = (b^k, 0^{n-k})$. Then there is a finite regular toroid $\mathcal{H}_{\mathbf{b}} = \{4, 3^{n-2}, 4\}_{\mathbf{b}}$ whose facets are isomorphic to the n-cube and whose vertex-figures are isomorphic to the n-cross-polytope. Moreover, $\mathcal{H}_{\mathbf{b}}$ is self-dual.*

The automorphism group $\Gamma(\mathcal{H}_{\mathbf{b}})$ is obtained from $\widetilde{C}_n$ by factoring out the single extra relation

$$(\rho_0\rho_1 \cdots \rho_{n-1}\rho_n\rho_{n-1} \cdots \rho_k)^{bk} = 1. \tag{2.8}$$

This group has order $2^{n+k-1} \cdot b^n \cdot n!$; and the toroid $\mathcal{H}_{\mathbf{b}}$ has $2^{k-1} \cdot b^n$ vertices.

Remark 2.2. When $n = 2$ there are actually only two families of 3-toroids, with $\mathbf{b} = (b, 0)$ or (b, b). If, say, in the second case we choose $b = 1$, then Λ is generated by translations with vectors $(1, 1)$ and $(1, -1)$. If we identify the opposite edges of the square spanned by these two vectors, then we do get a perfectly good map on the torus, with 2 vertices, 4 edges and 2 faces. However, this map is not polytopal, since each vertex lies on all four edges of each square. We refer here to the so-called 'diamond property' for abstract polytopes, which demands that exactly *two* j-faces be trapped between any incident $(j-1)$-face and $(j+1)$-face [12, page 25]. This accounts (in all ranks) for the restriction $b \geqslant 2$.

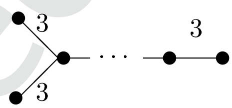
Notice that with $b = 1 = k$ in (2.8) we get $\rho_n = \rho_0^{\rho_1 \cdots \rho_{n-1}}$. We labelled this element σ_n in (2.3) and identified it as reflection in the nth coordinate hyperplane. Even though the toroid $\mathcal{H}_{(1,0,\ldots,0)}$ is non-polytopal, it delivers a useful

Lemma 2.3. *The group $B_n = \langle \rho_0, \ldots, \rho_{n-1}, \sigma_n \rangle$, with the redundant generator $\sigma_n := \rho_0^{\rho_1 \cdots \rho_{n-1}}$ has type $\{4, 3^{n-2}, 4\}$ and is isomorphic to $\Gamma(\mathcal{H}_{(1,0,\ldots,0)})$.*

To conclude this section we gather together various properties of the cubical groups B_n. Consider first the subgroup $D_n = \langle \lambda, \rho_1, \ldots, \rho_{n-1} \rangle$, where

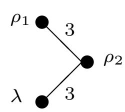
$$\lambda := \rho_1^{\rho_0} = \rho_0\rho_1\rho_0.$$

In fact [7, Chapter 1.1], this is the Coxeter group with diagram



(2.9)

Notice that when we truncate the diagrams in (2.2) and (2.9) from the right, we pass from B_n, D_n to B_{n-1}, D_{n-1}. When $n = 3$, B_3 is the group of the cube, and $D_3 \simeq A_3$ has diagram



Hence D_3 is isomorphic to the group of the regular tetrahedron, that is, to the symmetric group S_4. When $n = 2$, $B_2 \simeq \mathbb{I}_4$, the group of the square, and $D_2 \simeq C_2 \times C_2$, with diagram

Here are some useful properties of these groups.

Proposition 2.4. *Suppose* $n \geqslant 2$.

(a) *Both* D_n *and the rotation subgroup* B_n^+ *have index* 2 *in* B_n *and hence order* $2^{n-1} \cdot n!$. *Moreover, for odd* $n \geqslant 3$, $D_n \simeq B_n^+$.

(b) *The center of* B_n *has order* 2 *and is generated by the involution* ζ. *Also,* $\zeta \in D_n$ *(and* B_n^+*) if and only if* n *is even.*

(c) *For n=2,* B_2 *has cyclic rotation group* $B_2^+ \simeq C_4$*; and* $D_2 \simeq C_2 \times C_2$ *is not isomorphic to* B_2^+.

Proof. The calculations here are easy with the factorization $\gamma = \alpha\mu$ for $\gamma \in B_n$. Recall that we may interpret μ as an $n \times n$ diagonal matrix of signs ± 1, and α as a permutation matrix. From the structure of B_n as a semidirect product, we find that the map

$$\begin{array}{rcl} B_n & \to & C_2 \times C_2 \\ \alpha\mu & \mapsto & (\det \alpha, \det \mu) \end{array}$$

is an epimorphism. The groups D_n and B_n^+ correspond to two subgroups of index 2 generated by $(-1, +1)$ and $(-1, -1)$, respectively, in the image $C_2 \times C_2$. (The third subgroup consists of all sign changes together with even permutations.)

Part (b) is well-known; and (c) follows by inspection.

To finish part (a) for odd $n \geqslant 3$, observe that an isomorphism from D_n to B_n^+ is induced by the mapping

$$\alpha\mu \mapsto \begin{cases} \alpha\mu, & \text{if } \det \alpha = +1, \\ \alpha\mu\zeta, & \text{if } \det \alpha = -1. \end{cases}.$$

□

From Proposition 2.4(b) the centre of B_n is generated by

$$\zeta = (\rho_{n-1} \cdots \rho_1 \rho_0)^n = (\kappa\rho_0)^n = \sigma_1 \cdots \sigma_n,$$

following the notation in (2.3). We call the quotient $\overline{B}_n := B_n / \langle \zeta \rangle$ a *hemi-cubical group*. Indeed, $\overline{B}_n$ is also a string C-group; the corresponding regular n-polytope is the *hemi-n-cube* $\{4, 3^{n-2}\}_n$. Typically we use φ to indicate an element of $\overline{B}_n$. But it makes for smoother notation if we casually let ρ_j, σ_j refer to either the element of B_n or its image in $\overline{B}_n$, depending on context. Keep in mind, therefore, that $\sigma_1 \cdots \sigma_n = 1$ in $\overline{B}_n$.

Later we need a subgroup of index n in $\overline{B}_n$. A natural starting point is the stabilizer in B_n of the x_1-axis, say. This is the subgroup

$$\langle \sigma_1, \ldots, \sigma_n, \rho_2, \ldots, \rho_{n-1} \rangle = \langle \sigma_1, \sigma_n, \rho_2, \ldots, \rho_{n-1} \rangle$$

of order $2^n \cdot (n-1)!$. In the quotient $\overline{B}_n$, where the element $\sigma_1 = \sigma_2 \cdots \sigma_n$ is redundant, we get the subgroup

$$L_n := \langle \sigma_n, \rho_2, \ldots, \rho_{n-1} \rangle,$$

with order $2^{n-1} \cdot (n-1)!$. We need, in turn, its subgroup

$$M_n = \langle \sigma_2\sigma_n, \rho_2, \ldots, \rho_{n-1} \rangle.$$

One can think of M_n as containing sign changes in even numbers of the positions 2, ..., n.

Proposition 2.5. (a) $\overline{B}_n \simeq S_n \ltimes C_2^{n-1}$, *where* $S_n \simeq \langle \rho_1, \ldots, \rho_{n-1}\rangle$, *and* $C_2^{n-1} = \langle \sigma_2, \ldots, \sigma_n\rangle$.

(b) [3, page 128] *For odd* $n \geqslant 3$, $\overline{B}_n \simeq D_n (\simeq B_n^+)$.

(c) *For* $n \geqslant 2$, *the group* L_n *has index* n *in* $\overline{B}_n$, *with coset representatives* κ^j, $1 \leqslant j \leqslant n$.

(d) *The group* M_n *has index* 2 *in* $L_n = M_n \sqcup M_n\sigma_n$.

Proof. In (a), $\overline{B}_n$ inherits its semidirect product structure from B_n; note that $1 = \sigma_1 \cdots \sigma_n$ is invariant under permutation of subscripts.

For (b) we first recall that S_n has trivial centre for $n \geqslant 3$. Thus a central element in D_n can only be a pure sign change, indeed only 1 or ζ. When n is odd, D_n therefore has trivial centre and $B_n = D_n \times \langle \zeta \rangle$. Hence, $D_n \simeq \overline{B}_n$ (see Proposition 2.4(a)).

Parts (c) and (d) are clear from our earlier discussion of L_n. Note that products of *even* numbers of the commuting involutions $\sigma_2, \ldots, \sigma_n$ form a group of order 2^{n-2} which does not contain σ_n.

The case $\overline{B}_2$ is special. Here $\kappa = \rho_1 = (1,2)$; and since $\sigma_1\sigma_2 = 1$ we have $\sigma_2 = \sigma_1 = \rho_0$. Thus $L_2 = \langle \sigma_2 \rangle$, with order 2; and $M_2 = \langle 1 \rangle$. □

We also need some finicky properties of the sign change subgroup $\langle \sigma_2, ..., \sigma_n\rangle$. For more clarity, let us identify it with the additive group $\mathbb{Z}_2^{n-1}$, matching $\sigma = \sigma_2^{a_2} \cdots \sigma_n^{a_n}$ to $(a_2, ..., a_n)$. We require the following subgroup of sign changes:

$$Q := \begin{cases} \langle \sigma_3, \ldots, \sigma_n \rangle, & \text{for even } n \geqslant 2; \text{ so } a_2 = 0; \\ \langle \sigma_k\sigma_m \,:\, 2 \leqslant k < m \leqslant n\rangle, & \text{for odd } n \geqslant 3; \text{ so } a_2 + \cdots + a_n = 0. \end{cases} \tag{2.10}$$

In either case, $Q\sigma_2$ is a coset containing half the elements of $\langle \sigma_2, ..., \sigma_n\rangle$. Also let $\kappa_k = \rho_{k-1} \cdots \rho_1$, for $1 \leqslant k \leqslant n$. Thus $\kappa_n = \kappa$, $\kappa_1 = 1$ and we may identify κ_k with the cycle $(1, 2, ..., k)$.

Proposition 2.6. *Suppose* $\sigma, \lambda \in \langle \sigma_2, ..., \sigma_n\rangle$.

(a) *Suppose* $\sigma^{\kappa_k}, \lambda^{\kappa_k}$ *lie in the same coset of* Q *for* $2 \leqslant k \leqslant n$. *Then* $\sigma = \lambda$.

(b) *Each* σ *belongs to an odd number of the sets*

$$\langle \sigma_3, \sigma_4, ..., \sigma_n\rangle,\ \langle \sigma_2\sigma_3, \sigma_4, ..., \sigma_n\rangle,\ \langle \sigma_2, \sigma_3\sigma_4, ..., \sigma_n\rangle,$$
$$\ldots \langle \sigma_2, \sigma_3, ..., \sigma_{n-1}\sigma_n\rangle,\ \langle \sigma_2, \sigma_3, ..., \sigma_{n-1}\rangle\sigma_n.$$

(*Note that the last is a proper coset.*)

Proof. (a) Since Q has index 2 in $\langle \sigma_2, ..., \sigma_n\rangle$, all $(\varphi\lambda^{-1})^{\kappa_k} \in Q$, so that

$$\varphi\lambda^{-1} \in Q^{\rho_1} \cap \ldots \cap Q^{\rho_1 \cdots \rho_{n-1}}. \tag{2.11}$$

Say $(a_2, ..., a_n)$ represents $\varphi\lambda^{-1}$. When n is even, $Q^{\rho_1\cdots\rho_{k-1}}$ is determined by $a_k = 0$, for $2 \leqslant k \leqslant n$, so $a = (0, ..., 0)$ and the intersection in (2.11) is trivial. When n is odd, $Q^{\rho_1\cdots\rho_{k-1}}$ is now given by $a_2 + \cdots + a_{k-1} + a_{k+1} + \cdots a_n = 0$. Again $a = (0, ..., 0)$. In all cases, $\sigma = \lambda$.

(b) These n sets are determined by setting to 0 the quantities

$$a_2,\ a_2 + a_3,\ \ldots\ a_{n-1} + a_n,\ a_n + 1$$

in turn. But the sum of these terms is 1. □

3 Realizations of regular polytopes

We have already encountered 'realizations' of abstract polytopes: the convex n-cube $\mathbf{P}$, with base vertex $\mathbf{p}$, is a faithful realization (of dimension n) of the abstract cube $\mathcal{P} = \{4, 3^{n-2}\}$. Likewise, the tessellation $\mathbf{H}$ is a faithful realization of $\mathcal{H} = \{4, 3^{n-2}, 4\}$. Our main goal in this paper is to classify realizations of the cubical toroids $\mathcal{H}_{\mathbf{b}}$. First, we must carefully summarize the general theory of realizations.

Although it is often necessary to abandon concrete geometric figures (such as a convex polytope) when thinking of an abstract polytope $\mathcal{Q}$, it is still interesting to try to *realize* $\mathcal{Q}$ in a natural way in some Euclidean space $\mathbb{E}$. For an up-to-date and comprehensive survey of realizations and the *geometric regular polytopes* that then arise, we refer to [11]. Here we assume that $\mathcal{Q}$ is finite. This lets us modify slightly the discussion in [9, 10, 11].

Suppose then that $\mathcal{Q}$ is a finite regular n-polytope with group $\Gamma = \Gamma(\mathcal{Q}) = \langle \gamma_0, \ldots, \gamma_{n-1} \rangle$. Fixing an origin $\mathbf{0} \in \mathbb{E}$, we consider any homomorphism

$$\begin{aligned} f\colon \Gamma(\mathcal{Q}) &\rightarrow O(\mathbb{E}) \\ \gamma_j &\mapsto r_j \end{aligned}$$

(into the orthogonal group on $\mathbb{E}$). Note that each r_j is an isometry of period 2 (or 1). It is convenient to indicate the action of $\Gamma(\mathcal{Q})$ on points of $\mathbb{E}$ by writing $\mathbf{x}^\gamma := (\mathbf{x})(\gamma f)$. For example, $\mathbf{x}^{\gamma_j} := (\mathbf{x}) r_j$. Typically we use

$$G = G(\mathbf{Q}) := (\Gamma(\mathcal{Q}))f = \langle r_0, \ldots, r_{n-1} \rangle$$

to denote the image group of linear isometries.

Next define the *Wythoff space* for f to be

$$\mathbb{W} = \mathbb{W}_f := \{\mathbf{x} \in \mathbb{E} : \mathbf{x}^{\gamma_j} = \mathbf{x},\ 1 \leqslant j \leqslant n-1\} \tag{3.1}$$

(a linear subspace of $\mathbb{E}$). Let $w_G := \dim(\mathbb{W})$ be its dimension.

A *realization* $\mathbf{Q} := [f, \mathbf{q}]$ of $\mathcal{Q}$ is now defined by the homomorphism f, together with a choice of (geometric) base vertex $\mathbf{q} \in \mathbb{W}$.

Recall that, as an abstract regular polytope, $\mathcal{Q}$ has (abstract) base vertex $\Gamma_0 = \langle \gamma_1, \ldots, \gamma_{n-1} \rangle$. The (abstract) vertex set of $\mathcal{Q}$ is $\mathcal{Q}_0 := \{\Gamma_0 \gamma : \gamma \in \Gamma(\mathcal{Q})\}$, the set of right cosets of Γ_0 in Γ. The number of vertices in $\mathcal{Q}$ is just $v = |\mathcal{Q}_0|$, the index of Γ_0 in Γ.

Observe that the map

$$\begin{aligned} h\colon \quad \mathcal{Q}_0 &\rightarrow \mathbb{E} \\ \Gamma_0 \gamma &\mapsto \mathbf{q}^\gamma \end{aligned}$$

is well-defined. Each $\gamma \in \Gamma(\mathcal{Q})$ thereby induces an isometric permutation on

$$V = V(\mathbf{Q}) := (\mathcal{Q}_0)h,$$

the (geometric) vertex set of the realization. Note that $G = G(\mathbf{Q})$ leaves invariant the subspace $\mathbb{E}'$ spanned by the vertex set V. The *dimension* of the realization is $d_G := \dim(\mathbb{E}')$.

Remark 3.1. As we will soon see, it is quite possible to have $|V| < v = |\mathcal{Q}_0|$. In this and other ways, a realization can fail to be *faithful*. However, beyond this warning, we may put aside any worries as to just how a representation f induces a transfer of combinatorial structure from $\mathcal{Q}$ to some sort of geometric object (typically not a convex polytope) in $\mathbb{E}$. See below for examples and [11] for much more. It is convenient to use $\mathbf{Q}$ to also suggest the geometric object, just as we did for the n-cube $\mathbf{P}$ in Section 2.

Two realizations of $\mathcal{Q}$, say $\mathbf{Q}_j = [f_j, \mathbf{q}_j]$ in $\mathbb{E}_j$ for $j = 1, 2$, are *congruent* if there is an isometry $g\colon \mathbb{E}_1 \to \mathbb{E}_2$ such that

$$(\mathbf{q}_1)g = \mathbf{q}_2 \text{ and } (\gamma f_1)g = g(\gamma f_2), \quad \text{for all } \gamma \in \Gamma.$$

Now by *diagonal* we mean an unordered pair of *distinct* vertices in $\mathcal{Q}_0$. Suppose that there are r classes of diagonals under the action of $\Gamma(\mathcal{Q})$. If the jth diagonal class is represented by $\mathbf{q}, \mathbf{q}_j \in V(\mathbf{Q})$, and $||\mathbf{q} - \mathbf{q}_j||^2 = \delta_j$, then $\mathbf{Q}$ is determined up to congruence by the *diagonal vector* $\Delta(\mathbf{Q}) = (\delta_1, \ldots, \delta_r)$; see [12, Lemma 5A13]. Through this device we also find that the congruence classes of realizations have the structure of a convex r-dimensional cone [12, Theorem 5B2]. Moreover, by rescaling in an obvious way, we obtain *similar* realizations.

If $G(\mathbf{Q})$ acts reducibly on $\mathbb{E}'$, then in a natural way $\mathbf{Q}$ is congruent to a *blend* of lower dimensional realizations, say $\mathbf{Q}_1$ and $\mathbf{Q}_2$; and we write $\mathbf{Q} \equiv \mathbf{Q}_1 \# \mathbf{Q}_2$ [12, Section 5A]. On the other hand, if this does not happen, that is, if $G(\mathbf{Q})$ acts irreducibly on $\mathbb{E}'$, then $\mathbf{Q}$ is said to be a *pure* realization. The diagonal vectors of pure realizations span the extreme rays in the realization cone.

Every $\mathcal{Q}$ admits the *trivial realization* induced by the trivial representation

$$\begin{array}{ccc} \Gamma(\mathcal{Q}) & \to & O(\mathbb{E}^1) \\ \gamma & \mapsto & I \end{array}$$

We might take $\mathbf{q} = 1$, giving a realization of dimension $d_G = 1$; this convention suits us. However, it is sometimes more convenient to shift the base vertex to $\mathbf{q} = \mathbf{0}$, giving $d_G = 0$ and $w_G = 0$ [12, 5B]. This conforms to another situation, in which a higher dimensional representation G has Wythoff space $\mathbb{W} = \{\mathbf{0}\}$, thereby also yielding the trivial realization of $\mathcal{Q}$ (up to congruence).

Let us consider another extreme case. For $v = |\mathcal{Q}_0|$, let $\overline{\mathbb{E}}$ be $(v-1)$-dimensional Euclidean space. Clearly, $\mathcal{Q}$ has a *simplex realization* $\overline{\mathbb{E}}$, obtained by letting $\Gamma(\mathcal{Q})$ act in a natural way on the vertex set of a regular simplex in $\overline{\mathbb{E}}$. Let $\overline{w}$ be the dimension of its Wythoff space.

We can compute the important parameters r and $\overline{w}$ directly from the group by counting double cosets [12, Theorem 5B17]:

$$\begin{array}{rcl} \overline{w} & = & |\{\Gamma_0\gamma\Gamma_0 \ : \ \gamma \in \Gamma \backslash \Gamma_0\}| \\ r & = & |\{\Gamma_0\gamma\Gamma_0 \cup \Gamma_0\gamma^{-1}\Gamma_0 \ : \ \gamma \in \Gamma \backslash \Gamma_0\}| \end{array}$$

Let us now restrict our attention to the finitely many distinct, irreducible real representations $\Gamma \xrightarrow{f} G$ of $\Gamma(\mathcal{Q})$. (Any such f leaves invariant a positive definite, symmetric, bilinear form and so may be viewed as orthogonal.) The character for f (or more loosely, for G) has *norm*

$$\chi_G = |G|^{-1} \sum_{g \in G} \chi(g)\overline{\chi(g)}.$$

(Recall that $\chi(g)$ is the trace of the linear isometry g. See [12, page 131].)

In order not to overcount congruence classes of the realizations afforded by G, we must take into account the centralizer of G in its orthogonal group. For real irreducible representations G, we have three possibilities [12, Section 5B]:

1. $\chi_G = 1$; G is absolutely irreducible (i.e. remaining so upon extension of scalars to $\mathbb{C}$); the centralizer consists of unit reals $\{\pm 1\}$.

2. $\chi_G = 2$; $G = H + \overline{H}$, where $H, \overline{H}$ are conjugate but non-isomorphic, non-real irreducibles; the centralizer consists of unit complex numbers.

3. $\chi_G = 4$; $G = 2H$, where H is a non-real irreducible with real character; the centralizer consists of unit quaternions.

Discounting the action of the appropriate centralizer on the Wythoff space for G, we find that the dimension of the *essential Wythoff space* is the integer $w_G^* := w_G/\chi_G$.

Putting all this machinery to use, we get some very useful character-like results. These appeared first in [9], were corrected in [10], and were then re-corrected a final time and broadly generalized in [8]. The sums appearing in Theorem 3.2 are over all distinct irreducible, orthogonal representations G of $\Gamma(\mathcal{Q})$. In these sums we may ignore all representations in which $w_g = 0\ (= w_G^*)$. But, as mentioned earlier, we do take the trivial realization to have $d_G = w_G = w_G^* = 1$. This accounts for the small differences between parts (a) and (c) here and in [12, Theorem 5B14]. Part (b) appears in [8, Corollary 3.9]. The correction mentioned earlier is the term χ_G.

Recall that the regular polytope $\mathcal{Q}$ has v vertices, r diagonal classes and a simplex realization with Wythoff space dimension $\overline{w}$.

Theorem 3.2 ([8, 9, 10]). (a) $\sum_G w_G^* d_G = v$.

(b) $\sum_G w_G^* + \frac{1}{2} w_G^*(w_G^* - 1)\chi_G = r + 1$.

(c) $\sum_G w_G^* w_G = \overline{w} + 1$.

We will make good use of the following simple observation concerning the first kind of real representation. For a moment, we reconsider a general, not necessarily irreducible, orthogonal representation $f : \Gamma(\mathcal{Q}) \to G$ with realization $\mathbf{Q} = [f, \mathbf{q}]$.

Proposition 3.3. *If $w_G = 1$, then G is (absolutely) irreducible on $\mathbb{E}'$ (the subspace of $\mathbb{E}$ spanned by the vertex set V); and the corresponding realization $\mathbf{Q}$ is pure.*

Proof. As before, $G = \langle r_0, r_1, \ldots, r_{n-1} \rangle$ acts on the Euclidean space $\mathbb{E}'$ spanned by V, the G-orbit of the base vertex $\mathbf{q} \neq \mathbf{0}$. Suppose G admits a proper invariant subspace $\mathbb{E}_1$, so that $\mathbb{E}' = \mathbb{E}_1 \oplus \mathbb{E}_1^\perp$. We have $\mathbf{q} = \mathbf{q} + \mathbf{q}^\perp$, with $\mathbf{q} \in \mathbb{E}_1, \mathbf{q}^\perp \in \mathbb{E}_1^\perp$. But the subgroup $G_0 = \langle r_1, \ldots, r_{n-1} \rangle$ fixes the one-dimensional subspace $\mathbb{W} = \mathbb{R}\mathbf{q}$ of $\mathbb{E}'$. Since G_0 also leaves $\mathbb{E}_1$ and $\mathbb{E}_1^\perp$ invariant, both $\mathbf{q}$ and $\mathbf{q}^\perp$ are fixed by G_0. Since the two components cannot both be $\mathbf{0}$, V is contained in either $\mathbb{E}_1$ or $\mathbb{E}_1^\perp$, a contradiction.

Since $w_G = 1$, we can only have $w_G^* = 1$ and thus $\chi_G = 1$. □

We now determine the crucial parameter r for $\mathcal{H}_\mathbf{b}$, when $\mathbf{b} = (b, 0, \ldots, 0)$. Recall from Section 2 that the facets of the tessellation $\mathbf{H}$ of $\mathbb{E}^n$ are n-cubes, like $\mathbf{P}$, with edge length 2. We may thus envision the toroid $\mathcal{H}_\mathbf{b}$ as a $b \times \cdots \times b$ block of such cubes, with pairs of supporting hyperplanes $x_j = -b, x_j = b$ identified. It is convenient to describe the b^n vertices of $\mathcal{H}_{\mathbf{b}}$ in the familiar way by n-tuples $\boldsymbol{\ell} = (l_1, ..., l_n) \in \mathbb{Z}_b^n$. (See Figures 4 and 7 below for the cases $n = 2, 3$.) These modular coordinates are still symmetric under signed permutations.

We now compute the number r of proper diagonal classes in $\mathcal{H}_{\mathbf{b}}$. This is the number of distinct unordered pairs $\mathbf{0}, \boldsymbol{\ell}$, with $\boldsymbol{\ell} \in \mathbb{Z}_b^n$ and $\mathbf{0} \neq \boldsymbol{\ell}$. Note that the "half-turn" $\mathbf{x} \mapsto \boldsymbol{\ell} - \mathbf{x}$ is a symmetry of $\mathbb{Z}_b^n$ which sends $\mathbf{0}, \boldsymbol{\ell}$ to $\boldsymbol{\ell}, \mathbf{0}$. Thus all diagonal classes are symmetric. From [12, Theorem 5B18], this means that $r = \overline{w}$. However, we will later see this in a different way using Theorem 3.2(b),(c) and our Proposition 4.10.

Since we may take $\boldsymbol{\ell}$ modulo b and apply signed permutations, it follows that we can assume with no loss of generality that

$$0 \leqslant l_1 \leqslant l_2 \leqslant ... \leqslant l_n \leqslant \lfloor \frac{b}{2} \rfloor. \tag{3.2}$$

It is clear that different n-tuples of this type cannot be equivalent under a non-trivial signed permutation. Let us say that $\boldsymbol{\ell}$ has *standard form* if it satisfies (3.2).

The case $\boldsymbol{\ell} = \mathbf{0}$ was excluded when counting r, so we have the main part of

Proposition 3.4. *The toroid $\mathcal{H}_{\mathbf{b}}$, with $\mathbf{b} = (b, 0, ..., 0)$ and $b \geqslant 2$, has r proper diagonal classes, where $r + 1$ is the number of integer n-tuples $\boldsymbol{\ell} = (l_1, ..., l_n)$ satisfying $0 \leqslant l_1 \leqslant l_2 \leqslant ... \leqslant l_n \leqslant \lfloor \frac{b}{2} \rfloor$. In fact,*

$$r + 1 = \binom{n + \lfloor \frac{b}{2} \rfloor}{n}. \tag{3.3}$$

Proof. The sum of the $n+1$ non-negative increments $l_1 = l_1 - 0, l_2 - l_1, ..., l_n - l_{n-1}, \lfloor \frac{b}{2} \rfloor - l_n$ is $\lfloor \frac{b}{2} \rfloor$. Any such choice of increments specifies a unique $\boldsymbol{\ell}$. Essentially, we must split $\lfloor \frac{b}{2} \rfloor$ counters into $n + 1$ piles using n dividers. □

4 Realizations of the cubical toroids

We have seen that the problem of constructing the pure realizations of the cubical toroids $\mathcal{H}_{\mathbf{b}}$ becomes a matter of constructing the right family of real irreducible representations G of the abstract automorphism group $\Gamma(\mathcal{H}_{\mathbf{b}})$. Before magically pulling these G out of thin air, let us give some motivation.

Pure realizations in the case $n = 2$ were completely described in [14] (which, because of Proposition 3.3, does survive the correction in Theorem 3.2(b)). From that and from evidence gathered with GAP [5] for $n = 3, 4$, we guess that realizations for the $(n + 1)$-toroids $\mathcal{H}_b$, $n \geqslant 2$, can be described as follows.

For fixed $\mathbf{b} = (b, 0, \ldots, 0)$, the pure realizations $\mathbf{H}_{\mathbf{b}}$ of $\mathcal{H}_{\mathbf{b}}$ will be indexed by independent integer parameters $l_1, \ldots, l_n \pmod b$, where we may even take $0 \leqslant l_1 \leqslant \ldots \leqslant l_n \leqslant \lfloor \frac{b}{2} \rfloor$. As in Proposition 3.4, these parameters correspond to integer points $\boldsymbol{\ell}$ in a fundamental region for the natural action of B_n on $\mathcal{H}_{\mathbf{b}}$ (here understood to be a $b \times \cdots \times b$ block of n-cubes, with outer facets identified in pairs). The dimension (or degree) of a pure realization $\mathbf{H}_{\mathbf{b}}$ seems then to be the size of the orbit of $\boldsymbol{\ell} = (l_1, \ldots, l_n)$ under the action of B_n on $\mathcal{H}_{\mathbf{b}}$. We thus expect the maximal dimension to be $2^n \cdot n!$, which should also be generic when b is large.

Let us abuse notation by using τ_j to refer to both the translation in $\widetilde{C_n}$ and its image in $\Gamma(\mathcal{H}_{\mathbf{b}})$. However, we do use t_j for the image in any linear representation $G = G(\mathbf{H_b})$. In the latter group, we expect that the translation subgroup be realized as a direct product of copies of the cyclic group C_b. Since the toroid has cubical facets, we also expect the point group to be a faithful copy of B_n, at least for generic realizations.

We seek a linear group G which holds all this information. We further speculate that a typical translation will be a product of b-fold rotations in orthogonal 2-spaces in some suitable Euclidean space $\mathbb{E}$. Likely these come from dihedral groups $\mathbb{I}_b$, of order $2b$, acting on the individual 2-spaces. Therefore, since $\dim(\mathbb{E}) = 2^n \cdot n!$, we require $2^{n-1} \cdot n!$ copies of $\mathbb{I}_b$.

This motivates us to let K be the Coxeter group which is the direct product of $2^{n-1} \cdot n!$ copies of $\mathbb{I}_b$. The corresponding diagram has $2^n \cdot n!$ nodes, which can be parametrized by the elements B_n. Our first thought was to pair these off by taking the $2^{n-1} \cdot n!$ branches of the diagram to connect elements $\gamma, \rho_0\gamma$ (corresponding to 0-adjacent flags in the n-cube $\mathbf{P}$). The subgroup D_n then acts on K; and this indeed will work for n odd. But this is a dead end when n is even; we were led astray by the isomorphism in Theorem 2.5(b). The key is to exploit the group $\overline{B}_n$ for the hemi-cube in all dimensions. After a great deal of experiment we arrived at the construction which follows.

Construction 4.1. *Suppose $n \geqslant 2$ and fix an integer $b \geqslant 2$. Let $\mathbb{E}$ be a Euclidean space of dimension $2^n \cdot n!$ with orthonormal basis $\{\mathbf{e}_\varphi, \mathbf{f}_\varphi \ : \ \varphi \in \overline{B}_n\}$. (Recall that $\overline{B}_n$ has order $2^{n-1} \cdot n!$.) Thus $\mathbb{E}$ is the orthogonal sum of the planes $F_\varphi := \mathbb{R}\mathbf{e}_\varphi \oplus \mathbb{R}\mathbf{f}_\varphi$.*

For each $\varphi \in \overline{B}_n$ define two reflections on $\mathbb{E}$: r_φ whose mirror is the hyperplane normal to $\mathbf{e}_\varphi$, and s_φ with normal $-\cos(\pi/b)\mathbf{e}_\varphi - \sin(\pi/b)\mathbf{f}_\varphi$. Observe that $(\mathbf{e}_\varphi)r_\varphi = -\mathbf{e}_\varphi$, but r_φ fixes all other basis vectors, including $\mathbf{f}_\varphi$. The dihedral group $\mathbb{I}_b(\varphi) := \langle r_\varphi, s_\varphi \rangle$ has order $2b$ and acts as usual on the plane F_φ, while fixing pointwise all other such planes.

Let $K := \otimes_{\varphi \in \overline{B}_n} \mathbb{I}_b(\varphi)$ be the direct product of these dihedral groups. In fact, K is a (linear) Coxeter group; and an action of $\overline{B}_n$ on K is induced by permuting its generators. For $\gamma \in \overline{B}_n$, we have

$$r_\varphi \mapsto r_{\varphi\cdot\gamma} \text{ and } s_\varphi \mapsto s_{\varphi\cdot\gamma}.$$

(This is an example of the twisting discussed in [12, Section 8A].*) In similar fashion, $\overline{B}_n$ permutes basis vectors and so also acts as a group of isometries on $\mathbb{E}$:*

$$\mathbf{e}_\varphi \mapsto \mathbf{e}_{\varphi\cdot\gamma},\ \mathbf{f}_\varphi \mapsto \mathbf{f}_{\varphi\cdot\gamma}. \tag{4.1}$$

The semidirect product $\overline{B}_n \ltimes K$ is thus represented as a group of isometries in $O(\mathbb{E})$. □

Remark 4.2. The group K is crucial (and enormous, although in the end we use only a modest part of it). In the plane F_φ, the mirrors for r_φ, s_φ are configured as shown in Figure 2.

As a convenient notation for the b reflections in $\mathbb{I}_b(\varphi)$, let

$$[l, \varphi] = (r_\varphi s_\varphi)^l r_\varphi. \tag{4.2}$$

For example, $[0, \varphi] = r_\varphi$. Note that the integer parameter l can be taken mod b.

Now any subgroup G of $\overline{B}_n \ltimes K$ projects onto a subgroup of $\overline{B}_n$. To force G to have type $\{4, 3^{n-2}, 4\}$, we exploit Lemma 2.3, not forgetting that somehow the K components

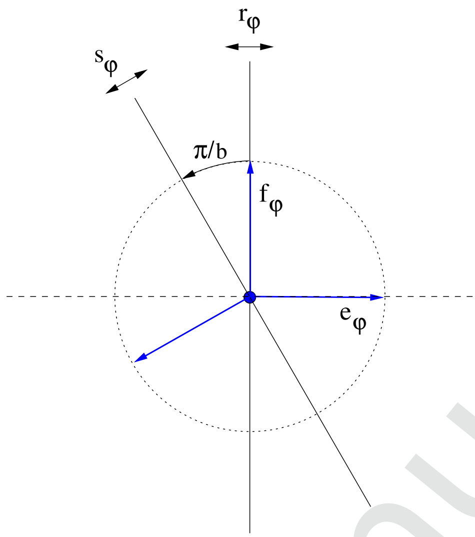


Figure 2: Generators for the dihedral group $\mathbb{I}_b(\varphi)$ acting on the plane F_φ.

should force the facet subgroup of G to be cubical, not hemi-cubical. From K we start with products of reflections like

$$x = \otimes_{\varphi\in\overline{B}_n}[l_\varphi, \varphi].$$

If $\gamma \in \overline{B}_n$, then

$$x^\gamma = \otimes_{\varphi\in\overline{B}_n}[l_\varphi, \varphi\cdot\gamma] = \otimes_{\varphi\in\overline{B}_n}[l_{\varphi\cdot\gamma^{-1}}, \varphi].$$

Recall from Proposition 2.5 that the subgroup M_n of $\overline{B}_n$ has index $2n$, with transversal elements $\kappa^j\sigma_n^i$, with $1 \leqslant j \leqslant n$ and $i = 0, 1$. Notice that in $L_n = M_n \sqcup M_n\sigma_n$, we have $\sigma_n M_n = M_n\sigma_n$. When n is even, we will also need the subgroup $Q := \langle\sigma_3, \ldots, \sigma_n\rangle$ of order 2^{n-2} in $\overline{B}_n$ (see equation (2.10)). Taking $S = \langle\rho_1, \ldots, \rho_{n-1}\rangle \simeq S_n$ as well, we form the set product $W := SQ\sigma_2$. Thus W has size $2^{n-2}\cdot n!$. When $n = 2$ we have $Q\sigma_2 = \{\sigma_2\}$.

Definition 4.3. For $n \geqslant 2$, $b \geqslant 2$, let $\mathbf{b} = (b, 0^{n-1})$; and fix any integers $l_1, \cdots, l_n$, which can be taken mod b. First let

$$x = \otimes_{\varphi\in\overline{B}_n}[l_\varphi, \varphi], \text{ where } l_\varphi = \begin{cases} l_j, & \text{if } \varphi \in \kappa^j M_n, \\ (-1)^n l_j, & \text{if } \varphi \in \kappa^j\sigma_n M_n. \end{cases} \tag{4.3}$$

(Thus $x \in K$; and the terms l_j and $(-1)^n l_j$ each occur $2^{n-2}\cdot(n-1)!$ times.)

(a) When n is odd, take

$$z := \otimes_{\varphi\in\overline{B}_n}[0, \varphi] = \otimes_{\varphi\in\overline{B}_n} r_\varphi \in K.$$

In this case, define elements of $\overline{B}_n \ltimes K$ as follows:

$$\begin{aligned} g_0 &= \rho_0 x \\ g_j &= \rho_j,\ 1 \leqslant j \leqslant n-1 \\ g_n &= \sigma_n z, \end{aligned}$$

(b) When n is even, the element in (4.3) reduces to $x = \otimes_{\varphi \in \overline{B}_n} [l_\varphi, \varphi]$, where $l_\varphi := l_j$ when $\varphi \in \kappa^j L_n$. Next let

$$w := \otimes_{\varphi \in W} [0, \varphi] = \otimes_{\varphi \in W} r_\varphi,$$

where $W = SQ\sigma_2 = \langle \rho_1, \ldots, \rho_{n-1} \rangle \langle \sigma_3, \ldots, \sigma_n \rangle \sigma_2$. In this case, define these elements of $\overline{B}_n \ltimes K$:

$$\begin{aligned} g_0 &= \rho_0 x \\ g_1 &= \rho_1 w \\ g_j &= \rho_j,\ 2 \leqslant j \leqslant n-1 \\ g_n &= \sigma_n. \end{aligned}$$

Finally, for each n let $GR_{\mathbf{b}}(l_1, \ldots, l_n) = \langle g_0, g_1, \ldots, g_{n-1}, g_n \rangle$, a subgroup of $K \rtimes \overline{B}_n$. We often briefly write instead $GR_{\mathbf{b}}(\boldsymbol{\ell})$, where $\boldsymbol{\ell} = (l_1, \ldots, l_n)$.

Remark 4.4. It is important to keep in mind that $\gamma \in \overline{B}_n$ acts on $\mathbb{E}$ by (regularly) permuting the planes F_φ through their subscripts. In contrast, $y \in K$ preserves each such plane, acting on it as a reflection or rotation.

We put an 'R' in $GR_{\mathbf{b}}$ to distinguish the group from a later subrepresentation of more interest. The σ_n may seem out of place; but here it does need to be distinguished from ρ_n.

The splitting of the definition according the parity of n is irksome, though perhaps inevitable. Certainly, a lot of effort yielded no fix. Peter McMullen notes that the range of toroidal quotients admitted by $\mathcal{H}_{\mathbf{b}}$ does depend on the parity of n [12, Theorem 6F1]. It is not clear why that could matter for our purposes, and in any case, our key results, Theorems 4.16 and 5.4 below, do not depend on the parity of n.

In Figure 3 we at first see the 3-cube $\mathbf{P}$, along with the barycentric subdivision of its boundary [12, Section 2C]. The generators ρ_0, ρ_1, ρ_2 for the symmetry group B_3 are indicated. The (shaded) base triangle which they enclose serves as a fundamental region for B_3 and corresponds to the base flag in the abstract 3-cube $\mathcal{P}$. After identifying antipodal points on the rim, we can just as well view Figure 3 as the hemi-cube of rank 3 (with its order complex). The 24 automorphisms $\varphi \in \overline{B}_3$ now correspond exactly to the 24 flags visible in the Figure 3. We have labelled these $\pm l_1, \pm l_2, \pm l_3$ as dictated by equation (4.3).

It is useful now to gather together some special calculations before proceeding to a key theorem.

Lemma 4.5. *Suppose $n \geqslant 4$ is even. Then*

(a) $w = \otimes_{\varphi \in SQ\sigma_2} [0, \varphi] = \otimes_{\varphi \in SQ\sigma_2} r_\varphi$ *satisfies* $ww^{\rho_2\rho_1}w^{\rho_2} = 1$.

(b) *for* $x = \otimes_{\varphi \in \overline{B}_n} [l_\varphi, \varphi]$ *as in* Definition 4.3(b), *we have*

$$xw^{\rho_0}x^{\rho_1\rho_0}w^{\rho_0\rho_1\rho_0}x^{\rho_1\rho_0\rho_1}w^{\rho_0\rho_1}x^{\rho_1}w = 1.$$

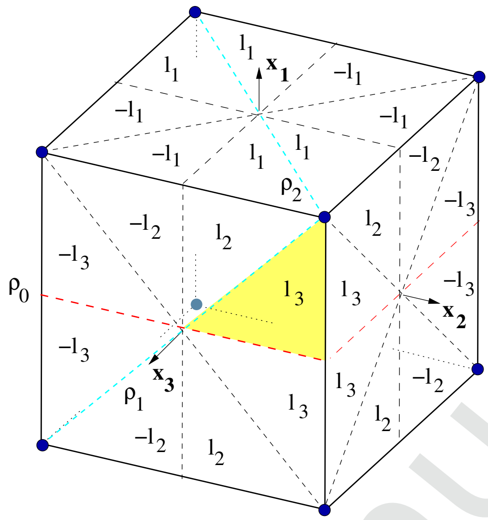


Figure 3: The hemi-cube ($n = 3$), with its order complex.

Proof. Notice that $\overline{B}_n = SQ \sqcup SQ\sigma_2$. As a temporary notation, let $[\infty, \varphi] = 1 \in \mathbb{I}_b(\varphi)$, then take

$$m_\varphi = \begin{cases} 0, & \text{if } \varphi \in SQ\sigma_2, \\ \infty, & \text{if } \varphi \in SQ. \end{cases}$$

Thus $w = \otimes_{\varphi \in \overline{B}_n} [m_\varphi, \varphi]$. The φ-component for $ww^{\rho_2\rho_1}w^{\rho_2}$, which is also an element of the direct product K, is then

$$[m_\varphi, \varphi] \cdot [m_{\varphi\cdot\rho_1\rho_2}, \varphi] \cdot [m_{\varphi\cdot\rho_2}, \varphi].$$

Now $U := \langle \sigma_4, \ldots, \sigma_n \rangle$ has coset representatives $1, \sigma_2, \sigma_3, \sigma_2\sigma_3$ in $\langle \sigma_1, \ldots, \sigma_n \rangle$. Notice that U is invariant under conjugation by ρ_1, ρ_2 and that $Q = U \sqcup U\sigma_3$. We now can easily compute the values in the following chart:

	$\varphi \in SU$	$\varphi \in SU\sigma_2$	$\varphi \in SU\sigma_3$	$\varphi \in SU\sigma_2\sigma_3$
$m_\varphi =$	∞	0	∞	0
$m_{\varphi\cdot\rho_1\rho_2} =$	∞	0	0	∞
$m_{\varphi\cdot\rho_2} =$	∞	∞	0	0

For example, if $\varphi \in SU\sigma_2$, then

$$\varphi \cdot \rho_2 \in SU\sigma_2^{\rho_2} = SU\sigma_3 \in SQ,$$

so $m_{\varphi\cdot\rho_2} = \infty$. Since there are an even number of 0's in each column, we get $ww^{\rho_2\rho_1}w^{\rho_2} = 1$. Actually part (a) holds for odd n, too.

With the conventions $1/\infty = 0,\ 1/0 = \infty$, we similarly find that

$$m_{\varphi\cdot\rho_1\rho_0} = 1/m_\varphi = m_{\varphi\cdot\rho_0}, \text{ for all } \varphi \in \overline{B}_n.$$

In part (b), $l_\varphi = l_j$ when $\varphi \in \kappa^j L$. Since $\rho_1\rho_0\rho_1 = \sigma_1^{\rho_1} = \sigma_2 \in L$, we always have $l_{\varphi\cdot\rho_1\rho_0\rho_1} = l_\varphi$ and $l_{\varphi\cdot\rho_1} = l_{\varphi\cdot\rho_0\rho_1}$. The φ-component of $xw^{\rho_0}\cdots x^{\rho_1}w$ in part (b) is then (condensing the notation)

$$\begin{aligned}&[l_\varphi][m_{\varphi\cdot\rho_0}][l_{\varphi\cdot\rho_0\rho_1}][m_{\varphi\cdot\rho_0\rho_1\rho_0}][l_{\varphi\cdot\rho_1\rho_0\rho_0}][m_{\varphi\cdot\rho_1\rho_0}][l_{\varphi\cdot\rho_1}][m_\varphi]\\ =\ &[l_\varphi][\frac{1}{m_\varphi}][l_{\varphi\cdot\rho_1}][m_\varphi][l_\varphi][\frac{1}{m_\varphi}][l_{\varphi\cdot\rho_1}][m_\varphi].\end{aligned}$$

If $m_\varphi = 0$, then $[\frac{1}{m_\varphi}, \varphi] = [\infty, \varphi] = 1$, and the previous product is

$$[l_\varphi][l_{\varphi\cdot\rho_1}][0][l_\varphi][l_{\varphi\cdot\rho_1}][0] = [l_\varphi][l_{\varphi\cdot\rho_1}][-l_\varphi][-l_{\varphi\cdot\rho_1}] = (r_\varphi s_\varphi)^{l_\varphi - l_{\varphi\cdot\rho_1}}(r_\varphi s_\varphi)^{-l_\varphi + l_{\varphi\cdot\rho_1}} = 1.$$

(Note that $[0][l_\varphi][0] = r_\varphi(r_\varphi s_\varphi)^{l_\varphi} r_\varphi = (r_\varphi s_\varphi)^{-l_\varphi} = [-l_\varphi]$.) The case $m_\varphi = \infty$ is similar. □

Now we can prove our main

Theorem 4.6. *For $n \geqslant 2$, $b \geqslant 2$, $\mathbf{b} = (b, 0^{n-1})$ and integer vector $\boldsymbol{\ell} = (l_1, \ldots, l_n)$, we have a representation*

$$\begin{aligned}\Gamma(\mathcal{H}_\mathbf{b}) &\to GR_\mathbf{b}(\boldsymbol{\ell})\\ \rho_j &\mapsto g_j,\ 0 \leqslant j \leqslant n.\end{aligned}$$

Proof. Considering Theorem 2.1, we must show that the g_j's satisfy the defining relations for the Coxeter group $\widetilde{C}_n$, along with the toroidal relation (2.8), taking $k = 1$. First of all, we have from Lemma 2.3 that the components ρ_j, σ_n from $\overline{B}_n$ (a quotient of B_n) do satisfy the relations for $\widetilde{C}_n$. (In Proposition 4.7, we dispel any worry that that the facet or vertex-figure subgroup of $\Gamma(\mathcal{H}_\mathbf{b})$ collapses in $GR_\mathbf{b}(\boldsymbol{\ell})$.) For now, we need only understand how x and z (or w) behave and check that they do not 'ruin' desired relations.

Suppose that n is odd. Notice that

$$z^\gamma = (\otimes_\varphi r_\varphi)^\gamma = \otimes_\varphi r_{\varphi\cdot\gamma} = z,$$

for all $\gamma \in B_n$ (all such products taken over $\overline{B}_n$). Thus, for example,

$$(g_n g_{n-1})^4 = (\sigma_n\rho_{n-1})^4 z^{\sigma_n} z^{\sigma_n\rho_{n-1}\sigma_n} z^{\rho_{n-1}\sigma_n\rho_{n-1}} z^{\rho_{n-1}} = z^4 = 1.$$

We also find that $g_n^2 = 1$ and g_n commutes with $g_1, \ldots, g_{n-2}$.

The dual calculations involving $g_0 = \rho_0 x$ fall out nicely. Consider the description of x in (4.3), with integers l_φ specified as in Definition 4.3. Since n is odd, $\rho_0 = \sigma_1 = (\sigma_2\sigma_3)\cdots(\sigma_{n-1}\sigma_n) \in M_n$. Thus $\varphi \in \kappa^j\sigma_n^i M_n$ implies $\varphi\cdot\rho_0 \in \kappa^j\sigma_n^i M$. It follows that $l_\varphi = l_{\varphi\cdot\rho_0}$ for all φ, giving $x^{\rho_0} = x$ and $g_0^2 = 1$. Similarly, for $2 \leqslant j \leqslant n$, $\rho_j \in M_n$ and $(g_0 g_j)^2 = 1$.

In the same way we find that $l_{\varphi\cdot\sigma_n} = -l_\varphi$. Compare this with

$$[0,\varphi]\,[l_\varphi,\varphi]\,[0,\varphi] = r_\varphi\cdot(r_\varphi s_\varphi)^{l_\varphi} r_\varphi\cdot r_\varphi = (r_\varphi s_\varphi)^{-l_\varphi} r_\varphi = [-l_\varphi, \varphi].$$

We conclude that $x^{\sigma_n} = zxz$, giving

$$(g_0 g_n)^2 = x^{\rho_0} z^{\rho_0\sigma_n} x^{\sigma_n} z = xzx^{\sigma_n}z = 1.$$

Similarly, $\rho_1\rho_0\rho_1 = \rho_0^{\rho_1} = \sigma_1^{\rho_1} = \sigma_2 \in \sigma_n M_n$, so that $l_\varphi = -l_{\varphi\cdot\rho_1\rho_0\rho_1}$ and $l_{\varphi\cdot\rho_1} = l_{\varphi\cdot\rho_1\rho_1\rho_0\rho_1} = -l_{\varphi\cdot\rho_0\rho_1}$. Now we have

$$(g_0g_1)^4 = x^{\rho_0}x^{\rho_0\rho_1\rho_0}x^{\rho_1\rho_0\rho_1}x^{\rho_1} = xx^{\rho_1\rho_0}x^{\rho_1\rho_0\rho_1}x^{\rho_1}.$$

But in the latter product, the φ-component is

$$[l_\varphi, \varphi]\,[l_{\varphi\cdot\rho_0\rho_1}, \varphi]\,[l_{\varphi\cdot\rho_1\rho_0\rho_1}, \varphi]\,[l_{\varphi\cdot\rho_1}, \varphi] = (r_\varphi s_\varphi)^{(l_\varphi - l_{\varphi\cdot\rho_1})} \cdot (r_\varphi s_\varphi)^{(-l_\varphi + l_{\varphi\cdot\rho_1})} = 1.$$

Thus $(g_0g_1)^4 = 1$. We are left to deal with the toroidal relation (2.8), so we examine

$$\begin{aligned}
g_0g_1\cdots g_{n-1}g_ng_{n-1}\cdots g_1 &= (\rho_0x)\cdot(\sigma_nz)^\kappa \quad (\kappa = \rho_{n-1}\cdots\rho_1)\\
&= \rho_0x\cdot\sigma_n^\kappa z^\kappa\\
&= \rho_0x\sigma_1z \quad (\sigma_1 = \rho_0)\\
&= x^{\rho_0}z\\
&= xz = \otimes_{\varphi\in\overline{B}_n}(r_\varphi s_\varphi)^{l_\varphi}.
\end{aligned}$$

But this product of rotations in the several planes F_φ has order dividing b, so relation (2.8) holds in $GR_{\mathbf{b}}(\boldsymbol{\ell})$. This finishes the job for n odd.

Now let n be even. We have $\rho_0(=\sigma_1), \sigma_n \in L$, so that $x = x^{\rho_0} = x^{\sigma_n}$, giving $g_0^2 = (g_0g_n)^2 = 1$. Similarly, $(g_0g_j)^2 = 1$ for $2 \leqslant n-1$. We can focus on g_1.

Notice that $\rho_1 = (1,2)$ leaves invariant Q, as well as the coset $Q\sigma_2$. This gives $w^{\rho_1} = w$ and $g_1^2 = 1$. Similarly, $(g_1g_j)^2 = 1$ for $3 \leqslant j \leqslant n-1$.

For the moment, let us continue with $n \geqslant 4$. Since $\sigma_n \in Q$, we get $(g_1g_n)^2 = 1$. Next we have

$$(g_1g_2)^3 = (\rho_1\rho_2)^3w^{\rho_2\rho_1\rho_2\rho_1\rho_2}w^{\rho_2\rho_1\rho_2}w^{\rho_2} = w^{\rho_1}w^{\rho_1\rho_2\rho_1}w^{\rho_2} = ww^{\rho_2\rho_1}w^{\rho_2} = 1,$$

by Lemma 4.5(a). To verify that $(g_0g_1)^4 = 1$, we use Lemma 4.5(b).

For the toroidal relation (2.8) we consider

$$\begin{aligned}
g_0g_1\cdots g_{n-1}g_ng_{n-1}\cdots g_1 &= \rho_0x\rho_1w\sigma_n^{\rho_{n-1}\cdots\rho_2}\rho_1w\\
&= \rho_0x\rho_1w\sigma_2\rho_1w\\
&= \sigma_1xw^{\rho_1}\sigma_2^{\rho_1}w = \sigma_1xw\sigma_1w\\
&= x^{\sigma_1}\cdot(w^{\sigma_1}w) = x(w^{\sigma_1}w)\\
&= \otimes_{\varphi\in\overline{B}_n}(r_\varphi s_\varphi)^{l_\varphi}.
\end{aligned}$$

But x is a product of reflections over *all* $\varphi \in \overline{B}_n$; and so is $w^{\sigma_1}w$, since $\overline{B}_n = W \sqcup W\sigma_1$. Note that $W\sigma_1 = SQ\sigma_2\sigma_1 = SQ(\sigma_3\cdots\sigma_n) = SQ$. Once more $g_0g_1\cdots g_{n-1}g_ng_{n-1}\cdots g_1$ is a product of rotations acting on the whole set of planes F_φ.

We are left with the case $n = 2$, which is a bit degenerate. Nevertheless, much the same calculations as above give $(g_0g_1)^4 = (g_1g_2)^4 = 1$ and $g_0g_1g_2g_1 = x(w^{\sigma_1}w)$. □

The next result is comforting, though strictly speaking not necessary for enumerating the realizations of $\mathcal{H}_{\mathbf{b}}$.

Proposition 4.7. *The subgroups $\langle\rho_0, ..., \rho_{n-1}\rangle$ and $\langle\rho_1, ..., \rho_n\rangle$ of $\Gamma(\mathcal{H}_{\mathbf{b}})$ are represented faithfully in the group $GR_{\mathbf{b}}(\boldsymbol{\ell})$ of* Theorem 4.6.

Proof. Recall from Definition 4.3 the description of the generators of $GR_{\mathbf{b}}(\boldsymbol{\ell})$ as elements in the semidirect product $\overline{B}_n \ltimes K$. Clearly there is an epimorphism $\langle g_0, ..., g_{n-1}\rangle \to \overline{B}_n$. Now consider $h = g_0 \cdots g_{n-1}$. Certainly $h^{2n} = 1$, so we must show h^n has period 2. Let $\alpha = \rho_0\rho_1 \cdots \rho_{n-1}$, which has period n in $\overline{B}_n$. When n is odd, $h = x\alpha$, so that

$$h^n = x \cdot x^{\alpha^{-1}} \cdots x^{\alpha^{-(n-1)}}.$$

Since x is 'supported' on all of $\overline{B}_n$, this product contains an odd number n of reflections from every $\mathbb{I}_b(\varphi)$, so h^n has period 2.

When n is even, $h = xy\alpha$, where $y = w^{\rho_0} = w^{\sigma_1}$ is supported on $SQ = \langle\rho_1, ..., \rho_{n-1}\rangle\langle\sigma_3, ..., \sigma_n\rangle$. Now the products of the $x^{\alpha^{-k}}$ contribute even numbers of reflections to each $\mathbb{I}_b(\varphi)$, so we need to show that

$$y \cdot y^{\alpha^{-1}} \cdots y^{\alpha^{-(n-1)}}$$

contains an odd number of reflections from each $\mathbb{I}_b(\varphi)$. The permutation component of φ (from S) can be discounted. After some calculation, we find that we must show that each sign change σ belongs to an odd number of the n sets listed in Lemma 2.6(b). Thus, for any n we have $\langle\rho_0, ..., \rho_{n-1}\rangle \simeq \langle g_0, ..., g_{n-1}\rangle$.

The dual argument that $\langle\rho_1, ..., \rho_n\rangle \simeq \langle g_1, ..., g_n\rangle$ is quite similar, after a natural modification of Lemma 2.6(b), again when n is even. □

We have taken a big step toward understanding the pure realizations of $\mathcal{H}_{\mathbf{b}}$. However, we still need to find a base vertex $\mathbf{v}$ for the realization.

Lemma 4.8. *Up to scale, the only non-zero point fixed by the vertex-figure subgroup* $\langle g_1, \ldots, g_n\rangle$ *of* $GR_{\mathbf{b}}(\boldsymbol{\ell})$ *is*

$$\mathbf{v} = \sum_{\varphi \in \overline{B}_n} \mathbf{f}_\varphi. \tag{4.4}$$

Proof. Suppose $\mathbf{v} = \sum_{\varphi\in\overline{B}_n} x_\varphi \mathbf{e}_\varphi + y_\varphi \mathbf{f}_\varphi$, with $x_\varphi, y_\varphi \in \mathbb{R}$. The coefficients $\rho_1, \ldots, \rho_{n-1}, \sigma_n$ for $g_1, \ldots, g_n$ generate $\overline{B}_n$. Since z (or w) fixes all $\mathbf{f}_\varphi$, we see that $y_\varphi = y$, a common value for all $\varphi \in \overline{B}_n$. Similarly, all $|x_\varphi| = x$, say. But since there are sign changes, $x = 0$. Thus $\mathbf{v} = y\sum_\varphi \mathbf{f}_\varphi$. □

For certain choices of $\boldsymbol{\ell} = (l_1, \ldots, l_n)$, it can happen that $GR_{\mathbf{b}}(\boldsymbol{\ell})$ does not act irreducibly on $\mathbb{E}$. We are interested in the action on the orbit of the base vertex.

Definition 4.9. Suppose $n \geqslant 2$, $b \geqslant 2$, $\mathbf{b} = (b, 0^{n-1})$, and $\boldsymbol{\ell} = (l_1, \ldots, l_n) \in \mathbb{Z}^n$. Let $G_{\mathbf{b}}(\boldsymbol{\ell})$ be the isometry group defined by restricting $GR_{\mathbf{b}}(\boldsymbol{\ell})$ to the *(*invariant*)* linear subspace $\mathbb{E}_{\mathbf{b}}(\boldsymbol{\ell})$ of $\mathbb{E}$ spanned by the orbit of $\mathbf{v}$.

We denote the corresponding realization of $\mathcal{H}_{\mathbf{b}}$ by either $\mathbf{H}_{\mathbf{b}}(l_1, \ldots, l_n)$ or $\mathbf{H}_{\mathbf{b}}(\boldsymbol{\ell})$.

In what follows we maintain our assumptions for $\mathbf{b} = (b, 0^{n-1})$ and $\boldsymbol{\ell} = (l_1, \ldots, l_n)$, but often use the abbreviations G for $G_{\mathbf{b}}(\boldsymbol{\ell})$ and $\mathbb{E}'$ for $\mathbb{E}_{\mathbf{b}}(\boldsymbol{\ell})$. Likewise we abuse notation slightly by using $g_0, \ldots, g_n$ for the generators of G. (To better conform with the notation of Section 3, we should use r_j for the restriction of g_j to $\mathbb{E}'$.)

From Lemma 4.8 and Proposition 3.3, we immediately get

Proposition 4.10. *The group* $G = G_{\mathbf{b}}(\boldsymbol{\ell})$ *is absolutely irreducible (on* $\mathbb{E}'$*). Thus* $w_G = w^*_G = 1$. *The corresponding realization* $\mathbf{H}_{\mathbf{b}}(\boldsymbol{\ell})$ *for* $\mathcal{H}_{\mathbf{b}}$ *is pure.*

Example 4.11. If all parameters $l_1 = \cdots = l_n = 0$, then $g_0 = \rho_0 \cdot \otimes_{\varphi \in \overline{B}_n} r_\varphi$ also fixes $\mathbf{v}$. In fact, the realization $\mathbf{H_b}(l_1, \ldots, l_n)$ is trivial if and only if $l_1 \equiv \cdots \equiv l_n \equiv 0 \pmod{b}$.

Proposition 4.12. *Each realization $\mathbf{H_b}(\boldsymbol{\ell})$ is congruent to a realization of this type with $\boldsymbol{\ell} = (l_1, \ldots, l_n)$ in standard form:*

$$0 \leqslant l_1 \leqslant \ldots \leqslant l_n \leqslant \lfloor \frac{b}{2} \rfloor. \tag{4.5}$$

Proof. Under the signed permutations in B_n, any integer point is equivalent to a unique point satisfying $0 \leqslant l_1 \leqslant \ldots \leqslant l_n$. But here $l_1, \ldots, l_n$ are parameters in the definition of $x = \otimes_{\varphi \in \overline{B}_n} [l_\varphi, \varphi]$ in (4.3), so we require a new action of B_n on $\mathbb{E}$.

First, fix j and consider the isometry g of $\mathbb{E}$ induced by mapping $e_\varphi \mapsto -e_\varphi$ just for $\varphi \in \kappa^j L_n$, while fixing all other basis vectors. (The effect is to reverse orientation in the corresponding planes F_φ.) Since $(\mathbf{v})g = \mathbf{v}$, we conclude that the realizations $\mathbf{H_b}(l_1, \ldots, l_j, \ldots, l_n)$ and $\mathbf{H_b}(l_1, \ldots, -l_j, \ldots, l_n)$ are congruent. Thus any l_j can be replaced by $-l_j$. In fact, since we can also reduce mod b, we can assume $0 \leqslant l_j \leqslant \lfloor b/2 \rfloor$, for $1 \leqslant j \leqslant n$.

Similarly, suppose g transposes $\mathbf{e}_{\kappa^j \mu}, \mathbf{e}_{\kappa^{j+1} \mu}$ and $\mathbf{f}_{\kappa^j \mu}, \mathbf{f}_{\kappa^{j+1} \mu}$, for $\mu \in L_n$, while fixing all remaining basis vectors. Then the isometry g lets us transpose l_j and l_{j+1}. We can permute the l_j as we wish. $\square$

Now we must address uniqueness. To do so, we might compute diagonal vectors for $\mathbf{H_b}(\boldsymbol{\ell})$ and $\mathbf{H_b}(\mathbf{k})$, with parameters $\boldsymbol{\ell}, \mathbf{k}$ in standard form (3.2). We instead take a related approach which makes for easier calculation. It is clear that congruent realizations have the same angle between the base vertex and its image under a particular translation. We therefore take a closer look at the translations in $GR_{\mathbf{b}}(\boldsymbol{\ell})$.

We earlier concluded from the semidirect decomposition in (2.7) that the translation subgroup $T = \langle \tau_1, \ldots, \tau_n \rangle$ acts transitively on the vertex set of the tessellation $\mathbf{H}$ of $\mathbb{E}^n$. This behaviour persists for the toroids $\mathcal{H}_\mathbf{b}$ and their realizations $\mathbf{H_b}$.

The first translation $\tau_1 := \rho_0 \rho_1 \cdots \rho_{n-1} \rho_n \rho_{n-1} \cdots \rho_1$; and for $2 \leqslant k \leqslant n$, the translation $\tau_k = \tau_1^{\rho_1 \cdots \rho_{k-1}}$. We have seen in the proof of Theorem 4.6 that τ_1 maps to

$$t_1 := g_0 g_1 \cdots g_{n-1} g_n g_{n-1} \cdots g_1 = \otimes_{\varphi \in \overline{B}_n} (r_\varphi s_\varphi)^{l_\varphi} \in GR_{\mathbf{b}}(\boldsymbol{\ell}),$$

regardless of whether n is even or odd. The image of τ_k is likewise $t_k := t_1^{g_1 \cdots g_{k-1}}$. Our task is to determine the φ-component for t_k, that is, the corresponding rotation in the plane F_φ.

When n is odd, $g_1 \cdots g_{k-1} = \rho_1 \cdots \rho_{k-1} = \kappa_k^{-1}$, again taking $\kappa_k = \rho_{k-1} \cdots \rho_1$. We have

$$t_k = [\otimes_{\varphi \in \overline{B}_n} (r_\varphi s_\varphi)^{l_\varphi}]^{\kappa_k^{-1}} = \otimes_{\varphi \in \overline{B}_n} (r_{\varphi \cdot \kappa_k^{-1}} s_{\varphi \cdot \kappa_k^{-1}})^{l_\varphi} = \otimes_{\varphi \in \overline{B}_n} (r_\varphi s_\varphi)^{l_{\varphi \cdot \kappa_k}}.$$

When n is odd, half the l_φ's have the form $-l_j$.

When n is even, each l_φ instead equals some l_j. But now $g_1 = \rho_1 w = w \rho_1$, where w is the product of the commuting reflections r_ψ, as ψ runs through $W = SQ\sigma_2 = \langle \rho_1, \ldots, \rho_{n-1} \rangle \langle \sigma_3, \ldots, \sigma_n \rangle \sigma_2$ (Definition 4.3). So before conjugating t_1 by κ_k^{-1}, we first conjugate by w, giving the preliminary translation

$$u = [\otimes_\varphi (r_\varphi s_\varphi)^{l_\varphi}]^w = \otimes_{\varphi \notin W} (r_\varphi s_\varphi)^{l_\varphi} \otimes_{\varphi \in W} (r_\varphi s_\varphi)^{-l_\varphi}$$

(which might not lie in $GR_{\mathbf{b}}(\boldsymbol{\ell})$).

Now we can put the cases n even or odd on equal footing simply by letting $u = t_1$ when n is odd. In both cases, $t_k = u^{\kappa_k^{-1}}$ for $2 \leqslant k \leqslant n$; and if we let $u = \otimes_{\varphi \in \overline{B}_n} (r_\varphi s_\varphi)^{m_\varphi}$, we get further unification. We always have

$$m_\varphi := \begin{cases} l_j, & \text{if } \varphi \in \kappa^j \langle \rho_2, \ldots, \rho_{n-1} \rangle Q; \\ -l_j, & \text{if } \varphi \in \kappa^j \langle \rho_2, \ldots, \rho_{n-1} \rangle Q \sigma_2, \end{cases} . \tag{4.6}$$

where Q was defined in (2.10). This needs a little calculation when n is even; and when n is odd, this is equivalent to Definition 4.3(a), giving $m_\varphi = l_\varphi$, as expected. We have

$$t_k = u^{\kappa_k^{-1}} = \otimes_{\varphi \in \overline{B}_n} (r_\varphi s_\varphi)^{m_{\varphi \cdot \kappa_k}}.$$

We must get a handle on $m_{\varphi \cdot \kappa_k}$ and equally $l_{\varphi \cdot \kappa_k}$. It follows from (4.6) that the parameters $m_{\varphi \cdot \kappa_k}$ and $l_{\varphi \cdot \kappa_k}$ can differ only in sign.

For each $\varphi \in \overline{B}_n$ we want to understand $\mathbf{j}(\varphi) = (l_\varphi, m_{\varphi \cdot \kappa_2}, \ldots, m_{\varphi \cdot \kappa_n})$. Because of the semidirect product structure of $\overline{B}_n$, there is a unique factorization $\varphi = \alpha\sigma$, where the permutation $\alpha \in \langle \rho_1, \ldots, \rho_{n-1} \rangle \simeq S_n$ and the sign change $\sigma \in \langle \sigma_1, \ldots, \sigma_n \rangle \simeq C_n^{n-1}$. Furthermore, there are unique $1 \leqslant j \leqslant n$ and $\mu \in \langle \rho_2, \ldots, \rho_{n-1} \rangle \simeq S_{n-1}$ such that $\alpha = \kappa^j \mu$. Then $l_\varphi = \pm l_j$, depending on σ.

Since $\varphi\kappa_k = \alpha\kappa_k \cdot \sigma^{\kappa_k}$, we can separately track the effect of κ_k on α and σ. As an aid we use the special permutation $\xi = \kappa_2 \cdots \kappa_n = (1, n)(2, n-1) \cdots$, so that

$$\xi \colon j \mapsto 1 - j \pmod{n}.$$

Suppose $\alpha = \kappa^j \mu$, with $\mu \in \langle \rho_2, \ldots, \rho_{n-1} \rangle$. Then $\alpha\kappa_k = \kappa^j \mu \kappa_k = \kappa^m \tilde{\mu}$, say, where $\tilde{\mu}$ fixes 1. This means that

$$(1)\kappa^{j-m}\mu = (1)\tilde{\mu}\kappa_k^{-1} = k \pmod{n},$$

so

$$1 + j - m = (k)\mu^{-1} = (k)\alpha^{-1}\kappa^j = (k)\alpha^{-1} + j \pmod{n},$$

whence

$$m = 1 - (k)\alpha^{-1} = (k)\alpha^{-1}\xi \pmod{n}.$$

It follows that for any $\varphi = \alpha\sigma$, $\mathbf{j}(\varphi)$ looks like $(l_{(1)\alpha^{-1}\xi}, l_{(2)\alpha^{-1}\xi}, \ldots, l_{(n)\alpha^{-1}\xi})$ (signs still to be inserted).

Example 4.13. When $\varphi = 1$ we obtain $\mathbf{j}(1) = (l_n, l_{n-1}, ..., l_2, l_1)$ (with no sign changes required). Similarly, $\mathbf{j}(\xi) = \boldsymbol{\ell} = (l_1, l_2, \ldots, l_n)$ in order.

Suppose now that we fix the permutation component $\alpha = \kappa^j \mu$ but vary the sign component $\sigma \in Q\sigma_2^i$, with $i \in \{0, 1\}$. It follows at once from Proposition 2.6(a) that $\sigma^{\kappa_2}, ..., \sigma^{\kappa_n}$ induce all 2^{n-1} possible sign changes $e_2, \ldots, e_n$ for the last $n-1$ positions in the permutation $l_{(1)\alpha^{-1}\xi}, l_{(2)\alpha^{-1}\xi}, \ldots, l_{(n)\alpha^{-1}\xi}$. Keeping close track of the first sign change e_1, we actually get

Proposition 4.14. *Suppose $\boldsymbol{\ell}$ is fixed as above. For each $\varphi = \alpha\sigma \in \overline{B}_n$, the translation t_i, $1 \leqslant i \leqslant n$, has φ-component $(r_\varphi s_\varphi)^{j_i}$, where $\mathbf{j}(\varphi) = (j_1, \ldots, j_n)$ is some signed permutation $(e_1 l_{(1)\alpha^{-1}\xi}, e_2 l_{(2)\alpha^{-1}\xi}, ..., e_n l_{(n)\alpha^{-1}\xi})$ of $\boldsymbol{\ell} = (l_1, \ldots, l_n)$. There are 2^{n-1} sign changes $(e_1, \ldots, e_n)$. When n is even, $e_1 = +1$ and the signs occur in all possible ways for $e_2, \ldots, e_n$. But when n is odd, $e_1 = -1$ half the time, always with $e_1 e_2 \cdots e_n = +1$.*

Finally, we can return to realizations like $\mathbf{H_b}(\boldsymbol{\ell})$ and $\mathbf{H_b}(\mathbf{k})$. Note that they share the same base vertex $\mathbf{v}$. If these pure realizations were congruent, then certainly $\mathbf{v}\cdot(\mathbf{v})t_1 \cdots t_k$ must be the same for them, for $1 \leqslant k \leqslant n$. (Of course, the actual translations $t_1 \cdots t_k$ do depend on $\boldsymbol{\ell}$ and $\mathbf{k}$.)

Let us focus on $\mathbf{H_b}(\boldsymbol{\ell})$. Recall that the component of $\mathbf{v}$ in F_φ is just $\mathbf{f}_\varphi$. If some translation t has rotational component $(r_\varphi s_\varphi)^l$ in this plane, then the contribution of φ to $\mathbf{v} \cdot (\mathbf{v})t$ is simply $\cos(2\pi l/b)$. Consulting Definition 4.3 again, we can already compute

$$\begin{aligned}
\mathbf{v} \cdot (\mathbf{v})t_1 &= \sum_{\varphi} \cos(\frac{2\pi l_\varphi}{b}) \\
&= 2^{n-2}(n-1)! \sum_{j=1}^{n} \cos(\frac{2\pi l_j}{b}) + \cos(\frac{(-1)^n 2\pi l_j}{b}) \\
&= 2^{n-1}(n-1)!(c_1 + \cdots + c_n),
\end{aligned}$$

where $c_j := \cos(2\pi l_j/b)$. The last sum involves the first elementary symmetric function of the cosines.

Now consider $t = t_1 \cdots t_k$ for $k \geqslant 2$. By our earlier calculations, the rotational component $(r_\varphi s_\varphi)^l$ when $\varphi = \alpha\sigma$ has $l = e_1 l_{(1)\alpha^{-1}\xi} + e_2 l_{(2)\alpha^{-1}\xi} \cdots + e_k l_{(k)\alpha^{-1}\xi}$, for some specific choice of signs e_i. The sign selections $e_2, \ldots, e_k$ come in 2^{k-2} pairs of opposites, though in slightly different ways depending on the parity of n; see Proposition 4.14. However, since the cosine function is even, we can assume $e_1 = 1$ when n is odd, taking care to adjust the subsequent counting. In this way we may ignore the parity of n.

The remaining subscripts and signs (for positions $k+1$ through n) now ensure that a specific choice for l occurs with multiplicity $2^{n-k} \cdot (n-k)!$ in the following calculation. For easier notation, we let $m_i = (i)\alpha^{-1}\xi$. We get

$$\begin{aligned}\mathbf{v}\cdot(\mathbf{v})t_1\cdots t_k &= 2^{n-k}(n-k)! \sum_{\substack{m_1,m_2,\ldots,m_k\\ e_2,\ldots,e_k=\pm 1}} \cos(\frac{2\pi[l_{m_1}+e_2l_{m_2}+\cdots+e_kl_{m_k})]}{b}) \\ &= 2^{n-k+1}(n-k)! \sum_{\substack{m_1,m_2,\ldots,m_k\\ e_3,\ldots,e_k=\pm 1}} \cos(\frac{2\pi l_{m_1}}{b}) \\ &\qquad\qquad \cos(\frac{2\pi[l_{m_2}+e_3l_{m_3}+\cdots+e_kl_{m_k})]}{b}) \\ \ldots &= 2^{n-1}(n-k)! \sum_{m_1,m_2,\ldots,m_k} \cos(\frac{2\pi l_{m_1}}{b})\cos(\frac{2\pi l_{m_2}}{b})\cdots\cos(\frac{2\pi l_{m_k}}{b}) \\ &= 2^{n-1}(n-k)!k! \sum_{1\leqslant m_1<m_2<\cdots<m_k\leqslant n} c_{m_1}c_{m_2}\cdots c_{m_k}\ .\end{aligned}$$

Here we repeatedly use the identity $\cos(\theta_1+\theta_2)+\cos(\theta_1-\theta_2)=2\cos(\theta_1)\cos(\theta_2)$. We have

Proposition 4.15. *For $\boldsymbol{\ell}=(l_1,\ldots,l_n)$, let $c_j := \cos(2\pi l_j/b)$. For $1\leqslant k\leqslant n$, the realization $\mathbf{H_b}(\boldsymbol{\ell})$ has dot product $\mathbf{v}\cdot(\mathbf{v})t_1\cdots t_k$ equal $2^{n-1}(n-k)!k!$ times the k-th elementary symmetric function in $c_1,\ldots,c_n$.*

Theorem 4.16. *Each pure realization of $\mathcal{H}_\mathbf{b}$ is similar to exactly one realization of the type $\mathbf{H_b}(\boldsymbol{\ell})$, with $\boldsymbol{\ell}$ is standard form (3.2).*

Proof. Two congruent realizations $\mathbf{H_b}(\boldsymbol{\ell})$ and $\mathbf{H_b}(\mathbf{k})$ certainly give equal inner products $\mathbf{v}\cdot(\mathbf{v})t_1\cdots t_k$, since they have common base vertex $\mathbf{v}$. From Proposition 4.15, the corresponding elementary symmetric functions of cosines are equal. But these functions determine the coefficients of the polynomial

$$\prod_{j=1}^{n}(x-c_j),$$

which must therefore be shared by $\mathbf{H_b}(\boldsymbol{\ell})$ and $\mathbf{H_b}(\mathbf{k})$. One set of cosines is thus a permutation of the other. Since $\boldsymbol{\ell},\mathbf{k}$ are in standard form, so that $c_1\geqslant c_2\geqslant\ldots\geqslant c_n$, for instance, we have $\boldsymbol{\ell}=\mathbf{k}$.

We have $\mathbf{H_b}(\boldsymbol{\ell})$ congruent to $\mathbf{H_b}(\mathbf{k})$ if and only if $\boldsymbol{\ell}=\mathbf{k}$ when in standard form. Thus the number of distinct pure realizations of this type is the number of $\boldsymbol{\ell}$ satisfying (3.2) in Proposition 4.12. By Proposition 3.4, this is the same number as the parameter $r+1$ for the abstract toroid $\mathcal{H}_\mathbf{b}$. We are done by Theorem 3.2(b) and Proposition 4.10. □

5 The dimension of $\mathbf{H_b}(\boldsymbol{\ell})$

The most interesting (and frustrating) part of this investigation has been to determine the dimension $d_\mathbf{b}(\boldsymbol{\ell})$ of $\mathbf{H_b}(\boldsymbol{\ell})$, that is, $d=\dim(\mathbb{E}')$ in our brief notation. To do this we first draw on the geometric ideas underlying *Clifford's Theorem* (see [4, Theorem 2.2]).

We know that $G=G_\mathbf{b}(\boldsymbol{\ell})$ acts irreducibly on $\mathbb{E}'$ and has the abelian normal subgroup $T=\langle t_1,\ldots,t_n\rangle$. (Recall that we abuse notation with T referring to the image of $\langle\tau_1,\ldots,\tau_n\rangle$ in G.)

Now T admits a 1- or 2-dimensional irreducible subspace $Y \subset \mathbb{E}'$. For each $h \in G$, Yh is also an irreducible T-space; and two of these either coincide or meet trivially. From this ensues a direct sum decomposition

$$\mathbb{E}' = Yh_1 \oplus \cdots \oplus Yh_p, \tag{5.1}$$

for certain $h_1, ..., h_p \in G$. Of course, the T-spaces Yh_j have the same (real) dimension 1 or 2. To cover both cases at once, we extend scalars to $\mathbb{C}$. Thus, for a bit, we work in the unitary spaces $\mathbb{E}_{\mathbb{C}}$ and $\mathbb{E}'_{\mathbb{C}}$.

The group T is a representation of $\langle \tau_1, \ldots, \tau_n \rangle \simeq \mathbb{Z}_b^n$. The latter abelian group has b^n irreducibles over $\mathbb{C}$, all 1-dimensional [4, Corollary 2.6]. Suppose $\omega = e^{\frac{2\pi \imath}{b}}$. Then the irreducibles are parametrized by all $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_b^n$ itself. For each $\mathbf{k}$, the corresponding representation maps

$$\tau_1^{j_1} \cdots \tau_n^{j_n} \mapsto \omega^{\mathbf{k} * \mathbf{j}},$$

as $\mathbf{j} = (j_1, \ldots, j_n)$ runs through $\mathbb{Z}_b^n$. It is very convenient here to treat $\mathbb{Z}_b^n$ as an additive group, equipped with the symmetric bilinear form $\mathbf{k} * \mathbf{j} := k_1 j_1 + \cdots + k_n j_n$. Note that the complex number $\omega^{\mathbf{k} * \mathbf{j}}$ is then well-defined. We next indulge in a little discrete Fourier analysis on $\mathbb{Z}_b^n$.

Lemma 5.1. *For fixed* $\mathbf{q} \in \mathbb{Z}_b^n$,

$$\frac{1}{b^n} \sum_{\mathbf{m} \in \mathbb{Z}_b^n} \omega^{\mathbf{q} * \mathbf{m}} = \begin{cases} 1, \text{ if } \mathbf{q} = \mathbf{0}; \\ 0, \text{ otherwise.} \end{cases}$$

Proof. The sum factors as a product of n geometric series indexed by $m_1, ..., m_n$, respectively. □

Now our goal must be to find a vector $\mathbf{y} \in \mathbb{E}'_{\mathbb{C}}$ such that the line $\mathbb{C}\mathbf{y}$ is T-invariant. Motivated by Schur's Lemma, we eventually think to try

$$\mathbf{y} := \frac{1}{b^n} \sum_{\mathbf{m} \in \mathbb{Z}_b^n} \omega^{-\mathbf{l} * \mathbf{m}} (\mathbf{v}) t^{\mathbf{m}}, \tag{5.2}$$

where for brevity we take $t^{\mathbf{m}} := t_1^{m_1} \cdots t_n^{m_n}$. It is easy to check that

$$t_i : \mathbf{y} \mapsto \omega^{l_i} \mathbf{y},\ 1 \leqslant i \leqslant n.$$

Thus $\mathbb{C}\mathbf{y}$ is indeed a T-invariant line in $\mathbb{E}'_{\mathbb{C}}$, so long as $\mathbf{y} \neq \mathbf{0}$. In fact, we could replace $\boldsymbol{\ell}$ in (5.2) by any $\mathbf{k} \in \mathbb{Z}_b^n$. But we will soon see why $\mathbf{y} = \mathbf{0}$ unless $\mathbf{k}$ is one of the signed permutations of $\boldsymbol{\ell}$ enumerated in Proposition 4.14.

We must determine the component $\mathbf{y}_\varphi$ of $\mathbf{y}$ in $\mathbb{C}F_\varphi$, for all $\varphi \in \overline{B}_n$. Recall from Proposition 4.14 that, for $1 \leqslant i \leqslant n$, t_i acts on F_φ as the rotation $(r_\varphi s_\varphi)^{j_i}$, where $\mathbf{j} = (j_1, \ldots, j_n)$ is some signed permutation of $\boldsymbol{\ell}$ (depending on φ). Now for fixed φ and $\mathbf{m} \in \mathbb{Z}_b^n$, the vector $(\mathbf{v})t^{\mathbf{m}}$ has this component in $\mathbb{C}F_\varphi$:

$$\begin{aligned}(\mathbf{f}_\varphi)(r_\varphi s_\varphi)^{\mathbf{j} * \mathbf{m}} &= -\sin(\frac{2\pi(\mathbf{j} * \mathbf{m})}{b})\mathbf{e}_\varphi + \cos(\frac{2\pi(\mathbf{j} * \mathbf{m})}{b})\mathbf{f}_\varphi \\ &= (\frac{\omega^{-\mathbf{j} * \mathbf{m}} - \omega^{\mathbf{j} * \mathbf{m}}}{2\imath})\mathbf{e}_\varphi + (\frac{\omega^{\mathbf{j} * \mathbf{m}} + \omega^{-\mathbf{j} * \mathbf{m}}}{2})\mathbf{f}_\varphi\end{aligned}$$

Next sum such terms weighted by $\omega^{-\mathbf{k}*\mathbf{m}}$ in (5.2) and simplify using Lemma 5.1. We conclude that $\mathbf{y}_\varphi$ is as described in the first column of Table 1.

$\mathbf{y}_\varphi$	$\mathbf{y}_{1,\varphi}$	$\mathbf{y}_{2,\varphi}$	$\mathbf{j} = \mathbf{j}(\varphi)$
$\frac{\imath}{2}\mathbf{e}_\varphi + \frac{1}{2}\mathbf{f}_\varphi$	$\frac{1}{2}\mathbf{f}_\varphi$	$\frac{1}{2}\mathbf{e}_\varphi$	$\mathbf{j} = \mathbf{l}, \mathbf{j} \neq -\mathbf{l}$
$\frac{-\imath}{2}\mathbf{e}_\varphi + \frac{1}{2}\mathbf{f}_\varphi$	$\frac{1}{2}\mathbf{f}_\varphi$	$\frac{-1}{2}\mathbf{e}_\varphi$	$\mathbf{j} \neq \mathbf{l}, \mathbf{j} = -\mathbf{l}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{j} \neq \mathbf{l}, \mathbf{j} \neq -\mathbf{l}$
$\mathbf{f}_\varphi$	$\mathbf{f}_\varphi$	$\mathbf{0}$	$\mathbf{j} = \mathbf{l} = -\mathbf{l}$

Table 1: Real components for $\mathbf{y}_1, \mathbf{y}_2$ spanning Y.

Certainly $\mathbf{y} \neq \mathbf{0}$, since at least $\mathbf{j}(\varphi) = \mathbf{l}$ for $\varphi = \xi$ (Example 4.13). We need to understand the 'support' $\mathrm{sp}(\mathbf{y}) = \{\varphi \in \overline{B}_n : \mathbf{y}_\varphi \neq \mathbf{0}\}$.

Since the group G is real, the vector $\overline{\mathbf{y}}$ also spans a T-invariant subspace; and clearly $\mathrm{sp}(\mathbf{y}) = \mathrm{sp}(\overline{\mathbf{y}})$. Now we can examine the real representation T. We have computed in Table 1 the components $\mathbf{y}_{1,\varphi}$ and $\mathbf{y}_{1,\varphi}$ in F_φ for the *two* real vectors $\mathbf{y}_1 := \frac{1}{2}(\mathbf{y} + \overline{\mathbf{y}})$ and $\mathbf{y}_2 := \frac{1}{2\imath}(\mathbf{y} - \overline{\mathbf{y}})$. In order to pin down the T-invariant subspace $Y = \mathbb{R}\mathbf{y}_1 + \mathbb{R}\mathbf{y}_2$, we must determine $\mathrm{sp}(\mathbf{y}_j) = \{\varphi \in \overline{B}_n : \mathbf{y}_{j,\varphi} \neq \mathbf{0}\}$. For instance, when $\mathbf{l} = -\mathbf{l}$, we have $\mathrm{sp}(\mathbf{y}_2) = \emptyset$. Otherwise, $\mathrm{sp}(\mathbf{y}_1) = \mathrm{sp}(\mathbf{y}_2) = \mathrm{sp}(\mathbf{y}) \neq \emptyset$.

Over $\mathbb{C}$ or $\mathbb{R}$, our goal is to count the distinct T-invariant subspaces Yh, with $h \in G$. But $G = TG_0$, where $G_0 = \langle g_1, \ldots, g_n \rangle$, so we can assume $h \in G_0$. Thus $h = \lambda y$, where $\lambda \in \overline{B}_n$ and $y = \otimes_{\beta \in A} r_\beta \in K$, for some subset $A \subseteq \overline{B}_n$. (For $h \in G_0$, the factor y has no s_β terms. This follows from Definition 4.3, though the details depend on the parity of n.) Thus y fixes all $\mathbf{f}_\varphi$ and all $\mathbf{e}_\varphi$, $\varphi \notin A$, but swaps $\mathbf{e}_\varphi, -\mathbf{e}_\varphi$ when $\varphi \in A$.

Lemma 5.2. *If $h = \lambda y \in G_0$, where $\lambda \in \overline{B}_n, y \in K$, then*

$$\mathrm{sp}((\mathbf{y})h) = (\mathrm{sp}(\mathbf{y}))\lambda.$$

A similar result holds for the real vectors $\mathbf{y}_1$ and $\mathbf{y}_2$.

Proof. If $\varphi \in \mathrm{sp}(\mathbf{y})$, then $\mathbf{y}_\varphi = a\mathbf{e}_\varphi + b\mathbf{f}_\varphi$, where the scalars a, b are not both 0. Thus

$$(\mathbf{y}_\varphi)h = (a\mathbf{e}_{\varphi\lambda} + b\mathbf{f}_{\varphi\lambda})y = \pm a\mathbf{e}_{\varphi\lambda} + b\mathbf{f}_{\varphi\lambda},$$

so $\varphi\lambda \in \mathrm{sp}((\mathbf{y})h)$. Thus $(\mathrm{sp}(\mathbf{y}))\lambda \subseteq \mathrm{sp}((\mathbf{y})h)$. The reverse inclusion follows in the same way by applying $h^{-1} = \lambda^{-1}y^{\lambda^{-1}}$ to $(\mathbf{y})h$. The calculations for $\mathbf{y}_1, \mathbf{y}_2$ are immediate. □

Let us now dispense with the special case that $\boldsymbol{\ell} = -\boldsymbol{\ell}$, so that only the last two lines of Table 1 are relevant. Thus $\mathbf{y}_2 = \mathbf{0}$ and $Y = \mathbb{R}\mathbf{y}_1$ is the real line spanned by $\mathbf{y}_1 = \sum_{\varphi \in \mathrm{sp}(\mathbf{y}_1)} \mathbf{f}_\varphi$.

For $\boldsymbol{\ell} = \mathbf{0}$, we have already observed in Example 4.11 that $\mathbf{H}_\mathbf{b}(\mathbf{0})$ is the trivial realization of $\mathcal{H}_\mathbf{b}$, with $d_\mathbf{b}(\mathbf{0}) = 1$, according to our conventions.

Suppose $\boldsymbol{\ell} \neq \mathbf{0}$ but $2\boldsymbol{\ell} = \mathbf{0} \pmod{b}$. Then it must be that b is even and we may take

$$\boldsymbol{\ell} = (0, ..., 0, \frac{b}{2}, ..., \frac{b}{2}) = (0^m, (\frac{b}{2})^{n-m}),$$

for some $0 \leqslant m < n$. Again, since $\boldsymbol{\ell} = -\boldsymbol{\ell}$ we can ignore the complications due to signs. Referring to Proposition 4.14, we must determine all $\varphi = \alpha\sigma$ for which $\mathbf{j}(\varphi) = \mathbf{l}$.

Now σ can be any of the 2^{n-1} sign changes in $\overline{B}_n$; and $\alpha \in S_n$ is any permutation such that $\alpha^{-1}\xi$ preserves the set $\{1, \ldots, m\}$, hence also $\{m+1, \ldots, n\}$. (Recall that $\xi = \kappa_2 \cdots \kappa_n = (1, n)(2, n-1) \cdots$.) Thus $\alpha^{-1}\xi$ lies in

$$D = \langle \rho_1, \ldots, \rho_{m-1}, \rho_{m+1}, \ldots, \rho_{n-1}, \sigma_2, \ldots, \sigma_n \rangle,$$

a subgroup of order $2^{n-1}m!(n-m)!$ in $\overline{B}_n$. We have $\mathrm{sp}(\mathbf{y}_1) = \xi D$. But from Lemma 5.2, the subspaces Yh, $h \in G_0$, are spanned by vectors $(\mathbf{y}_1)h$, which in turn are supported on subsets $\xi D\lambda$. These fixed left translates of the various right cosets $D\lambda$ partition $\overline{B}_n$. The distinct vectors $(\mathbf{y}_1)h$ are linearly independent, being supported on disjoint subsets of basis vectors $\mathbf{f}_\varphi$ in $\mathbb{E}$. Furthermore, they lie in $\mathbb{E}'$ and their sum is the base vertex $\mathbf{v}$. Thus the dimension of $\mathbb{E}'$ is the index of D in $\overline{B}_n$. To sum up, we have

Theorem 5.3. *Suppose b is even and $\boldsymbol{\ell} = (0^m, (\frac{b}{2})^{n-m})$, for some $0 \leqslant m \leqslant n$. Then the realization $\mathbf{H}_{\mathbf{b}}(\boldsymbol{\ell})$ has dimension*

$$d_{\mathbf{b}}(\boldsymbol{\ell}) = \binom{n}{m}.$$

It remains to consider the case $\boldsymbol{\ell} \neq -\boldsymbol{\ell}$, so that just the first three lines of Table 1 are needed. Now $\mathbf{y}_1, \mathbf{y}_2$ are orthogonal and T acts irreducibly on the plane $Y = \mathbb{R}\mathbf{y}_1 \oplus \mathbb{R}\mathbf{y}_2$. (By a variant of the Jordan-Hölder Theorem, the dimension of the summands in (5.1) is uniquely determined by T; there can be no alternative decomposition of $\mathbb{E}'$ into invariant lines.)

Again suppose $\boldsymbol{\ell}$ is in standard form, so that $0 \leqslant l_1 \leqslant l_2 \leqslant \ldots \leqslant l_n \leqslant \lfloor \frac{b}{2} \rfloor$; but now some $0 < l_i < b/2$. For $0 \leqslant k \leqslant \lfloor \frac{b}{2} \rfloor$, let m_k be the number of (necessarily consecutive) l_i's which equal k. Note that $m_k = 0$ is quite possible and that

$$n = m_0 + m_1 + \cdots + m_{\lfloor \frac{b}{2} \rfloor}.$$

It is useful to let $m_* = m_{\lfloor \frac{b}{2} \rfloor}$ for b even, and $m_* = 0$ when b is odd.

Recall from Proposition 4.14 the way that $\varphi = \alpha\sigma \in \overline{B}_n$ acts on $\boldsymbol{\ell}$: $\alpha \in \langle \rho_1, \ldots, \rho_{n-1} \rangle$ induces the subscript permutation $\alpha^{-1}\xi$, while $\sigma \in C_2^{n-1}$ induces the sign change $(e_1, \ldots, e_n)$. Clearly the blocks of equal parameters l_i are preserved by some subgroup of S_n. To these permutations adjoin all sign changes in the first m_0 or last m_* slots giving a subgroup D of $\overline{B}_n$ with order

$$st_{\boldsymbol{\ell}} := 2^{m_0+m_*} \cdot m_0! \cdot m_1! \cdots m_{\lfloor \frac{b}{2} \rfloor}! \,. \tag{5.3}$$

Quite possibly $m_0 + m_* = 0$; but in any case, the sign change subgroup is normalized by the permutation subgroup.

Again we use Proposition 4.14 to compute $\mathrm{sp}(\mathbf{y}) = \{\varphi = \alpha\sigma \in \overline{B}_n : \mathbf{j}(\varphi) = \pm\mathbf{1}\}$. Once more $\alpha \in \xi D$. The analysis for σ is more involved, with eight similar cases depending on the parity of n and which of $m_0, m_* \neq 0$.

Suppose, for example, that $m_0 = m_* = 0$, so $0 < l_1 \leqslant \ldots \leqslant l_n < b/2$. If n is even, $e_1 = 1$ so $\mathbf{j}(\varphi) = -\mathbf{1}$ is inadmissible ($l_1 \neq -l_1$). If n is odd, $\mathbf{j}(\varphi) = -\mathbf{1}$ is again impossible since an even number of $e_i = -1$. In either case, $\mathbf{j}(\varphi) = \mathbf{1}$ forces $\sigma = 1$, so the sign change subgroup has order $1 = 2^{m_0+m_*}$. We conclude that $\mathrm{sp}(\mathbf{y}) = \xi D$.

At the other extreme, $m_0, m_* > 0$, with b even, although $u = n - m_0 - m_* > 0$. Now

$$\mathbf{l} = (0, 0, \ldots, 0; l_{m_0+1}, \ldots, l_{n-m_*}; \frac{b}{2}, \ldots, \frac{b}{2})$$

has the three segments indicated. None of the middle u terms equals its negative, so all signs assigned there must be $+1$, when $\mathbf{j}(\varphi) = \mathbf{l}$ or all -1 when $\mathbf{j}(\varphi) = -\mathbf{l}$. Entries in the first and last segments can then be negated in various ways.

Continuing with this case, suppose, for instance, that n is odd. Just an even number of sign changes is allowed. If u is odd as well, then $m_0 + m_*$ is even. For $\boldsymbol{\ell}$, any even number of changes amongst these $m_0 + m_*$ is allowed, and for $-\boldsymbol{\ell}$ any odd number is allowed (because there are already u changes). Thus the first and last segments can be given signs in all $2^{m_0+m_*}$ ways. If u is even, the allocation of signs is a little different, but there are still $2^{m_0+m_*-1}$ ways for each of $\boldsymbol{\ell}$ and $-\boldsymbol{\ell}$. If n is even, the counting is a bit simpler but with the same end result.

It is easy to adapt the argument when $m_0 = 0$ or $m_* = 0$. In all cases, $\mathrm{sp}(\mathbf{y}) = \xi D$.

Now we proceed as we did for Theorem 5.3. By Lemma 5.2, the plane Yh, $h \in G_0$ has a basis $(\mathbf{y}_1)h, (\mathbf{y}_2)h$ which in turn is supported on some subset $\xi D\lambda$, where $\lambda \in \overline{B}_n$. These fixed left translates of the various right cosets $D\lambda$ partition $\overline{B}_n$. Once more, the distinct vectors $(\mathbf{y}_1)h$ are linearly independent, they lie in $\mathbb{E}'$ and their sum is now half the base vertex $\mathbf{v}$. Since $\dim(Y) = 2$, the dimension of $\mathbb{E}'$ is *twice* the index of D in $\overline{B}_n$. We have

Theorem 5.4. *Assume, as usual,* $\mathbf{b} = (b, 0, \ldots, 0)$*,* $b \geqslant 2$*. Suppose* $\boldsymbol{\ell}$ *is in standard form, and for* $0 \leqslant k \leqslant \lfloor \frac{b}{2} \rfloor$*, let* m_k *be the number of* l_i*'s which equal* k*. Also let* $m_* = m_{\lfloor \frac{b}{2} \rfloor}$ *(respectively,* $m_* = 0$*) for* b *even (respectively,* b *odd). Then the realization* $\mathbf{H_b}(\boldsymbol{\ell})$ *has dimension*

$$d_{\mathbf{b}}(\boldsymbol{\ell}) = \frac{2^n \cdot n!}{st_{\boldsymbol{\ell}}} = 2^{n-m_0-m_*} \binom{n}{m_0, m_1, m_2, \ldots, m_{\lfloor \frac{b}{2} \rfloor}}. \tag{5.4}$$

Remark 5.5. In fact, Theorem 5.3 is also covered by the expression on the right in (5.4).

Recall from Section 2 our topological description of the toroid $\mathcal{H}_{\mathbf{b}}$, still with $\mathbf{b} = (b, 0, \ldots, 0)$. We pack a $b \times \cdots \times b$ block of n-cubes into a larger n-cube whose outermost facets are identified in pairs. The b^n vertices of $\mathcal{H}_{\mathbf{b}}$ are parametrized by the n-tuples $\mathbf{m} \in \mathbb{Z}_b^n$. We have been much concerned with the role played by $\overline{B}_n$ in Proposition 4.14. But put that aside and recall from Section 2 how B_n itself acts by signed permutations on $\mathbb{Z}_b^n$.

Suppose $\boldsymbol{\ell} \in \mathbb{Z}_b^n$ has standard form. It is easy to see that its stabilizer in B_n has the order

$$st_{\boldsymbol{\ell}} = 2^{m_0+m_*} \cdot m_0! \cdot m_1! \cdots m_{\lfloor \frac{b}{2} \rfloor}!,$$

as computed before. (This does not contradict (5.3), where $st_{\boldsymbol{\ell}}$ was the order of a subgroup $D \subseteq \overline{B}_n$. However, we had assumed there that $m_0 + m_* \leqslant n - 1$.) We immediately get from equation (5.4) the following

Corollary 5.6. *Suppose* $\mathbf{b} = (b, 0, \ldots, 0)$*,* $b \geqslant 2$ *and* $\boldsymbol{\ell} \in \mathbb{Z}_b^n$*. Then the dimension* $d_{\mathbf{b}}(\boldsymbol{\ell})$ *of the realization* $\mathbf{H_b}(\boldsymbol{\ell})$ *of* $\mathcal{H}_{\mathbf{b}}$ *is simply the size of the orbit of* $\boldsymbol{\ell}$ *in* $\mathbb{Z}_b^n$ *under signed permutations.*

Remark 5.7. The calculations for the Corollary are easily visualized when $n = 2$ or 3. Various examples are described in Section 7. For $n = 2$, the Corollary appeared piecemeal in [13, Theorem 3.1]. The several orbits partition $\mathbb{Z}_b^n$, of course, so we have confirmed Theorem 3.2(a) for these toroids.

Next we make a note about the dimension in 'most' cases. Let us say that $\boldsymbol{\ell}$ (in standard form) is *generic* for b if $0 < l_1 < l_2 < \cdots < l_n < \frac{b}{2}$. A non-generic $\boldsymbol{\ell}$ thus has some l_i either repeated or equal 0 or $b/2$. But this just means that $\boldsymbol{\ell}$ is a point on a hyperplane of symmetry or on the outer surface of the large n-cube which holds the b^n smaller cubes used in the construction of $\mathcal{H}_{\mathbf{b}}$. There are n^2 hyperplanes of symmetry, each containing order $O(b^{n-1})$ integer vertices from the b^n smaller cubes. We therefore see that most $\boldsymbol{\ell}$ will be generic (for fixed n and large enough b). From Theorem 5.4 we immediately have

Corollary 5.8. *If $\boldsymbol{\ell}$ is generic for b, then $\mathbf{H}_{\mathbf{b}}(\boldsymbol{\ell})$ has dimension $2^n \cdot n!$ (so that $\mathbb{E}' = \mathbb{E}$).*

Remark 5.9. Several years ago I showed our original paper [13] on 3-toroids to Keith Taylor, who noted that our construction must relate to an application of the Mackey Theorems from group representation theory [6, Section 6.4]. I have been unable to make the connection, though there should be one. That approach resides in fairly heavy-duty character theory, whereas here we want and do describe the actual representations. The Mackey Theorems also produce a full character table for $\Gamma(\mathcal{H}_{\mathbf{b}})$, but we are concerned with just with those irreducibles having non-zero Wythoff space. Still, it would be nice to have a more direct proof of Corollary 5.6.

We close this section by using the machinery at hand to compute the actual number of vertices realized in $\mathbf{H}_{\mathbf{b}}(\boldsymbol{\ell})$. We know that $\mathbb{Z}_b^n$ acts transitively on the vertex set of the realization. The stabilizer of $\mathbf{v}$ can be described as

$$A := \{\mathbf{m} \in \mathbb{Z}_b^n : \mathbf{m} * \mathbf{j}(\varphi) = 0 \pmod{b}, \text{ for all } \varphi \in \overline{B}_n\}.$$

Using Proposition 4.14 and the fact that here $\mathbf{m} * (-\mathbf{j}(\varphi)) = 0$, we conclude that A is closed under *all* signed permutations of $\mathbf{m}$. But $A \simeq \Lambda/(b\mathbb{Z}^n)$, for some submodule Λ of $\mathbb{Z}^n$ which contains $b\mathbb{Z}^n$ and is itself closed under signed permutations. This means that we can adapt an argument from [12, p. 166]. Let s be the minimal positive integer among the coordinates of vectors $\mathbf{x} \in \Lambda$. Certainly $1 \leqslant s \leqslant b$. There then exists a minimal k such that Λ is generated by all signed permutations of $\mathbf{s} = (s^k, 0^{n-k})$. In fact, $k = 1, 2$ or n; and in the current situation, s must be a divisor of b. We have $b\mathbb{Z}^n \subseteq \Lambda \subseteq s\mathbb{Z}^n \subseteq \mathbb{Z}^n$.

To pin down s and k we must finally involve $\boldsymbol{\ell} = (l_1, ..., l_n)$. For instance, if $k = 1$ then $\mathbf{x} = (s, 0, ..., 0) \in \Lambda$, so that $s\, l_i \equiv 0 \pmod{b}$ for all i. A bit of number theory gives $s = s_1 = b/\gcd(b, l_1, ..., l_n)$. We summarize this and the other cases in the following chart:

k	$s \geqslant 1$
1	$s_1 = b/(\gcd(b, l_1, ..., l_n)$
2	$s_2 = b/(\gcd(b, l_1 + l_2, l_2 + l_3, ..., l_{n-1} + l_n, 2l_n)$
n	$s_n = b/(\gcd(b, l_1 + l_2 + \cdots l_n, 2l_2, ..., 2l_n)$

Given $\boldsymbol{\ell} = (l_1, ..., l_n)$, we have $s = \min\{s_1, s_2, s_n\}$. For ties take k to be the smallest subscript.

But Λ has index 2^{k-1} in $s\mathbb{Z}^n$, so we easily compute the index

$$[\mathbb{Z}_b^n : A] = [\mathbb{Z}^n : \Lambda] = [\mathbb{Z}^n : s\mathbb{Z}^n] \cdot [s\mathbb{Z}^n : \Lambda] = 2^{k-1} s^n.$$

We have

Proposition 5.10. *Suppose* $\mathbf{b} = (b, 0, \ldots, 0)$*. Using the parameters computed above, we have that the number of vertices in the realization* $\mathbf{H_b}(\boldsymbol{\ell})$ *is* $2^{k-1}s^n$*.*

Example 5.11. If b is odd, then it follows that $k = 1$. In fact, if b is an odd prime, then $\mathbf{H_b}(\boldsymbol{\ell})$ has either all b^n vertices or is trivial ($\boldsymbol{\ell} = \mathbf{0}$).

6 Toroids of type $\mathbf{H}_{(c,c,0,\ldots,0)}$ or $\mathbf{H}_{(c,c,\ldots,c)}$

We move to the toroids $\mathcal{H}_\mathbf{c}$, with $\mathbf{c} = (c^2, 0^{n-2})$ or (c^n). If we represent the translation subgroup of $\widetilde{C}_n$ by $\mathbb{Z}^n$, then $\Gamma(\mathcal{H}_\mathbf{c}) \simeq \widetilde{C}_n/\Lambda$, where $\Lambda \leqslant \mathbb{Z}^n$ is a subgroup symmetric under signed permutations of coordinates, just as in the previous section; and Λ is generated by $\mathbf{c}$. But Λ contains $(c, -c, 0, ..., 0)$, respectively $(c, -c, -c, ..., -c)$, so that hence $2c\mathbb{Z}^n \subseteq \Lambda$. Taking $b = 2c$ and $\mathbf{b} = (b, 0, ..., 0)$, this means that there is an epimorphism

$$\Gamma(\mathcal{H}_\mathbf{b}) \xrightarrow{h} \Gamma(\mathcal{H}_\mathbf{c})$$

mapping the specified generators of $\Gamma(\mathcal{H}_\mathbf{b})$ to those of $\Gamma(\mathcal{H}_\mathbf{c})$. On the other hand, a pure realization of $\mathcal{H}_\mathbf{c}$ is determined by a map $f : \Gamma(\mathcal{H}_\mathbf{c}) \to G$, acting on the space $\mathbb{D}$, say, with base vertex $\mathbf{w} \neq \mathbf{0}$ (fixed by the vertex-figure subgroup of $\Gamma(\mathcal{H}_\mathbf{c})$). Thus $[hf, \mathbf{w}]$ is a pure realization of $\mathcal{H}_\mathbf{b}$. Rescale $\mathbf{w}$ so that $\mathbf{w} \cdot \mathbf{w} = 2^{n-1}n! = \mathbf{v} \cdot \mathbf{v}$. By Theorem 4.16, the induced realization of $\mathcal{H}_\mathbf{b}$ is congruent to a unique $\mathbf{H_b}(\boldsymbol{\ell})$ with $\boldsymbol{\ell}$ in standard form.

It is easy to see that this means that $\ker h$ lies in the kernel of the epimorphism $\Gamma(\mathcal{H}_\mathbf{b}) \to G_\mathbf{b}(\boldsymbol{\ell})$. Conversely, if this condition holds, then there is induced a map $\Gamma(\mathcal{H}_\mathbf{c}) \to G_\mathbf{b}(\boldsymbol{\ell})$, which in turn gives a pure realization of $\mathcal{H}_\mathbf{c}$. We conclude from relation (2.8) in Theorem 2.1, that the pure realizations of $\mathcal{H}_\mathbf{c}$, in the cases $k = 2, n$, are precisely those $\mathbf{H_b}(\boldsymbol{\ell})$ for which $(g_0 g_1 \cdots g_{n-1} g_n g_{n-1} \cdots g_k)^{ck} = 1$ in $G_\mathbf{b}(\boldsymbol{\ell})$.

For example, when $n = k = 2$, the pure realizations of $\mathcal{H}_{(c,c)}$ are given by those $\mathcal{H}_{(2c,0)}$ in which

$$\begin{aligned} 1 &= (g_0 g_1 g_2)^{2c} \\ &= (t_1 t_2)^c \\ &= (r_1 s_1)^{c(l_2+l_1)} (r_{\rho_0} s_{\rho_0})^{c(l_2-l_1)} (r_{\rho_1} s_{\rho_1})^{c(l_1+l_2)} (r_{\rho_0\rho_1} s_{\rho_0\rho_1})^{c(l_1-l_2)}. \end{aligned}$$

We require that $\boldsymbol{\ell} = (l_1, l_2)$ satisfy $c(l_1 \pm l_2) \equiv 0 \pmod{2c}$, or more simply,

$$l_1 \equiv l_2 \pmod 2$$

(cf. [14, Theorem 4.1]). In fact, for any $n \geqslant 2$, it is straightforward to check that

(a) for $k = 2$,

$$(g_0 g_1 \cdots g_{n-1} g_n g_{n-1} \cdots g_2)^{2c} = (t_1 t_2)^c$$

(b) for $k = n \geqslant 3$,

$$(g_0 g_1 \cdots g_{n-1} g_n)^{nc} = (t_1 t_2 \cdots t_n)^c$$

To see this last equality, it is easiest to pull back to $\widetilde{C}_n$, where

$$\rho_0 \rho_1 \cdots \rho_n : (x_1, ..., x_n) \mapsto (x_2, x_2, ..., x_n, x_1 + 2)$$

We saw earlier that the φ-component of $t_1 t_2$ is $(r_\varphi s_\varphi)^{\pm l_j \pm l_m}$, for $1 \leqslant j \neq m \leqslant n$. All parameter choices occur, as do half the choices of opposite signs (Proposition 4.14). When $k = 2$ we thus get

Theorem 6.1. *Each pure realization of $\mathcal{H}_{(c^2,0^{n-2})}$ is similar to exactly one realization of the type $\mathbf{H}_{(2c,0,0,\ldots,0)}(\boldsymbol{\ell})$, where $\boldsymbol{\ell} = (l_1, \ldots, l_n)$ is in standard form and $l_j \equiv l_m \pmod 2$, for all $1 \leqslant j < m \leqslant n$.*

Similarly, for $k = n$ we have

Theorem 6.2. *Each pure realization of $\mathcal{H}_{(c^n)}$ is similar to exactly one realization of the type $\mathbf{H}_{(2c,0,0,\ldots,0)}(\boldsymbol{\ell})$, where $\boldsymbol{\ell} = (l_1, \ldots, l_n)$ is in standard form and*

$$l_1 + l_2 + \cdots + l_n \equiv 0 \pmod 2.$$

7 Toroids of rank 3 or 4 ($n = 2$ or 3)

In this section we consider some of the more accessible examples.

When $n = 2$, it is quite easy to interpret Theorems 4.16 and 5.4 using Corollary 5.6. In Figure 4 we display the (topological) toroids $\mathcal{H}_{(b,0)}$, for $b = 4$ and 5. The opposite (outer) edges of each square are identified in the usual manner. We have noted how the vertices of $\mathcal{H}_{(b,0)}$ correspond to points of $\mathbb{Z}_b^2$. The points $\boldsymbol{\ell}$ in turn serve double duty by parametrizing the realizations of $\mathcal{H}_{(b,0)}$ (and $\mathcal{H}_{(b/2,b/2)}$, when b is even). With the l_1 and l_2 axes labelled as shown, the $\boldsymbol{\ell}$ in standard form are the circled grid points in the shaded region. This triangle is a fundamental region for the action of $B_2 = \langle \gamma_0, \gamma_1 \rangle$ by signed permutations. Here $\gamma_0 : (l_1, l_2) \mapsto (-l_1, l_2)$ and $\gamma_1 : (l_1, l_2) \mapsto (l_2, l_1)$. (We use new generators γ_i to avoid confusion with the ρ_i so often used to generate $\overline{B}_n$.)

The dimension of $\mathbf{H_b}(\boldsymbol{\ell})$ is easily computed from Corollary 5.6 and is entered inside the corresponding circle. Since the point $\boldsymbol{\ell} = \mathbf{0}$ is fixed by B_2, it has orbit size 1; we get the correct dimension 1 for the trivial realization.

Suppose $b = 2c$ is even. Then the point $\boldsymbol{\ell} = (c, c)$ also has orbit size 1, since the corners of the square coincide under the toroidal identifications (Figure 4, with $b = 4$ on the left). In this case, the 1-skeleton of $\mathcal{H}_{(b,0)}$ is bipartite, and the toroid can be collapsed onto a segment. (Proposition 5.10 correctly predicts the $2^{2-1} \cdot 1^2 = 2$ vertices.) Similarly, from $\boldsymbol{\ell} = (0, c)$, we get orbit size 2; the toroid can be collapsed to a square.

Whenever $1 \leqslant a < b/2$, both $\boldsymbol{\ell} = (a, a)$ and $(0, a)$ have orbit size 4. The only generic case so far is $\boldsymbol{\ell} = (1, 2)$ when $b = 5$ (on the right of Figure 4). When $b = 2c$, there are $(c+1)(c+2)/2$ grid points in the fundamental triangle; and $3c$ of these are on one of its sides. If $b > 594$, then 99% of the pure realizations are generic and have dimension 8.

The 4-dimensional pure realizations of $\mathcal{H}_{(4,0)}$ are particularly interesting, since we can situate them inside the regular convex 4-cube $\mathbf{P}_4$. The actual calculations (here, at least) are not hard with the aid of Table 1 and the other machinery in Section 5.

For instance, suppose $\boldsymbol{\ell} = (0, 1)$. Then $\mathbf{j}(\varphi) = \boldsymbol{\ell}$ (or $-\boldsymbol{\ell}$) only for $\varphi = \rho_1$ (or $\rho_0\rho_1$). Ordering the planes $F_\varphi = \mathbb{R}\mathbf{e}_\varphi \oplus \mathbb{R}\mathbf{f}_\varphi$ by the list $1, \rho_0, \rho_1, \rho_0\rho_1$, we conclude that the basic T-space Y has basis

$$\begin{aligned} \mathbf{y}_1 &= (0, 0, 0, 0, 0, 1/2, 0, 1/2) \\ \mathbf{y}_2 &= (0, 0, 0, 0, 1/2, 0, -1/2, 0) \end{aligned}$$

Then $\mathbb{E}' = Y \oplus (Y g_1)$ is the 4-dimensional Euclidean space on which $G = G_{(4,0)}((0, 1))$ acts. Furthermore, t_1, t_2 are simple rotations through 90° about the axes $Y, Y g_1$, respectively. The 16 vertices of the realization $\mathbf{H}_{(4,0)}((0, 1))$ are therefore the vertices of the

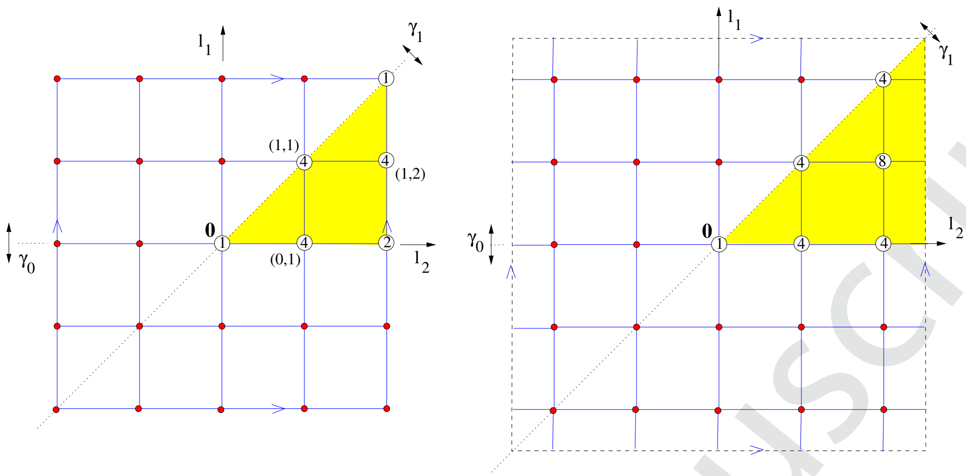


Figure 4: $\mathcal{H}_{(b,0)}$, for $b = 4$ (left) and $b = 5$ (right).

double prism $\{4\} \times \{4\}$, or equally well, the vertices of the regular convex 4-cube $\mathbf{P}_4$ [2, Section 4.5].

The most symmetric planar projection of $\mathbf{P}_4$ appears on the left of Figure 5. The realization $\mathbf{H}_{(4,0)}((0,1))$ contains all 16 vertices and 32 edges of $\mathbf{P}_4$, but just 16 of the 24 square faces. Recall that each facet of $\mathbf{P}_4$ is a 3-cube which belongs in three ways to a belt of 4 in which consecutive faces share a square. Indeed, all eight facets lie in two complementary belts. If we discard the eight intermediary squares from the two belts, we are left with the 2-faces of $\mathbf{H}_{(4,0)}((0,1))$. These 2-faces are here realized by actual squares. Since all proper faces of $\mathcal{H}_{(4,0)}$ appear as distinct objects in $\mathbf{H}_{(4,0)}((0,1))$, we say that this realization is *faithful*. Of course, this was not so for $\boldsymbol{\ell} = (0,0), (2,2)$ or $(0,2)$.

Actually, for any $b \geqslant 3$, $\mathbf{H}_{(b,0)}((0,1))$ provides the (planar) square faces of the double prism $\{b\} \times \{b\}$. (See [2, page 37], where $\mathbf{H}_{(b,0)}((0,1))$ appeared as the metrically regular skew polyhedron $\{4, 4 \,|\, b\}$).

On the right of Figure 5 we display $\mathbf{H}_{(4,0)}((1,2))$. Again we get all 16 vertices of $\mathbf{P}_4$. But now the edges of $\mathbf{H}_{(4,0)}((1,2))$ are realized as the 32 main diagonals of the 8 cubical facets of $\mathbf{P}_4$. Each 2-face of $\mathbf{H}_{(4,0)}((1,2))$ is a skew quadrilateral inscribed in one of the above-mentioned belts of 4 facets. This realization is also faithful.

Finally, in $\mathbf{H}_{(4,0)}((1,1))$, the edges appear as certain diagonals in the square faces of $\mathbf{P}_4$. There is a 2 to 1 collapse of the vertices of $\mathcal{H}_{(4,0)}$ onto a set of 8 alternate vertices of $\mathbf{P}_4$. In fact, these 8 points are the vertices of a (convex) regular cross-polytope $\mathbf{Q}_4 = \{3,3,4\}$ inscribed in $\mathbf{P}_4$. In Figure 6 we have used a different orthogonal projection for this cross-polytope [2, Figure 4.2A]. The dotted lines indicate the removal from $\mathbf{Q}_4$ of the edges of two equatorial squares in orthogonal planes.

The realization $\mathbf{H}_{(4,0)}((1,1))$ is not faithful. However, according to Theorem 6.1, we also have here a realization of the different toroid $\mathcal{H}_{(2,2)}$, which does turn out to be faithful. The eight 2-faces of $\mathcal{H}_{(2,2)}$ are certain Petrie polygons belonging to the (eight) tetrahedral facets of $\mathbf{Q}_4$ (see [12, Section 7B]). Adjacent 2-faces of $\mathcal{H}_{(2,2)}$ map to Petrie polygons in tetrahedra which share just an edge of $\mathbf{Q}_4$.

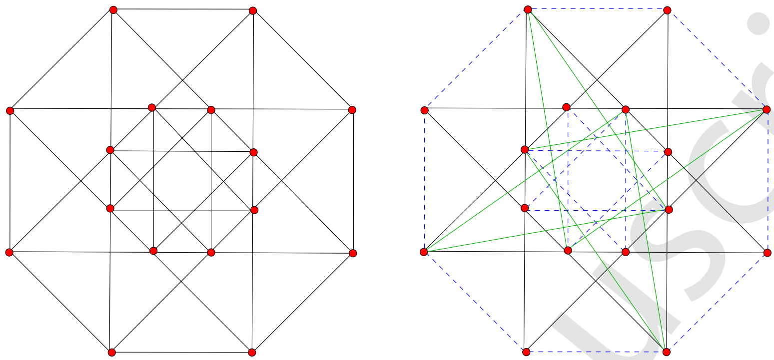


Figure 5: $\mathbf{H}_{(4,0)}(\boldsymbol{\ell})$, for $\boldsymbol{\ell} = (0, 1)$ (left) and $\boldsymbol{\ell} = (1, 2)$ (right).

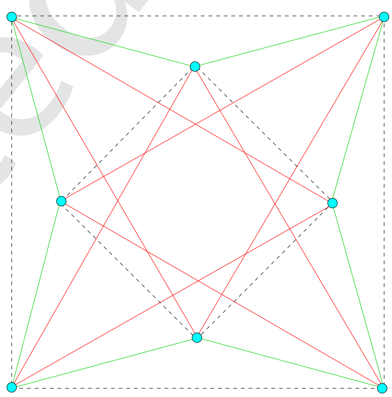


Figure 6: $\mathbf{H}_{(4,0)}(\boldsymbol{\ell})$, with $\boldsymbol{\ell} = (1, 1)$, also gives a pure realization of $\mathcal{H}_{(2,2)}$.

Now let us increase the rank. For $n = 3$, we can still visualize the topological toroid. A portion of $\mathcal{H}_{(4,0,0)}$ is shown in Figure 7. Using Corollary 5.6, it is again easy to enumerate the distinct pure realizations and their dimensions. There are ten pure realizations, none generic. (For that we need $b > 6$.) For instance, $\boldsymbol{\ell} = (0, 0, 1)$ has 2 images on each of the 12 edges of the cube under the action of B_3; but these are identified in fours to give dimension 6.

From Theorems 6.1 and 6.2 we can also locate here the five pure realizations for $\mathcal{H}_{(2,2,0)}$ and the six for $\mathcal{H}_{(2,2,2)}$.

Notice that both $\boldsymbol{\ell} = (0, 0, 2)$ and $\boldsymbol{\ell} = (0, 2, 2)$ give 3-dimensional realizations. But these must be different! In fact, from Proposition 5.10 we find 8 and 4 vertices, respectively. $\mathbf{H}_{(4,0,0)}((0, 0, 2))$ describes a very believable collapse of each of $\mathcal{H}_{(4,0,0)}$, $\mathcal{H}_{(2,2,0)}$, $\mathcal{H}_{(2,2,2)}$ onto a 3-cube. On the other hand, $\mathbf{H}_{(4,0,0)}((0, 2, 2))$ describes a further collapse onto a hemi-cube, here realized as the Petrie dual of a regular tetrahedron.

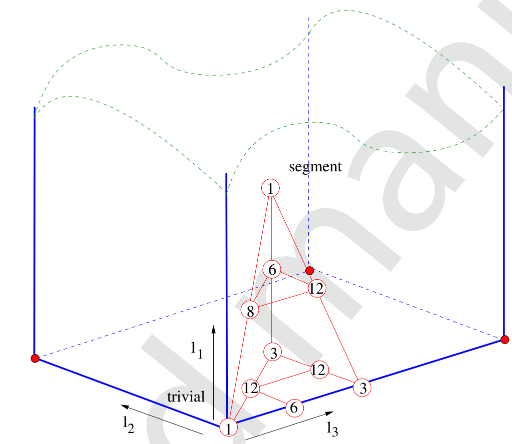


Figure 7: A fragment of the 4-toroid $\mathcal{H}_{(4,0,0)}$, showing the dimensions of $\mathbf{H}_{(4,0,0)}(\boldsymbol{\ell})$ for $\boldsymbol{\ell}$ in standard form.

We conclude this section with a look at the two 6-dimensional realizations of $\mathcal{H}_{(4,0,0)}$, again using the tools of Section 5, with some help from GAP [5]. Both $\mathbf{H}_{(4,0,0)}((0, 0, 1))$ and $\mathbf{H}_{(4,0,0)}((1, 2, 2))$ can in some fashion be 'inscribed' in a regular convex 6-cube $\mathbf{P}_6$; and in each case $\Gamma(\mathcal{H}_{(4,0,0)})$ is faithfully represented (but in different ways!) as a subgroup of index 15 in the group B_6 (of signed permutations).

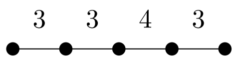
First of all, $\mathbf{H}_{(4,0,0)}((0, 0, 1))$ is a natural generalization of the realization $\mathbf{H}_{(4,0)}((0, 1))$ described earlier for $n = 2$. Now t_1, t_2, t_3 act as simple 90° rotations in three mutually orthogonal planes spanning $\mathbb{E}'$ (each having the two remaining planes as axis). This realization finds all 64 vertices of $\mathbf{P}_6$ and all its 192 edges, but omits 48 squares and 96 3-cubes from $\mathbf{P}_6$. The angle between any vertex and an adjacent vertex is $\arccos(2/3)$. Since $t_1^2\, t_2^2\, t_3^2 = -1$, the realization is centrally symmetric.

In the second realization $\mathbf{H}_{(4,0,0)}((1, 2, 2))$, we again achieve the 64 vertices. But now each vertex is joined to the former neighbors of its antipode, giving the complementary angle $\arccos(-2/3)$. The translations here are $\tilde{t}_j = -t_{4-j}^{-1}$.

8 Conclusion

Looking back, our results may seem more interesting for the group-theoretic techniques involved, rather than for the geometric objects which arise. It is the nature of regular toroids that small dimensional examples are few in number, often with considerable collapse of the structure in $\mathcal{H}_{\mathbf{b}}$.

Our title is also slightly dodgy, as we have not considered *all* the toroids of higher rank. We have indeed omitted the 5-toroids $\mathcal{F}_{\mathbf{b}} = \{3, 4, 3, 3\}_{\mathbf{b}}$ and their duals (for which $\mathbf{b} = (b^k, 0^{4-k})$, $k = 1$ or 2 and $b \geqslant 2$; see [12, Section 6E]). The starting point here is the Coxeter group $\widetilde{F_4}$ with diagram



Notice that $B_4 = \langle \rho_3, \rho_2, \rho_1, \rho_0 \rangle$ appears in dual fashion as the facet group for $\widetilde{F_4}$. We then get $\widetilde{F_4}$ by adjoining a new, naturally situated, reflection ρ_{-1} not fixing the origin $\mathbf{0}$. Then $\widetilde{F_4}$ is infinite, acts discretely on $\mathbb{E}^4$ and is the symmetry group of the dual regular tessellations of $\mathbb{E}^4$ by cross-polytopes $\{3, 3, 4\}$ or by 24-cells $\{3, 4, 3\}$. After adjusting our usual representation of B_4 a little, we can take the vertex set of the tessellation $\{3, 4, 3, 3\}$ to be the D_4 lattice

$$\Lambda = \mathbb{Z}^4 \cup (\mathbb{Z}^4 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})).$$

(See [12, Section 6E].) We find that $\widetilde{F_4}$ has a subgroup of translations isomorphic to $\mathbb{Z}^4$ and normalized by $F_4 = \langle \rho_2, \rho_1, \rho_0, \rho_{-1} \rangle \simeq [3, 4, 3]$. The toroids $\{3, 4, 3, 3\}_{\mathbf{b}}$ can be constructed much as in the cubical case.

Based on our previous work, including [13], we now expect various things. First, the distinct pure realizations of $\mathcal{F}_{\mathbf{b}}$ will be parametrized by vectors $\boldsymbol{\ell}$ describing vertices of the toroid but confined to some fundamental region for the action of the point group F_4 on the toroid itself. The dimension of a realization will equal the size of the $\boldsymbol{\ell}$-orbit, so generic pure realizations will have dimension 1152. (In fact, we have calculated that this first happens for $\mathbf{b} = (12, 0, 0, 0)$. The single generic pure realization is accompanied by fourteen of dimension 576.) Accessible examples of small dimension will be sparse.

On the other hand, we also expect that the same group-theoretic methods as used in this paper (twisting, Clifford's Theorem, etc.) will be enough to do the classification. But that we leave to someone else. It would be unfair to call our neglect of these toroids 'laziness', although 'exhaustion' is accurate.

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