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High order Morley elements for biharmonic equations on polytopal partitions

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ABSTRACT

This paper introduces an extension of the Morley element for approximating solutions to biharmonic equations. Traditionally limited to piecewise quadratic polynomials on triangular elements, the extension leverages weak Galerkin finite element methods to accommodate higher degrees of polynomials and the flexibility of general polytopal elements. By utilizing the Schur complement of the weak Galerkin method, the extension allows for fewest local degrees of freedom while maintaining sufficient accuracy and stability for the numerical solutions. The numerical scheme incorporates locally constructed weak tangential derivatives and weak second order partial derivatives, resulting in an accurate approximation of the biharmonic equation. Optimal order error estimates in both a discrete H^2 norm and the usual L^2 norm are established to assess the accuracy of the numerical approximation. Additionally, numerical results are presented to validate the developed theory and demonstrate the effectiveness of the proposed extension.

1. Introduction

This paper is concerned with the new development of high order Morley elements for the biharmonic equation by using the weak Galerkin (WG) method. For simplicity, we consider the biharmonic equation that seeks an unknown function u satisfying

$$\begin{aligned} \Delta^2 u &= g, \quad \text{in } \Omega, \\ u &= \zeta, \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= \xi, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is a bounded polytopal domain with Lipschitz continuous boundary $\partial\Omega$, and $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$.

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A weak formulation of (1.1) seeks $u \in H^2(\Omega)$ satisfying $u|_{\partial\Omega} = \zeta$ and $\frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} = \xi$ such that

$$\sum_{i,j=1}^{d} (\partial_{ij}^2 u, \partial_{ij}^2 v) = (g, v), \qquad \forall v \in H_0^2(\Omega), \tag{1.2}$$

where $H_0^2(\Omega) = \{v \in H^2(\Omega) : v|_{\partial\Omega} = 0, \nabla v|_{\partial\Omega} = \mathbf{0}\}$.

The H^2-conforming finite element method for the biharmonic equation is well-known but requires a C^1-continuity of piecewise polynomials on simplicial elements, which poses practical difficulties. To address this issue, various nonconforming finite element methods were introduced. Among these methods, the Morley element has the fewest degrees of freedom on each triangular element, making it not only a popular research topic but also a practically useful method. Previous works such as [1–3] extended the Morley element to higher dimensions. Other works, including [4–8], proposed generalizations of the Morley element for different types of meshes. Parallel algorithms and multigrid methods for the Morley element were developed in [9–12]. Since then, rapid progress has been made in various numerical methods for the biharmonic equation on polytopal meshes, such as discontinuous Galerkin finite element methods [13–15], virtual element methods [16,17], and weak Galerkin methods [18–27]. The WG finite element method was first proposed for second-order elliptic problems in [28]. The WG method is a natural generalization of classical finite element methods as it relaxes the continuity requirement for the approximating functions. This weak continuity of the numerical approximation allows for high flexibility in constructing weak finite elements with any desired order of convergence. To the best of our knowledge, no high-order extension has been developed that combines the advantages of the Morley element, including its minimal degrees of freedom, with the ability to handle general polytopal partitions.

The objective of this paper is to present a high-order generalization of the Morley element using the weak Galerkin method. Inspired by the de Rham complexes for weak Galerkin spaces [29], we propose innovations to the original weak finite element procedures. These innovations involve the introduction of additional approximating functions defined on the $(d-2)$-dimensional sub-polytopes and $(d-1)$-dimensional sub-polytopes of d-dimensional polytopal elements, resulting in a reduction of the degrees of freedom. To enhance the numerical scheme, we incorporate a locally designed weak tangential derivative operator and a weak second-order partial derivative operator. Furthermore, we establish optimal order error estimates for the resulting numerical approximations in both the energy norm and the L^2 norm.

The main contributions of this paper can be summarized as follows. Firstly, unlike the original Morley element, the proposed WG extension allows for higher-order polynomial approximation with the local minimum number of degrees of freedom, while also being applicable to general polytopal elements. This extension broadens the scope of problems that can be effectively addressed. Secondly, in comparison to existing results on WG methods, we introduce a novel technique within the WG framework that significantly reduces the number of unknowns. This advancement enhances the efficiency and computational feasibility of the method. Finally, the versatility of the new WG method enables its application to various modeling problems, including those that involve the Hessian operator in their weak formulation.

The paper is structured as follows. In Section 2, we provide a review of the definitions of the discrete weak tangential derivative and the discrete weak second-order partial derivatives. Section 3 presents the weak Galerkin scheme and introduces its Schur complement. Section 4 establishes the solution existence and uniqueness for this new scheme. Section 5 is devoted to the derivation of an error equation for the weak Galerkin scheme, providing insights into the accuracy of the method. In Section 6, we present some technical results that are utilized in the subsequent section. Section 7 is dedicated to establishing error estimates for the numerical approximation, considering both the energy norm and the L^2 norm. Finally, in Section 8, we present numerical results that demonstrate the effectiveness of the developed theory.

This paper will follow the standard notations for the Sobolev space. For an open bounded domain $D \subset \mathbb{R}^d$ with Lipschitz continuous boundary ∂D, we denote by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$ the norm and semi-norm in the Sobolev space $H^s(D)$ for any $s \geq 0$. When $s = 0$, we use $(\cdot,\cdot)$ and $|\cdot|_D$ to denote the usual integral inner product and semi-norm, respectively. The subscript will be omitted when $D = \Omega$. Moreover, we use "$A \lesssim B$" to denote the inequality "$A \leq CB$" where C stands for a generic constant independent of the meshsize or the functions appearing in the inequality.

2. Discrete weak derivatives

Let $\mathcal{T}_h$ be a polytopal partition of Ω satisfying the shape regular assumptions described in [30]. For $T \in \mathcal{T}_h$, denote by ∂T the boundary of T consisting of $(d-1)$-dimensional polytopal elements (called "face" for simplicity). For each face $\mathcal{F} \subset \partial T$, denote by $\partial\mathcal{F}$ the boundary of $\mathcal{F}$ consisting of $(d-2)$-dimensional polytopal elements (called "edge" for simplicity). Denote by $\mathcal{F}_h$ the set of all faces for all elements in $\mathcal{T}_h$ and $\mathcal{F}_h^0 = \mathcal{F}_h \setminus \partial\Omega$ the set of all interior faces. Analogously, denote by $\mathcal{E}_h$ the set of all edges for all elements in $\mathcal{T}_h$ and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ the set of all interior edges. Moreover, denote by h_T the meshsize of T and $h = \max_{T\in\mathcal{T}_h} h_T$ the meshsize of $\mathcal{T}_h$. For any given integer $r \geq 0$, denote by $P_r(T)$ and $P_r(\partial T)$ the space of polynomials on T and ∂T with degrees no more than r, respectively.

For each element $T \in \mathcal{T}_h$, we introduce a weak function $v = \{v_0, v_{b,e}, v_{b,f}, v_n\mathbf{n}_f\}$, where v_0 represents the value of v in the interior of T, $v_{b,e}$ and $v_{b,f}$ represent the values of v on the edge e and face $\mathcal{F}$ respectively, $\mathbf{n}_f$ is the assigned unit normal vector to $\mathcal{F}$, and v_n represents the normal derivative of v on ∂T along the direction $\mathbf{n}_f$.

For any given integer $k \geq 3$, denote by $V_k(T)$ the discrete space of local weak functions given by

$$\begin{aligned} V_k(T) = \{\{v_0, v_{b,e}, v_{b,f}, v_n\mathbf{n}_f\} :\ & v_0 \in P_k(T), v_{b,e} \in P_{k-2}(e), v_{b,f} \in P_{k-3}(\mathcal{F}), \\ & v_n \in P_{k-2}(\mathcal{F}), \mathcal{F} \subset \partial T, e \subset \partial\mathcal{F}\}. \end{aligned}$$

It should be pointed out that $v_{b,e} = const$ from problems in 2D.

On each face $\mathcal{F}$, we introduce a finite element space $\mathcal{W}_{k-2}(\mathcal{F})$ as polynomial vectors of degree $k-2$ tangential to $\mathcal{F}$:

$$\mathcal{W}_{k-2}(\mathcal{F}) = \{\boldsymbol{\psi} : \ \boldsymbol{\psi} \in [P_{k-2}(\mathcal{F})]^d, \ \boldsymbol{\psi} \cdot \mathbf{n}_f = 0\}.$$

Definition 2.1 ([29]).(Discrete weak tangential derivative) The discrete weak tangential derivative for any weak function $v \in V_k(T)$, denoted by $\nabla_{w,\boldsymbol{\tau},k-2,T} v$, is defined as the unique polynomial in $\mathcal{W}_{k-2}(\mathcal{F})$ satisfying

$$\langle \nabla_{w,\boldsymbol{\tau},k-2,T} v, \boldsymbol{\psi} \times \boldsymbol{n}_f \rangle_{\mathcal{F}} = -\langle v_{b,f}, (\nabla \times \boldsymbol{\psi}) \cdot \boldsymbol{n}_f \rangle_{\mathcal{F}} + \langle v_{b,e}, \boldsymbol{\psi} \cdot \boldsymbol{\tau} \rangle_{\partial \mathcal{F}} \tag{2.1}$$

for all $\boldsymbol{\psi} \in \mathcal{W}_{k-2}(\mathcal{F})$. Here, $\boldsymbol{\tau}$ represents the tangential unit vector on $\partial\mathcal{F}$ that is set such that $\boldsymbol{\tau}$ and $\boldsymbol{n}_f$ obey the right hand rule.

With the normal derivative v_n and the discrete weak tangential derivative $\nabla_{w,\boldsymbol{\tau},k-2,T} v$, we can define the weak gradient of v on the face $\mathcal{F}$ as follows:

$$\boldsymbol{v}_g = v_n \mathbf{n}_f + \nabla_{w,\boldsymbol{\tau},k-2,T} v. \tag{2.2}$$

Definition 2.2 ([19]). (Discrete weak second order partial derivative) For any $v \in V_k(T)$, the discrete weak second order partial derivative, denoted by $\partial^2_{ij,w,k-2,T} v$, is defined as a unique polynomial in $P_{k-2}(T)$ satisfying

$$(\partial^2_{ij,w,k-2,T} v, \varphi)_T = (v_0, \partial^2_{ji}\varphi)_T - \langle v_{b,f} n_i, \partial_j \varphi \rangle_{\partial T} + \langle v_{gi}, \varphi n_j \rangle_{\partial T} \tag{2.3}$$

for any $\varphi \in P_{k-2}(T)$. Here, $\mathbf{n} = (n_1, \ldots, n_d)$ represents the unit outward normal vector to ∂T, and v_{gi} is the ith component of the vector $\boldsymbol{v}_g$ given in (2.2).

By utilizing the integration by parts to the first term on the right-hand side of (2.3) we obtain

$$(\partial^2_{ij,w,k-2,T} v, \varphi)_T = (\partial^2_{ij} v_0, \varphi)_T + \langle (v_0 - v_{b,f}) n_i, \partial_j \varphi \rangle_{\partial T} - \langle \partial_i v_0 - v_{gi}, \varphi n_j \rangle_{\partial T} \tag{2.4}$$

for any $\varphi \in P_{k-2}(T)$.

3. Weak Galerkin schemes

We construct a global finite element space V_h by patching $V_k(T)$ over all the elements $T \in \mathcal{T}_h$ through common values $v_{b,e}$ on $\mathcal{E}_h^0$, $v_{b,f}$ and $v_n \mathbf{n}_f$ on $\mathcal{F}_h^0$; i.e.,

$$V_h = \{v = \{v_0, v_{b,e}, v_{b,f}, v_n \mathbf{n}_f\} : \ v|_T \in V_k(T), \ T \in \mathcal{T}_h\}.$$

Denote by V_h^0 the subspace of V_h given by

$$V_h^0 = \{v : v \in V_h, \ v_{b,e}|_e = 0, \ v_{b,f}|_{\mathcal{F}} = 0, \ v_n|_{\mathcal{F}} = 0, \ e \subset \partial\Omega, \ \mathcal{F} \subset \partial\Omega\}.$$

For convenience, denote by $\nabla_{w,\boldsymbol{\tau}} v$ the discrete weak tangential derivative $\nabla_{w,\boldsymbol{\tau},k-2,T} v$ and $\partial^2_{ij,w} v$ the discrete weak second order partial derivative $\partial^2_{ij,w,k-2,T} v$; i.e.,

$$(\nabla_{w,\boldsymbol{\tau}} v)|_T = \nabla_{w,\boldsymbol{\tau},k-2,T}(v|_T), \quad (\partial^2_{ij,w} v)|_T = \partial^2_{ij,w,k-2,T}(v|_T), \quad v \in V_h.$$

Denote by Q_b, Q_f and Q_n the usual L^2 projection operators onto $P_{k-2}(e)$, $P_{k-3}(\mathcal{F})$ and $P_{k-2}(\mathcal{F})$, respectively. In $V_h \times V_h$, we introduce the following bilinear forms:

$$\begin{aligned}
(\partial^2_w w, \partial^2_w v) &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^{d} (\partial^2_{ij,w} w, \partial^2_{ij,w} v)_T, \\
s(w, v) &= \sum_{T \in \mathcal{T}_h} h_T^{-2} \langle Q_b w_0 - w_{b,e}, Q_b v_0 - v_{b,e} \rangle_{\partial\mathcal{F}} \\
&\quad + \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle Q_f w_0 - w_{b,f}, Q_f v_0 - v_{b,f} \rangle_{\partial T} \\
&\quad + \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_n(\nabla w_0) \cdot \boldsymbol{n}_f - w_n, Q_n(\nabla v_0) \cdot \boldsymbol{n}_f - v_n \rangle_{\partial T} \\
&\quad + \delta_{k,3} \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_n D_{\boldsymbol{\tau}} w_0 - \nabla_{w,\boldsymbol{\tau}} w, Q_n D_{\boldsymbol{\tau}} v_0 - \nabla_{w,\boldsymbol{\tau}} v \rangle_{\partial T}, \\
a_s(w, v) &= (\partial^2_w w, \partial^2_w v) + s(w, v),
\end{aligned}$$

where $Q_n D_{\boldsymbol{\tau}} w_0 = Q_n(\mathbf{n}_f \times (\nabla w_0 \times \mathbf{n}_f))$ and $\delta_{k,3}$ is the usual Kronecker's delta with value 1 when $k = 3$ and value 0 otherwise.

Weak Galerkin Algorithm 1. *A numerical approximation for the model Eq. (1.1) based on the weak formulation (1.2) can be obtained by seeking* $u_h = \{u_0, u_{b,e}, u_{b,f}, u_n \mathbf{n}_f\} \in V_h$ *satisfying* $u_{b,e} = Q_b \zeta$ *on* $e \subset \partial\Omega$, $u_{b,f} = Q_f \zeta$ *and* $u_n = Q_n \xi$ *on* $\mathcal{F} \subset \partial\Omega$ *and the following equation*

$$a_s(u_h, v) = (g, v_0), \qquad \forall v \in V_h^0. \tag{3.1}$$

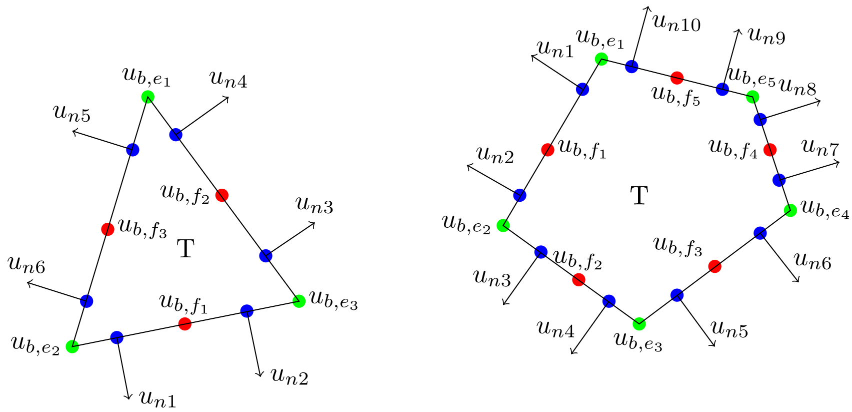

Fig. 3.1. Local degrees of freedom for the finite element space $V_3(T)$ on a triangular element (left) and a pentagonal element (right).

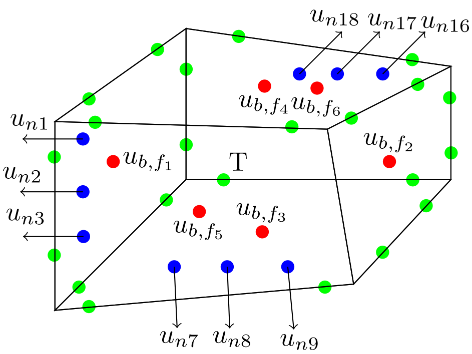

Fig. 3.2. Local degrees of freedom for the finite element space $V_3(T)$ on a hexahedral element.

One may apply the Schur complement approach to the weak Galerkin scheme (3.1), yielding an equivalent formulation with reduced number of unknowns in the resulting linear system. More specifically, the Schur complement for (3.1) seeks $u_h = \{D(u_{b,e}, u_{b,f}, u_n, g), u_{b,e}, u_{b,f}, u_n \mathbf{n}_f\} \in V_h$ such that $u_{b,e} = Q_b\zeta$ on $e \subset \partial\Omega$, $u_{b,f} = Q_f\zeta$ and $u_n = Q_n\xi$ on $\mathcal{F} \subset \partial\Omega$ satisfying

$$a_s(\{D(u_{b,e}, u_{b,f}, u_n, g), u_{b,e}, u_{b,f}, u_n \mathbf{n}_f\}, v) = 0 \tag{3.2}$$

for all $v = \{0, v_{b,e}, v_{b,f}, v_n \mathbf{n}_f\} \in V_h^0$, where $u_0 = D(u_{b,e}, u_{b,f}, u_n, g)$ is obtained by solving the following equation

$$a_s(\{u_0, u_{b,e}, u_{b,f}, u_n \mathbf{n}_f\}, v) = (g, v_0) \tag{3.3}$$

for all $v = \{v_0, 0, 0, \mathbf{0}\} \in V_h^0$.

Remark 3.1. The weak Galerkin scheme (3.1) is equivalent to its Schur complement (3.2)–(3.3). The proof is similar to that in [31]. As an illustration, when $k = 3$, the degrees of freedom on a triangular element, a pentagonal element and a hexahedral element are shown in Figs. 3.1 and 3.2, respectively.

4. Solution existence and uniqueness

On each element $T \in \mathcal{T}_h$, denote by Q_0 the usual L^2 projection operator onto $P_k(T)$. For any $\phi \in H^2(\Omega)$, let

$$Q_h\phi = \{Q_0\phi, Q_b\phi, Q_f\phi, Q_n(\nabla\phi \cdot \mathbf{n}_f)\mathbf{n}_f\}.$$

Similarly, denote by $\mathbb{Q}_h$ the L^2 projection operator onto $P_{k-2}(T)$.

Lemma 4.1. *For any $\phi \in H^2(T)$, the following commutative property holds true*

$$\nabla_{w,\tau} Q_h\phi = Q_n(\mathbf{n}_f \times (\nabla\phi \times \mathbf{n}_f)), \tag{4.1}$$

$$\partial^2_{ij,w}(Q_h\phi) = \mathbb{Q}_h(\partial^2_{ij}\phi), \quad i,j = 1,\ldots,d. \tag{4.2}$$

Proof. First of all, the identity (4.1) has been established in [29]. Hence, the gradient representation (2.2) for $(Q_h\phi)_g$ has the following form

$$\begin{aligned}(Q_h\phi)_g &= Q_n(\nabla\phi\cdot\mathbf{n}_f)\mathbf{n}_f + Q_n(\mathbf{n}_f\times(\nabla\phi\times\mathbf{n}_f))\\ &= Q_n(\nabla\phi).\end{aligned} \tag{4.3}$$

In other words, the weak gradient of $Q_h\phi$ is the L^2 projection of the classical gradient of ϕ on each face $\mathcal{F}\subset\partial T$. Thus, from (2.3) and the usual integration by parts we obtain

$$\begin{aligned}&(\partial^2_{ij,w}(Q_h\phi),\varphi)_T\\ =&(Q_0\phi,\partial^2_{ji}\varphi)_T - \langle Q_f\phi n_i,\partial_j\varphi\rangle_{\partial T} + \langle Q_n(\nabla\phi)_i,\varphi n_j\rangle_{\partial T}\\ =&(\phi,\partial^2_{ji}\varphi)_T - \langle\phi n_i,\partial_j\varphi\rangle_{\partial T} + \langle(\nabla\phi)_i,\varphi n_j\rangle_{\partial T}\\ =&(\partial^2_{ij}\phi,\varphi)_T\\ =&(\mathbb{Q}_h(\partial^2_{ij}\phi),\varphi)_T\end{aligned}$$

for all $\varphi\in P_{k-2}(T)$. This verifies the identity (4.2).

Observe that the bilinear form $a_s(v,v)$ induces a semi-norm in the finite element space V_h given by

$$|||v||| = \left(a_s(v,v)\right)^{1/2}. \tag{4.4}$$

Lemma 4.2. *The semi-norm $|||v|||$ defined by (4.4) is a norm in the subspace V_h^0.*

Proof. It suffices to show that $|||v|||=0$ implies $v=0$. To this end, assume $|||v|||=0$ for some $v\in V_h^0$. From (4.4) we have $\partial^2_w v=0$ and $s(v,v)=0$, which implies $\partial^2_{ij,w}v=0$ for $i,j=1,\ldots,d$ on each T, $Q_bv_0=v_{b,e}$ on each $\partial\mathcal{F}$, $Q_fv_0=v_{b,f}$ and $Q_n(\nabla v_0)\cdot\mathbf{n}_f=v_n$ on each ∂T. Thus, on each element $T\in\mathcal{T}_h$ we have $Q_hv_0=v$ so that by using (4.2)

$$\partial^2_{ij}v_0 = \mathbb{Q}_h\partial^2_{ij}v_0 = \partial^2_{ij,w}(Q_hv_0) = \partial^2_{ij,w}v = 0, \quad i,j=1,\ldots,d.$$

Hence, $\nabla v_0 = const$ on each $T\in\mathcal{T}_h$. Note that on each face $\mathcal{F}\in\partial T$ we have

$$\nabla v_0 = (\nabla v_0\cdot\mathbf{n}_f)\mathbf{n}_f + \mathbf{n}_f\times(\nabla v_0\times\mathbf{n}_f),$$

which, together with $Q_n(\mathbf{n}_f\times(\nabla v_0\times\mathbf{n}_f)) = \nabla_{w,\tau}v$ and $Q_n(\nabla v_0)\cdot\mathbf{n}_f = v_n$, gives rise to $\nabla v_0 = v_n\mathbf{n}_f + \nabla_{w,\tau}v$ on each face $\mathcal{F}\in\mathcal{F}_h$ and hence $\nabla v_0\in C^0(\Omega)$. Next, with $v_{b,f}=0$ on each $\mathcal{F}\subset\partial\Omega$ and $v_{b,e}=0$ on each $e\subset\partial\Omega$ we have from (2.1) that $\nabla_{w,\tau}v=0$ on each $\mathcal{F}\subset\partial\Omega$. This, together with $v_n=0$ on each $\mathcal{F}\subset\partial\Omega$, gives $\nabla v_0=0$ on $\mathcal{F}\subset\partial\Omega$ and further $\nabla v_0=0$ in the domain Ω since $\nabla v_0=const$ on each T and $\nabla v_0\in C^0(\Omega)$. Hence, $v_n=0$ on each $\mathcal{F}$ and $v_0=const$ on each T. This further leads to $v_0=Q_bv_0=v_{b,e}$ on each $\partial\mathcal{F}$ and $v_0=Q_fv_0=v_{b,f}$ on each ∂T, and hence $v_0\in C^0(\Omega)$. From $v_{b,e}=0$ on $e\subset\partial\Omega$ and $v_{b,f}=0$ on each $\mathcal{F}\subset\partial\Omega$ we have $v_0=0$ in Ω. Finally, from $v_{b,e}=Q_bv_0$ on each $\partial\mathcal{F}$ and $v_{b,f}=Q_fv_0$ on each ∂T we have $v_{b,e}=0$ on each $\partial\mathcal{F}$ and $v_{b,f}=0$ on each ∂T. This completes the proof of the lemma.

Lemma 4.3. *The weak Galerkin scheme (3.1) has one and only one numerical approximation.*

Proof. It suffices to verify the uniqueness of the numerical approximation. To this end, assume that $u_h^{(1)}$ and $u_h^{(2)}$ are two solutions of (3.1). It is clear that

$$a_s(u_h^{(1)}-u_h^{(2)},v)=0, \quad \forall v\in V_h^0. \tag{4.5}$$

By letting $v=u_h^{(1)}-u_h^{(2)}\in V_h^0$ in (4.5) we obtain

$$|||u_h^{(1)}-u_h^{(2)}|||=0,$$

which implies $u_h^{(1)}=u_h^{(2)}$ from Lemma 4.2. This completes the proof of the lemma.

5. Error equations

Let u be the exact solution of the model Eq. (1.1) and $u_h\in V_h$ be the numerical solution of the WG scheme (3.1), respectively. Denote by

$$e_h = Q_hu - u_h \tag{5.1}$$

the error function between the L^2 projection of the exact solution and its WG finite element approximation u_h.

Lemma 5.1. *The error function e_h defined in (5.1) satisfies the following error equation*

$$a_s(e_h, v) = \zeta_u(v), \qquad \forall v \in V_h^0, \tag{5.2}$$

where $\zeta_u(v)$ is given by

$$\begin{aligned}\zeta_u(v) =& s(Q_h u, v) + \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle v_0 - v_{b,f}, \partial_j(\mathbb{Q}_h(\partial^2_{ij}u) - \partial^2_{ij}u)n_i\rangle_{\partial T} \\ &+ \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle \partial_i v_0 - v_{gi}, (\partial^2_{ij}u - \mathbb{Q}_h(\partial^2_{ij}u))n_j\rangle_{\partial T}.\end{aligned} \tag{5.3}$$

Proof. Let $v \in V_h^0$. On any face $\mathcal{F} \subset \partial\Omega$, we have $v_{b,f} = 0$ and $v_{b,e} = 0$ on $e \subset \partial\mathcal{F}$. Thus, from (2.1) we have

$$\langle \nabla_{w,\tau} v, \boldsymbol{\psi} \times \boldsymbol{n}_f\rangle_{\mathcal{F}} = -\langle v_{b,f}, (\nabla\times\boldsymbol{\psi})\cdot \boldsymbol{n}_f\rangle_{\mathcal{F}} + \langle v_{b,e}, \boldsymbol{\psi}\cdot\boldsymbol{\tau}\rangle_{\partial\mathcal{F}} = 0$$

for any $\boldsymbol{\psi} \in \mathcal{W}_{k-2}(\mathcal{F})$. Hence, $\nabla_{w,\tau} v = 0$ on $\partial\Omega$. This, together with (2.2) and $v_n = 0$ on $\partial\Omega$, gives rise to $\boldsymbol{v}_g = 0$ on $\partial\Omega$.

By testing the model Eq. (1.1) against v_0 and then using the usual integration by parts we have

$$\begin{aligned}(g, v_0) =& \sum_{T\in\mathcal{T}_h}(\Delta^2 u, v_0)_T \\ =& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}(\partial^2_{ij}u, \partial^2_{ij}v_0)_T - \langle \partial^2_{ij}u, \partial_i v_0 n_j\rangle_{\partial T} + \langle \partial_j(\partial^2_{ij}u)n_i, v_0\rangle_{\partial T} \\ =& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}(\partial^2_{ij}u, \partial^2_{ij}v_0)_T - \langle \partial^2_{ij}u, (\partial_i v_0 - v_{gi})n_j\rangle_{\partial T} \\ &+ \langle \partial_j(\partial^2_{ij}u)n_i, v_0 - v_{b,f}\rangle_{\partial T},\end{aligned} \tag{5.4}$$

where we used the fact that

$$\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle \partial^2_{ij}u, v_{gi}n_j\rangle_{\partial T} = 0,$$

$$\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle \partial_j(\partial^2_{ij}u)n_i, v_{b,f}\rangle_{\partial T} = 0,$$

and $v_{b,f} = 0$, $\mathbf{v}_g = 0$ on $\mathcal{F} \subset \partial\Omega$.

To handle the first term on last line in (5.4), we choose $\varphi = \mathbb{Q}_h(\partial^2_{ij}u) \in P_{k-2}(T)$ in (2.4) and then use Lemma 4.1 to obtain

$$\begin{aligned}(\partial^2_{ij}v_0, \partial^2_{ij}u)_T =& (\partial^2_{ij}v_0, \mathbb{Q}_h(\partial^2_{ij}u))_T \\ =& (\partial^2_{ij,w}v, \mathbb{Q}_h(\partial^2_{ij}u))_T - \langle (v_0 - v_{b,f})n_i, \partial_j(\mathbb{Q}_h(\partial^2_{ij}u))\rangle_{\partial T} \\ &+ \langle \partial_i v_0 - v_{gi}, \mathbb{Q}_h(\partial^2_{ij}u)n_j\rangle_{\partial T} \\ =& (\partial^2_{ij,w}v, \partial^2_{ij,w}Q_h u)_T - \langle (v_0 - v_{b,f})n_i, \partial_j(\mathbb{Q}_h(\partial^2_{ij}u))\rangle_{\partial T} \\ &+ \langle \partial_i v_0 - v_{gi}, \mathbb{Q}_h(\partial^2_{ij}u)n_j\rangle_{\partial T}.\end{aligned} \tag{5.5}$$

Substituting (5.5) into (5.4) gives

$$\begin{aligned}(g, v_0) =& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}(\partial^2_{ij,w}v, \partial^2_{ij,w}Q_h u)_T + \langle v_0 - v_{b,f}, \partial_j(\partial^2_{ij}u - \mathbb{Q}_h(\partial^2_{ij}u))n_i\rangle_{\partial T} \\ &+ \langle \partial_i v_0 - v_{gi}, (\mathbb{Q}_h(\partial^2_{ij}u) - \partial^2_{ij}u)n_j\rangle_{\partial T}.\end{aligned} \tag{5.6}$$

Subtracting (3.1) from (5.6) gives rise to Lemma 5.1.

6. Technical results

Note that for any $T \in \mathcal{T}_h$ and $\phi \in H^1(T)$, the following trace inequality [30] holds true:

$$\|\phi\|^2_{\partial T} \lesssim h_T^{-1}\|\phi\|^2_T + h_T\|\nabla\phi\|^2_T. \tag{6.1}$$

If ϕ is a polynomial on the element $T \in \mathcal{T}_h$, we have from the inverse inequality that

$$\|\phi\|^2_{\partial T} \lesssim h_T^{-1}\|\phi\|^2_T. \tag{6.2}$$

Lemma 6.1. *Assume that $\mathcal{T}_h$ is a finite element partition satisfying the regular assumptions described in [30]. Then, for any $0 \le s \le 2$, the following error estimates [30,32] hold true:*

$$\sum_{T\in\mathcal{T}_h} h_T^{2s}\|\phi - Q_0\phi\|_{s,T}^2 \lesssim h^{2(k+1)}\|\phi\|_{k+1}^2, \tag{6.3}$$

$$\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d} h_T^{2s}\|\partial_{ij}^2\phi - \mathbb{Q}_h(\partial_{ij}^2\phi)\|_{s,T}^2 \lesssim h^{2(k-1)}\|\phi\|_{k+1}^2. \tag{6.4}$$

Lemma 6.2. *For any $v \in V_h$, there holds*

$$\Big(\sum_{T\in\mathcal{T}_h}\sum_{i=1}^{d} h_T^{-1}\|Q_n(\partial_i v_0) - v_{gi}\|_{\partial T}^2\Big)^{\frac{1}{2}} \lesssim |||v|||. \tag{6.5}$$

Proof. From $\nabla v_0 = (\nabla v_0\cdot\mathbf{n}_f)\mathbf{n}_f + \mathbf{n}_f\times(\nabla v_0\times\mathbf{n}_f)$ and (2.2) we have

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\sum_{i=1}^{d} h_T^{-1}\|Q_n(\partial_i v_0) - v_{gi}\|_{\partial T}^2\\
=&\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_n(\nabla v_0) - \boldsymbol{v}_g\|_{\partial T}^2\\
=&\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_n(\nabla v_0\cdot\mathbf{n}_f)\mathbf{n}_f + Q_n(\mathbf{n}_f\times(\nabla v_0\times\mathbf{n}_f)) - (v_n\mathbf{n}_f + \nabla_{w,\boldsymbol{\tau}}v)\|_{\partial T}^2\\
\lesssim&\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_n(\nabla v_0\cdot\mathbf{n}_f) - v_n\|_{\partial T}^2 + h_T^{-1}\|Q_n(\mathbf{n}_f\times(\nabla v_0\times\mathbf{n}_f)) - \nabla_{w,\boldsymbol{\tau}}v\|_{\partial T}^2\\
\lesssim&|||v|||^2 + \sum_{\mathcal{F}\in\mathcal{F}_h} h_T^{-1}\|Q_n(\mathbf{n}_f\times(\nabla v_0\times\mathbf{n}_f)) - \nabla_{w,\boldsymbol{\tau}}v\|_{\mathcal{F}}^2.
\end{aligned} \tag{6.6}$$

Next, from (2.1) and the Stokes theorem we have

$$\begin{aligned}
&|\langle Q_n(\mathbf{n}_f\times(\nabla v_0\times\mathbf{n}_f)) - \nabla_{w,\boldsymbol{\tau}}v, \boldsymbol{\psi}\times\mathbf{n}_f\rangle_{\mathcal{F}}|\\
=&|\langle Q_f v_0 - v_{b,f}, (\nabla\times\boldsymbol{\psi})\cdot\mathbf{n}_f\rangle_{\mathcal{F}} + \langle v_{b,e} - Q_b v_0, \boldsymbol{\psi}\cdot\boldsymbol{\tau}\rangle_{\partial\mathcal{F}}|\\
\le&\|Q_f v_0 - v_{b,f}\|_{\mathcal{F}}\|\nabla\times\boldsymbol{\psi}\|_{\mathcal{F}} + \|v_{b,e} - Q_b v_0\|_{\partial\mathcal{F}}\|\boldsymbol{\psi}\|_{\partial\mathcal{F}}\\
\lesssim&\|Q_f v_0 - v_{b,f}\|_{\mathcal{F}} h_T^{-1}\|\boldsymbol{\psi}\|_{\mathcal{F}} + \|v_{b,e} - Q_b v_0\|_{\partial\mathcal{F}} h_T^{-\frac{1}{2}}\|\boldsymbol{\psi}\|_{\mathcal{F}}
\end{aligned}$$

for all $\boldsymbol{\psi}\in\mathcal{W}_{k-2}(\mathcal{F})$. Hence,

$$\|Q_n(\mathbf{n}_f\times(\nabla v_0\times\mathbf{n}_f)) - \nabla_{w,\boldsymbol{\tau}}v\|_{\mathcal{F}} \lesssim h_T^{-1}\|Q_f v_0 - v_{b,f}\|_{\mathcal{F}} + h_T^{-\frac{1}{2}}\|v_{b,e} - Q_b v_0\|_{\partial\mathcal{F}}.$$

Substituting the above estimate into (6.6) gives rise to the desired inequality (6.5).

Lemma 6.3. *For any $v \in V_h$, there yields*

$$\sum_{T\in\mathcal{T}_h}|v_0|_{2,T}^2 \lesssim |||v|||^2. \tag{6.7}$$

Proof. By taking $\varphi = \partial_{ij}^2 v_0 \in P_{k-2}(T)$ in (2.4) we have

$$\begin{aligned}
&(\partial_{ij,w}^2 v, \partial_{ij}^2 v_0)_T\\
=&(\partial_{ij}^2 v_0, \partial_{ij}^2 v_0)_T + \langle (v_0 - v_{b,f})n_i, \partial_j(\partial_{ij}^2 v_0)\rangle_{\partial T} - \langle \partial_i v_0 - v_{gi}, \partial_{ij}^2 v_0 n_j\rangle_{\partial T}\\
=&(\partial_{ij}^2 v_0, \partial_{ij}^2 v_0)_T + \langle (Q_f v_0 - v_{b,f})n_i, \partial_j(\partial_{ij}^2 v_0)\rangle_{\partial T} - \langle Q_n(\partial_i v_0) - v_{gi}, \partial_{ij}^2 v_0 n_j\rangle_{\partial T}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h}|v_0|_{2,T}^2 \lesssim&\Big(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\|\partial_{ij,w}^2 v\|_T^2\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\|\partial_{ij}^2 v_0\|_T^2\Big)^{\frac{1}{2}}\\
&+\Big(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|Q_f v_0 - v_{b,f}\|_{\partial T}^2\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d} h_T^3\|\partial_j(\partial_{ij}^2 v_0)\|_{\partial T}^2\Big)^{\frac{1}{2}}\\
&+\Big(\sum_{T\in\mathcal{T}_h}\sum_{i=1}^{d} h_T^{-1}\|Q_n(\partial_i v_0) - v_{gi}\|_{\partial T}^2\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d} h_T\|\partial_{ij}^2 v_0\|_{\partial T}^2\Big)^{\frac{1}{2}}\\
\lesssim&|||v|||\Big(\sum_{T\in\mathcal{T}_h}|v_0|_{2,T}^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

This completes the proof of the lemma.

Denote by $[v_0]$ the jump of v_0 on the face $\mathcal{F}$ shared by two adjacent elements T_L and T_R, i.e., $[v_0] = v_0|_{T_L\cap\mathcal{F}} - v_0|_{T_R\cap\mathcal{F}}$.

Lemma 6.4. *Let $k \geq 3$. For any $v \in V_h$ and $\varphi \in H^{k+1}(\Omega)$, there holds*

$$|\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle v_0 - Q_f v_0, \partial_j(\mathbb{Q}_h(\partial_{ij}^2\varphi) - \partial_{ij}^2\varphi)n_i\rangle_{\partial T}| \lesssim h^{k-1}\|\varphi\|_{k+1}|\!|\!|v|\!|\!|, \tag{6.8}$$

$$\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_n(D_{\boldsymbol{\tau}}Q_0\varphi) - \nabla_{w,\boldsymbol{\tau}}Q_h\varphi\|_{\partial T}^2\Big)^{\frac{1}{2}} \lesssim h^{k-1}\|\varphi\|_{k+1}. \tag{6.9}$$

Proof. We first note the following identity

$$\begin{aligned}
J :=& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle v_0 - Q_f v_0, \partial_j(\partial_{ij}^2\varphi - \mathbb{Q}_h(\partial_{ij}^2\varphi))n_i\rangle_{\partial T}\\
=& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle v_0 - Q_f v_0, \partial_j\partial_{ij}^2\varphi n_i\rangle_{\partial T}\\
=& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle v_0 - Q_f v_0, (I-Q_f)\partial_j\partial_{ij}^2\varphi n_i\rangle_{\partial T}\\
=& \sum_{\mathcal{F}\in\mathcal{F}_h}\sum_{i,j=1}^{d}\langle [v_0] - Q_f[v_0], (I-Q_f)\partial_j\partial_{ij}^2\varphi n_i\rangle_{\mathcal{F}}.
\end{aligned}$$

For $k > 3$, the finite element space on face $\mathcal{F}$ consists of linear functions so that

$$|\langle [v_0] - Q_f[v_0], (I-Q_f)\partial_j\partial_{ij}^2\varphi n_i\rangle_{\mathcal{F}}| \leq Ch^2\|[v_0]\|_{2,\mathcal{F}}\|(I-Q_f)\partial_j\partial_{ij}^2\varphi\|_{0,\mathcal{F}},$$

which can be used to derive the desired inequality (6.8) without any difficulty.

For the case of $k = 3$, the finite element space on face $\mathcal{F}$ consists of constants only so that

$$|\langle [v_0] - Q_f[v_0], (I-Q_f)\partial_j\partial_{ij}^2\varphi n_i\rangle_{\mathcal{F}}| \leq Ch\|[D_{\boldsymbol{\tau}}v_0]\|_{0,\mathcal{F}}\|(I-Q_f)\partial_j\partial_{ij}^2\varphi\|_{0,\mathcal{F}},$$

where $D_{\boldsymbol{\tau}}v_0$ stands for the tangential derivative on $\mathcal{F}$. It follows from the trace inequalities (6.1)–(6.2) and the inverse inequality that

$$\begin{aligned}
|J| \lesssim& \Big(\sum_{\mathcal{F}\in\mathcal{F}_h} h_T^2\|[D_{\boldsymbol{\tau}}v_0] - Q_n([D_{\boldsymbol{\tau}}v_0])\|_{\mathcal{F}}^2 + h_T^2\|Q_n([D_{\boldsymbol{\tau}}v_0])\|_{\mathcal{F}}^2\Big)^{\frac{1}{2}}\\
&\cdot\Big(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d} h_T^{-1}\|(I-Q_f)\partial_j\partial_{ij}^2\varphi\|_T^2 + h_T|(I-Q_f)\partial_j\partial_{ij}^2\varphi|_{1,T}^2\Big)^{\frac{1}{2}}\\
\lesssim& \Big(\sum_{\mathcal{F}\in\mathcal{F}_h} h_T^4\|[D_{\boldsymbol{\tau}\boldsymbol{\tau}}v_0]\|_{\mathcal{F}}^2 + h_T^2\|Q_n([D_{\boldsymbol{\tau}}v_0]) - [\nabla_{w,\boldsymbol{\tau}}v]\|_{\mathcal{F}}^2\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h} h_T^{2k-5}\|\varphi\|_{k+1,T}^2\Big)^{\frac{1}{2}}\\
\lesssim& \Big(\sum_{T\in\mathcal{T}_h} h_T^3|v_0|_{2,T}^2 + \sum_{T\in\mathcal{T}_h} h_T^2\|Q_nD_{\boldsymbol{\tau}}v_0 - \nabla_{w,\boldsymbol{\tau}}v\|_{\partial T}^2\Big)^{\frac{1}{2}} h^{k-\frac{5}{2}}\|\varphi\|_{k+1}\\
\lesssim& h^{k-1}\|\varphi\|_{k+1}|\!|\!|v|\!|\!|,
\end{aligned}$$

which, together with (4.4), completes the proof of (6.8).

To verify (6.9), we recall that $Q_nD_{\boldsymbol{\tau}}w_0 = Q_n(\mathbf{n}_f\times(\nabla w_0\times\mathbf{n}_f))$. Hence, from (4.1), the trace inequality (6.1), and (6.3) we arrive at

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_n(D_{\boldsymbol{\tau}}Q_0\varphi) - \nabla_{w,\boldsymbol{\tau}}Q_h\varphi\|_{\partial T}^2\\
=& \sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_n(\mathbf{n}_f\times(\nabla Q_0\varphi\times\mathbf{n}_f)) - Q_n(\mathbf{n}_f\times(\nabla\varphi\times\mathbf{n}_f))\|_{\partial T}^2\\
\lesssim& \sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{n}_f\times(\nabla Q_0\varphi\times\mathbf{n}_{\mathcal{F}}) - \mathbf{n}_f\times(\nabla\varphi\times\mathbf{n}_f)\|_{\partial T}^2\\
\lesssim& \sum_{T\in\mathcal{T}_h} h_T^{-1}\|\nabla Q_0\varphi - \nabla\varphi\|_{\partial T}^2\\
\lesssim& h^{k-1}\|\varphi\|_{k+1}.
\end{aligned}$$

This completes the proof of the lemma.

7. Error estimates

The following is an error estimate for the numerical scheme (3.1) with respect to the natural "energy" norm.

Theorem 7.1. *Let u be the exact solution of Eq. (1.1) and $u_h \in V_h$ be its numerical approximation arising from the WG scheme (3.1). Under the assumption of $u \in H^{k+1}(\Omega)$, the following error estimate holds true:*

$$|||e_h||| \lesssim h^{k-1}\|u\|_{k+1}. \tag{7.1}$$

Proof. By taking $v = e_h \in V_h^0$ in (5.2) we have

$$\begin{aligned}
|||e_h|||^2 =& \zeta_u(e_h)\\
=& s(Q_h u, e_h) + \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle e_0 - e_{b,f}, \partial_j(\mathbb{Q}_h(\partial^2_{ij}u) - \partial^2_{ij}u)n_i\rangle_{\partial T}\\
&+ \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle \partial_i e_0 - e_{gi}, (\partial^2_{ij}u - \mathbb{Q}_h(\partial^2_{ij}u))n_j\rangle_{\partial T}\\
=& I_1 + I_2 + I_3.
\end{aligned} \tag{7.2}$$

For I_1, we have from the Cauchy–Schwarz inequality that

$$\begin{aligned}
|I_1| =& |s(Q_h u, e_h)|\\
\le& \sum_{T\in\mathcal{T}_h} h_T^{-2}|\langle Q_b(Q_0 u) - Q_b u, Q_b e_0 - e_{b,e}\rangle_{\partial\mathcal{F}}|\\
&+ \sum_{T\in\mathcal{T}_h} h_T^{-3}|\langle Q_f(Q_0 u) - Q_f u, Q_f e_0 - e_{b,f}\rangle_{\partial T}|\\
&+ \sum_{T\in\mathcal{T}_h} h_T^{-1}|\langle Q_n(\nabla Q_0 u)\cdot\mathbf{n}_f - Q_n(\nabla u\cdot\mathbf{n}_f), Q_n(\nabla e_0)\cdot\mathbf{n}_f - e_n\rangle_{\partial T}|\\
&+ \delta_{k,3}\sum_{T\in\mathcal{T}_h} h_T^{-1}|\langle Q_n D_{\boldsymbol{\tau}} Q_0 u - \nabla_{w,\boldsymbol{\tau}} Q_h u, Q_n D_{\boldsymbol{\tau}} e_0 - \nabla_{w,\boldsymbol{\tau}} e_h\rangle_{\partial T}|\\
\lesssim& \Big(\sum_{T\in\mathcal{T}_h} h_T^{-2}\|Q_0 u - u\|^2_{\partial\mathcal{F}}\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-2}\|Q_b e_0 - e_{b,e}\|^2_{\partial\mathcal{F}}\Big)^{\frac12}\\
&+ \Big(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|Q_0 u - u\|^2_{\partial T}\Big)^{\frac12}|||e_h||| + \Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\nabla Q_0 u - \nabla u\|^2_{\partial T}\Big)^{\frac12}|||e_h|||\\
&+ \delta_{k,3}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_n D_{\boldsymbol{\tau}} Q_0 u - \nabla_{w,\boldsymbol{\tau}} Q_h u\|^2_{\partial T}\Big)^{\frac12}|||e_h|||.
\end{aligned}$$

Next, using the trace inequality (6.1) and the estimates (6.3) and (6.9) we arrive at

$$\begin{aligned}
|I_1| \lesssim& \Big(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|Q_0 u - u\|^2_{\mathcal{F}} + h_T^{-1}\|\nabla Q_0 u - \nabla u\|^2_{\mathcal{F}}\Big)^{\frac12}|||e_h|||\\
&+ \Big(\sum_{T\in\mathcal{T}_h} h_T^{-4}\|Q_0 u - u\|^2_T + h_T^{-2}\|\nabla Q_0 u - \nabla u\|^2_T\Big)^{\frac12}|||e_h|||\\
&+ \Big(\sum_{T\in\mathcal{T}_h} h_T^{-2}\|\nabla Q_0 u - \nabla u\|^2_T + |Q_0 u - u|^2_{2,T}\Big)^{\frac12}|||e_h||| + \delta_{k,3}h^2\|u\|_4|||e_h|||\\
\lesssim& \Big(\sum_{T\in\mathcal{T}_h} h_T^{-4}\|Q_0 u - u\|^2_T + h_T^{-2}\|\nabla Q_0 u - \nabla u\|^2_T\Big)^{\frac12}|||e_h|||\\
&+ \Big(\sum_{T\in\mathcal{T}_h} h_T^{-2}\|\nabla Q_0 u - \nabla u\|^2_T + |Q_0 u - u|^2_{2,T}\Big)^{\frac12}|||e_h||| + \delta_{k,3}h^2\|u\|_4|||e_h|||\\
\lesssim& h^{k-1}\|u\|_{k+1}|||e_h|||.
\end{aligned} \tag{7.3}$$

For the term I_2, we have from (6.8), the Cauchy–Schwarz inequality, and the trace inequality (6.1) that

$$\begin{aligned} |I_2| =& \Big|\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle e_0-e_{b,f},\partial_j(\mathbb{Q}_h(\partial^2_{ij}u)-\partial^2_{ij}u)n_i\rangle_{\partial T}\Big| \\ =& \Big|\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle e_0-Q_f e_0,\partial_j(\mathbb{Q}_h(\partial^2_{ij}u)-\partial^2_{ij}u)n_i\rangle_{\partial T} \\ &+\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle Q_f e_0-e_{b,f},\partial_j(\mathbb{Q}_h(\partial^2_{ij}u)-\partial^2_{ij}u)n_i\rangle_{\partial T}\Big| \\ \lesssim& h^{k-1}\|u\|_{k+1}|||e_h|||+\Big(\sum_{T\in\mathcal{T}_h}h_T^{-3}\|Q_f e_0-e_{b,f}\|^2_{\partial T}\Big)^{\frac12} \\ &\cdot\Big(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}h_T^{3}\|\partial_j(\mathbb{Q}_h(\partial^2_{ij}u)-\partial^2_{ij}u)\|^2_{\partial T}\Big)^{\frac12} \\ \lesssim& h^{k-1}\|u\|_{k+1}|||e_h|||. \end{aligned} \tag{7.4}$$

As to I_3, we have from the Cauchy–Schwarz inequality, Lemmas 6.2–6.3, the trace inequality (6.1), and (6.4) that

$$\begin{aligned} |I_3| =& \Big|\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle \partial_i e_0-e_{gi},(\partial^2_{ij}u-\mathbb{Q}_h(\partial^2_{ij}u))n_j\rangle_{\partial T}\Big| \\ \lesssim&\Big(\sum_{T\in\mathcal{T}_h}\sum_{i=1}^{d}h_T^{-1}\|Q_n(\partial_i e_0)-e_{gi}\|^2_{\partial T}+h_T^{-1}\|\partial_i e_0-Q_n(\partial_i e_0)\|^2_{\partial T}\Big)^{\frac12} \\ &\cdot\Big(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}h_T\|\partial^2_{ij}u-\mathbb{Q}_h(\partial^2_{ij}u)\|^2_{\partial T}\Big)^{\frac12} \\ \lesssim&\Big(|||e_h|||^2+\sum_{T\in\mathcal{T}_h}|e_0|^2_{2,T}\Big)^{\frac12}h^{k-1}\|u\|_{k+1} \\ \lesssim& h^{k-1}\|u\|_{k+1}|||e_h|||. \end{aligned} \tag{7.5}$$

Substituting (7.3)–(7.5) into (7.2) gives rise to (7.1). This completes the proof of the theorem.

To establish an optimal order error estimate for the numerical solution in the L^2 norm, we consider the dual problem that seeks Φ satisfying

$$\begin{aligned} \Delta^2\Phi &= e_0, && \text{in } \Omega, \\ \Phi &= 0, && \text{on } \partial\Omega, \\ \frac{\partial\Phi}{\partial\mathbf{n}} &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{7.6}$$

Assume that the problem (7.6) has the H^4-regularity in the sense that there exists a constant C such that

$$\|\Phi\|_4\le C\|e_0\|. \tag{7.7}$$

Theorem 7.2. *Let $u\in H^{k+1}(\Omega)$ be the exact solution of the problem (1.1) and $u_h\in V_h$ be its numerical solution arising from the WG scheme (3.1). Under the H^4-regularity assumption (7.7), we have the following error estimate*

$$\|e_0\|\lesssim h^{k+1}\|u\|_{k+1}.$$

Proof. First, using (2.1) with $e_{b,f}=0$ on each $\mathcal{F}\subset\partial\Omega$ and $e_{b,e}=0$ on each $e\subset\partial\Omega$ gives $\nabla_{w,\tau}e_h=0$ on each $\mathcal{F}\subset\partial\Omega$. This, together with $e_n=0$ on $\partial\Omega$ and (2.2), gives $\mathbf{e}_g=0$ on $\partial\Omega$. Next, we test the dual problem (7.6) against e_0 and use the integration by parts to obtain

$$\begin{aligned} \|e_0\|^2 =&(\Delta^2\Phi,e_0) \\ =&\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}(\partial^2_{ij}\Phi,\partial^2_{ij}e_0)_T-\langle\partial^2_{ij}\Phi,\partial_i e_0 n_j\rangle_{\partial T}+\langle\partial_j(\partial^2_{ij}\Phi)n_i,e_0\rangle_{\partial T} \\ =&\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}(\partial^2_{ij}\Phi,\partial^2_{ij}e_0)_T-\langle\partial^2_{ij}\Phi,(\partial_i e_0-e_{gi})n_j\rangle_{\partial T} \\ &+\langle\partial_j(\partial^2_{ij}\Phi)n_i,e_0-e_{b,f}\rangle_{\partial T}, \end{aligned} \tag{7.8}$$

where we have used $\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle\partial^2_{ij}\Phi,e_{gi}n_j\rangle_{\partial T}=0$ and $\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle\partial_j(\partial^2_{ij}\Phi)n_i,e_{b,f}\rangle_{\partial T}=0$ since $e_{b,f}=0$ and $\mathbf{e}_g=0$ on $\partial\Omega$.

Analogous to (5.5), we have

$$
\begin{aligned}
(\partial_{ij}^2\Phi, \partial_{ij}^2 e_0)_T =&(\partial_{ij,w}^2 e_h, \partial_{ij,w}^2 Q_h\Phi)_T + \langle (e_{b,f}-e_0)n_i, \partial_j(\mathbb{Q}_h(\partial_{ij}^2\Phi))\rangle_{\partial T}\\
&+ \langle \partial_i e_0 - e_{gi}, \mathbb{Q}_h(\partial_{ij}^2\Phi)n_j\rangle_{\partial T},
\end{aligned}
$$

which, together with (5.2) and (7.8)–(5.3), leads to

$$
\begin{aligned}
\|e_0\|^2 =& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}(\partial_{ij,w}^2 e_h, \partial_{ij,w}^2 Q_h\Phi)_T + \langle (\partial_i e_0 - e_{gi})n_j, \mathbb{Q}_h(\partial_{ij}^2\Phi) - \partial_{ij}^2\Phi\rangle_{\partial T}\\
&+ \langle (e_0 - e_{b,f})n_i, \partial_j(\partial_{ij}^2\Phi - \mathbb{Q}_h(\partial_{ij}^2\Phi))\rangle_{\partial T}\\
=&\zeta_u(Q_h\Phi) - \zeta_\Phi(e_h)\\
=&\sum_{i=1}^{3} J_i - \zeta_\Phi(e_h),
\end{aligned}
\tag{7.9}
$$

where J_i are given as in (5.3) with $v = Q_h\Phi$.

The rest of the proof amounts to the estimate for each of the four terms on the last line in (7.9).

For J_1, we have from Cauchy–Schwarz inequality, (6.9), the trace inequality (6.1), (6.3), and the H^4 regularity assumption (7.7) that

$$
\begin{aligned}
&|J_1|\\
=&|\sum_{T\in\mathcal{T}_h} h_T^{-2}\langle Q_b(Q_0u) - Q_bu, Q_b(Q_0\Phi) - Q_b\Phi\rangle_{\partial\mathcal{F}}\\
&+ h_T^{-3}\langle Q_f(Q_0u) - Q_fu, Q_f(Q_0\Phi) - Q_f\Phi\rangle_{\partial T}\\
&+ h_T^{-1}\langle Q_n(\nabla Q_0u)\cdot\mathbf{n}_f - Q_n(\nabla u\cdot\mathbf{n}_f), Q_n(\nabla Q_0\Phi)\cdot\mathbf{n}_f - Q_n(\nabla\Phi\cdot\mathbf{n}_f)\rangle_{\partial T}\\
&+ \delta_{k,3}\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_nD_\tau Q_0u - \nabla_{w,\tau}Q_hu, Q_nD_\tau Q_0\Phi - \nabla_{w,\tau}Q_h\Phi\rangle_{\partial T}|\\
\lesssim&\Big(\sum_{T\in\mathcal{T}_h} h_T^{-2}\|Q_0u-u\|_{\partial\mathcal{F}}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-2}\|Q_0\Phi-\Phi\|_{\partial\mathcal{F}}^2\Big)^{\frac12}\\
&+\Big(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|Q_0u-u\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|Q_0\Phi-\Phi\|_{\partial T}^2\Big)^{\frac12}\\
&+\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\nabla Q_0u-\nabla u\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\nabla Q_0\Phi-\nabla\Phi\|_{\partial T}^2\Big)^{\frac12}\\
&+\delta_{k,3}h^4\|u\|_4\|\Phi\|_4\\
\lesssim& h^{k+1}\|u\|_{k+1}\|\Phi\|_4\\
\lesssim& h^{k+1}\|u\|_{k+1}\|e_0\|.
\end{aligned}
\tag{7.10}
$$

For the term J_2, we have

$$
\begin{aligned}
J_2 =& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle Q_0\Phi - Q_f\Phi, \partial_j(\mathbb{Q}_h(\partial_{ij}^2u) - \partial_{ij}^2u)n_i\rangle_{\partial T}\\
=& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle (Q_0\Phi - \Phi) + (\Phi - Q_f\Phi), \partial_j(\mathbb{Q}_h(\partial_{ij}^2u) - \partial_{ij}^2u)n_i\rangle_{\partial T}\\
=& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle Q_0\Phi - \Phi, \partial_j((\mathbb{Q}_h - I)\partial_{ij}^2u)n_i\rangle_{\partial T} + \langle \Phi - Q_f\Phi, \partial_j(\partial_{ij}^2u)n_i\rangle_{\partial T}\\
=& \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle Q_0\Phi - \Phi, \partial_j(\mathbb{Q}_h\partial_{ij}^2u - \partial_{ij}^2u)n_i\rangle_{\partial T},
\end{aligned}
$$

where we used the fact that $\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\langle \Phi - Q_f\Phi, \partial_j(\partial_{ij}^2u)n_i\rangle_{\partial T} = 0$. It follows that

$$
\begin{aligned}
|J_2| \lesssim&\Big(\sum_{T\in\mathcal{T}_h}\|Q_0\Phi-\Phi\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^{d}\|\partial_j(\mathbb{Q}_h(\partial_{ij}^2u) - \partial_{ij}^2u)\|_{\partial T}^2\Big)^{\frac12}\\
\lesssim& h^{k+1}\|\Phi\|_4\ \|u\|_{k+1}\\
\lesssim& h^{k+1}\|u\|_{k+1}\|e_0\|.
\end{aligned}
\tag{7.11}
$$

For the term J_3, we note that the weak gradient of the L^2 projection of a smooth function is the same as the L^2 projection of its classical gradient on the boundary of each element, see (4.3). Hence,

$$J_3 = \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} \langle \partial_i Q_0 \Phi - Q_n(\partial_i \Phi), (\partial_{ij}^2 u - \mathbb{Q}_h(\partial_{ij}^2 u))n_j \rangle_{\partial T}.$$

It follows from the Cauchy–Schwarz inequality, the trace inequality (6.1), Lemma 6.1, and the regularity assumption (7.7) that

$$\begin{aligned} |J_3| \lesssim & \Big(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|\nabla Q_0 \Phi - \nabla \Phi\|_T^2 + h_T |\nabla Q_0 \Phi - \nabla \Phi|_{1,T}^2 \Big)^{\frac{1}{2}} \\ & \cdot \Big(\sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{d} h_T^{-1} \|\partial_{ij}^2 u - \mathbb{Q}_h(\partial_{ij}^2 u)\|_T^2 + h_T \|\nabla(\partial_{ij}^2 u - \mathbb{Q}_h(\partial_{ij}^2 u))\|_T^2 \Big)^{\frac{1}{2}} \\ \lesssim & \, h^{k+1} \|\Phi\|_4 \, \|u\|_{k+1} \\ \lesssim & \, h^{k+1} \|u\|_{k+1} \|e_0\|. \end{aligned} \tag{7.12}$$

To deal with the last term, using the same arguments as in (7.3)–(7.5) with $u = \Phi$ and then combining (7.1) with (7.7), there yields

$$\begin{aligned} |\zeta_\Phi(e_h)| \lesssim & h^2 \|\Phi\|_4 |||e_h||| \\ \lesssim & h^{k+1} \|u\|_{k+1} \|\Phi\|_4 \\ \lesssim & h^{k+1} \|u\|_{k+1} \|e_0\|. \end{aligned} \tag{7.13}$$

Finally, substituting (7.10)–(7.13) into (7.9) completes the proof of the theorem.

We further introduce the following measure for the numerical solutions on element boundaries:

$$\begin{aligned} \|e_{b,e}\|_{\mathcal{E}_h} =& \Big(\sum_{T\in\mathcal{T}_h} h_T^2 \|e_{b,e}\|_{\partial \mathcal{F}}^2 \Big)^{\frac{1}{2}}, \\ \|e_{b,f}\|_{\mathcal{F}_h} =& \Big(\sum_{T\in\mathcal{T}_h} h_T \|e_{b,f}\|_{\partial T}^2 \Big)^{\frac{1}{2}}, \\ \|e_n\|_{\mathcal{F}_h} =& \Big(\sum_{T\in\mathcal{T}_h} h_T \|e_n\|_{\partial T}^2 \Big)^{\frac{1}{2}}. \end{aligned}$$

Theorem 7.3. *Under the assumptions of Theorem 7.2, there holds*

$$\|e_{b,e}\|_{\mathcal{E}_h} \lesssim h^{k+1} \|u\|_{k+1}, \tag{7.14}$$

$$\|e_{b,f}\|_{\mathcal{F}_h} \lesssim h^{k+1} \|u\|_{k+1}, \tag{7.15}$$

$$\|e_n\|_{\mathcal{F}_h} \lesssim h^{k} \|u\|_{k+1}. \tag{7.16}$$

Proof. From the triangular inequality, the trace inequality (6.2), (4.4), Theorems 7.1 and 7.2, there holds

$$\begin{aligned} \|e_{b,e}\|_{\mathcal{E}_h} =& \Big(\sum_{T\in\mathcal{T}_h} h_T^2 \|e_{b,e}\|_{\partial \mathcal{F}}^2 \Big)^{\frac{1}{2}} \\ \lesssim & \Big(\sum_{T\in\mathcal{T}_h} h_T^2 \|Q_b e_0\|_{\partial \mathcal{F}}^2 + h_T^2 \|e_{b,e} - Q_b e_0\|_{\partial \mathcal{F}}^2 \Big)^{\frac{1}{2}} \\ \lesssim & \Big(\sum_{T\in\mathcal{T}_h} h_T^2 h_T^{-1} \|e_0\|_{\partial T}^2 + h_T^2 h_T^2 |||e_h|||^2 \Big)^{\frac{1}{2}} \\ \lesssim & \Big(\sum_{T\in\mathcal{T}_h} h_T h_T^{-1} \|e_0\|_T^2 + h_T^4 h_T^{2(k-1)} \|u\|_{k+1}^2 \Big)^{\frac{1}{2}} \\ \lesssim & h^{k+1} \|u\|_{k+1}, \end{aligned}$$

which completes the proof for (7.14).

The proof for (7.15) and (7.16) can be obtained by using a similar argument.

8. Numerical experiments

In this section, the numerical scheme (3.1) will be implemented to verify the convergence theory established in the previous sections. To this end, we first solve the biharmonic Eq. (1.1) on the unit square $\Omega = (0,1)^2$, where g and the boundary conditions

Table 8.1
The error profile for solving (8.1) on square grids shown in Fig. 8.1.

Grid	$\|Q_h u - u_h\|$	Rate	$|||Q_h u - u_h|||$	Rate
		The P_3 WG finite element		
5	0.1486E−02	3.90	0.9339E+00	1.95
6	0.9595E−04	3.95	0.2373E+00	1.98
7	0.6092E−05	3.98	0.5981E−01	1.99
		The P_4 WG finite element		
3	0.3791E−01	3.85	0.3692E+01	2.86
4	0.1330E−02	4.83	0.4803E+00	2.94
5	0.4232E−04	4.97	0.6068E−01	2.98
		The P_5 WG finite element		
2	0.2460E+00	5.05	0.1823E+02	5.21
3	0.5110E−02	5.59	0.9983E+00	4.19
4	0.8558E−04	5.90	0.5589E−01	4.16

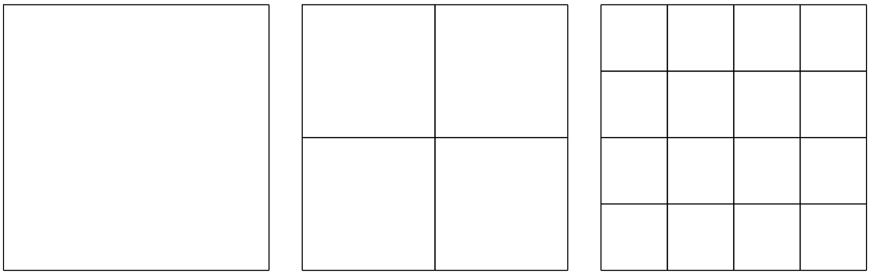

Fig. 8.1. The first three levels of square grids used in Table 8.1 computation.

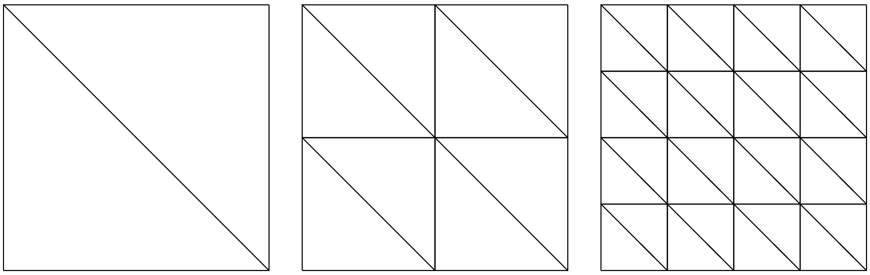

Fig. 8.2. The first three levels of triangular grids used in Table 8.2 computation.

are chosen so that the exact solution is

$$u(x, y) = 2^8(x - x^2)^2(y - y^2)^2. \tag{8.1}$$

Test Example 1. We take the square as the initial mesh, and subdivide each square into four to get subsequent meshes, as shown in Fig. 8.1. One can see from Table 8.1 that the optimal rates of convergence are obtained in the usual L^2 and H^2-like triple-bar norms for P_3, P_4 and P_5 WG methods.

Test Example 2. We take the uniform triangular meshes, as shown in Fig. 8.2. One can see from Table 8.2 that optimal rates of convergence are demonstrated in the usual L^2 and H^2-like triple-bar norms for P_3, P_4 and P_5 WG methods.

Test Example 3. We take polygonal meshes shown as in Fig. 8.3. Table 8.3 illustrates the corresponding numerical results which clearly demonstrate optimal rates of convergence in the usual L^2 and H^2-like triple-bar norms for P_3, P_4 and P_5 WG methods.

Test Example 4. We solve the biharmonic Eq. (1.1) on the unit cubic domain $\Omega = (0, 1)^3$, where g and the boundary conditions are chosen so that the exact solution is given by

$$u(x, y, z) = 2^{12}(x - x^2)^2(y - y^2)^2(z - z^2)^2. \tag{8.2}$$

In this test, we use the uniform cube meshes shown as in Fig. 8.4. The results from the P_3 and P_4 WG methods are shown in Table 8.4. The optimal order of convergence is achieved in all cases.

Table 8.2
The error profile for solving (8.1) on triangular grids shown in Fig. 8.2.

Grid	$\|Q_h u - u_h\|$	Rate	$\|\|\|Q_h u - u_h\|\|\|$	Rate
		The P_3 WG finite element		
5	0.8263E−03	3.97	0.7030E+00	1.98
6	0.5190E−04	3.99	0.1764E+00	2.00
7	0.3252E−05	4.00	0.4414E−01	2.00
		The P_4 WG finite element		
4	0.6526E−03	4.86	0.2874E+00	2.88
5	0.2088E−04	4.97	0.3666E−01	2.97
6	0.6563E−06	4.99	0.4606E−02	2.99
		The P_5 WG finite element		
3	0.2622E−02	5.59	0.3941E+00	3.65
4	0.4362E−04	5.91	0.2601E−01	3.92
5	0.6929E−06	5.98	0.1649E−02	3.98

Table 8.3
The error profile for solving (8.1) on polygonal grids shown in Fig. 8.3.

Grid	$\|Q_h u - u_h\|$	Rate	$\|\|\|Q_h u - u_h\|\|\|$	Rate
		The P_3 WG finite element		
4	0.5052E−02	3.92	0.1724E+01	1.94
5	0.3207E−03	3.98	0.4355E+00	1.98
6	0.1999E−04	4.00	0.1092E+00	2.00
		The P_4 WG finite element		
3	0.4732E−02	4.96	0.9861E+00	3.04
4	0.1473E−03	5.01	0.1213E+00	3.02
5	0.4974E−05	4.89	0.1510E−01	3.01
		The P_5 WG finite element		
1	0.1201E+01	0.00	0.3857E+02	0.00
2	0.1468E−01	6.36	0.1683E+01	4.52
3	0.2705E−03	5.76	0.9122E−01	4.21

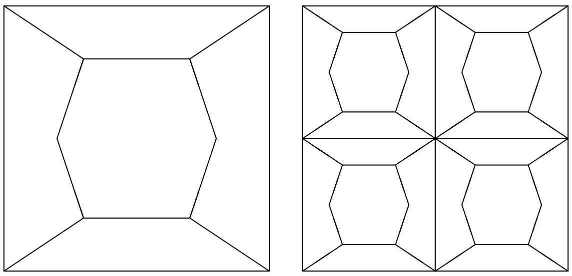

Fig. 8.3. The first two levels of quadrilateral–pentagon–hexagon grids used in Table 8.3 computation.

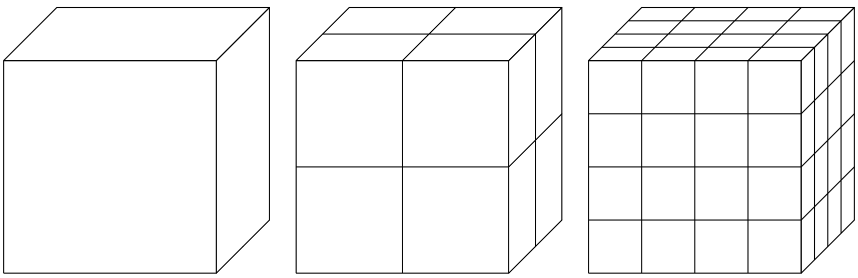

Fig. 8.4. The first three levels of cube grids used in the computation of Table 8.4.

Table 8.4
The error profile for solving (8.2) on cube grids shown in Fig. 8.4.

Grid	$\|Q_h u - u_h\|$	Rate	$\|\|\|Q_h u - u_h\|\|\|$	Rate
		The P_3 WG finite element		
2	0.8474E−01	6.2	0.1633E+01	4.5
3	0.2583E−02	5.0	0.1846E+00	3.1
4	0.2063E−03	3.6	0.4861E−01	1.9
		The P_4 WG finite element		
2	0.2247E−01	9.8	0.1373E+01	6.9
3	0.4988E−03	5.5	0.1079E+00	3.7
4	0.1705E−04	4.9	0.1009E−01	3.4

Data availability

Data will be made available on request.

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