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An E^3 embedding of Coxeter's regular map $\{8, 4 \mid 3\}$ results in a regular Leonardo polyhedron

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Abstract

This paper is an update of a result published by Schulte and Wills in 1986. It explains in detail the construction of the polyhedral embedding of Coxeter's regular map of type $\{8, 4 \mid 3\}$ in Euclidean 3-space. The explanation is supported by numerous computer generated pictures. This polyhedron has the full octahedral symmetry group of order 48, and its combinatorial automorphism group is of order 48^2. It should be considered in context to additional five Leonardo polyhedra described recently by the authors in this journal.

Keywords: Leonardo polyhedron, regular map.

Math. Subj. Class.: 52B70

1 Preliminary remarks

This paper is closely related to reference [3], it provides a solution to a problem that was explained in detail in this paper. We recommend the reader to read reference [3] first, because this article is like a lid to its pot, the reference [3]. Thus we do not have to repeat basic terminology and arguments why the construction of a polyhedral embedding of Coxeter's regular $\{8, 4|3\}$ in E^3 in which the faces are all convex polygons is an interesting aspect.

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Regular maps have been studied as a part of group theory or as a part of graph theory. Some work has been done to find topological embeddings of regular maps in dimension 3. There are also papers in which polyhedral embeddings of regular maps in dimension 3 were found with methods from computational synthetic geometry, [2]. Starting with a combinatorial input and searching for a corresponding polyhedral embedding is known to be a difficult problem. This contribution is part of the latter aspect with an emphasis of geometric symmetries of these polyhedral embeddings in dimension 3. We have explained in [3] that based on the very involved parallel projection of Coxeter's regular $\{8, 4|3\}$ in [15], we did not find an embedding in dimension 3. Section 5 describes the construction starting with Coxeter's 2-manifold $\{8, 4 \mid 3\}$ as a subset of the 2-skeleton of a truncated 24-cell, [8], to obtain a resulting 3-dimensional embedding that we see in various versions in Blender files. In Section 7 we provide coordinates for producing pictures of the 3-dimensional object with any other software.

2 A brief history

Polyhedral 2-manifolds of genus $g \geqslant 2$ were first introduced by Leonardo da Vinci in 1498-1502. In De Divina Proportione Leonardo da Vinci presented drawings of the one-skeleta of the regular Platonic solids in which the edges were expanded to polyhedral struts; these can be viewed as Leonardo polyhedra if the faces of these struts are viewed as convex faces of high-genus polyhedra. For recent videos of Leonardo's drawings we mention the work of Roberto Cardil Ricol [5].

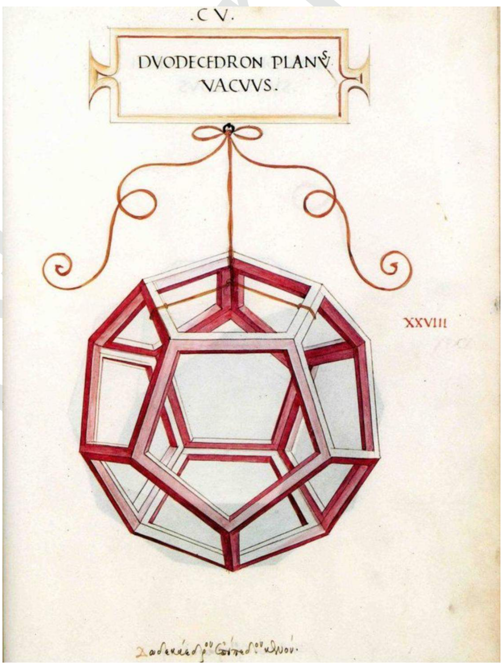

Figure 1: One of Leonardo da Vinci's drawings for Luca Pacioli's book, De Divina Proportione, [13].

The case $g = 1$, polyhedral tori, was already known 100 years before Leonardo. Leonardo's polyhedra have the rotation group or even the full symmetry group of a Platonic solid, but they have no regularity properties.

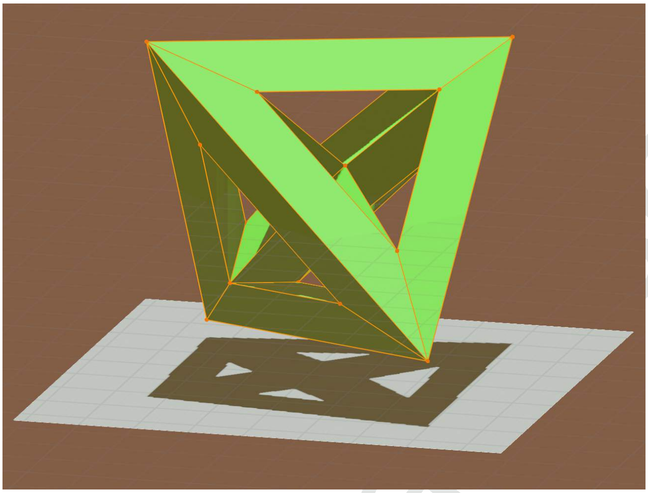

Figure 2: A first regular Leonardo polyhedron by Alicia Boole Stott of genus 6.

The first two regular Leonardo polyhedra were found by Alicia Boole Stott in 1910, one of genus $g = 6$ (Figure 2), one with $g = 73$, see Figure 3, [4] and [3].

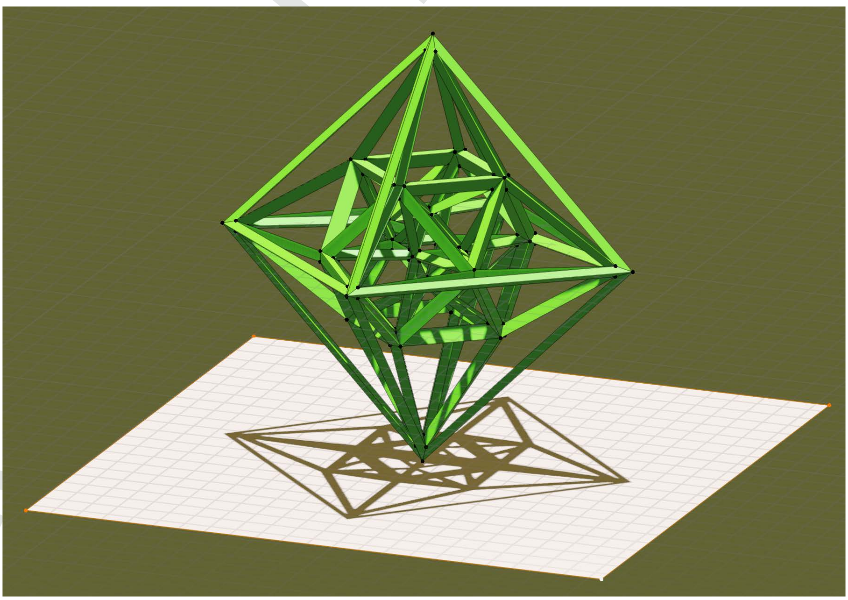

Figure 3: A second regular Leonardo polyhedron by Alicia Boole Stott of genus 73. The construction of its dual, embedded in E^3, will be the main result in this article.

In 1937 Coxeter found the duals of these regular Leonardo polyhedra in E^4 by an elegant and ingenious construction, [8], see also [15]. The smaller one can rather easily be projected into E^3, we use again a picture in Figure 4 from [3].

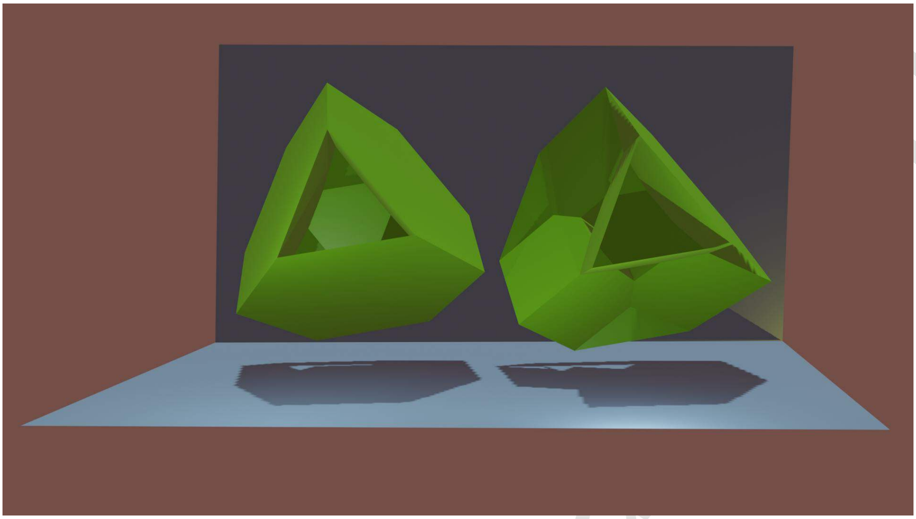

Figure 4: Coxeter's regular Leonardo Polyhedron $\{6, 4 \mid 3\}$, two hexagons missing (right).

For the projection of the large one with $g = 73$ we present in this paper a more detailed explanation of the 1986 construction in [15], supported by computer generated pictures, revealing many details of the rather complicated structure of the regular polyhedron (and of the underlying 4-polytope) in question. This has clarified issues that arose when trying to create a computer representation already in a previous release, [3].

3 Basic definitions

Definition 3.1. A *polyhedron* in this article is a compact topological 2-manifold without boundary, embedded in Euclidean 3-space E^3 and made up of convex polygons. The embedding means that no self-intersections are allowed. The convex polygons of the cells of the manifold exclude general topological embeddings, maintain a close relationship with the Platonic solids and form the faces, edges, and vertices of the polyhedron. Adjacent polygons are not coplanar.

Definition 3.2. For the numbers of vertices, edges, and faces, f_0, f_1, and f_2, we have Euler's polyhedron formula $f_2 - f_1 + f_0 = 2 - 2g$ in which g denotes the *genus* of the polyhedron. We consider also the combinatorial properties of a polyhedron in which the vertices are numbers, the edges are pairs of its vertices, and the faces are cyclic sequences of its vertices. This leads to the notion of the *combinatorial automorphism group* of the polyhedron as a permutation group of its vertices.

Definition 3.3. A triple consisting of a face, an edge of this face, and a vertex of this edge is called a *flag*.

Definition 3.4. A polyhedron is called *regular* if the permutation group of its vertices, i.e., the combinatorial automorphism group, acts transitively on its flags.

Definition 3.5. We call each Platonic solid and a regular polyhedron of genus $g \geqslant 2$ with the geometric rotational symmetry of a Platonic solid a *Leonardo polyhedron*.

4 Coxeter's 2-manifold $\{8, 4 \mid 3\}$

Coxeter's 2-manifold $\{8, 4 \mid 3\}$ is a subset of the 2-skeleton of a truncated 24-cell, [8]. For a better understanding we describe its construction in detail.

We start with the 24-cell in E^4. At each of the 24 vertices we put the tangent hyperplane with its normal through the vertex and the center of the 24-cell. If we move the hyperplanes simultaneously towards the center, then they intersect with the 24-cell in 24 regular 3-cubes. The 24 facets of the 24-cell (regular octahedra) become truncated octahedra. We continue this process until the 24 hyperplanes meet the midpoints of the 96 edges of the 24-cell. We then have a 4-polytope with two types of facets (in fact it is a 4-dimensional Archimedean solid). It has 48 facets: 24 of them are regular 3-cubes, each of them touching another cube at its vertex. The other 24 facets are cuboctahedra, whose 2-faces are six squares and 8 regular triangles. Each of these 24 facets shares a square with one of the cubes, and each of them shares a triangle with another facet of this type. If we move the 24 hyperplanes slightly towards the center of the 24-cell, then the 24 cubes become truncated cubes with 6 octagons and 8 triangles as facets. The other 24 facets become Archimedean solids with 84 triangles and 6 octagons each. They share the octagons with the 24 former truncated cubes. If we now delete all 2-faces except the octagons, we have a 2-manifold, built up of 144 octagons, 4 meeting at each vertex. The 192 triangles are the shortest non null-homotope ways, the geodesics or Coxeter-holes on this 2-manifold. So we have the regular skew polyhedron $(8, 4 \mid 3)$ in E^4. Of course one can continue the simultaneous shifting of the 24 hyperplanes until the 144 octagons are regular octagons.

The regularity of the Leonardo polyhedron can easily be described via the standard definition: flag transitivity of the (combinatorial) automorphism group. The projected polyhedron in E^3 has the full octahedral symmetry of order 48. This is true for all 24 truncated octahedra, derived from the 24 vertices of the 24-cell. By self-duality of the 24-cell we can interchange the role of vertices and cells of the 24-cell. This doubles the number of automorphisms, so we get $1152 \times 2 = 2304$ automorphisms. On the other hand the polyhedron has 144 octagons, hence $144 \times 16 = 48^2$ flags, which proves its combinatorial regularity.

Although $\{8, 4 \mid 3\}$ in E^4 and E^3 are isomorphic, there is a fundamental difference: In E^3 the polyhedron separates E^3 into an inner and an outer (unbounded) component: If we insert the 192 triangles, then the inner component is dissected into 24 truncated cubes, and the same is true for the outer component. In the last case one of the truncated cubes is unbounded. If we remove the 192 triangles, we have the $\{8, 4 \mid 3\}$ again. The remaining three edges of each triangle are the 192 geodesics (Coxeter holes). The 24 truncated octahedra of the inner component are of three congruent classes, six inner ones, six outer ones, and 12 in the middle layer. The 24 octahedra of the outer component fall into 5 classes: The (unbounded) outer one, the small inner one, then 8 with 3-fold symmetry, which are incident with the inner and 8 incident with the outer unbounded truncated cube. Finally 6 truncated cubes with 4-fold symmetry, which are in the middle layer. We end with a remark about the octahedral symmetry: The smallest circumscribed cube has the same symmetry as the $\{8, 4 \mid 3\}$. It has four 3-fold rotation axes and the polyhedron are disjoint (one can look through the polyhedron in these directions). The three 4-fold axes intersect the polyhedron in 8 faces each, altogether 24 faces. These 24 faces have 4-fold rotational symmetry (and are regular), if all edges have the same length.

5 Construction of the vertices of Coxeter's 2-manifold $\{8, 4 \mid 3\}$ in E^3

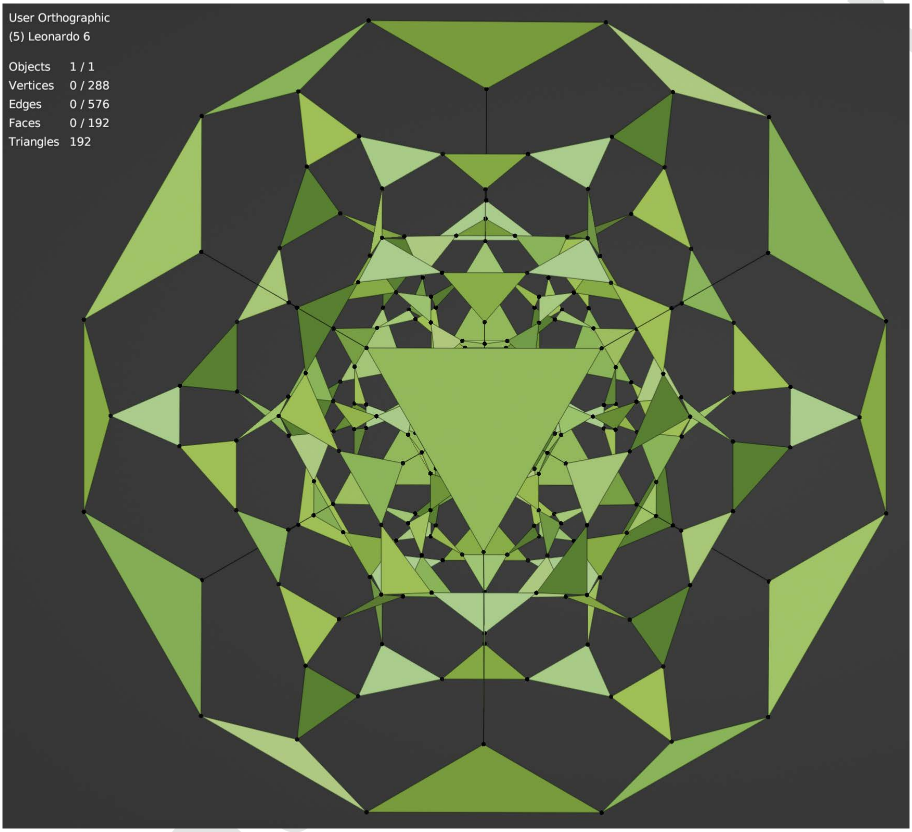

Figure 5: These 192 triangles in E^3 define the polyhedron $\{8,4 \mid 3\}$. Their boundaries define 144 octagons.

We start with the 24-cell. We use *the dodecagonal projection of the* 24*-cell* from Figure 7C12 in McMullen's book entitled *Geometric Regular Polytopes*, [12]. This projection of the 4-dimensional coordinates for all vertices of the 24-cell into the plane are seen in this Figure 6, upper part. Some edges have been omitted. We do not use them in the following construction. In a dynamical program like Cinderella, we obtain all possible projections into the plane, when we change the planar position of the 4 unit vectors in 4-space. Another projection with partial information about some cubes, Figure 6, lower part, explains e.g., that the symmetry of the upper part is very useful.

We provide in what follows the description of our 4-dimensional calculations.

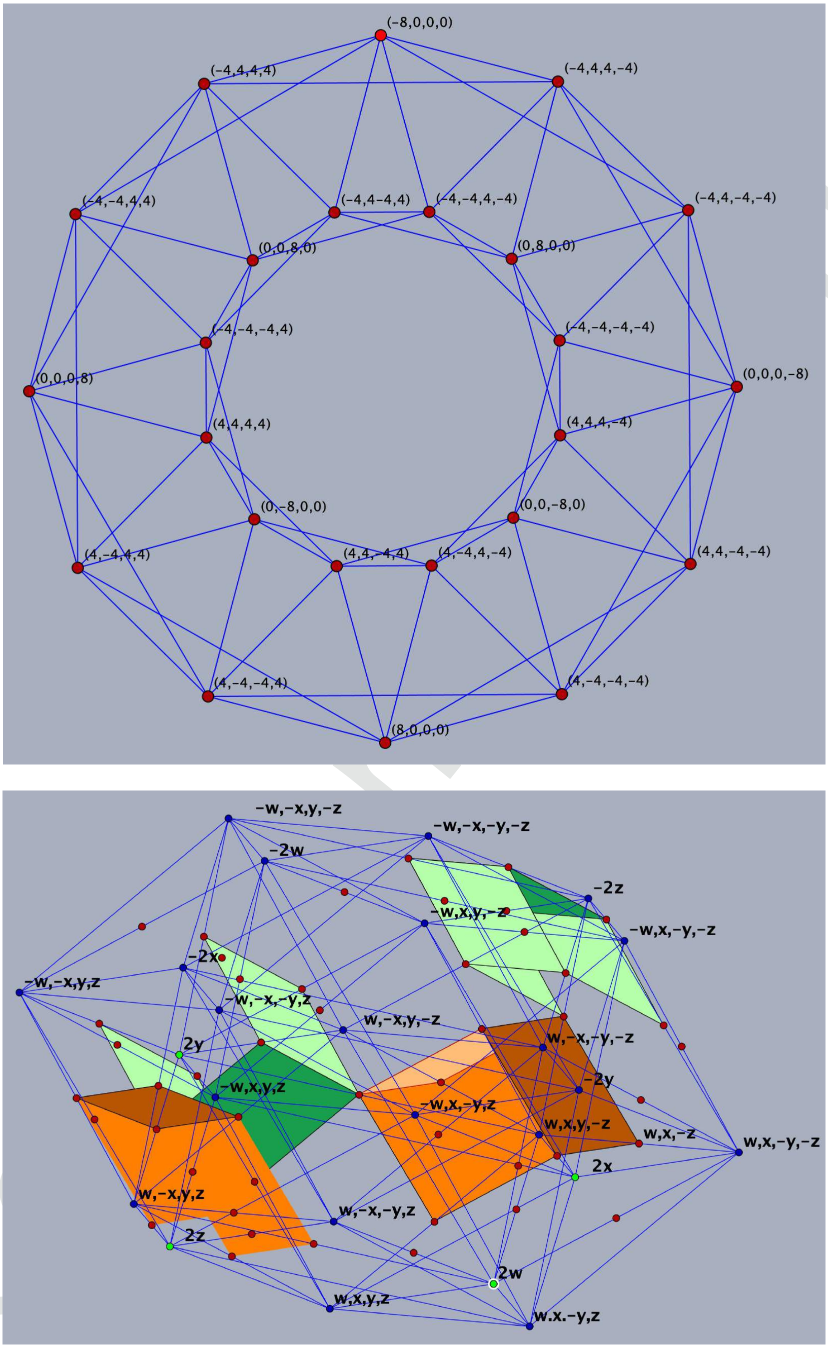

Figure 6: Projections of edges of a 24-cell for constructing the cubes.

We determine 96 midpoints of those edges of the 24-cell that lead to 24 cubes. They determine two circular sequences of 12 three-dimensional cubes in E^4, see Figure 7.

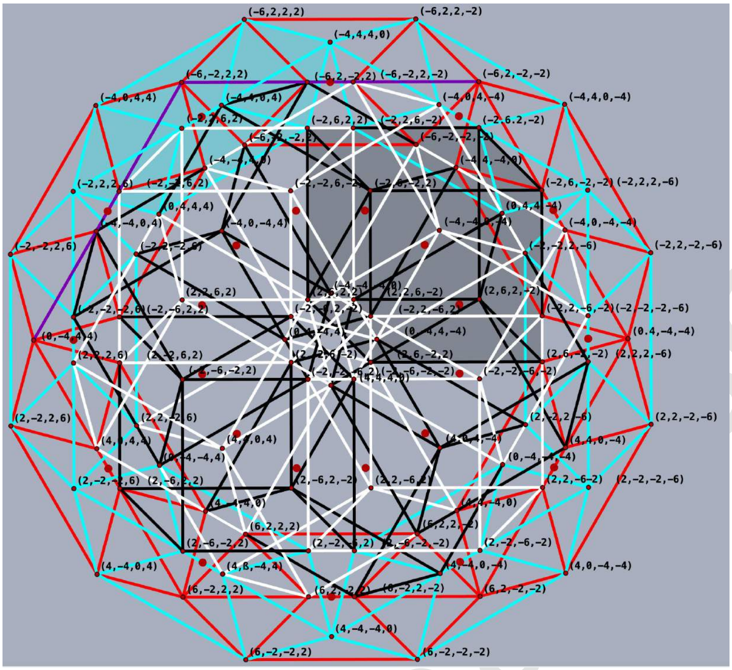

Figure 7: Dodecagonal projection of midpoints of some edges of a 24-cell. These midpoints form the vertices of 24 3-cubes. They can be seen with its projected edges in two circular sequences.

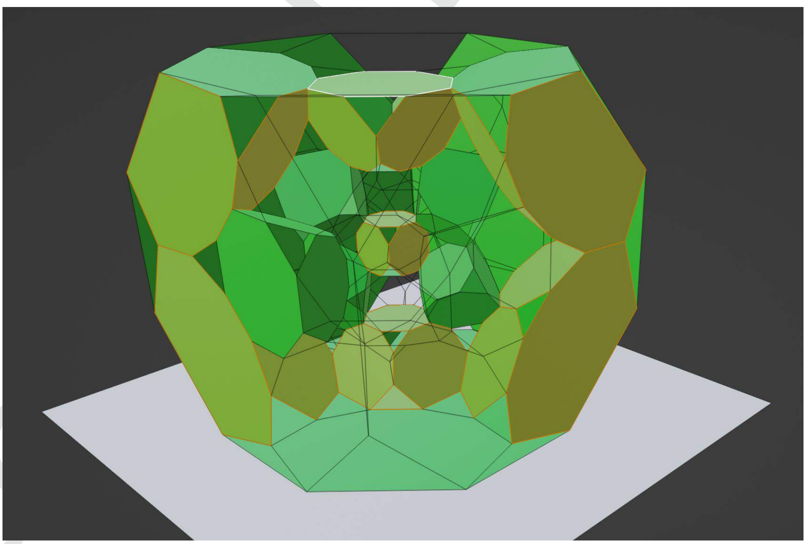

Figure 8: Only some octagons are depicted. All octagons would cover most of the rest. We show symmetric substructures in further figures.

Each vertex belongs to precisely two of these 24 3-cubes. We magnify all these cubes from their centers such that their intersections determine regular triangles and regular octagons with equal edge lengths and when we remove all triangular pyramids of these intersections.

Thus we obtain for each of the 96 vertices a regular triangle that belongs to the two magnified cubes of the former two cubes that had this vertex in common. All these 96 triangles form the vertices of 48 Archimedean bodies, i.e., truncated cubes, such that all edges have equal length. Picking one of these Archimedean bodies, and using a symmetric choice for constructing a Schlegel diagram, leads to a 3-dimensional embedding of the 2-manifold $\{8, 4 \mid 3\}$ with only octagons such that each vertex is incident with 4 of them, and such that a geometric octahedral symmetry occurs.

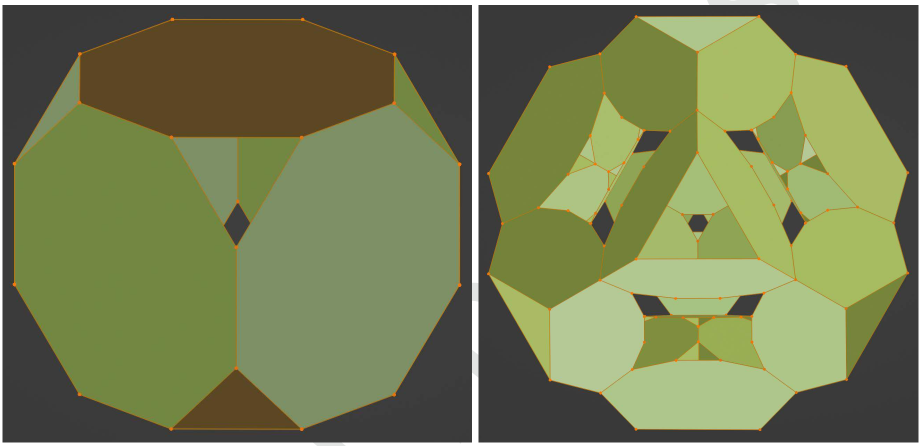

Figure 9: Outer Frame of 6 Octagons and adjacent eight former cubes with their octagons.

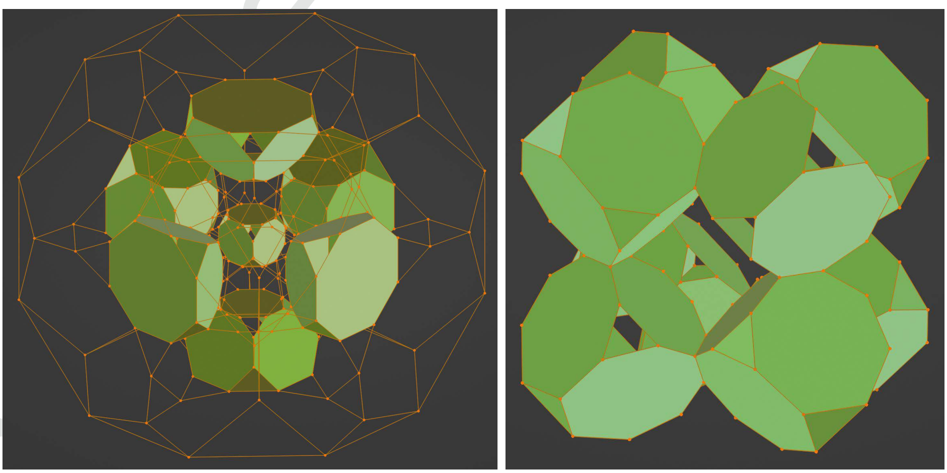

Figure 10: Six former cubes and center cube with their octagons and eight inner cubes with their octagons.

We describe this construction once more in more detail, so that it can easily be reconstructed. The convex hull of 8 vertices obtained by permuting the coordinates: $(8, 0, 0, 0)$ with positive and negative signs and 16 vertices obtained by permuting the coordinates $(4, 4, 4, 4)$, again with all choices for their signs, leads to a 24-cell. We have used these coordinates to have integers in later processing.

Starting from the projection of the four scaled unit vectors into a plane, all other vertices of the 24-cell for this selected projection can be entered using corresponding parallelograms in a dynamic geometry drawing program like Cinderella or Geometer's Sketchpad.

Then it is possible using a suitable shifting of the unit vectors to produce a special projection of the 24-cell like Figure 6.

This gives a good overview of the edge structure. The midpoints of the edges then emerge in this symmetrical view, and the 24 cubes, two examples of which are shown in Figure 7, are recognizable in this projection. Two orbits with a 12-fold symmetry show the 24 cubes. Each vertex of a cube is also the vertex of another cube of these 24 cubes. For each of these 96 cube vertices, we choose the center of one of the two cubes that have this vertex in common and the three adjacent vertices of this cube. From these 5 vertices, we determine the three vertices of the corresponding triangle. This can be done in a spreadsheet program such as *Google Sheet* or *Excel*. After choosing the cube, e.g., in the plane $w = -6$ and the projection center $(-8, 0, 0, 0)$ to generate a Schlegel diagram, the 3D coordinates were created, which can be entered into a 3D program like *Blender*. The 144 octagons of the projected cube facets result in the 2-dimensional manifold of type $\{8, 4 \mid 3\}$ in E^3 with 576 edges and 288 vertices. It has the rotational symmetry of a cube or an octahedron and its genus is 73. Two YouTube videos can be seen:

https://youtu.be/KWO9KIVjvtY
https://youtu.be/o-VSIqtmGNM

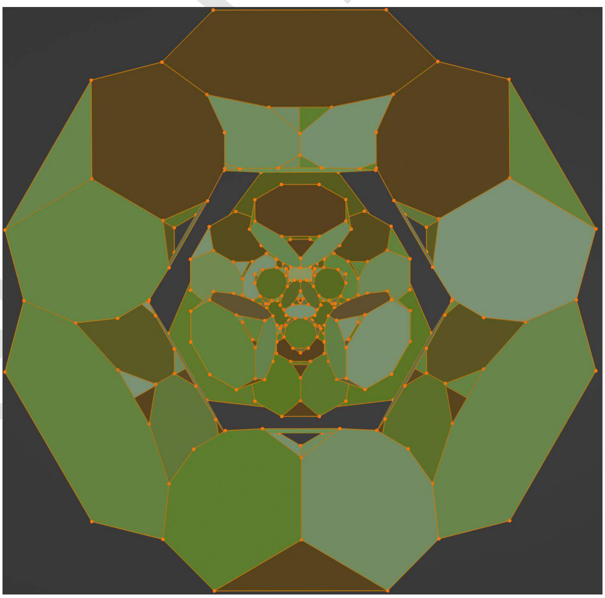

Figure 11: This exploded view is an additional attempt to explain the shape of this regular Leonardo polyhedron. Only six inner octagons and the six outer octagons are missing.

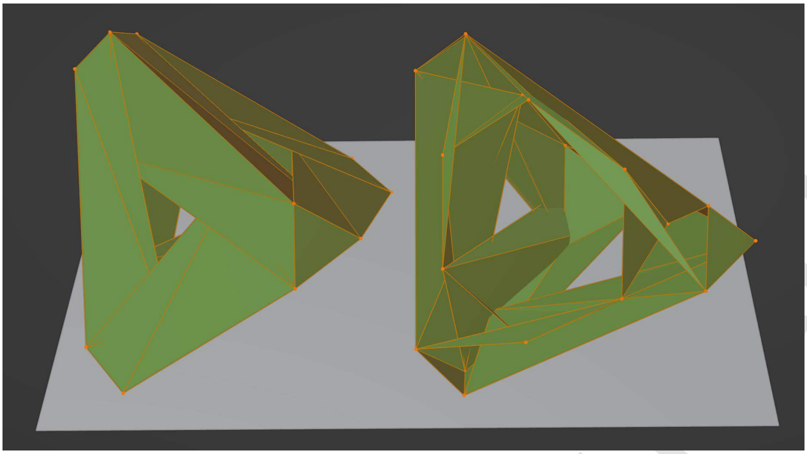

Figure 12: Regular Leonardo Polyeder due to Schulte and Wills [14], based on Klein's quartic.

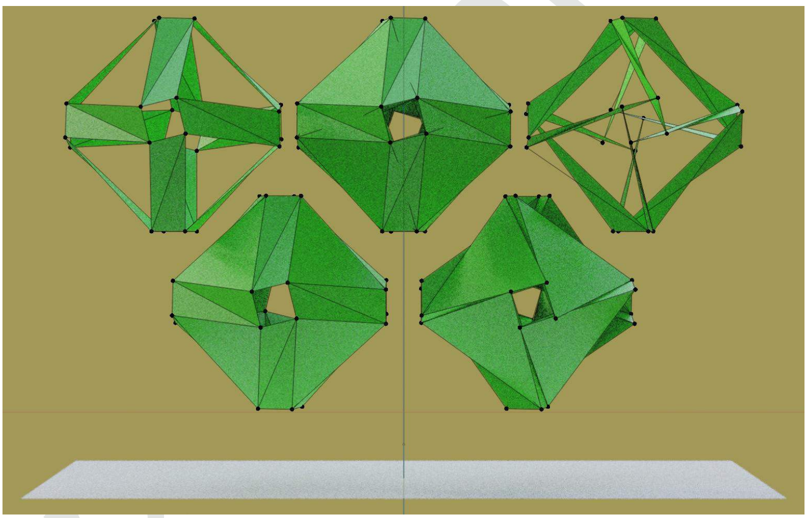

Figure 13: Regular Leonardo Polyeder based on Fricke-Klein, by Grünbaum et al., see [10].

6 Outlook

After the construction of $\{8,4 \mid 3\}$, there is the question, if there is a seventh or are there even more regular Leonardo Polyhedra. These are closely related to Platonic solids, they have strong restrictions and they seem to be rare objects. But regular Leonardo polyhedra are only part of a much more general problem: Polyhedral realizations of regular maps. Regular maps (Riemannian manifolds, algebraic curves) were investigated since Riemann, Poincare, Klein, Dyck, and Hurwitz. In the famous book by Coxeter and Moser, [9], there are infinite series and many sporadic regular maps. The best modern list of regular maps is by Marston Conder, [6, 7], where all regular maps up to $g = 301$ are listed (more than 3000). The natural question is: which of them are realizable as polyhedra in E^3 like Platonic solids? There are theorems which give restrictions for these realizations as e.g. the famous Ringel-Youngs theorem or Cauchy's classical result, that there are no equivelar polyhedra with $q = 3$. But there are no general criteria for the existence of combinatorially regular polyhedra. Such polyhedra are known for $p = 4$ and all $q \geqslant 5$ and for $q = 4$ and all $p \geqslant 5$. But nothing equivalent is known for $p = 3$ (triangular polyhedra). And, although there are infinitely many equivelar polyhedra with icosahedral symmetry, no regular polyhedron of this type is known. Finally: although there are infinitely many regular maps with $p \geqslant 5$ and $q \geqslant 5$ exist, no regular polyhedron and even no equivelar polyhedron of this type are known.

Not much is known about the first and probably best known infinite series of regular maps, namely Hurwitz's series of Schläfli type $\{3, 7\}$ with maximal number of automorphisms per genus g. The first one is Klein's quartic, see [15] and [11], and it is one of the Leonardo polyhedra. The second one, of genus 7, usually named by Hurwitz or Macbeath, has a polyhedral realization, see [1], but without any geometric symmetry. Nothing is known about polyhedral realizability of the others.

Finally we repeat the aim of our two pages in other words: Leonardo created geometric objects closely related to the Platonic solids but more complex than these, namely polyhedral embeddings of genus $g \geqslant 2$. He kept (preserved?) the symmetries, but lost regularities. In our two papers we present the known six regular Leonardo polyhedra and cast fresh-light on them.

7 Data

In this section we provide coordinates of the vertices for the embedding of Coxeter's regular map $\{8, 4 \mid 3\}$ in E^3. Because of the 3 symmetry planes defined by pairs of coordinate axes, we provide only a subset of coordinates for constructing the full orbit. Figure 14 shows the corresponding 66 labels.

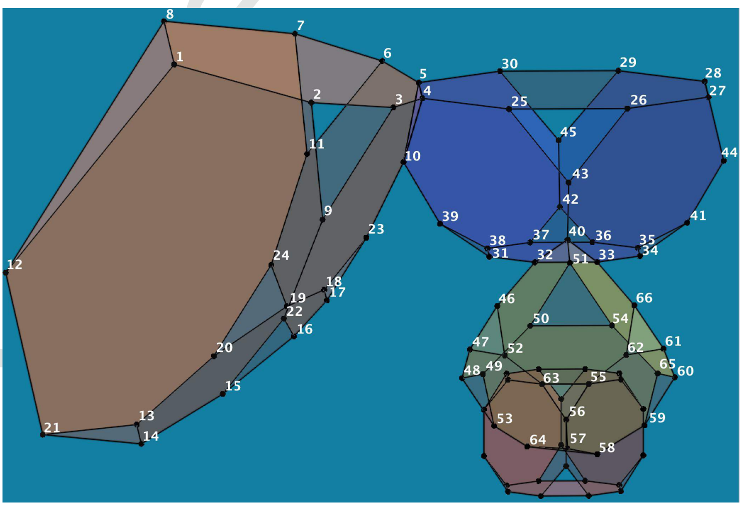

Figure 14: Labels of those vertices that lead to coordinates of all vertices.

The vertices 1, 8 and 12 are also vertices of the outer truncated cube, which are obtained in the same way as the vertices of the inner truncated cube through the three vertices 55, 56 and 63 of the green truncated cube due to the symmetry. Because of transparent faces all vertices can be seen in one image. Verticess 4, 5 and 10 show the common triangle of the brown and the green truncated cube, and vertices 32, 33 and 40 form the triangle of the blue and green truncated cube.

n	x	y	z	n	x	y	z	n	x	y	z
01	24.85	10.29	-24.85	23	8.787	8.787	-12.43	45	-7.132	7.132	-17.21
02	23.20	0	-23.20	24	3.733	21.76	-9.013	46	6.105	0	-8.634
03	15.67	0	-22.16	25	9.013	-3.733	-21.76	47	8.634	0	-6.105
04	9.013	3.733	-21.76	26	3.733	-9.013	-21.76	48	11.45	-1.964	-4.741
05	3.733	9.013	-21.76	27	-3.733	-9.013	-21.76	49	13.29	-5.505	-5.505
06	0	15.67	-22.16	28	-9.013	-3.733	-21.76	50	12.43	-8.787	-8.787
07	0	23.20	-23.20	29	-9.013	3.733	-21.76	51	8.787	-8.787	-12.43
08	10.29	24.85	-24.85	30	-3.733	9.013	-21.76	52	5.421	0	-5.421
09	22.16	0	-15.67	31	4.741	1.964	-11.45	53	11.45	-4.741	-1.964
10	7.132	7.132	-17.21	32	4.741	-1.964	-11.45	54	8.787	-12.43	-8.787
11	0	22.16	-15.67	33	1.964	-4.741	-11.45	55	1.471	-3.550	-3.550
12	24.85	24.85	-10.29	34	-1.964	-4.741	-11.45	56	3.550	-3.550	-1.471
13	15.67	22.16	0	35	-4.741	-1.964	-11.45	57	5.421	-5.421	0
14	22.16	15.67	0	36	-4.741	1.964	-11.45	58	6.105	-8.634	0
15	21.76	9.013	-3.733	37	-1.964	4.741	-11.45	59	4.741	-11.45	-1.964
16	17.21	7.132	-7.132	38	1.964	4.741	-11.45	60	1.964	-11.45	-4.741
17	12.43	8.787	-8.787	39	5.505	5.505	-13.29	61	0	-8.634	-6.105
18	8.787	12.43	-8.787	40	5.505	-5.505	-13.29	62	0	-5.421	-5.421
19	7.132	17.21	-7.132	41	-5.505	-5.505	-13.29	63	3.550	-1.471	-3.550
20	9.012	21.76	-3.733	42	-5.505	5,505	-13.29	64	9.421	-5.706	-0.5420
21	23.20	23.20	0	43	7.132	-7.132	-17.21	65	5.505	-13.29	-5.505
22	21.76	3.733	-9.013	44	-7.132	-7.132	-17.21	66	0	-6.105	-8.634

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