THE ART OF DISCRETE AND APPLIED MATHEMATICS

ISSN 2590-9770
The Art of Discrete and Applied Mathematics
https://doi.org/10.26493/2590-9770.1606.28a
(Also available at http://adam-journal.eu)

Generalizations of Cayley graphs to uniform hypergraphs

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Received 9 January 2023, accepted 13 February 2024

Abstract

A vertex-transitive graph is a Cayley graph if and only if it admits a group of automorphisms acting regularly on its set of vertices. We generalize Cayley graphs to k-uniform hypergraphs, i.e. pairs $(V, \mathcal{E})$ where $\mathcal{E}$ consists of k-subsets of V, $k \geq 2$. We call a k-uniform hypergraph Cayley if and only if the hypergraph admits a group of automorphisms acting regularly on the vertex set of the hypergraph. We compare our definition with two other generalizations of Cayley graphs to hypergraphs found in the literature, and argue that ours is the most general; the other generalizations are special cases of the one proposed in this article. We present some basic properties of Cayley hypergraphs, discuss the connection of this concept to some recent results concerning hypergraphs, as well as connections to our ongoing project of constructing k-uniform hypergraphs whose full automorphism group is isomorphic to a prescribed finite group G and acts regularly on the vertices of the hypergraph.

Keywords: Cayley graph, uniform hypergraph, regular automorphism group.

Math. Subj. Class.: 05C25, 05C65, 05E18

1 Introduction

Hypergraphs are natural generalizations of graphs in which instead of edges, which are subsets of the vertex set of size 2, one admits hyperedges, which are subsets of the vertex set of arbitrary sizes. If all the hyperedges of a hypergraph are of the same size k, one talks about k-uniform hypergraphs. Since Cayley graphs constitute an important subclass of the class of vertex-transitive graphs, it is reasonable to assume that generalizing Cayley

*Corresponding author. Supported by VEGA 1/0437/23 and APVV-19-0308.

†Supported by VEGA 1/0437/23, and APVV-19-0308.

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graphs to k-uniform hypergraphs might lead to interesting insights into the theory of vertex-transitive k-uniform hypergraphs (and possibly even to further insights into the theory of vertex-transitive graphs).

A large part of the importance of Cayley graphs within the class of vertex-transitive graphs stems from the fundamental fact that Cayley graphs are characterized by the existence of a regular group of automorphisms. Regular group actions are the most natural transitive group actions, and appeared already in the classical proof of Cayley's theorem: Given a finite group G, the *left regular permutation representation* $G_L = \{\sigma_g \mid g \in G\}$ of G is defined via left multiplication by the elements of G on itself, $\sigma_g(h) = gh$, for all $g, h \in G$. The key property of the left regular permutation representation of a group G is the existence, for each pair of elements $h, h' \in G$, of a unique σ_g mapping h to h'. In view of this, a transitive action of a group G on a set V is *regular* if and only if the stabilizer of any vertex is trivial and/or $|G| = |V|$. Furthermore, if G acts regularly on V, there exists a one-to-one correspondence $\Phi\colon V \longrightarrow G$ that allows one to identify the elements of V with the elements of G and the action of G on V with the left regular action of G on itself. Since our paper focuses on generalizations of Cayley graphs, and therefore on regular actions, we will always assume that regular actions are realized as the action of G on itself via left multiplication.

The definition of a Cayley graph used most widely today does not start off from a regular action. Instead, it starts from a (finite) group G and a set X of elements of G, $X \subset G$, closed under taking inverses, $X = X^{-1}$, and not containing 1_G. For a given group G and a subset S satisfying the above properties, the *Cayley graph* $\Gamma = \mathcal{C}(G, X)$ is the graph $(G, \mathcal{E})$ where $\mathcal{E} = \{\ \{g, g \cdot x\} \mid g \in G, x \in X\}$. Alternately, the set of edges of Γ is the set of (unordered) pairs $\{a, b\}$ of elements of G satisfying the property $a^{-1} \cdot b \in X$. It is clear that for any $g \in G$ the left-multiplication $\sigma_g \in G_L$ is a graph automorphism of $\mathcal{C}(G, X)$ mapping $\{a, ax\} \to \{ga, gax\}$, for all $a \in G$ and $x \in X$, and hence $G_L \leq \mathrm{Aut}(\mathcal{C}(G, X))$. This fundamental connection between Cayley graphs as defined above and regular group actions was first observed by Sabidussi in 1958 [11]:

Theorem 1.1 ([11]). *A graph $\Gamma = (V, \mathcal{E})$ is a Cayley graph if and only if it admits a group of automorphisms G acting regularly on V; in which case $\Gamma = \mathcal{C}(G, X)$, for some $X \subset G$ closed under taking inverses and not containing 1_G.*

One of the first to recognize the universality of structures admitting a group of automorphisms acting regularly on the set of vertices was Babai in [1], where he defined the *Cayley object* for any concrete category (a category admitting a faithful forgetful functor into the category of sets) precisely via the existence of a group of automorphisms acting regularly on the underlying set of the structure. This approach was later reinforced by Pálfy in [10] who considered Cayley objects in the category of relational structures. Nevertheless, attempts at generalizing Cayley graphs to hypergraphs available in the literature usually choose to generalize the concept of a Cayley graph using the language of the connecting set $X \subset G$. While objects defined this way tend to maintain the property of admitting a regular group of automorphisms, they usually do not cover all hypergraphs with this property. In our paper, we choose to follow Babai's approach and to make the existence of a regular group of automorphisms a characterizing property of Cayley hypergraphs:

Definition 1.2. A *Cayley hypergraph* $(V, \mathcal{H})$ is a k-uniform hypergraph for some $1 \leq k \leq |V|$, that admits a regular group of automorphisms.

Thus, our generalization of Cayley graphs generalizes the most important property of Cayley graphs equivalent (in the case of graphs) to a graph being Cayley. In what follows, we present some basic properties of automorphism groups of Cayley hypergraphs, and compare our definition with two other generalizations of Cayley graphs to hypergraphs. We also mention the connections of these concepts to some recent results concerning hypergraphs [5, 9], as well as connections to our project of constructing k-uniform hypergraphs whose full automorphism group is isomorphic to a prescribed finite group G and acts regularly on the vertices of the hypergraph [8].

2 Preliminaries

Throughout the paper, all groups considered are finite and so are the sets upon which they act.

A *hypergraph* $\Gamma = (V, \mathcal{H})$ consists of a set V and a collection $\mathcal{H}$ of subsets of V, $\mathcal{H} \subseteq \mathcal{P}(V)$. We will call the elements of $\mathcal{H}$ *hyperedges* or more specifically, k*-hyperedges*, if they are of size k. A hypergraph is said to be k*-uniform* (abbreviated to a k*-hypergraph*) if all the hyperedges in $\mathcal{H}$ are of the same size k, i.e., $\mathcal{H} \subseteq \mathcal{P}_k(V)$, where $\mathcal{P}_k(V)$ denotes the set of k-element subsets of V. A hypergraph is said to be r*-regular* if all vertices $v \in V$ are contained in exactly r hyperedges. A hypergraph $\Gamma = (V, \mathcal{H})$ can be alternately represented via its bipartite *incidence graph* (sometimes called *Levi graph*) $(V \cup \mathcal{H}, E)$ with edges corresponding to incidences between vertices in V and hyperedges in $\mathcal{H}$. A k-uniform r-regular hypergraph $(V, \mathcal{H})$ corresponds in this sense to a bipartite graph $(V \cup \mathcal{H}, E)$ in which all vertices belonging to V are of degree r and all vertices in $\mathcal{H}$ are of degree k. The *dual hypergraph* $\tilde{\Gamma} = \widetilde{(V, \mathcal{H})}$ to a k-uniform r-regular $(V, \mathcal{H})$ is the r-uniform k-regular hypergraph whose vertices are the elements of $\mathcal{H}$, hyperedges are the elements from V, and the incidence relation is the reverse of the incidence relation of $(V, \mathcal{H})$. An *automorphism* of a hypergraph $\Gamma = (V, \mathcal{H})$ is a permutation of the elements of V that preserves the hyperedges, i.e., a permutation $\varphi \in Sym_V$ with the property $\varphi(H) \in \mathcal{H}$ *if and only if* $H \in \mathcal{H}$. The group of all automorphisms of Γ will be denoted by $\mathrm{Aut}(\Gamma)$.

The following lemma was originally stated in a more general way in [7].

Lemma 2.1 ([7]). *Let* $\Gamma = (V, \mathcal{H})$ *be a* k*-uniform hypergraph,* $1 \leq k \leq |V|$. *Then*

(i) $\mathrm{Aut}(V, \mathcal{H}) = \mathrm{Aut}(V, \mathcal{P}_k(V) - \mathcal{H})$*;*

(ii) $\mathrm{Aut}(V, \mathcal{H}) = \mathrm{Aut}(V, \{H^c \mid H \in \mathcal{H}\})$*, where* $(V, \{H^c \mid H \in \mathcal{H}\})$ *is the* $(|V| - k)$*-uniform hypergraph whose hyperedges are the complements of the hyperedges in* $\mathcal{H}$ *with respect to the set* V*.*

A finite group G admits a *regular representation as the full automorphism group of a* k*-uniform hypergraph* if there exists a set of k-hyperedges $\mathcal{H} \subseteq \mathcal{P}_k(G)$ for which $\mathrm{Aut}(G, \mathcal{H}) = G_L$. In particular, G *admits a Graphical Regular Representation* (abbreviated to *GRR*) if there exists a system of 2-hyperedges (edges) $\mathcal{E} \subseteq \mathcal{P}_2(G)$ such that $\mathrm{Aut}(G, \mathcal{E}) = G_L$, and G *admits a Digraphical Regular Representation, DRR,* if there exists a system of directed edges $E \subseteq G \times G$ such that $\mathrm{Aut}(G, E) = G_L$. The problem we study in our on-going project [8] calls for finding all positive integers k, $1 \leq k \leq |G|$, for any given finite group G, for which there exists a regular representation of G as the full automorphism group of some k-hypergraph on G.

We conclude the section by stating the solutions to the much more specific GRR- and DRR-problems as well as the classification of finite groups that admit regular representation via (not necessarily k-uniform) hypergraphs which are essential with regard to the above mentioned project.

Theorem 2.2 ([6]). *Let G be a finite group that does not admit a GRR. Then G is an abelian group of exponent greater than 2 or G is a generalized dicyclic group or G is isomorphic to one of the 13 groups :* $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathbb{D}_3$, $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbb{A}_4$, $\mathbb{Q}_8 \times \mathbb{Z}_3$, $\mathbb{Q}_8 \times \mathbb{Z}_4$,
$\langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ abc = bca = cab \rangle$, $\langle a, b \mid a^8 = b^2 = 1,\ b^{-1}ab = a^5 \rangle$,
$\langle a, b, c \mid a^3 = b^3 = c^2 = 1,\ ab = ba,\ (ac)^2 = (bc)^2 = 1 \rangle$,
$\langle a, b, c \mid a^3 = b^3 = c^3 = 1,\ ac = ca,\ bc = cb,\ b^{-1}ab = ac \rangle$.

As a consequence of Sabidussi's characterization, if a finite group G admits a GRR $\Gamma = (V, \mathcal{E})$, then Γ must be a Cayley graph for G.

A *digraph* (V, E) is a pair that consists of vertices V and a set $E \subseteq V \times V$ of ordered pairs of distinct vertices. An *automorphism of a digraph* (V, E) is a permutation φ of V that preserves the set E. A *Cayley digraph* $\mathcal{CD}(G, X)$, $1_G \notin X \subseteq G$ (where we do not require X to be closed under taking inverses), has G for its vertex set and the set of directed edges $E = \{(g, gx) \mid g \in G, x \in X\}$. A digraph (V, E) admits a regular automorphism group G if and only if $(V, E) \cong \mathcal{CD}(G, X)$ for some X.

Theorem 2.3 ([2]). *The finite group G admits a DRR if and only if G is neither the quaternion group $\mathbb{Q}_8$ nor any of $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathbb{Z}_3^2$.*

Before stating the next theorem, we have to stress that the hypergraphs $(G, \mathcal{H})$ considered are not necessarily assumed to be k-uniform.

Theorem 2.4 ([7]). *A finite group G can be represented as a regular full automorphism group of some hypergraph $(G, \mathcal{H})$ if and only if G is not one of the groups $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ or $\mathbb{Z}_2^2$.*

3 Cayley hypergraphs

Recall that we have defined a Cayley hypergraph as a k-uniform hypergraph admitting a regular group of automorphisms, and not via a specific definition of its hyperedges. The following theorem provides us with this 'missing' definition and can also be viewed as a direct analogue of Sabidussi's characterization of Cayley graphs.

Theorem 3.1 ([7]). *Let $\Gamma = (V, \mathcal{H})$ be a vertex-transitive hypergraph. Then Γ admits a regular group G of automorphisms (and is therefore Cayley) if and only if there exists a collection of sets $B_i \in \mathcal{P}(G)$, $1 \le i \le s$, such that Γ is isomorphic to the hypergraph*

$$(G, \bigcup_{i=1}^{s} B_i^{G_L}).$$

Specifically, if $\Gamma = (V, \mathcal{H})$ is a vertex-transitive k-uniform hypergraph, then Γ admits a regular group G of automorphisms (and is therefore k-uniform Cayley) if and only if there exists a collection of k-sets $B_i \in \mathcal{P}_k(G)$, $1 \le i \le s$, such that Γ is isomorphic to $(G, \bigcup_{i=1}^{s} B_i^{G_L})$.

To pursue the analogy with Cayley graphs even further, let us point out that a Cayley graph $\mathcal{C}(G, X)$ can also be viewed as the graph $\Gamma = (G, \mathcal{E})$ defined by selecting elements $X \subseteq G$, $1_G \notin X, X = X^{-1}$, and completing the edge set in such a way that guarantees that all permutations $\sigma_g \in G_L$ are automorphisms of Γ, i.e.,

$$\mathcal{E} = \{\sigma_g(\{1_G, x\}) \mid g \in G, x \in X\}.$$

This observation can now analogously be applied to hypergraphs as follows.

Theorem 3.2. *If $\Gamma = (V, \mathcal{H})$ is a vertex-transitive k-uniform hypergraph, then Γ admits a regular group G of automorphisms (and is therefore a k-uniform Cayley graph) if and only if there exists a set $X \subseteq \mathcal{P}_{k-1}(G - \{1_G\})$ such that Γ is isomorphic to the hypergraph*

$$(G, \bigcup_{B \in X} (B \cup \{1_G\})^{G_L}).$$

The proof of this last statement is relatively simple and is left to the reader.

4 $t-$Cayley hypergraphs

As we have stated in the introduction, previous attempts at generalizing Cayley graphs to hypergraphs did not cover all hypergraphs admitting a regular group of automorphisms. The first of these generalizations can be found in [4], where Buratti considered (among others) the following type of hypergraphs admitting a group of automorphisms acting regularly on their vertex sets.

Definition 4.1. Let G be a finite group, X be a subset of $G \setminus \{1_G\}$, and t be a positive integer satisfying $2 \leq t \leq \max\{|x| \mid x \in X\}$. The *$t-$Cayley hypergraph* $t-\mathrm{Cay}(G, X)$ is the hypergraph $(G, \mathcal{H})$ with $\mathcal{H}$ consisting of hyperedges of the form $\{g, gx, \ldots, gx^{t-1}\}$, for all $g \in G$ and $x \in X$.

Clearly, $t-\mathrm{Cay}(G, X)$ is a t-uniform hypergraph if and only if $2 \leq t \leq \min\{|x| \mid x \in X\}$. It is also fairly easy to see that $G_L \leq \mathrm{Aut}(t-\mathrm{Cay}(G, X))$. Consequently, using any of the equivalent definitions of Cayley hypergraphs derived in the previous section, the class of t-uniform $t-$Cayley hypergraphs is a subclass of the class of t-uniform Cayley hypergraphs defined in this text.

It is a proper subclass, as, for example, the Fano plane $PG(2, 2)$ is a 3-uniform hypergraph admitting the regular action of $\mathbb{Z}_7$ but is not a $3-$Cayley hypergraph [4]. However, the two classes coincide for $t = 2$ as the class of $2-$Cayley hypergraphs is equal to the class of the classical Cayley graphs which is also equal to the class of 2-uniform Cayley hypergraphs. In this sense, Buratti is justified in his claim that $t-$Cayley hypergraphs constitute a generalization of Cayley graphs. Nevertheless, the generalization introduced in our paper is in our opinion a better match to Sabidussi's concept of Cayley graphs.

On the other hand, being a narrower class than that of Cayley hypergraphs, the class of $t-$Cayley hypergraphs is sometimes easier to use and search. For example, $t-$Cayley hypergraphs have been recently successfully used in the context of cages in [5] where they have been used to construct small 3-regular graphs of large girth as dual graphs to 3-uniform $3-$Cayley hypergraphs. The fact that the authors of [5] only considered $3-$Cayley hypergraphs, and avoided searching through all 3-uniform Cayley hypergraphs, sufficiently sped up their calculations and searches to allow them to find new record 3-regular graphs.

5 C_k-hypergraphs

We conclude our text by considering a second generalization of Cayley graphs to hypergraphs different from that of Cayley hypergraphs considered in our paper so far. To accommodate classification of finite groups G that allow for the existence of a k-hypergraph $(G, \mathcal{H})$, $k \geq 3$, satisfying $\mathrm{Aut}(G, \mathcal{H}) = G_L$, the paper [8] introduced the following generalization of Cayley graphs.

Definition 5.1. Let G be a (finite) group, and let $X_1, X_2, \ldots, X_{k-1}$ be subsets of G that do not contain the identity 1_G. The *C_k-hypergraph* $C_k(G; X_1, X_2, \ldots, X_{k-1})$ is the k-uniform hypergraph $(G, \mathcal{H})$ with $\mathcal{H}$ being the set of all k-subsets of the form

$$\{g, gx_1, gx_1x_2, \ \ldots, \ gx_1x_2 \ldots x_{k-1}\},$$

$g \in G$, and $x_i \in X_i$, for $1 \leq i \leq k-1$.

Note that we strictly require that the hyperedges have exactly k vertices in order to be included, i.e., all the vertices $g, gx_1, gx_1x_2, \ \ldots, \ gx_1x_2 \ldots x_{k-1}$ must be distinct. This is equivalent to saying $x_i x_{i+1} \ldots x_j \neq 1$, for all $1 \leq i \leq j \leq k-1$. (This requirement may sometimes force $\mathcal{H} = \emptyset$.) The 2-hypergraph $C_2(G; X)$ is the Cayley graph $\mathcal{C}(G, X)$, and in the case when $X = X_1 = X_2 = \ldots = X_{k-1}$, the resulting hyperedges of $C_k(G; X, X, \ldots, X)$ are the sets of vertices corresponding to the *k-arcs* of the Cayley graph $\mathcal{C}(G, X)$ *that do not contain repeated vertices* ([3, Chapter 17]).

Obviously, the automorphism group of a C_k-hypergraph $C_k(G; X_1, X_2, \ldots, X_{k-1})$ should be related to the groups $\mathrm{Aut}(\mathcal{C}(G, X_i))$. For instance, since graph automorphisms preserve k-arcs that do not contain repeated vertices,

$$\mathrm{Aut}(\mathcal{C}(G, X)) \leq \mathrm{Aut}(C_k(G; X, X, \ldots, X)).$$

The next lemma presents sufficient conditions for this inclusion to become an identity. The *girth* of a graph $\Gamma = (V, \mathcal{E})$ is the number of edges in a smallest cycle in Γ. Note that even though some of the results presented here already appeared in [8], they were presented without proofs.

Lemma 5.2. *Let $k \geq 2$ be an integer, and $\mathcal{C}(G, X)$ be a Cayley graph of girth $g > 2k-2$ and valency $|X| > k-1$. Then $\mathrm{Aut}(\mathcal{C}(G, X)) = \mathrm{Aut}(C_k(G; X, X, \ldots, X))$.*

Proof. We say that a word (or a product) $x_1 x_2 \ldots x_\ell \in X^*$ (i.e., $x_i \in X$, for all $1 \leq i \leq \ell$) is *reduced* if it does not contain a generator immediately followed by its inverse: $x_{i+1} \neq x_i^{-1}$, for $1 \leq i \leq \ell - 1$. The girth length assumption $g > 2k-2$ implies that the $\mathcal{C}(G, X)$ neighborhood $N_{C(G,X)}(g) = \{h \in G \mid d(g, h) \leq k-1\}$ of any vertex g is isomorphic to a regular tree of valency $|X|$ and depth $k-1$. Thus, reduced words $x_1 x_2 \ldots x_\ell$, $l \leq k-1$, represent vertices of distance ℓ from 1_G, and different reduced words $x_1 x_2 \ldots x_\ell$, $1 \leq \ell \leq k-1$, represent different vertices of $\mathcal{C}(G, X)$. The unique path of length ℓ between 1_G and $x_1 x_2 \ldots x_\ell$ is the path $1_G, x_1, x_1x_2, \ldots, x_1x_2 \ldots x_\ell$.

Suppose (by means of contradiction) that there exists $\varphi \in \mathrm{Aut}(C_k(G; X, X, \ldots, X))$ that does not belong to $\mathrm{Aut}(\mathcal{C}(G, X))$. Then there exists an $a \in G$ and $\hat{x} \in X$, such that $\varphi(a)^{-1}\varphi(a\hat{x}) \notin X$. Let ψ be the composition of φ with the left multiplications $\sigma_a, \sigma_{\varphi(a)^{-1}}$, $\psi = \sigma_{\varphi(a)^{-1}} \circ \varphi \circ \sigma_a$. Then $\psi \in \mathrm{Aut}(C_k(G; X, X, \ldots, X))$, $\psi(1_G) = \varphi(a)^{-1} \cdot \varphi(a \cdot 1_G) = 1_G$, and $\psi(\hat{x}) = \varphi(a)^{-1} \cdot \varphi(a \cdot \hat{x}) \notin X$. Each k-hyperedge of

$C_k(G;X,X,\ldots,X)$ that contains 1_G assumes one of the following forms (based on our arguments from the previous paragraph, all of these are indeed k-subsets):

$$\{1_G, y_1, y_1y_2, \ldots, y_1y_2\ldots y_{k-1}\}, \{y_1^{-1}, 1_G, y_2, \ldots, y_2y_3\ldots y_{k-1}\},$$

$$\{y_2^{-1}y_1^{-1}, y_2^{-1}, 1_G, y_3, \ldots, y_3\ldots y_{k-1}\}, \ldots, \{y_{k-1}^{-1}\ldots y_2^{-1}y_1^{-1}, y_{k-1}^{-1}\ldots y_2^{-1}, \ldots, y_{k-1}^{-1}, 1_G\},$$

$y_i \in X$, $1 \le i \le k-1$. The automorphism ψ fixes 1_G, and therefore must map the hyperedges

$$\{1_G, \hat{x}, \hat{x}y_2, \ldots, \hat{x}y_2y_3\ldots y_{k-1}\}, \tag{5.1}$$

$y_i \in X, 2 \le i \le k-1$, to hyperedges containing 1_G. This means, in particular, that $\psi(\hat{x})$, which is assumed not to be an element of X, must belong to one of these hyperedges as well, and therefore must be of the form $\psi(\hat{x}) = x_1x_2\ldots x_\ell$, for some $x_i \in X$, $2 \le \ell \le k-1$, which is an element of distance at least 2 from 1_G in $\mathcal{C}(G,X)$.

The rest of the argument relies on counting. There are $(|X|-1)^{k-2}$ hyperedges of the form (5.1): after choosing y_2 from $X - \{\hat{x}^{-1}\}$, we have $|X|-1$ choices to choose y_3 from $X - \{y_2^{-1}\}$, and so on. On the other hand, each hyperedge containing both 1_G and $\psi(\hat{x}) = x_1x_2\ldots x_\ell$ must contain the vertices $1_G, x_1, x_1x_2, \ldots, x_1x_2\ldots x_\ell$ of the unique path connecting 1_G and $x_1x_2\ldots x_l$. Therefore, inspecting the list of hyperedges containing $1_G, x_1, x_1x_2, \ldots, x_1x_2\ldots x_\ell$, it is easy to see that there are at most $(k-\ell)(|X|-1)^{k-1-\ell}$ hyperedges that contain both 1_G and $\psi(\hat{x})$ (where the number $(k-\ell)$ is added to possibly account for the different positions of 1_G in the hyperedges; which might, but does not have to, result in different blocks). Consequently, the $(|X|-1)^{k-2}$ distinct hyperedges of the form (5.1) must map *bijectively* onto *at most* $(k-\ell)(|X|-1)^{k-1-\ell}$ hyperedges containing 1_G and $\psi(\hat{x})$. Since $|X| > k-1$, $(|X|-1)^{k-2} > (k-\ell)(|X|-1)^{k-1-\ell}$, for all $2 \le \ell \le k-1$, which makes an one-to-one correspondence impossible, and leads to a contradiction. □

Corollary 5.3. *Let G be a finite group that admits a GRR of girth $g > 2m-2$ and valency r. Then G can be regularly represented as the full automorphism group of some k-hypergraph for all $2 \le k \le min\,\{m, r-1\}$.*

A symmetric subset X, $X = X^{-1}$, of a group G is *symmetrically irreducible* if $|\langle X - \{x, x^{-1}\}\rangle| < |\langle X\rangle|$ for all $x \in X$. The girth of the Cayley graph $\mathcal{C}(G,X)$ is larger than 4 if and only if $\mathcal{C}(G,X)$ does not contain triangles and cycles of length 4. This must be the case, for example, when $X = \{x_1, x_2, \ldots, x_r\}$ is symmetrically irreducible and satisfying the additional conditions $x_i^2 \ne x_j^2$, $1 \le i \ne j \le r$, and $x_ix_j \ne x_jx_i$ whenever $x_j \ne x_i^{-1}$. Based on these observations, the above lemma guarantees the existence of a 3-uniform regular representation $C_3(G,X,X)$ for all finite groups G that admit a GRR $\mathcal{C}(G,X)$ with X of order at least 4, symmetrically irreducible, and satisfying the above conditions. The high girth requirements imposed on the above GRR's are certainly quite restrictive. The following generalization of Lemma 5.2 presents a way of avoiding the need for high girth.

Lemma 5.4. *Let G be a finite group, $X_1, X_2, \ldots X_{k-1}$ be symmetric subsets of G not containing 1_G, $|X_i| \ge k$, for all $1 \le i \le k-1$, and suppose that all reduced products $x_1x_2\ldots x_\ell$, $x_i \in X_i$, $1 \le \ell \le k-1$, represent different elements of G. Then,* $\mathrm{Aut}(C_k(G;X_1,X_2,\ldots,X_{k-1})) \le \mathrm{Aut}(\mathcal{C}(G,X_1))$.

Proof. The counting argument is essentially identical to the one in the proof of Lemma 5.2. □

Corollary 5.5. *Let G be a finite group that admits a GRR $\mathcal{C}(G, X_1)$, $\mathrm{Aut}(\mathcal{C}(G, X_1)) = G_L$. If there exist symmetric subsets $X_2, X_3, \ldots, X_{k-1}$ of G not containing 1_G, and such that all the* reduced *words $x_1 x_2 \ldots x_l$, $x_i \in X_i$, $1 \le l \le k-1$, represent different elements of G and $|X_i| > k-1$, for all $1 \le i \le k-1$, then $\mathrm{Aut}(C_k(G; X_1, X_2, \ldots, X_{k-1})) = G_L$, and G admits a regular representation as the full automorphism group of an n-hypergraph, for all n in the range $2 \le n \le k$.*

Proof. $G_L \le \mathrm{Aut}(C_k(G; X_1, X_2, \ldots, X_{k-1}))$ by the definition of C_k hypergraphs.

$$\mathrm{Aut}(C_k(G; X_1, X_2, \ldots, X_{n-1})) \le \mathrm{Aut}(\mathcal{C}(G, X_1)) = G_L,$$

for all $2 \le n \le k$ by Lemma 5.4. □

Thus, given a finite group G and a GRR $C(G, X)$, to construct a 3-hypergraph with a regular automorphism group G we just need to find a symmetric set of at least 3 elements $y_1, y_2, y_3, \ldots, y_l$ in G such that all the elements $1_G, x, xy_j$, $x \in X$ and $1 \le j \le l$, are different (note that $|X| \ge 3$ as $\mathcal{C}(G, X)$ is a GRR). This seems to be reasonably easy and may lead to a 3-hypergraph for essentially all finite groups G that admit a GRR.

As was the case of the use of $t-$Cayley hypergraphs in [5], focusing on a narrower class of Cayley hypergraphs allows for faster exhaustive searches. Similarly, considering the $C_k(G; X_1, X_2, \ldots, X_{k-1})$ hypergraphs introduced in this section allows one to find regular representations of finite groups via hypergraphs in a narrower class of possible examples.

In the final result of our paper, we shall abandon the graphs $C_k(G; X_1, X_2, \ldots, X_{k-1})$ in favor of yet another class of k-uniform hypergraphs. We will rely on the concept of an *r-regular rooted oriented tree of depth d*, $\vec{\mathcal{T}}_{r,d}$, which is an oriented tree containing $1 + r + r^2 + \cdots + r^d$ vertices, has a root u of out-degree r and in-degree 0, all of its vertices of distance $1 \le i < d$ from u are of out-degree r and in-degree 1, and all of its vertices of distance d from u are of out-degree 0 and in-degree 1.

Definition 5.6. Let G be a (finite) group, let $\vec{\mathcal{T}}_{r,k-1}$ be an oriented rooted tree whose vertex set is a subset of G and whose root is the identity 1_G, and let $X_i = N_i(1_G)$, the set of vertices of $\vec{\mathcal{T}}_{r,k-1}$ of distance i from 1_G in $\vec{\mathcal{T}}_{r,k-1}$, $1 \le i \le k-1$. The k-hypergraph $\tilde{C}_k(G, \vec{\mathcal{T}}_{r,k-1})$ is the k-uniform hypergraph $(G, \mathcal{H})$ with $\mathcal{H}$ being the set of all k-subsets of the form

$$\{g, gx_1, gx_2, gx_3 \ \ldots, \ gx_{k-1}\},$$

$g \in G$, $x_i \in X_i$, for $1 \le i \le k-1$, and each (x_i, x_{i+1}) being an oriented edge of $\vec{\mathcal{T}}_{r,k-1}$.

As our first observation, we point out that none of the X_i's contain 1_G and each hyperedge $\{g, gx_1, gx_2, gx_3 \ \ldots, \ gx_{k-1}\}$ corresponds to an oriented path

$$(1_G, x_1), (x_1, x_2), \ldots, (x_{k-2}, x_{k-1})$$

of length $k-1$ in $\vec{\mathcal{T}}_{r,k-1}$ starting from 1_G and reaching $x_{k-1} \in X_{k-1}$ (which are outward oriented k-arcs with no repeated vertices starting in 1_G). Due to the fact that the sets X_i are disjoint, all the above sets are *necessarily* k-subsets of G. The next observation is a combination of results similar to those used in the proofs of Lemma 5.2 and Corollary 5.3.

Lemma 5.7. *Let G be a finite group, let $X = X^{-1}$ be a subset of G of cardinality r, let k be an integer, $1 \leq k \leq |X|$, and let $\vec{\mathcal{T}}_{r,k-1}$ be an oriented rooted tree in G with $X_1 = N_1(1_G) = X$. Furthermore, suppose that all the elements*

$$\{x \mid x \in X_i, 1 \leq i \leq k-1\}, \{x^{-1} \mid x \in X_i \text{ and } x^{-1} \notin X_i, 2 \leq i \leq k-1\},$$
$$\{x^{-1}y \mid x \in X_i, y \in X_j, 1 \leq i \neq j \leq k-1\} \tag{5.2}$$

are distinct. Then $\mathrm{Aut}(\tilde{C}_k(G, \vec{\mathcal{T}}_{r,k-1})) \leq \mathrm{Aut}(\mathcal{C}(G, X_1))$.

Proof. Assume all the above and suppose that there exists $\varphi \in \mathrm{Aut}(\tilde{C}_k(G, \vec{\mathcal{T}}_{r,k-1}))$ that does not belong to $\mathrm{Aut}(\mathcal{C}(G, X_1))$. Then there exists an $a \in G$ and $\hat{x} \in X_1$, such that $\varphi(a)^{-1}\varphi(a\hat{x}) \notin X_1$. Let ψ be the composition of φ with the left multiplications $\sigma_a, \sigma_{\varphi(a)^{-1}}$, $\psi = \sigma_{\varphi(a)^{-1}} \circ \varphi \circ \sigma_a$. Then $\psi \in \mathrm{Aut}(\tilde{C}_k(G, \vec{\mathcal{T}}_{r,k-1}))$, $\psi(1_G) = \varphi(a)^{-1} \cdot \varphi(a \cdot 1_G) = 1_G$, and $\psi(\hat{x}) = \varphi(a)^{-1} \cdot \varphi(a \cdot \hat{x}) \notin X_1$. Each k-hyperedge of $\tilde{C}_k(G, \vec{\mathcal{T}}_{r,k-1})$ that contains 1_G must be one of the following

$$\{1_G, x_1, x_2, x_3, \ldots, x_{k-1}\}, \{x_1^{-1}, 1_G, x_1^{-1}x_2, x_1^{-1}x_3, \ldots, x_1^{-1}x_{k-1}\},$$
$$\{x_2^{-1}, x_2^{-1}x_1, 1_G, x_2^{-1}x_3, \ldots, x_2^{-1}x_{k-1}\}, \ldots, \{x_{k-1}^{-1}, x_{k-1}^{-1}x_1, x_{k-1}^{-1}x_2, x_{k-1}^{-1}x_3, \ldots, 1_G\},$$

$x_i \in X_i$, $1 \leq i \leq k-1$. The automorphism ψ fixes 1_G, and therefore must map the $2r^{k-2}$ hyperedges $\{1_G, \hat{x}, x_2, x_3, \ldots, x_{k-1}\}$, $\{\hat{x}, 1_G, \hat{x}x_2, \hat{x}x_3, \ldots, \hat{x}x_{k-1}\}$, $x_i \in X_i$, $2 \leq i \leq k-1$, containing both 1_G and $\hat{x}$, to hyperedges containing 1_G and $\psi(\hat{x})$. This means, in particular, that $\psi(\hat{x})$, which is assumed not to be an element of $X_1 = X$, must belong to one of these hyperedges as well, and therefore must be equal to x or x^{-1}, for some $x \in X_i$, $2 \leq i \leq k-1$, or must be equal to some $x^{-1}y$, $x \in X_i$, $y \in X_j$, $1 \leq i \neq j \leq k-1$. We shall consider each possibility separately.

First assume that $\psi(\hat{x}) = x$ or x^{-1}, for some $x \in X_i$, $2 \leq i \leq k-1$, and recall that we assume that all the elements listed in (5.2) are distinct. Hence, the only hyperedges containing both 1_G and x are the hyperedges of the form $\{1_G, x_1, x_2, x_3, \ldots, x, \ldots, x_{k-1}\}$ or hyperedges of the form $\{x, xx_1, xx_2, \ldots, 1_G, \ldots, xx_{k-1}\}$ (in case x^{-1} also belongs to X_i). Since $i \geq 2$, there are at most r^{k-i-1}, $i > 1$, hyperedges of the first kind (for example, x_1 is the uniquely determined element from X_1 in $\vec{\mathcal{T}}_{r,k-1}$ on the path between 1_G and x) and there are at most r^{k-2} hyperedges of the second kind. Hence, there are not enough hyperedges to map the $2r^{k-2}$ hyperedges containing 1_G and $\hat{x}$ onto, and therefore $\psi(\hat{x}) \neq x \in X_i$, $i \geq 2$. Similarly, the only hyperedges containing 1_G and x^{-1} are the hyperdges of the form $\{x^{-1}, x^{-1}x_1, x^{-1}x_2, \ldots, 1_G, \ldots, x^{-1}x_{k-1}\}$, with at most r^{k-2} of them, and the hyperedges of the form $\{1_G, x_1, x_2, x_3, \ldots, x, \ldots, x_{k-1}\}$, of which there are at most r^{k-i-1}. Again, there are not enough hyperedges to serve as the images of the hyperedges containing 1_G and $\hat{x}$.

Next consider the possibility that $\psi(\hat{x}) = x^{-1}y$, $x \in X_i$, $y \in X_j$, $1 \leq i \neq j \leq k-1$. The only hyperedges containig 1_G and $x^{-1}y$ are the hyperedges $\{x^{-1}, x^{-1}x_1, x^{-1}x_2, \ldots, 1_G, \ldots, x^{-1}x_{k-1}\}$ (with one of the products equal to $x^{-1}y$), and there are at most r^{k-3} of them.

We conclude that $\psi(\hat{x})$ must belong to $X_1 = X$, and therefore

$$\mathrm{Aut}(\tilde{C}_k(G, \vec{\mathcal{T}}_{r,k-1})) \leq \mathrm{Aut}(\mathcal{C}(G, X_1)).$$

□

Corollary 5.8. *Let G be a finite group, let $X = X^{-1}$ be a subset of G of cardinality r making $\mathcal{C}(G, X_1)$ into a GRR. Let k be an integer, $1 \leq k \leq |X|$, let $\vec{\mathcal{T}}_{r,k-1}$ be an oriented rooted tree in G with $X_1 = N_1(1_G) = X$, and suppose that all the elements*

$$\{x \mid x \in X_i, 1 \leq i \leq k-1\}, \{x^{-1} \mid x \in X_i \text{ and } x^{-1} \notin X_i, 2 \leq i \leq k-1\},$$

$$\{x^{-1}y \mid x \in X_i, y \in X_j, 1 \leq i \neq j \leq k-1\}$$

are distinct. Then $\mathrm{Aut}(\tilde{C}_k(G, \vec{\mathcal{T}}_{r,k-1})) = G_L$.

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