



Counting special Lagrangian classes and semistable mukai vectors for K3 surfaces

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Abstract

Motivated by the study of the growth rate of the number of geodesics in flat surfaces with bounded lengths, we study generalizations of such problems for K3 surfaces. In one generalization, we give a result regarding the upper bound on the asymptotics of the number of classes of irreducible special Lagrangians in K3 surfaces with bounded period integrals. In another generalization, we give the exact leading term in the asymptotics of the number of Mukai vectors of semistable coherent sheaves on algebraic K3 surfaces with bounded central charges, with respect to generic Bridgeland stability conditions.

Contents

1	Introduction	1
1.1	Motivation and problems	1
1.2	Results	1
2	Background	2
2.1	Lattice terminology	2
2.2	K3 surfaces	2
2.2.1	K3 lattice	2
2.2.2	Weight-two Hodge structure and the period domain	2
2.2.3	Néron–Severi lattice	2
2.2.4	Twistor family	2
2.3	Coherent sheaves	2
2.3.1	Mukai lattice	2
2.3.2	Stability conditions	2
2.4	Special Lagrangian classes	2
3	Counting semistable Mukai vectors	3
4	Counting special Lagrangian classes	3
	References	3

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1 Introduction

1.1 Motivation and problems

For a Riemann surface, a holomorphic 1-form Ω (also called an Abelian differential) with finitely many zeros endows the surface a flat metric given by $g = \frac{1}{2}\Omega\bar{\Omega}$, with conical singularities at the zeros of Ω , where the cone angles are integer multiples of 2π . The Abelian differential also gives an area form $\omega = \frac{i}{2}\Omega \wedge \bar{\Omega}$. Such flat surfaces have nontrivial but restricted holonomy that the parallel transport of a vector along a small closed path around a conical singularity turns the vector by exactly the cone angle. Studying flat surfaces reveals many interesting mathematical phenomena as reviewed in [26]. We are interested in studying generalizations of problems on flat surfaces to other settings.

Having a metric, it is natural to study closed geodesics and saddle connections, which are geodesic segments connecting pairs of conical points. A geodesics can be described as a curve on which $\text{Im}(e^{i\phi}\Omega)$ restricts to zero, for some angle ϕ . Indeed, in a local coordinate chart away from the zeros of Ω , we can express Ω as $\Omega = dz$, then straight lines of angle ϕ are the loci where $\text{Im}(e^{i\phi}z) = 0$. For a flat surface (S, Ω) with area normalized to one, one of the interesting problems is to understand the growth rate, as $R \rightarrow \infty$, of the counting function $N_{S,\Omega}(R)$ of the number of saddle connections, or the number of maximal cylinders filled with closed geodesics, of length at most R . This has been intensively studied. Masur [16] showed the existence of quadratic upper and lower bounds for $N_{S,\Omega}(R)$ for all flat surfaces, and Eskin–Masur [8], building on work of Veech [24], showed the exact quadratic asymptotics for almost all flat surfaces with respect to the natural Masur–Smillie–Veech measure on the space of quadratic differentials. More recently, Eskin–Mirazkhan–Mohammadi [9] showed a Cesàro-type quadratic asymptotic for every flat surface.

One example of generalization of geodesics on flat surfaces is to consider special Lagrangian submanifolds in Calabi–Yau manifolds. A Calabi–Yau manifold of complex dimension n is a complex Kähler manifold that admits a nowhere vanishing holomorphic top form (i.e. n -form). By [25], a compact Calabi–Yau manifold admits a Ricci-flat Kähler metric. While this is not an exact higher dimensional generalization of flat surfaces, it provides a class of examples in a similar spirit since the Ricci-flat metric also has nontrivial but restricted holonomy. In this paper, we specifically look at K3 surfaces. A K3 surface is a compact complex surface that admits a nowhere vanishing holomorphic 2-form Ω and is simply connected. By [19], all K3 surfaces are Kähler, so they are Calabi–Yau.

Given a Calabi–Yau manifold X of real dimension $2n$ with a Ricci-flat Kähler form ω , this symplectic form ω is used to define Lagrangian submanifolds, which are real n dimensional submanifolds on which ω restricts to 0. (On a Riemann surface, any curve is a Lagrangian.) The holomorphic top form Ω is used in addition to define special Lagrangian submanifolds similar to how it is used to define geodesics on Riemann surfaces. A Lagrangian submanifold L is special of phase ϕ if $\text{Im}(e^{i\phi}\Omega)|_L = 0$. Analogous to the counting function of geodesics on a flat surface, here we can consider a lattice theoretic counting function of irreducible special Lagrangian (sLag) classes with bounded period integral

$$SL_{\omega,\Omega}(R) = \# \left\{ \gamma \in H^n(X, \mathbb{Z}) : \exists \text{ irreducible sLag } L \text{ s.t. } [L]^{Pd} = \gamma, |\gamma \cdot \Omega| \leq R \right\}, \quad (1.1)$$

where $[L]^{Pd}$ denotes the Poincaré dual of the homology class of L , and $\gamma \cdot \Omega$ denotes the intersection pairing which is the period integral $\int_L \Omega$. In this paper, we focus on K3 surfaces where $n = 2$.

Let X be a K3 surface with a Ricci-flat metric g . There is a 2-sphere family $(X, J_t), t \in \mathbb{S}^2$, known as the twistor family, of K3 surfaces that are all compatible with the metric g . Their associated Kähler forms ω_t , and the real and imaginary parts of their (normalized) nowhere vanishing holomorphic 2-forms Ω_t , all lie in a positive definite 3-plane $P \subset H^2(X, \mathbb{R})$. The counting function of interest for special Lagrangian classes in this twistor family $\mathbb{S}^2(P)$ formulation is

$$SL_P(R) = \# \left\{ \gamma \in H^2(X, \mathbb{Z}) : \exists \omega_t \in \mathbb{S}^2(P), L \text{ irreducible sLag} \right. \\ \left. \text{w.r.t. } (\omega_t, \Omega_t), [L]^{Pd} = \gamma, |\gamma \cdot \Omega_t| \leq R \right\}. \quad (1.2)$$

As pointed out in [11], the Riemann surface analogue of the twistor sphere is a circle in a cylinder, for which rotating along a circle does not change the metric nor the complex structure. The difference for a K3 surface is that the complex structure changes as we vary along the twistor sphere. The work of [11, 14] study the count of special Lagrangian tori in K3 surfaces in this twistor formulation, and [14] also explains the relevance to physics. Very briefly, special Lagrangians in K3 correspond to BPS states via compactifying M-theory on K3, and their count is relevant to understanding the Bekenstein–Hawking entropy of simple black holes via counting microstates [22].

Another generalization of geodesics that we can consider are semistable coherent sheaves, which correspond to Lagrangian submanifolds via homological mirror symmetry (HMS). The HMS conjecture was first formulated by [13] to fully capture the phenomenon of mirror symmetry, which is a nontrivial duality between complex geometry and symplectic geometry, as an equivalence of triangulated categories. For a mirror pair of Calabi-Yau manifolds (X, ω_X, J_X) and (Y, ω_Y, J_Y) , it predicts the following equivalences

$$D^\pi \text{Fuk}(X, \omega_X) \cong D^b \text{Coh}(Y, J_Y) \quad \text{and} \quad D^b \text{Coh}(X, \omega_X) \cong D^\pi \text{Fuk}(Y, J_Y), \quad (1.3)$$

where $D^\pi \text{Fuk}$ is the split-closed derived Fukaya category and $D^b \text{Coh}$ is the bounded derived category of coherent sheaves. For K3 surfaces, HMS is proved in the case of Greene-Plesser mirrors by [21].

The Fukaya category depends on the symplectic structure but not on the complex structure, and its objects are Lagrangian submanifolds (other objects that might not be geometric are added in the derived category). As we mentioned in the above, if in addition we consider a complex structure and a holomorphic top form, we can define special Lagrangian submanifolds that are the stable objects. On the other hand, the category of coherent sheaves depends on the complex structure but not on the symplectic structure. If in addition we consider the symplectic structure, we can define stable coherent sheaves. For example, for a vector bundle E on a complex curve, its slope is defined to be $\mu(E) := \deg E / \text{rk} E$, and E is stable (semistable) if every subbundle F satisfies $\mu(F) < \mu(E)$ ($\mu(F) \leq \mu(E)$). These considerations are behind Bridgeland's definition of stability conditions building on Douglas' work [5, 6] on Π -stability for D-branes.

The space of Bridgeland stability condition $\text{Stab}(\mathcal{D})$ is defined [3] on a triangulated category $\mathcal{D}$, e.g. the derived category of coherent sheaves $\mathcal{D} = \mathcal{D}^b \text{Coh}(X)$ for a Calabi–Yau manifold X . When X is an algebraic K3 surface and $\mathcal{D} = \mathcal{D}^b \text{Coh}(X)$, Bridgeland [4] gives a detailed description of one connected component of $\text{Stab}(\mathcal{D})$. Denote by $K(\mathcal{D})$ its Grothendieck group consisting of objects in $\mathcal{D}$ up to some equivalence relations given by the triangulated structure. For an element $E \in K(\mathcal{D})$, its Mukai vector $v(E)$ is a cohomology class element in $H^*(X, \mathbb{Z})$ (see Eq. (2.16) for a more precise definition). The set of Mukai vectors are in bijection with the numerical Grothendieck group $N(\mathcal{D})$, which is the

Table 1 Analogy with flat surfaces

Flat surfaces	Calabi–Yau manifolds	Triangulated categories
Abelian differentials	Holomorphic top forms	Stability conditions
Geodesics	Special Lagrangians	Semistable objects
Lengths	Period integrals	Central charges

Table 2 Correspondence between mirror pairs of K3 surfaces (X, Y)

K3 surface X	K3 surface Y
Symplectic form ω_X , which defines Lagrangian	Holomorphic 2-form Ω_Y , which gives $\mathcal{D}^b\text{Coh}(Y, \Omega_Y)$
Holomorphic 2-form Ω_X , which defines special Lagrangian	Symplectic form ω_Y , which gives $\sigma_\omega \in \text{Stab}(\mathcal{D}^b(Y))$
$\int_- \Omega_X$	$Z_{\sigma_\omega}(-) = (\exp(\omega), -)_{\text{Muk}}$

quotient $K(\mathcal{D})/\ker \chi(-, -)$, where χ is the Euler pairing. The space of Bridgeland stability conditions $\text{Stab}(\mathcal{D})$ consists of all pairs $\sigma = (Z_\sigma, \mathcal{P}_\sigma)$, where $Z_\sigma : N(\mathcal{D}) \rightarrow \mathbb{C}$ is a group homomorphism called the central charge and $\mathcal{P}_\sigma(\phi)$ is a full additive subcategory of $\mathcal{D}$ consisting of σ -semistable objects of phase ϕ for each $\phi \in \mathbb{R}$ (together with a few axioms that these need to satisfy). In this setting, the counting function of interest is the number of σ -semistable Mukai vectors with bounded central charge

$$N_\sigma(R) = \#\{\gamma \in N(\mathcal{D}) : |Z_\sigma(\gamma)| \leq R \text{ and } \exists \sigma\text{-semistable object } E \text{ with } v(E) = \gamma\}. \quad (1.4)$$

We summarize in Table 1 the analogies between geodesics and saddle connections on flat Riemann surfaces, special Lagrangians in Calabi–Yau manifolds, and semistable objects in triangulated categories. We summarize in Table 2 the correspondence between a mirror pair of K3 surfaces.

1.2 Results

We begin by stating our result on the growth rate of the number of semistable classes $N_\sigma(R)$ as introduced in Eq. (1.4). In general, [3] showed that $\text{Stab}(\mathcal{D})$ is a complex manifold, with a left action by the group of triangulated autoequivalences $\text{Aut}(\mathcal{D})$. There is a subset $U(\mathcal{D}) \subset \text{Stab}(\mathcal{D})$ known as the geometric stability conditions (see Definition 2.1), and denote by $\text{Stab}^\dagger(\mathcal{D})$ the connected component of $\text{Stab}(\mathcal{D})$ that contains $U(\mathcal{D})$. For $\mathcal{D} := D^b\text{Coh}(X)$ being the derived category of coherent sheaves on an algebraic K3 surface, [4] showed that there is some autoequivalence of $\mathcal{D}$ that maps $\text{Stab}(\mathcal{D})^\dagger$ into the closure of a subset $U(\mathcal{D})$. Therefore, for almost every $\sigma \in \text{Stab}^\dagger(\mathcal{D})$, we can find a representative $\sigma' \in U(\mathcal{D})$ such that $N_\sigma(R) = N_{\sigma'}(R)$. Furthermore, there is a subset $V(\mathcal{D}) \subset U(\mathcal{D})$ that is the set of geometric stability conditions of phase 1 (more precisely defined in Eq. (2.25)). It is shown in [4] that for any $\sigma' \in U(\mathcal{D})$, $Z_{\sigma'}$ can be obtained from Z_σ , for some $\sigma \in V(\mathcal{D})$, via a $\text{GL}^+(2, \mathbb{R})$ action. More precisely, there is a $g \in \text{GL}^+(2, \mathbb{R})$ such that $Z_{\sigma'} = Z_\sigma g$, where g acts by multiplication to the image of Z_σ in $\mathbb{C} \cong \mathbb{R}^2$. Any element of $\text{GL}^+(2, \mathbb{R}) = \mathbb{R}^+ \times \text{SL}(2, \mathbb{R})$ is a composition of three different types of elements:

- $g \in \mathbb{R}^+$,
- $g \in \mathrm{SO}(2, \mathbb{R})$, i.e. in the rotational part of $\mathrm{SL}(2, \mathbb{R})$, and
- $g = \begin{pmatrix} 1 & \kappa \\ 0 & \lambda \end{pmatrix}$ is a shear by $\kappa + i\lambda$.

For a geometric stability of phase 1, its central charge is of the form $Z_\sigma = \exp(B + i\omega)$, where B, ω are real divisor classes and ω is ample.

Theorem 1.1 *Let X be a complex algebraic K3 surface. Then for almost every $\sigma \in \mathrm{Stab}^\dagger(\mathcal{D}^b(X))$,*

$$N_\sigma(R) = C(\sigma) \cdot R^{\rho+2} + o(R^{\rho+2}),$$

where ρ is the Picard rank of X , and $C(\sigma)$ is a constant depending on σ . For $\sigma \in V(\mathcal{D})$,

$$C(\sigma) = \frac{2\pi^{(\rho+2)/2}}{(\rho+2)\Gamma(\frac{\rho}{2}+1)(\omega^2)^{(\rho+2)/2}\sqrt{|\mathrm{DiscNS}(X)|}}.$$

For $\sigma' \in U(\mathcal{D})$, $Z_{\sigma'} = Z_\sigma g$ for some $\sigma \in V(\mathcal{D})$ and $g \in \mathrm{GL}^+(2, \mathbb{R})$. There are three cases

- $g \in \mathbb{R}^+$, then $C(\sigma') = \frac{C(\sigma)}{g^{\rho+2}}$.
- g is in the rotation part of $\mathrm{SL}(2, \mathbb{R})$, then $C(\sigma') = C(\sigma)$.
- g is a shear by $\kappa + i\lambda$, $\lambda > 0$, then

$$C(\sigma') = \frac{\pi^{\rho/2} \int_0^{2\pi} \left(\cos^2 \theta + \frac{1}{\lambda^2} (\sin \theta - \kappa \cos \theta)^2 \right)^{\rho/2} d\theta}{(\rho+2)\Gamma(\frac{\rho}{2}+1)\lambda(\omega^2)^{(\rho+2)/2}\sqrt{|\mathrm{DiscNS}(X)|}}.$$

One can see that the coefficient $C(\sigma)$ of the leading term does not depend on B . As stated in the theorem, this formula applies to “almost every”, or generic, stability condition, which make up the complement of a measure zero set (more precisely, see Definition 3.1 and the paragraph below that).

Now we turn to the counting function $SL_{\omega, \Omega}(R)$ of special Lagrangian classes as introduced in Eq. (1.1) (with $n = 2$ for K3 surfaces). Denote by $\mathrm{Lag}(X, \omega) := H^2(X, \mathbb{Z}) \cap [\omega]^\perp \subset H^2(X, \mathbb{Z})$ be the lattice of consisting of cohomology classes of Lagrangian submanifolds (see the paragraph containing Eq. (2.33) for more explanation). We have the following result.

Theorem 1.2 *Assuming $\mathrm{Lag}(X, \omega)$ has signature $(2, 19)$, then*

$$SL_{\omega, \Omega}(R) \leq C(\omega, \Omega) R^{21} + o(R^{21})$$

where

$$C(\omega, \Omega) = \frac{2\pi^{21/2}}{21\Gamma(\frac{21}{2})K_\Omega^{21/2}\sqrt{|\mathrm{DiscLag}(X, \omega)|}}.$$

Note that $\mathrm{Lag}(X, \omega)$ has signature $(2, 19)$ if ω is a rational Kähler class, so the statement holds in particular for polarized K3 surfaces.

In the twistor formulation as introduced in Eq. (1.2), we have the following result.

Theorem 1.3 *Let P be a positive definite 3-plane in $H^2(X, \mathbb{R})$ parametrizing a twistor family $\{(X, \omega_t, \Omega_t)\}_{t \in \mathbb{S}^2(P)}$. Then*

$$SL_P(R) \leq C \cdot R^{22} + o(R^{22})$$

where C is a constant independent of the choice of the twistor family $\mathbb{S}^2(P)$.

2 Background

2.1 Lattice terminology

A lattice $\mathbb{L}$ is a finitely generated free $\mathbb{Z}$ -module, together with a nondegenerate symmetric bilinear form $(-, -) : \mathbb{L} \times \mathbb{L} \rightarrow \mathbb{Z}$. With respect to a basis on the $\mathbb{Z}$ -module, the bilinear form can be represented by a matrix. The signature (n_+, n_-) of $\mathbb{L}$ is the number n_+ of positive and the number n_- of negative eigenvalues of that matrix. The signature is independent of the choice of basis.

2.2 K3 surfaces

A K3 surface is a compact complex surface X that admits a nowhere vanishing holomorphic 2-form and is simply connected. Denote the complex structure of X by J , and let Ω_J be a nowhere vanishing holomorphic 2-form, whose choice is unique up to scaling. In some occasions in this paper, we would like to only consider algebraic K3 surfaces, which coincides with the subset of all complex analytic K3 surfaces that are projective.

2.2.1 K3 lattice

All K3 surfaces are diffeomorphic, so they have the same cohomology groups, which are $H^0(X, \mathbb{Z}) \cong H^4(X, \mathbb{Z}) \cong \mathbb{Z}$, $H^1(X, \mathbb{Z}) \cong H^3(X, \mathbb{Z}) = 0$, and $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{22}$. Because $H^2(X, \mathbb{Z})$ is free, the intersection pairing,

$$(-, -) : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}, \quad (2.1)$$

given by the cup product is a nondegenerate symmetric bilinear form. We will always use $a.b$ or $(a.b)$ to denote intersection pairing.

A K3 lattice is $H^2(X, \mathbb{Z})$ together with the intersection pairing above. For any K3 surface, there is a choice of basis for $H^2(X, \mathbb{Z})$ with respect to which the intersection pairing corresponds to a block matrix of the form

$$\Lambda := U^{\oplus 3} \oplus (-E_8)^{\oplus 2}, \quad (2.2)$$

where U represents $\mathbb{Z}^2 \cong \mathbb{Z}\langle e \rangle \oplus \mathbb{Z}\langle f \rangle$ with the bilinear form given by $e^2 = f^2 = 0$ and $e.f = 1$. So U is called the hyperbolic plane, and it has signature $(1, 1)$. The E_8 lattice is positive definite, so altogether Λ has signature $(3, 19)$. An isometry

$$\varphi : H^2(X, \mathbb{Z}) \rightarrow \Lambda \quad (2.3)$$

induced by choosing a basis is called a marking of X .

2.2.2 Weight-two Hodge structure and the period domain

The intersection pairing on $H^2(X, \mathbb{Z})$ extends complex linearly to the intersection pairing on $H^2(X, \mathbb{C}) = H^2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ that is the same as

$$(-, -) : H^2(X, \mathbb{C}) \otimes H^2(X, \mathbb{C}) \rightarrow \mathbb{C}, \quad a.b = \int_X a \wedge b. \quad (2.4)$$

The isometry $\varphi : H^2(X, \mathbb{Z}) \rightarrow \Lambda$ extends to

$$\varphi_{\mathbb{C}} := \varphi \otimes \mathbb{C} : H^2(X, \mathbb{C}) \rightarrow \Lambda_{\mathbb{C}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{C}. \quad (2.5)$$

Because K3 surfaces are Kähler [19], we have the following weight-two Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X). \quad (2.6)$$

This decomposition is determined by the complex line $H^{2,0}(X) \cong \mathbb{C}[\Omega_J]$. This is because $H^{0,2}(X)$ is complex conjugate to $H^{2,0}(X)$, and $H^{1,1}(X)$ is orthogonal to $H^{2,0}(X) \oplus H^{0,2}(X)$. We have $h^{2,0}(X) = h^{0,2}(X) = 1$ and $h^{1,1}(X) = 20$.

Two K3 surfaces, X_1 and X_2 , are said to be Hodge isometric if there is an isomorphism $\phi : H^2(X_1, \mathbb{Z}) \rightarrow H^2(X_2, \mathbb{Z})$ such that ϕ preserves the intersection form and $(\phi \otimes \mathbb{C})(H^{2,0}(X_1)) = H^{2,0}(X_2)$, hence preserving the weight-two Hodge structure. Torelli theorem for K3 surfaces says that X_1 and X_2 , are isomorphic as complex manifolds if and only if they are Hodge isometric. This is proved by [17] in the algebraic case and by [2] in the non-algebraic case, and see [12, Chapter 2] for an exposition.

Note that Ω_J satisfies $\Omega_J \cdot \Omega_J = 0$ and $\Omega_J \cdot \overline{\Omega}_J > 0$. We call the following set the period domain,

$$\begin{aligned} \Omega(\Lambda_{\mathbb{C}}) &:= \{[\Omega] \in \mathbb{P}(\Lambda_{\mathbb{C}}) : \Omega \cdot \Omega = 0, \Omega \cdot \overline{\Omega} > 0\} \\ &= \{[\Omega] \in \mathbb{P}(\Lambda_{\mathbb{C}}) : K_{\Omega} = (\operatorname{Re} \Omega)^2 = (\operatorname{Im} \Omega)^2 > 0, (\operatorname{Re} \Omega) \cdot (\operatorname{Im} \Omega) = 0\}. \end{aligned} \quad (2.7)$$

The map $X \mapsto [\varphi_{\mathbb{C}}(H^{2,0}(X))] \in \Omega(\Lambda_{\mathbb{C}})$ is called the period mapping, and it takes a $(2, 0)$ -form Ω to its marking $\varphi_{\mathbb{C}}(\Omega) \in \Lambda_{\mathbb{C}}$, which corresponds to the line $[\varphi_{\mathbb{C}}(\Omega)] \in \mathbb{P}(\Lambda_{\mathbb{C}})$. This map is surjective but not injective on the moduli space of marked K3 surfaces [12, Chapter 6].

2.2.3 Néron–Severi lattice

For compact Kähler manifolds, the image of the first Chern class map $c_1 : \operatorname{Pic}(X) \rightarrow H^2(X, \mathbb{C})$ is known as the Néron–Severi group, $\operatorname{NS}(X)$, i.e. numerical classes of line bundles classified by the first Chern class. By Lefschetz theorem on $(1, 1)$ -classes, this image is equal to

$$H^{1,1}(X, \mathbb{Z}) := H^{1,1}(X) \cap \operatorname{Image}(H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{C})). \quad (2.8)$$

The kernel of this map is $H^1(X, \mathcal{O}_X)$, which is 0 for K3 surfaces, so this map is injective. Hence for K3 surfaces, the c_1 map gives the identification

$$\operatorname{Pic}(X) = \operatorname{NS}(X) = H^{1,1}(X, \mathbb{Z}). \quad (2.9)$$

Because $H^{1,1}(X)$ has dimension 20, we get that the Picard rank, $\rho(X) := \operatorname{rk}(\operatorname{Pic}(X))$, is in the range $0 \leq \rho(X) \leq 20$. For projective K3 surfaces, $1 \leq \rho(X) \leq 20$.

Hodge index theorem says that for a compact Kähler surface, the intersection pairing

$$H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \rightarrow \mathbb{R}, \quad a \cdot b \mapsto \int_X a \wedge b \quad (2.10)$$

restricted to $H^{1,1}(X, \mathbb{R})$ has signature $(1, h^{1,1}(X) - 1)$. One can also define an intersection pairing

$$\operatorname{Pic}(X) \times \operatorname{Pic}(X) \rightarrow \mathbb{Z}, \quad L \cdot L' = \int_X c_1(L) \wedge c_1(L'). \quad (2.11)$$

A line bundle $L \in \text{Pic}(X)$ is called numerically trivial if $L \cdot L' = 0$ for all line bundles L' . For complex projective surfaces, a line bundle is numerically trivial if and only if $c_1(L) = 0$. So for projective K3 surfaces, since the kernel of c_1 is trivial, only the trivial bundle is numerically trivial. Hence for projective K3 surfaces, the intersection pairing (2.11) on the Néron-Severi group is nondegenerate and has signature $(1, \rho(X) - 1)$.

2.2.4 Twistor family

Let (X, J) be a K3 surface, by [25], any Kähler class in $H^{1,1}(X, \mathbb{R}) \subset H^2(X, \mathbb{R}) \cong \Lambda_{\mathbb{R}}$ has a unique representative ω whose associated metric $g(\cdot, \cdot) = \omega_J(\cdot, J \cdot)$ is Ricci-flat, i.e. with vanishing Ricci curvature.

Denote by Ω a nowhere vanishing holomorphic 2-form. Note that $\text{Re}\Omega$ and $\text{Im}\Omega$ are orthogonal to ω because $\Omega \in H^{2,0}(X)$, $\overline{\Omega} \in H^{0,2}(X)$, and $H^{2,0}(X) \oplus H^{0,2}(X)$ is orthogonal to $H^{1,1}(X)$. Also, as in Eq. (2.7), $(\text{Re}\Omega)^2 = (\text{Im}\Omega)^2 > 0$ and $\text{Re}\Omega \cdot \text{Im}\Omega = 0$. So

$$P := \text{Span}\{\text{Re}\Omega, \text{Im}\Omega, \omega\} \subset H^2(X, \mathbb{R}) \cong \Lambda_{\mathbb{R}} \quad (2.12)$$

is a positive-definite 3-plane in $\Lambda_{\mathbb{R}} \cong \mathbb{R}^{3,19}$. If we assume the normalization $d\text{vol} := \frac{1}{2}\Omega \wedge \overline{\Omega}$ and $\int_X d\text{vol} = 1$, then this is equivalent to $K_{\Omega} = (\text{Re}\Omega)^2 = (\text{Im}\Omega)^2 = 1$. If furthermore the Kähler class $[\omega]$ we have satisfies $\int_X \omega^2 = 1$, then $\{\text{Re}\Omega, \text{Im}\Omega, \omega\} \subset \mathbb{S}^2(P)$ form an orthonormal basis for P .

The Ricci-flatness means that the holonomy group of g is in $\text{SU}(2)$. Because $\text{SU}(2)$ is isomorphic to the group $\text{Sp}(1)$ of unitary quaternions, g is a hyperkähler metric. Consequently, X admits a two-sphere $\mathbb{S}^2$ family, known as the twistor family, of complex structures that are compatible with g and we now describe them. Denote by ω_J and Ω_J the forms on (X, J) , and assume they are normalized as above. Let $\omega_K = \text{Re}(\Omega_J)$ and $\omega_I = \text{Im}(\Omega_J)$. Let I be the complex structure on X that is compatible with (g, ω_I) , i.e. $\omega_I(\cdot, \cdot) = g(I \cdot, \cdot)$. Similarly, let K be the complex structure compatible with (g, ω_K) . With respect to I , $\Omega_I = \omega_J + i\omega_K$ is a holomorphic 2-form. Similarly, with respect to K , $\Omega_K = \omega_I + i\omega_J$ is a holomorphic 2-form. The complex structures I, J, K satisfies the quaternionic commutation relation $I^2 = J^2 = K^2 = I \cdot J \cdot K = -1$. Then, any complex structure J_t , $t = (x, y, z)$, in the following two sphere

$$\{J_t = xI + yJ + zK \mid x^2 + y^2 + z^2 = 1\} = \mathbb{S}^2 \quad (2.13)$$

is compatible with the metric g , i.e. $g(\cdot, \cdot) = g(J_t \cdot, J_t \cdot)$. For each $t \in \mathbb{S}^2$, denote by ω_t the associated Kähler form given by $\omega_t(\cdot, \cdot) = g(J_t \cdot, \cdot)$, and denote by Ω_t a normalized nowhere vanishing holomorphic 2-form with respect to J_t (there is a $\mathbb{S}^1$ -worth of compatible choices for such Ω_t). Then $\{\omega_t, \text{Re}\Omega_t, \text{Im}\Omega_t\} \subset \mathbb{S}^2(P)$, for P in Eq. (2.12) and P is known as the twistor plane.

Conversely, given any positive definite 3-plane $P \subset H^2(X, \mathbb{R})$, there is an associated twistor family on X .

2.3 Coherent sheaves

In this section, we assume X is projective. Denote by $\mathcal{D} := \mathcal{D}^b\text{Coh}(X)$ the bounded derived category of coherent sheaves on X .

2.3.1 Mukai lattice

Denote by $K(\mathcal{D})$ the Grothendieck group, which admits an Euler pairing

$$\chi(E, F) := \sum_j (-1)^j \dim \operatorname{Hom}_{\mathcal{D}}^j(E, F). \quad (2.14)$$

Serre duality implies that the Euler pairing is symmetric: $\chi(E, F) = \chi(F, E)$.

The Mukai vector is a map

$$v : K(\mathcal{D}) \rightarrow H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \quad (2.15)$$

given by

$$v(E) := \operatorname{ch}(E) \sqrt{\operatorname{td}(X)} = (\operatorname{rk}(E), c_1(E), \chi(E) - \operatorname{rk}(E)). \quad (2.16)$$

The Hirzebruch-Riemann-Roch formula implies

$$\chi(E, F) = -\langle v(E), v(F) \rangle, \quad (2.17)$$

where

$$\begin{aligned} \langle -, - \rangle : H^*(X, \mathbb{Z}) \times H^*(X, \mathbb{Z}) &\rightarrow \mathbb{Z}, \\ \langle (r_1, D_1, s_1), (r_2, D_2, s_2) \rangle &= D_1 \cdot D_2 - r_1 \cdot s_2 - r_2 \cdot s_1 \end{aligned} \quad (2.18)$$

is the Mukai pairing. Notation-wise, we will always use $(-, -)$ to denote the intersection pairing and $\langle -, - \rangle$ to denote the Mukai pairing. Note that the Mukai pairing on $H^*(X, \mathbb{Z})$ restricted to $H^2(X, \mathbb{Z})$ is the same as the intersection pairing.

We call $H^*(X, \mathbb{Z})$, equipped with the Mukai pairing, the Mukai lattice. According to (2.17), the Mukai vector given by (2.16) is a map

$$v : (K(\mathcal{D}), -\chi(-, -)) \longrightarrow (H^*(X, \mathbb{Z}), \langle -, - \rangle). \quad (2.19)$$

The Mukai lattice is an extension of the K3 lattice on $H^2(X, \mathbb{Z})$ so that $H^0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$ is isometric to a hyperbolic plane. Hence, the Mukai lattice is isometric to

$$\Lambda_{\operatorname{Muk}} = U \oplus \Lambda = U^{\oplus 4} \oplus (-E_8)^{\oplus 2}, \quad (2.20)$$

which has signature $(4, 20)$.

The numerical Grothendieck group is defined to be the quotient

$$N(\mathcal{D}) = K(\mathcal{D}) / \ker \chi(-, -). \quad (2.21)$$

By this definition, the Mukai pairing on $N(\mathcal{D})$ is nondegenerate. When $\dim_{\mathbb{C}}(X) = 2$, it is known that the Chern character map, and so the Mukai vector map v , descends to a map on $N(\mathcal{D})$, and this map is injective. So it identifies $N(\mathcal{D})$ with its image

$$(N(\mathcal{D}), -\chi(-, -)) \cong (H^0(X, \mathbb{Z}) \oplus \operatorname{NS}(X) \oplus H^4(X, \mathbb{Z}), \langle -, - \rangle) \subset (H^*(X, \mathbb{Z}), \langle -, - \rangle). \quad (2.22)$$

The numerical Grothendieck group with the Mukai pairing is a lattice with signature $(2, \rho(X))$, where $\rho(X) = \operatorname{rk}(\operatorname{NS}(X))$ is the Picard rank.

2.3.2 Stability conditions

The space of locally finite numerical Bridgeland stability conditions, $\text{Stab}(\mathcal{D})$, on $\mathcal{D} := \mathcal{D}^b\text{Coh}(X)$ consists of all pairs $\sigma = (Z, \mathcal{P})$, where

- $Z : N(\mathcal{D}) \rightarrow \mathbb{C}$ is a group homomorphism called the central charge, and
- $\mathcal{P} := \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$ is a collection of full additive subcategories of $\mathcal{D}$, one for each $\phi \in \mathbb{R}$ (objects in $\mathcal{P}(\phi)$ are called the σ -semistable objects of phase ϕ),

such that:

- (1) if $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$;
- (2) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$;
- (3) if $\phi_1 > \phi_2$ and $E_j \in \mathcal{P}(\phi_j)$, then $\text{Hom}(E_1, E_2) = 0$;
- (4) (Harder-Narasimhan filtration) for each $0 \neq E \in \mathcal{D}$, there exists a collection of triangles

$$0 = E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_{k-1} \longrightarrow E$$

with $B_j \in \mathcal{P}(\phi_j)$ and $\phi_1 > \phi_2 > \cdots > \phi_k$;

- (5) (support property) there is a constant $C > 0$ and a norm $\|\cdot\|$ on $N(\mathcal{D}) \otimes_{\mathbb{Z}} \mathbb{R}$ such that for any semistable object E , we have $\|E\| \leq C|Z(E)|$.

Note that the support property implies the set $\{Z(E)\}$ of central charges of σ -semistable objects are discrete in $\mathbb{C}$ and has no accumulation point. Indeed, otherwise there would be a sequence of classes $[E_n] \in N(\mathcal{D})$ supporting σ -semistable objects such that $\lim_{n \rightarrow \infty} |Z(E_n)|$ is finite. Together with $\limsup_{n \rightarrow \infty} \|[E_n]\| = \infty$, this contradicts with the support property.

There is a left action on $\text{Stab}(\mathcal{D})$ by the group of triangulated autoequivalences $\text{Aut}(\mathcal{D})$ of $\mathcal{D}$, and there is a right action on $\text{Stab}(\mathcal{D})$ by $\text{GL}^+(2; \mathbb{R})$ (the universal cover of $\text{GL}^+(2, \mathbb{R})$) which descends to a $\text{GL}^+(2, \mathbb{R})$ action on $Z : N(\mathcal{D}) \rightarrow (\mathbb{C} \cong \mathbb{R}^2)$ by post-composition. Note that by the Gram-Schmidt procedure, $\text{GL}^+(2, \mathbb{R})$ deformation retracts onto $SO(2, \mathbb{R}) \cong S^1$, so $\pi_1(\text{GL}^+(2, \mathbb{R})) \cong \mathbb{Z}$, with the generator of the fundamental group acting by $e^{2\pi i}$.

The non-degeneracy of the Mukai pairing gives an identification

$$\text{Hom}(N(\mathcal{D}), \mathbb{C}) \cong N(\mathcal{D})_{\mathbb{C}} := N(\mathcal{D}) \otimes \mathbb{C}, \quad Z \mapsto \varphi, \quad Z(v) = \langle \varphi, v \rangle. \quad (2.23)$$

Bridgeland [3] showed that $\text{Stab}(\mathcal{D})$ has the structure of a complex manifold, and the forgetful map,

$$\pi : \text{Stab}(\mathcal{D}) \rightarrow \text{Hom}(N(\mathcal{D}), \mathbb{C}) \cong N(\mathcal{D})_{\mathbb{C}}, \quad \sigma = (Z, \mathcal{P}) \mapsto Z, \quad (2.24)$$

gives a local isomorphism.

Of special interest is the subset $U(\mathcal{D}) \subset \text{Stab}(\mathcal{D})$ of geometric stability conditions.

Definition 2.1 A *geometric stability condition* is a $\sigma \in \text{Stab}(\mathcal{D})$ for which all skyscraper sheaves, $\mathcal{O}_x$ where $x \in X$, are σ -stable and of the same phase.

Note that $U(\mathcal{D})$ is connected, due to [4, Proposition 11.2 and the discussion after Corollary 11.3]. Let $\text{Stab}^+(\mathcal{D})$ be the connected component of $\text{Stab}(\mathcal{D})$ that contains $U(\mathcal{D})$. Bridgeland [4] showed that there is some autoequivalence of $\mathcal{D}$ that maps $\text{Stab}^+(\mathcal{D})$ into the closure of $U(\mathcal{D}) \subset \text{Stab}^+(\mathcal{D})$.

Furthermore, [4] showed that for any $\sigma \in U(\mathcal{D})$, there is a unique $\tilde{g} \in \widetilde{\mathrm{GL}^+(2, \mathbb{R})}$ such that $\sigma \cdot \tilde{g} \in V(\mathcal{D})$, where

$$V(\mathcal{D}) = \{\sigma \in U(\mathcal{D}) \mid \pi(\sigma) \in \mathcal{Q}(X), \mathcal{O}_X \text{ is } \sigma\text{-stable of phase 1 } \forall x \in X\}, \quad (2.25)$$

where $\mathcal{Q}(X) \subset N(\mathcal{D}) \otimes \mathbb{C}$ is the set of vectors whose real and imaginary part spans positive 2-planes in $N(\mathcal{D}) \otimes \mathbb{R}$. It is worth mentioning that $\mathcal{Q}(X)$ is closely related to the period domain associated to $N(\mathcal{D})$, and can be identified with

$$\mathcal{Q}(X) = \{\varphi = \exp(B + i\omega) \mid B + i\omega \in \mathrm{NS} \otimes \mathbb{C}, \omega^2 > 0\} \subset N(\mathcal{D})_{\mathbb{C}}. \quad (2.26)$$

Note that

$$\varphi = \exp(B + i\omega) = \left(1, B, \frac{B^2 - \omega^2}{2}\right) + i(0, \omega, B.\omega). \quad (2.27)$$

Bridgeland [4] also showed that the forgetful map π is injective when restricted to $V(\mathcal{D})$. The image of

$$\pi|_{V(\mathcal{D})} : V(\mathcal{D}) \xrightarrow{\sim} L(\mathcal{D}) \quad (2.28)$$

is

$$L(\mathcal{D}) = \{\varphi \in \mathcal{K}(X) \mid \langle \varphi, \delta \rangle \notin \mathbb{R}_{\leq 0} \text{ for all } \delta \in \Delta^+(N(\mathcal{D}))\} \subset \mathcal{K}(X), \quad (2.29)$$

where

$$\mathcal{K}(X) := \{\varphi = \exp(B + i\omega) \in \mathcal{Q}(X) \mid \omega \in \mathrm{Amp}(X)\} \subset \mathcal{Q}(X) \quad (2.30)$$

and

$$\Delta^+(N(\mathcal{D})) = \{\delta = (r, D, s) \in N(\mathcal{D}) \mid \langle \delta, \delta \rangle = -2, r > 0\}. \quad (2.31)$$

So the forgetful map identifies stability conditions $\sigma = (Z, \mathcal{P}) \in V(\mathcal{D})$ with its image $\varphi = \exp(B + i\omega) \in L(\mathcal{D})$ for some $B, \omega \in \mathrm{NS}(X)_{\mathbb{R}}$ with $\omega \in \mathrm{Amp}(X)$. By (2.23), this $\varphi = \exp(B + i\omega)$ is identified with the central charge homomorphism

$$Z(v) = \langle \mathrm{Re}(\varphi), v \rangle + i \langle \mathrm{Im}(\varphi), v \rangle = \left\langle \left(1, B, \frac{B^2 - \omega^2}{2}\right), v \right\rangle + i \langle (0, \omega, B.\omega), v \rangle. \quad (2.32)$$

2.4 Special Lagrangian classes

Given a Kähler class in $H^{1,1}(X, \mathbb{R})$, it is uniquely represented by a Ricci-flat Kähler form ω . Below, we use the same notation ω to denote either the Kähler form or the Kähler class, depending on the context. For a Lagrangian submanifold L in (X, ω) , let $[L]^{Pd} \in H^2(X, \mathbb{Z})$ be the Poincaré dual of the homology class of L , then we have the intersection pairing $[L]^{Pd}.\omega = \int_L \omega|_L = 0$. Conversely, by ([23], Corollary 2.4), if $\gamma \in H^2(X, \mathbb{Z})$ such that $\gamma.\omega = 0$, then it represents an immersed Lagrangian submanifold. Hence $\gamma \in H^2(X, \mathbb{Z})$ represents an immersed Lagrangian submanifold if and only if γ belongs to

$$\mathrm{Lag}(X, \omega) := H^2(X, \mathbb{Z}) \cap \omega^\perp \subset H^2(X, \mathbb{Z}). \quad (2.33)$$

We call $\mathrm{Lag}(X, \omega)$, with the intersection pairing, the Lagrangian class lattice.

In addition to the Kähler form ω , if we also take a nonvanishing holomorphic 2-form Ω on X into account, then we can define special Lagrangian (sLag) submanifolds of phase ϕ ,

which are Lagrangian submanifolds L such that $\iota^*\Omega = e^{i\phi}d\text{vol}|_L$, where ϕ is a constant and $d\text{vol}$ is the volume form induced by the Kähler metric determined by ω . Define the set of irreducible special Lagrangian classes with respect to ω, Ω to be

$$\text{SLag}(X, \omega, \Omega) = \{\gamma \in H^2(X, \mathbb{Z}) : \exists \text{ irreducible sLag } L \text{ with } [L]^{Pd} = \gamma\}. \quad (2.34)$$

Using the hyperkähler rotation and standard facts about holomorphic curves, one has the following lemma.

Lemma 2.2 (Lai–Lin–Schaffler, [15, Lemma 2.2])

$$\text{SLag}(X, \omega, \Omega) \subset \{\gamma \in \text{Lag}(X, \omega) : \gamma^2 \geq -2\}.$$

Note that notation-wise for us $\text{SLag}(X, \omega, \Omega)$ consists of only the classes represented by special Lagrangians that are irreducible. The same notation in [15] does not require irreducibility, but the above lemma is indeed about irreducible Lagrangians.

Remark 2.3 By [15, Proposition 3.3], given a lattice $\Lambda' \subsetneq \Lambda \cong H^2(X, \mathbb{Z})$, there exists a Kähler K3 surface (X, ω) such that $\text{Lag}(X, \omega) \cong \Lambda'$ if and only if Λ' is proper, saturated, and such that $(\Lambda')^\perp \subseteq \Lambda$ contains a vector with positive self-intersection. Hence in general we don't have control on the signature of $\text{Lag}(X, \omega)$; also, the lattice could be degenerate.

Remark 2.4 In this remark, we briefly describe the numerical Grothendieck group of the derived Fukaya category of X , which is expected to correspond to $N(\mathcal{D}(Y))$ if X and Y are mirror K3s. See [20] for a more detailed exposition. Denote by $\text{Fuk}(X)$ the Fukaya category of X , whose objects are unobstructed immersed Lagrangian submanifolds. It is an A_∞ -category that is linear over the Novikov field $\mathbb{F}$ over $\mathbb{C}$ given by $\mathbb{F} := \left\{ \sum_{j=0}^{\infty} a_j q^{\lambda_j} \mid a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, \lim_{j \rightarrow \infty} \lambda_j = +\infty \right\}$. Denote by $\mathcal{F} := \mathcal{F}(X)$ the triangulated category given by the split-closures of the category of twisted complexes on the Fukaya category. Let $K(\mathcal{F})$ be the Grothendieck group, then there is a composition of the Chern character map and the open-closed map

$$K(\mathcal{F}) \xrightarrow{\text{ch}} HH_0(\mathcal{F}) \xrightarrow{OC} H^2(X, \mathbb{F}). \quad (2.35)$$

Shklyanov [18] introduced Mukai pairing for dg categories and proved that the Mukai pairing on $HH_0(\mathcal{F})$ satisfies

$$\langle \text{ch}(E), \text{ch}(F) \rangle = -\chi(E, F). \quad (2.36)$$

An object in $K(\mathcal{F})$ needs not be geometric, but when it is, i.e. when it can be represented by a unobstructed immersed Lagrangian submanifold, [20, Lemma 5.13] shows that (2.35) is the map $[L] \mapsto [L]^{Pd} \in \text{Lag}(X, \omega) \subset H^2(X, \mathbb{Z})$. When L_1, L_2 are immersed Lagrangian submanifolds, their Euler pairing is the Euler characteristics of the Floer cohomology, and we see that the Mukai pairing coincides with the intersection pairing on $H^2(X, \mathbb{Z})$

$$\langle \text{ch}(L_1), \text{ch}(L_2) \rangle = -\chi(L_1, L_2) = -\chi(HF^*(L_1, L_2)) = [L_1].[L_2]. \quad (2.37)$$

Let $N(\mathcal{F}) := K(\mathcal{F})/\ker \chi(-, -)$ be the numerical Grothendieck group, then the map $K(\mathcal{F}) \rightarrow H^2(X, \mathbb{F})$ descends to an injective map $N(\mathcal{F}) \rightarrow H^2(X, \mathbb{F})$. If X and Y are mirror K3 surfaces, then the expectation is that

$$N(\mathcal{F}(X)) = N(\mathcal{D}(Y)), \quad (2.38)$$

so then $N(\mathcal{F}(X))$ would have signature $(2, \rho)$ where $\rho = \text{rk}(\text{Pic}(Y))$. It is not known whether $N(\mathcal{F}(X))$ can be identified with $\text{Lag}(X, \omega)$, see [20, Remark 7.6] for more detailed discussions.

3 Counting semistable Mukai vectors

In this section we explain Theorem 1.1. Let X be a complex algebraic $K3$ surface, denote by $\mathcal{D} := D^b\text{Coh}(X)$ be the bounded derived category of coherent sheaves on X , and denote by $N(\mathcal{D})$ the numerical Grothendieck group, which we view as the signature $(2, \rho(X))$ Mukai lattice $H^*(X, \mathbb{Z})$ via the identification given by Eq. (2.22). To proceed, please refer to the notations introduced in Sect. 2.3.2. Fix a Bridgeland stability condition $\sigma \in \text{Stab}^\dagger(\mathcal{D})$ with central charge $Z_\sigma: N(\mathcal{D}) \rightarrow \mathbb{C}$. We are interested in the following counting function of semistable classes with bounded charges

$$N_\sigma(R) = \#\{v \in N(\mathcal{D}) : |Z_\sigma(v)| \leq R \text{ and } \exists \sigma\text{-semistable object } E \text{ with } [E] = v\}, \quad (3.1)$$

where $[E]$ is the image of E under the Mukai map in Eq. (2.19). Because by [4], there is some autoequivalence of $\mathcal{D}$ that maps $\text{Stab}^\dagger(\mathcal{D})$ into the closure of $U(\mathcal{D})$, we will just do the count for geometric stability conditions. As mentioned in Sect. 2.3.2, for any stability condition $\sigma \in U(\mathcal{D})$, there is a unique $\tilde{g} \in \text{GL}^+(2, \mathbb{R})$ such that $\sigma \cdot \tilde{g} \in V(\mathcal{D})$, which corresponds to $\pi(\sigma \cdot \tilde{g}) \in L(\mathcal{D})$ via the isomorphism $\pi|_{V(\mathcal{D})}$. So for each element $\pi(\sigma) \in \pi(U(\mathcal{D}))$, there is a unique $g = \pi(\tilde{g}) \in \text{GL}^+(2, \mathbb{R})$ such that $\pi(\sigma) \cdot g \in L(\mathcal{D})$. So in this section, we will first look into $N_\sigma(R)$ for $\sigma \in V(\mathcal{D})$ and then consider other stability conditions in $U(\mathcal{D})$ obtained via $\text{GL}^+(2, \mathbb{R})$ action.

Given a Mukai vector v , by [4] there is a locally finite set of walls (which are real codimension one submanifolds) in $\text{Stab}^\dagger(\mathcal{D})$ such that so long as σ varies within a chamber, the set of σ -semistable objects with class v does not change. Following [1], we make the following definition.

Definition 3.1 A stability condition is *generic* with respect to v if it does not lie on a wall.

Almost every point $\sigma \in \text{Stab}^\dagger(\mathcal{D})$ is generic with respect to all Mukai vectors. Indeed, each Mukai vector gives a locally finite set of walls, and there are countably many Mukai vectors, so locally the union of all walls is a measure zero set in $\text{Stab}^\dagger(\mathcal{D})$. The following useful theorem characterizes Mukai vectors of semistable objects.

Theorem 3.2 (Bayer–Macrì, [1, Theorem 2.15]) *Let $v = mv_0$ be a Mukai vector with v_0 primitive and $m > 0$, and let $\sigma \in \text{Stab}^\dagger(\mathcal{D})$ be a generic stability condition with respect to v , then v supports a semistable object if and only if $v_0^2 \geq -2$.*

For a $\sigma \in \text{Stab}^\dagger(\mathcal{D})$ that is generic with respect to all Mukai vectors, Theorem 3.2 immediately gives that

$$N_\sigma(R) = \#\{v \in N(\mathcal{D}) : v = mv_0, m \in \mathbb{Z}_+, v_0 \text{ primitive}, v_0^2 \geq -2, |Z_\sigma(v)| \leq R\}. \quad (3.2)$$

Note that there is no $v \in N(\mathcal{D})$ with $v^2 = -1$ because the Mukai lattice is even, so

$$\begin{aligned} N_\sigma(R) = & (\#\{v \in N(\mathcal{D}) : v^2 \geq 0, |Z_\sigma(v)| \leq R\}) \\ & + (\#\{v \in N(\mathcal{D}) : v = mv_0, m \in \mathbb{Z}_+, v_0^2 = -2, |Z_\sigma(v)| \leq R\}). \end{aligned} \quad (3.3)$$

In the equation above, we didn't explicitly specify that v_0 is primitive, but it is, since any v_0 such that $v_0^2 = -2$ is primitive.

Let us first discuss the second term in Eq. (3.3). There is actually an upper bound on how large m can be. Due to the support property in the definition of locally finite numerical Bridgeland stability conditions introduced in Sect. 2.3.2, the systole introduced in [10], which is defined as

$$\text{sys}(\sigma) := \min\{|Z_\sigma(v(E))| : E \text{ is } a\sigma\text{-semistable object}\}, \quad (3.4)$$

is attained and is a positive number, see [10, Remark 2.5]. Consequently, if $v = mv_0$ and $|Z_\sigma(v)| < R$, then we must have $m < \frac{R}{\text{sys}(\sigma)}$ since $|Z_\sigma(v)| = m|Z_\sigma(v_0)|$ and $|Z_\sigma(v_0)| \geq \text{sys}(\sigma)$. So the second term in Eq. (3.3) is bounded by

$$\begin{aligned} \#\{v \in N(\mathcal{D}) : v = mv_0, m \in \mathbb{Z}_+, v_0^2 = -2, |Z_\sigma(v)| \leq R\} \\ < \frac{R}{\text{sys}(\sigma)} \cdot \#\{v \in N(\mathcal{D}) : v^2 = -2, |Z_\sigma(v)| \leq R\}. \quad (3.5) \\ = o(R^{\rho+2}) \end{aligned}$$

The last line of the above equation is due to the fact that $\#\{v \in N(\mathcal{D}) : v^2 = -2, |Z_\sigma(v)| \leq R\}$ is of order R^ρ as explained in Lemma 3.3 below. So when multiplied with $\frac{R}{\text{sys}(\sigma)}$, overall we have $o(R^{\rho+2})$.

Lemma 3.3 *The count $\#\{v \in N(\mathcal{D}) : v^2 = -2, |Z_\sigma(v)| \leq R\}$ is of order R^ρ .*

Proof This is a consequence of the main theorem of [7, Theorem 1.2], and let us explain that we are indeed in the correct setting to this theorem. For $v = (r, D = (D_1, \dots, D_\rho), s) \in H^0(X, \mathbb{Z}) \oplus \text{NS}(X) \oplus H^4(X, \mathbb{Z})$, we have $F(v) = v^2 = D^2 - 2rs$ is an integral polynomial. So we are looking at the asymptotics as $R \rightarrow \infty$ of the number of integer points in the affine homogeneous variety $V = \{x \in \mathbb{C}^{\rho+2} : F(x) = -2\}$ and such that the positive semidefinite norm given by $|Z_\sigma(x)|^2$ is less than R^2 . Note that F is a quadratic form of signature $(2, \rho)$. (This is exactly the setting of [7, Example 1.5], which start out with F being a quadratic form of an arbitrary signature and then go on to analyze a more specialized case where the signature of F is different from ours.) The variety V is a symmetric space of the form $\text{SO}(2, \rho - 1) \backslash \text{SO}(2, \rho)$. Denote by $H = \text{SO}(2, \rho - 1)$. By the main theorem of [7], if $\text{vol}(H(\mathbb{Z}) \backslash H(\mathbb{R})) < \infty$, then $\#\{v \in V(\mathbb{Z}) : |Z_\sigma(v)|^2 < R^2\}$ is of order R^ρ ; otherwise, it is of order $R^\rho \log R$. So, in either case, this count is of order $o(R^{\rho+1})$. For our H , we in fact have $\text{vol}(H(\mathbb{Z}) \backslash H(\mathbb{R})) < \infty$ due to the fundamental theorem of Borel and Harish-Chandra. $\square$

Below we will discuss the first term in Eq. (3.3). We will use Lemma 3.4 below, which is a classical result.

Lemma 3.4 (Gauss Circle Problem) *Let $\Upsilon(1)$ be a closed subset of $\mathbb{R}^n$ with a piecewise smooth boundary of measure zero. Let $\Upsilon(R) = \{Rx \mid x \in \Upsilon(1)\}$ be the R -dilate of $\Upsilon(1)$. Let $N_\Upsilon(R) = \#\{\Upsilon(R) \cap \mathbb{Z}^n\}$ be the number of lattice points inside $\Upsilon(R)$. Then*

$$\lim_{R \rightarrow \infty} \frac{N_\Upsilon(R)}{R^n} = \text{Vol}(\Upsilon(1)).$$

We see that the first term in Eq. (3.3) is the count of the number of lattice points in the set

$$\Upsilon_\sigma(R) = \{v \in N(\mathcal{D})_{\mathbb{R}} \cong \mathbb{R}^{\rho+2}, v^2 \geq 0, |Z_\sigma(v)|^2 \leq R^2\}.$$

Because $|Z_\sigma(v)|^2$ is a positive semidefinite quadratic form, $\Upsilon_\sigma(1)$ is a closed set with piecewise smooth boundary of measure zero. Using Lemma 3.4 with $\Upsilon = \Upsilon_\sigma$ and $n = \rho + 2$, we get that $\lim_{R \rightarrow \infty} \frac{N_{\Upsilon_\sigma}(R)}{R^{\rho+2}} = \text{Vol}(\Upsilon_\sigma(1))$.

Putting the two terms of Eq. (3.3) together, we have

$$N_{\sigma}(R) = C(\sigma)R^{\rho+2} + o(R^{\rho+2}), \quad (3.6)$$

where

$$C(\sigma) = \text{Vol}(\Upsilon_{\sigma}(1)) = \text{Vol}\{v \in N(\mathcal{D})_{\mathbb{R}} : v^2 \geq 0, |Z_{\sigma}(v)|^2 \leq 1\}. \quad (3.7)$$

Geometric stability conditions of phase 1

For $\sigma \in V(\mathcal{D})$ a geometric stability condition of phase 1, $Z_{\sigma}(v)$ is given by Eq. (2.32), so according to Eq. (3.7), the leading term of $N_{\sigma}(R)$ is equal to $C(\sigma)R^{\rho+2}$, where

$$C(\sigma) = \text{Vol}\left(\left\{v \in N(\mathcal{D})_{\mathbb{R}} : v^2 \geq 0, \langle \text{Re}(\varphi), v \rangle^2 + \langle \text{Im}(\varphi), v \rangle^2 \leq 1\right\}\right) \quad (3.8)$$

and

$$\varphi = \exp(B + i\omega) = \left(1, B, \frac{B^2 - \omega^2}{2}\right) + i(0, \omega, B \cdot \omega) \in N(\mathcal{D})_{\mathbb{C}}. \quad (3.9)$$

We certainly have $\text{Re}(\varphi), \text{Im}(\varphi) \in N(\mathcal{D})_{\mathbb{R}}$, so we can write any $v \in N(\mathcal{D})_{\mathbb{R}}$ as

$$v = \alpha_1 \text{Re}(\varphi) + \alpha_2 \text{Im}(\varphi) + \beta, \quad (3.10)$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\beta \in \langle \text{Re}(\varphi), \text{Im}(\varphi) \rangle^{\perp_{N(\mathcal{D})_{\mathbb{R}}}}$.

Recall that $\{\text{Re}(\varphi), \text{Im}(\varphi)\} \subseteq N(\mathcal{D})_{\mathbb{R}}$ forms an orthogonal basis of a positive definite 2-plane, hence $\langle \text{Re}(\varphi), \text{Im}(\varphi) \rangle^{\perp_{N(\mathcal{D})_{\mathbb{R}}}}$ is of signature $(0, \rho)$. Let $\{w_1, \dots, w_{\rho}\}$ be a basis of $\langle \text{Re}(\varphi), \text{Im}(\varphi) \rangle^{\perp_{N(\mathcal{D})_{\mathbb{R}}}}$ such that $\langle w_i, w_j \rangle = -\delta_{ij}$. Then we can write β as

$$\beta = \beta_1 w_1 + \dots + \beta_{\rho} w_{\rho} \text{ for } \beta_j \in \mathbb{R}. \quad (3.11)$$

Then

$$v^2 \geq 0 \iff \omega^2(\alpha_1^2 + \alpha_2^2) \geq \beta_1^2 + \dots + \beta_{\rho}^2, \quad (3.12)$$

and

$$\langle \text{Re}(\varphi), v \rangle^2 + \langle \text{Im}(\varphi), v \rangle^2 \leq 1 \iff \frac{1}{(\omega^2)^2} \geq \alpha_1^2 + \alpha_2^2. \quad (3.13)$$

To compute the volume of $\{\alpha_1, \alpha_2, \beta_1, \dots, \beta_{\rho}\}$ satisfying Eqs. (3.12) and (3.13), we introduce a coordinate r such that $\alpha_1^2 + \alpha_2^2 = r^2$, then

$$0 \leq r \leq \frac{1}{\omega^2} \text{ and } \beta_1^2 + \dots + \beta_{\rho}^2 \leq \omega^2 r^2. \quad (3.14)$$

Hence the volume with respect to $\{\alpha_1, \alpha_2, \beta_1, \dots, \beta_{\rho}\}$ is:

$$\begin{aligned} & 2\pi \int_0^{\frac{1}{\omega^2}} r \cdot \text{Vol} B_{\rho}(r\sqrt{\omega^2}) dr \\ &= 2\pi \int_0^{\frac{1}{\omega^2}} r \cdot \frac{\pi^{\rho/2} (\omega^2)^{\rho/2} r^{\rho}}{\Gamma(\frac{\rho}{2} + 1)} dr \\ &= \frac{2\pi^{(\rho+2)/2}}{(\rho+2)\Gamma(\frac{\rho}{2} + 1)(\omega^2)^{(\rho+4)/2}}. \end{aligned} \quad (3.15)$$

In order to get the volume in Eq. (3.8), we need to analyze the effect of the change-of-basis from the standard basis $\mathbf{e} = (e_1, \dots, e_{\rho+2})$ of $H^0(X, \mathbb{Z}) \oplus NS(X) \oplus H^4(X, \mathbb{Z})$ to the basis

$\mathbf{f} = (f_1, \dots, f_{\rho+2})$ where $f_1 = \operatorname{Re}(\varphi)$, $f_2 = \operatorname{Im}(\varphi)$, and $f_j = w_{j-2}$ for $j = 3, \dots, \rho + 2$. Denote by A the change of basis matrix, i.e. $A\mathbf{e} = \mathbf{f}$. Then

$$\langle \mathbf{f}_i, \mathbf{f}_j \rangle = \left\langle \sum_k a_{ik} e_k, \sum_\ell a_{j\ell} e_\ell \right\rangle = \sum_{k,\ell} a_{ik} a_{j\ell} \langle e_k, e_\ell \rangle, \text{ i.e. } A[\langle e_i, e_j \rangle] A^T = [\langle f_i, f_j \rangle],$$

where $[\langle e_i, e_j \rangle]$ and $[\langle f_i, f_j \rangle]$ denote the Gram matrices for $\mathbf{e}$ and $\mathbf{f}$, respectively. So then

$$(\det A)^2 = \frac{\det[\langle f_i, f_j \rangle]}{\det[\langle e_i, e_j \rangle]}.$$

Since $\det[\langle f_i, f_j \rangle] = (\omega^2)^2$ and $\det[\langle e_i, e_j \rangle] = \operatorname{DiscNS}(X)$, we get that the determinant of the change of basis matrix is

$$\det(A) = \frac{\omega^2}{\sqrt{|\operatorname{DiscNS}(X)|}}. \quad (3.16)$$

Hence the final result is

$$C(\sigma) = \frac{2\pi^{(\rho+2)/2}}{(\rho+2)\Gamma(\frac{\rho}{2}+1)(\omega^2)^{(\rho+2)/2}\sqrt{|\operatorname{DiscNS}(X)|}}. \quad (3.17)$$

We see that the leading coefficient depends on the following geometric data

- rank and discriminant of $\operatorname{NS}(X) = \langle \Omega \rangle^{\perp_{H^2(X, \mathbb{Z})}}$, and
- ω^2 (the “B-field” B does not matter).

Geometric stability conditions of any phase

For a geometric stability condition $\sigma \in U(\mathcal{D})$ of phase other than 1, we know there is a unique $g \in \operatorname{GL}^+(2, \mathbb{R})$ such that $Z_\sigma \cdot g = Z_{\sigma'} \in L(\mathcal{D})$ for some $\sigma' \in V(\mathcal{D})$. Note that $\operatorname{GL}^+(2, \mathbb{R}) \cong \mathbb{R}^+ \times \operatorname{SL}(2, \mathbb{R})$. If $g \in \mathbb{R}^+$, it is simply a scaling $Z_{\sigma'} = gZ_\sigma$, so $N_{\sigma'}(R) = N_\sigma(R/g)$, and so $C(\sigma') = C(\sigma)/g^{\rho+2}$. If g belongs to the rotation part of $\operatorname{SL}(2, \mathbb{R})$ doesn’t change the norm $|Z_{\sigma'}| = |Z_\sigma|$, so $N_{\sigma'}(R) = N_\sigma(R)$. So the only thing left to consider a shear by $\kappa + i\lambda$ with $\lambda > 0$, which is represented by the matrix $\begin{pmatrix} 1 & \kappa \\ 0 & \lambda \end{pmatrix}$ so $Z_{\sigma'} = \operatorname{Re} Z_\sigma + i(\kappa \operatorname{Re} Z_\sigma + \lambda \operatorname{Im} Z_\sigma)$. Then Eq. (3.7) in this context is

$$C(\sigma) = \operatorname{Vol}\{v \in N(\mathcal{D})_{\mathbb{R}}, v^2 \geq 0, \langle \operatorname{Re}(\varphi), v \rangle^2 + (\langle \operatorname{Re}(\varphi), v \rangle \kappa + \langle \operatorname{Im}(\varphi), v \rangle \lambda)^2 \leq 1\}. \quad (3.18)$$

The computation is very similar to what we did above in the phase 1 case. In exactly the same way, we write $v = \alpha_1 \operatorname{Re}(\varphi) + \alpha_2 \operatorname{Im}(\varphi) + \beta$, where $\beta = \beta_1 w_1 + \dots + \beta_\rho w_\rho$ as in Eq. (3.11). So the determinant of the change of basis matrix for $\{\operatorname{Re} \varphi, \operatorname{Im} \varphi, w_1, \dots, w_\rho\}$ is again $\frac{\omega^2}{\sqrt{|\operatorname{DiscNS}(X)|}}$. Then the $v^2 \geq 0$ condition is the same as in Eq. (3.12), and Eq. (3.13) in this context is

$$\langle \operatorname{Re}(\varphi), v \rangle^2 + (\langle \operatorname{Re}(\varphi), v \rangle \kappa + \langle \operatorname{Im}(\varphi), v \rangle \lambda)^2 \leq 1 \Leftrightarrow \alpha_1^2 + (\kappa \alpha_1 + \lambda \alpha_2)^2 \geq \left(\frac{1}{\omega^2}\right)^2. \quad (3.19)$$

Again, we can compute $C(\sigma)$ given in Eq. (3.18) by computing the volume of $\{\alpha_1, \alpha_2, \beta_1, \dots, \beta_\rho\}$ satisfying Eqs. (3.12) and (3.19), and then multiply by the determinant of the change of basis matrix for $\{\operatorname{Re}(\varphi), \operatorname{Im}(\varphi), w_1, \dots, w_\rho\}$. To compute the volume of

$\{\alpha_1, \alpha_2, \beta_1, \dots, \beta_\rho\}$ satisfying the two conditions, we change to polar coordinates by letting

$$\alpha_1 = r \cos \theta, \quad \kappa \alpha_1 + \lambda \alpha_2 = r \sin \theta. \quad (3.20)$$

Then $0 \leq r \leq \frac{1}{\omega^2}$ and

$$\beta_1^2 + \dots + \beta_\rho^2 \leq \omega^2(\alpha_1^2 + \alpha_2^2) = r^2 \omega^2 \left(\cos^2 \theta + \frac{1}{\lambda^2} (\sin \theta - \kappa \cos \theta)^2 \right). \quad (3.21)$$

From Eq. (3.20), we see that the Jacobian determinant for the change of variable between (α_1, α_2) and (r, θ) is

$$\left| \frac{\partial(\alpha_1, \alpha_2)}{\partial(r, \theta)} \right| = \det \begin{bmatrix} \cos \theta & -r \sin \theta \\ \frac{1}{\lambda} (\sin \theta - \kappa \cos \theta) & \frac{r}{\lambda} (\cos \theta + \kappa \sin \theta) \end{bmatrix} = \frac{r}{\lambda}. \quad (3.22)$$

So the volume with respect to $\{\alpha_1, \alpha_2, \beta_1, \dots, \beta_\rho\}$ is:

$$\begin{aligned} & \int_0^{\frac{1}{\omega^2}} \int_0^{2\pi} \left| \frac{\partial(\alpha_1, \alpha_2)}{\partial(r, \theta)} \right| \text{Vol} B_\rho \left(r \sqrt{\omega^2 \left(\cos^2 \theta + \frac{1}{\lambda^2} (\sin \theta - \kappa \cos \theta)^2 \right)} \right) d\theta dr \\ &= \int_0^{\frac{1}{\omega^2}} \int_0^{2\pi} \frac{r}{\lambda} \frac{\pi^{\rho/2} (\omega^2)^{\rho/2} r^\rho}{\Gamma(\frac{\rho}{2} + 1)} \left(\cos^2 \theta + \frac{1}{\lambda^2} (\sin \theta - \kappa \cos \theta)^2 \right)^{\rho/2} d\theta dr \\ &= \frac{\pi^{\rho/2} (\omega^2)^{\rho/2}}{\lambda \Gamma(\frac{\rho}{2} + 1)} \int_0^{\frac{1}{\omega^2}} r^{\rho+1} dr \int_0^{2\pi} \left(\cos^2 \theta + \frac{1}{\lambda^2} (\sin \theta - \kappa \cos \theta)^2 \right)^{\rho/2} d\theta \\ &= \frac{\pi^{\rho/2}}{(\rho+2) \Gamma(\frac{\rho}{2} + 1) \lambda (\omega^2)^{(\rho+4)/2}} \int_0^{2\pi} \left(\cos^2 \theta + \frac{1}{\lambda^2} (\sin \theta - \kappa \cos \theta)^2 \right)^{\rho/2} d\theta. \end{aligned} \quad (3.23)$$

Then multiplying by $\frac{\omega^2}{\sqrt{|\text{Disc}(\text{NS}(X))|}}$ gives

$$C(\sigma) = \frac{\pi^{\rho/2} \int_0^{2\pi} \left(\cos^2 \theta + \frac{1}{\lambda^2} (\sin \theta - \kappa \cos \theta)^2 \right)^{\rho/2} d\theta}{(\rho+2) \Gamma(\frac{\rho}{2} + 1) \lambda (\omega^2)^{(\rho+2)/2} \sqrt{|\text{Disc}(\text{NS}(X))|}}. \quad (3.24)$$

One can see that for $\kappa = 0, \lambda = 1$, this matches the phase 1 case. Again we see that this leading coefficient does not depend on the B -field just like the phase 1 case.

4 Counting special Lagrangian classes

In this section, we explain Theorems 1.2 and 1.3. First, we consider

$$SL_{\omega, \Omega}(R) = \#\{\gamma \in \text{SLag}(X, \omega, \Omega) : |\gamma \cdot \Omega| \leq R\} \subset H^2(X, \mathbb{Z}). \quad (4.1)$$

By Lemma 2.2,

$$SL_{\omega, \Omega}(R) \leq \#\{\gamma \in \text{Lag}(X, \omega) : \gamma^2 \geq -2, |\gamma \cdot \Omega| \leq R\}. \quad (4.2)$$

Since γ^2 is an indefinite quadratic form and $|\gamma \cdot \Omega|^2$ is a positive semidefinite quadratic form, we can use Lemma 4.1 below to conclude that

$$SL_{\omega, \Omega}(R) = C(\omega, \Omega) R^{\rho+2} + o(R^{\rho+2}). \quad (4.3)$$

where

$$C(\omega, \Omega) = \text{Vol}\{\gamma \in \text{Lag}(X, \omega)_{\mathbb{R}} : \gamma^2 \geq 0, (\gamma \cdot \Omega)^2 = (\gamma \cdot \text{Re}\Omega)^2 + (\gamma \cdot \text{Im}\Omega)^2 \leq 1\}. \quad (4.4)$$

Note that even though $\text{Re}\Omega$ and $\text{Im}\Omega$ are orthogonal to ω , they might not be in $\text{Lag}(X, \omega)_{\mathbb{R}}$. This is because $\text{Lag}(X, \omega) \subset H^2(X, \mathbb{Z})$ consists of integral vectors orthogonal to ω , which might not generate the space that $\text{Re}\Omega$ and $\text{Im}\Omega$ live in. If we work with the case where (X, ω) is such that $\text{Lag}(X, \omega)$ has full rank, i.e. with signature $(2, 19)$, then $\text{Re}\Omega$ and $\text{Im}\Omega$ are both in $\text{Lag}(X, \omega)_{\mathbb{R}}$. In this case the calculation is exactly the same as the calculation we did in Sect. 3 for the geometric stability conditions of phase 1, the only difference being that we replace ρ in that calculation by 19 here, and replace ω^2 there by $K_{\Omega} = (\text{Re}\Omega)^2 = (\text{Im}\Omega)^2$ here. We get

$$C(\omega, \Omega) = \frac{2\pi^{21/2}}{21\Gamma(\frac{21}{2})K_{\Omega}^{21/2}\sqrt{|\text{DiscLag}(X, \omega)|}}. \quad (4.5)$$

This case where $\text{Lag}(X, \omega)$ having signature $(2, 19)$ can be achieved when ω is rational, i.e. in $H^2(X, \mathbb{Q})$; in particular, it can be achieved for polarized K3 surfaces (X, ω) .

Lemma 4.1 *Suppose P is a positive semidefinite quadratic form and Q is any quadratic form. Then for*

$$N_{P,Q}(R) = \#\{x \in \mathbb{Z}^n : P(x) \leq R^2, Q(x) \geq -c\},$$

where c is a positive constant, we have

$$N_{P,Q}(R) = C_{P,Q}R^n + o(R^n),$$

where

$$C_{P,Q} = \text{Vol}\{x \in \mathbb{R}^n : P(x) \leq 1, Q(x) \geq 0\}.$$

Proof Let

$$\Upsilon_{P,Q}(R) = \{x \in \mathbb{R}^n : P(x) \leq R^2, Q(x) \geq 0\}.$$

Using Lemma 3.4 with $\Upsilon = \Upsilon_{P,Q}$, we get that

$$\lim_{R \rightarrow \infty} \frac{\#\{x \in \mathbb{Z}^n \cap \Upsilon_{P,Q}(R)\}}{R^n} = \text{Vol}(\Upsilon_{P,Q}(1))$$

Now consider the remaining points $\mathbb{Z}^n \cap \Upsilon_{P,Q}^c(R)$, where

$$\Upsilon_{P,Q}^c(R) := \{x \in \mathbb{R}^n : P(x) \leq R^2, Q(x) \in [-c, 0)\}.$$

Note that

$$\Upsilon_{P,Q}^c(R) = R \cdot \tilde{\Upsilon}_{P,Q}^c(R), \text{ where } \tilde{\Upsilon}_{P,Q}^c(R) = \left\{x \in \mathbb{R}^n : P(x) \leq 1, Q(x) \in \left[-\frac{c}{R^2}, 0\right]\right\}.$$

Whenever $R > R_0$ for some fixed R_0 , we have $\Upsilon_{P,Q}^c(R) \subset R \cdot \tilde{\Upsilon}_{P,Q}^c(R_0)$, so

$$\lim_{R \rightarrow \infty} \frac{\#\{x \in \Upsilon_{P,Q}^c(R) \cap \mathbb{Z}^n\}}{R^n} \leq \text{Vol}(\tilde{\Upsilon}_{P,Q}^c(R_0)).$$

Because the boundary $\{x \in \mathbb{R}^n : P(x) \leq 1, Q(x) = 0\}$ has measure zero, the $\text{Vol}(\tilde{\Upsilon}_{P,Q}^c(R_0))$ can be made arbitrarily small by taking R_0 to be arbitrarily large. So the number of remaining points $\#\{x \in \Upsilon_{P,Q}^c(R) \cap \mathbb{Z}^n\} = o(R^n)$. $\square$

Now let us consider the twistor plane

$$P := \langle \omega, \operatorname{Re}\Omega, \operatorname{Im}\Omega \rangle \subset H^2(X, \mathbb{R}), \quad (4.6)$$

where $\omega, \operatorname{Re}\Omega, \operatorname{Im}\Omega$ are normalized so they form an orthonormal basis of P as introduced in Sect. 2.2.4. The twistor plane defines a seminorm $\|\cdot\|_P$ on $H^2(X, \mathbb{Z})$, which is for $\gamma \in H^2(X, \mathbb{Z})$,

$$\|\gamma\|_P := \sup_{u \in P, u^2=1} \gamma \cdot u. \quad (4.7)$$

Since $H^2(X, \mathbb{R}) = P \oplus P^\perp$, for any $\gamma \in H^2(X, \mathbb{Z})$, we can write $\gamma = \gamma_P \oplus \gamma_{P^\perp}$. Then $\|\gamma\|_P$ is the norm of the projection γ_P of γ onto the 3-plane P .

For a fixed P , instead of fixing an ω like we did in the count of $SL_{\omega, \Omega}(R)$, here we incorporate all $\omega_t \in \mathbb{S}^2(P)$ by considering

$$SL_P(R) = \# \{ \gamma \in H^2(X, \mathbb{Z}) : \exists \omega_t \in \mathbb{S}^2(P), \gamma \in \operatorname{SLag}(X, \omega_t, \Omega_t), \text{ and } |\gamma \cdot \Omega_t| \leq R \}. \quad (4.8)$$

Note that $|\gamma \cdot \Omega_t|$ is the length of the projection of γ to the positive 2-plane spanned by orthonormal vectors $\{\operatorname{Re}\Omega_t, \operatorname{Im}\Omega_t\}$. So if there is a t such that $\gamma \cdot \omega_t = 0$, i.e. γ is a Lagrangian with respect to ω_t , then

$$|\gamma \cdot \Omega_t| = \|\gamma\|_P. \quad (4.9)$$

The characterization given by Lemma 2.2 for $\gamma \in \operatorname{SLag}(X, \omega_t, \Omega_t)$ implies that

$$\begin{aligned} SL_P(R) &\leq \# \{ \gamma \in H^2(X, \mathbb{Z}) : \gamma^2 \geq -2, \exists \omega_t \in \mathbb{S}^2(P) \text{ s.t. } \gamma \cdot \omega_t = 0, \text{ and } \|\gamma\|_P \leq R \} \\ &= \# \{ \gamma \in H^2(X, \mathbb{Z}) : \gamma^2 \geq -2, \|\gamma\|_P \leq R \}. \end{aligned} \quad (4.10)$$

The last equality is due to the fact that for any $\gamma \in H^2(X, \mathbb{Z})$, it is always in the Lagrangian class for some ω_t , i.e. $\gamma \cdot \omega_t = 0$ since we can always project γ onto P and then choose ω_t to be orthogonal to that.

We can then use Lemma 4.1, since $\|\gamma\|_P^2$ is positive semidefinite, to conclude that the leading term of $\# \{ \gamma \in H^2(X, \mathbb{Z}) : \gamma^2 \geq -2, \|\gamma\|_P^2 \leq R^2 \}$ is given by the leading term of the counting function $\# \{ \gamma \in H^2(X, \mathbb{Z}) : \gamma^2 \geq 0, \|\gamma\|_P^2 \leq R^2 \}$. Then using Lemma 4.2 below, and combining with Eq. (4.10), we can conclude that

$$SL_P(R) \leq C \cdot R^{22} + o(R^{22}), \quad (4.11)$$

where C is a constant independent of the choice of $P \subset \Lambda_{\mathbb{R}}$.

Lemma 4.2 *Let Q be a signature (m, n) quadratic form on $\mathbb{R}^d$, with $d = m + n$. Given $v \in \mathbb{R}^d$ and an m -dimensional subspace $P \subset \mathbb{R}^d$ that is positive definite (i.e. for all nonzero $u \in P$, we have $Q(u, u) > 0$), define*

$$\|v\|_P = \max \{ Q(v, u) : u \in P, Q(u, u) = 1 \}.$$

Then the asymptotics of the counting function

$$N_P(R) = \# \{ v \in \mathbb{Z}^d : Q(v, v) \geq 0, \|v\|_P \leq R \}$$

is given by

$$N_P(R) = c_P R^d + o(R^d),$$

where the constant

$$c_P = \text{Vol}\{v \in \mathbb{R}^d : Q(v, v) \geq 0, \|v\|_P \leq 1\}$$

is independent of the choice of m -dimensional positive definite subspace P .

Proof Because P is positive definite, the set

$$\Upsilon_P(R) := \{v \in \mathbb{R}^d : Q(v, v) \geq 0, \|v\|_P \leq R\} = \{Rv : v \in \Upsilon_P(1)\} = R \cdot \Upsilon_P(1)$$

is a R -dilate of $\Upsilon_P(1)$. Using Lemma 3.4 with $\Upsilon = \Upsilon_P$ and $n = d$, we get that $N_P(R) = c_P R^d + o(R^d)$, where $c_P = \text{Vol}(\Upsilon_P(1))$.

Now we show that c_P is independent of the choice of P . Let $\mathcal{P}$ denote the collection of m -dimensional positive definite subspaces P . Our claim will follow from the fact that $G = SO(Q)$ acts transitively on $\mathcal{P}$. Note that if $P \in \mathcal{P}$, gP is also in $\mathcal{P}$ since g preserves Q . This is due to Sylvester's law of inertia.

Next note that

$$\begin{aligned} \|v\|_{gP} &= \max\{Q(v, u) : u \in gP, Q(u, u) = 1\} \\ &= \max\{Q(v, gu) : u \in P, Q(u, u) = 1\} \\ &= \max\{Q(g^{-1}v, u) : u \in P, Q(u, u) = 1\} \\ &= \|g^{-1}v\|_P. \end{aligned}$$

Therefore,

$$\begin{aligned} \{v \in \mathbb{R}^d : Q(v, v) \geq 0, \|v\|_{gP} \leq 1\} \\ &= \{v \in \mathbb{R}^d : Q(v, v) \geq 0, \|g^{-1}v\|_P \leq 1\} \\ &= \{gw \in \mathbb{R}^d : Q(gw, gw) \geq 0, \|w\|_P \leq 1\} \\ &= \{gw \in \mathbb{R}^d : Q(w, w) \geq 0, \|w\|_P \leq 1\} \\ &= g\{w \in \mathbb{R}^d : Q(w, w) \geq 0, \|w\|_P \leq 1\}, \end{aligned}$$

and so $c_{gP} = c_P$, since the action of $G = SO(Q)$ preserves Lebesgue measure on $\mathbb{R}^d$. $\square$

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Declarations

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