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Efficient valuation of guaranteed minimum maturity benefits in regime switching jump diffusion models with surrender risk

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ABSTRACT

We present an efficient valuation approach for guaranteed minimum maturity benefits (GMMBs) embedded in variable annuity (VA) contracts in a regime-switching jump diffusion model. We allow early surrender of the VA contract and impose surrender charges, which are important in practice to discourage early termination/lapse of the contract. We consider both continuously-monitored and discretely-monitored surrender behaviors before maturity, and utilize an intensity-based framework. Based on the continuous-time Markov chain (CTMC) approximation combined with the Fourier cosine series expansion method, we find that the valuation problem can be solved under a regime-switching jump diffusion framework. Both error analysis and numerical experiments demonstrate the accuracy and efficiency of the proposed method.

1. Introduction

With the increased life expectancy, there is an increasing need for the management of longevity risks and investment post retirement. A variable annuity (VA) is a contract between the insurance company and the policyholder in which the policyholder pays premium during the accumulation phase and the insurer guarantees minimum periodic payments. It is a long-term insurance contact aimed at meeting retirement and other long-term investment planning goals. The VA contracts can be roughly divided into two major categories: guaranteed minimum death benefits (GMDBs) and guaranteed minimum living benefits (GMLBs). GMDBs provide guaranteed payments of the accumulated premium values to the beneficiaries in the event of death of policyholder. GMLBs provide living protection of the policyholder's income against market risk during either the accumulation phase and/or the annuitization phase. Within GMLBs there are several sub-categories, with one particular popular product being the guaranteed minimum maturity benefits (GMMBs), which at maturity provide the greater of the accumulated account value and the guaranteed minimum benefits.

VA contracts embedded with GMMBs have received increasing attention in recent literature. Under the Black–Scholes model, where asset value follows a geometric Brownian motion (GBM), Shen et al. [1] proposed a numerical approach for the pricing of GMMBs, derived the corresponding pricing partial differential equation (PDE) and proposed an integral representation of the solution using Dunhamel principle. Ng et al. [2] utilized the conditional Esscher transform to study the valuation of investment guarantees in a GARCH-type model. In a Heston-type stochastic volatility setting, Cui et al. [3] priced a GMMB with VIX-linked fees. Kang and Ziveyi [4] extended the framework presented in Bernard and Kwak [5], and designed a dynamic hedging algorithm to relieve the insurance company of the net liability. Feng and Volkmer

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[6] presented a new application of the spectral expansion method for the quantitative risk management in GMMB, which was found to be an efficient and accurate method for the computation of risk measures.

Another strand of literature focuses on the policyholder's behavior since the surrender of the VA contract has a great impact on both the insurer and the policyholder. On one hand, for the insurer, the surrender of the contract causes loss of premium revenue, and consequently affect its asset–liability structure and solvency. On the other hand, for the policyholder, the surrender will make her lose the protection function of the insurance component and affects the long-term financial planning. Therefore, the problem of surrender risk has become an active area of research. As pointed out in Shen et al. [1], Kang and Ziveyi [4], Bernard et al. [7], Jeon and Kwak [8], the *surrender option* embedded in VAs has great similarity with and hence can be treated as an American put option. Equivalently, the optimal surrender problem in VA contracts can be represented as optimal stopping problems. A common theme of the above literature is that the surrender option is the consequence of endogenous rational choice. However, the surrender option is also influenced by exogenous factors such as personal motivations outside the financial consideration, i.e. surrender decisions in reality are usually not fully financially rational. This exogenous viewpoint is shared by Consiglio and De Giovanni [9], De Giovanni [10] and Russo et al. [11]. Inspired by Russo et al. [11], we shall adopt an intensity-based framework for modeling the surrender behaviors in this paper.

In general, the maturities of VAs usually span for several decades, hence potential regime shifts in social policy and economic development have a great impact on the valuation of VAs. This motivates us to cast the valuation problem under the *regime-switching* jump diffusion model, which involves Markov-modulated geometric Brownian motions with jumps. Under the regime-switching jump diffusion model, Buffington and Elliott [12] priced the American option. Since the market is incomplete, Elliott et al. [13] and Lin et al. [14] used Esscher transform to determine an equivalent martingale measure. Siu et al. [15] adopt the Laplace transform to value the contingent options. Ignatieva et al. [16] used Fourier space time-stepping (FST) algorithm to price and hedge the guaranteed minimum benefits (GMBs) embedded in VAs. Mamon et al. [17] obtained the semi-closed-form solution of GMMB price by a series of measure changes, and employed the Fourier transform to carry out numerical approximation of the price. In addition, many researches have studied the valuation of related financial products in the regime-switching jump diffusion model. See, for example, Zhang and Guo [18], Cui et al. [19], Cui et al. [20], Kang et al. [21], Kirkby et al. [22], Kirkby and Nguyen [23], Wang et al. [24], Ai and Zhang [25], etc.

In light of the above, we shall focus on the pricing problems of VAs under the regime-switching jump diffusion model and with surrender risk modeled based on the intensity-based framework. We consider GMMB benefits with both continuous and discrete surrender behaviors. Due to the change in the cash flows of its riders from surrender, this puts forward higher requirements for the insurance company to price the product. To model the dynamics of the underlying fund, we use regime-switching jump diffusion model, which is more general than pure Lévy processes. For the determination of the surrender time, we apply a continuous-time Markov chain (CTMC) method to approximate the corresponding intensity process. The CTMC has recently gained popularity as an efficient and accurate method for approximating diffusion processes. The CTMC method discretizes the state space, but preserves the continuous time nature of the original diffusion model, which has several clear advantages. One clear advantage is the avoidance of time discretization and also the need for recursive valuations such as in a discrete-time lattice such as binomial tree. The CTMC approximation method was proposed by Mijatović and Pistorius [26] to price barrier options. In the one-dimensional case, many scholars use the CTMC method to approximate Markov process and carry out research on the valuation of various financial products. Lo and Skindilias [27] generalized the CTMC approximation method, and proposed a nonuniform grid design for stochastic differential equations with jumps. The grid design has been adopted in Ding et al. [28], Kirkby and Nguyen [23], Cui et al. [29], Kirkby et al. [22]. Cai et al. [30] priced both continuously and discretely monitored Asian options and obtained the closed-form double transform approximation formulas. Cui et al. [31] generalized the results of Cai et al. [30], and they obtained explicit single Laplace transforms for Asian options. Zhang et al. [32] developed two algorithms for solving the perpetual optimal stopping problem. An updated account of the extant literature on the applications of the CTMC method can be found in Cui et al. [33] and the references therein. Recently, Zhang and Li [34] and Li and Zhang [35] established the precise convergence rate of the CTMC method with a newly designed grid scheme. Other methods of stochastic approximation can be found in Kushner and Yin [36]. In order to handle the early surrender feature, we also utilize the Fourier cosine (COS) series expansion method. The COS method was first proposed in Fang and Oosterlee [37], and has consequently been applied in finance and insurance, see Fang and Oosterlee [37], Ruijter and Oosterlee [38], Zhang and Oosterlee [39], Zhang and Oosterlee [40], Chau et al. [41], Zhang [42], Xie and Zhang [43], etc.

The contributions of the paper are three-fold: First, we consider a general regime-switching jump diffusion model with consideration of the surrender option, and this generalizes previous literature which primarily considers diffusion models without one or all of the three features: regime switching, jumps, and the surrender option. Second, with the previous literature considering only continuous surrender, our framework also incorporates the more practical case of discrete surrender structure. Third, we present explicit closed-form formulas for the value of the GMMB contract by novelly combining the method of CTMC approximation and Fourier cosine expansions. The closed-form valuation formulas allow us to also compute the hedging parameters, i.e. Greeks, in closed-form, which is relevant to the sensitivity analysis and risk management of VAs under surrender risk.

The remainder of the paper is organized as follows. In Section 2, we present the regime switching jump diffusion model with the surrender option. Section 3 introduces the CTMC approximation method. Sections 4 and 5 discuss the valuation of the surrender option in continuous and discrete monitoring cases, respectively. Error analysis of our method is carried out in Section 6. In Section 7, our method is shown to be both efficient and accurate. Finally, Section 8 concludes the paper.

2. Problem formulation

2.1. The model

In this paper, all stochastic quantities are considered in a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{Q})$, where $\mathbb{F} = (\mathcal{G}_t)_{t\geq 0}$ is a filtration augmented in the usual way, and $\mathbb{Q}$ is the risk-neutral probability measure. The underlying price process is described by a regime-switching jump diffusion process as follows. Suppose $\{\alpha_1(t), t \geq 0\}$ is a continuous-time Markov chain (CTMC), which makes transitions between states in a finite state space $\mathcal{M}_1 = \{1, 2, \dots, m_1\}$. The transition law of $\alpha_1(t)$ is governed by an intensity matrix $\mathbf{\Lambda} = (\lambda_{jk})_{m_1\times m_1}$. The elements λ_{jk} of $\mathbf{\Lambda}$ satisfy: (i) $\lambda_{jj} \leq 0$, and $\lambda_{jk} \geq 0$, if $j \neq k$, and (ii) $\sum_k \lambda_{jk} = 0$, $\forall\, j \in \mathcal{M}_1$. In terms of λ_{jk}, $\alpha_1(t)$ makes transitions according to

$$\mathbb{Q}(\alpha_1(t+\Delta_t) = k|\alpha_1(t) = j, \alpha_1(t'), 0 \leq t' \leq t) = \lambda_{jk}\Delta_t + o(\Delta_t), \qquad \forall j \neq k,$$

for some small time increment $\Delta_t > 0$.

The underlying price process $(S_t)_{t\geq 0}$ is modeled by

$$\frac{dS_t}{S_{t-}} = (r - \kappa_{\alpha_1(t)})dt + \sigma_{\alpha_1(t)}dW_t^1 + \int_{x\in\mathbb{R}} [e^x - 1]\mathcal{N}_{\alpha_1(t)}(dx, dt),$$

where r is the risk-free interest rate and $(W_t^1)_{t\geq 0}$ is a standard $\mathbb{Q}$-Brownian motion. Here for each regime $j \in \mathcal{M}_1$, $\sigma_j > 0$ is the diffusion volatility parameter, $\mathcal{N}_j(dx, dt)$ is a Poisson random jump measure with Lévy measure Π_j, and $\kappa_j = \int [e^x - 1]\Pi_j(dx)$ is finite. The log price process $X_t = \log[S_t/S_0]$ is a regime-switching jump diffusion process that satisfies the dynamics

$$dX_t = \left(r - \kappa_{\alpha_1(t)} - \frac{1}{2}\sigma^2_{\alpha_1(t)}\right)dt + \sigma_{\alpha_1(t)}dW_t^1 + \int_{x\in\mathbb{R}} x\mathcal{N}_{\alpha_1(t)}(dx, dt).$$

For $\Delta_t > 0$ and $\omega \in \mathbb{R}$, the risk-neutral characteristic function of X_{Δ_t} is given by

$$\mathbb{E}[e^{i\omega X_{\Delta_t}}|\alpha_1(0 \leq s \leq \Delta_t) = j] := e^{\psi_j^X(\omega)t}, \qquad \omega \in \mathbb{R}, \qquad j \in \mathcal{M}_1,$$

where $\mathbb{E}(\cdot)$ denotes the expectation operator associated with $\mathbb{Q}$ and

$$\psi_j^X(\omega) = i\omega(r - \kappa_j - \frac{1}{2}\sigma_j^2) - \frac{1}{2}\sigma_j^2\omega^2 + \int (e^{i\omega x} - 1)\Pi_j(dx)$$

is called the characteristic exponent.

In this paper, matrix and vector symbols will be used frequently in our analysis. We use $\boldsymbol{e}_j$ to denote a column vector of zeros with appropriate dimension, except for the value 1 in the position j. We use $\boldsymbol{I}$ to denote the identity matrix with appropriate dimension. For any matrix $\boldsymbol{A}$ and vector $\boldsymbol{a}$, we use $\boldsymbol{A}'$ and vector $\boldsymbol{a}'$ to denote their transposes. For a square matrix $\boldsymbol{A}$, the corresponding matrix exponential is defined by $\mathrm{Exp}(\boldsymbol{A}) = \sum_{n=0}^{\infty} \frac{\boldsymbol{A}^n}{n!}$.

2.2. GMMB with surrender option

In this subsection, we introduce the VA contract with GMMB riders under the above regime-switching models. Assume that the insurer continuously takes out a constant rate $c > 0$ from the policyholder's account as compensation for the VA contract provided. Then the policyholder's account value at time t is given by

$$F_t = e^{-ct}S_t, \qquad t \geq 0.$$

Let $T > 0$ denote the maturity of the VA contract, the value of a VA contract without the surrender option at initial time is given by

$$\mathbb{E}[e^{-rT}\max(F_T, G_T)] = e^{-rT}\mathbb{E}[F_T] + e^{-rT}\mathbb{E}[(G_T - F_T)_+],$$

where G_T is the guaranteed minimum payoff at time T, and for any real number x, $x_+ := \max(x, 0)$.

Jeon and Kwak [8] consider a VA contract with surrender option, in which the policyholder is allowed to surrender the embedded guarantee and receive a surrender benefit with a penalty at any time before maturity, and the policyholder will receive his/her account value at maturity. In this paper, we shall also assume that the policyholder could exercise the surrender option at time t before maturity T, with surrender benefit

$$P(t)(G_t - F_t)_+, \qquad t > 0,$$

where G_t denotes the guarantee level at time t, and $P(\cdot)$ is an increasing penalty function satisfying $0 < P(t) < 1$ for $0 \leq t < T$, and $P(t) = 1$ for $t \geq T$. For this GMMB with surrender option, we assume that surrender is only possible after a pre-specified time $t_1 > 0$, and denote the random variable τ as the policyholder's surrender time. Then the value of VA with surrender option is given by

$$e^{-rT}\mathbb{E}[F_T] + \mathbb{E}\left[e^{-r(\tau\wedge T)}P(\tau \wedge T)(G_{\tau\wedge T} - F_{\tau\wedge T})_+\right], \tag{2.1}$$

where the first term in (2.1) is the expected discounted account value, and is given by

$$e^{-rT}\mathbb{E}[F_T] = e^{-cT}F_0,$$

since the discounted process $e^{-rt}S_t$ is a $\mathbb{Q}$-martingale. Hence, it remains to compute the second term in (2.1) which corresponds to the value of the surrender option.

For the surrender risk, we use an intensity-based approach to model the surrender time. Let ν_t denote the surrender intensity, which satisfies the following stochastic differential equation

$$d\nu_t = \hat{\mu}(\nu_t)dt + \hat{\sigma}(\nu_t)dW_t^2, \tag{2.2}$$

where $(W_t^2)_{t\geq 0}$ is a standard Brownian motion satisfying $\mathbb{E}[dW_t^1 dW_t^2] = \rho dt$ for some $\rho \in (-1, 1)$. We suppose that the coefficients $\hat{\mu}$ and $\hat{\sigma}$ satisfy the following Lipschitz condition and linear growth condition, i.e. there exist some constants $C_1, C_2 > 0$ satisfying

$$|\hat{\mu}(x) - \hat{\mu}(y)| + |\hat{\sigma}(x) - \hat{\sigma}(y)| \leq C_1|x - y|, \quad |\hat{\mu}(x)|^2 + |\hat{\sigma}(x)|^2 \leq C_2(1 + x^2)$$

for all x, y in the state space of ν_t. Note that the above conditions ensure that there exists a unique strong solution ν_t satisfying the strong Markov property. Throughout this paper, we suppose that $\hat{\sigma}(\cdot) > 0$ on the domain of ν_t.

In this paper, we shall consider the following two types of surrender structures:

- (**continuous surrender**): The surrender is possible continuously on the interval $[t_1, T)$, and the conditional probability that surrender does not occur before time t is

$$\mathbb{Q}(\tau \geq t|\mathcal{G}_t) = e^{-\int_0^t \nu_s ds}, \quad t \geq 0.$$

- (**discrete surrender**): The surrender is possible only at a sequence of pre-specified payment times $0 < t_1 < \cdots < t_{M-1} < t_M = T$, where $t_{m+1} - t_m = \Delta_t$ for $m = 1, \ldots, M-1$ and $\Delta_t > 0$. In this case, we suppose that for $m = 1, \ldots, M-1$, $\mathbf{1}_{\{\tau = t_m\}} = \mathbf{1}_{\{\tau \geq t_m\}} - \mathbf{1}_{\{\tau \geq t_{m+1}\}}$ with

$$\mathbb{Q}(\tau \geq t_m|\mathcal{G}_{t_m}) = e^{-\int_0^{t_m} \nu_s ds}, \quad m = 1, 2, \ldots, M.$$

Under above assumptions, we shall pay attention to computing the following conditional expectation

$$\mathcal{V}_j(\nu_0) = \mathbb{E}\left[e^{-r(\tau\wedge T)}P(\tau \wedge T) \cdot (G_{\tau\wedge T} - F_{\tau\wedge T})_+ \middle| \nu_0, \alpha_1(0) = j\right], \quad \nu_0 > 0,\ j \in \mathcal{M}_1, \tag{2.3}$$

which is the value of the surrender option.

3. CTMC approximation

In this section, we shall apply the CTMC method to approximate the intensity process ν_t, so that our valuation problem can be solved under a regime-switching framework. Note that the two Brownian motions W_t^1 and W_t^2 are not independent. In order to use the CTMC approximation method, it is more convenient to separate these two Brownian motions into two independent components. For this purpose, we adapt the strategy in Kirkby et al. [22].

First, define the following auxiliary functions under regime $j \in \mathcal{M}_1$,

$$\hat{f}_j(x) = \int_{\cdot}^{x} \frac{\sigma_j}{\hat{\sigma}(u)}du, \quad h_j(x) = \hat{\mu}(x)\hat{f}_j'(x) + \frac{1}{2}\hat{\sigma}^2(x)\hat{f}_j''(x), \quad f_j(\nu_0, \nu_t) = \rho[\hat{f}_j(\nu_t) - \hat{f}_j(\nu_0)].$$

By Ito's lemma, we have

$$df_j(\nu_0, \nu_t) = \rho\sigma_j dW_t^2 + \rho h_j(\nu_t)dt. \tag{3.1}$$

Furthermore, define

$$W_t^* := \frac{W_t^1 - \rho W_t^2}{\sqrt{1-\rho^2}}, \qquad t \geq 0,$$

which is also a standard $\mathbb{Q}$-Brownian motion, and independent of W_t^2. Using W_t^* and formula (3.1) we obtain

$$\begin{aligned}
dX_t &= \left(r - \kappa_{\alpha_1(t)} - \frac{1}{2}\sigma_{\alpha_1(t)}^2\right)dt + \sigma_{\alpha_1(t)}\left(\rho dW_t^2 + \sqrt{1-\rho^2}dW_t^*\right) + \int_{x\in\mathbb{R}} x\mathcal{N}_{\alpha_1(t)}(dx, dt) \\
&= \left(r - \kappa_{\alpha_1(t)} - \frac{1}{2}\sigma_{\alpha_1(t)}^2\right)dt + df_{\alpha_1(t)}(\nu_0, \nu_t) - \rho h_{\alpha_1(t)}(\nu_t)dt \\
&\quad + \sigma_{\alpha_1(t)}\sqrt{1-\rho^2}dW_t^* + \int_{x\in\mathbb{R}} x\mathcal{N}_{\alpha_1(t)}(dx, dt),
\end{aligned}$$

which yields, for $\widetilde{X}_t := \log(S_t/S_0) - f_{\alpha_1(t)}(\nu_0, \nu_t)$,

$$d\widetilde{X}_t = \left(r - \kappa_{\alpha_1(t)} - \frac{1}{2}\sigma^2_{\alpha_1(t)} - \rho h_{\alpha_1(t)}(\nu_t)\right) dt + \sigma_{\alpha_1(t)}\sqrt{1-\rho^2}dW_t^* + \int_{x\in\mathbb{R}} x\mathcal{N}_{\alpha_1(t)}(dx, dt). \tag{3.2}$$

Now the stock price process can be expressed as

$$S_t = S_0 e^{\widetilde{X}_t + f_{\alpha_1(t)}(\nu_0,\nu_t)}, \qquad t \geq 0. \tag{3.3}$$

Next, we construct the CTMC approximation of the intensity process ν_t. Let $(\alpha_2(t))_{t\geq 0}$ be a CTMC that makes transitions in the state space $\mathcal{M}_2 := \{1, 2, \dots, m_2\}$. The transition dynamics of $\alpha_2(t)$ are determined by the generator matrix $\mathbf{Q} = (q_{jk})_{m_2\times m_2}$. The elements q_{jk} of $\mathbf{Q}$ satisfy: (i) $q_{jj} \leq 0$, and $q_{jk} \geq 0$, if $j \neq k$, and (ii) $\sum_k q_{jk} = 0$, $\forall\, j \in \mathcal{M}_2$. Starting from a state $j \in \mathcal{M}_2$, $\alpha_2(t)$ makes transitions according to

$$\mathbb{Q}(\alpha_2(t + \Delta_t) = k|\alpha_2(t) = j, \alpha_2(t'), 0 \leq t' \leq t) = q_{jk}\Delta_t + o(\Delta_t), \qquad \forall k \neq j, \tag{3.4}$$

for a small time increment $\Delta_t > 0$.

For the intensity process ν_t, we approximate it by a CTMC $\bar{\nu}_{\alpha_2(t)}$, which has a state space $\boldsymbol{\nu} = \{\bar{\nu}_1, \dots, \bar{\nu}_{m_2}\}$ with $\bar{\nu}_i < \bar{\nu}_{i+1}$ and $\mathbf{k} = \{k_1, \dots, k_{m_2-1}\}$, with the set of grid spacing $k_i = \bar{\nu}_{i+1} - \bar{\nu}_i$. Assume that the initial value $\nu_0 \in \boldsymbol{\nu}$, and inspired by Lo and Skindilias [27], we construct the entries in the generator matrix $\mathbf{Q}$ as

$$q_{ij} = \begin{cases} \dfrac{\hat{\mu}^-(\bar{\nu}_i)}{k_{i-1}} + \dfrac{\hat{\sigma}^2(\bar{\nu}_i) - (k_{i-1}\hat{\mu}^-(\bar{\nu}_i) + k_i\hat{\mu}^+(\bar{\nu}_i))}{k_{i-1}(k_{i-1}+k_i)}, & \text{if } j = i-1, \\ \dfrac{\hat{\mu}^-(\bar{\nu}_i)}{k_i} + \dfrac{\hat{\sigma}^2(\bar{\nu}_i) - (k_{i-1}\hat{\mu}^-(\bar{\nu}_i) + k_i\hat{\mu}^+(\bar{\nu}_i))}{k_i(k_{i-1}+k_i)}, & \text{if } j = i+1, \\ -q_{i,i-1} - q_{i,i+1}, & \text{if } j = i, \\ 0, & \text{if } j \neq i-1, i, i+1. \end{cases}$$

In addition, if the set $\mathbf{k}$ satisfies

$$0 < \max_{1\leq i\leq m_2-1}\{k_i\} \leq \min_{1\leq i\leq m_2}\left\{\frac{\hat{\sigma}^2(\bar{\nu}_i)}{\mid \hat{\mu}(\bar{\nu}_i)\mid}\right\},$$

then, we have

$$\hat{\sigma}^2(\bar{\nu}_i) \geq \max_{1\leq i\leq m_2-1}\{k_i\}\cdot \mid \hat{\mu}(\bar{\nu}_i)\mid \geq \max_{1\leq i\leq m_2-1}\{k_i\}\cdot(\hat{\mu}^+(\bar{\nu}_i) - \hat{\mu}^-(\bar{\nu}_i)) \geq k_{i-1}\hat{\mu}^-(\bar{\nu}_i) + k_i\hat{\mu}^+(\bar{\nu}_i).$$

Therefore, $q_{ij} \geq 0$ for $1 \leq i \neq j \leq m_2$, and $\sum_{j=1}^{m_2} q_{ij} = 0$, $i = 1, \dots, m_2$.

In order to apply the CTMC approximation, we determine the variance grids $\{\bar{\nu}_j\}_{j=1}^{m_2}$ as follows. According to Tavella and Randall [44], we define a non-uniform grid as

$$\bar{\nu}_j = \nu_0 + \bar{\alpha}\sinh\left(c_2\frac{j}{m_2} + c_1\left(1 - \frac{j}{m_2}\right)\right),\ j = 2, \dots, m_2 - 1,$$

where

$$c_1 = \operatorname{arcsinh}\left(\frac{\bar{\nu}_1 - \nu_0}{\bar{\alpha}}\right),$$

and

$$c_2 = \operatorname{arcsinh}\left(\frac{\bar{\nu}_{m_2} - \nu_0}{\bar{\alpha}}\right).$$

A smaller $\bar{\alpha}$ can make the grid more dense, and in this paper we set $\bar{\alpha} = \frac{\bar{\nu}_{m_2} - \bar{\nu}_1}{2}$. Further, for grid boundaries $\bar{\nu}_1$ and $\bar{\nu}_{m_2}$, we define $\bar{\mu}(t) = \mathbb{E}[\nu_t|\nu_0]$ and $\bar{\sigma}(t)$ is the standard deviation under ν_0. If ν_t is a positive process, then we let $\bar{\nu}_1 = \max\{\bar{\nu}, \bar{\mu}(t) - \gamma\bar{\sigma}(t)\}$, and otherwise, $\bar{\nu}_1 = \bar{\mu}(t) - \gamma\bar{\sigma}(t)$, where the constant $t = \frac{T}{2}$. Finally, we set constants $\gamma = 3 \sim 5$ and $0 < \bar{\nu} \ll 1$.

Remark 1. Unlike other discrete approximations, such as the lattice methods, the CTMC method eliminates time-discretization error. From Cui et al. [45], when the scheme satisfies certain regularity conditions, it is weakly convergent, and achieves a convergence order $\mathcal{O}(m_2^{-2})$. Furthermore, for a continuous function of ν_t, the convergence order in the spatial variable is of second order. Similar convergence order results appear in Ma et al. [46].

Based on the CTMC approximation $\bar{\nu}_{\alpha_2(t)}$, we can further approximate $\widetilde{X}_t$ by a regime-switching jump diffusion process $\bar{X}_t$, whose dynamics are given by

$$d\bar{X}_t = \left(r - \kappa_{\alpha_1(t)} - \frac{1}{2}\sigma^2_{\alpha_1(t)} - \rho h_{\alpha_1(t)}(\bar{\nu}_{\alpha_2(t)})\right) dt + \sigma_{\alpha_1(t)}\sqrt{1-\rho^2}dW_t^* + \int_{x\in\mathbb{R}} x\mathcal{N}_{\alpha_1(t)}(dx, dt).$$

It follows from formula (3.3) that

$$S_t = S_0 e^{\tilde{X}_t + f_{\alpha_1(t)}(v_0, v_t)} \approx \bar{S}_t := S_0 e^{\bar{X}_t + f_{\alpha_1(t)}(\bar{v}_{\alpha_2(0)}, \bar{v}_{\alpha_2(t)})}.$$

Note that the process $\bar{X}_t$ is driven by a two-dimensional CTMC $(\alpha_1(t), \alpha_2(t))$. For the convenience of the following analysis, we introduce a new CTMC with state space $\mathcal{M}_\beta = \{1, 2, \dots, m_1 \cdot m_2\}$ and generator matrix

$$\boldsymbol{G} = \begin{pmatrix} \lambda_{11}\boldsymbol{I} + \boldsymbol{Q} & \lambda_{12}\boldsymbol{I} & \cdots & \lambda_{1m_1}\boldsymbol{I} \\ \lambda_{21}\boldsymbol{I} & \lambda_{22}\boldsymbol{I} + \boldsymbol{Q} & \cdots & \lambda_{2m_1}\boldsymbol{I} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{m_1 1}\boldsymbol{I} & \lambda_{m_1 2}\boldsymbol{I} & \cdots & \lambda_{m_1 m_1}\boldsymbol{I} + \boldsymbol{Q} \end{pmatrix}.$$

Define two mappings $\pi_i : \mathcal{M}_\beta \to \mathcal{M}_i$, $i = 1, 2$, by

$$\pi_1(n) = j, \quad \pi_2(n) = k, \quad n \in \mathcal{M}_\beta,$$

where $j \in \mathcal{M}_1$, $k \in \mathcal{M}_2$, and $n = (j-1)m_2 + k$. It is easy to see that $\alpha_j(t)$ is identical in law to $\pi_j(\beta(t))$, $j = 1, 2$.

Now for each $n \in \mathcal{M}_\beta$, let

$$\bar{\mu}_n = r - \kappa_{\pi_1(n)} - \frac{1}{2}\sigma^2_{\pi_1(n)} - \rho h_{\pi_1(n)}(\nu_{\pi_2(n)}),$$
$$\bar{\sigma}_n = \sigma_{\pi_1(n)}\sqrt{1-\rho^2}, \quad \bar{\mathcal{N}}_n(dx, dt) = \mathcal{N}_{\pi_1(n)}(dx, dt),$$

and define a new regime-switching jump diffusion process Y_t by

$$dY_t = \bar{\mu}_{\beta(t)}dt + \bar{\sigma}_{\beta(t)}dW_t^* + \int_{x\in\mathbb{R}} x\bar{\mathcal{N}}_{\beta(t)}(dx, dt),$$

For $n_1, n_2 \in \mathcal{M}_\beta$, let

$$\tilde{f}_{n_1,n_2} = f_{\pi_1(n_2)}(\bar{\nu}_{\pi_2(n_1)}, \bar{\nu}_{\pi_2(n_2)}).$$

It is easy to see that $\bar{X}_t$ and Y_t have the same probability law, which further imply that the processes $\bar{S}_t$ and $S_0 e^{Y_t + \tilde{f}_{\beta(0),\beta(t)}}$ have the same probability law.

For any real numbers θ_1, θ_2, we have

$$d\left(\theta_1 \int_0^t \tilde{\nu}_{\beta(s)}ds + \theta_2 Y_t\right) = (\theta_1\tilde{\nu}_{\beta(t)} + \theta_2\bar{\mu}_{\beta(t)})dt + \theta_2\bar{\sigma}_{\beta(t)}dW_t^* + \int_{x\in\mathbb{R}} \theta_2 x\bar{\mathcal{N}}_{\beta(t)}(dx, dt).$$

Hence, the process $\theta_1 \int_0^t \tilde{\nu}_{\beta(s)}ds + \theta_2 Y_t$ is also a regime-switching jump diffusion process. By Asmussen [47], we know that the conditional moment generating function of $\left(\int_0^t \tilde{\nu}_{\beta(s)}ds, Y_t\right)$ is given by

$$\mathbb{E}[e^{\theta_1 \int_0^t \tilde{\nu}_{\beta(s)}ds + \theta_2 Y_t}; \beta(t) = n_1 | \beta(0) = n_0] = \boldsymbol{e}'_{n_0}\mathrm{Exp}(t\boldsymbol{K}(\theta_1, \theta_2))\boldsymbol{e}_{n_1}, \tag{3.5}$$

where

$$\boldsymbol{K}(\theta_1, \theta_2) = \boldsymbol{G} + \mathrm{diag}(\boldsymbol{\psi}_{11}(\theta_1, \theta_2), \dots, \psi_{1m_2}(\theta_1, \theta_2), \dots, \psi_{m_1 1}(\theta_1, \theta_2), \dots, \boldsymbol{\psi}_{m_1 m_2}(\theta_1, \theta_2))$$

with

$$\boldsymbol{\psi}_{jk}(\theta_1, \theta_2) = \theta_1\bar{\nu}_k - \rho\theta_2 h_j(\bar{\nu}_k) + \psi_j^X(-i\theta_2).$$

By analytic continuation, we find that the formula (3.5) still holds true when θ_1 and θ_2 are complex numbers.

4. Valuation under continuous surrender

In this section, we pay attention to the valuation of surrender options under continuous monitoring. Throughout this section, suppose that the guarantee level at time t is given by

$$G_t = G_0 e^{wt}, \qquad t \geq 0,$$

where $G_0 > 0$ is the initial guarantee and w is the rolled-up rate of guarantee. We further suppose that the penalty function $P(t)$ is also an exponential function taking the form

$$P(t) = e^{-\delta(T-t)}, \quad 0 \leq t \leq T, \ \delta \geq 0.$$

Under the above assumptions, the conditional expectation in (2.3) becomes

$$\begin{aligned}\mathcal{V}_j(\nu_0) &= \int_{t_1}^{T} e^{-rt} e^{-\delta(T-t)} \mathbb{E}\left[\nu_t e^{-\int_0^t \nu_s ds} \cdot \left(G_0 e^{wt} - F_t\right)_+ \Big| \nu_0, \alpha_1(0) = j\right] dt \\ &\quad + e^{-rT} \mathbb{E}\left[e^{-\int_0^T \nu_s ds} \cdot \left(G_0 e^{wT} - F_T\right)_+ \Big| \nu_0, \alpha_1(0) = j\right] \\ &= \int_{t_1}^{T} e^{(w-r)t} e^{-\delta(T-t)} \mathbb{E}\left[\nu_t e^{-\int_0^t \nu_s ds} \cdot \left(G_0 - e^{-(w+c)t} S_t\right)_+ \Big| \nu_0, \alpha_1(0) = j\right] dt \\ &\quad + e^{(w-r)T} \mathbb{E}\left[e^{-\int_0^T \nu_s ds} \cdot \left(G_0 - e^{-(w+c)T} S_T\right)_+ \Big| \nu_0, \alpha_1(0) = j\right]. \end{aligned} \tag{4.1}$$

In the remainder of this section, we shall focus on how to compute $\mathcal{V}_j(\nu_0)$ based on the above formulas.

4.1. Valuation by CTMC approximation

In order to use the CTMC approximation, suppose that $\nu_0 = \bar{\nu}_{\alpha_2(0)} = \bar{\nu}_k$ for some $k \in \mathcal{M}_2$. Then we have

$$\begin{aligned} &\mathcal{V}_j(\nu_0) \approx \widehat{\mathcal{V}}_j(\nu_0) \\ &:= \int_{t_1}^{T} e^{(w-r)t} e^{-\delta(T-t)} \mathbb{E}\left[\bar{\nu}_{\alpha_2(t)} e^{-\int_0^t \bar{\nu}_{\alpha_2(s)} ds} \cdot \left(G_0 - S_0 e^{\bar{X}_t - (w+c)t + f_{\alpha_1(t)}(\bar{\nu}_{\alpha_2(0)}, \bar{\nu}_{\alpha_2(t)})}\right)_+ \Big| \alpha_1(0) = j, \alpha_2(0) = k\right] dt \\ &\quad + e^{(w-r)T} \mathbb{E}\left[e^{-\int_0^T \bar{\nu}_{\alpha_2(s)} ds} \cdot \left(G_0 - S_0 e^{\bar{X}_T - (w+c)T + f_{\alpha_1(T)}(\bar{\nu}_{\alpha_2(0)}, \bar{\nu}_{\alpha_2(T)})}\right)_+ \Big| \alpha_1(0) = j, \alpha_2(0) = k\right] \\ &= \int_{t_1}^{T} e^{(w-r)t} e^{-\delta(T-t)} \mathbb{E}\left[\tilde{\nu}_{\beta(t)} e^{-\int_0^t \tilde{\nu}_{\beta(s)} ds} \cdot \left(G_0 - S_0 e^{\bar{Y}_t + \tilde{f}_{\beta(0),\beta(t)}}\right)_+ \Big| \beta(0) = (j-1)m_2 + k\right] dt \\ &\quad + e^{(w-r)T} \mathbb{E}\left[e^{-\int_0^T \tilde{\nu}_{\beta(s)} ds} \cdot \left(G_0 - S_0 e^{\bar{Y}_T + \tilde{f}_{\beta(0),\beta(T)}}\right)_+ \Big| \beta(0) = (j-1)m_2 + k\right], \end{aligned}$$

where for $t \geq 0$, $\bar{Y}_t = \bar{X}_t - (w+c)t$, and for $n \in \mathcal{M}_\beta$, $\tilde{\nu}_n = \bar{\nu}_{\pi_2(n)}$. It follows that we only need to compute the following functions: for $n_0 \in \mathcal{M}_\beta$,

$$\begin{aligned} \widehat{\mathcal{V}}_{n_0,1} &= \int_{t_1}^{T} e^{(w-r)t} e^{-\delta(T-t)} \mathbb{E}\left[\tilde{\nu}_{\beta(t)} e^{-\int_0^t \tilde{\nu}_{\beta(s)} ds} \cdot \left(G_0 - S_0 e^{\bar{Y}_t + \tilde{f}_{\beta(0),\beta(t)}}\right)_+ \Big| \beta(0) = n_0\right] dt, \\ \widehat{\mathcal{V}}_{n_0,2} &= e^{(w-r)T} \mathbb{E}\left[e^{-\int_0^T \tilde{\nu}_{\beta(s)} ds} \cdot \left(G_0 - S_0 e^{\bar{Y}_T + \tilde{f}_{\beta(0),\beta(T)}}\right)_+ \Big| \beta(0) = n_0\right]. \end{aligned}$$

For $\widehat{\mathcal{V}}_{n_0,1}$, we have

$$\begin{aligned} \widehat{\mathcal{V}}_{n_0,1} &= \sum_{n_1 \in \mathcal{M}_\beta} e^{-\delta T} \tilde{\nu}_{n_1} \int_{t_1}^{T} e^{(w+\delta-r)t} \mathbb{E}\left[e^{-\int_0^t \tilde{\nu}_{\beta(s)} ds} \cdot \left(G_0 - S_0 e^{\bar{Y}_t + \tilde{f}_{n_0,n_1}}\right)_+ ; \beta(t) = n_1 \Big| \beta(0) = n_0\right] dt \\ &= \sum_{n_1 \in \mathcal{M}_\beta} e^{-\delta T} \tilde{\nu}_{n_1} \int_{t_1}^{T} e^{(w+\delta-r)t} \int_{-\infty}^{+\infty} \left(G_0 - S_0 e^{x + \tilde{f}_{n_0,n_1}}\right)_+ \cdot g_{n_0,n_1}(x;t) dx dt, \end{aligned} \tag{4.2}$$

where for each $t > 0$ and $n_0, n_1 \in \mathcal{M}_\beta$, $g_{n_0,n_1}(x;t)$ is a conditional density function satisfying

$$g_{n_0,n_1}(x;t)dx = \mathbb{E}\left[e^{-\int_0^t \tilde{\nu}_{\beta(s)} ds}, \bar{Y}_t \in dx; \beta(t) = n_1 \Big| \beta(0) = n_0\right], \qquad x \in \mathbb{R}.$$

Similarly, we have

$$\begin{aligned} \widehat{\mathcal{V}}_{n_0,2} &= \sum_{n_1 \in \mathcal{M}_\beta} e^{(w-r)T} \mathbb{E}\left[e^{-\int_0^T \tilde{\nu}_{\beta(s)} ds} \cdot \left(G_0 - S_0 e^{\bar{Y}_T + \tilde{f}_{\beta(0),\beta(T)}}\right)_+ ; \beta(T) = n_1 \Big| \beta(0) = n_0\right] \\ &= \sum_{n_1 \in \mathcal{M}_\beta} e^{(w-r)T} \int_{-\infty}^{+\infty} \left(G_0 - S_0 e^{x + \tilde{f}_{n_0,n_1}}\right)_+ \cdot g_{n_0,n_1}(x;T) dx. \end{aligned} \tag{4.3}$$

Hence, we still need to compute the integrals in (4.2) and (4.3), which will be discussed in details in the next subsection.

4.2. Valuation by Fourier cosine expansions

In this subsection, we apply the Fourier cosine expansion of Fang and Oosterlee [37,48] to compute the integrals in (4.2) and (4.3). This method is very efficient for approximating an integrable function as long as it has a closed-form

Fourier transform. For an integrable function $g(\cdot)$, define its Fourier transform by

$$\mathcal{F}g(\omega)=\int_{-\infty}^{+\infty}e^{i\omega x}g(x)dx,\ \ \omega\in\mathbb{R}.$$

On a finite interval $[a,b]$, we can approximate the density function g by its Fourier-cosine series expansion as follows,

$$g(x)\approx\widetilde{g}(x):=\sum_{l=0}^{L-1}{}^{\prime}A_l(g)\cdot\cos\left(l\pi\frac{x-a}{b-a}\right),\ \ x\in[a,b],$$

where the positive integer L is the truncation parameter, and the $\sum'$ indicates that the first term in summation is multiplied by $1/2$. The COS series coefficients $A_l(g)$ are given by

$$A_l(g)=\frac{2}{b-a}\Re\left\{\mathcal{F}g\left(\frac{l\pi}{b-a}\right)e^{-il\pi\frac{a}{b-a}}\right\},\ \ l=0,1,\dots,L-1,$$

where $\Re(\cdot)$ means taking the real part of the complex number in bracket.

For the conditional density function $g_{n_0,n_1}(x;t)$, it does not have a closed-form expression. However, it has a closed-form Fourier transform given by

$$\begin{aligned}\mathcal{F}g_{n_0,n_1}(\omega;t)&=\int_{-\infty}^{+\infty}e^{i\omega x}g_{n_0,n_1}(x;t)dx\\&=\mathbb{E}\left[e^{-\int_0^t\tilde{\nu}_{\beta(s)}ds+i\omega\bar{Y}_t};\beta(t)=n_1\middle|\beta(0)=n_0\right]\\&=\mathbb{E}\left[e^{-\int_0^t\tilde{\nu}_{\beta(s)}ds+i\omega Y_t-i\omega(w+c)t};\beta(t)=n_1\middle|\beta(0)=n_0\right]\\&=\boldsymbol{e}_{n_0}'\mathrm{Exp}(t\,(\boldsymbol{K}(-1,i\omega)-i\omega(w+c)\boldsymbol{I})\boldsymbol{e}_{n_1}.\end{aligned}$$

By the COS method, we can approximate $g_{n_0,n_1}(x;t)$ by

$$\widetilde{g}_{n_0,n_1}(x;t):=\sum_{l=0}^{L-1}{}^{\prime}A_l(g_{n_0,n_1}(\cdot;t))\cdot\cos\left(l\pi\frac{x-a}{b-a}\right),\ \ x\in[a,b],$$

where for $l=0,1,\dots,L-1$,

$$\begin{aligned}&A_l(g_{n_0,n_1}(\cdot;t))\\=&\frac{2}{b-a}\Re\left\{\mathcal{F}g_{n_0,n_1,l}\left(\frac{l\pi}{b-a};t\right)e^{-il\pi\frac{a}{b-a}}\right\}\\=&\frac{2}{b-a}\Re\left\{\boldsymbol{e}_{n_0}'\mathrm{Exp}\left(t\left(\boldsymbol{K}\left(-1,i\frac{l\pi}{b-a}\right)-i\frac{l\pi}{b-a}(w+c)\boldsymbol{I}\right)\right)\boldsymbol{e}_{n_1}e^{-il\pi\frac{a}{b-a}}\right\}.\end{aligned}\tag{4.4}$$

Using the COS approximation $\widetilde{g}_{n_0,n_1}(x;t)$, we can approximate $\widehat{\mathcal{V}}_{n_0,1}$ as follows

$$\widehat{\mathcal{V}}_{n_0,1}\approx\widetilde{\mathcal{V}}_{n_0,1}\tag{4.5}$$

$$:=\sum_{n_1\in\mathcal{M}_\beta}e^{-\delta T}\tilde{\nu}_{n_1}\int_{t_1}^{T}e^{(w+\delta-r)t}\int_a^b\left(G_0-S_0e^{x+\tilde{f}_{n_0,n_1}}\right)_+\cdot\widetilde{g}_{n_0,n_1}(x;t)dxdt$$

$$=e^{-\delta T}\sum_{l=0}^{L-1}{}^{\prime}\sum_{n_1\in\mathcal{M}_\beta}\tilde{\nu}_{n_1}\int_{t_1}^{T}e^{(w+\delta-r)t}A_l(g_{n_0,n_1}(\cdot;t))dtV_{n_0,n_1,l},\tag{4.6}$$

where

$$V_{n_0,n_1,l}=\int_a^b\left(G_0-S_0e^{x+\tilde{f}_{n_0,n_1}}\right)_+\cdot\cos\left(l\pi\frac{x-a}{b-a}\right)dx,\qquad n_0,n_1\in\mathcal{M}_\beta,\ l=0,1,\dots,L-1.$$

By formula (4.4), the integral in (4.5) can be explicitly computed as follows

$$\begin{aligned}&\int_{t_1}^{T}e^{(w+\delta-r)t}A_l(g_{n_0,n_1}(\cdot;t))dt\\=&\frac{2}{b-a}\int_{t_1}^{T}e^{(w+\delta-r)t}\Re\left\{\boldsymbol{e}_{n_0}'\mathrm{Exp}\left(t\left(\boldsymbol{K}\left(-1,i\frac{l\pi}{b-a}\right)-i\frac{l\pi}{b-a}(w+c)\boldsymbol{I}\right)\right)\boldsymbol{e}_{n_1}e^{-il\pi\frac{a}{b-a}}\right\}dt\end{aligned}$$

$$= \frac{2}{b-a}\boldsymbol{e}_{n_0}'\Re\left\{e^{-il\pi\frac{a}{b-a}}\int_{t_1}^{T}\mathrm{Exp}(t\boldsymbol{P}_l)dt\right\}\boldsymbol{e}_{n_1}$$

$$= \frac{2}{b-a}\boldsymbol{e}_{n_0}'\Re\left\{e^{-il\pi\frac{a}{b-a}}\boldsymbol{P}_l^{-1}[\mathrm{Exp}(T\boldsymbol{P}_l)-\mathrm{Exp}(t_1\boldsymbol{P}_l)]\right\}\boldsymbol{e}_{n_1},$$

where

$$\boldsymbol{P}_l = \boldsymbol{K}\Big(-1, i\frac{l\pi}{b-a}\Big) - i\frac{l\pi}{b-a}(w+c)\boldsymbol{I} + (w+\delta-r)\boldsymbol{I}.$$

Hence, formula (4.5) becomes

$$\widetilde{\mathcal{V}}_{n_0,1} \approx \frac{2e^{-\delta T}}{b-a}\sum_{l=0}^{L-1}{}'\boldsymbol{e}_{n_0}'\Re\left\{e^{-il\pi\frac{a}{b-a}}\boldsymbol{P}_l^{-1}[\mathrm{Exp}(T\boldsymbol{P}_l)-\mathrm{Exp}(t_1\boldsymbol{P}_l)]\right\}\boldsymbol{a}_{1,l},$$

where for $l = 0, 1, \ldots, L-1$,

$$\boldsymbol{a}_{1,l} = [\tilde{\nu}_1 V_{n_0,1,l}, \tilde{\nu}_2 V_{n_0,2,l}, \ldots, \tilde{\nu}_{m_1m_2} V_{n_0,m_1m_2,l}]'.$$

Similarly, $\widehat{\mathcal{V}}_{n_0,2}$ in (4.3) can be approximated as follows

$$\begin{aligned}
&\widetilde{\mathcal{V}}_{n_0,2} \approx \widehat{\mathcal{V}}_{n_0,2}\\
&:= \sum_{n_1\in\mathcal{M}_\beta} e^{(w-r)T}\int_a^b \Big(G_0 - S_0e^{x+\tilde{f}_{n_0,n_1}}\Big)_+ \cdot \widetilde{g}_{n_0,n_1}(x;T)dx\\
&= e^{(w-r)T}\sum_{l=0}^{L-1}{}'\sum_{n_1\in\mathcal{M}_\beta} A_l(g_{n_0,n_1}(\cdot;T))V_{n_0,n_1,l}\\
&= \frac{2e^{(w-r)T}}{b-a}\sum_{l=0}^{L-1}{}'\boldsymbol{e}_{n_0}'\Re\left\{e^{-il\pi\frac{a}{b-a}}\mathrm{Exp}\left(T\left(\boldsymbol{P}_l-(w+\delta-c)\boldsymbol{I}\right)\right)\right\}\boldsymbol{a}_{2,l},
\end{aligned}$$

where for $l = 0, 1, \ldots, L-1$,

$$\boldsymbol{a}_{2,l} = [V_{n_0,1,l}, V_{n_0,2,l}, \ldots, V_{n_0,m_1m_2,l}]'.$$

Remark 2. The integral in $V_{n_0,n_1,l}$ can also be explicitly computed. Usually, the domain truncation parameters a and b satisfy $a < \ln(\frac{G_0}{S_0}) - \tilde{f}_{n_0,n_1} < b$. In this case, we have

$$\begin{aligned}
V_{n_0,n_1,l} &= \int_a^{\ln(\frac{G_0}{S_0})-\tilde{f}_{n_0,n_1}}\Big(G_0 - S_0e^{x+\tilde{f}_{n_0,n_1}}\Big)\cdot\cos\left(l\pi\frac{x-a}{b-a}\right)dx\\
&= G_0\int_a^{\ln(\frac{G_0}{S_0})-\tilde{f}_{n_0,n_1}}\cos\left(l\pi\frac{x-a}{b-a}\right)dx - S_0e^{\tilde{f}_{n_0,n_1}}\int_a^{\ln(\frac{G_0}{S_0})-\tilde{f}_{n_0,n_1}} e^x\cdot\cos\left(l\pi\frac{x-a}{b-a}\right)dx.
\end{aligned}$$

Furthermore, for $x_1 < x_2$, set

$$\begin{aligned}
\chi_l(x_1,x_2) &:= \int_{x_1}^{x_2} e^y\cos\left(l\pi\frac{y-a}{b-a}\right)dy\\
&= \frac{1}{1+(\frac{l\pi}{b-a})^2}\Bigg[\cos\left(l\pi\frac{x_2-a}{b-a}\right)e^{x_2} - \cos\left(l\pi\frac{x_1-a}{b-a}\right)e^{x_1}\\
&\quad + \frac{l\pi}{b-a}\sin\left(l\pi\frac{x_2-a}{b-a}\right)e^{x_2} - \frac{l\pi}{b-a}\sin\left(l\pi\frac{x_1-a}{b-a}\right)e^{x_1}\Bigg]
\end{aligned}$$

and

$$\zeta_l(x_1,x_2) := \int_{x_1}^{x_2}\cos\left(l\pi\frac{y-a}{b-a}\right)dy = \begin{cases} \frac{b-a}{l\pi}\left[\sin\left(l\pi\frac{x_2-a}{b-a}\right) - \sin\left(l\pi\frac{x_1-a}{b-a}\right)\right], & l \neq 0,\\ x_2 - x_1, & l = 0.\end{cases}$$

Then we have

$$V_{n_0,n_1,l} = G_0\zeta_l\Big(a, \ln(\frac{G_0}{S_0}) - \tilde{f}_{n_0,n_1}\Big) - S_0e^{\tilde{f}_{n_0,n_1}}\chi_l\Big(a, \ln(\frac{G_0}{S_0}) - \tilde{f}_{n_0,n_1}\Big).$$

Remark 3. The Greeks for the GMMB contract can be straightforwardly computed as follows:

$$\begin{aligned}\Delta \approx & \frac{2e^{-\delta T}}{b-a} {\sum_{l=0}^{L-1}}' \boldsymbol{e}'_{n_0} \Re \left\{ e^{-il\pi \frac{a}{b-a}} \boldsymbol{P}_l^{-1} [\mathrm{Exp}(T\boldsymbol{P}_l) - \mathrm{Exp}(t_1 \boldsymbol{P}_l)] \right\} \boldsymbol{a}^{\Delta}_{1,l} \\ & + \frac{2e^{(w-r)T}}{b-a} {\sum_{l=0}^{L-1}}' \boldsymbol{e}'_{n_0} \Re \left\{ e^{-il\pi \frac{a}{b-a}} \mathrm{Exp}\left(T\left(\boldsymbol{P}_l - (w+\delta-c)\boldsymbol{I}\right)\right) \right\} \boldsymbol{a}^{\Delta}_{2,l},\end{aligned}$$

where

$$\boldsymbol{a}^{\Delta}_{1,l} = [\tilde{\nu}_1 V^{\Delta}_{n_0,1,l}, \tilde{\nu}_2 V^{\Delta}_{n_0,2,l}, \ldots, \tilde{\nu}_{m_1 m_2} V^{\Delta}_{n_0,m_1 m_2,l}]'$$

and

$$\boldsymbol{a}^{\Delta}_{2,l} = [V^{\Delta}_{n_0,1,l}, V^{\Delta}_{n_0,2,l}, \ldots, V^{\Delta}_{n_0,m_1 m_2,l}]'$$

with

$$V^{\Delta}_{n_0,n_1,l} = G_0 \zeta_l^{\Delta}\left(a, \ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1}\right) - e^{\tilde{f}_{n_0,n_1}} \chi_l\left(a, \ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1}\right) + S_0 e^{\tilde{f}_{n_0,n_1}} \chi_l^{\Delta}\left(a, \ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1}\right),$$

$$\zeta_l^{\Delta}\left(a, \ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1}\right) = \begin{cases} -\frac{1}{S_0} \cos\left(l\pi \frac{\ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1} - a}{b-a}\right), & l \neq 0, \\ -\frac{1}{S_0}, & l = 0 \end{cases}$$

and

$$\begin{aligned}& \chi_l^{\Delta}\left(a, \ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1}\right) \\ = & -\frac{1}{S_0} \frac{1}{1 + \left(\frac{l\pi}{b-a}\right)^2} \left[\cos\left(l\pi \frac{\ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1} - a}{b-a}\right) \cdot \left(e^{\ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1}} + \left(\frac{l\pi}{b-a}\right)^2\right) \right. \\ & \left. - \frac{l\pi}{b-a} \sin\left(l\pi \frac{\ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1} - a}{b-a}\right) e^{\ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1}} \right].\end{aligned}$$

Meanwhile, we can also calculate Γ, which is shown to be

$$\begin{aligned}\Gamma \approx & \frac{2e^{-\delta T}}{b-a} {\sum_{l=0}^{L-1}}' \boldsymbol{e}'_{n_0} \Re \left\{ e^{-il\pi \frac{a}{b-a}} \boldsymbol{P}_l^{-1} [\mathrm{Exp}(T\boldsymbol{P}_l) - \mathrm{Exp}(t_1 \boldsymbol{P}_l)] \right\} \boldsymbol{a}^{\Gamma}_{1,l} \\ & + \frac{2e^{(w-r)T}}{b-a} {\sum_{l=0}^{L-1}}' \boldsymbol{e}'_{n_0} \Re \left\{ e^{-il\pi \frac{a}{b-a}} \mathrm{Exp}\left(T\left(\boldsymbol{P}_l - (w+\delta-c)\boldsymbol{I}\right)\right) \right\} \boldsymbol{a}^{\Gamma}_{2,l},\end{aligned}$$

where

$$\boldsymbol{a}^{\Gamma}_{1,l} = [\tilde{\nu}_1 V^{\Gamma}_{n_0,1,l}, \tilde{\nu}_2 V^{\Gamma}_{n_0,2,l}, \ldots, \tilde{\nu}_{m_1 m_2} V^{\Gamma}_{n_0,m_1 m_2,l}]'$$

and

$$\boldsymbol{a}^{\Gamma}_{2,l} = [V^{\Gamma}_{n_0,1,l}, V^{\Gamma}_{n_0,2,l}, \ldots, V^{\Gamma}_{n_0,m_1 m_2,l}]'$$

with

$$\begin{aligned}& V^{\Gamma}_{n_0,n_1,l} \\ = & G_0 \zeta_l^{\Gamma}\left(a, \ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1}\right) + 2e^{\tilde{f}_{n_0,n_1}} \chi_l^{\Delta}\left(a, \ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1}\right) + S_0 e^{\tilde{f}_{n_0,n_1}} \chi_l^{\Gamma}\left(a, \ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1}\right),\end{aligned}$$

$$\zeta_l^{\Gamma}\left(a, \ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1}\right) = \begin{cases} \frac{1}{S_0^2} \left(\cos\left(l\pi \frac{\ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1} - a}{b-a}\right) - \frac{l\pi}{b-a} \sin\left(l\pi \frac{\ln\left(\frac{G_0}{S_0}\right) - \tilde{f}_{n_0,n_1} - a}{b-a}\right)\right), & l \neq 0, \\ \frac{1}{S_0^2}, & l = 0 \end{cases}$$

and

$$
\begin{aligned}
&\chi_l^{\Gamma}\Big(a, \ln\big(\frac{G_0}{S_0}\big) - \tilde{f}_{n_0,n_1}\Big)\\
=& -\frac{1}{S_0}\chi_l^{\Delta}\Big(a, \ln\big(\frac{G_0}{S_0}\big) - \tilde{f}_{n_0,n_1}\Big) + \frac{1}{{S_0}^2}\frac{1}{1+\big(\frac{l\pi}{b-a}\big)^2}\\
&\times\left[\cos\left(l\pi\,\frac{\ln\big(\frac{G_0}{S_0}\big) - \tilde{f}_{n_0,n_1} - a}{b-a}\right)e^{\ln\big(\frac{G_0}{S_0}\big) - \tilde{f}_{n_0,n_1}}\left(1-\left(\frac{l\pi}{b-a}\right)^2\right)\right.\\
&\quad\left.+\sin\left(l\pi\,\frac{\ln\big(\frac{G_0}{S_0}\big) - \tilde{f}_{n_0,n_1} - a}{b-a}\right)\left(-2\frac{l\pi}{b-a}e^{\ln\big(\frac{G_0}{S_0}\big) - \tilde{f}_{n_0,n_1}} - \left(\frac{l\pi}{b-a}\right)^3\right)\right].
\end{aligned}
$$

5. Valuation under discrete surrender structure

In this section, we consider the valuation of surrender option under discrete surrender. First, for $j \in \mathcal{M}_1$ we have

$$
\begin{aligned}
\mathcal{V}_j(\nu_0) =& \sum_{m=1}^{M} e^{-rt_m}P(t_m)\mathbb{E}\left[\mathbf{1}_{\{\tau=t_m\}}\cdot(G_{t_m}-F_{t_m})_+|\nu_0,\alpha_1(0)=j\right]\\
=& \sum_{m=1}^{M} e^{-rt_m}P(t_m)\mathbb{E}\left[\mathbf{1}_{\{\tau\geq t_m\}}\cdot(G_{t_m}-F_{t_m})_+|\nu_0,\alpha_1(0)=j\right]\\
&-\sum_{m=1}^{M-1} e^{-rt_m}P(t_m)\mathbb{E}\left[\mathbf{1}_{\{\tau\geq t_{m+1}\}}\cdot(G_{t_m}-F_{t_m})_+|\nu_0,\alpha_1(0)=j\right]\\
=& \sum_{m=1}^{M} e^{-rt_m}P(t_m)\mathbb{E}\left[e^{-\int_0^{t_m}\nu_s ds}\cdot\left(G_{t_m}-S_0e^{\widetilde{X}_{t_m}-ct_m+f_{\alpha_1(t_m)}(\nu_0,\nu_{t_m})}\right)_+\Big|\nu_0,\alpha_1(0)=j\right]\\
&-\sum_{m=1}^{M-1} e^{-rt_m}P(t_m)\mathbb{E}\left[e^{-\int_0^{t_{m+1}}\nu_s ds}\cdot\left(G_{t_m}-S_0e^{\widetilde{X}_{t_m}-ct_m+f_{\alpha_1(t_m)}(\nu_0,\nu_{t_m})}\right)_+\Big|\nu_0,\alpha_1(0)=j\right].
\end{aligned}
\tag{5.1}
$$

In order to apply the CTMC approximation, we again assume that $\nu_0 = \bar{\nu}_{\alpha_2(0)} = \bar{\nu}_k$ for some $k \in \mathcal{M}_2$. Then by formula (5.1) we obtain

$$
\begin{aligned}
&\mathcal{V}_j(\nu_0) \approx \widehat{\mathcal{V}}_j(\nu_0)\\
:=& \sum_{m=1}^{M} e^{-rt_m}P(t_m)\mathbb{E}\left[e^{-\int_0^{t_m}\bar{\nu}_{\alpha_2(s)}ds}\cdot\left(G_{t_m}-S_0e^{\bar{X}_{t_m}-ct_m+f_{\alpha_1(t_m)}(\bar{\nu}_{\alpha_2(0)},\bar{\nu}_{\alpha_2(t_m)})}\right)_+\Big|\alpha_1(0)=j,\alpha_2(0)=k\right]\\
&-\sum_{m=1}^{M-1} e^{-rt_m}P(t_m)\mathbb{E}\left[e^{-\int_0^{t_{m+1}}\bar{\nu}_{\alpha_2(s)}ds}\cdot\left(G_{t_m}-S_0e^{\bar{X}_{t_m}-ct_m+f_{\alpha_1(t_m)}(\bar{\nu}_{\alpha_2(0)},\bar{\nu}_{\alpha_2(t_m)})}\right)_+\Big|\alpha_1(0)=j,\alpha_2(0)=k\right]\\
=& \sum_{m=1}^{M} e^{-rt_m}P(t_m)\mathbb{E}\left[e^{-\int_0^{t_m}\bar{\nu}_{\beta(s)}ds}\cdot\left(G_{t_m}-S_0e^{\bar{Y}_{t_m}+\tilde{f}_{\beta(0),\beta(t_m)}}\right)_+\Big|\beta(0)=(j-1)m_2+k\right]\\
&-\sum_{m=1}^{M-1} e^{-rt_m}P(t_m)\mathbb{E}\left[e^{-\int_0^{t_{m+1}}\bar{\nu}_{\beta(s)}ds}\cdot\left(G_{t_m}-S_0e^{\bar{Y}_{t_m}+\tilde{f}_{\beta(0),\beta(t_m)}}\right)_+\Big|\beta(0)=(j-1)m_2+k\right],
\end{aligned}
\tag{5.2}
$$

where $\bar{Y}_{t_m} = \bar{X}_{t_m} - ct_m$. For the second expectation, we have

$$
\begin{aligned}
&\mathbb{E}\left[e^{-\int_0^{t_{m+1}}\bar{\nu}_{\beta(s)}ds}\cdot\left(G_{t_m}-S_0e^{\bar{Y}_{t_m}+\tilde{f}_{n_0,\beta(t_m)}}\right)_+\Big|\beta(0)=(j-1)m_2+k\right]\\
=& \sum_{n_1\in\mathcal{M}_\beta}\mathbb{E}\left[e^{-\int_0^{t_{m+1}}\bar{\nu}_{\beta(s)}ds}\cdot\left(G_{t_m}-S_0e^{\bar{Y}_{t_m}+\tilde{f}_{n_0,n_1}}\right)_+;\beta(t_m)=n_1\Big|\beta(0)=(j-1)m_2+k\right]\\
=& \sum_{n_1\in\mathcal{M}_\beta}\mathbb{E}\left[e^{-\int_0^{t_m}\bar{\nu}_{\beta(s)}ds}\cdot\left(G_{t_m}-S_0e^{\bar{Y}_{t_m}+\tilde{f}_{n_0,n_1}}\right)_+;\beta(t_m)=n_1\Big|\beta(0)=(j-1)m_2+k\right]\\
&\qquad\times\mathbb{E}\left[e^{-\int_{t_m}^{t_{m+1}}\bar{\nu}_{\beta(s)}ds}\Big|\beta(t_m)=n_1\right].
\end{aligned}
$$

Hence, formula (5.2) yields

$$\begin{aligned}\widehat{\mathcal{V}}_j(\nu_0) =& \sum_{m=1}^{M}\sum_{n_1\in\mathcal{M}_\beta} e^{-rt_m}P(t_m)\widehat{U}_{(j-1)m_2+k,n_1}(t_m)\\ &-\sum_{m=1}^{M-1}\sum_{n_1\in\mathcal{M}_\beta} e^{-rt_m}P(t_m)\widehat{U}_{(j-1)m_2+k,n_1}(t_m)\times\mathbb{E}\left[e^{-\int_{t_m}^{t_{m+1}}\tilde{\nu}_{\beta(s)}ds}\Big|\beta(t_m)=n_1\right],\end{aligned} \tag{5.3}$$

where for $n_0, n_1 \in \mathcal{M}_\beta$,

$$\widehat{U}_{n_0,n_1}(t_m) = \mathbb{E}\left[e^{-\int_0^{t_m}\tilde{\nu}_{\beta(s)}ds}\cdot\left(G_{t_m}-S_0e^{\bar{Y}_{t_m}+\tilde{f}_{\beta(0),\beta(t_m)}}\right)_+;\beta(t_m)=n_1\Big|\beta(0)=n_0\right].$$

Since the conditional expectation in (5.3) is given by

$$\mathbb{E}\left[e^{-\int_{t_m}^{t_{m+1}}\tilde{\nu}_{\beta(s)}ds}\Big|\beta(t_m)=n_1\right] = \mathbb{E}\left[e^{-\int_0^{\Delta_t}\tilde{\nu}_{\beta(s)}ds}\Big|\beta(0)=n_1\right] = \boldsymbol{e}'_{n_1}\mathrm{Exp}(\Delta_t\boldsymbol{K}(-1,0)),$$

then we only need to determine the conditional expectation $\widehat{U}_{n_0,n_1}(t_m)$.

5.1. Deterministic guarantee

In this subsection, we compute the conditional expectation $\widehat{U}_{n_0,n_1}(t_m)$ under the condition that G_t is a deterministic function of t. For $m = 1,\ldots,M$, let $p_{n_0,n_1}(x,t_m)$ denote the conditional density function satisfying

$$p_{n_0,n_1}(x;t_m)dx = \mathbb{E}\left[e^{-\int_0^{t_m}\tilde{\nu}_{\beta(s)}ds}, \bar{Y}_{t_m}\in dx;\beta(t_m)=n_1\Big|\beta(0)=n_0\right], \qquad x\in\mathbb{R}.$$

Then we have

$$\widehat{U}_{n_0,n_1}(t_m) \approx \widetilde{U}_{n_0,n_1}(t_m) := \int_a^b\left(G_{t_m}-S_0e^{x+\tilde{f}_{n_0,n_1}}\right)_+\widetilde{p}_{n_0,n_1}(x;t_m)dx, \tag{5.4}$$

where the COS approximation $\widetilde{p}_{n_0,n_1}(x;t_m)$ is given by

$$\begin{aligned}&\widetilde{p}_{n_0,n_1}(x;t_m)\\ =&\frac{2}{b-a}\sum_{l=0}^{L-1}{}'\Re\left\{\mathcal{F}p_{n_0,n_1}\left(\frac{l\pi}{b-a};t_m\right)e^{-il\pi\frac{a}{b-a}}\right\}\cos\left(l\pi\frac{x-a}{b-a}\right)\\ =&\frac{2}{b-a}\sum_{l=0}^{L-1}{}'\Re\left\{\boldsymbol{e}'_{n_0}\mathrm{Exp}\left(\Delta_t\left(\boldsymbol{K}\left(-1,i\frac{l\pi}{b-a}\right)-i\frac{l\pi}{b-a}c\boldsymbol{I}\right)\right)\boldsymbol{e}_{n_1}e^{-il\pi\frac{a}{b-a}}\right\}\cos\left(l\pi\frac{x-a}{b-a}\right).\end{aligned} \tag{5.5}$$

Substituting formula (5.5) into (5.4), we obtain

$$\begin{aligned}&\widetilde{U}_{n_0,n_1}(t_m)\\ =&\frac{2}{b-a}\sum_{l=0}^{L-1}{}'\Re\left\{\boldsymbol{e}'_{n_0}\mathrm{Exp}\left(\Delta_t\left(\boldsymbol{K}\left(-1,i\frac{l\pi}{b-a}\right)-i\frac{l\pi}{b-a}c\boldsymbol{I}\right)\right)\boldsymbol{e}_{n_1}e^{-il\pi\frac{a}{b-a}}\right\}V_{n_0,n_1,l}(t_m),\end{aligned}$$

where for $l = 0, 1, \ldots, L-1$,

$$V_{n_0,n_1,l}(t_m) = \int_a^b\left(G_{t_m}-S_0e^{x+\tilde{f}_{n_0,n_1}}\right)_+\cos\left(l\pi\frac{x-a}{b-a}\right)dx.$$

In particular, for $a < \ln(\frac{G_{t_m}}{S_0}) - \tilde{f}_{n_0,n_1} < b$, we have

$$V_{n_0,n_1,l}(t_m) = G_0\zeta_l\left(a,\ln\left(\frac{G_{t_m}}{S_0}\right)-\tilde{f}_{n_0,n_1}\right) - S_0e^{\tilde{f}_{n_0,n_1}}\chi_l\left(a,\ln\left(\frac{G_{t_m}}{S_0}\right)-\tilde{f}_{n_0,n_1}\right).$$

5.2. Geometric average guarantee

In this subsection, the guarantee is assumed to be the geometric average of the account value process given by

$$G_{t_m} = \kappa\cdot\left(\prod_{l=0}^{m}F_{t_l}\right)^{\frac{1}{m+1}},$$

where $\kappa > 0$ is a constant.

Using the CTMC approximation, we have

$$\begin{aligned}
&\widehat{U}_{n_0,n_1}(t_m)\\
&= \mathbb{E}\left[e^{-\int_0^{t_m}\tilde{\nu}_{\beta(s)}ds}\left(\kappa\cdot\left(\prod_{l=0}^{m}S_0e^{\bar{Y}_{t_l}+\tilde{f}_{n_0,\beta(t_l)}}\right)^{\frac{1}{m+1}} - S_0e^{\bar{Y}_{t_m}+\tilde{f}_{n_0,n_1}}\right)_+ ; \beta(t_m)=n_1\middle|\beta(0)=n_0\right]\\
&= \kappa S_0\mathbb{E}\left[e^{-\int_0^{t_m}\tilde{\nu}_{\beta(s)}ds+\bar{Y}_{t_m}}\left(\left(\prod_{l=0}^{m}e^{\bar{Y}_{t_l}+\tilde{f}_{n_0,\beta(t_l)}}\right)^{\frac{1}{m+1}}e^{-\bar{Y}_{t_m}} - \kappa^{-1}e^{\tilde{f}_{n_0,n_1}}\right)_+ ; \beta(t_m)=n_1\middle|\beta(0)=n_0\right],
\end{aligned}$$

where

$$\begin{aligned}
\left(\prod_{l=0}^{m}e^{\bar{Y}_{t_l}+\tilde{f}_{n_0,\beta(t_l)}}\right)^{\frac{1}{m+1}}e^{-\bar{Y}_{t_m}} &= \exp\left\{\frac{1}{m+1}\sum_{l=1}^{m}[\bar{Y}_{t_l}+\tilde{f}_{n_0,\beta(t_l)}]-\bar{Y}_{t_m}\right\}\\
&:= \exp\{Z_m\}. \qquad (5.6)
\end{aligned}$$

For $m = 1,\dots,M$, let $B_{n_0,n_1}(x;t_m)$ denote the conditional density function satisfying

$$B_{n_0,n_1}(x;t_m)dx = \mathbb{E}\left[e^{-\int_0^{t_m}\tilde{\nu}_{\beta(s)}ds+\bar{Y}_{t_m}}, Z_m\in dx; \beta(t_m)=n_1\middle|\beta(0)=n_0\right], \qquad x\in\mathbb{R},$$

then we have

$$\widehat{U}_{n_0,n_1}(t_m) = \kappa S_0\int_{-\infty}^{\infty}\left(e^x-\kappa^{-1}e^{\tilde{f}_{n_0,n_1}}\right)_+ B_{n_0,n_1}(x;t_m)dx. \qquad (5.7)$$

For $m = 1,\dots,M$, let

$$\widetilde{B}_{n_0,n_1}(x;t_m) = \frac{2}{b-a}\sum_{l=0}^{L-1}{}'\Re\left\{\mathcal{F}B_{n_0,n_1}\left(\frac{l\pi}{b-a};t_m\right)e^{-il\pi\frac{a}{b-a}}\right\}\cos\left(l\pi\frac{x-a}{b-a}\right)$$

be its COS approximation, where the Fourier transform of $B_{n_0,n_1}(x;t_m)$ is given by

$$\begin{aligned}
\mathcal{F}B_{n_0,n_1}(\omega;t_m) &= \int_{-\infty}^{+\infty}e^{i\omega x}B_{n_0,n_1}(x;t_m)dx\\
&= \mathbb{E}\left[e^{i\omega Z_m-\int_0^{t_m}\tilde{\nu}_{\beta(s)}ds+\bar{Y}_{t_m}}; \beta(t_m)=n_1\middle|\beta(0)=n_0\right].
\end{aligned}$$

The above Fourier transform can be determined as follows, first, it follows from (5.6) that

$$\begin{aligned}
&e^{i\omega Z_m-\int_0^{t_m}\tilde{\nu}_{\beta(s)}ds+\bar{Y}_{t_m}}\\
&= \exp\left\{\frac{i\omega}{m+1}\sum_{l=1}^{m}[\bar{Y}_{t_l}+\tilde{f}_{n_0,\beta(t_l)}]-i\omega\bar{Y}_{t_m}-\int_0^{t_m}\tilde{\nu}_{\beta(s)}ds+\bar{Y}_{t_m}\right\}\\
&= \exp\left\{-\int_0^{t_m}\tilde{\nu}_{\beta(s)}ds+\sum_{l=1}^{m}\frac{i\omega}{m+1}\tilde{f}_{n_0,\beta(t_l)}+\Big(1-i\omega\frac{m}{m+1}\Big)\bar{Y}_{t_m}+\sum_{l=1}^{m-1}\frac{i\omega}{m+1}\bar{Y}_{t_l}\right\}\\
&= \exp\left\{-\sum_{l=1}^{m}\int_{t_{l-1}}^{t_l}\tilde{\nu}_{\beta(s)}ds+\sum_{l=1}^{m}\frac{i\omega}{m+1}\tilde{f}_{n_0,\beta(t_l)}+\sum_{l=1}^{m}\Big(1-i\omega\frac{l}{m+1}\Big)(\bar{Y}_{t_l}-\bar{Y}_{t_{l-1}})\right\}\\
&= \exp\left\{\sum_{l=1}^{m}\left[-\int_{t_{l-1}}^{t_l}\tilde{\nu}_{\beta(s)}ds+\frac{i\omega}{m+1}\tilde{f}_{n_0,\beta(t_l)}+\Big(1-i\omega\frac{l}{m+1}\Big)(\bar{Y}_{t_l}-\bar{Y}_{t_{l-1}})\right]\right\}.
\end{aligned}$$

Next, by conditioning on the transition states of the CTMC $\beta(t)$ at times $t_0, t_1, \dots, t_m$, we can obtain the following explicit expression

$$\mathcal{F}B_{n_0,n_1}(\omega;t_m) = \boldsymbol{e}'_{n_0}[\boldsymbol{N}_{m,1}(\omega)\overline{\boldsymbol{H}}_m(\bar{\nu}_0,\omega)]\times\cdots\times[\boldsymbol{N}_{m,m}(\omega)\overline{\boldsymbol{H}}_m(\bar{\nu}_0,\omega)]\boldsymbol{e}_{n_1}, \qquad (5.8)$$

where for $l = 1,\dots,m$,

$$\boldsymbol{N}_{m,l}(\omega) = \mathrm{Exp}\left(\Delta_t\left(\boldsymbol{K}\Big(-1, 1-i\omega\frac{l}{m+1}\Big)-c\Big(1-i\omega\frac{l}{m+1}\Big)\boldsymbol{I}\right)\right)$$

and

$$\overline{\boldsymbol{H}}_m(\bar{v}_0,\omega)=\operatorname{diag}\left(e^{\frac{i\omega}{m+1}f_1(\bar{v}_0,\bar{v}_1)},\ldots,e^{\frac{i\omega}{m+1}f_1(\bar{v}_0,\bar{v}_{m_2})},\ldots,e^{\frac{i\omega}{m+1}f_{m_1}(\bar{v}_0,\bar{v}_{m_2})}\right).$$

Finally, replacing $B_{n_0,n_1}(x;t_m)$ by its COS approximation in (5.7), we obtain

$$\begin{aligned}&\widehat{U}_{n_0,n_1}(t_m)\approx\widetilde{U}_{n_0,n_1}(t_m)\\&:=\kappa S_0\int_a^b\left(e^x-\kappa^{-1}e^{\tilde{f}_{n_0,n_1}}\right)_+\widetilde{B}_{n_0,n_1}(x;t_m)dx\\&=\frac{2\kappa S_0}{b-a}\sum_{l=0}^{L-1}{}'\Re\left\{\mathcal{F}B_{n_0,n_1}\left(\frac{l\pi}{b-a};t_m\right)e^{-il\pi\frac{a}{b-a}}\right\}\int_a^b\left(e^x-\kappa^{-1}e^{\tilde{f}_{n_0,n_1}}\right)_+\cos\left(l\pi\frac{x-a}{b-a}\right)dx,\end{aligned}\tag{5.9}$$

where for $a<\tilde{f}_{n_0,n_1}-\ln\kappa<b$, we have

$$\int_a^b\left(e^x-\kappa^{-1}e^{\tilde{f}_{n_0,n_1}}\right)_+\cos\left(l\pi\frac{x-a}{b-a}\right)dx=\chi_l(\tilde{f}_{n_0,n_1}-\ln\kappa,b)-\kappa^{-1}e^{\tilde{f}_{n_0,n_1}}\zeta_l(\tilde{f}_{n_0,n_1}-\ln\kappa,b).$$

6. Error analysis

In this section, we shall present an error analysis of our method proposed in Sections 4 and 5. In the remainder of this section, we use C to denote a positive generic constant, which may vary at different steps. Note that for both continuous surrender and discrete surrender, we have the following error decomposition:

$$\textbf{Total Error}=\textbf{CTMC Error}+\textbf{COS Error},$$

where the analysis of the COS error is very standard and we shall give a brief analysis as follows.

Let us consider continuous surrender. Recall that we use the COS method to approximate the density function $g_{n_0,n_1}(x;t)$ by $\widetilde{g}_{n_0,n_1}(x;t)$, and then approximate the following integral

$$\int_{-\infty}^{+\infty}\left(G_0-S_0e^{x+\tilde{f}_{n_0,n_1}}\right)_+\cdot g_{n_0,n_1}(x;t)dx$$

via replacing $g_{n_0,n_1}(x;t)$ by $\widetilde{g}_{n_0,n_1}(x;t)$. In the above approximation procedure, there exist two types of errors. The first type is the integration range truncation error given by

$$\epsilon_1:=\int_{\mathbb{R}\backslash[a,b]}\left(G_0-S_0e^{x+\tilde{f}_{n_0,n_1}}\right)_+\cdot g_{n_0,n_1}(x;t)dx.$$

The second type error comes from the approximation of $g_{n_0,n_1}(x;t)$, which is given by

$$\begin{aligned}\epsilon_2:=&\int_a^b\left(G_0-S_0e^{x+\tilde{f}_{n_0,n_1}}\right)_+\cdot|g_{n_0,n_1}(x;t)-\widetilde{g}_{n_0,n_1}(x;t)|dx\\\le& G_0\cdot(b-a)\cdot\sup_{x\in[a,b]}|g_{n_0,n_1}(x;t)-\widetilde{g}_{n_0,n_1}(x;t)|.\end{aligned}$$

It is known that the error $\epsilon_{\text{cos}}:=\sup_{x\in[a,b]}|g_{n_0,n_1}(x;t)-\widetilde{g}_{n_0,n_1}(x;t)|$ depends on the smoothness of the density function $g_{n_0,n_1}(x;t)$; see e.g. Section 4 in Fang and Oosterlee [48]. If the density function is infinitely times differentiable, we can obtain exponential decay rate given by

$$\epsilon_{\text{cos}}\le P_{n_0,n_1}(L)\exp\left(-(L-1)\upsilon_{n_0,n_1}\right),$$

where $\upsilon_{n_0,n_1}>0$ is constant, and the term $P_{n_0,n_1}(L)$ varies less than exponentially with respect to L. Otherwise, the error can be bounded by

$$\epsilon_{cos}\le\frac{\bar{P}_{n_0,n_1}(L)}{(L-1)^{\lambda_{n_0,n_1}-1}},$$

where $\bar{P}_{n_0,n_1}(L)$ is a constant and $\lambda_{n_0,n_1}\ge 1$. In our paper, if the Brownian motion term exists in each regime, we can obtain exponential decay rate for ϵ_2, and the first error will not dominate the COS error.

Because the analysis of other errors due to the COS method is exactly as above, we shall omit the detailed arguments. In the next two subsections, we shall pay attention to the CTMC error.

6.1. CTMC error under the continuous surrender

In this subsection, we study the CTMC error under the continuous surrender. Before performing the analysis, we need to make the following assumptions.

Assumption 1. Suppose that the surrender intensity ν_t takes values in a compact set $\mathcal{B}$ with reflecting or absorbing boundaries, and $\hat{\mu}(\cdot) \in C^3(\mathcal{B})$, $\hat{\sigma}(\cdot) \in C^4(\mathcal{B})$.

Assumption 2. Suppose that for each $j \in \mathcal{M}_1$, $h_j(x)$ satisfies the following Lipschitz condition

$$|h_j(x) - h_j(y)| \le C|x-y|.$$

It follows from (3.2) that $(\widetilde{X}_t)_{t\ge 0}$ can be expressed as

$$\widetilde{X}_t = J_t - \int_0^t \rho h_{\alpha_1(s)}(\nu_s)ds,$$

where J_t, independent of the surrender intensity process ν_t, is given by

$$J_t = \int_0^t \left(r - \kappa_{\alpha_1(s)} - \frac{1}{2}\sigma^2_{\alpha_1(s)}\right) ds + \int_0^t \sigma_{\alpha_1(s)}\sqrt{1-\rho^2}dW_s^* + \int_0^t \int_{x\in\mathbb{R}} x\mathcal{N}_{\alpha_1(s)}(dx, ds). \tag{6.1}$$

Now the stock price process can be expressed as

$$S_t = S_0 e^{J_t + f_{\alpha_1(t)}(\nu_0,\nu_t) - \int_0^t \rho h_{\alpha_1(s)}(\nu_s)ds}.$$

To highlight the dependence on the intensity process ν_t, we put

$$\Gamma\nu(t) = \nu_t e^{-\int_0^t \nu_s ds}\cdot\left(G_0 - e^{-(w+c)t}S_0 e^{J_t + f_{\alpha_1(t)}(\nu_0,\nu_t) - \int_0^t \rho h_{\alpha_1(s)}(\nu_s)ds}\right)_+, \quad 0<t<T, \tag{6.2}$$

and

$$\widetilde{\Gamma}\nu(T) = e^{-\int_0^T \nu_s ds}\cdot\left(G_0 - e^{-(w+c)T}S_0 e^{J_T + f_{\alpha_1(T)}(\nu_0,\nu_T) - \int_0^T \rho h_{\alpha_1(s)}(\nu_s)ds}\right)_+, \tag{6.3}$$

so that we can write (4.1) as

$$\mathcal{V}_j(\nu_0) = \int_{t_1}^T e^{(w-r)t}e^{-\delta(T-t)}\mathbb{E}[\Gamma\nu(t)|\nu_0, \alpha_1(0)=j]dt + e^{(w-r)T}\mathbb{E}[\widetilde{\Gamma}\nu(T)|\nu_0,\alpha_1(0)=j].$$

Similarly, after replacing ν_t by its CTMC approximation $\bar{\nu}_{\alpha_2(t)}$ in (6.2) and (6.3), we find that the CTMC approximation $\widehat{V}_j(\nu_0)$ can be expressed as

$$\widehat{\mathcal{V}}_j(\nu_0) = \int_{t_1}^T e^{(w-r)t}e^{-\delta(T-t)}\mathbb{E}[\Gamma\bar{\nu}_{\alpha_2(\cdot)}(t)|\nu_0, \alpha_1(0)=j]dt + e^{(w-r)T}\mathbb{E}[\widetilde{\Gamma}\bar{\nu}_{\alpha_2(\cdot)}(T)|\nu_0,\alpha_1(0)=j].$$

Hence, the CTMC approximation error can be bounded as follows,

$$|\mathcal{V}_j(\nu_0) - \widehat{\mathcal{V}}_j(\nu_0)| \le \int_{t_1}^T e^{(w-r)t}e^{-\delta(T-t)}\mathcal{E}_{1,j}(t;\nu_0)dt + e^{(w-r)T}\mathcal{E}_{2,j}(T;\nu_0), \tag{6.4}$$

where

$$\begin{aligned}
\mathcal{E}_{1,j}(t;\nu_0) &= |\mathbb{E}[\Gamma\nu(t)|\nu_0,\alpha_1(0)=j] - \mathbb{E}[\Gamma\bar{\nu}_{\alpha_2(\cdot)}(t)|\nu_0,\alpha_1(0)=j]|,\\
\mathcal{E}_{2,j}(t;\nu_0) &= |\mathbb{E}[\widetilde{\Gamma}\nu(T)|\nu_0,\alpha_1(0)=j] - \mathbb{E}[\widetilde{\Gamma}\bar{\nu}_{\alpha_2(\cdot)}(T)|\nu_0,\alpha_1(0)=j]|.
\end{aligned}$$

In the remainder of this subsection, we are devoted to the error $\mathcal{E}_{1,j}(t;\nu_0)$, since the analysis of $\mathcal{E}_{2,j}$ is very similar. Next, we discretize the processes ν_t and $\nu_{\alpha_2(t)}$ by defining

$$\hat{\nu}_s = \nu_{k\Delta}, \quad \hat{\nu}_{\alpha_2(s)} = \bar{\nu}_{\alpha_2(k\Delta)}, \qquad k\Delta \le s < (k+1)\Delta, \quad k = 0, 1, \ldots, n_\nu - 1,$$

where $\Delta = t/n_\nu$ and n_ν is a large integer. Note that the above discretely modified processes $\hat{\nu}_s$ and $\hat{\nu}_{\alpha_2(s)}$ are right continuous and satisfy $\hat{\nu}_t = \nu_t$, $\hat{\nu}_{\alpha_2(t)} = \nu_{\alpha_2(t)}$. Furthermore, define $\Gamma\hat{\nu}(t)$ and $\Gamma\hat{\nu}_{\alpha_2(\cdot)}(t)$ via replacing ν by $\hat{\nu}$ and $\hat{\nu}_{\alpha_2(\cdot)}$ in (6.2), respectively. Then by triangle inequality we obtain

$$\begin{aligned}
\mathcal{E}_{1,j}(t;\nu_0) \le\ & \mathbb{E}[|\Gamma\hat{\nu}(t) - \Gamma\nu(t)||\nu_0,\alpha_1(0)=j] + \mathbb{E}[|\Gamma\bar{\nu}_{\alpha_2(\cdot)}(t) - \Gamma\hat{\nu}_{\alpha_2(\cdot)}(t)||\nu_0,\alpha_1(0)=j]\\
&+ |\mathbb{E}[\Gamma\hat{\nu}(t)|\nu_0,\alpha_1(0)=j] - \mathbb{E}[\Gamma\hat{\nu}_{\alpha_2(\cdot)}(t)|\nu_0,\alpha_1(0)=j]|\\
:=\ & \mathcal{E}_{1,j,1}(t;\nu_0) + \mathcal{E}_{1,j,2}(t;\nu_0) + \mathcal{E}_{1,j,3}(t;\nu_0).
\end{aligned} \tag{6.5}$$

Now we study the error $\mathcal{E}_{1,j,1}(t;\nu_0)$. First, $|\Gamma\nu(t) - \Gamma\hat{\nu}(t)|$ can be bounded as follows,

$$\begin{aligned}
&|\Gamma\nu(t) - \Gamma\hat{\nu}(t)|\\
\le&\Bigg|\nu_t e^{-\int_0^t \nu_s ds}\cdot\left(G_0 - S_0 e^{-(w+c)t + J_t + f_{\alpha_1(t)}(\nu_0,\nu_t) - \int_0^t \rho h_{\alpha_1(s)}(\nu_s)ds}\right)_+\\
&-\ \hat{\nu}_t e^{-\int_0^t \hat{\nu}_s ds}\cdot\left(G_0 - S_0 e^{-(w+c)t + J_t + f_{\alpha_1(t)}(\nu_0,\nu_t) - \int_0^t \rho h_{\alpha_1(s)}(\nu_s)ds}\right)_+\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&+ \Big|\hat{\nu}_t e^{-\int_0^t \hat{\nu}_s ds} \cdot \Big(G_0 - S_0 e^{-(w+c)t+J_t+f_{\alpha_1(t)}(\nu_0,\nu_t)-\int_0^t \rho h_{\alpha_1(s)}(\nu_s)ds}\Big)_+ \\
&- \hat{\nu}_t e^{-\int_0^t \hat{\nu}_s ds} \cdot \Big(G_0 - S_0 e^{-(w+c)t+J_t+f_{\alpha_1(t)}(\hat{\nu}_0,\hat{\nu}_t)-\int_0^t \rho h_{\alpha_1(s)}(\hat{\nu}_s)ds}\Big)_+\Big| \\
\le & G_0 \nu_t \Big| e^{-\int_0^t \nu_s ds} - e^{-\int_0^t \hat{\nu}_s ds}\Big| + C \cdot \nu_t e^{-(w+c)t+J_t+f_{\alpha_1(t)}(\nu_0,\nu_t)} \Big| e^{-\int_0^t \rho h_{\alpha_1(s)}(\hat{\nu}_s)ds} - e^{-\int_0^t \rho h_{\alpha_1(s)}(\nu_s)ds}\Big| \\
\le & C \cdot \nu_t \Big| \int_0^t (\nu_s - \hat{\nu}_s)ds\Big| + C \cdot \nu_t e^{-(w+c)t+J_t+f_{\alpha_1(t)}(\nu_0,\nu_t)} \Big| \int_0^t h_{\alpha_1(s)}(\hat{\nu}_s)ds - \int_0^t h_{\alpha_1(s)}(\nu_s)ds\Big|.
\end{aligned} \tag{6.6}
$$

By the definition of $\hat{\nu}$, we have

$$
\Big|\int_0^t (\nu_s - \hat{\nu}_s)ds\Big| \le \sum_{k=0}^{n_\nu - 1} \Big| \int_{k\Delta}^{(k+1)\Delta} (\nu_s - \nu_{k\Delta})ds\Big| \le \sum_{k=0}^{n_\nu-1} \max_{k\Delta \le s \le (k+1)\Delta} \Big|\nu_s - \nu_{k\Delta}\Big| \cdot \Delta. \tag{6.7}
$$

Similarly, by Assumption 2 we can obtain

$$
\begin{aligned}
& \Big| \int_0^t h_{\alpha_1(s)}(\hat{\nu}_s)ds - \int_0^t h_{\alpha_1(s)}(\nu_s)ds\Big| \\
\le & \sum_{j\in\mathcal{M}_1} \int_0^t \Big|h_j(\hat{\nu}_s) - h_j(\nu_s)\Big| \mathbf{1}_{\{\alpha_1(s)=j\}} ds \\
\le & \sum_{j\in\mathcal{M}_1} \int_0^t \Big|h_j(\hat{\nu}_s) - h_j(\nu_s)\Big| ds \\
\le & \sum_{j\in\mathcal{M}_1} C \cdot \int_0^t \Big|\hat{\nu}_s - \nu_s\Big| ds \\
\le & C \cdot \sum_{k=0}^{n_\nu-1} \max_{k\Delta\le s\le(k+1)\Delta} \Big|\nu_s - \nu_{k\Delta}\Big| \cdot \Delta.
\end{aligned} \tag{6.8}
$$

Recall that Assumption 1 states the intensity process ν_t takes values in a compact set, then by (6.6)–(6.8) we have

$$
\begin{aligned}
\mathcal{E}_{1,j,1}(t;\nu_0) \le & \; C \sum_{k=0}^{n_\nu-1} \Delta \mathbb{E}[\nu_t \cdot \max_{k\Delta\le s\le(k+1)\Delta} |\nu_s - \nu_{k\Delta}| | \nu_0] \\
& + C \sum_{k=0}^{n_\nu-1} \Delta \mathbb{E}[\nu_t e^{-(w+c)t+J_t+f_{\alpha_1(t)}(\nu_0,\nu_t)} \cdot \max_{k\Delta\le s\le(k+1)\Delta} |\nu_s - \nu_{k\Delta}| | \nu_0] \\
\le & \; C \sum_{k=0}^{n_\nu-1} \Delta \left[\mathbb{E}\left[\max_{k\Delta\le s\le(k+1)\Delta} |\nu_s - \nu_{k\Delta}|^2\right]\right]^{\frac{1}{2}},
\end{aligned} \tag{6.9}
$$

where the second step follows from Cauchy–Schwarz inequality.

Next, for $k\Delta \le s \le (k+1)\Delta$, we have

$$
\begin{aligned}
(\nu_s - \nu_{k\Delta})^2 \le & \left(\int_{k\Delta}^s \hat{\mu}(\nu_t)dt + \int_{k\Delta}^s \hat{\sigma}(\nu_t)dW_t\right)^2 \\
\le & \; 2\left(\int_{k\Delta}^s \hat{\mu}(\nu_t)dt\right)^2 + 2\left(\int_{k\Delta}^s \hat{\sigma}(\nu_t)dW_t\right)^2,
\end{aligned}
$$

which together with the Lipschitz condition of $\hat{\mu}$ and $\hat{\sigma}$ yields

$$
\begin{aligned}
& \mathbb{E}\left[\max_{k\Delta\le s\le(k+1)\Delta} |\nu_s - \nu_{k\Delta}|^2\right] \\
\le & 2\mathbb{E}\left[\max_{k\Delta\le s\le(k+1)\Delta} \left(\int_{k\Delta}^s \hat{\mu}(\nu_t)dt\right)^2 + \max_{k\Delta\le s\le(k+1)\Delta} \left(\int_{k\Delta}^s \hat{\sigma}(\nu_t)dW_t\right)^2\right] \\
\le & C \cdot \Delta^2 + 2\mathbb{E}\left[\max_{k\Delta\le s\le(k+1)\Delta} \left(\int_{k\Delta}^s \hat{\sigma}(\nu_t)dW_t\right)^2\right].
\end{aligned}
$$

From Doob's inequality in Chung and Williams [49], we have

$$\mathbb{E}\left[\max_{k\Delta\le s\le(k+1)\Delta}\left(\int_{k\Delta}^{s}\hat{\sigma}(\nu_t)dW_t\right)^2\right]\le 2\mathbb{E}\left[\int_{k\Delta}^{(k+1)\Delta}\hat{\sigma}^2(\nu_t)dt\right]\le C\cdot\Delta.$$

Therefore, we obtain

$$\mathbb{E}\left[\max_{k\Delta\le s\le(k+1)\Delta}|\nu_s-\nu_{k\Delta}|^2\right]\le C\Delta^2+C\Delta,$$

which together with (6.9) gives

$$\mathcal{E}_{1,j,1}(t;\nu_0)\le Cn_\nu\Delta^{\frac{3}{2}}=Ct\Delta^{\frac{1}{2}}. \tag{6.10}$$

By exactly the same arguments leading to (6.9), we can obtain

$$\mathcal{E}_{1,j,2}(t;\nu_0)\le C\sum_{k=0}^{n_\nu-1}\Delta\left[\mathbb{E}\left[\max_{k\Delta\le s\le(k+1)\Delta}|\bar{\nu}_{\alpha_2(s)}-\bar{\nu}_{\alpha_2(k\Delta)}|^2\right]\right]^{\frac{1}{2}}. \tag{6.11}$$

For each k we have

$$\mathbb{E}\left[\max_{k\Delta\le s\le(k+1)\Delta}|\bar{\nu}_{\alpha_2(s)}-\bar{\nu}_{\alpha_2(k\Delta)}|^2\right]\le C\cdot[1-\mathbb{Q}(\alpha_2(s)=\alpha_2(k\Delta);k\Delta\le s\le(k+1)\Delta)]\le C\Delta.$$

Consequently, (6.11) gives

$$\mathcal{E}_{1,j,2}(t;\nu_0)\le Cn_\nu\Delta^{\frac{3}{2}}=Ct\Delta^{\frac{1}{2}}. \tag{6.12}$$

Finally, we consider the error $\mathcal{E}_{1,j,3}(t;\nu_0)$. By the definition of process $\{\hat{\nu}_s\}_{0\le s\le t}$, we have $\nu_{k\Delta}=\hat{\nu}_{k\Delta}$ for $k=0,1,\ldots,n_\nu$. Using the tower property of conditional expectation we can obtain

$$\mathbb{E}[\Gamma\hat{\nu}(t)|\nu_0,\alpha_1(0)=j]=\mathbb{E}[H_j(\nu_0,\nu_\Delta,\ldots,\nu_{n_\nu\Delta})|\nu_0],$$

where

$$H_j(\nu_0,\nu_\Delta,\ldots,\nu_{n_\nu\Delta})=\mathbb{E}[\Gamma\hat{\nu}(t)|\alpha_1(0)=j,\nu_0,\nu_\Delta,\ldots,\nu_{n_\nu\Delta}].$$

Let $p(\Delta,x,y)$ be the transition kernel for the diffusion process ν_t which satisfies $\mathbb{Q}(\nu_\Delta\in dy|\nu_0=x)=p(\Delta,x,y)dy$. Then we have

$$\begin{aligned}&\mathbb{E}[H_j(\nu_0,\nu_\Delta,\ldots,\nu_{n_\nu\Delta})|\nu_0]\\&=\int\cdots\iint H_j(\nu_0,x_1,\ldots,x_{n_\nu})p(\Delta,x_{n_\nu-1},x_{n_\nu})dx_{n_\nu}\cdot p(\Delta,x_{n_\nu-2},x_{n_\nu-1})dx_{n_\nu-1}\cdots p(\Delta,\nu_0,x_1)dx_1.\end{aligned} \tag{6.13}$$

After discretizing the above integrals by the rectangle rule, we obtain

$$\begin{aligned}&\mathbb{E}[H_j(\nu_0,\nu_\Delta,\ldots,\nu_{n_\nu\Delta})|\nu_0]\\&=\sum_{x_1\in\boldsymbol{\nu}}\cdots\sum_{x_{n_\nu-1}\in\boldsymbol{\nu}}\sum_{x_{n_\nu}\in\boldsymbol{\nu}}H_j(\nu_0,x_1,\ldots,x_{n_\nu})p(\Delta,x_{n_\nu-1},x_{n_\nu})\delta x_{n_\nu}+O(\delta x_{n_\nu})\\&\quad\cdot p(\Delta,x_{n_\nu-2},x_{n_\nu-1})\delta x_{n_\nu-1}+O(\delta x_{n_\nu-1})\cdots p(\Delta,\nu_0,x_1)\delta x_1+O(\delta x_1)\\&=\sum_{x_1\in\boldsymbol{\nu}}\cdots\sum_{x_{n_\nu-1}\in\boldsymbol{\nu}}\sum_{x_{n_\nu}\in\boldsymbol{\nu}}H_j(\nu_0,x_1,\ldots,x_{n_\nu})p(\Delta,x_{n_\nu-1},x_{n_\nu})\delta x_{n_\nu}\\&\quad\cdot p(\Delta,x_{n_\nu-2},x_{n_\nu-1})\delta x_{n_\nu-1}\cdots p(\Delta,\nu_0,x_1)\delta x_1+O(h),\end{aligned} \tag{6.14}$$

where $h=\max_{i=1,\ldots,m_2-1}(k_i)$ and $\delta\bar{\nu}_i=\frac{1}{2}(\delta^+\bar{\nu}_i+\delta^-\bar{\nu}_i)$ with $\delta^+\bar{\nu}_i=\bar{\nu}_{i+1}-\bar{\nu}_i$ and $\delta^-\bar{\nu}_i=\bar{\nu}_i-\bar{\nu}_{i-1}$.

Let $\bar{p}(\Delta,x,y)$ denote the transition kernel of the CTMC model which satisfies $\mathbb{Q}(\bar{\nu}_{\alpha_2(\Delta)}=y|\bar{\nu}_{\alpha_2(0)}=x)=\bar{p}(\Delta,x,y)\delta y$ for $x,y\in\boldsymbol{\nu}$. Then we have

$$\begin{aligned}&\mathbb{E}[\Gamma\hat{\nu}_{\alpha_2(\cdot)}(t)|\nu_0]\\&=\sum_{x_1\in\boldsymbol{\nu}}\cdots\sum_{x_{n_\nu-1}\in\boldsymbol{\nu}}\sum_{x_{n_\nu}\in\boldsymbol{\nu}}H_j(\nu_0,x_1,\ldots,x_{n_\nu})\bar{p}(\Delta,x_{n_\nu-1},x_{n_\nu})\delta x_{n_\nu}\cdot\bar{p}(\Delta,x_{n_\nu-2},x_{n_\nu-1})\delta x_{n_\nu-1}\cdots\bar{p}(\Delta,\nu_0,x_1)\delta x_1.\end{aligned}$$

From Proposition 4 in Zhang and Li [34], we know that for $x,y\in\boldsymbol{\nu}$

$$\bar{p}(\Delta,x,y)=p(\Delta,x,y)+p(\Delta,x,y)\frac{\hat{\mu}(y)}{\hat{\sigma}(y)}(\delta^+y-\delta^-y)+O(h^2),$$

which together with Assumption 1 yields

$$
\begin{aligned}
&\mathbb{E}[\Gamma \hat{\nu}_{\alpha_2(\cdot)}(t)|\nu_0] \\
=&\sum_{x_1\in\boldsymbol{\nu}}\cdots\sum_{x_{n_\nu-1}\in\boldsymbol{\nu}}\sum_{x_{n_\nu}\in\boldsymbol{\nu}} H_j(\nu_0,x_1,\ldots,x_{n_\nu}) \\
&\left(p(\Delta,x_{n_{\nu-1}},x_{n_\nu})+p(\Delta,x_{n_{\nu-1}},x_{n_\nu})\frac{\hat{\mu}(x_{n_\nu})}{\hat{\sigma}(x_{n_\nu})}(\delta^+x_{n_\nu}-\delta^-x_{n_\nu})+O(h^2)\right)\delta x_{n_\nu} \\
&\cdot\left(p(\Delta,x_{n_{\nu-2}},x_{n_{\nu-1}})+p(\Delta,x_{n_{\nu-2}},x_{n_{\nu-1}})\frac{\hat{\mu}(x_{n_\nu-1})}{\hat{\sigma}(x_{n_\nu-1})}(\delta^+x_{n_\nu-1}-\delta^-x_{n_\nu-1})+O(h^2)\right)\delta x_{n_{\nu-1}} \\
&\cdots\left(p(\Delta,\nu_0,x_1)+p(\Delta,\nu_0,x_1)\frac{\hat{\mu}(x_1)}{\hat{\sigma}(x_1)}(\delta^+x_1-\delta^-x_1)+O(h^2)\right)\delta x_1 \\
=&\sum_{x_1\in\boldsymbol{\nu}}\cdots\sum_{x_{n_\nu-1}\in\boldsymbol{\nu}}\sum_{x_{n_\nu}\in\boldsymbol{\nu}} H_j(\nu_0,x_1,\ldots,x_{n_\nu})p(\Delta,x_{n_{\nu-1}},x_{n_\nu})\delta x_{n_\nu} \\
&\cdot p(\Delta,x_{n_{\nu-2}},x_{n_{\nu-1}})\delta x_{n_{\nu-1}}\cdots p(\Delta,\nu_0,x_1)\delta x_1+O(h^2).
\end{aligned}
\tag{6.15}
$$

By (6.14) and (6.15), we have

$$\mathcal{E}_{1,j,3}(t;\nu_0)=O(h). \tag{6.16}$$

Finally, from (6.10), (6.12) and (6.16), we have

$$\mathcal{E}_{1,j}(t;\nu_0)=O(t\Delta^{\frac{1}{2}})+O(h). \tag{6.17}$$

Similarly, we can obtain $\mathcal{E}_{1,j}(t;\nu_0)=O(t\Delta^{\frac{1}{2}})+O(h)$. As a result, (6.4) yields that the CTMC error has the following convergence rate

$$\mathcal{V}_j(\nu_0)-\widehat{\mathcal{V}}_j(\nu_0)=O(\Delta^{\frac{1}{2}})+O(h). \tag{6.18}$$

6.2. CTMC error under the discrete surrender

For deterministic guarantee, its error is similar to that of continuous guarantee. In this subsection, we only consider error under geometric average guarantee. Similar to Section 6.1, we put

$$
\begin{aligned}
\Upsilon\nu(t_m)&=e^{-\int_0^{t_m}\nu_s ds}\cdot\left(G_{t_m}-S_0e^{\widetilde{X}_{t_m}-ct_m+f_{\alpha_1(t_m)}(\nu_0,\nu_{t_m})}\right)_+,\quad 1\le m\le M,\\
\widetilde{\Upsilon}\nu(t_{m+1})&=e^{-\int_0^{t_{m+1}}\nu_s ds}\cdot\left(G_{t_m}-S_0e^{\widetilde{X}_{t_m}-ct_m+f_{\alpha_1(t_m)}(\nu_0,\nu_{t_m})}\right)_+,\quad 1\le m\le M-1.
\end{aligned}
$$

Then, (5.1) can be reexpressed by

$$
\begin{aligned}
\mathcal{V}_j(\nu_0)=&\sum_{m=1}^{M}e^{-rt_m}P(t_m)\mathbb{E}\left[\Upsilon\nu(t_m)\middle|\nu_0,\alpha_1(0)=j\right]\\
&-\sum_{m=1}^{M-1}e^{-rt_m}P(t_m)\mathbb{E}\left[\widetilde{\Upsilon}\nu(t_{m+1})\middle|\nu_0,\alpha_1(0)=j\right].
\end{aligned}
$$

Replacing ν_{t_m} by its CTMC approximation $\bar{\nu}_{\alpha_2(t_m)}$ in the above formula, we have

$$
\begin{aligned}
\widehat{\mathcal{V}}_j(\nu_0)=&\sum_{m=1}^{M}e^{-rt_m}P(t_m)\mathbb{E}\left[\Upsilon\bar{\nu}_{\alpha_2(\cdot)}(t_m)\middle|\nu_0,\alpha_1(0)=j\right]\\
&-\sum_{m=1}^{M-1}e^{-rt_m}P(t_m)\mathbb{E}\left[\widetilde{\Upsilon}\bar{\nu}_{\alpha_2(\cdot)}(t_{m+1})\middle|\nu_0,\alpha_1(0)=j\right].
\end{aligned}
$$

Hence, the error caused by CTMC approximation is

$$|\mathcal{V}_j(\nu_0)-\widehat{\mathcal{V}}_j(\nu_0)|\le\sum_{m=1}^{M}e^{-rt_m}P(t_m)\bar{\mathcal{E}}_{1,j}(t;\nu_0)+\sum_{m=1}^{M-1}e^{-rt_m}P(t_m)\bar{\mathcal{E}}_{2,j}(t;\nu_0), \tag{6.19}$$

where

$$\bar{\mathcal{E}}_{1,j}(t_m;\nu_0)=\left|\mathbb{E}\left[\Upsilon\nu(t_m)\middle|\nu_0,\alpha_1(0)=j\right]-\mathbb{E}\left[\Upsilon\bar{\nu}_{\alpha_2(\cdot)}(t_m)\middle|\nu_0,\alpha_1(0)=j\right]\right|,$$

$$\bar{\mathcal{E}}_{2,j}(t_m;\nu_0)=\left|\mathbb{E}\left[\widetilde{\Upsilon}\nu(t_{m+1})\Big|\nu_0,\alpha_1(0)=j\right]-\mathbb{E}\left[\widetilde{\Upsilon}\bar{\nu}_{\alpha_2(\cdot)}(t_{m+1})\Big|\nu_0,\alpha_1(0)=j\right]\right|.$$

In the following, we shall calculate $\bar{\mathcal{E}}_{1,j}(t_m;\nu_0)$ since the analysis of $\bar{\mathcal{E}}_{2,j}(t_m;\nu_0)$ is similar. First, we discretize the processes ν_{t_m} and $\nu_{\alpha_2(t_m)}$ according to

$$\hat{\nu}_s=\nu_{k\Delta_M},\quad \hat{\nu}_{\alpha_2(s)}=\nu_{\alpha_2(k\Delta_M)},\quad k\Delta_M\le s<(k+1)\Delta_M,\quad k=0,1,\dots,n_M-1,$$

where $\Delta_M=t_M/n_M$ and n_M is a large integer. Here without loss of generality we assume $\hat{\nu}_{t_m}=\nu_{t_m}$, $\hat{\nu}_{\alpha_2(t_m)}=\nu_{\alpha_2(t_m)}$, for $m=1,\dots,M$.

Similar to (6.5), we have

$$\begin{aligned}\bar{\mathcal{E}}_{1,j}(t_m;\nu_0)\le\ &\mathbb{E}\left[|\Upsilon\hat{\nu}(t_m)-\Upsilon\nu(t_m)||\nu_0,\alpha_1(0)=j\right]+\mathbb{E}[|\Upsilon\bar{\nu}_{\alpha_2(\cdot)}(t_m)-\Upsilon\hat{\nu}_{\alpha_2(\cdot)}(t_m)||\nu_0,\alpha_1(0)=j]\\&+|\mathbb{E}[\Upsilon\hat{\nu}(t_m)|\nu_0,\alpha_1(0)=j]-\mathbb{E}[\Upsilon\hat{\nu}_{\alpha_2(\cdot)}(t_m)|\nu_0,\alpha_1(0)=j]|\\:=\ &\bar{\mathcal{E}}_{1,j,1}(t_m;\nu_0)+\bar{\mathcal{E}}_{1,j,2}(t_m;\nu_0)+\bar{\mathcal{E}}_{1,j,3}(t_m;\nu_0),\end{aligned}$$

were $\Upsilon\hat{\nu}(t_m)$, $\Upsilon\bar{\nu}_{\alpha_2(\cdot)}(t_m)$, and $\Upsilon\hat{\nu}_{\alpha_2(\cdot)}(t_m)$ are defined similar to $\Upsilon\nu(t_m)$.

Now we analyze the error $\bar{\mathcal{E}}_{1,j,1}(t_m;\nu_0)$. First, using (5.6) and (6.1), we can bound $\left|\Upsilon\hat{\nu}(t_m)-\Upsilon\nu(t_m)\right|$ as

$$\begin{aligned}&\left|\Upsilon\hat{\nu}(t_m)-\Upsilon\nu(t_m)\right|\\
\le&\Bigg|S_0e^{-\int_0^{t_m}\nu_sds+J_{t_m}-\int_0^{t_m}\rho h_{\alpha_1(s)}(\nu_s)ds+f_{\alpha_1(t_m)}(\nu_0,\nu_{t_m})}\Bigg(\kappa e^{\frac{1}{m+1}\sum_{l=1}^{m-1}\left[J_{t_l}-\int_0^{t_l}\rho h_{\alpha_1(s)}(\nu_s)ds-ct_l+f_{\alpha_1(t_l)}(\nu_0,\nu_{t_l})\right]}\\
&\cdot e^{-\frac{m}{m+1}(J_{t_m}-\int_0^{t_m}\rho h_{\alpha_1(s)}(\nu_s)ds)+f_{\alpha_1(t_m)}(\nu_0,\nu_{t_m})-\frac{1}{m+1}ct_m}-e^{-ct_m}\Bigg)_+\\
&-S_0e^{-\int_0^{t_m}\nu_sds+J_{t_m}-\int_0^{t_m}\rho h_{\alpha_1(s)}(\nu_s)ds+f_{\alpha_1(t_m)}(\nu_0,\nu_{t_m})}\Bigg(\kappa e^{\frac{1}{m+1}\sum_{l=1}^{m-1}\left[J_{t_l}-\int_0^{t_l}\rho h_{\alpha_1(s)}(\hat{\nu}_s)ds-ct_l+f_{\alpha_1(t_l)}(\hat{\nu}_0,\hat{\nu}_{t_l})\right]}\\
&\cdot e^{-\frac{m}{m+1}(J_{t_m}-\int_0^{t_m}\rho h_{\alpha_1(s)}(\hat{\nu}_s)ds)+f_{\alpha_1(t_m)}(\hat{\nu}_0,\hat{\nu}_{t_m})-\frac{1}{m+1}ct_m}-e^{-ct_m}\Bigg)_+\Bigg|\\
+&\Bigg|S_0e^{-\int_0^{t_m}\nu_sds+J_{t_m}-\int_0^{t_m}\rho h_{\alpha_1(s)}(\nu_s)ds+f_{\alpha_1(t_m)}(\nu_0,\nu_{t_m})}\Bigg(\kappa e^{\frac{1}{m+1}\sum_{l=1}^{m-1}\left[J_{t_l}-\int_0^{t_l}\rho h_{\alpha_1(s)}(\hat{\nu}_s)ds-ct_l+f_{\alpha_1(t_l)}(\hat{\nu}_0,\hat{\nu}_{t_l})\right]}\\
&\cdot e^{-\frac{m}{m+1}(J_{t_m}-\int_0^{t_m}\rho h_{\alpha_1(s)}(\hat{\nu}_s)ds)+f_{\alpha_1(t_m)}(\hat{\nu}_0,\hat{\nu}_{t_m})-\frac{1}{m+1}ct_m}-e^{-ct_m}\Bigg)_+\\
&-S_0e^{-\int_0^{t_m}\hat{\nu}_sds+J_{t_m}-\int_0^{t_m}\rho h_{\alpha_1(s)}(\hat{\nu}_s)ds+f_{\alpha_1(t_m)}(\hat{\nu}_0,\hat{\nu}_{t_m})}\Bigg(\kappa e^{\frac{1}{m+1}\sum_{l=1}^{m-1}\left[J_{t_l}-\int_0^{t_l}\rho h_{\alpha_1(s)}(\hat{\nu}_s)ds-ct_l+f_{\alpha_1(t_l)}(\hat{\nu}_0,\hat{\nu}_{t_l})\right]}\\
&\cdot e^{-\frac{m}{m+1}(J_{t_m}-\int_0^{t_m}\rho h_{\alpha_1(s)}(\hat{\nu}_s)ds)+f_{\alpha_1(t_m)}(\hat{\nu}_0,\hat{\nu}_{t_m})-\frac{1}{m+1}ct_m}-e^{-ct_m}\Bigg)_+\Bigg|\\
:=&\varepsilon_1+\varepsilon_2.\end{aligned}$$

Next, we study the errors ε_1 and ε_2, respectively. On the one hand, for the error ε_1, by Lipschitz condition and Assumption 2, we have

$$\begin{aligned}\varepsilon_1\le&C\cdot e^{J_{t_m}-\int_0^{t_m}\rho h_{\alpha_1(s)}(\nu_s)ds+f_{\alpha_1(t_m)}(\nu_0,\nu_{t_m})}\cdot\Bigg|\frac{1}{m+1}\sum_{l=1}^{m-1}\left(\int_0^{t_l}h_{\alpha_1(s)}(\hat{\nu}_s)-h_{\alpha_1(s)}(\nu_s)ds\right)\\
&+\frac{m}{m+1}\int_0^{t_m}h_{\alpha_1(s)}(\hat{\nu}_s)-h_{\alpha_1(s)}(\nu_s)ds\Bigg|\\
\le&C\cdot e^{J_{t_m}-\int_0^{t_m}\rho h_{\alpha_1(s)}(\nu_s)ds+f_{\alpha_1(t_m)}(\nu_0,\nu_{t_m})}\sum_{k=0}^{n_M-1}\max_{k\Delta_M\le s\le(k+1)\Delta_M}\left|\nu_s-\nu_{k\Delta_M}\right|\cdot\Delta_M.\end{aligned}$$

On the other hand, for the error ε_2, by Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\varepsilon_2\le&C\cdot e^{2\max_{0\le l\le m}\left(J_{t_l}-\int_0^{t_l}\rho h_{\alpha_1(s)}ds+f_{\alpha_1(t_l)}(\hat{\nu}_0,\hat{\nu}_{t_l})\right)}\cdot\left(\int_0^{t_m}\left|\nu_s-\hat{\nu}_s\right|ds+\int_0^{t_m}\left|h_{\alpha_1(s)}(\hat{\nu}_s)-h_{\alpha_1(s)}(\nu_s)\right|ds\right)\\
\le&C\cdot e^{2\max_{0\le l\le m}\left(J_{t_l}-\int_0^{t_l}\rho h_{\alpha_1(s)}ds+f_{\alpha_1(t_l)}(\hat{\nu}_0,\hat{\nu}_{t_l})\right)}\cdot\sum_{k=0}^{n_M-1}\max_{k\Delta_M\le s\le(k+1)\Delta_M}\left|\nu_s-\nu_{k\Delta_M}\right|\cdot\Delta_M.\end{aligned}$$

Hence, by exactly the same arguments as in the previous subsection, we get

$$\bar{\mathcal{E}}_{1,j,1}(t_m;\nu_0)=O(t_m\Delta_M^{\frac{1}{2}}).$$

Table 1
Model parameters for CIR and Vasicek model.

Model	η_ν	θ_ν	σ_ν	ν_0	m_2
CIR	0.5	0.02	0.15	0.05	95
Vasicek	3	0.05	0.15	0.05	15

For the errors $\bar{\mathcal{E}}_{1,j,2}(t_m;\nu_0)$ and $\bar{\mathcal{E}}_{1,j,3}(t_m;\nu_0)$, we can prove that they have convergence order $O(t_m\Delta_M^{\frac{1}{2}})$ and $O(h)$, respectively. Hence, we obtain

$$\bar{\mathcal{E}}_{1,j}(t_m;\nu_0) = O(t_m\Delta_M^{\frac{1}{2}}) + O(h).$$

Similarly, we have $\bar{\mathcal{E}}_{2,j}(t_m;\nu_0) = O(t_m\Delta_M^{\frac{1}{2}}) + O(h)$. Hence, (6.19) yields that the CTMC error has the following convergence rate

$$\mathcal{V}_j(\nu_0) - \widehat{\mathcal{V}}_j(\nu_0) = O(\Delta_M^{\frac{1}{2}}) + O(h).$$

7. Numerical results

In this section, we shall provide some numerical experiments to illustrate the effectiveness of the proposed framework. All experiments are assessed relative to the benchmark Monte Carlo simulation. This section is organized into three parts. In Section 7.1, we present some model parameters. Comparisons of our method with the Monte Carlo simulations are given in Section 7.2. In Section 7.3, we show the effects of some parameters through sensitivity analysis. All experiments are conducted in Matlab 2019 on a personal computer with Intel(R) Core(TM) i7-10700F CPU @2.90 GHz.

7.1. Model parameters

In order to verify the effectiveness of our method comprehensively, we consider four popular regime-switching processes, one of which is the Black–Scholes model (BSM) without jumps. The remaining three models are all with jumps, including the double-exponential jump diffusion model (Kou, Kou [50]), Merton's normal jump diffusion model (MJD, Merton [51]) and the mixture of two normal jump diffusion model (MNJD, Florescu et al. [52]).

We consider two states regime-switching models with intensity matrix $\mathbf{\Lambda}$ given by

$$\mathbf{\Lambda} = \begin{bmatrix} -0.4 & 0.4 \\ 0.5 & -0.5 \end{bmatrix}.$$

We set $t_1 = 5$, $r = 0.01$, $\rho = -0.7$, $F_0 = S_0 = G_0 = 100$, $w = 0.1$, $c = 0.01$, $\delta = 0.01$ and $\Delta_t = 1$. In particular, we use affine stochastic processes for modeling stochastic surrender intensity. In order to explain the dynamics of the stochastic surrender intensity in the analysis, we assume that surrender intensity satisfies Vasicek model by Vasicek [53],

$$d\nu_t = \eta_\nu(\theta_\nu - \nu_t)dt + \sigma_\nu dW_t^2.$$

Another important financial model is the Cox–Ingersoll–Ross (CIR) model proposed by Cox et al. [54],

$$d\nu_t = \eta_\nu(\theta_\nu - \nu_t)dt + \sigma_\nu\sqrt{\nu_t}dW_t^2.$$

The latter model takes non-negative values under the Feller condition $2\eta_\nu\theta_\nu > \sigma_\nu^2$. Table 1 summarizes the parameter settings used in the CIR and Vasicek models.

Since the interval of the integral has a great influence on the COS method, in order to have a better approximation effect, we need to select these parameters for continuous surrender and discrete surrender with deterministic guarantee. By Fang and Oosterlee [48], we define cumulants to determine truncation interval $[a, b]$ with

$$a = \min_{j\in\mathcal{M}_1, k\in\mathcal{M}_2}(a_{j,k}), \quad b = \max_{j\in\mathcal{M}_1, k\in\mathcal{M}_2}(b_{j,k}),$$

where

$$[a_{j,k}, b_{j,k}] = \left[\gamma_{j,k}^1 T - \bar{L}\sqrt{\gamma_{j,k}^2 T + \sqrt{\gamma_{j,k}^4 T}},\ \ \gamma_{j,k}^1 T + \bar{L}\sqrt{\gamma_{j,k}^2 T + \sqrt{\gamma_{j,k}^4 T}}\ \right],\ j\in\mathcal{M}_1,\ k\in\mathcal{M}_2$$

with $\gamma_{j,k}^n$, $n = 1, 2, 4$, are the n^{th} cumulant. When the characteristic exponents are known, the cumulants are calculated as

$$\gamma_{j,k}^n = \frac{1}{i^n}\frac{\partial^n(\psi_{j,k}^g(\omega))}{\partial\omega^n}\bigg|_{\omega=0},\ \ n = 1, 2, 4,\ \ j\in\mathcal{M}_1,\ k\in\mathcal{M}_2,$$

Table 2
The characteristic exponent $\psi^g_{j,k}(\omega)$ under four regime-switching processes.

Model	Characteristic exponent $\psi^g_{j,k}(\omega)$
BSM	$\psi^g_{j,k}(\omega) = i\omega\zeta_{j,k} - \frac{1}{2}(1-\rho^2)\sigma_j^2\omega^2 \qquad \zeta_{j,k} = r - \frac{1}{2}\sigma_j^2 - \rho h_j^k(\nu_t) - w - c$
MJD	$\psi^g_{j,k}(\omega) = i\omega\zeta_{j,k} - \frac{1}{2}(1-\rho^2)\sigma_j^2\omega^2 + \lambda_j\left(e^{i\omega\mu_j^J - \frac{1}{2}{\sigma_j^J}^2\omega^2} - 1\right)$ $\zeta_{j,k} = r - \frac{1}{2}\sigma_j^2 - \rho h_j^k(\nu_t) - w - c - \lambda_j\left(e^{\mu_j^J + \frac{1}{2}{\sigma_j^J}^2} - 1\right)$
Kou	$\psi^g_{j,k}(\omega) = i\omega\zeta_{j,k} - \frac{1}{2}(1-\rho^2)\sigma_j^2\omega^2 + \lambda_j\left(\frac{p_j\eta_{j1}}{\eta_{j1}-i\omega} + \frac{(1-p_j)\eta_{j2}}{\eta_{j2}+i\omega} - 1\right)$ $\zeta_{j,k} = r - \frac{1}{2}\sigma_j^2 - \rho h_j^k(\nu_t) - w - c - \lambda_j\left(\frac{p_j\eta_{j1}}{\eta_{j1}-1} + \frac{(1-p_j)\eta_{j2}}{\eta_{j2}+1} - 1\right)$
MNJD	$\psi^g_{j,k}(\omega) = i\omega\zeta_{j,k} - \frac{1}{2}(1-\rho^2)\sigma_j^2\omega^2$ $+\lambda_j\left(p_j\exp\left(i\omega\mu_{j1}^J - \frac{{\sigma_{j1}^J}^2}{2}\omega^2\right) + (1-p_j)\exp\left(i\omega\mu_{j2}^J - \frac{{\sigma_{j2}^J}^2}{2}\omega^2\right) - 1\right)$ $\zeta_{j,k} = r - \frac{1}{2}\sigma_j^2 - \rho h_j^k(\nu_t) - w - c - \lambda_j\left(p_j\exp\left(\mu_{j1}^J + \frac{{\sigma_{j1}^J}^2}{2}\right) + (1-p_j)\exp\left(\mu_{j2}^J + \frac{{\sigma_{j2}^J}^2}{2}\right) - 1\right)$

Table 3
The cumulants $\gamma^n_{j,k}$.

Model	Cumulant $\gamma^n_{j,k}$
BSM	$\gamma^1_{j,k} = r - \frac{1}{2}\sigma_j^2 - \rho h_j^k(\nu_t) - w - c, \quad \gamma^2_{j,k} = (1-\rho^2)\sigma_j^2, \quad \gamma^4_{j,k} = 0$
MJD	$\gamma^1_{j,k} = r - \frac{1}{2}\sigma_j^2 - \rho h_j^k(\nu_t) - w - c - \lambda_j\left(e^{\mu_j^J + \frac{1}{2}{\sigma_j^J}^2} - 1\right) + \lambda_j\mu_j^J$ $\gamma^2_{j,k} = (1-\rho^2)\sigma_j^2 + \lambda_j\left({\mu_j^J}^2 + {\sigma_j^J}^2\right)$ $\gamma^4_{j,k} = \lambda_j\left({\mu_j^J}^4 + 6{\mu_j^J}^2{\sigma_j^J}^2 + 3{\sigma_j^J}^4\right)$
Kou	$\gamma^1_{j,k} = r - \frac{1}{2}\sigma_j^2 - \rho h_j^k(\nu_t) - w - c - \lambda_j\left(\frac{p_j\eta_{j1}}{\eta_{j1}-1} + \frac{(1-p_j)\eta_{j2}}{\eta_{j2}+1} - 1\right) + \lambda_j\left(\frac{p_j}{\eta_{j1}-1} - \frac{(1-p_j)}{\eta_{j2}+1}\right)$ $\gamma^2_{j,k} = (1-\rho^2)\sigma_j^2 + 2\lambda_j\left(\frac{p_j}{\eta_{j1}^2} + \frac{(1-p_j)}{\eta_{j2}^2}\right)$ $\gamma^4_{j,k} = 24\lambda_j\left(\frac{p_j}{\eta_{j1}^4} + \frac{(1-p_j)}{\eta_{j2}^4}\right)$
MNJD	$\gamma^1_{j,k} = r - \frac{1}{2}\sigma_j^2 - \rho h_j^k(\nu_t) - w - c - \lambda_j\left(p_j\exp\left(\mu_{j1}^J + \frac{{\sigma_{j1}^J}^2}{2}\right)(\mu_{j1}^J + {\sigma_{j1}^J}^2)\right.$ $\left.+(1-p_j)\exp\left(\mu_{j2}^J + \frac{{\sigma_{j2}^J}^2}{2}\right)(\mu_{j2}^J + {\sigma_{j2}^J}^2)\right)$ $\gamma^2_{j,k} = (1-\rho^2)\sigma_j^2 + \lambda_j p_j\left({\mu_{j1}^J}^2 + {\mu_{j1}^J}^2\right) + \lambda_j(1-p_j)\left({\mu_{j2}^J}^2 + {\mu_{j2}^J}^2\right)$ $\gamma^4_{j,k} = \lambda_j p_j\left({\mu_{j1}^J}^4 + 6{\mu_{j1}^J}^2{\sigma_{j1}^J}^2 + 3{\sigma_{j1}^J}^4\right) + \lambda_j(1-p_j)\left({\mu_{j2}^J}^4 + 6{\mu_{j2}^J}^2{\sigma_{j2}^J}^2 + 3{\sigma_{j2}^J}^4\right)$

where $\psi^g_{j,k}(\omega)$ is the characteristic exponents of $g_{n_0,n_1}(\omega; t)$ and $h_j^k(\nu_t)$ represents the k^{th} component in $h_j(\nu_t)$. The characteristic exponents and cumulants for the models are given in Tables 2 and 3.

Due to the unavailability of defining characteristic exponent under discrete surrender with geometric average guarantee, we cannot follow Fang and Oosterlee [48] to determine truncation interval. By (5.8), we let $C^l_{j,k}(\omega)$ denote the $((j-1)m_2+k)^{\text{th}}$ component of the diagonal matrix $\boldsymbol{N}_{m,l}(\omega)\overline{\boldsymbol{H}}_m(\bar{\nu}_0,\omega)$ for $j \in \mathcal{M}_1$ and $k \in \mathcal{M}_2$. Similar to the method above, we set

$$\gamma^n_{j,k,l} = \frac{1}{i^n}\frac{\partial^n(C^l_{j,k}(\omega))}{\partial\omega^n}\bigg|_{\omega=0}, \quad n = 1,2,4, \quad l = 1,\ldots,m, \quad j \in \mathcal{M}_1, \quad k \in \mathcal{M}_2.$$

Then, we let

$$\overline{\gamma}^n_{j,k} = \frac{\sum_{l=1}^m \gamma^n_{j,k,l}}{T}, n = 1,2,4, \quad j \in \mathcal{M}_1, \quad k \in \mathcal{M}_2$$

denote cumulants. For truncation interval $[\bar{a}, \bar{b}]$, we define

$$\bar{a} = \min_{j\in\mathcal{M}_1, k\in\mathcal{M}_2}(\bar{a}_{j,k}), \quad \bar{b} = \max_{j\in\mathcal{M}_1, k\in\mathcal{M}_2}(\bar{b}_{j,k}),$$

Table 4
Jump distribution parameters.

Regime	BSM	MJD	Kou	MNJD
1	$\sigma_1 = 0.15$	$\sigma_1 = 0.15$ $\lambda_1 = 1$ $\mu_1^J = 0.1$ $\sigma_1^J = 0.3$	$\sigma_1 = 0.15$ $\lambda_1 = 1$ $p_1 = 0.2$ $\eta_{11} = 25,\ \eta_{12} = 15$	$\sigma_1 = 0.15$ $\lambda_1 = 1$ $p_1 = 0.6$ $\mu_{11}^J = -0.05,\ \mu_{12}^J = 0.07$ $\sigma_{11}^J = 0.02,\ \sigma_{12}^J = 0.03$
2	$\sigma_2 = 0.3$	$\sigma_2 = 0.3$ $\lambda_2 = 1$ $\mu_2^J = -0.2$ $\sigma_2^J = 0.15$	$\sigma_2 = 0.3$ $\lambda_2 = 1$ $p_2 = 0.7$ $\eta_{21} = 15,\ \eta_{22} = 5$	$\sigma_2 = 0.3$ $\lambda_2 = 1$ $p_2 = 0.2$ $\mu_{21}^J = 0.02,\ \mu_{22}^J = 0.02$ $\sigma_{21}^J = 0.01,\ \sigma_{22}^J = 0.04$

Table 5
COS method vs. MC method for valuation under continuous surrender when $\alpha(0) = 1$.

Model	T	CIR model				Vasicek model			
		MC	T_{mc}	COS	T_{cos}	MC	T_{mc}	COS	T_{cos}
BSM	10	130.8873	121.2461	129.4130	1.2434	113.4110	115.7205	113.3833	0.1969
	15	235.9550	185.0392	235.8604	1.3256	189.0535	175.3153	189.1344	0.2040
	20	385.3857	246.8375	385.3661	1.2569	279.9099	236.6649	280.0656	0.2094
MJD	10	144.1303	290.2876	143.1235	1.2543	125.5012	287.0285	125.4994	0.2056
	15	249.7653	436.6214	249.8048	1.2478	201.6240	433.0064	201.6298	0.2030
	20	397.5098	585.0833	398.9102	1.2452	292.3263	572.2915	292.4554	0.2388
Kou	10	131.1799	245.2560	131.6643	1.4753	115.3802	238.1089	115.3625	0.2147
	15	238.0126	362.3474	237.8999	1.3269	190.9995	359.3161	191.0250	0.2131
	20	386.6004	500.2120	387.1210	1.2528	281.7193	468.3871	281.8088	0.2134
MNJD	10	130.0062	241.3256	129.8440	1.2471	114.1069	244.0717	114.0746	0.2116
	15	236.6377	370.5664	236.2167	1.2365	189.7301	366.6698	189.7553	0.2077
	20	384.1844	495.1514	384.2412	1.2856	280.4855	491.0821	280.6285	0.2119

where

$$[\bar{a}_{j,k}, \bar{b}_{j,k}] = \left[\overline{\gamma}_{j,k}^1 T - \bar{L}\sqrt{\overline{\gamma}_{j,k}^2 T + \sqrt{\overline{\gamma}_{j,k}^4 T}},\ \ \overline{\gamma}_{j,k}^1 T + \bar{L}\sqrt{\overline{\gamma}_{j,k}^2 T + \sqrt{\overline{\gamma}_{j,k}^4 T}} \right],\ j \in \mathcal{M}_1,\ \ k \in \mathcal{M}_2.$$

According to testing, we found that $\bar{L} = 6 \sim 12$ has the best approximation effect. Without loss of generality, we shall choose $\bar{L} = 8$ in this section.

7.2. Computational efficiency analysis

For all analysis in this subsection, we compare the performance of our COS method against the Monte Carlo (MC) simulation under four regime-switching models, and list corresponding parameters for models in Table 4.

We use T_{cos} and T_{mc} to represent the time required for the calculation of COS method and Monte Carlo simulation, respectively, and their units are seconds. Let $L = 2^6$ denote the number of grid points. In order to make the results of Monte Carlo simulation more accurate, we generate 10^6 paths to approximate value.

We first consider the valuation under continuous surrender. Table 5 presents efficiency comparisons between the Monte Carlo simulation and COS method. For Monte Carlo simulation, we discover that models with jumps require more time to fit the corresponding stochastic process. However, for our method, the complexity of the model does not increase the computational cost greatly, and it usually requires one second to get the accurate prices of VAs. In particular, it can be controlled within one second under Vasicek model for all regime-switching model. Moreover, we also find that when T increases, the time consumed by the Monte Carlo simulation increases, but the time consumed by the COS method remains basically unchanged.

Tables 6 and 7 show price comparisons and the corresponding computational costs of the two methods under discrete surrender. Same as the continuous case, the computational cost of the COS method is very low as compared to the Monte Carlo simulation method, which takes at least 63 s to generate comparable prices. For the geometric average guarantee, it increases the computation time because it has a special form of characteristic exponent. In this case, we find that the time consumed by the COS method also increases slightly as T increases. From Tables 5–7, it can be seen that the method we propose can be used to compute the value of VA within regime-switching models with/without jumps with very high accuracy. In the following subsection, we will consider the sensitivity analysis with respect to the model parameters.

Table 6
COS method vs. MC method for valuation under discrete surrender with deterministic guarantee when $\alpha(0) = 1$ and $G_{t_m} = 100$.

Model	T	CIR model				Vasicek model			
		MC	T_{mc}	COS	T_{cos}	MC	T_{mc}	COS	T_{cos}
BSM	5	15.4213	67.9699	15.5905	1.3866	14.8563	65.2140	14.9538	0.1613
	10	21.7243	133.8588	21.6379	2.6252	19.8510	122.5423	19.0888	0.2050
	15	25.2807	200.9649	25.2908	4.1373	22.3758	191.9369	22.4153	0.2658
MJD	5	28.0338	152.0890	27.9193	1.3503	27.1350	148.0092	27.0599	0.1499
	10	36.4535	303.0014	36.3897	2.6879	33.2461	297.1753	33.0936	0.2157
	15	40.8381	458.1935	40.7871	4.1362	35.1904	446.4713	35.3547	0.2680
Kou	5	18.4501	129.5355	18.5417	1.3465	19.5383	124.6834	19.4813	0.1388
	10	25.3738	257.5540	25.4078	2.6770	24.4726	249.8069	24.5717	0.1893
	15	29.3699	376.1345	29.3800	4.1192	26.7075	374.5000	26.5675	0.2316
MNJD	5	16.3801	129.9332	16.2979	1.3600	17.3941	119.8989	17.5416	0.1434
	10	22.4620	246.8446	22.4670	2.6622	22.2495	242.1594	22.1346	0.1861
	15	26.1263	311.1033	26.1655	4.1081	23.7556	362.9118	23.9387	0.2355

Table 7
COS method vs. MC method for valuation under discrete surrender with geometric average guarantee when $\alpha(0) = 1$ and $\kappa = 1.8$.

Model	T	CIR model				Vasicek model			
		MC	T_{mc}	COS	T_{cos}	MC	T_{mc}	COS	T_{cos}
BSM	5	74.7293	129.1535	74.9312	3.8210	75.2719	125.1641	75.5890	0.5203
	10	70.2679	257.9193	71.1045	15.6325	72.2903	252.1618	71.7265	1.7337
	15	66.5078	387.3206	65.1776	33.2185	69.7250	378.6062	69.7919	3.4712
MJD	5	71.5840	212.6325	71.4799	4.0329	73.1529	209.8843	74.0648	0.5396
	10	67.7355	425.5256	67.0198	15.4547	70.1301	420.6285	70.0963	1.7821
	15	64.2167	650.2159	63.8972	35.3946	67.2005	632.4209	67.3369	3.5871
Kou	5	73.3209	190.2563	73.5670	4.2541	74.4803	185.5614	73.6968	0.5657
	10	69.2713	381.2412	69.7132	14.3296	71.3548	371.8221	71.5923	1.7812
	15	66.1491	570.1978	66.1420	34.2690	68.9974	559.3843	68.7291	3.6095
MNJD	5	74.0556	180.4639	74.3407	4.2560	75.5104	180.5931	75.1703	0.5640
	10	70.0206	330.2563	70.1201	15.2140	72.1343	363.6367	72.1693	1.8028
	15	66.9273	450.2106	66.9494	33.6307	69.8865	545.2393	69.6251	3.6048

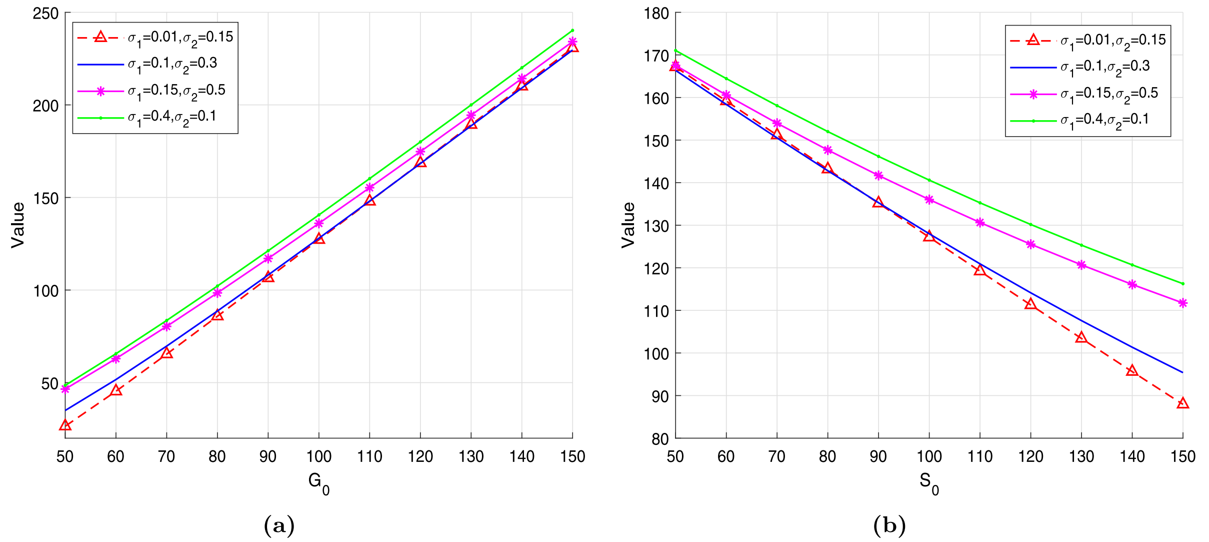

Fig. 1. The sensitivity of valuation under continuous surrender with regime-switching BSM model when T = 10. Left: $S_0 = 100$. Right: $G_0 = 100$.

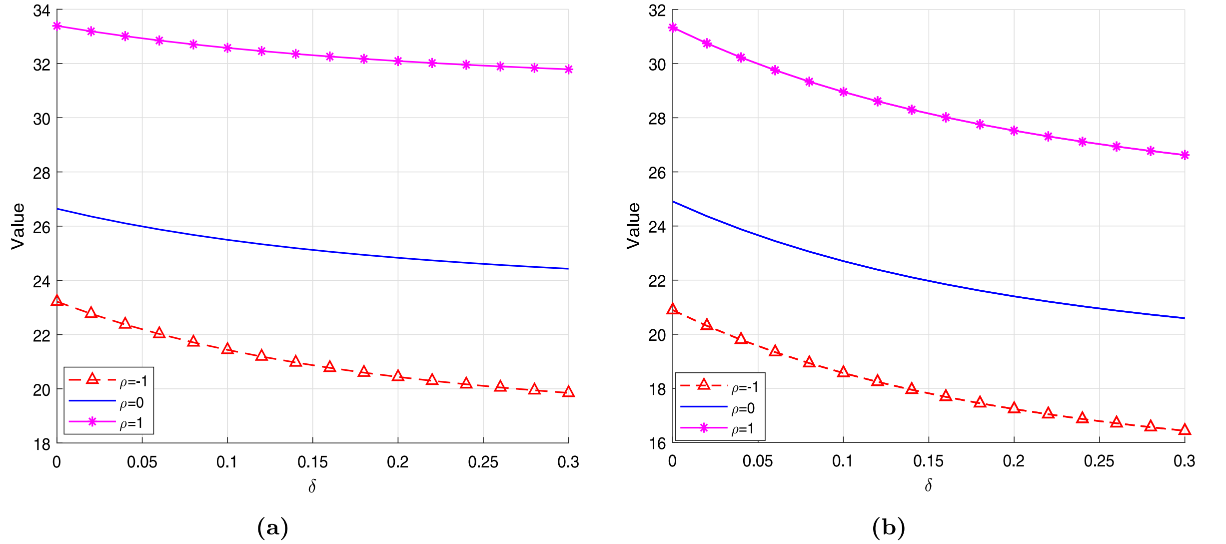

Fig. 2. The sensitivity of valuation under discrete surrender with regime-switching BSM model when $G_{t_m} = 100$, $T = 10$ and $\alpha(0) = 1$. Left: CIR model. Right: Vasicek model.

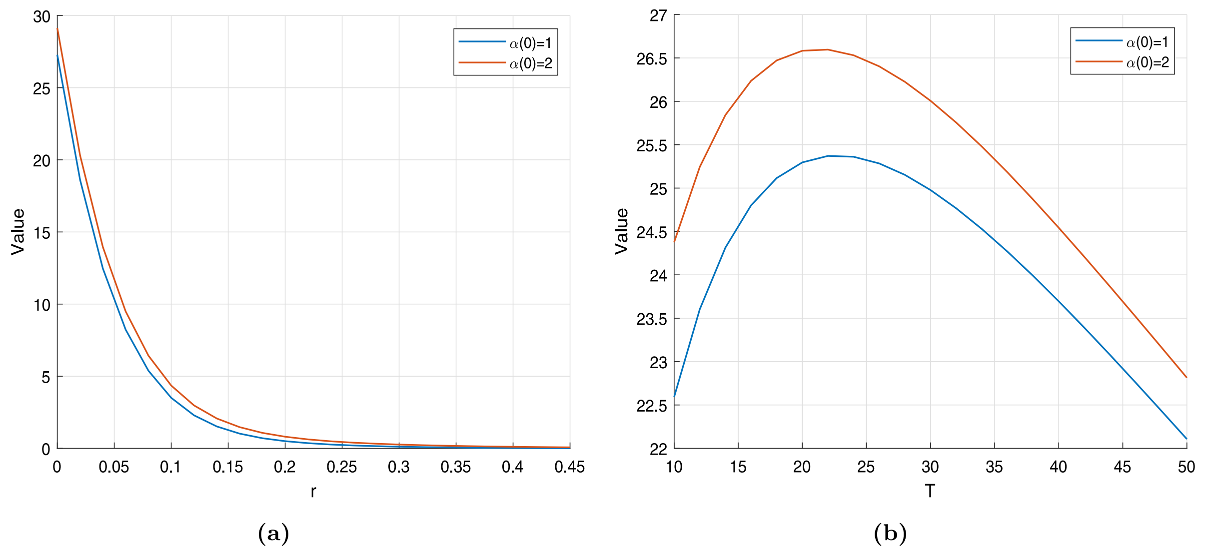

Fig. 3. The sensitivity of valuation under discrete surrender with regime-switching Kou's jump diffusion model when $G_{t_m} = 100$. Left: $T = 10$. Right: $r = 0.1$.

7.3. Sensitivity analysis

In the above subsection, we compared the COS method with Monte Carlo simulation. Here we illustrate the impact of model parameters on the prices of VAs. Without loss of generality, we assume that the initial state is 1 in Figs. 1 and 2 .

In Fig. 1, for continuous surrender, we consider the VA price as a function of guarantee level G_0 (Left) and price process at initial time S_0 (Right). In the BSM case, we can discover that the monotone impact of increasing G_0, for any fixed σ_1 and σ_2 in the left plot. Interestingly, for a guarantee level at initial time, the price of VA is not necessarily monotonically increasing in σ_1 and σ_2. From the right plot of Fig. 1, we also find a similar phenomenon. The difference is that as S_0 increases, the price of VA decreases monotonically.

In Fig. 2, we consider the VA price as a function of the penalty factor δ. We plot the effect of three different correlation coefficients on the price of VA under the CIR and Vasicek model. From Fig. 2, we find that the monotone decreasing impact of increasing δ, for any fixed ρ in the left plot. And we also find that the correlation coefficient ρ has a greater impact on the Vasicek model than the CIR model. Fig. 3 contains subfigures of the changes in VA prices with respect to changes in the risk-free interest r and maturity of the contract T, respectively. From Fig. 3, we note that the VA price is a decreasing function of risk-free interest which is in line with increasing risk from the VA providers perspective. When the risk is high, insurance companies need to increase the value of VA to deal with complex situations. From right of Fig. 3, regardless of the initial state, we find that the value of VA is very sensitive to the contract maturity. From the above tables, we also found this phenomenon. Some scholars have found that the price of VA is concave with contract maturity in the previous literature (2.1). This is because we decomposed the value of VA into two parts in Section 2.2, one part of which increases as maturity increases and the other part decreases as maturity increases. Finally, the two parts are combined to present the effect of Fig. 3.

8. Conclusion

In this paper, we have proposed an efficient method to valuate VA contracts embedded with GMMBs in a regime-switching jump diffusion model with surrender risk. For regime-switching jump diffusion process, each regime has different characteristics, which naturally needs a Markov chain to drive transitions between regimes. For the surrender risk, we use an intensity-based approach to model the surrender time, and apply the CTMC method to approximate the intensity process. We have considered two cases: (i) continuous surrender; (ii) discrete surrender, where deterministic guarantee and geometric average guarantee are included in discrete surrender. In the above cases, we use the Fourier cosine expansion method to arrive at explicit closed-form expressions given the knowledge of the Fourier transform. In numerical experiments, our method is compared with the Monte Carlo simulation method, and is demonstrated to be highly efficient and accurate.

Data availability

No data was used for the research described in the article.

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