

# COUNTING PAIRS OF SADDLE CONNECTIONS

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ABSTRACT. We show that for almost every translation surface the number of pairs of saddle connections with bounded virtual area has asymptotic growth like  $cR^2$  where the constant  $c$  depends only on the area and the connected component of the stratum. The proof techniques combine classical results for counting saddle connections with the crucial result that the Siegel-Veech transform is in  $L^2$ . In order to capture information about pairs of saddle connections, we consider pairs with bounded virtual area since the set of such pairs can be approximated by a fibered set which is equivariant under geodesic flow. In the case of lattice surfaces, small virtual area is equivalent to counting parallel pairs of saddle connections, which also have a quadratic growth of  $cR^2$  where  $c$  depends in this case on the given lattice surface.

## 1. INTRODUCTION

A translation surface  $(X, \omega)$  is a pair consisting of a compact Riemann surface  $X$  and  $\omega$ , a non-zero holomorphic one-form. For succinctness we denote a translation surface by  $\omega$  where the underlying Riemann surface is understood. A *saddle connection* on  $\omega$  is a geodesic in the flat metric determined by  $\omega$  connecting two zeros of  $\omega$  with no zeros in its interior. Let  $\text{SC}_\omega$  be the set of saddle connections on  $\omega$ . For  $\gamma \in \text{SC}_\omega$ , the associated *holonomy vector* is given by

$$z_\gamma = \int_\gamma \omega \in \mathbb{C}.$$

Let

$$\Lambda_\omega = \{z_\gamma : \gamma \in \text{SC}_\omega\}$$

denote the set of holonomy vectors of saddle connections on  $\omega$ . This is a countable discrete subset of the plane  $\mathbb{C}$ . The *length*  $\ell(\gamma)$  of a saddle connection  $\gamma$  is

$$\ell(\gamma) = |z_\gamma|.$$

For  $R > 0$ , let  $\Lambda_\omega(R) = \Lambda_\omega \cap B(0, R)$  be the collection of holonomy vectors of saddle connections with length at most  $R$ . We are interested in the distribution of *pairs* of saddle connections, in particular the growth rate of the following counting function, the count of pairs of bounded *virtual area*. Fix  $A > 0$ , and define

$$N_A(\omega, R) = \#\{(z, w) \in \Lambda_\omega(R)^2 : |z \wedge w| \leq A, |w| \leq |z|\},$$

where for  $z = x + iy, w = u + iv$ , the signed area of the parallelogram spanned by the column vectors in  $\mathbb{R}^2$  associated to  $z$  and  $w$  is denoted by  $\det(z|w) = xv - yu$ , and the area is denoted by

$$|z \wedge w| = |\det(z|w)| = |xv - yu| = |\text{Im}(\bar{z}w)|.$$

The moduli space  $\Omega_g$  of compact genus  $g$  area 1 translation surfaces (where  $(X_1, \omega_1) \sim (X_2, \omega_2)$  if there is a biholomorphism  $f : X_1 \rightarrow X_2$  with  $f_*\omega_2 = \omega_1$ ) is stratified by integer partitions of  $2g - 2$  (fixing the orders of the zeros of  $\omega$ ). The area of a surface  $\omega$  is given by

$$\text{Area}(\omega) = \frac{i}{2} \int_X \omega \wedge \bar{\omega}.$$

These strata have at most 3 connected components [KZ03], and each connected component  $\mathcal{H}$  carries a natural Lebesgue probability measure  $\mu = \mu_{\mathcal{H}}$  [Mas82, Vee82]. We fix  $\mathcal{H}$  to be a connected component of a stratum. Our main result is an almost sure asymptotic growth result for the set of pairs of saddle connections with bounded virtual area.

**Theorem 1.1.** *There is a constant  $c_A = c_A(\mu)$  such that for  $\mu$ -almost every  $\omega \in \mathcal{H}$*

$$\lim_{R \rightarrow \infty} \frac{N_A(\omega, R)}{R^2} = c_A.$$

**1.1. History and prior results.** The study of counting problems for saddle connections is very active, and connected to many different areas of mathematics, from low-dimensional dynamical systems to algebraic geometry. Motivated by problems in counting special trajectories for billiards in rational polygons, Masur [Mas90] proved that the counting function

$$N(\omega, R) = \#\Lambda_\omega(R)$$

has quadratic upper and lower bounds for all  $\omega$ , that is, there are  $0 < c_1 = c_1(\omega) < c_2 = c_2(\omega)$  so that for all  $R$ ,

$$(1.1) \quad c_1 R^2 \leq N(\omega, R) \leq c_2 R^2.$$

Subsequently, Veech [Vee98] showed there is a constant  $c = c(\mathcal{H})$  such that

$$\lim_{R \rightarrow \infty} \int_{\mathcal{H}} \left| \frac{N(\omega, R)}{R^2} - c \right| d\mu(\omega) = 0,$$

an  $L^1$ -quadratic asymptotic result. Inspired by Veech's approach, Eskin-Masur [EM01] adapted ideas from homogeneous dynamics (specifically, the work of Eskin-Margulis-Mozes [EMM95, EMM98] on quantitative versions of Oppenheim's conjecture) and an ergodic theorem of Nevo [Nev17] to improve this to a pointwise asymptotic result, showing that for  $\mu$ -almost every  $\omega \in \mathcal{H}$ ,

$$\lim_{R \rightarrow \infty} \frac{N(\omega, R)}{R^2} = c.$$

More recently, Nevo-Rühr-Weiss [NRW20], using error term estimates in Nevo's ergodic theorem coming from mixing properties of the *Teichmüller geodesic flow*, showed that there is an  $\alpha < 2$  such that for almost every  $\omega \in \mathcal{H}$ ,

$$N(\omega, R) = cR^2 + o(R^\alpha).$$

Our approach uses ideas from all of these results: we will, using ideas similar to Eskin-Masur [EM01], set up our counting problem as an integral over a piece of an  $SL(2, \mathbb{R})$ -orbit on  $\mathcal{H}$ , and then apply the ergodic theorem of Nevo [Nev17]. To implement our strategy, we will need upper bounds in the spirit of [Mas90], and approximation ideas carefully implemented in [NRW20].

**1.2. The Siegel-Veech transform.** A crucial ingredient in the work of Veech [Vee98] is the *Siegel-Veech transform*. Let  $B_c(X)$  be the space of bounded measurable functions with compact support on a space  $X$ . For  $f \in B_c(\mathbb{C})$ , we define a function  $\hat{f}$  on  $\mathcal{H}$  by

$$\hat{f}(\omega) = \sum_{z \in \Lambda_\omega} f(z).$$

For example, if  $f = \mathbf{1}_{B(0, R)}$  is the indicator function of  $B(0, R)$ ,

$$\hat{f}(\omega) = N(\omega, R).$$

A beautiful result of Veech [Vee98] is the *Siegel-Veech formula*, which states that there is a  $c = c_{SV}$  so that for  $f \in B_c(\mathbb{R})$ ,  $\hat{f} \in L^1(\mathcal{H}, \mu)$  and

$$\int_{\mathcal{H}} \hat{f} d\mu = c \int_{\mathbb{C}} f(z) dz.$$

In fact a crucial ingredient in Eskin-Masur's asymptotic result is that  $\hat{f} \in L^{1+\beta}$  for some  $\beta > 0$ . We will need similar results for a generalized Siegel-Veech transform. Given  $h \in B_c(\mathbb{C}^2)$ , we define a function  $\hat{h}$  on  $\mathcal{H}$  by

$$\hat{h}(\omega) = \sum_{z_1, z_2 \in \Lambda_\omega} h(z_1, z_2).$$

For example, if  $h = \mathbf{1}_{D_A(R)}$  is the indicator function of the set

$$D_A(R) = \{(z, w) \in \mathbb{C}^2 : |w| \leq |z| \leq R, |z \wedge w| \leq A\},$$

then

$$\widehat{h}(\omega) = N_A(\omega, R).$$

In our proof of Theorem 1.1, we rely on a result of Athreya-Cheung-Masur [ACM19] which shows that  $h \in L^{1+\beta}$  for  $h \in B_c(\mathbb{C}^2)$  (which is equivalent to showing that for  $f \in B_c(\mathbb{C})$ ,  $\widehat{f} \in L^{2+\beta}(\mathcal{H})$ ).

**1.2.1. Notation.** Since the functions we are taking transforms of will be sometimes quite complicated to write down, we introduce the following notation given bounded compactly supported functions  $f \in B_c(\mathbb{C})$  or  $h \in B_c(\mathbb{C}^2)$  we write

$$\widehat{f}(\omega) = f^{\text{SV}}(\omega) \text{ and } \widehat{h}(\omega) = h^{\text{SV}}(\omega).$$

**1.3. The  $SL(2, \mathbb{R})$ -action on strata.** There is an action of the group  $SL(2, \mathbb{R})$  on strata. A translation surface  $\omega$  gives an atlas of charts from  $X \setminus \{\omega^{-1}(0)\}$  to  $\mathbb{C}$  whose transition maps are translations: the atlas around a point  $p_0$  is given by

$$z(p) = \int_{p_0}^p \omega.$$

In these coordinates,  $\omega = dz$ . Equivalently, such an atlas of charts determines a pair  $(X, \omega)$ . The group  $GL^+(2, \mathbb{R})$  acts by  $\mathbb{R}$ -linear-postcomposition with charts, and the group  $SL(2, \mathbb{R})$  preserves the set of surfaces with area 1. The measure  $\mu_{\mathcal{H}}$  constructed by Masur and Veech is ergodic and invariant under the  $SL(2, \mathbb{R})$ -action, and is locally given by Lebesgue measure in appropriate coordinates on  $\mathcal{H}$ . Note that the assignment

$$\omega \mapsto \Lambda_\omega$$

is  $SL(2, \mathbb{R})$ -equivariant, that is

$$\Lambda_{g\omega} = g\Lambda_\omega.$$

**1.4. Strategy of proof.** We now outline the strategy of proof of Theorem 1.1. First, we recall the strategy of Eskin-Masur for understanding the counting function  $N(\omega, R)$ : they construct a function  $f \in B_c(\mathbb{C})$  (essentially the indicator function of a trapezoid), which satisfies

$$\frac{1}{2\pi} \int_0^{2\pi} f(g_t r_\theta z) d\theta \approx e^{-2t} \mathbf{1}_{A(\frac{e^t}{2}, e^t)}(z),$$

where the matrices

$$(1.2) \quad g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad r_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

act  $\mathbb{R}$ -linearly on  $\mathbb{C}$ , and for  $0 < R_1 < R_2$ ,

$$A(R_1, R_2) = \{z \in \mathbb{C} : R_1 < |z| < R_2\}.$$

Putting  $e^t = R$ , and adding the above expression over all  $z \in \Lambda_\omega$ , we obtain

$$\frac{1}{R^2} (N(\omega, R) - N(\omega, R/2)) \approx \frac{1}{2\pi} \int_0^{2\pi} \widehat{f}(g_t r_\theta \omega) d\theta.$$

This reduces the counting problem to a problem of understanding the sequence of integrals

$$\frac{1}{2\pi} \int_0^{2\pi} \widehat{f}(g_t r_\theta \omega) d\theta.$$

Nevo's ergodic theorem (Theorem 2.1) deals precisely with integrals of this form, but with some compactness and smoothness assumptions on the integrand. Theorem 2.1 gives that almost surely the the integrals converge to  $\int \widehat{f} d\mu$ . The Siegel-Veech formula is then applied to say that this last integral is  $c \int_{\mathbb{C}} f(z) dz$ .

1.4.1. *Pairs.* In our case we will construct as the main part of the proof, a function  $h_A \in B_c(\mathbb{C}^2)$  so that

$$(1.3) \quad \frac{1}{2\pi} \int_0^{2\pi} h_A(g_t r_\theta(z, w)) \approx \frac{e^{-2t}}{\pi} \mathbf{1}_{D_A(e^t/2, e^t)}(z, w),$$

where for  $R_1 < R_2$ ,

$$D_A(R_1, R_2) = \{(z, w) \in \mathbb{C}^2 : |z \wedge w| \leq A, |w| \leq |z|, R_1 \leq |z| \leq R_2\}$$

and the action of  $SL(2, \mathbb{R})$  on  $\mathbb{C}^2$  is the diagonal  $\mathbb{R}$ -linear action; that is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x + iy \\ u + iv \end{pmatrix} = \begin{pmatrix} (ax + by) + i(cx + dy) \\ (au + bv) + i(cu + dv) \end{pmatrix}.$$

Adding (1.3) over all  $(z, w) \in \Lambda_\omega^2$ , we will prove

$$(1.4) \quad \lim_{R \rightarrow \infty} \left( \frac{1}{\pi R^2} N_A^*(\omega, R) - \frac{1}{2\pi} \int_0^{2\pi} \widehat{h}_A(g_{\log(R)} r_\theta \omega) d\theta \right) = 0,$$

where

$$N_A^*(\omega, R) = N_A(\omega, R) - N_A(\omega, R/2).$$

Once again we will need to show that the limit of the circle averages

$$\lim_{t \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \widehat{h}_A(g_t r_\theta \omega) d\theta$$

exists. To do that we again will implement Nevo's theorem (Theorem 2.1) along with careful analysis of the boundary of the support of  $h_A$ . We will also rely on [ACM19, Theorems 1.2 and 3.4] which shows that there is a  $\kappa > 0$  such that

$$\widehat{h} \in L^{1+\kappa}(\mathcal{H}, \mu),$$

and provides a version of the Siegel-Veech formula which works for functions defined on  $\mathbb{C}^2$ .

**1.5. Lattice surfaces.** Given a surface  $\omega$ , we define its *Veech group*  $SL(\omega)$  to be its stabilizer under the  $SL(2, \mathbb{R})$  action. A class of surfaces where counting problems are well-understood are *lattice surfaces*, surfaces  $\omega$  whose stabilizer  $\Gamma = SL(\omega)$  under the  $SL(2, \mathbb{R})$ -action is a lattice. These are also known as *Veech surfaces*. While lattice surfaces are rare, in the sense they form a set of measure 0 in each stratum, they are a *dense* set in each stratum. See [SW10] and the references within for more details.

1.5.1. *Counting and orbits.* Veech [Vee98] showed that in this setting the set of holonomy vectors  $\Lambda_\omega$  is a finite union of orbits of the Veech group. That is, there is a finite collection of complex numbers  $z_1, z_2, \dots, z_m$  such that

$$(1.5) \quad \Lambda_\omega = \bigcup_{i=1}^m \Gamma z_i.$$

Using this, and techniques from homogeneous dynamics, he proved, for each  $i$ , there is a  $c_i$  so that

$$\#(\Gamma z_i \cap B(0, R)) \sim c_i R^2,$$

and thus overall quadratic asymptotics for  $N(\omega, R)$ .

1.5.2. *No small triangles.* Subsequently, Smillie-Weiss [SW10] gave many equivalent characterizations of lattice surfaces. In particular, they showed that  $(X, \omega)$  is a lattice surface if and only if it satisfies the *no small virtual triangles (NSVT)* condition: there is an  $A_0 > 0$  so that for any non-parallel  $z, w \in \Lambda_\omega$ ,

$$|z \wedge w| > A_0.$$

So for  $A < A_0$ , the problem of understanding  $N_A(\omega, R)$  becomes the problem of counting parallel pairs of vectors in  $\Lambda_\omega$ .

1.5.3. *Counting parallel pairs.* We write

$$N_0(\omega, R) = \#\{(z, w) : |z \wedge w| = 0, |z| \leq |w| \leq R\}.$$

**Theorem 1.2.** *Let  $\omega$  be a lattice surface. There is a constant  $c = c(\omega)$  such that*

$$\lim_{R \rightarrow \infty} \frac{N_0(\omega, R)}{R^2} = c.$$

We prove this result in §6, and show how to compute  $c$  using the decomposition of  $\Lambda_\omega$  into orbits of  $SL(X, \omega)$ , the structure of the cusps of the Fuchsian group  $\Gamma = SL(X, \omega)$ , and Veech's counting results. We note that a generic (in the sense of Masur-Smillie-Veech-almost every) surface has *no* pairs of parallel holonomy vectors, as the existence of such a pair is a closed and positive codimension condition.

**1.6. Organization of the paper.** In §2, we state Nevo's ergodic theorem and a version of a Siegel-Veech type formula from [ACM19], which we use to prove convergence of circle averages of Siegel-Veech transforms for continuous functions. In §3, we construct our function  $h_A$  and show it has our desired properties, state our main technical result Theorem 3.1, Proposition 3.2 about almost sure quadratic upper bounds, and Proposition 3.3 about convergence of circle averages for  $h_A$ , all of which are used to prove Theorem 1.1. In §4 we give sufficient conditions to prove Theorem 3.1. In §5, we prove Proposition 3.2 and a lemma necessary for Proposition 3.3, along with a key modification to prove Proposition 5.1, which show the conditions from §4 indeed hold. Finally, in §6 we prove our results on lattice surfaces.

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## 2. NEVO'S ERGODIC THEOREM AND SIEGEL-VEECH MEASURES

We state the results of [Nev17] and [ACM19] needed to precisely move between circle averages and counting asymptotics. We then combine these results to show convergence of averaging operators of Siegel-Veech transforms.

**2.1. Averaging operators.** Suppose  $SL(2, \mathbb{R})$  acts on a space  $X$  (here our spaces will be  $\mathbb{C}$ ,  $\mathbb{C}^2$ , and connected components of strata  $\mathcal{H}$ , all with the natural  $\mathbb{R}$ -linear actions). Given a function  $h$  on  $X$ , and  $p \in X$ , we define

$$(A_t h)(p) = \frac{1}{2\pi} \int_0^{2\pi} h(g_t r_\theta p) d\theta.$$

Note that for  $f \in B_c(\mathbb{C})$  or  $h \in B_c(\mathbb{C}^2)$ , we can interchange sum and integral to obtain

$$(A_t \widehat{f})(\omega) = \widehat{(A_t f)}(\omega) \text{ and } (A_t \widehat{h})(\omega) = \widehat{(A_t h)}(\omega).$$

A key tool is Nevo's ergodic theorem for the operators  $A_t$  acting on  $\mathcal{H}$ .

**Theorem 2.1.** [Nev17, Theorem 1.1] *Suppose  $\mu$  is an ergodic  $SL(2, \mathbb{R})$ -invariant probability measure on  $\mathcal{H}$ . Assume  $f \in L^{1+\kappa}(\mathcal{H}, \mu)$  for some  $\kappa > 0$ , and that  $f$  is  $K$ -finite, that is, if  $f_\theta(\omega) = f(r_\theta \omega)$ , the span of the functions  $\{f_\theta : \theta \in [0, 2\pi)\}$  is finite-dimensional. Let  $\eta \in C_c(\mathbb{R})$  be a continuous non-negative bump function with compact support and of unit integral. Then for  $\mu$ -almost every  $\omega \in \mathcal{H}$ ,*

$$\lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} \eta(t-s) (A_s f)(\omega) ds = \int_{\mathcal{H}} f d\mu.$$

**2.2. Siegel-Veech measures.** We will apply Nevo's theorem to the Siegel-Veech transforms of functions defined on  $\mathbb{C}^2$ . By [ACM19, Theorem 3.4], for any  $f \in B_c(\mathbb{C})$ , for some  $\kappa > 0$ ,  $\widehat{f} \in L^{2+2\kappa}(\mathcal{H}, \mu)$ . Then, for any  $h \in B_c(\mathbb{C}^2)$ ,  $\widehat{h} \in L^{1+\kappa}(\mathcal{H}, \mu)$ , since we can dominate

$$\widehat{h}(\omega) = \sum_{v_1, v_2 \in \Lambda_\omega} h(v_1, v_2)$$

by  $(\widehat{f})^2$  where  $f = \|h\|_\infty \chi_H$ , where  $H$  denotes the union of the projections of the support of  $h$  via the coordinate projection maps. By the invariance of  $\mu$ , and the integrability condition,

$$h \mapsto \int_{\mathcal{H}} \widehat{h}(\omega) d\tau(\omega)$$

is an  $SL(2, \mathbb{R})$ -invariant linear functional on  $C_c(\mathbb{C}^2)$ . Therefore, there is an  $SL(2, \mathbb{R})$ -invariant measure  $m = m(\mu)$  (a *Siegel-Veech measure*) on  $\mathbb{C}^2$  so that

$$\int_{\mathcal{H}} \widehat{h}(\omega) d\mu(\omega) = \int_{\mathbb{C}^2} h dm$$

By the monotone convergence theorem, we can extend the class of  $h$  for all  $h \in B_c^{SC}(\mathbb{C}^2)$ , which are those  $h \in B_c(\mathbb{C}^2)$  which are either upper or lower semi-continuous. In particular  $B_c^{SC}$  will include characteristic functions of the compact closed sets defined in Section 3. To describe the possible  $SL(2, \mathbb{R})$ -invariant measures on  $\mathbb{C}^2$ , we need to understand  $SL(2, \mathbb{R})$ -orbits on  $\mathbb{C}^2$ . For  $t \in \mathbb{R}$ , let

$$D_t = \{(z, w) \in \mathbb{C}^2 : |\det(z|w)| = t\},$$

and notice we can identify  $D_1$  with  $SL(2, \mathbb{R})$ . For  $t \neq 0$ ,  $D_t$  is an  $SL(2, \mathbb{R})$ -orbit.  $D_0$  decomposes further. For  $s \in \mathbb{P}^1(\mathbb{R})$ , let

$$L_s = \{(z, sz) : z \in \mathbb{C} \setminus \{0\}\},$$

with

$$L_\infty = \{(0, w) : w \in \mathbb{C} \setminus \{0\}\}.$$

$D_t$  and  $L_s$  are the non-trivial  $SL(2, \mathbb{R})$  orbits on  $\mathbb{C}^2$ , and each carries a unique (up to scaling)  $SL(2, \mathbb{R})$ -invariant measure. These are the (non-atomic) ergodic invariant measures for  $SL(2, \mathbb{R})$  action on  $\mathbb{C}^2$ . On  $D_t$ , the measure is Haar measure (which we denote  $\lambda$ ) on  $SL(2, \mathbb{R})$ , and on  $L_s$  it is Lebesgue on  $\mathbb{C}$ . Thus, associated to any  $SL(2, \mathbb{R})$  invariant measure  $m$  on  $\mathbb{C}^2$  we have measures  $\nu = \nu(m)$  and  $\rho = \rho(m)$  so that we have [ACM19, Theorem 1.2]

$$(2.1) \quad \int_{\mathbb{C}^2} h dm = \int_{\mathbb{R} \setminus \{0\}} \left( \int_{SL(2, \mathbb{R})} h(tz, w) d\lambda(z, w) \right) d\nu(t) + \int_{\mathbb{P}^1(\mathbb{R})} \left( \int_{\mathbb{C}} h(z, sz) dz \right) d\rho(s).$$

### 2.3. Convergence of averaging operators for continuous functions.

**Proposition 2.2.** *Suppose  $\varphi \in C_c(\mathbb{C}^2)$ . Then for  $\mu$ -almost every  $\omega \in \mathcal{H}$ , the circle averages of  $\widehat{\varphi}$  converge*

$$\lim_{\tau \rightarrow \infty} A_\tau \widehat{\varphi}(\omega) = \int_{\mathcal{H}} \widehat{\varphi} d\mu = \int_{\mathbb{C}^2} \varphi dm.$$

*Proof.* By [ACM19, Theorem 3.4],  $\widehat{\varphi} \in L^{1+\kappa}(\mathcal{H}, \mu)$  for some  $\kappa > 0$ . We want to construct  $K$ -finite functions which sufficiently approximate  $\widehat{\varphi}$ , which we do by constructing a family of  $K$ -finite functions which are dense in the continuous functions.

Define  $H = \overline{B(0, l_\varphi)} \times \overline{B(0, l_\varphi)}$ , the product of closed balls with radius chosen so that  $\varphi \leq \|\varphi\|_\infty \mathbf{1}_H$ . We will also consider also the slightly larger set  $H_1 = \overline{B(0, l_\varphi + 1)}^2$ . Notice that  $H, H_1$  are rotation invariant subsets of  $\mathbb{C}^2$  under the diagonal action  $r_\theta(v, w) = (r_\theta v, r_\theta w)$ .

Consider the following family of functions in  $C(H)$  defined by  $\mathcal{F} = \{f_{m_1, n_1, m_2, n_2} : m_i, n_i \in \mathbb{Z}\}$  where for  $z = r_1 e^{i\theta_1}, w = r_2 e^{i\theta_2}$ ,

$$f_{m_1, n_1, m_2, n_2}(z, w) = r_1^{m_1} r_2^{m_2} e^{in_1 \theta_1} e^{in_2 \theta_2}.$$

We consider a subalgebra  $\mathcal{A}$  of  $C(H)$  given by the  $\mathbb{C}$ -linear span of  $\mathcal{F} \cup \{1\}$ , where 1 is the constant unit function on  $H$ . Then by definition,  $\mathcal{A}$  is closed under addition and multiplication by complex scalars.

Moreover  $\mathcal{F}$  is closed under multiplication and complex conjugation, so  $\mathcal{A}$  is an algebra. Except for zero, the elements of  $\mathcal{F}$  separate points, and the inclusion of 1 guarantees separation of all points. Lastly, to see that Siegel-Veech transforms of elements of  $\mathcal{A}$  are  $K$ -finite, note that for any  $f_{m_1, n_1, m_2, n_2} \in \mathcal{F}$ ,

$$(f_{m_1, n_2, m_2, n_2} \circ r_\theta)(z, w) = r_1^{m_1} r_2^{m_2} e^{in_1(\theta_1 + \theta)} e^{in_2(\theta_2 + \theta)} = e^{(n_1 + n_2)i\theta} f_{m_1, n_1, m_2, n_2}(z, w).$$

By linearity,  $\widehat{f \circ r_\theta} = e^{(n_1 + n_2)i\theta} \widehat{f}$ , so the  $\mathbb{C}$ -linear span of  $\{\widehat{f_\theta} : \theta \in [0, 2\pi)\}$  is exactly the  $\mathbb{C}$ -linear span of  $\widehat{f}$ . Hence extending by linearity again for each  $f \in \mathcal{A}$ , both  $f$  and  $\widehat{f}$  are  $K$ -finite. Thus by the Stone-Weierstrass Theorem,  $K$ -finite functions are dense in the uniform topology in  $C(H)$ .

Thus we can choose a sequence of  $K$ -finite functions  $(f_n)_{n \in \mathbb{N}}$  which converge uniformly to  $\varphi$ . Since the convergence is uniform,  $\text{Im}(f_n)$  converges uniformly to zero, and  $\text{Re}(f_n)$  converges uniformly to  $\varphi$ . So replacing  $f_n$  with  $\text{Re}(f_n)$ , we will assume that each  $f_n$  is real valued. When necessary we will extend the functions  $f_n$  on  $H$  to functions on  $\mathbb{C}$  by considering them as a product with  $\mathbf{1}_H$  which still gives a semi-continuous  $K$ -finite function on  $\mathbb{C}^2$  with compact support.

Fix  $\eta \in C_c^\infty(\mathbb{R})$  a positive mollifier, with  $\eta(t) \geq 0$ ,

$$\int_{-\infty}^{\infty} \eta(t) dt = 1,$$

and support of  $\eta$  in  $[-1, 1]$ . For  $f \in C_0^\infty(\mathbb{C}^2)$ , and  $(z, w) \in \mathbb{C}^2$ , denote the convolution by

$$(\eta * f)(z, w) := \int_{-\infty}^{\infty} \eta(t) f(g_{-t}(z, w)) dt.$$

We will use the notation  $\eta_\gamma(t) = \gamma^{-1} \eta(t/\gamma)$  which has the property that the support of  $\eta_\gamma$  is in  $[-\gamma, \gamma]$  and

$$\lim_{\gamma \rightarrow 0} \eta_\gamma(t) = \delta(t)$$

where  $\delta$  is the Dirac delta distribution. We claim  $\eta_\gamma * \varphi$  converges uniformly to  $\varphi$  on  $\mathbb{C}^2$  as  $\gamma \rightarrow 0$ . To see this, for any  $\epsilon > 0$  choose  $\gamma_0$  so that whenever  $\gamma \leq \gamma_0$ , since the support of  $\eta_\gamma$  is contained in  $[-\gamma, \gamma]$ , by uniform continuity of  $\varphi$  on  $H$ , whenever  $|t| < \gamma$ ,

$$|\varphi(g_{-t}(z, w)) - \varphi(z, w)| < \epsilon.$$

Thus for any  $(z, w) \in \mathbb{C}$ ,

$$|(\eta_\gamma * \varphi)(z, w) - \varphi(z, w)| \leq \int_{-\gamma}^{\gamma} \eta_\gamma(t) |\varphi(g_{-t}(z, w)) - \varphi(z, w)| dt < \epsilon.$$

We will also use the fact that there is some  $\gamma_0$  so that for  $\gamma < \gamma_0$ , for any  $(z, w) \in H$ ,  $g_{-\gamma}(z, w) \in H_1$ , so

$$(2.2) \quad \mathbf{1}_H \leq \eta_{\gamma_0} * \mathbf{1}_{H_1}.$$

By Theorem 2.1, for each  $f_n$  and almost every  $\omega$ ,

$$\begin{aligned} \lim_{\tau \rightarrow \infty} A_\tau(\widehat{\eta * f_n})(\omega) &= \lim_{\tau \rightarrow \infty} \int_{-\infty}^{\infty} \eta(t) \left( A_{\tau-t} \widehat{f_n} \right)(\omega) dt \\ &= \lim_{\tau \rightarrow \infty} \int_{-\infty}^{\infty} \eta(\tau - s) \left( A_s \widehat{f_n} \right)(\omega) ds \\ &= \int_{\mathcal{H}} \widehat{f_n} d\mu = \int_{\mathbb{C}^2} f_n dm, \end{aligned}$$

where  $m$  is the Siegel-Veech measure as in Equation 2.1.

Let  $\epsilon > 0$ . Choose  $H$  slightly larger if necessary so all  $\omega$  have a saddle connection in  $B(0, l_\varphi)$ , so

$$1 \leq \widehat{\mathbf{1}_H}(\omega) < \infty.$$

Since  $\mathbf{1}_H \in B_c^{SC}(\mathbb{C}^2)$ ,  $\widehat{\mathbf{1}}_H \in L^{1+\kappa}(\mu)$  so

$$\int_{\mathcal{H}} \widehat{\mathbf{1}}_H d\mu = m(H) < \infty.$$

By uniform convergence, choose  $N$  so that for all  $n \geq N$  and a corresponding sequence  $\gamma_n$ , for all  $(z, w) \in H$ ,

$$(2.3) \quad |f_n(z, w) - \varphi(z, w)| < \epsilon \text{ and } |(\eta_{\gamma_n} * \varphi)(z, w) - \varphi(z, w)| < \epsilon.$$

Since the support of  $f_n$  and  $\varphi$  are contained in  $H$ , summing over  $\Lambda_{\omega'}$  for any  $\omega' \in \mathcal{H}$  gives the pointwise bounds

$$(2.4) \quad |\widehat{f_n}(\omega') - \widehat{\varphi}(\omega')| < \epsilon \widehat{\mathbf{1}}_H(\omega') \text{ and } |(\eta_{\gamma_n} * \varphi)^{SV}(\omega') - \widehat{\varphi}(\omega')| < \epsilon \widehat{\mathbf{1}}_H(\omega').$$

Now fix  $n$  large. Since  $\mathbf{1}_{H_1}$  is  $K$ -finite, we apply Theorem 2.1. So for almost every  $\omega \in \mathcal{H}$  choose  $T = T(\omega)$  large enough so that for all  $\tau \geq T$ ,

$$(2.5) \quad |A_\tau(\eta_{\gamma_n} * \mathbf{1}_{H_1})^{SV}(\omega) - m(H_1)| < \epsilon.$$

Choose  $\tau \geq T$  for  $T = T(\eta_{\gamma_0}, \omega)$  so that again by Theorem 2.1

$$(2.6) \quad A_\tau \widehat{\mathbf{1}}_H(\omega) \leq A_\tau(\eta_{\gamma_n} * \mathbf{1}_{H_1})^{SV}(\omega) \leq m(H_1) + \epsilon.$$

For almost every  $\omega$ , and  $\tau \geq T(\omega)$ , we apply Equation (2.4) for each  $\omega' = g_\tau r_\theta \omega$ , and then by Equation (2.6)

$$(2.7) \quad |A_\tau(\widehat{\eta_{\gamma_n} * f_n})(\omega) - A_\tau(\eta_{\gamma_n} * \varphi)^{SV}(\omega)| \leq \epsilon A_\tau(\eta_{\gamma_n} * \mathbf{1}_{H_1})^{SV}(\omega) < \epsilon[\epsilon + m(H_1)].$$

Equation 2.4 says for each  $\omega' = g_\tau r_\theta \omega$ , we obtain

$$(2.8) \quad \begin{aligned} |A_\tau(\eta_{\gamma_n} * \varphi)^{SV}(\omega) - A_\tau \widehat{\varphi}(\omega)| &\leq A_\tau |(\eta_{\gamma_n} * \varphi)^{SV}(\omega) - \widehat{\varphi}(\omega)| \\ &\leq \epsilon A_\tau(\widehat{\mathbf{1}}_H)(\omega) \\ &\leq \epsilon[\epsilon + m(H_1)]. \end{aligned}$$

Next again applying Theorem 2.1, for a.e.  $\omega$  choose  $T = T(\omega)$  large enough so that for all  $\tau \geq T$ ,

$$(2.9) \quad \left| A_\tau(\eta_{\gamma_n} * f_n)^{SV}(\omega) - \int_{\mathbb{C}^2} f_n dm \right| < \epsilon.$$

Next we can again use Equation 2.3 and since all of the functions have support in  $H_1$  to see that

$$(2.10) \quad \left| \int_{\mathbb{C}^2} f_n dm - \int_{\mathbb{C}^2} \varphi dm \right| \leq \epsilon m(H_1).$$

By the triangle inequality combined with Equations (2.7), (2.8), (2.9), (2.10) we conclude that for almost every  $\omega$  and each  $n$ , there is  $T = T(\varphi, \omega)$  so that for all  $\tau \geq T$ ,

$$(2.11) \quad \left| A_\tau \widehat{\varphi}(\omega) - \int_{\mathbb{C}^2} \varphi dm \right| \leq 2\epsilon[\epsilon + m(H_1)] + \epsilon + \epsilon m(H_1).$$

Since  $m(H)$  and  $m(H_1)$  are fixed constants we conclude for almost every  $\omega$ ,

$$(2.12) \quad \lim_{\tau \rightarrow \infty} A_\tau \widehat{\varphi}(\omega) = \int_{\mathbb{C}^2} \varphi dm.$$

□

### 3. APPROXIMATION FUNCTION AND PROPERTIES

In this section, we construct the function  $h_A$  satisfying (1.3).



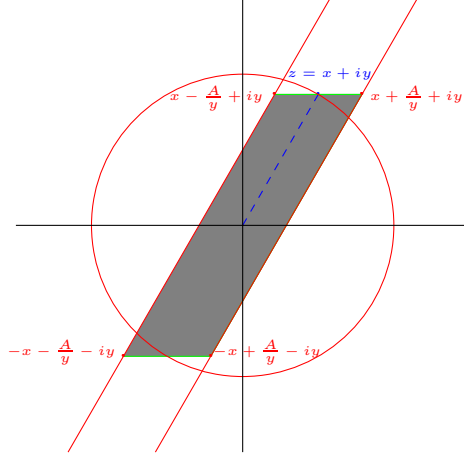
**3.1. Fibered sets.** Fix  $A > 0$ . Given  $z \in \mathbb{C}$ , define the approximating parallelogram

$$R_A(z) = \{w \in \mathbb{C} : |w \wedge z| \leq A, |\operatorname{Im} w| \leq |\operatorname{Im} z|\}$$

which we will use to approximate the desired set

$$D_A(z) = \{w \in \mathbb{C} : |w \wedge z| \leq A, |w| \leq |z|\}.$$

FIGURE 1. The sets  $R_A(z)$  and  $D_A(z)$ .  $R_A(z)$  is the shaded parallelogram, and  $D_A(z)$  is the region inside the red circle bounded by the two parallel red lines. This is the picture for  $x > A/y$ , and there is a corresponding figure for  $x < A/y$ .



Given  $S \subset \mathbb{C}$ , we define the *fibered* parallelogram and desired set by

$$R_A(S) = \{(z, w) \in \mathbb{C}^2 : z \in S, w \in R_A(z)\} \text{ and } D_A(S) = \{(z, w) \in \mathbb{C}^2 : z \in S, w \in D_A(z)\}.$$

Note that for  $z = x + iy$ ,  $R_A(z)$  is a parallelogram with vertices  $x \pm \frac{A}{y} \pm iy$  (see Figure 1). From this observation, we have  $g_t$ -equivariance of  $R_A$ : for  $t \in \mathbb{R}$ ,

$$g_t(R_A(z)) = R_A(g_t z),$$

so for  $S \subset \mathbb{C}$ ,

$$g_t(R_A(S)) = R_A(g_t S).$$

We will be particularly interested in two families of fibered sets. First, define the trapezoid  $\mathcal{T}$  by

$$\mathcal{T} = \left\{ z = x + iy \in \mathbb{C} : \frac{1}{2} \leq y \leq 1, |x| \leq y \right\}$$

We set  $h_A = \mathbf{1}_{R_A(\mathcal{T})}$  to be the indicator function of the fibered set  $R_A(\mathcal{T})$ . Next, for  $R_2 > R_1 > 0$  set

$$\begin{aligned} B(R_1) &= B(0, R_1) = \{z \in \mathbb{C} : |z| < R_1\} \\ A(R_1, R_2) &= B(R_2) \setminus B(R_1) = \{z \in \mathbb{C} : R_1 < |z| < R_2\} \end{aligned}$$

We define

$$\begin{aligned} D_A(R) &= D_A(B(R)) \\ D_A(R_1, R_2) &= D_A(A(R_1, R_2)). \end{aligned}$$

We have  $N_A(\omega, R) = (\mathbf{1}_{D_A(R)})^{\text{SV}}(\omega)$  and  $N_A^*(\omega, R) = (\mathbf{1}_{D_A(R/2, R)})^{\text{SV}}(\omega)$ .

Our main goal is to prove

**Theorem 3.1.**

$$(3.1) \quad \left| N_A^*(\omega, e^t) - \pi e^{2t} \left( A_t \widehat{h_A} \right) (\omega) \right| = o(e^{2t}).$$

In order to obtain Theorem 3.1, we will need to use quadratic upper bounds to control error terms.

**Proposition 3.2.** *Given  $A > 0$ , for a.e.  $(X, \omega)$  there exists  $C$  and  $T > 0$  such that for all  $t > T$ ,*

$$N_A^*(\omega, e^t) \leq Ce^{2t}.$$

We will also need the following

**Proposition 3.3.** *For almost every  $\omega$ ,*

$$\lim_{t \rightarrow \infty} (A_t \widehat{h_A})(\omega) = \int_{\mathcal{H}} \widehat{h_A} d\mu.$$

To prove this proposition, we make use of the following lemma proved in Section 5.

**Lemma 3.4.** *For all  $\epsilon$  there exists a function  $g_\epsilon \in C_c(\mathbb{C}^2)$  and  $T \geq 0$  so that for all  $\tau \geq T$ , and  $\mu$ -a.e.  $\omega$ ,*

$$(3.2) \quad \left| A_\tau(\widehat{g_\epsilon} - \widehat{h_A})(\omega) \right| < \epsilon \quad \text{and} \quad \left| \int_{\mathcal{H}} \widehat{g_\epsilon} - \widehat{h_A} d\mu \right| < \epsilon.$$

*Proof of Proposition 3.3.* Let  $\epsilon > 0$ . Choose a function  $g_\epsilon \in C_c(\mathbb{C}^2)$  and a constant  $T$  so that for all  $\tau \geq T$ , Equation 3.2 holds. Also choosing  $T = T(\epsilon)$  larger if necessary so that by Proposition 2.2,

$$\left| A_\tau \widehat{g_\epsilon}(\omega) - \int_{\mathcal{H}} \widehat{g_\epsilon} d\mu \right| < \epsilon.$$

Thus by the triangle inequality

$$\left| A_\tau \widehat{h_A}(\omega) - \int_{\mathcal{H}} \widehat{h_A} d\mu \right| \leq \left| A_\tau \widehat{h_A}(\omega) - A_\tau \widehat{g_\epsilon}(\omega) \right| + \left| A_\tau \widehat{g_\epsilon}(\omega) - \int_{\mathcal{H}} \widehat{g_\epsilon} d\mu \right| + \left| \int_{\mathcal{H}} \widehat{g_\epsilon} - \widehat{h_A} d\mu \right| < 3\epsilon.$$

□

We conclude this section by proving Theorem 1.1 assuming Theorem 3.1 and Proposition 3.2.

*Proof of Theorem 1.1.* Combining Theorem 3.1 and Theorem 3.3 we have, that for  $\mu$ -almost every  $\omega \in \mathcal{H}$

$$(3.3) \quad \lim_{t \rightarrow \infty} \frac{N_A^*(\omega, e^t)}{\pi e^{2t}} = c_0$$

where

$$c_0 = \int_{\mathbb{R} \setminus \{0\}} \left( \int_{SL(2, \mathbb{R})} h_A(tz, w) d\lambda(z, w) \right) d\nu(t) + \int_{\mathbb{P}^1(\mathbb{R})} \left( \int_{\mathbb{C}} h_A(z, sz) dx \right) d\rho(s).$$

Notice that  $c_0$  only depends on  $A$  and  $(\mathcal{H}, \mu)$ .

To extend to  $N_A(\omega, e^t)$  we use a geometric series argument along with the dominated convergence theorem giving upper bounds via Proposition 3.2. Specifically for each fixed  $j$  setting  $s = \frac{e^t}{2^j}$  and using Equation 3.3, we have pointwise convergence

$$\lim_{t \rightarrow \infty} \frac{N_A^*(\omega, \frac{e^t}{2^j})}{e^{2t}} = \lim_{s \rightarrow \infty} \frac{N_A^*(\omega, e^s)}{2^{2j} e^{2s}} = \frac{\pi c_0}{2^{2j}}.$$

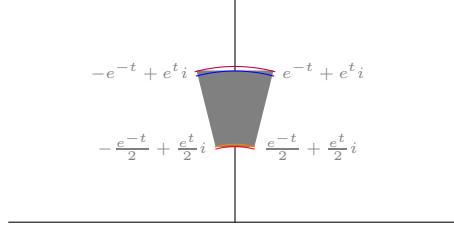
We also have a dominating integrable function

$$\frac{N_A^*(\omega, \frac{e^t}{2^j})}{e^{2t}} \leq \frac{cT^2}{2^{2j}},$$

where we are without loss of generality assuming  $T > 1$  is the constant from Proposition 3.2. Namely for each  $j$  whenever  $e^t > T2^j$ , Proposition 3.2 gives an upper bound of  $C2^{-2j} < cT^2 2^{-2j}$ . If  $e^t \leq T2^j$ , using quadratic upper bounds from Equation 1.1,

$$\frac{N_A^*(\omega, \frac{e^t}{2^j})}{e^{2t}} \leq \frac{N(\omega, \frac{e^t}{2^j})^2}{e^{2t}} \leq c_2 \frac{e^{2t}}{2^{4j}} \leq c_2 \frac{T^2}{2^{2j}}.$$

FIGURE 2. For  $z \in A\left(\sqrt{\frac{\cosh(2t)}{2}}, e^t\right)$ ,  $|\Theta_t(z)| = 2 \arctan(e^{-2t})$ .



Therefore by the dominated convergence theorem and the fact that for each fixed  $t$ , the tail of the telescoping series gives

$$\lim_{j \rightarrow \infty} N_A\left(\omega, \frac{e^t}{2^{j+1}}\right) = 0,$$

so

$$\lim_{t \rightarrow \infty} \frac{N_A(\omega, e^t)}{e^{2t}} = \lim_{t \rightarrow \infty} \sum_{j=0}^{\infty} \frac{N_A^*(\omega, \frac{e^t}{2^j})}{e^{2t}} = \sum_{j=0}^{\infty} \frac{\pi c_0}{2^{2j}} = \frac{4}{3} \pi c_0.$$

□

#### 4. COUNTING AND ERRORS

In this section we derive estimates that are necessary for the proof of Theorem 3.1. Note that (see Figure 2) the trapezoid  $g_{-t}\mathcal{T}$  has vertices

$$\pm \frac{e^{-t}}{2} + i \frac{e^t}{2}, \quad \pm e^{-t} + i e^t.$$

Given  $z, w \in \mathbb{C}$ , define

$$\begin{aligned} \Theta_t(z) &= \{\theta \in [0, 2\pi) : g_t r_\theta z \in \mathcal{T}\} \\ \Theta_t(z, w) &= \{\theta \in [0, 2\pi) : g_t r_\theta(z, w) \in R_A(\mathcal{T})\}. \end{aligned}$$

For  $f = \mathbf{1}_{\mathcal{T}}$  and  $h = \mathbf{1}_{R_A(\mathcal{T})}$ , we have

$$\begin{aligned} |\Theta_t(z)| &= 2\pi(A_t f)(z) \\ |\Theta_t(z, w)| &= 2\pi(A_t h_A)(z, w) \end{aligned}$$

**Lemma 4.1.** *For  $t > 0$  and  $(z, w) \in \mathbb{C}^2$ ,*

$$A_t(h_A(z, w)) \leq A_t(f(z)) \leq \frac{\arctan(e^{-2t})}{\pi}.$$

*Proof.* The first inequality follows from the fact that  $h_A(z, w) = f(z)\mathbf{1}_{R_A(z)}(w)$ . By our computation of the endpoints,

$$g_{-t}\mathcal{T} \subset \left\{ r e^{i\theta} \in \mathbb{C} : \theta \in \left[ \frac{\pi}{2} - \arctan(e^{-2t}), \frac{\pi}{2} + \arctan(e^{-2t}) \right] \right\}.$$

Since  $r_\theta z = e^{i\theta} z$ , we have

$$(A_t h_A)(z, w) \leq (A_t f)(z) = \frac{1}{2\pi} |\Theta_t(z)| \leq \frac{\arctan(e^{-2t})}{\pi}.$$

□

Our next lemma captures the fact that for  $(z, w)$  in a set that is only *slightly* smaller than  $D_A(e^t/2, e^t)$   $(A_t h_A)(z, w)$  captures a fixed contribution of order  $e^{-2t}$ .

**Lemma 4.2.** *For  $t > 0$  if*

$$(4.1) \quad (z, w) \in D_A \left( \sqrt{\frac{\cosh(2t)}{2}}, e^t \right) \text{ and } |w| \leq \frac{|z|}{\sqrt{1 + e^{-4t}}}$$

*then*

$$(4.2) \quad (A_t h_A)(z, w) = \frac{\arctan(e^{-2t})}{\pi}.$$

*Moreover, for all  $(z, w)$  such that  $(A_t h_A)(z, w) > 0$ ,*

$$(4.3) \quad |w| \leq \sqrt{1 + (8A + 16A^2)e^{-4t}} |z|.$$

*Proof.* To show (4.2), we must show

$$|\Theta_t(z, w)| = 2 \arctan(e^{-2t}).$$

Note that (see Figure 2)

$$z \in A \left( \sqrt{\frac{\cosh(2t)}{2}}, e^t \right) \implies |\Theta_t(z)| = 2 \arctan(e^{-2t}).$$

We now consider the second component. For any  $\theta$ ,

$$|r_\theta z \wedge r_\theta w| = |z \wedge w| \leq A.$$

For  $\theta \in \Theta(z)$ , we need to verify  $r_\theta w \in R_A(r_\theta z)$ . That is, we need to check that

$$|\operatorname{Im}(r_\theta w)| \leq |\operatorname{Im}(r_\theta z)|$$

Since  $|w| \leq \frac{|z|}{\sqrt{1 + e^{-4t}}}$ ,

$$|\operatorname{Im}(r_\theta w)| \leq \frac{|z|}{\sqrt{1 + e^{-4t}}}.$$

We claim that for any  $\theta \in \Theta_t(z)$ ,

$$\operatorname{Im}(r_\theta z) \geq \frac{|z|}{\sqrt{1 + e^{-4t}}}.$$

Indeed,  $\operatorname{Im}(r_\theta z)$  is minimized over  $\theta \in \Theta_t(z)$  when  $\theta = \theta_0$  so that  $r_{\theta_0} z$  is on (either) non-horizontal edge of the trapezoid, that is

$$r_{\theta_0} z = |z| e^{i(\pi/2 - \arctan(e^{-2t}))},$$

Note that for  $p = u + iv$  on either such edge,  $|u| = e^{-2t}v$ , so

$$|p| = \sqrt{1 + e^{-4t}} \operatorname{Im}(p).$$

Thus

$$\operatorname{Im}(r_\theta z) \geq \operatorname{Im} \left( |z| e^{i(\pi/2 - \arctan(e^{-2t}))} \right) = \frac{|z|}{\sqrt{1 + e^{-4t}}}$$

as desired. To show (4.3), consider  $\theta$  so that  $g_t r_\theta(z, w) \in R_A(\mathcal{T})$ . So  $r_\theta z = x + iy \in g_{-t}\mathcal{T}$ . Thus  $e^t/2 \leq |z| = |r_\theta z| \leq \sqrt{2 \cosh(2t)}$ ,  $|x|/y \leq e^{-2t}$ , and  $\frac{e^t}{2} \leq y \leq e^t$ . Since  $r_\theta w \in R_A(z)$ ,

$$\begin{aligned}
|w| = |r_\theta w| &\leq \sqrt{\left(|x| + \frac{A}{y}\right)^2 + y^2} \\
&= \sqrt{|z|^2 + 2\frac{A|x|}{y} + \frac{A^2}{y^2}} \\
&\leq \sqrt{|z|^2 + 2Ae^{-2t} + 4A^2e^{-2t}} \\
&= |z| \sqrt{1 + \frac{(2A + 4A^2)e^{-2t}}{|z|^2}} \\
&\leq |z| \sqrt{(1 + 4(2A + 4A^2)e^{-4t})},
\end{aligned}$$

where in the last line we are using that

$$|z| > e^t/2 \implies |z|^{-2} \leq 4e^{-2t}.$$

□

**4.1. Proving Theorem 3.1.** To prove Theorem 3.1 we break the left-hand side of 3.1 into several terms. First note we have

$$\begin{aligned}
(4.4) \quad \left| N_A^*(\omega, e^t) - \pi e^{2t} \left( A_t \widehat{h_A} \right) (\omega) \right| &= \left| \sum_{(z,w) \in \Lambda_\omega^2} \left( \mathbf{1}_{D_A(e^t/2, e^t)}(z, w) - \pi e^{2t} (A_t h_A)(z, w) \right) \right| \\
&= \left| \left( \mathbf{1}_{D_A(e^t/2, e^t)} - \pi e^{2t} A_t h_A \right)^{\text{SV}} (\omega) \right|.
\end{aligned}$$

We break (4.4) into several pieces: a *main term* discussed in (§4.2) and four *error terms* in (§4.3). We will show in §5 how to control the error terms. Note that for any  $h \in B_c(\mathbb{C}^2)$ ,  $S \subset \mathbb{C}^2$ , we can write

$$\begin{aligned}
\sum_{(z,w) \in \Lambda_\omega \cap S} h(z, w) &= \sum_{(z,w) \in \Lambda_\omega} h(z, w) \mathbf{1}_S(z, w) \\
&= (h \cdot \mathbf{1}_S)^{\text{SV}}(\omega),
\end{aligned}$$

**4.2. Main term.** Let

$$M_t = \left\{ (z, w) \in D_A \left( \sqrt{\frac{\cosh(2t)}{2}}, e^t \right) : |w| < |z|(1 + e^{-4t})^{-1/2} \right\}.$$

That is  $M_t$  is the collection of pairs satisfying (4.1). Our *main term* will be

$$\begin{aligned}
(4.5) \quad m_t(\omega) &= \sum_{(z,w) \in \Lambda_\omega^2 \cap M_t} \left( \mathbf{1}_{D_A(e^t/2, e^t)}(z, w) - \pi e^{2t} (A_t h_A)(z, w) \right) \\
&= (\mathbf{1}_{M_t} \cdot (\mathbf{1}_{D_A(e^t/2, e^t)} - \pi e^{2t} A_t h_A))^{\text{SV}}(\omega) \\
&= (\mathbf{1}_{M_t} - \pi e^{2t} (A_t h_A) \cdot \mathbf{1}_{M_t})^{\text{SV}}(\omega)
\end{aligned}$$

where the last line follows from the fact that  $M_t \subset D_A(e^t/2, e^t)$ .

**4.3. Error terms.**

4.3.1. *Error term 1: Bottom of trapezoid.* We define

$$E_t^1 = D_A \left( e^t/2, \sqrt{\frac{\cosh(2t)}{2}} \right).$$

In particular, for  $(z, w) \in E_t^1$   $r_\theta z$  hits only the *bottom* of the trapezoid  $g_{-t}\mathcal{T}$ , and the arc  $\Theta_t(z)$  is not the full possible arc of width  $2 \arctan(e^{-2t})$ . Note that the smallest possible length of a vector in  $g_{-t}\mathcal{T}$  is  $e^t/2$ , and for  $z = |z|e^{i\varphi}$  with

$$\begin{aligned} e^t/2 < |z| < \sqrt{\frac{\cosh(2t)}{2}} &= \sqrt{\frac{e^{2t} + e^{-2t}}{4}} \\ \Theta_t(z) &= (\arcsin(e^t/2|z|) - \varphi, \pi - \arcsin(e^t/2|z|) - \varphi), \end{aligned}$$

so

$$|\Theta_t(z)| = \pi - 2 \arcsin(e^t/2|z|) = 2 \arccos(e^t/2|z|).$$

We define

$$\begin{aligned} (4.6) \quad e_t^1(\omega) &= \sum_{(z,w) \in \Lambda_\omega^2 \cap E_t^1} ((\mathbf{1}_{D_A(e^t/2, e^t)}(z, w) - \pi e^{2t}(A_t h_A)(z, w))) \\ &= \left( \mathbf{1}_{E_t^1} \cdot (\mathbf{1}_{D_A(e^t/2, e^t)} - \pi e^{2t} A_t h_A) \right)^{\text{SV}}(\omega) \\ &= \left( \mathbf{1}_{E_t^1} - \pi e^{2t}(A_t h_A) \cdot \mathbf{1}_{E_t^1} \right)^{\text{SV}}(\omega) \end{aligned}$$

where the last line follows from the fact that  $E_t^1 \subset D_A(e^t/2, e^t)$ .

4.3.2. *Error term 2:  $|w| > |z|(1 + e^{-4t})^{-1/2}$ .* Our second error term consists of pairs  $(z, w)$  for which

$$|\Theta_t(z)| = 2 \arctan(e^{-2t}) \text{ but } |w| > |z|(1 + e^{-4t})^{-1/2},$$

so (4.1) is not satisfied. That is,

$$E_t^2 = \left\{ (z, w) \in D_A \left( \sqrt{\frac{\cosh(2t)}{2}}, e^t \right) : |w| > |z|(1 + e^{-4t})^{-1/2} \right\},$$

and we define the counting function

$$\begin{aligned} (4.7) \quad e_t^2(\omega) &= \sum_{(z,w) \in \Lambda_\omega^2 \cap E_t^2} (\mathbf{1}_{D_A(e^t/2, e^t)}(z, w) - \pi e^{2t}(A_t h_A)(z, w)) \\ &= \left( \mathbf{1}_{E_t^2} \cdot (\mathbf{1}_{D_A(e^t/2, e^t)} - \pi e^{2t} A_t h_A) \right)^{\text{SV}}(\omega) \end{aligned}$$

4.3.3. *Error term 3: The top of the trapezoid.* Our third error term is based on the set

$$E_t^3 = \{(z, w) \in \mathbb{C}^2 : (A_t h_A)(z, w) > 0, |z| > e^t\},$$

that is, where  $z$  is in the *top* of the trapezoid, and  $(z, w) \notin D_A(e^t/2, e^t)$ . We set

$$\begin{aligned}
(4.8) \quad e_t^3(\omega) &= \sum_{(z,w) \in \Lambda_\omega^2 \cap E_t^3} (\mathbf{1}_{D_A(e^t/2, e^t)}(z, w) - \pi e^{2t}(A_t h_A)(z, w)) \\
&= \left( \mathbf{1}_{E_t^3} \cdot (\mathbf{1}_{D_A(e^t/2, e^t)} - \pi e^{2t} A_t h_A) \right)^{\text{SV}}(\omega) \\
&= - \left( \mathbf{1}_{E_t^3} \cdot \pi e^{2t} A_t h_A \right)^{\text{SV}}(\omega)
\end{aligned}$$

where the last line follows from the fact that  $E_t^3$  is disjoint from  $D_A(e^t/2, e^t)$ .

4.3.4. *Error term 4:*  $|w| > |z|$ . Our fourth and final error term is based on the set where the averaging operator is positive, but  $(z, w) \notin D_A(e^t/2, e^t)$ . We define

$$\begin{aligned}
(4.9) \quad E_t^4 &= \{(z, w) \in \mathbb{C}^2 : (A_t h_A)(z, w) > 0, e^t/2 \leq |z| \leq e^t, |w| > |z|\}, \\
e_t^4(\omega) &= \sum_{(z,w) \in \Lambda_\omega \cap E_t^4} (\mathbf{1}_{D_A(e^t/2, e^t)}(z, w) - \pi e^{2t}(A_t h_A)(z, w)) \\
&= \left( \mathbf{1}_{E_t^4} \cdot (\mathbf{1}_{D_A(e^t/2, e^t)} - \pi e^{2t} A_t h_A) \right)^{\text{SV}}(\omega) \\
&= - \left( \mathbf{1}_{E_t^4} \cdot \pi e^{2t} A_t h_A \right)^{\text{SV}}(\omega)
\end{aligned}$$

where the last line follows from the fact that  $E_t^4$  is disjoint from  $D_A(e^t/2, e^t)$ .

4.3.5. *Decomposition.* By construction

$$D_A(e^t/2, e^t) \cup \{(z, w) : (A_t h_A)(z, w) > 0\} = M_t \cup \bigcup_{i=1}^4 E_t^i,$$

and the sets  $M_t$  and  $E_t^i$  are pairwise disjoint. Therefore

$$\begin{aligned}
(4.10) \quad \left| N_A^*(\omega, e^t) - \pi e^{2t} \widehat{(A_t h_A)}(\omega) \right| &= \left| \sum_{(z,w) \in \Lambda_\omega^2} (\mathbf{1}_{D_A(e^t/2, e^t)}(z, w) - \pi e^{2t}(A_t h_A)(z, w)) \right| \\
&= \left| (\mathbf{1}_{D_A(e^t/2, e^t)} - \pi e^{2t} A_t h_A)^{\text{SV}}(\omega) \right| \\
&= \left| m_t(\omega) + \sum_{i=1}^4 e_t^i(\omega) \right|.
\end{aligned}$$

## 5. UPPER BOUNDS

5.1. **Almost sure bounds.** We now show our key almost sure quadratic upper bound for pairs of saddle connections with bounded virtual area (Proposition 3.2), then show how to modify the proof to give Proposition 5.1, which controls the main and error terms defined in the previous section.

**Proposition 5.1.** *For almost every  $\omega$ ,*

$$(5.1) \quad |m_t(\omega)| = o(e^{2t})$$

and for  $i = 1, 2, 3, 4$

$$(5.2) \quad |e_t^i(\omega)| = o(e^{2t})$$

5.1.1. *Proving Theorem 3.1.* To prove Theorem 3.1, we combine (4.10) and Prop 5.1 to get

$$\begin{aligned}
 (5.3) \quad \left| N_A^*(\omega, e^t) - \pi e^{2t} \left( A_t \widehat{h_A} \right) (\omega) \right| &= \left| m_t(\omega) + \sum_{i=1}^4 e_t^i(\omega) \right| \\
 &\leq |m_t(\omega)| + \sum_{i=1}^4 |e_t^i(\omega)| \\
 &= o(e^{2t}).
 \end{aligned}$$

5.1.2. *Proving the estimate (5.1).* For  $(z, w) \in M_t$ , we have, by Lemma 4.2:

$$\begin{aligned}
 (\mathbf{1}_{D_A(e^t/2, e^t)} - (A_t h_A)) (z, w) &= 1 - \pi e^{2t} \cdot \frac{\arctan(e^{-2t})}{\pi} \\
 &= 1 - 1 + e^{2t} \cdot O(e^{-6t}) \\
 &= O(e^{-4t})
 \end{aligned}$$

Combining this with Proposition 3.2, we have

$$\begin{aligned}
 (5.4) \quad |m_t(\omega)| &\leq O(e^{-4t}) \cdot N_A^*(\omega, e^t) \\
 &= O(e^{-4t}) O(e^{2t}) \\
 &= O(e^{-2t}) = o(e^{2t}).
 \end{aligned}$$

**5.2. Notation.** In the remainder of the section we prove Propositions 3.2 and 5.1. We adopt the following notation:

*Systoles.* If  $\gamma$  is a saddle connection on  $\omega$ , we write  $\ell(\gamma)$  for the length of  $\gamma$ , i.e.,

$$\ell(\gamma) = |z_\gamma|, \text{ where } z_\gamma = \int_\gamma \omega \text{ is the holonomy vector of } \gamma.$$

We write  $\ell(\omega) = \ell(\gamma_0)$  for the length of the shortest saddle connection  $\gamma_0(\omega)$  on  $\omega$ , and  $\tilde{\ell}(\omega) = \ell(\gamma_1)$  for the length of the shortest saddle connection  $\gamma_1(\omega)$  not homologous to the shortest saddle connection  $\gamma_0(\omega)$ . We define

$$C = \{\omega \in \mathcal{H} : \gamma_0(\omega) \text{ bounds a cylinder}\}$$

and

$$BC_\epsilon = \{\omega \in \mathcal{H} : \gamma_0(\omega) \text{ bounds a cylinder of width } \leq \epsilon\}.$$

Note that

$$BC = \bigcup_{\epsilon > 0} BC_\epsilon.$$

Scales. Fix  $0 < \sigma < 1$ . Let

$$s_\sigma(\omega) = \frac{\log \ell(\omega)}{\log \sigma}, \tilde{s}(\omega) = \frac{\log \tilde{\ell}(\omega)}{\log \sigma}$$

*Angles.* On a base surface  $\omega$  we refer to a holonomy vector  $z$  of a saddle connection  $\gamma$  without subscripts. On the surface  $g_t r_\theta \omega$  the image holonomy vector  $g_t r_\theta z$  will be denoted  $z_{\theta, t}$ . If  $z = |z|e^{i\varphi}$ , we define  $\theta_z = \pi/2 - \varphi$  to be the angle so that  $r_{\theta_z} z$  is vertical, that is  $r_{\theta_z} = |z|i$ .



*Delaunay triangulations.* Following [MS91, §4], we define a *Delaunay triangulation* of a translation surface  $\omega \in \mathcal{H}$ . Consider the Voronoi decomposition of the translation surface with respect to the flat metric: each zero of  $\omega$  determines a cell given by points which are closer to it than to any other zero, and have a unique shortest geodesic connecting the two. The dual decomposition is the Delaunay decomposition of the surface, and any triangulation given by a further dissection of this is a *Delaunay triangulation* of  $\omega$ .

Let  $N = N(\mathcal{H})$  be the maximum number of edges in a Delaunay triangulation by saddle connections of any  $\omega \in \mathcal{H}$ . This  $N$  is finite, since there are a bounded number of vertices since they are at the zeroes of  $\omega$ . Given any pair of zeroes the saddle connections joining them are in different homotopy classes. There are a bounded number of arcs joining two points when the arcs are in different homotopy classes and they do not intersect. Thus we have a uniformly bounded number of edges. For the rest of this section, we fix  $\delta < \frac{1}{2N}$ .

*The set.* Define

$$P_A(\omega, e^t) = \Lambda_\omega^2 \cap D_A(e^t/2, e^t),$$

so  $|P_A(\omega, e^t)| = N_A^*(\omega, e^t)$ .

*Partitioning the circle.* For each  $t > 0$  partition  $[0, 2\pi)$  into  $\lfloor e^{2t} \rfloor$  intervals  $I(\theta_i)$  of radius  $\frac{\pi}{\lfloor e^{2t} \rfloor}$  centered at points  $\theta_i$  for  $i = 1, \dots, \lfloor e^{2t} \rfloor$ . We will look at counting on the finite set of surfaces  $\{\omega_i = g_t r_{\theta_i} \omega\}$  and then prove Lemma 5.6 to observe that in each of these intervals centered at  $\theta_i$  lengths change by at most a multiplicative constant which will be absorbed in our estimates.

*Ratios.* For each  $i$ , define

$$j = j(i) = \lfloor s_\sigma(\omega_i) - \tilde{s}_\sigma(\omega_i) \rfloor,$$

so

$$\sigma^{j+1} < \frac{\ell(\omega_i)}{\tilde{\ell}(\omega_i)} \leq \sigma^j.$$

5.2.1. *Measure bounds.* Next we state a result originally due to Masur-Smillie [MS91], estimating the measure of the set of surfaces with two non-homologous short saddle connections.

**Lemma 5.2.** [MS91, Equation 7] *For all  $\epsilon, \kappa > 0$ , the Masur-Smillie-Veech measure of the set  $V_1(\epsilon, \kappa) \subset \mathcal{H}$  of  $\omega$  which have a saddle connection of length at most  $\epsilon$ , and a non-homologous saddle connection with length at most  $\kappa$  is  $O(\epsilon^2 \kappa^2)$ .*

5.2.2. *Counting lemmas.* Finally we state a counting lemma which is summarized from [ACM19, §3.6.2 and 3.6.3].

**Lemma 5.3.** *Fix  $\omega \in \mathcal{H}$  with shortest saddle connection  $\gamma$  and second shortest saddle connection  $\gamma'$ . If either  $\gamma$  does not bound a cylinder, or  $\gamma$  bounds a cylinder of width at least  $\epsilon_0$  then*

$$N(\omega, \epsilon_0)^2 = O(|\ell(\gamma')|^{-2N}).$$

5.3. **Counting bounds with systoles.** The following is a slight modification of [EM01, Theorem 5.1].

**Lemma 5.4.** *For any  $L_0 > 0$  and  $\delta > 0$  there exists  $C = C(\delta, L_0)$  such that for any  $L < L_0$  and any surface  $(X, \omega)$  in the stratum*

$$(5.5) \quad N(\omega, L) \leq C \left( \frac{L}{\ell(\omega)} \right)^{1+\delta}$$

*Proof.* Fix  $L_0 > 0$ ,  $\delta > 0$ , let  $L < L_0$ , and let  $\omega \in \mathcal{H}$ . [EM01, Theorem 5.1] states that there is a  $\kappa = \kappa(\mathcal{H}) > 0$  and  $C' = C(\delta, \mathcal{H})$  so that for  $L < \kappa$ ,

$$N(\omega, L) \leq C' \left( \frac{L}{\ell(\omega)} \right)^{1+\delta}.$$

Thus, if  $L_0 < \kappa$  we are done. We now consider the case  $L_0 > \kappa$ . Divide  $[0, 2\pi)$  into equally sized intervals  $J_i$  of radius  $\frac{\kappa^2}{4L_0^2}$  and let  $\phi_i$  be the center of  $J_i$ . Note that there are  $O(L_0^2)$  such intervals. For any  $z \in \Lambda_\omega(L)$  choose  $\phi_i$  with angle  $\theta_z \in J_i$ . We rotate  $z$  to almost vertical via  $r_{-\phi_i}$ , and then shrink  $r_{-\phi_i} z$  to have length

less than  $\kappa$  by applying  $g_t r_{-\phi_i}$  where  $e^t = \frac{2L_0}{\kappa}$ . The largest possible imaginary component for  $r_{-\phi_i} z$  is bounded above by  $|z| < L < L_0$ , so

$$\operatorname{Im}(z_{-\phi_i, t}) = \frac{\kappa}{2L_0} \operatorname{Im}(r_{-\phi_i} z) \leq \frac{\kappa}{2}.$$

After rotating by  $-\phi_i$ ,  $r_{-\phi_i} z$  must lie in  $B(0, L)$  with angle with the vertical in  $\left(-\frac{\kappa^2}{4L_0^2}, \frac{\kappa^2}{4L_0^2}\right)$ , so the real component of  $r_{-\phi_i} z$  must satisfy

$$\operatorname{Re}(z_{-\phi_i, t}) = \frac{2L_0}{\kappa} \operatorname{Re}(r_{-\phi_i} z) \leq \frac{2L_0}{\kappa} \operatorname{Im}(r_{-\phi_i} z) \tan\left(\frac{\kappa^2}{4L_0^2}\right) \leq \frac{2L_0^2}{\kappa} \frac{\kappa^2}{4L_0^2} = \frac{\kappa}{2},$$

where we used  $\tan(x) < x$  for  $0 < x < \frac{1}{4}$ . Thus

$$|g_t r_{-\phi_i} z| \leq \kappa.$$

Moreover, for each  $i$  the systoles satisfy

$$\ell(\omega) \leq \ell(g_t r_{\phi_i} \omega) \cdot \frac{2L_0}{\kappa}.$$

So

$$N(\omega, L) = \sum_{i=1}^{O(L_0^2)} C' \left( \frac{L}{\ell(g_t r_{-\phi_i} \omega)} \right)^{1+\delta} \leq O(L_0^2) \left( \frac{2L_0}{\kappa} \right)^{1+\delta} C \left( \frac{L}{\ell(\omega)} \right)^{1+\delta}.$$

Combining the terms  $O(L_0^2) \left( \frac{2L_0}{\kappa} \right)^{1+\delta} C'$ , we get a constant  $C = C(\delta, L_0)$  as desired.  $\square$

**5.4. Integral bounds and counting in thin part of the stratum.** Choose  $\delta < 1/2N$ . Given  $\epsilon$ , let

$$\mathcal{H}_\epsilon = \{\omega \in \mathcal{H} : \ell(\omega) \leq \epsilon\}$$

denote the  $\epsilon$ -thin part of the stratum. Fix  $L_0 > 0$ . For  $\omega \in \mathcal{H}_\epsilon$  let

$$\psi_{L_0, \epsilon}(\omega) = \chi_{B(0, L_0)^2}^{SV}(\omega) \chi_{\mathcal{H}_\epsilon}(\omega).$$

$\psi_{L_0, \epsilon}(\omega)$  counts the number of pairs of saddle connections of length at most  $L_0$  if  $\omega \in \mathcal{H}_\epsilon$ , and if  $\omega \notin \mathcal{H}_\epsilon$ ,  $\psi_{L_0, \epsilon}(\omega) = 0$ . For ease of notation we write  $\psi = \psi_{L_0, \epsilon}$ .

**Lemma 5.5.**

$$\int_{\mathcal{H}} \psi d\mu = O(\epsilon^{\frac{1}{N}-2\delta}),$$

where the implied bound depends on  $L_0$ .

*Proof.* We define three families of sets exhausting the thin part of the stratum, in terms of the length of the shortest and second shortest non-homologous saddle connections. Recall we have fixed  $0 < \sigma < 1$ . For  $0 < j < N$ , let  $F(j)$  be the set of  $\omega$  with a shortest saddle connection  $\gamma_0(\omega)$  with length between  $\sigma^{j+1}$  and  $\sigma^j$  and a non-homologous saddle connection of length at most  $\sigma^{\frac{j}{2N}}$ . That is

$$(5.6) \quad F(j) = \left\{ \omega \in \mathcal{H} : \lfloor s_\sigma(\omega) \rfloor = j, \tilde{s}_\sigma(\omega) < \frac{j}{2N} \right\}.$$

By Lemma 5.2,

$$\mu(F(j)) = O\left(\sigma^{2j+2\frac{j}{2N}}\right) = O\left(\sigma^{2j}\sigma^{\frac{j}{N}}\right).$$

We now apply Lemma 5.4 which says for each  $\omega \in F(j)$  we have  $\psi(\omega) = O(\sigma^{-j(2+2\delta)})$ . This gives

$$\int_{F(j)} \psi d\mu = O(\sigma^{j(\frac{1}{N}-2\delta)}).$$

Next, let  $\epsilon_0 := 8\pi\widehat{c}_A$ . Define  $G(j)$  to be the set of  $\omega$  where the shortest saddle connection  $\gamma_0(\omega)$  has length between  $\sigma^{j+1}$  and  $\sigma^j$ , the shortest non-homologous saddle connection  $\gamma_1(\omega)$  has length at least  $\sigma^{\frac{j}{2N}}$ , and  $\gamma_0(\omega)$  either does not bound a cylinder or bounds a cylinder of width at least  $\epsilon_0 := 8\pi\widehat{c}_A$ . That is,

$$G(j) = \left\{ \omega \notin BC_{\epsilon_0} : \lfloor s_\sigma(\omega) \rfloor = j, \tilde{s}_\sigma(\omega) > \frac{j}{2N} \right\}$$

By Lemma 5.3, for each  $\omega \in G(j)$  we have  $\psi(\omega) = O\left((\sigma^{j/2N})^{-2N}\right) = O(\sigma^{-j})$ . We also have by Lemma 5.2 that  $\mu(G(j)) = O(\sigma^{2j})$  giving

$$\int_{G(j)} \psi d\mu = O(\sigma^j).$$

Finally let  $H(j)$  be the set of  $\omega$  where the shortest saddle connection  $\gamma_0(\omega)$  has length between  $\sigma^{j+1}$  and  $\sigma^j$  and bounds a cylinder of width at most  $\epsilon_0$ ,

$$H(j) = \{\omega \in BC_{\epsilon_0} : \lfloor s_\sigma(\omega) \rfloor = j\}.$$

Then again if  $\omega \in H(j)$   $\psi(\omega) = O(\sigma^{-j(2+2\delta)})$  and now

$$\mu(H(j)) = O(\sigma^{3j}),$$

where the implied constant depends on  $\epsilon_0$ . This gives

$$\int_{H(j)} \psi d\mu = O(\sigma^{j(1-2\delta)}).$$

Finally choose  $j_0$  so that  $\sigma^{j_0+1} \leq \epsilon < \sigma^{j_0}$ . Thus

$$\mathcal{H}_\epsilon \subset \bigcup_{j \geq j_0} F(j) \cup G(j) \cup H(j)$$

and so

$$\int_{\mathcal{H}_\epsilon} \psi d\mu = O\left(\sum_{j \geq j_0} \sigma^j + \sigma^{j(1/N-2\delta)} + \sigma^{j(1-2\delta)}\right) = O\left(\sigma^{j_0(1/N-2\delta)}\right) = O\left(\epsilon^{1/N-2\delta}\right).$$

□

**5.5. Reducing to finitely many surfaces.** This next lemma shows that within a fixed range of angles for  $\theta$  and times for  $t$ ,  $|g_t r_\theta z|$  cannot change too much, which allows us to reduce to our finite collection of surfaces.

**Lemma 5.6.** *Given  $t > 0$ ,  $\theta_0 \in [0, 2\pi)$ , define*

$$I_t(\theta_0) = (\theta_0 - \pi e^{-2t}, \theta_0 + \pi e^{-2t}).$$

*Then for any  $t > 0$ , any  $z \in \mathbb{C}$ ,  $\theta \in I(\theta_0)$ , and  $s \in (t, t + \log 2)$  we have*

$$\frac{1}{8\pi} \leq \frac{|g_s r_\theta z|}{|g_t r_{\theta_0} z|} \leq 8\pi.$$

*Proof.* First write

$$g_s r_\theta z = g_s r_{\theta - \theta_0} g_{-t}(g_t r_{\theta_0} z).$$

We need to control the operator norm of  $g_s r_\psi g_{-t}$  where  $|\psi| = |\theta - \theta_0| \leq \pi e^{-2t}$ . We note that  $g_s r_\psi g_{-t} = g_s r_\psi g_{-s} g_{s-t}$ . Note that  $\|g_{s-t}\|_{op} \leq 2$ , and

$$g_s r_\psi g_{-s} = \begin{pmatrix} \cos \psi & -e^{2s} \sin \psi \\ e^{-2s} \sin \psi & \cos \psi \end{pmatrix}.$$

The operator norm of a matrix  $A$  is the square root of the largest eigenvalue of  $A^T A$ . In this case

$$(g_s r_\psi g_{-s})^T (g_s r_\psi g_{-s}) = \begin{pmatrix} \cos^2 \psi + e^{-4s} \sin^2 \psi & -\sin(2\psi) \sinh(2s) \\ -\sin(2\psi) \sinh(2s) & \cos^2 \psi + e^{4s} \sin^2 \psi \end{pmatrix}.$$

For a determinant 1 matrix  $M$ , with  $|\operatorname{tr}(M)| > 2$  the largest eigenvalue  $\lambda^+(M)$  is given by

$$\lambda^+(M) = \frac{\operatorname{tr}(M)}{2} + \sqrt{\left(\frac{\operatorname{tr}(M)}{2}\right)^2 - 1}.$$

Applying this to  $M = (g_s r_\psi g_{-s})^T (g_s r_\psi g_{-s})$ , we have

$$\frac{\operatorname{tr}(M)}{2} = \cos^2 \psi + \cosh(4s) \sin^2 \psi = 1 + (\cosh(4s) - 1) \sin^2 \psi = 1 + 2 \sinh^2(2s) \sin^2 \psi,$$

so

$$\lambda^+(B) = 1 + 2 \sinh^2(2s) \sin^2 \psi + \sqrt{2} \sinh(2s) \sin \psi.$$

Since  $|\psi| < \pi e^{-2t}$ ,  $|\sin \psi| < \pi e^{-2t}$ , and  $\sinh(2s) < 2e^{2t}$ , so

$$\begin{aligned} \lambda^+(M) &\leq 1 + 2\pi^2 e^{-4t} e^{4t} + \sqrt{2} 2e^{2t} \pi e^{-2t} \\ &\leq 1 + 8\pi^2 + 2\sqrt{2}\pi. \end{aligned}$$

Therefore,

$$\|g_s r_\psi g_{-s}\|_{op} = \sqrt{\lambda^+(M)} \leq \sqrt{1 + 8\pi^2 + 2\sqrt{2}\pi} \leq 4\pi,$$

and so

$$\|g_s r_\psi g_{-t}\|_{op} \leq \|g_s r_\psi g_{-s}\|_{op} \|g_{s-t}\|_{op} \leq 8\pi.$$

□

**5.6. Geodesic flow length bounds for pairs.** In this section we collect bounds that hold under certain assumptions for pairs of holonomy vectors. Note that, by (4.3), if  $A_t h_A(z, w) > 0$ , then

$$|w| \leq \sqrt{1 + c_A e^{-4t}} |z|,$$

where  $c_A = (8A + 16A^2)$ . We define the *large set*

$$L_A(e^t) = \left\{ (z, w) : \frac{e^t}{2} \leq |z| \leq \sqrt{2 \cosh(2t)}, |w \wedge z| \leq A, |w| \leq \sqrt{1 + c_A e^{-4t}} |z| \right\},$$

which satisfies, for  $1 \leq i \leq 4$ ,

$$E_t^i \subseteq L_A(e^t) \quad \text{and} \quad P_A(\omega, e^t) \subseteq L_A(e^t) \cap \Lambda_\omega^2.$$

Recall that for  $z \in \mathbb{C}$ ,  $\theta_z = \pi/2 - \arg(z)$  is the angle so that  $r_{\theta_z} z$  is vertical.

**Lemma 5.7.** *There exists  $\widehat{c}_A$  so that for all  $(z, w) \in L_A(e^t)$*

$$\frac{1}{2} \leq |g_t r_{\theta_z} z| \leq \sqrt{2},$$

and

$$|g_t r_{\theta_z} w| \leq \widehat{c}_A.$$

*Proof.* By definition  $r_{\theta_z} z = |z|i$ . Since

$$\frac{e^t}{2} \leq |z| \leq e^t \sqrt{1 + e^{-4t}},$$

we have

$$|g_t r_{\theta_z} z| = |e^{-t} |z| i| = e^{-t} |z|,$$

so

$$\frac{1}{2} \leq |g_t r_{\theta_z} z| \leq \sqrt{1 + e^{-4t}} \leq \sqrt{2}.$$

By Lemma 4.2,  $|w| \leq |z| \sqrt{1 + c_A e^{-4t}}$ , so the vertical component of  $r_{\theta_z} w$  is at most  $|z| \sqrt{1 + c_A e^{-4t}}$ , so

$$\operatorname{Im}(g_t r_{\theta_z} w) \leq |g_t r_{\theta_z} z| \sqrt{1 + c_A e^{-4t}}.$$

Since  $|z \wedge w| \leq A$ ,  $|\operatorname{Re}(g_t r_{\theta_z} w)| \cdot |g_t r_{\theta_z} z| \leq A$ . Hence

$$|g_t r_{\theta_z} w| \leq |\operatorname{Im}(g_t r_{\theta_z} w)| + |\operatorname{Re}(g_t r_{\theta_z} w)| \leq |g_t r_{\theta_z} z| \sqrt{1 + c_A e^{-4t}} + \frac{A}{|g_t r_{\theta_z} z|} \leq \sqrt{2} \sqrt{1 + c_A} + 2A.$$

where we now define  $\widehat{c}_A = \sqrt{2}\sqrt{1+c_A} + 2A$ .  $\square$

### 5.7. Proof of Proposition 3.2.

*Proof.* We partition the circle into  $e^t$  intervals centered at angles  $\theta_i$ . We will break up the set of  $\theta_i$  into three subsets depending on the lengths of shortest and second shortest saddle connections of  $g_t r_{\theta_i} \omega$ . In the case of  $F(j, \frac{j}{2N})$  for  $j$  large we will use Nevo's ergodic theorem (Theorem 2.1) to show that the number of  $\theta_i$  is small. We again will use Lemma 5.4 to count pairs of saddle connections for this small set of  $\theta_i$ . In the case of  $G(j, \frac{j}{2N})$  as we count pairs exactly as before using Lemma 5.3 and then use some standard quadratic estimates to count the number of possible such  $\theta_i$ . The case of  $H(j, \epsilon_0)$  will be handled similarly.

**Case I.** Following the notation from Section 5.2, for each  $j$  define

$$N_j^I = \# \left\{ i \in \{1, \dots, e^{2t}\} : g_t r_{\theta_i} \omega \in F\left(j, \frac{j}{2N}\right) \right\}.$$

We first bound  $N_j^I$  using Theorem 2.1. Similar to our definition of the set  $F(j, \frac{j}{2N})$  define  $E(j, \frac{j}{2N})$  to be the set of  $\omega$  with a saddle connection of length between  $8\pi\sigma^{j+1}$  and  $8\pi\sigma^j$ , and a nonhomologous saddle connection of length at most  $8\pi\sigma^{\frac{j}{2N}}$ . Again by Lemma 5.2,

$$\mu\left(E\left(j, \frac{j}{2N}\right)\right) = O\left(\sigma^{2j+2\frac{j}{2N}}\right) = O\left(\sigma^{2j}\sigma^{\frac{j}{N}}\right).$$

Let  $h$  be the characteristic function of  $E(j, \frac{j}{2N})$ . The function  $h$  is  $K$ -finite since  $E(j, \frac{j}{2N})$  is invariant under rotations. Choose  $\eta \in C_0^\infty(\mathbb{R})$  be such that  $\eta(t) = 1$  on  $[-\log(2), 0]$  and  $\int \eta(t)dt = 2$ . By Theorem 2.1, there is some  $t_0$  large enough so that for  $t > t_0$

$$\int_{-\infty}^{\infty} \eta(t-s)(A_s h)(\omega)ds \leq 3 \int_{\mathcal{H}} h d\mu = O\left(\sigma^{2j}\sigma^{\frac{j}{N}}\right).$$

Switching the order of integration

$$(5.7) \quad \sum_{i=1}^{e^{2t}} \int_{I(\theta_i)} \int_t^{t+\log(2)} h(g_s r_{\theta} \omega) ds d\theta = \int_t^{t+\log(2)} (A_s h)(\omega) ds \leq \int_{-\infty}^{\infty} \eta(t-s)(A_s h)(\omega) ds = O\left(\sigma^{2j}\sigma^{\frac{j}{N}}\right).$$

Now if there is some  $i \in \{1, \dots, e^{2t}\}$  so that  $g_t r_{\theta_i}(X, \omega) \in F(j, \frac{j}{2N})$ , so  $i$  contributes to  $N_j^I$ , then Lemma 5.6 guarantees that  $g_s r_{\theta} \omega \in E(j, \frac{j}{2N})$  whenever

$$(\theta, s) \in I(\theta_i) \times [t, t + \log(2)].$$

Since for each  $i$  contributing to  $N_j^I$  we have a full annulus of integration, (5.7) gives an upper bound

$$N_j^I \frac{2\pi}{e^{2t}} \cdot [t + \log(2) - t] = O\left(\sigma^{2j}\sigma^{\frac{j}{N}}\right).$$

Hence

$$N_j^I = O\left(e^{2t}\sigma^{2j}\sigma^{\frac{j}{N}}\right).$$

Now suppose we are given  $(z, w) \in L_A(e^t)$ . Choose  $i$  so that  $\theta_z \in I(\theta_i)$ . Lemma 5.7 and Lemma 5.6 show that  $|g_t r_{\theta_i} z| \leq 8\pi$  and  $|g_t r_{\theta_i} w| \leq 8\pi\widehat{c}_A$ . Apply Lemma 5.4 with  $L_0 = 8\pi\widehat{c}_A$ . Thus for each  $j$  and  $i$ , if  $g_t r_{\theta_i} \omega \in F(j, \frac{j}{2N})$ , then

$$\# \{(z, w) \in L_A(e^t) \cap \Lambda_\omega^2 : \theta_z \in I(\theta_i)\} \leq N(g_t r_{\theta_i} \omega, L_0)^2 = O\left(\left(\frac{1}{|g_t r_{\theta_i} z_i|}\right)^{2(1+\delta)}\right) = O\left(\sigma^{-j(2+2\delta)}\right).$$

The number of possible  $i$  is given by  $N_j^I$ , so for  $t$  large enough and any  $j$

$$\begin{aligned} \# \left\{ (z, w) \in L_A(e^t) \cap \Lambda_\omega^2 : \exists i \text{ so that } \theta_z \in I(\theta_i), g_t r_{\theta_i}(X, \omega) \in F\left(j, \frac{j}{2N}\right) \right\} &\leq N_j^I O\left(\sigma^{-j(2+2\delta)}\right) \\ &= O\left(e^{2t}\sigma^{j(\frac{1}{N}-2\delta)}\right). \end{aligned}$$

Summing over all possible  $j$ , since  $\frac{1}{N} - 2\delta > 0$  and  $\sigma < 1$ ,

$$(5.8) \quad \begin{aligned} & \# \left\{ (z, w) \in L_A(e^t) \cap \Lambda_\omega^2 : \exists i, \text{ and } j(i) \text{ so that } \theta_z \in I(\theta_i), g_t r_{\theta_i}(X, \omega) \in F\left(j, \frac{j}{2N}\right) \right\} \\ &= O\left(e^{2t} \sum_{j=1}^{\infty} \sigma^{j(\frac{1}{N}-2\delta)}\right) = O(e^{2t}). \end{aligned}$$

**Case II.** Define

$$N_j^{II} = \# \left\{ i \in \{1, \dots, e^{2t}\} : g_t r_{\theta_i} \omega \in G\left(j, \frac{j}{2N}\right) \right\}.$$

For any holonomy vector  $z$ , define

$$N_j^{II}(z) = \# \left\{ i : g_t r_{\theta_i} \omega \in G\left(j, \frac{j}{2N}\right) \text{ and } \sigma^{j+1} < |g_t r_{\theta_i} z| \leq \sigma^j \right\}.$$

Specifically, the holonomy vector  $z$  which becomes short after rotating and flowing satisfies  $|r_{\theta_i} z| = |z|$  and

$$\frac{|r_{\theta_i} z|}{e^t |g_t r_{\theta_i} z|} e^t \sigma^{j+1} \leq |z| \leq \frac{|r_{\theta_i} z|}{e^t |g_t r_{\theta_i} z|} e^t \sigma^j.$$

Notice  $|r_{\theta_i} z| \leq e^t |g_t r_{\theta_i} z|$ , so we can choose  $k \geq j$  so that

$$\sigma^{k+1} e^t \leq |z| \leq \sigma^k e^t.$$

The horizontal component  $\text{Re}(r_{\theta_i} z)$  satisfies

$$e^t \text{Re}(r_{\theta_i} z) \leq |g_t r_{\theta_i} z| \leq \sigma^j.$$

So if  $\phi_i = \arg(z) - \pi/2$  is the angle which makes  $r_{\phi_i} z$  vertical, then

$$(5.9) \quad |\sin(\phi_i - \theta_i)| = \frac{|\text{Re}(r_{\theta_i} z)|}{|r_{\theta_i} z|} \leq \frac{\sigma^j}{e^t} \cdot \frac{1}{\sigma^{k+1} e^t} = e^{-2t} \sigma^{j-k-1}.$$

By symmetry,

$$|\{\theta \in [0, 2\pi) : |\sin(\phi_i - \theta)| \leq e^{-2t} \sigma^{j-k-1}\}| = 4|\{\theta \in [\phi_i, \pi/2 + \phi_i) : |\sin(\phi_i - \theta)| \leq e^{-2t} \sigma^{j-k-1}\}|.$$

Then when  $|\phi_i - \theta| \leq \frac{\pi}{2}$  we have  $|\sin(\phi_i - \theta)| > \frac{2}{3}|\phi_i - \theta|$ . Since the  $\theta_i$  are spread evenly over  $[0, 2\pi)$ , we can estimate

$$\begin{aligned} N_j^{II}(z) &= e^{2t} \frac{|\{\theta \in [0, 2\pi) : |\sin(\phi_i - \theta)| \leq e^{-2t} \sigma^{j-k-1}\}|}{2\pi} \leq \frac{4e^{2t}}{2\pi} \left| \left\{ \theta : |\phi_i - \theta| \leq \frac{3}{2} e^{-2t} \sigma^{j-k-1} \right\} \right| \\ &= O(e^{2t} \cdot e^{-2t} \sigma^{j-k}) = O(\sigma^{j-k}). \end{aligned}$$

Using quadratic growth from [Mas90], the number of possible  $z$  with length at most  $\sigma^k e^t$  is  $O(\sigma^{2k} e^{2t})$ . Since  $k \geq j$ ,

$$N_j^{II} \leq O\left(\sum_{k \geq j} \sigma^{2k} e^{2t} \cdot \sigma^{j-k}\right) \leq O(\sigma^{2j} e^{2t}).$$

For each  $i$  in  $N_j^{II}$ , by Lemma 5.3, the number of pairs of saddle connections of length at most  $8\pi\hat{c}_A$  is  $O(\sigma^{-j})$ . Multiplying by  $N_j^{II}$ , the number of pairs of length at most  $8\pi\hat{c}_A$  is

$$O(\sigma^j e^{2t}).$$

Then we have

$$(5.10) \quad \# \left\{ (z, w) \in P_A(\omega, e^t) : \exists i, j(i) \text{ so that } \theta_z \in I(\theta_i), g_t r_{\theta_i} \omega \in G\left(j, \frac{j}{2N}\right) \right\} \leq O\left(\sum_{j=0}^{\infty} O(\sigma^j e^{2t})\right) = O(e^{2t})$$

**Case III.**

$$N_j^{III} = \#\{i \in \{1, \dots, e^{2t}\} : g_t r_{\theta_i} \omega \in H(\epsilon_0, j)\}$$

To finish the proof we need to cover the last case from Lemma 5.3 concerning the set  $H(\epsilon_0, \sigma^j)$ . As we have seen

$$\mu(H(\epsilon_0, \sigma^j)) = O(\sigma^{3j}) = O\left(\sigma^{2j + \frac{j}{N}}\right).$$

In this case we can use the estimate (5.8), and follow the exact method as in Case I to get a bound of  $O(e^{2t})$  on the number of pairs.

This concludes the proof of Proposition 3.2, where we note  $T$  in the statement must be large enough to apply Theorem 5.4 in both Case I and Case II. Moreover when applying Lemma 5.4 and Lemma 5.3, the constant  $C$  depends on  $A$ .  $\square$

The next corollary is a small modification of Proposition 3.2 which bounds the number of pairs  $(z, w) \in L_A(e^t) \cap \Lambda_\omega^2$  where flowing in an appropriate direction lands in the  $\hat{\epsilon}$ -thin part

**Corollary 5.8.**

$$\#\{(z, w) \in L_A(e^t) \cap \Lambda_\omega^2 : \exists i, \text{ and } j(i) \text{ so that } \theta_z \in I(\theta_i), g_t r_{\theta_i} \omega \in \mathcal{H}_{\hat{\epsilon}}\} = O(\hat{\epsilon}^{\frac{1}{N} - 2\delta} e^{2t}).$$

*Proof.* We follow exactly the proof of Proposition 3.2 where we have the additional assumption now that  $\sigma^j \leq \hat{\epsilon}$ . This comes into effect in Equations (5.8) and (5.10) and in Case III when we restrict  $\sigma^j < \hat{\epsilon}$  which limits the set of possible  $j$  to  $j \geq \frac{\log(\hat{\epsilon})}{\log(\sigma)}$ . So then the geometric series when altering Equation (5.8) yields

$$O\left(e^{2t} \sum_{j=\frac{\log \hat{\epsilon}}{\log \sigma}}^{\infty} \sigma^{j(\frac{1}{N} - 2\delta)}\right) = O(\hat{\epsilon}^{\frac{1}{N} - 2\delta} e^{2t})$$

and similarly for Equation (5.10) and Case III.  $\square$

**5.8. Subquadratic decay in the thick part of the stratum.** As a complement to Corollary 5.8, we want to understand decay of pairs of saddle connections in the thick part of the stratum, that is the set of surfaces where the systole has length at least  $\epsilon$ . To this end we will prove the following lemma which gives measure bounds for pairs of saddle connections near the boundary of  $R_A(\mathcal{T})$ . Given  $L, \hat{\epsilon}, \epsilon' > 0$ , and  $L' \in \{\frac{1}{2}, 1\}$  define  $\Omega(\hat{\epsilon}, \epsilon', L, L')$  to be the set of surfaces  $\omega$  such that  $\omega$  is  $\hat{\epsilon}$ -thick, and there are  $(z, w) \in \Lambda_\omega^2 \cap B(0, L)^2$  where at least one of the following holds:

- (1)  $1 - \epsilon' \leq \frac{|\operatorname{Im} w|}{|\operatorname{Im} z|} \leq 1 + \epsilon'$
- (2)  $(1 - \epsilon')A \leq |z \wedge w| \leq (1 + \epsilon')A$
- (3)  $|\operatorname{Im}(z) - L'| < \epsilon'$
- (4)  $(1 - \epsilon')\operatorname{Im} z \leq |\operatorname{Re} z| \leq (1 + \epsilon')\operatorname{Im} z$ .

**Lemma 5.9.** *Let  $D$  be the complex dimension of the stratum. For each  $L$ , there exists  $C$  so that for all  $\hat{\epsilon}, \epsilon'$*

$$\mu(\Omega(\hat{\epsilon}, \epsilon', L, L')) \leq C \left( \frac{\epsilon'}{\hat{\epsilon}^{2+2D}} \right).$$

*Proof.* Using the Delaunay triangulation [MS91, §4] for each  $\omega$  we take a basis  $\{\gamma_i\}_{i=1}^D$  for  $H_1(X, \Sigma, \mathbb{Z})$  of saddle connections, where the lengths are all bounded below by  $\hat{\epsilon}$ . Here  $\Sigma$  is the set of zeroes of  $\omega$ . Let  $z_i = x_i + iy_i = \int_{\gamma_i} \omega$  be the holonomy vectors of the basis. The assumption of thickness says  $|z_i| \geq \hat{\epsilon}$ . It follows from [MS91, §4] that the lengths are also bounded above by  $\frac{1}{\epsilon}$ . Since the thick part of the stratum is compact we can cover it by finitely many coordinates maps via the map

$$\eta \mapsto \left( \int_{\gamma_1} \eta, \dots, \int_{\gamma_D} \eta \right)$$

which gives local coordinates on  $\mathcal{H}$  in a neighborhood of  $\omega$ . For the remainder of the proof we will work in a single chart and to simplify notation we will use  $(z_1, \dots, z_D)$  as the coordinate charts. Our measure  $\mu$  arises from Lebesgue measure in these period coordinates. Since there are a bounded number of possible Delaunay triangulations that cover the stratum it is enough to bound the measure for each Delaunay triangulation. In [ACM19, §2] it was shown that Delaunay triangulations are *efficient*. By this we mean that up to a fixed multiplicative constant, the length of any saddle connection is bounded below by the length of a homologous path of edges in the triangulation joining the same endpoints. It follows that since  $|z|, |w| \leq L$ , we can write the holonomies  $z, w$  in this basis as  $\sum_{i=1}^D n_i z_i$  and  $\sum_{i=1}^D m_i z_i$  respectively with coefficients  $n_i, m_i \in \mathbb{Z}$  that are  $O(\frac{1}{\epsilon})$ , where the implied bound depends on  $L$ . We wish to compute the Lebesgue measure of the subset of  $\mathbb{C}^D$  where we have one of the relations.

For the first where we have a relation between  $|\operatorname{Im} z|$  and  $|\operatorname{Im} w|$  the assumption that the ratio of the imaginary parts of the holonomy  $z, w$  are within  $\epsilon'$  of each other gives for some  $m_i, n_i, z_i$

$$\left| \frac{\sum_{i=1}^D n_i y_i}{\sum_{i=1}^D m_i y_i} - 1 \right| < \epsilon'$$

which by the bound on  $m_i, n_i$  and  $y_i$  implies

$$(5.11) \quad \left| \sum_{i=1}^D (n_i - m_i) y_i \right| = O\left(\frac{\epsilon'}{\epsilon^2}\right).$$

For each fixed  $D$ -tuple of integers  $n_i - m_i$  the set of  $(z_1, \dots, z_D)$  which satisfy Equation 5.11 forms a neighborhood of width  $O(\frac{\epsilon'}{\epsilon^2})$  around the  $\mathbb{R}^{2D-1}$  hyperplane formed by the linear condition on the imaginary parts of  $z_i = x_i + iy_i$ . So the Lebesgue measure in  $\mathbb{C}^D$  within this fixed chart is  $O(\frac{\epsilon'}{\epsilon^2})$ . Since  $|n_i - m_i| = O(\frac{1}{\epsilon})$  there are  $O(\frac{1}{\epsilon^D})$  integer tuples when taking the differences and so the total measure is  $O(\frac{\epsilon'}{\epsilon^{2+D}})$ . This gives a bound for the measure of the set of  $\omega$  satisfying the first statement in the definition of  $\Omega(\hat{\epsilon}, \epsilon', L, L')$ .

For the second we have

$$\left| \left( \sum_{i=1}^D n_i x_i \right) \left( \sum_{i=1}^D m_i y_i \right) - \left( \sum_{i=1}^D m_i x_i \right) \left( \sum_{i=1}^D n_i y_i \right) - A \right| < \epsilon'.$$

For fixed  $D$  tuples  $(n_1, \dots, n_D)$  and  $(m_1, \dots, m_D)$  the set of possible  $z_i = x_i + iy_i$  lives in the  $\epsilon'$  neighborhood of a hyperplane determined by the determinant condition, and thus for fixed  $A$  has Lebesgue measure  $O(\epsilon')$ . In this case there are  $O(\frac{1}{\epsilon^{2D}})$  such tuples and so the result follows.

The proof of the measure bound for  $\omega$  satisfying the third and fourth statements are similar. Write  $z = \sum n_i z_i$ . The third assumption is

$$\left| \sum_{i=1}^D n_i y_i - L' \right| < \epsilon'.$$

The measure of the set of  $z$  that satisfy this inequality for some  $n_i$  is  $O(\frac{\epsilon'}{\epsilon^D})$ , which implies the desired bound since  $\hat{\epsilon} < 1$ . The proof of the last is similar.

□

**5.9. Proof of Proposition 5.1.** In this section we will apply Lemma 5.9 and Corollary 5.8 to prove Proposition 5.1.

*Proof of Proposition 5.1.* Let  $\epsilon'' > 0$ . Choose  $\hat{\epsilon}$  so that  $\epsilon'' = \hat{\epsilon}^{\frac{1}{N}-2\delta}$ . Let  $L = \sqrt{8\pi}\hat{\epsilon}_A$ , and choose  $\epsilon'$  small enough so that

$$\frac{\epsilon'}{\hat{\epsilon}^{2+2D}} \left( \frac{L}{\hat{\epsilon}} \right)^{1+\delta} < \epsilon''.$$

Choose  $T_0$  large enough so that whenever  $t \geq T_0$ , Corollary 5.8 holds. Corollary 5.8 shows that the number of pairs (without restrictions to be in  $E_t^j$ ) such that after rotating and flowing land us in the  $\hat{\epsilon}$ -thin part



(where the shortest curve has length at most  $\sigma^j < \hat{\epsilon}$ ) is  $O(\epsilon'' e^{2t})$ . Since we have the desired bounds in the thin part, we now only need to consider the thick part of the stratum, so we suppose  $|g_t r_{\theta_i} z_i| > \hat{\epsilon}$ .

**Bounds on  $z$  in trapezoid.** Suppose  $\theta \in I(\theta_i)$  satisfies  $r_\theta z \in g_{-t}\mathcal{T}$ , where we first choose  $t$  large enough so

$$\tan(\pi e^{-2t}) \leq 6\pi e^{-2t}.$$

Notice that  $\theta_z$ , the angle which makes  $r_{\theta_z}$  vertical has  $r_{\theta_z} z, r_\theta z \in g_{-t}\mathcal{T}$ , so  $|\theta - \theta_z| \leq e^{-2t}$ . Then for any  $\theta' \in I(\theta_i)$ , since  $|\theta - \theta'| \leq 2\pi e^{-2t}$ , we have

$$(5.12) \quad |\theta' - \theta_z| \leq 2\pi e^{-2t} + e^{-2t} \leq 3\pi e^{-2t}.$$

Notice  $\text{Im}(r_{\theta'} z) \leq |r_{\theta_z} z|$ , so we now want a lower bound for  $\text{Im}(r_{\theta'} z)$ . Indeed since

$$|\text{Re}(r_{\theta'} z)| = \text{Im}(r_{\theta'} z) \tan(|\theta_z - \theta'|) \leq \text{Im}(r_{\theta'} z) \tan(3\pi e^{-2t}) \leq \text{Im}(r_{\theta'} z) 6\pi e^{-2t},$$

this implies

$$\text{Im}(r_{\theta'} z)^2 (1 + 36\pi^2 e^{-4t}) \geq \text{Im}(r_{\theta'} z)^2 + \text{Re}(r_{\theta'} z)^2 \geq |z|^2 = |\text{Im}(r_{\theta_z} z)|^2.$$

Thus for any  $\theta' \in I(\theta_i)$ ,

$$(5.13) \quad \frac{|\text{Im}(r_{\theta_z} z)|}{\sqrt{1 + 36\pi^2 e^{-4t}}} \leq \text{Im}(r_{\theta'} z) \leq |\text{Im}(r_{\theta_z} z)|$$

**Bounds on  $w$ .** We claim the imaginary parts of  $r_{\theta_z} w$  and  $r_{\theta'} w$  don't differ too much. To do this we will use polar coordinates, so set

$$r_{\theta_z} w = |w| e^{i\psi_z} \quad \text{and} \quad r_{\theta'} w = |w| e^{i\psi'}.$$

Recall by Equation (5.12),  $|\psi' - \psi_z| = |\theta_z - \theta'| \leq 3\pi e^{-2t}$ . Then since circumference  $|\psi' - \psi_z|$  is bigger than the length of a chord on the unit circle,

$$3\pi e^{-2t} \geq |\theta' - \theta_z| = |\psi' - \psi_z| \geq |e^{i\psi'} - e^{i\psi_z}| \geq |\sin(\psi') - \sin(\psi_z)|.$$

Dividing by  $\sin(\psi_z)$ , we have

$$(5.14) \quad \left| \frac{\sin(\psi')}{\sin(\psi_z)} - 1 \right| \leq \frac{3\pi e^{-2t}}{|\sin(\psi_z)|}.$$

**Error term  $E_t^1$ .** Choose  $T_1 \geq T_0$  large enough so that for  $t \geq T_1$ ,

$$\max \left\{ \left( \frac{1}{2} - \frac{1}{2\sqrt{1 + 36\pi^2 e^{-4t}}} \right), \left( \frac{\sqrt{1 + e^{-4t}}}{2} - \frac{1}{2} \right) \right\} \leq \epsilon'.$$

Combining Equation (5.13) with the fact that

$$\frac{e^t}{2} \leq \text{Im}(r_{\theta_z} z) = |z| \leq \frac{e^t}{2} \sqrt{1 + e^{-4t}},$$

$$\frac{1}{2\sqrt{1 + 36\pi^2 e^{-4t}}} \leq \text{Im}(g_t r_{\theta'} z) \leq \frac{\sqrt{1 + e^{-4t}}}{2}.$$

By our choice of  $T_1$ ,  $\frac{1}{2} - \epsilon' \leq \text{Im}(g_t r_{\theta'} z) \leq \frac{1}{2} + \epsilon'$ . Thus the resulting surface by Lemma 5.6 satisfies

$$g_t r_{\theta'}(X, \omega) \in \Omega \left( \frac{\hat{\epsilon}}{\sqrt{8\pi}}, \epsilon', L, \frac{1}{2} \right).$$

**Error term  $E_t^2$ .** Choose  $T_2 \geq T_0$  large enough so that for  $t \geq T_2$ ,

$$\max \left\{ \sqrt{1 + 36\pi^2 e^{-4t}} \left( 1 + 3\pi e^{-2t} \left( \frac{\sqrt{1+e^{-4t}}}{1-4Ae^{-2t}} \right) \right) - 1, \right. \\ \left. 1 - \left( 1 - 3\pi e^{-2t} \left( \frac{\sqrt{1+e^{-4t}}}{1-4Ae^{-2t}} \right) \right) \frac{1-4Ae^{-2t}}{\sqrt{1+e^{-4t}}} \right\} \leq \epsilon'.$$

Since  $|z| = \text{Im}(r_{\theta_z} z)$ ,

$$|\text{Im}(r_{\theta_z} z)| \frac{1}{\sqrt{1+e^{-4t}}} \leq |w| \leq |\text{Im}(r_{\theta_z} z)|.$$

The determinant condition guarantees small angle. That is if  $\psi$  is the angle of  $r_{\theta_z} w$  from the vertical, then

$$\frac{|\text{Re}(r_{\theta_z} w)|}{|w|} = |\sin(\psi)| \leq \frac{A}{|w| \cdot |z|} \leq \frac{A\sqrt{1+e^{-4t}}}{|z|^2} \leq \frac{4Ae^{-2t}}{\sqrt{1+e^{-4t}}}.$$

Thus

$$\frac{|\text{Im}(r_{\theta_z} z)|}{\sqrt{1+e^{-4t}}} \leq |\text{Re}(r_{\theta_z} w)| + |\text{Im}(r_{\theta_z} w)| \leq |\text{Im}(r_{\theta_z} w)| + \frac{4Ae^{-2t}}{\sqrt{1+e^{-4t}}} |\text{Im}(r_{\theta_z} z)|,$$

which yields

$$(5.15) \quad \frac{|\text{Im}(r_{\theta_z} w)|}{|\text{Im}(r_{\theta_z} z)|} \geq \frac{1}{\sqrt{1+e^{-4t}}} - \frac{4Ae^{-2t}}{\sqrt{1+e^{-4t}}} = \frac{1-4Ae^{-2t}}{\sqrt{1+e^{-4t}}}.$$

In the other direction we have the easier inequality that

$$(5.16) \quad \frac{|\text{Im}(r_{\theta_z} w)|}{|\text{Im}(r_{\theta_z} z)|} \leq \frac{|w|}{|z|} \leq 1.$$

Thus by our assumption on  $|w|$ , Equation (5.14) and Equation (5.15),

$$(5.17) \quad \left| \frac{\text{Im}(r_{\theta'} w)}{\text{Im}(r_{\theta_z} w)} - 1 \right| \leq 3\pi e^{-2t} \frac{|w|}{|\text{Im}(r_{\theta_z} w)|} \leq 3\pi e^{-2t} \frac{\text{Im}(r_{\theta_z} z)}{|\text{Im}(r_{\theta_z} w)|} \leq 3\pi e^{-2t} \left( \frac{\sqrt{1+e^{-4t}}}{1-4Ae^{-2t}} \right).$$

In the first direction we combine (5.17), (5.13), and (5.16) to obtain

$$\frac{|\text{Im}(r_{\theta'} w)|}{|\text{Im}(r_{\theta'} z)|} \leq \sqrt{1 + 36\pi^2 e^{-4t}} \left( 1 + 3\pi e^{-2t} \left( \frac{\sqrt{1+e^{-4t}}}{1-4Ae^{-2t}} \right) \right) \leq 1 + \epsilon'.$$

In the other direction, combine (5.17), (5.13), and (5.15) to obtain

$$\frac{|\text{Im}(r_{\theta'} w)|}{|\text{Im}(r_{\theta'} z)|} \geq \left( 1 - 3\pi e^{-2t} \left( \frac{\sqrt{1+e^{-4t}}}{1-4Ae^{-2t}} \right) \right) \frac{1-4Ae^{-2t}}{\sqrt{1+e^{-4t}}} \geq 1 - \epsilon'.$$

Thus by Lemma 5.6, and noting that flowing by  $g_t$  does not change the ratio of imaginary parts,

$$g_t r_{\theta'} \omega \in \Omega \left( \frac{\hat{\epsilon}}{\sqrt{8\pi}}, \epsilon', L, 1 \right).$$

**Error term  $E_t^3$ .** Choose  $T_3 \geq T_0$  large enough so that for  $t \geq T_3$ ,

$$\max \left\{ \left( 1 - \frac{1}{\sqrt{1+36\pi^2 e^{-4t}}} \right), \left( \sqrt{1+e^{-4t}} - 1 \right) \right\} \leq \epsilon'.$$

For any  $\theta' \in I(\theta_i)$ , combining 5.13 with the fact that in this case

$$e^t \leq \text{Im}(r_{\theta_z} z) = |z| \leq e^t \sqrt{1+e^{-4t}},$$

we have

$$\frac{1}{\sqrt{1+36\pi^2 e^{-4t}}} \leq \text{Im}(g_t r_{\theta'} z) \leq \sqrt{1+e^{-4t}}.$$

By our choice of  $T_3$ ,  $1 - \epsilon' \leq \text{Im}(g_t r_{\theta'} z) \leq 1 + \epsilon'$ . Thus the resulting surface by Lemma 5.6 satisfies

$$g_t r_{\theta'} \omega \in \Omega \left( \frac{\hat{\epsilon}}{\sqrt{8\pi}}, \epsilon', L, 1 \right).$$

**Error term  $E_t^4$ .** Choose  $T_4 \geq T_0$  large enough so that for  $t \geq T_4$ ,

$$\max \left\{ \begin{aligned} & \sqrt{1+c_A e^{-4t}} \sqrt{1+36\pi^2 e^{-4t}} \left( 1 + 3\pi e^{-2t} \left( \frac{\sqrt{1+c_A e^{-4t}}}{1-4Ae^{-2t}\sqrt{1+c_A e^{-4t}}} \right) \right) - 1 \\ & 1 - (1-4Ae^{-2t}\sqrt{1+c_A e^{-4t}}) \left( 1 - 3\pi e^{-2t} \left( \frac{\sqrt{1+c_A e^{-4t}}}{1-4Ae^{-2t}\sqrt{1+c_A e^{-4t}}} \right) \right) \end{aligned} \right\} \leq \epsilon'.$$

Since  $|z| = \text{Im}(r_{\theta_z} z)$ ,

$$|\text{Im}(r_{\theta_z} z)| \leq |w| \leq |\text{Im}(r_{\theta_z} z)| \sqrt{1+c_A e^{-4t}}.$$

The determinant condition guarantees small angle. That is if  $\psi$  is the angle of  $r_{\theta_z} w$  from the vertical, then

$$\frac{|\text{Re}(r_{\theta_z} w)|}{|w|} = |\sin(\psi)| \leq \frac{A}{|w| \cdot |z|} \leq \frac{A}{|z|^2} \leq 4Ae^{-2t}.$$

Thus

$$|\text{Im}(r_{\theta_z} z)| \leq |w| \leq |\text{Re}(r_{\theta_z} w)| + |\text{Im}(r_{\theta_z} w)| \leq |\text{Im}(r_{\theta_z} w)| + 4Ae^{-2t} |\text{Im}(r_{\theta_z} z)| \sqrt{1+c_A e^{-4t}},$$

which yields

$$(5.18) \quad \frac{|\text{Im}(r_{\theta_z} w)|}{|\text{Im}(r_{\theta_z} z)|} \geq 1 - 4Ae^{-2t} \sqrt{1+c_A e^{-4t}}.$$

In the other direction we have the easier inequality that

$$(5.19) \quad \frac{|\text{Im}(r_{\theta_z} w)|}{|\text{Im}(r_{\theta_z} z)|} \leq \frac{|w|}{|z|} \leq \sqrt{1+c_A e^{-4t}}.$$

Thus by our assumption on  $|w|$ , Equation (5.14) and Equation (5.18),

$$(5.20) \quad \left| \frac{\text{Im}(r_{\theta'} w)}{\text{Im}(r_{\theta_z} w)} - 1 \right| \leq 3\pi e^{-2t} \frac{|w|}{|\text{Im}(r_{\theta_z} w)|} \leq 3\pi e^{-2t} \frac{\text{Im}(r_{\theta_z} z)}{|\text{Im}(r_{\theta_z} w)|} \sqrt{1+c_A e^{-4t}} \\ \leq 3\pi e^{-2t} \left( \frac{\sqrt{1+c_A e^{-4t}}}{1-4Ae^{-2t}\sqrt{1+c_A e^{-4t}}} \right).$$

Now in the first direction we combine Equations (5.13), (5.20) and (5.19) to obtain

$$\frac{|\text{Im}(r_{\theta'} w)|}{|\text{Im}(r_{\theta'} z)|} \leq \sqrt{1+c_A e^{-4t}} \sqrt{1+36\pi^2 e^{-4t}} \left( 1 + 3\pi e^{-2t} \left( \frac{\sqrt{1+c_A e^{-4t}}}{1-4Ae^{-2t}\sqrt{1+c_A e^{-4t}}} \right) \right) \leq 1 + \epsilon'.$$

In the other direction, combine Equations (5.20), (5.13), and (5.18) to obtain

$$\frac{|\text{Im}(r_{\theta'} w)|}{|\text{Im}(r_{\theta'} z)|} \geq (1-4Ae^{-2t}\sqrt{1+c_A e^{-4t}}) \left( 1 - 3\pi e^{-2t} \left( \frac{\sqrt{1+c_A e^{-4t}}}{1-4Ae^{-2t}\sqrt{1+c_A e^{-4t}}} \right) \right) \geq 1 - \epsilon'.$$

Thus by Lemma 5.6, and noting that flowing by  $g_t$  does not change the ratio of imaginary parts,

$$g_t r_{\theta'} \omega \in \Omega \left( \frac{\hat{\epsilon}}{\sqrt{8\pi}}, \epsilon', L, L' \right).$$

**Combining cases.** We conclude by Lemma 5.9, that

$$\mu \left( \Omega \left( \frac{\hat{\epsilon}}{\sqrt{8\pi}}, \epsilon', L, L' \right) \right) = O \left( \frac{\epsilon'}{\hat{\epsilon}^{2+2D}} \right).$$

Using the relative homology coordinates given by the Delaunay triangulation we can identify  $\Omega \left( \frac{\hat{\epsilon}}{\sqrt{8\pi}}, \epsilon', L, L' \right)$  with a domain in  $\mathbb{C}^D$ .

Let  $h$  be the characteristic function of the compact set  $\Omega \left( \frac{\hat{\epsilon}}{\sqrt{8\pi}}, \epsilon', L, L' \right)$ . Choose a small neighborhood of  $\Omega \left( \frac{\hat{\epsilon}}{\sqrt{8\pi}}, \epsilon', L, L' \right)$  and a continuous  $g \in C_0^\infty(\mathbb{C}^D)$  such that  $h \leq g$ , and

$$\int_{\mathcal{H}} g d\mu = O \left( \frac{\epsilon'}{\hat{\epsilon}^{2+2D}} \right).$$

We can consider a family of functions  $\mathcal{F}$  as in the proof of Proposition 2.2 in this case defined on  $\mathbb{C}^D$ . Let  $\phi$  be a radially symmetric continuous function of compact support which is identically 1 on  $\Omega\left(\frac{\hat{\epsilon}}{\sqrt{8\pi}}, \epsilon', L, L'\right)$ . and consider the family  $\bar{\mathcal{F}} = \{\phi f : f \in \mathcal{F}\}$ . This is a  $K$ -finite compactly supported family and so there is  $\bar{f} \in \bar{\mathcal{F}}$  uniformly close to  $g$ . Since we only need upper bounds, we can choose  $g$  larger if necessary so that we can choose a smoothing function  $\eta$  with

$$\int_{-\infty}^{\infty} \eta(u) du = 1$$

and support close enough to 0 so that  $h \leq (\eta * \bar{f})$ . By Theorem 2.1,

$$\lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} \eta(t-s)(A_s \bar{f})(\omega) ds = \int_{\mathcal{H}} \bar{f} d\mu = O\left(\frac{\epsilon'}{\hat{\epsilon}^{2+2D}}\right).$$

which for  $t$  large enough gives

$$A_t h(\omega) = O\left(\frac{\epsilon'}{\hat{\epsilon}^{2+2D}}\right).$$

Then as in Equation (5.7), by Lemma 5.6, for  $t$  sufficiently large

$$\# \{I(\theta_i) : \exists \theta \in I(\theta_i) \text{ so that } r_{\theta} z \in g_{-t} \mathcal{T} \text{ for } z \in E_t^k\} = O\left(\frac{\epsilon'}{\hat{\epsilon}^{2+2D}} e^{2t}\right).$$

Moreover by Lemma 5.4 for each  $I(\theta_i)$  the number of possible  $z \in E_t^k$  is  $O\left(\left[\frac{L}{\hat{\epsilon}}\right]^{1+\delta}\right)$ . Thus by our choice of  $\epsilon'$ , as desired

$$|E_t^k| = O\left(\frac{\epsilon'}{\hat{\epsilon}^{2+2D}} e^{2t} \left[\frac{L}{\hat{\epsilon}}\right]^{1+\delta}\right) = O(\epsilon'' e^{2t}).$$

□

*Proof of Lemma 3.4.* For any  $\epsilon'$  consider the set  $U$  of points  $(z, w)$  such that  $1/4 \leq \text{Im}(z) \leq 2$ ,  $|\text{Re}(z)| \leq \text{Im}(z) + 1$ , and  $|w| \leq 3\sqrt{1 + (8A + 16A^2)}$  (c. f. Equation 4.3), and at least one of

- (1)  $A(1 - \epsilon') \leq |z \wedge w| \leq (1 + \epsilon')A$
- (2)  $(1 - \epsilon') \text{Im } z \leq |\text{Im } w| \leq (1 + \epsilon') \text{Im } z$
- (3)  $(1 - \epsilon') \text{Im } z \leq |\text{Re } z| \leq (1 + \epsilon') \text{Im } z$
- (4)  $(1 - \epsilon') \leq |\text{Im}(z)| \leq 1 + \epsilon'$
- (5)  $(\frac{1}{2} - \epsilon') \leq |\text{Im}(z)| \leq \frac{1}{2} + \epsilon'$

holds.

The set  $U$  describes a neighborhood of  $\partial R_A(\mathcal{T})$ . Choose  $g = g(\epsilon')$  continuous such that

- (1)  $g(z, w) \leq 1$  and is supported in  $R_A(\mathcal{T}) \cup U$
- (2)  $g(z, w) = 1$  for  $(z, w) \in R_A(\mathcal{T})$

Let  $\phi = \phi(\epsilon')$  continuous, supported on  $U$ , such that  $\phi \leq 1$ , and  $\phi = g$  on  $U \setminus R_A(\mathcal{T})$ . (The point is that  $\phi = 1$  on  $\partial R_A(\mathcal{T})$ ). Then

$$g - h_A \leq \phi.$$

By Proposition 2.2,

$$\lim_{\tau \rightarrow \infty} (A_{\tau} \hat{\phi})(\omega) ds = \int_{\mathcal{H}} \hat{\phi} d\mu = \int_{\mathcal{H}_{\hat{\epsilon}}} \hat{\phi} d\mu + \int_{\mathcal{H} \setminus \mathcal{H}_{\hat{\epsilon}}} \hat{\phi} d\mu,$$

where  $\mathcal{H}_{\hat{\epsilon}}$  is the  $\hat{\epsilon}$ -thin part. By Lemma 5.5 the first term on the right is

$$O\left(\hat{\epsilon}^{\frac{1}{N} - 2\delta}\right).$$

By Lemma 5.4, each  $\omega$  in the  $\widehat{\epsilon}$  thick part has  $O\left(\frac{1}{\widehat{\epsilon}^{2+2\delta}}\right)$  pairs of saddle connections of bounded length. This together with Lemma 5.9 says the second term is

$$O\left(\frac{\epsilon'}{\widehat{\epsilon}^{4+2D+2\delta}}\right).$$

These two inequalities imply that for  $\tau$  large enough,

$$A_\tau(\widehat{g} - \widehat{h}_A)(\omega) = O(\widehat{\epsilon}^{\frac{1}{N}-2\delta}) + O\left(\frac{\epsilon'}{\widehat{\epsilon}^{4+2D+2\delta}}\right).$$

Recall we fixed  $\delta$  so that  $\frac{1}{N} > 2\delta$ . Then given  $\epsilon$ , choose  $\widehat{\epsilon}$  so the first term is at most  $\epsilon/2$ . Then choose  $\epsilon'$  so the second term is also at most  $\epsilon/2$ . The first conclusion of the Lemma follows. The second conclusion follows directly from Lemma 5.9 and Lemma 5.5.  $\square$

## 6. PARALLEL SADDLE CONNECTIONS ON LATTICE SURFACES

In this section, we will prove Theorem 1.2. Let  $\omega$  be a lattice surface. We recall first further details of Veech's result on the decomposition of  $\Lambda_\omega$  into finitely many orbits of the Fuchsian group  $\Gamma = SL(\omega)$  acting  $\mathbb{R}$ -linearly on  $\mathbb{C}$ .

**6.1. Cusps and orbits.** We denote the finite collection of *cusps* of  $\Gamma$  by  $[\Gamma_1], [\Gamma_2], \dots, [\Gamma_n]$ , where each  $[\Gamma_i]$  is a distinct conjugacy class of a parabolic subgroup of  $\Gamma$ . To each  $[\Gamma_i]$  we can choose a direction in  $\mathbb{R} \cup \{\infty\}$  stabilized by a representative of  $[\Gamma_i]$ , and in this direction, there will be a finite set of parallel saddle connections  $\gamma_{i,1}, \dots, \gamma_{i,m_i}$  with

$$\ell(\gamma_{i,1}) \geq \ell(\gamma_{i,2}) \geq \dots \geq \ell(\gamma_{i,m_i}).$$

Let

$$z_{i,j} = \int_{\gamma_{i,j}} \omega, r_{i,k} = \frac{z_{i,1}}{z_{i,k}}.$$

Note that for a fixed  $i$ , the  $z_{i,j}$  are parallel, so the  $r_{i,k} = \frac{z_{i,1}}{z_{i,k}} = \frac{\ell(\gamma_{i,1})}{\ell(\gamma_{i,k})}$  are real numbers greater than 1 for  $k \geq 1$ . By [Vee98, Theorem 16.1], there is a  $c_{i,k}$  such that

$$\#(\Gamma \cdot z_{i,k} \cap B(0, R)) \sim c_{i,k} R^2.$$

Note that since

$$\#(\Gamma \cdot z_{i,k} \cap B(0, R)) = \#(\Gamma \cdot z_{i,1} \cap B(0, Rr_{i,k})),$$

we have

$$c_{i,k} = c_{i,1} r_{i,k}^2.$$

Therefore, for  $j < k$ ,

$$\#(\Gamma \cdot (z_{i,k}, z_{i,j}) \cap B(0, R)^2) = \#(\Gamma \cdot z_{i,j} \cap B(0, R)) \sim c_{i,1} r_{i,j}^2 R^2.$$

**6.2. Completing the proof.** To complete the proof of Theorem 1.2, we put these together to obtain

$$N_0(\omega, R) = \sum_{i=1}^n \sum_{j=1}^{m_j-1} \sum_{k=1}^{j-1} \#(\Gamma \cdot (z_{i,k}, z_{i,j}) \cap B(0, R)^2) \sim \sum_{i=1}^n \sum_{j=1}^{m_j-1} \sum_{k=1}^{j-1} c_{i,1} r_{i,j}^2 R^2.$$

This proves Theorem 1.2, with

$$c = \sum_{i=1}^n \sum_{j=1}^{m_j-1} \sum_{k=1}^{j-1} c_{i,1} r_{i,j}^2 = \sum_{i=1}^n c_{i,1} \sum_{j=1}^{m_j-1} (j-1) r_{i,j}^2.$$

$\square$

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