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Half-arc-transitive graphs of arbitrarily large girth*

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Abstract

We show that for any $k \geq 2$ and $g \geq 3$ there are infinitely many finite $2k$-valent half-arc-transitive graphs of girth g with cyclic vertex-stabiliser of order k.

Keywords: Graph, automorphism, group.

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1 Introduction

A group of automorphisms $G \leq \mathrm{Aut}(X)$ of a graph X is *half-arc-transitive* if it is transitive on vertices and edges but not on arcs of X; the *graph* X itself is *half-arc-transitive* if $G = \mathrm{Aut}(X)$. It is well known that a finite half-arc-transitive is $2k$-valent for some $k \geq 2$. There are numerous constructions of half-arc-transitive graphs available, see for instance [1, 2, 11, 12, 14, 17, 18].

The smallest non-trivial valency of a half-transitive graph is 4. Somewhat surprisingly, all constructions of 4-valent half-arc-transitive graphs known to the date of submission of this note have bounded girth (length of the shortest cycle). This led Primož Šparl in his invited talk at the SIGMAP 2022 workshop to state the following question.

Question: Are there 4-valent half-arc-transitive graphs of arbitrarily large girth?

We answer this question in the affirmative as a consequence of the following more general result.

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Theorem 1.1. *For any integers $k \geq 2$ and $g \geq 3$ there are infinitely many finite half-arc-transitive graphs of valency $2k$, girth g and with the cyclic vertex-stabilisers of order k.*

The proof, presented in Section 3, relies on a combination of some known facts from the theory of regular hypermaps and their relationship to half-transitivity of their medial graphs, the existence of such hypermaps of a given type and arbitrarily large edge-width, and the existence of their chiral covers; all these notions and relevant results will be introduced and summed up in Section 2.

2 Hypermaps: orientable regularity, chirality, and large edge-width

We begin with a very brief outline of some aspects of the theory of hypermaps and their symmetries needed in this note. For more information on combinatorial models of hypermaps see [4, 6, 16]. A recent account can be found in [8, Section 7.6].

A 3-*edge-colouring* of a cubic graph X is a mapping from the edge set of X to $\{0, 1, 2\}$ such that, for each vertex of X, the three edges incident to the vertex receive pairwise different colours. A *topological hypermap* $\mathcal{H}$ is a 2-cell-embedding of a connected cubic 3-edge-colored graph on an orientable surface, such that the boundary of each face is a bi-colored cycle of even length greater than 2. The hypermap has type $\{k, m, n\}$ if $2n$, $2m$ and $2k$ are the least common multiples of the lengths of all facial alternating $0-1$ cycles, $0-2$ cycles and $1-2$ cycles, respectively.

The three colours determine three perfect matchings of the underlying graph X of our hypermap $\mathcal{H}$, and each of the three perfect matchings defines an involution r_j, $j \in \{0, 1, 2\}$, on the set of vertices of X (the vertices are *flags* of the hypermap), with r_j interchanging the vertices incident to edges coloured j. The group $\mathrm{Mon}(\mathcal{H}) = \langle r_0, r_1, r_2\rangle$ generated by the three involutions is the *monodromy group* of the hypermap. The assumed orientability of the supporting surface of $\mathcal{H}$ implies that the 'even-length-word' subgroup $\mathrm{Mon}^+(\mathcal{H}) = \langle r_0r_1, r_1r_2\rangle$ has index two in $\mathrm{Mon}(\mathcal{H})$.

Based on the above, by a *combinatorial hypermap* we will understand the quadruple $\mathcal{H} = (F; r_0, r_1, r_2)$, where F is its set of flags and r_i, $i = 0, 1, 2$, are fixed-point-free involutory permutations of F such that the group $H = \langle r_0, r_1, r_2\rangle$ is transitive on F and its even-length-word subgroup $H^+ = \langle r_1r_2, r_2r_0\rangle$ is of index 2 in H and acts on F with two orbits, say, F^+ and F^-, with $F^+ \cup F^- = F$. We have deliberately used the same symbol for a hypermap introduced at the beginning of this section and its 'combinatorial mate', the latter being an algebraic equivalent of the former.

Addressing the issue of symmetries of a combinatorial hypermap, an *automorphism* of $\mathcal{H}$ is a permutation $\psi \in \mathrm{Sym}(F)$ of its flag-set which centralizes the group $\mathrm{Mon}(\mathcal{H}) = \langle r_0, r_1, r_2\rangle$. Recalling our assumption of orientability, the automorphism ψ is *orientation-preserving* if it preserves both sets F^+ and F^-, and *orientation-reversing* if it swaps the two sets. The hypermap $\mathcal{H}$ is *orientably-regular* if the group of all its orientation preserving automorphisms $\mathrm{Aut}^+(\mathcal{H})$ is a regular permutation group on F^+ (and hence also on F^-).

By well known general facts from the theory of permutation groups, in the case of an orientably-regular hypermap the group $\mathrm{Aut}^+(\mathcal{H})$ is isomorphic to the even-length-word subgroup $\mathrm{Mon}^+(\mathcal{H}) = \langle r_1r_2, r_2r_0\rangle \cong H^+$. In addition, if such a hypermap $\mathcal{H}$ admits an orientation-reversing automorphism, then the group $\mathrm{Aut}(\mathcal{H})$ of *all* automorphisms of $\mathcal{H}$ is isomorphic to its monodromy group $\mathrm{Mon}(\mathcal{H}) = \langle r_0, r_1, r_2\rangle$, and the hypermap is said to be *reflexible*.

In an orientably-regular and reflexible hypermap $\mathcal{H} = (F; r_0, r_1, r_2)$ of type $\{k, m, n\}$,

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the orders of the products r_1r_2, r_2r_0 and r_0r_1 of automorphism are, respectively, equal to n, m and k. The group $\mathrm{Aut}(\mathcal{H}) \cong \mathrm{Mon}(\mathcal{H})$ then has a presentation of the form

$$\mathrm{Aut}(\mathcal{H}) = \langle r_0, r_1, r_2 \mid r_0^2, r_1^2, r_2^2, (r_1r_2)^k, (r_2r_0)^m, (r_0r_1)^n, \ldots \rangle. \tag{2.1}$$

If only orientable regularity of $\mathcal{H} = (F; r_0, r_1, r_2)$ is assumed, then, using the notation $R = r_1r_2$, $L = r_2r_0$ and $T = r_0r_1 = (RL)^{-1}$, the group $\mathrm{Aut}^+(\mathcal{H}) \cong \mathrm{Mon}^+(\mathcal{H})$ admit a presentation

$$\mathrm{Aut}^+(\mathcal{H}) = \langle R, L, T \mid R^k, L^m, T^n, RLT, \ldots \rangle \,. \tag{2.2}$$

Let $\Delta(k, m, n)$ and $\Delta^+(k, m, n)$ denote the extended and the ordinary (k, m, n)-triangle groups with presentations

$$\Delta(k, m, n) = \langle a, b, c \mid a^2, b^2, c^2, (bc)^k, (ca)^m, (ab)^n \rangle, \text{ and} \tag{2.3}$$

$$\Delta^+(k, m, n) = \langle x, y, z \mid x^k, y^m, z^n, xyz \rangle. \tag{2.4}$$

It is obvious that the groups $\mathrm{Aut}(\mathcal{H})$ and $\mathrm{Aut}^+(\mathcal{H})$ are smooth quotients of $\Delta(k, m, n)$ and $\Delta^+(k, m, n)$ induced by the epimorphisms taking the ordered triples (a, b, c) and (x, y, z) to the triples (r_0, r_1, r_2) and (R, L, T), respectively. Note that the group $\Delta^+(k, l, m)$ forms the 'even-length-word' subgroup of $\Delta(k, m, n)$ of index 2 by taking $x = bc$, $y = ca$ and $z = ab$.

To explain the topological counterpart of the above quotients and epimorphisms, consider the orientably-regular and reflexible hypermap $\mathcal{U} = (\Delta(k, m, n); a, b, c)$; notice that the flag-set of $\mathcal{U}$ is identified with the group $\Delta(k, m, n) = \langle a, b, c \rangle$ itself. The hypermap $\mathcal{U}$ is known as the *universal hypermap* of type $\{k, m, n\}$. Its supporting surface is a sphere, an Euclidean plane, and a hyperbolic plane, depending on whether $1/k+1/m+1/n$ is greater than, equal to, and smaller than 1. Its universality is given by the fact that it smoothly covers *any* orientably-regular and regular hypermap of type $\{k, m, n\}$; in particular, the above smooth group epimorphism $\Delta(k, m, n) \to \mathrm{Aut}(\mathcal{H})$ extends to a smooth covering of the orientably-regular hypermap $\mathcal{H}$ by the universal hypermap $\mathcal{U}$.

By the same token, the hypermap $\mathcal{U}^+$ that one obtains from $\mathcal{U}$ by taking into account only its orientation-preserving automorphism group $\Delta^+(k, m, n)$, serves as a smooth cover of every orientably-regular hypermap of type $\{k, m, n\}$. This is equivalent to the fact that the projection $\pi\colon\ \Delta^+(k, m, n) \to \mathrm{Aut}^+(\mathcal{H})$ such that $\pi(x, y, z) = (R, L, T)$ extends to a smooth cover of the orientably-regular hypermap $\mathcal{H}$ by $\mathcal{U}^+$. However, the epimorphism π need not project reflexibility of $\mathcal{U}^+$ down to $\mathcal{H}$ in general; the orientably-regular hypermap $\mathcal{H} = \pi(\mathcal{U}^+)$ is reflexible if and only if the kernel $K = \ker(\pi)$, which is a normal subgroup of $\Delta^+(k, m, n)$, is also normal in the extended triangle group $\Delta(k, m, n)$. If this is not the case, then the hypermap $\mathcal{H}$ is commonly called *chiral* (irreflexible); equivalently, $\mathcal{H}$ is chiral if $\mathrm{Aut}(\mathcal{H}) = \mathrm{Aut}^+(\mathcal{H})$.

An important ingredient for the proof of our main result in Section 3 is the following non-trivial theorem proved by G. Jones [9].

Theorem 2.1 (Jones 2015)**.** *For every finite orientably-regular hypermap $\mathcal{H}$ of type $\{k, m, n\}$ with $1/k + 1/m + 1/n \le 1$ there exist infinitely many finite orientably-regular but chiral maps of type $\{k, m, n\}$ which are smooth covers $\mathcal{H}$.*

For many more details on (orientable) hypermaps we refer the reader to [6, 7]. In the remaining part we address hypermaps with 'large edge-width' and we confine ourselves to

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the case of orientable regularity. Let $\mathcal{H} = (F; r_0, r_1, r_2)$ be an orientably-regular hypermap of type $\{k, m, n\}$ with $1/k + 1/m + 1/n \leq 1$, which is equivalent to assuming that the (orientable) supporting surface of the underlying 3-edge-coloured graph X of $\mathcal{M}$ is not a sphere. The *edge-width* $\mathrm{ew}(\mathcal{H})$ of $\mathcal{H}$ is the length of the shortest non-contractible cycle in the embedding of X. We say that $\mathcal{H}$ is a hypermap of *large edge-width* if $\mathrm{ew}(\mathcal{H}) > \max\{2k, 2m, 2n\}$; hence, for a hypermap with large edge-width, $\mathrm{ew}(\mathcal{H})$ exceeds the length of any closed walk bounding a face of $\mathcal{H}$.

Edge-width has been thoroughly studied in [13] in a much more general setting for arbitrary cellular embeddings of graphs. The main motivation was the fact that embeddings of 3-connected graphs with edge-width exceeding the length of any face-bounding cycle share a number of properties with embeddings of 3-connected graphs in a plane; in particular, one can generalize Whitney's theorem relating map and graph automorphisms in the planar case. Here we need a consequence for regular hypermaps that follows from [13, Sections 5.1 and 5.2].

Theorem 2.2. *Let $\mathcal{H}$ be an orientably-regular hypermap with an underlying 3-edge-coloured graph X embedded in an orientable surface that is not a sphere. If $\mathcal{H}$ has large edge-width and if X is 3-connected, then $\mathrm{Aut}(X) = \mathrm{Aut}(\mathcal{H})$; moreover, a shortest facial cycle of the embedding of X is a girth cycle of X.*

The fact that non-spherical orientably-regular maps of arbitrary types exist may be surprising at a first glance; it is a consequence of Maltsev's theorem on residual finiteness applied to triangle groups.

Theorem 2.3. *For every $\{k, m, n\}$ such that $1/k + 1/m + 1/n \leq 1$ there exist finite hypermaps of type $\{k, m, n\}$ of edge-width larger than any $j > \max\{2k, 2m, 2n\}$, and consequently with a 3-connected underlying graph.*

Proof. The proof is an adaptation of that of Proposition 2 and Theorem 1 of [15] and we give here only a sketch. Let $\mathcal{U}^+$ be the universal orientably-regular map of type $\{k, m, n\}$, $1/k + 1/m + 1/n \leq 1$, its orientation-preserving automorphism group being the ordinary $\{k, m, n\}$-group $\Delta^+(k, m, n) = \langle x, y, z \mid x^k, y^m, z^n, xyz \rangle$. An element of $\Delta^+(k, m, n)$, expressed as a word of length ℓ in the generators $\{x, y, z\}$, is *irreducible* if it is not possible to reduce it by means of the relators $\{x^k, y^m, z^n, xyz\}$ to a word of a smaller length.

Let $j > \max\{2k, 2m, 2n\}$ and let $\mathcal{W}_j$ be the finite set of irreducible (non-identity) words over the alphabet $\{x, y, z\}$ of length less than or equal to j. By residual finiteness, the group $\Delta^+(k, m, n)$ contains a normal subgroup N of finite index avoiding the set $\mathcal{W}_j$. It follows that, in the finite quotient $\Delta^+(k, m, n)/N$, every irreducible word of length at most j represents a non-identity element. But such irreducible elements correspond to non-contractible closed curves of length at most j in the supporting surface of the finite orientably-regular hypermap $\mathcal{H}$ with orientation-preserving automorphism group $\Delta^+(k, m, n)/N$. We conclude that the hypermap $\mathcal{H}$ with $\mathrm{Aut}^+(\mathcal{H}) = \Delta^+(k, m, n)/N$ has edge-width larger than j; this also implies that the underlying graph of $\mathcal{H}$ is 3-connected, see [13, Proposition 5.5.12]. □

3 The main result

In accordance with [3], for any given orientable hypermap $\mathcal{H} = (F; r_0, r_1, r_2)$ and any i from the index set $\{0, 1, 2\}$ identified with $\mathbb{Z}_3$, we may introduce the (orientable) *medial*

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map $\mathrm{Med}_i(\mathcal{H})$ of $\mathcal{H}$ by contracting every face-bounding cycle alternately edge-coloured $i-1$ and $i+1$ onto a new vertex, leaving the edges coloured i intact.

To describe an algebraic counterpart of the presented topological construction, recall that to describe an orientable map (a cellular embedding of a connected graph on an orientable surface) one only needs to specify at each vertex a cyclic permutation of arcs emanating from the vertex and consistent with the chosen surface orientation. The product ρ of all such 'local' cyclic orientations, taken over all vertices, is the *rotation* of the map, and together with the arc-reversing involution (usually denoted by λ) they completely determine the map. In such an algebraic way the medial map $\mathrm{Med}_i(\mathcal{H})$ of an orientable hypermap $\mathcal{H} = (F; r_0, r_1, r_2)$ is obtained by taking its arc set equal to $F = F^+ \cup F^-$ and its arc-reversing involution λ equal to r_i; its rotation ρ is then defined by $\rho(x) = r_{i+1}x$ if $x \in F^+$ and $\rho(x) = r_{i-1}$ if $x \in F^-$.

The following statement proved by Breda and Nedela in [3] shows that there is a correspondence between orientably regular hypermaps and half-arc-transitive groups of automorphisms of graphs.

Theorem 3.1. *Let $\mathcal{H} = (F; r_0, r_1, r_2)$ be an orientably-regular hypermap of type $\{k_0, k_1, k_2\}$ for $k_i = \mathrm{ord}(r_{i-1}r_{i+1})$ and $i \in \mathbb{Z}_3$, with $\mathrm{Aut}^+(\mathcal{H}) = G$. Then, for any $i \in \mathbb{Z}_3$, its medial map $\mathcal{M} = \mathrm{Med}_i(\mathcal{H})$ is an edge-transitive map on the same surface, with the group $G \le \mathrm{Aut}^+(\mathcal{M})$ acting half-arc-transitively on the $2k_i$-valent underlying graph of the medial map, with cyclic vertex-stabilisers of order k_i.*

Conversely, if X is a graph and $G \le \mathrm{Aut}(X)$ is a group acting half-arc-transitively on edges of X with cyclic vertex stabilisers, then there exists an orientably regular hypermap $\mathcal{H}$ with $\mathrm{Aut}^+(\mathcal{H}) \cong G$ such that X is the underlying graph of a medial map $\mathrm{Med}_i(\mathcal{H})$ for some $i \in \mathbb{Z}_3$.

Now we are ready to prove our main result.

Proof of Theorem 1.1. Let $n > 1$ be an integer. Observe that assuming $n > g$, we have always $1/k + 1/g + 1/n \le 1$, except when $(k, g, n) = (2, 3, 4)$ and $(k, g, n) = (2, 3, 5)$. Hence, in the case $k = 2$ and $g = 3$ we assume $n \ge 6$. By Theorem 2.3, for any $n > g$ (and for any $n \ge 6$ if $k = 2$ and $g = 3$), there exist orientably regular hypermaps $\mathcal{K}$ of type $\{k, g, n\}$ and with arbitrarily large edge-width. Throughout this proof we do not assume $k \le g$, i.e. both possibilities $k \le g$ and $k > g$ will be allowed. According to Theorem 2.1 there are infinitely many finite orientably-regular but chiral hypermaps $\mathcal{H}$ of type $\{k, g, n\}$ covering $\mathcal{K}$. Set $\mathcal{M} = \mathrm{Med}_0(\mathcal{H})$. Since $\mathcal{H}$ is a smooth cover of $\mathcal{K}$, the hypermap $\mathcal{H}$ is of large edge-width and 3-connected, and hence so is its medial map $\mathcal{M}$. Theorem 2.2 then implies that $\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}(X)$, where X is the underlying graph of $\mathcal{M}$.

Following Theorem 3.1, the group $G = \mathrm{Aut}^+(\mathcal{H})$ can be considered to be a subgroup of $H = \mathrm{Aut}(\mathcal{M})$ of index at most 2, with G acting half-arc-transitively on X with cyclic stabiliser of order k. We aim to prove that $G = H$; this together with $H \cong \mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}(X)$ will establish our result. As G is transitive on vertices, it will be sufficient to prove that, for some vertex v of $\mathcal{M}$, the vertex stabilizers G_v and H_v coincide.

First, recall that the group $G_v \cong \mathbb{Z}_k$ contains only orientation-preserving automorphisms. Further, in the medial map $\mathcal{M}$ there are $2k$ faces incident to v; among them k are of face size g and the other k are of face size n, and they appear alternately around v on the (orientable) supporting surface of $\mathcal{M}$. Since $g < n$ and $G_v \cong \mathbb{Z}_k$ is orientation-preserving, the stabilizer H_v cannot contain any more orientation-preserving elements except those

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already in G. But since H_v stabilizes a vertex *as part of a map automorphism group*, the assumption $G_v \neq H_v$ would imply existence of a reflection in H_v. Hence, both the orientable map $\mathcal{M}$ and the orientable hypermap $\mathcal{H}$ would be reflexible. This, however, contradicts chirality of $\mathcal{H}$.

It follows that the groups $G = \mathrm{Aut}^+(\mathcal{H})$ and $H = \mathrm{Aut}(\mathcal{M})$ can be identified, as claimed. Finally, we prove that the girth of the underlying graph X_{Med} of the medial map $\mathcal{M} = \mathrm{Med}_0(\mathcal{H})$ is equal to g. But the way a medial map was introduced implies that X_{Med} contains facial cycles of length only g and n, where $g < n$, and by Theorem 2.2 the graph X_{Med} has girth g. □

We note that Theorem 1.1 has predecessors dealing with special subcases in [3, 5, 11, 10]. For $k = 1$ Theorem 1.1 does not hold, since there are no half-arc-transitive graphs of valency 2.

References

[1] B. Alspach, D. Marušič and L. Nowitz, Constructing graphs which are 1/2-transitive, *J. Aust. Math. Soc., Ser. A* **56** (1994), 391–402.

[2] B. Alspach and M.-Y. Xu, 1/2-transitive graphs of order $3p$, *J. Algebr. Comb.* **3** (1994), 347–355, doi:10.1023/A:1022466626755, https://doi.org/10.1023/A:1022466626755.

[3] A. Breda D'Azevedo and R. Nedela, Half-arc-transitive graphs and chiral hypermaps., *Eur. J. Comb.* **25** (2004), 423–436, doi:10.1016/j.ejc.2003.07.003, https://doi.org/10.1016/j.ejc.2003.07.003.

[4] R. P. Bryant and D. Singerman, Foundations of the theory of maps on surfaces with boundary, *Q. J. Math., Oxf. II. Ser.* **36** (1985), 17–41, doi:10.1093/qmath/36.1.17, https://doi.org/10.1093/qmath/36.1.17.

[5] M. Conder and A. Žitnik, Half-arc-transitive graphs of arbitrary even valency grater than 2, *Eur. J. Comb.* **54** (2016), 177–186.

[6] R. Cori and A. Machi, Maps, hypermaps and their automorphisms: A survey. I., II., III., *Expo. Math.* **10** (1992), 403–427, 429–447,449–467.

[7] D. Corn and D. Singerman, Regular hypermaps, *Eur. J. Comb.* **9** (1988), 337–351, doi:10.1016/S0195-6698(88)80064-7, https://doi.org/10.1016/S0195-6698(88)80064-7.

[8] J. L. Gross, J. Yellen and P. Zhang, *Handbook of Graph Theory*, CRC Press, 2nd edition, 2013.

[9] G. A. Jones, Chiral covers of hypermaps, *Ars Math. Contemp.* **8** (2015), 425–431, doi:10.26493/1855-3974.587.3eb, https://doi.org/10.26493/1855-3974.587.3eb.

[10] D. Marušič, Quartic half-arc-transitive graphs with large vertex stabilizers, *Discrete Math.* **299** (2005), 180–193, doi:10.1016/j.disc.2004.02.025, https://doi.org/10.1016/j.disc.2004.02.025.

[11] D. Marušič and R. Nedela, Maps and half-transitive graphs of valency 4, *Eur. J. Comb.* **19** (1998), 345–354, doi:10.1006/eujc.1998.0187, https://doi.org/10.1006/eujc.1998.0187.

[12] D. Marušič and M. Y. Xu, A 1/2-transitive graph of valency 4 with a nonsolvable goup of automorphisms, *J. Graph Theory* **25** (1997), 133–138.

[13] B. Mohar and C. Thomassen, *Graphs on Surfaces*, volume 10, J. Hopkins Univ. press, 2001.

Accepted manuscript

[14] A. Ramos Rivera and P. Šparl, New structural results on tetravalent half-arc-transitive graphs, *J. Comb. Theory, Ser. B* **135** (2019), 256–278, doi:10.1016/j.jctb.2018.08.006, https://doi.org/10.1016/j.jctb.2018.08.006.

[15] J. Širáň, Triangle group representations and constructions of regular maps, *Proc. Lond. Math. Soc. (3)* **82** (2001), 513–532, doi:10.1112/plms/82.3.513, https://doi.org/10.1112/plms/82.3.513.

[16] A. Vince, Combinatorial maps, *J. Comb. Theory, Ser. B* **34** (1983), 91–21, doi:10.1016/0095-8956(83)90002-3, https://doi.org/10.1016/0095-8956(83)90002-3.

[17] R.-J. Wang, Half-transitive graphs of order a product of two distinct primes, *Commun. Algebra* **22** (1994), 915–927, doi:10.1080/00927879408824885, https://doi.org/10.1080/00927879408824885.

[18] M. Xu, Half-transitive graphs of prime-cube order, *J. Algebr. Comb.* **1** (1992), 275–282, doi:10.1023/A:1022440002282, https://doi.org/10.1023/A:1022440002282.

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