EXISTENTIAL CLOSEDNESS AND THE STRUCTURE OF BIMODULES OF II_1 FACTORS

ADRIAN IOANA AND HUI TAN

Abstract. We prove that if a separable II_1 factor M is existentially closed, then every M-bimodule is weakly contained in the trivial M-bimodule, $\mathrm{L}^2(M)$, and, equivalently, every normal completely positive map on M is a pointwise 2-norm limit of maps of the form $x \mapsto \sum_{i=1}^k a_i^* x a_i$, for some $k \in \mathbb{N}$ and $(a_i)_{i=1}^k \subset M$. This provides the first examples of non-hyperfinite separable II_1 factors M with the latter properties. We also obtain new characterizations of M-bimodules which are weakly contained in the trivial or coarse M-bimodule and of relative amenability inside M. Additionally, we give an operator algebraic presentation of the proof of the existence of existentially closed II_1 factors. While existentially closed II_1 factors have property Gamma, by adapting this proof we construct non-Gamma II_1 factors which are existentially closed in every weakly coarse extension.

1. Introduction and statement of main results

In this paper, motivated by the notion of existentially closed II_1 factors in continuous model theory, we investigate the structure of bimodules of II_1 factors. The model theoretic study of II_1 factors was initiated by Farah, Hart and Sherman in [FHS11] (see [GH22] for a recent survey). Existentially closed II_1 factors were first considered in [GHS12, FGHS13] and have since been studied in [GHS12, FGHS13, Go18, AGKE20, GH21, CDI22a, CDI22b, GJKEP23], while bimodules of II_1 factors were studied from the perspective of model theory in [GHS18]. Existentially closed II_1 factors were shown to be McDuff in [GHS12] and to only have approximately inner automorphisms in [FGHS13, Proposition 3.1]. The original, model theoretic definition of existentially closed II_1 factors involves existential formulae, see [FGHS13, Definition 1.1]. In this paper, we will restrict to separable II_1 factors and work with the following equivalent operator algebraic definition (see, e.g., [GH21, Remarks 3.2] and the comments following [GH22, Lemma 5.2]).

Definition 1.1. If $M \subset N$ are separable II_1 factors, we say that M is *existentially closed* in N if there is a $*$-homomorphism $\pi : N \to M^{\mathcal{U}}$, for a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, such that the restriction of π to M is the diagonal embedding of M into $M^{\mathcal{U}}$. A separable II_1 factor M is called *existentially closed* if it is existentially closed in any separable II_1 factor N containing it.

Our first main result shows that being existentially closed has strong consequences to the structure of bimodules and completely positive maps of a II_1 factor.

Theorem A. *Consider the following conditions for a separable II_1 factor M.*

1. *M is existentially closed.*
2. *Every M-bimodule $\mathcal{H}$ satisfies $\mathcal{H} \subset \mathrm{L}^2(M^{\mathcal{U}})$, for some ultrafilter $\mathcal{U}$ on a set I. Moreover, if $\mathcal{H}$ is separable, then we can take $I = \mathbb{N}$.*
3. *Every M-bimodule $\mathcal{H}$ belongs to the closure of $\mathrm{L}^2(M)$, in the Fell topology.*
4. *Every M-bimodule $\mathcal{H}$ is weakly contained in $\mathrm{L}^2(M)$.*

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(5) *For every normal, completely positive map* $\Phi : M \to M$*, there exists a sequence* $(\Phi_n) \subset \mathcal{P}_M$ *such that* $\lim_{n\to\infty} \|\Phi_n(x) - \Phi(x)\|_2 = 0$*, for every* $x \in M$*. Moreover, if* Φ *is subunital and subtracial, then we can take* $(\Phi_n) \subset \mathcal{S}_M$*.*

Then (1) $\Rightarrow$ *(2)* $\Rightarrow$ *(3)* $\Rightarrow$ *(4)* $\Leftrightarrow$ *(5), for every separable* II_1 *factor* M*.*

Here, we denote by $\mathcal{P}_M$ the set of maps $\Phi : M \to M$ of the form $\Phi(x) = \sum_{i=1}^k a_i^* x a_i$, for some $(a_i)_{i=1}^k \subset M$, and by $\mathcal{S}_M$ the set of maps $\Phi : M \to M$ of the form $\Phi(x) = \sum_{i=1}^k a_i^* x a_i$, for some $(a_i)_{i=1}^k \subset M$ with $\sum_{i=1}^k a_i^* a_i \leq 1$ and $\sum_{i=1}^k a_i a_i^* \leq 1$. Every $\Phi \in \mathcal{P}_M$ is normal and completely positive, and every $\Phi \in \mathcal{S}_M$ is normal, subunital, subtracial and completely positive. For the other terminology used in the statement of Theorem A, we refer to Section 2.

Before commenting on the proof of Theorem A, let us highlight that it leads to a new phenomenon for II_1 factors. Let $\mathcal{A}$ be the family of separable II_1 factors M which satisfy the equivalent conditions (4) and (5) from Theorem A. If M is a separable II_1 factor, then every M-bimodule weakly contains the coarse bimodule, $\text{L}^2(M) \otimes \text{L}^2(M)$ (see [Po86, Proposition 2.3.2]). On the other hand, there exists a (unique up to weak equivalence) separable M-bimodule, denoted $\mathcal{H}_{\max}$, which weakly contains every M-bimodule. Class $\mathcal{A}$ consists of II_1 factors M for which $\mathcal{H}_{\max} = \text{L}^2(M)$. If R is the hyperfinite II_1 factor [MvN43], then any two R-bimodules are weakly equivalent (see [Po86, Proposition 3.1.4]), and therefore $R \in \mathcal{A}$. Also, $\mathcal{A}$ is closed under amplifications and inductive limits, and thus we have $M\overline{\otimes}R \in \mathcal{A}$, for every $M \in \mathcal{A}$ (see Proposition 2.6). Theorem A provides the first examples of non-hyperfinite separable II_1 factors $M \in \mathcal{A}$. In fact, it implies that $\mathcal{A}$ contains uncountably many non-isomorphic such II_1 factors, since there are uncountably many non-isomorphic separable existentially closed II_1 factors (see [FGHS13, Corollary 1.3]).

To prove that (1) implies (2) in Theorem A, we use Shlyakhtenko's M-valued semicircular systems [Sh97], see Section 3 for the definition. This construction, which associates a tracial von Neumann algebra $\Gamma(M, \mathcal{H})''$ containing M to any symmetric M-bimodule $\mathcal{H}$, allows us to realize every M-bimodule as a sub-bimodule of $\text{L}^2(N)$, for a tracial von Neumann algebra N containing M. The proofs of the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are immediate, and moreover show that these implications hold for every fixed M-bimodule $\mathcal{H}$.

Remark 1.2. We do not know if conditions (2)-(5) in Theorem A are equivalent for an arbitrary separable II_1 factor M. If M satisfies (2) or (3), then M must have property Gamma. Indeed, Connes' spectral gap theorem [Co75] implies that if the M-bimodule $\text{L}^2(M) \oplus \text{L}^2(M)$ belongs to the closure of $\text{L}^2(M)$, then M has property Gamma, see [Po86, Theorem 3.3.1] and Lemma 2.1. However, we do not know if the equivalent conditions (4) and (5) imply property Gamma.

If a separable II_1 factor M satisfies conditions (4) and (5) (i.e., $M \in \mathcal{A}$ in the above notation) but fails property Gamma, then M must have property (T) in the sense of [Co80, CJ83]. In particular, the free group factors $\text{L}(\mathbb{F}_n)$, for $n \geq 2$, do not belong to $\mathcal{A}$, which answers a question asked by Jesse Peterson. To justify our claim, let $\mathcal{H}$ be an M-bimodule which weakly contains $\text{L}^2(M)$. Since M satisfies (4), $\mathcal{H}$ is weakly contained in $\text{L}^2(M)$. Since M does not have property Gamma, [BMO19, Proposition 3.2] shows that any M-bimodule which is weakly equivalent to $\text{L}^2(M)$ must contain $\text{L}^2(M)$. This implies that $\mathcal{H}$ must contain $\text{L}^2(M)$ and thus M has property (T).

Remark 1.3. Partially addressing the question posed in Remark 1.2 in the first version of the paper, Amine Marrakchi proved that conditions (2)-(4) are equivalent for every separable II_1 factor M with property Gamma. More precisely, [Ma23, Corollary 3.9] shows that if M has Gamma, then for every M-bimodule $\mathcal{H}$ we have $\mathcal{H} \subset \text{L}^2(M^{\mathcal{U}})$ if and only if $\mathcal{H}$ belongs to the closure of $\text{L}^2(M)$ and if and only if $\mathcal{H}$ is weakly contained in $\text{L}^2(M)$. Since any separable II_1 factor satisfying (3) must have property Gamma, it follows that (2) $\Leftrightarrow$ (3), for arbitrary separable II_1 factors M.

Let $\mathcal{B} \subset \mathcal{A}$ be the family of separable II_1 factors M which satisfy the equivalent conditions (2) and (3) from Theorem A. Then the previous paragraph implies that $\mathcal{B}$ consists of all $M \in \mathcal{A}$ with property Gamma. By Remark 1.2, if $M \in \mathcal{A} \setminus \mathcal{B}$ (and thus M does not have Gamma), then M has property (T). It is an open question whether there are II_1 factors $M \in \mathcal{A}$ with property (T).

Another natural question is whether conditions (1) and (2) from Theorem A are equivalent. Our next result shows that (2) is satisfied by the hyperfinite II_1 factor R. By [FGHS13, Corollary 2.2], R is existentially closed if and only if the Connes Embedding Problem (CEP) has a positive answer. A negative answer to the CEP has been announced in the preprint [JNVWY20].

Corollary B. *Let R be the hyperfinite II_1 factor. Then every R-bimodule $\mathcal{H}$ satisfies $\mathcal{H} \subset \mathrm{L}^2(R^{\mathcal{U}})$, for some ultrafilter $\mathcal{U}$ on a set I. Moreover, if $\mathcal{H}$ is separable, then we can take $I = \mathbb{N}$.*

Corollary B is a consequence of the following result on (approximate) embeddings of Shlyakhtenko's M-valued semicircular systems [Sh97]. It can be alternatively deduced by using that R satisfies condition (4) from Theorem A and that conditions (2) and (4) are equivalent for II_1 factors with property Gamma by [Ma23, Corollary 3.9].

Theorem C. *Let (M,τ) be a tracial von Neumann algebra and $\mathcal{H},\mathcal{K}$ be two symmetric M-bimodules such that $\mathcal{H}$ is weakly contained in $\mathcal{K}$. Then there exists a trace preserving $*$-homomorphism*

$$\pi : \Gamma(M,\mathcal{H})'' \to (\Gamma(M,\mathcal{K}\otimes \ell^2 S)'')^{\mathcal{U}},$$

for some ultrafilter $\mathcal{U}$ on a set I and a set S, whose restriction to M is the diagonal embedding of M into $(\Gamma(M,\mathcal{K}\otimes \ell^2 S)'')^{\mathcal{U}}$. Moreover, if M and $\mathcal{H}$ are separable, then we can take $I = S = \mathbb{N}$.

Theorem C is of independent interest. In addition to Corollary B, Theorem C and its proof lead to characterizations of weak containment in the trivial or coarse bimodule and of relative amenability for inclusions, in the sense of Ozawa and Popa [OP07].

Corollary D. *Let (M,τ) be a separable tracial von Neumann algebra, $\mathcal{H}$ a separable M-bimodule, $P,Q \subset M$ von Neumann subalgebras, and $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$. Denote $\widehat{M} = M\overline{\otimes}\mathrm{L}(\mathbb{F}_\infty)$, $\overline{M} = M * \mathrm{L}(\mathbb{F}_\infty)$, and $\widetilde{M} = M *_Q (Q\overline{\otimes}\mathrm{L}(\mathbb{Z}))$. Then the following hold:*

1. *$\mathcal{H}$ is weakly contained in $\mathrm{L}^2(M)$ if and only if $\mathcal{H} \subset \mathrm{L}^2(\widehat{M}^{\mathcal{U}})$.*
2. *$\mathcal{H}$ is weakly contained in $\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)$ if and only if $\mathcal{H} \subset \mathrm{L}^2(\overline{M}^{\mathcal{U}}) \ominus \mathrm{L}^2(M^{\mathcal{U}})$.*
3. *P is amenable relative to Q inside M if and only if there exists $x \in P' \cap \widetilde{M}^{\mathcal{U}}$ such that $\mathrm{E}_{M^{\mathcal{U}}}(x) = 0$ and $\mathrm{E}_{M^{\mathcal{U}}}(x^*x) = 1$.*
4. *The M-bimodule $\mathrm{L}^2(M)\otimes_Q \mathrm{L}^2(M)$ is weakly contained in $\mathrm{L}^2(M)$ if and only if there exists a $*$-homomorphism $\pi : \widetilde{M} \to \widehat{M}^{\mathcal{U}}$ whose restriction to M is the diagonal embedding of M.*

Remark 1.4. Using a method from [Ha84, IV14], Marrakchi showed that if an M-bimodule $\mathcal{H}$ is weakly contained in $\mathrm{L}^2(M)$, then $\mathcal{H} \subset \mathrm{L}^2((M\overline{\otimes}\mathrm{L}(\mathbb{Z}))^{\mathcal{U}})$, see [Ma23, Theorem 2.1]. This improves part (1) of Corollary D. We also note that by [Ma23, Theorem 3.3] the M-bimodule $\mathrm{L}^2(M)\otimes_Q\mathrm{L}^2(M)$ is weakly contained in $\mathrm{L}^2(M)$ (i.e., Q is *weakly bicentralized* in M, in the sense of [BMO19, IM19]) if and only if $(Q' \cap M^{\mathcal{U}})' \cap M = Q$.

Although existential closedness is a strong property, it was noted in [GHS12, FGHS13] that the class of existentially closed separable II_1 factors is embedding universal in the following sense:

Theorem E ([GHS12, FGHS13])**.** *Any separable II_1 factor M_0 is contained in an existentially closed separable II_1 factor M.*

Theorem E was noted in [FGHS13] as a consequence of results from [Us08]. Its proof consists of verifying the model-theoretic definition of existential closedness given in [FGHS13, Definition 1.1]. In Section 4, we give an operator algebraic presentation of the proof of Theorem E, in which we instead directly verify Definition 1.1. We hope that this will both make the class of existentially closed II_1 factors more accessible to the operator algebra community and allow to prove new results. In fact, by adapting our operator algebraic presentation of the proof of Theorem E, we obtain the following new result:

Theorem F. *Any separable II_1 factor M_0 is contained in a non-Gamma separable II_1 factor M which satisfies the following: M is existentially closed in any II_1 factor $N \supset M$ such that the M-bimodule $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M)$ is weakly contained in $\mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$.*

By [GHS12], all existentially closed II_1 factors are McDuff and hence have property Gamma. Consequently, a non-Gamma II_1 factor M cannot be existentially closed in all larger II_1 factors. Moreover, as we will see below, M cannot even be existentially closed in all larger non-Gamma II_1 factors.

These facts suggest the following question: how "close" can a non-Gamma separable II_1 factor M be to being existentially closed? To make this question more precise, denote by $\mathcal{E}_M$ the family of separable II_1 factors $N \supset M$ in which M is existentially closed. We would like to find M for which $\mathcal{E}_M$ is as large as possible. If $N \in \mathcal{E}_M$, then since $M \subset N \subset M^{\mathcal{U}}$ and M is non-Gamma, we get that $M' \cap N = \mathbb{C}1$ and moreover that $M' \cap N^{\mathcal{U}} = \mathbb{C}1$. Hence, N is also non-Gamma. As a consequence, $\mathcal{E}_M$ is a proper subset of the set of all non-Gamma separable II_1 factors which contain M. In particular, this shows that there is no non-Gamma II_1 factor which is existentially closed in all non-Gamma II_1 factors containing it. Indeed, the separable II_1 factor $P = (M\overline{\otimes}\mathrm{L}(\mathbb{Z})) * \mathrm{L}(\mathbb{Z})$ is non-Gamma and contains M, but does not belong to $\mathcal{E}_M$ since $M' \cap P \neq \mathbb{C}1$. More generally, if $N \in \mathcal{E}_M$, then any II_1 subfactor $Q \subset M$ with $Q' \cap M^{\mathcal{U}} = \mathbb{C}1$, must satisfy that $Q' \cap N^{\mathcal{U}} = \mathbb{C}1$. This altogether shows that for any non-Gamma II_1 factor M, inclusions of the form $M \subset N$, with $N \in \mathcal{E}_M$, are significantly constrained.

Nevertheless, Theorem F shows that there exist non-Gamma II_1 factors M for which $\mathcal{E}_M$ is a large family which contains all II_1 factors $N \supset M$ such that $(\star)$ *the M-bimodule $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M)$ is weakly contained in $\mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$.*

Remark 1.5. Condition $(\star)$ in Theorem F is optimal in the following sense: there are non-Gamma II_1 factors M such that $(\star)$ holds for every $N \in \mathcal{E}_M$. This fact follows from two recent papers.

First, a result of Amine Marrakchi (see [Ma23, Corollary 3.5]) implies that if $M = \mathrm{L}(\Gamma)$, where Γ is a countable icc group which is biexact in the sense of Ozawa, then the M-bimodule $\mathrm{L}^2(M^{\mathcal{U}}) \ominus \mathrm{L}^2(M)$ is weakly contained in $\mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$. In Lemma 4.6, we give a direct proof of this when $M = \mathrm{L}(\mathbb{F}_n)$, for $n \geq 2$. For such a II_1 factor M, condition $(\star)$ is satisfied by any $N \in \mathcal{E}_M$ since the M-bimodule $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M)$ is contained in $\mathrm{L}^2(M^{\mathcal{U}}) \ominus \mathrm{L}^2(M)$ and thus weakly contained in $\mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$.

Secondly, Changying Ding and Jesse Peterson define in [DP23] a notion of biexactness for arbitrary von Neumann algebras M. They prove that, for tracial von Neumann algebras M, the condition $\mathrm{L}^2(M^{\mathcal{U}}) \ominus \mathrm{L}^2(M) \subset_{\text{weak}} \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$ is equivalent to the W*AO property defined in [Ca22, Definition 2.1] and is implied by biexactnenss, see [DP23, Theorems 7.19 and 7.20].

Our last main result provides a spectral gap characterization of Haagerup's approximation property for separable II_1 factors, paralleling the characterization of property (T) obtained in [Ta22]. A separable II_1 factor M is said to have *Haagerup's (approximation) property* [Ch83, CJ83] if there exists a sequence of completely positive maps $\Phi_n : M \to M$ such that $\|\Phi_n(x) - x\|_2 \to 0$, for every $x \in M$, and Φ_n is subtracial ($\tau \circ \Phi_n \leq \tau$) and its extension to $\mathrm{L}^2(M)$ is a compact operator, for every n. By [BF07, OOT15], Haagerup's property for M is equivalent to the existence of a strictly mixing M-bimodule (see Definition 2.7 for this notion) that weakly contains $\mathrm{L}^2(M)$.

Theorem G. *A separable II_1 factor M has Haagerup's property if and only if there exists a separable II_1 factor $\widetilde{M}$ which contains M such that the M-bimodule $\mathrm{L}^2(\widetilde{M}) \ominus \mathrm{L}^2(M)$ is strictly mixing and $M' \cap \widetilde{M}^{\mathcal{U}} \not\subset M^{\mathcal{U}}$, for a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$.*

Organization of the paper. Besides the introduction, this paper consists of three other sections. Sections 2 and 3 are devoted to von Neumann algebras preliminaries and M-valued semicircular systems, respectively. In Section 4, we present the proofs of the results stated in the introduction.

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2. Preliminaries

In this section, we record some terminology and basic results concerning tracial von Neumann algebras, bimodules and completely positive maps. We refer to [AP18] for more information.

2.1. **Tracial von Neumann algebras.** A tracial von Neumann algebra is a pair (M, τ) consisting of a von Neumann algebra M and a distinguished normal tracial state $\tau : M \to \mathbb{C}$. In order to emphasize the dependence of τ on M, we will sometimes write τ_M instead of τ. We assume that all inclusions of tracial von Neumann algebras $N \subset M$ are trace preserving, i.e., satisfy $\tau_N = (\tau_M)_{|N}$. We let $\|x\|_2 = \tau(x^*x)^{1/2}$, for every $x \in M$, and denote by $\mathrm{L}^2(M)$ the Hilbert space obtained by taking the closure of M with respect to $\|\cdot\|_2$. For an ultrafilter $\mathcal{U}$ on a set I, we denote by $M^{\mathcal{U}}$ the tracial ultraproduct von Neumann algebra endowed with the trace given by $\tau_{\mathcal{U}}(x) = \lim_{i\to\mathcal{U}} \tau(x_i)$, for every $x = (x_i) \in M^{\mathcal{U}}$. For a von Neumann subalgebra $N \subset M$, we denote by $M \ominus N$ the set $\{x \in M \mid \mathrm{E}_N(x) = 0\}$, where $\mathrm{E}_N : M \to N$ denotes the conditional expectation onto N.

2.2. **Bimodules.** Let (M, τ) be a tracial von Neumann algebra. An *M-bimodule* is a Hilbert space $\mathcal{H}$ equipped with two normal $*$-homomorphisms $\pi_1 : M \to \mathbb{B}(\mathcal{H})$ and $\pi_2 : M^{\mathrm{op}} \to \mathbb{B}(\mathcal{H})$ whose images commute. We write $x\xi y = \pi_1(x)\pi_2(y^{\mathrm{op}})\xi$ for $\xi \in \mathcal{H}$ and $x, y \in M$, and define a $*$-homomorphism $\pi_{\mathcal{H}} : M \otimes_{\mathrm{alg}} M^{\mathrm{op}} \to \mathbb{B}(\mathcal{H})$ by letting $\pi_{\mathcal{H}}(x \otimes y^{\mathrm{op}}) = \pi_1(x)\pi_2(y^{\mathrm{op}})$. Examples of bimodules include the *trivial* M-bimodule $\mathrm{L}^2(M)$, with the left and right actions given by extending the left and right multiplication actions of M on itself, and the *coarse* M-bimodule $\mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$ with the left and right actions given by $x(\xi \otimes \eta)y = x\xi \otimes \eta y$, for all $x, y \in M$ and $\xi, \eta \in \mathrm{L}^2(M)$.

Given two M-bimodules $\mathcal{H}, \mathcal{K}$, we say that $\mathcal{H}$ is *contained* in $\mathcal{K}$ and write $\mathcal{H} \subset \mathcal{K}$ if there is an M-bimodular isometry $T : \mathcal{H} \to \mathcal{K}$. A basis of neighborhoods for the *Fell topology* on the space of all M-bimodules is given by $V(\mathcal{H}, F, S, \varepsilon)$, where $\mathcal{H}$ is an M-bimodule, $F \subset M, S \subset \mathcal{H}$ are finite sets and $\varepsilon > 0$. Here, $V(\mathcal{H}, F, S, \varepsilon)$ is defined as the set of all M-bimodules $\mathcal{K}$ for which we can find a map $T : S \to \mathcal{K}$ such that

$$|\langle xT(\xi)y, T(\eta)\rangle - \langle x\xi y, \eta\rangle| < \varepsilon, \text{ for every } x, y \in F, \text{ and } \xi, \eta \in S.$$

If $\mathcal{H}, \mathcal{K}$ are M-bimodules, we say that $\mathcal{H}$ is *weakly contained* in $\mathcal{K}$ and write $\mathcal{H} \subset_{\mathrm{weak}} \mathcal{K}$ if $\mathcal{H}$ belongs to the closure of $\mathcal{K}^{\oplus\infty} := \mathcal{K} \otimes \ell^2(\mathbb{N})$ in the Fell topology.

We next recall [Po86, Theorem 3.3.1], which gives a bimodule characterization of property Gamma for separable II$_1$ factors (cf. [BMO19, Proposition 3.2]). For the reader's convenience, we include a brief proof.

Lemma 2.1 ([Po86]). *A separable II_1 factor M has property Gamma if and only if* $\mathrm{L}^2(M)\oplus\mathrm{L}^2(M)$ *belongs to the closure of* $\mathrm{L}^2(M)$*, in the Fell topology.*

Proof. Denote $\xi_1 = 1\oplus 0, \xi_2 = 0\oplus 1 \in \mathrm{L}^2(M)\oplus\mathrm{L}^2(M)$.

First, assume that M has property Gamma. Let $u_n\in M$ be a sequence of trace zero unitaries such that $\|u_nx - xu_n\|_2\to 0$, for every $x\in M$. We claim that (u_n) admits a subsequence which converges weakly to 0. To find such a subsequence, let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and define a normal linear functional $\varphi: M\to\mathbb{C}$ by letting $\varphi(x) = \lim_{n\to\mathcal{U}}\tau(xu_n)$, for every $x\in M$. Let $z = \mathrm{E}_M(u)$, where $u = (u_n)\in M^{\mathcal{U}}$ and E_M is the conditional expectation from $M^{\mathcal{U}}$ onto the diagonal copy of M. Then $\varphi(x) = \tau_{\mathcal{U}}(xu) = \tau(xz)$, for every $x\in M$. Since $\varphi(xy) = \varphi(yx)$, for every $x, y\in M$, and M is a factor we get that $z = \tau(z)1\in\mathbb{C}1$. Since $\tau(z) = \varphi(1) = 0$, we get that $\varphi(x) = \lim_{n\to\mathcal{U}}\tau(xu_n) = 0$, for every $x\in M$. This implies our claim.

After replacing (u_n) with its subsequence given by the above claim, we have that $u_n\to 0$, weakly. Put $\xi_1^n = 1\in\mathrm{L}^2(M)$ and $\xi_2^n = u_n\in\mathrm{L}^2(M)$. Then we have $\lim_n\langle x\xi_i^n y, \xi_j^n\rangle = \langle x\xi_i y, \xi_j\rangle$, for every $x, y\in M$ and $i, j\in\{1,2\}$. Thus, $\mathrm{L}^2(M)\oplus\mathrm{L}^2(M) = \overline{\mathrm{span}(M\xi_1M)}\oplus\overline{\mathrm{span}(M\xi_2M)}$ lies in the closure of $\mathrm{L}^2(M)$.

Conversely, assume that $\mathrm{L}^2(M)\oplus\mathrm{L}^2(M)$ belongs to the closure of $\mathrm{L}^2(M)$ in the Fell topology. Then we can find nets $(\xi_1^n), (\xi_2^n)\subset\mathrm{L}^2(M)$ such that $\lim_n\langle x\xi_i^n y, \xi_j^n\rangle = \langle x\xi_i y, \xi_j\rangle$, for every $x, y\in M$ and $i, j\in\{1,2\}$. Suppose by contradiction that M does not have property Gamma. Since $x\xi_i = \xi_i x$ and $\|\xi_i\|_2 = 1$, we get that $\|x\xi_i^n - \xi_i^n x\|_2\to 0$ and $\|\xi_i^n\|_2\to 1$, for every $x\in M$ and $i\in\{1,2\}$. Since M does not have property Gamma, [Co75, Theorem 2.1] implies the existence of a net $(\alpha_i^n)\subset\mathbb{T} = \{z\in\mathbb{C}\mid |z| = 1\}$ such that $\|\xi_i^n - \alpha_i^n 1\|_2\to 0$, for every $i\in\{1,2\}$. Thus, $|\langle\xi_1^n, \xi_2^n\rangle - \alpha_1^n\overline{\alpha_2^n}|\to 0$. Since $\langle\xi_1^n, \xi_2^n\rangle\to\langle\xi_1, \xi_2\rangle = 0$ and $(\alpha_1^n\overline{\alpha_2^n})\subset\mathbb{T}$, this is a contradiction. □

Definition 2.2. A *symmetric* M-bimodule $(\mathcal{H}, J)$ is an M-bimodule $\mathcal{H}$ equipped with an anti-unitary involution $J:\mathcal{H}\to\mathcal{H}$ such that $J(x\xi y) = y^*(J\xi)x^*$, for every $x, y\in M$ and $\xi\in\mathcal{H}$.

Remark 2.3. Let $\mathcal{H}$ be an M-bimodule. Denote by $\overline{\mathcal{H}}$ the conjugate Hilbert space endowed with the M-bimodule structure $x\overline{\xi}y = \overline{y^*\xi x^*}$ (see [Po86, 1.3.7]). Then $\mathcal{K} := \mathcal{H}\oplus\overline{\mathcal{H}}$ is a symmetric Hilbert M-bimodule, as witnessed by the anti-unitary involution $J:\mathcal{K}\to\mathcal{K}$ given by $J(\xi\oplus\overline{\eta}) = \eta\oplus\overline{\xi}$.

Let $\mathcal{H}$ be an M-bimodule. A vector $\xi\in\mathcal{H}$ is called *tracial* if $\langle x\xi, \xi\rangle = \langle\xi x, \xi\rangle = \tau(x)$, for every $x\in M$, and *subtracial* if $\langle x\xi, \xi\rangle, \langle\xi x, \xi\rangle\leq\tau(x)$, for every $x\in M$. A vector $\xi\in\mathcal{H}$ is called *left* (respectively, *right*) *bounded* if there $C>0$ such that $\|\xi x\|\leq C\|x\|_2$ (respectively, $\|x\xi\|\leq C\|x\|_2$) for every $x\in M$. A vector is *bounded* if it is left and right bounded. Note that the subspace of bounded vectors is dense in $\mathcal{H}$ (see [AP18], Proposition 8.4.4). If $\xi\in\mathcal{H}$ is left bounded, we denote by $T_\xi:\mathrm{L}^2(M)\to\mathcal{H}$ the bounded operator uniquely determined by setting $T_\xi(x) = \xi x$ for every $x\in M$. If $\xi, \eta\in\mathcal{H}$ are left bounded vectors, then $\langle\xi, \eta\rangle_M := T_\xi^* T_\eta\in\mathbb{B}(\mathrm{L}^2(M))$ belongs to M. Note that

$$\langle\eta x, \xi\rangle = \tau(\langle\xi, \eta\rangle_M x), \text{ for every } x\in M.$$

For M-bimodules $\mathcal{H}, \mathcal{K}$, we denote by $\mathcal{H}\otimes_M\mathcal{K}$ their *Connes tensor product* (see [Co94], V, Appendix B). Let $\mathcal{H}^0\subset\mathcal{H}$ be the subspace of left bounded vectors. Then $\mathcal{H}\otimes_M\mathcal{K}$ is obtained by separation and completion from the algebraic tensor product $\mathcal{H}^0\otimes\mathcal{K}$ endowed with the sesquilinear form

$$\langle\xi_1\otimes\eta_1, \xi_2\otimes\eta_2\rangle = \langle\eta_1, \langle\xi_1, \xi_2\rangle_M\eta_2\rangle.$$

If $(M,\tau), (N,\tau)$ are tracial von Neumann algebras, then an M-N-bimodule is any Hilbert space equipped with normal $*$-homomorphisms of M and N^{op} which have commuting images. If $\mathcal{H}$ is an

M-N-bimodule and $\mathcal{K}$ is an N-P-bimodule, let $\mathcal{H}^0$ be the subspace of $\mathcal{H}$ of left N-bounded vectors. Then the Connes tensor product $\mathcal{H}\otimes_N\mathcal{K}$, obtained by separation and completion from the algebraic tensor product $\mathcal{H}^0\otimes\mathcal{K}$ endowed with the sesquilinear form $\langle\xi_1\otimes\eta_1,\xi_2\otimes\eta_2\rangle=\langle\eta_1,\langle\xi_1,\xi_2\rangle_N\eta_2\rangle$, is an M-P-bimodule. For more information on bimodules we refer the reader to [AP18, Chapter 13].

Remark 2.4. Let $\mathcal{H},\mathcal{K}$ be M-bimodules, $\mathcal{H}^0\subset\mathcal{H}$ the subspace of left bounded vectors and ${}^0\mathcal{K}\subset\mathcal{K}$ the subspace of right bounded vectors. Let $\xi\in\mathcal{H}^0$ and $(\eta_i)\subset\mathcal{K}$ be a net with $\|\eta_i\|\to 0$. Then $\|\xi\otimes\eta_i\|=\|\langle\eta_i,\langle\xi,\xi\rangle_M\eta_i\rangle\|^{1/2}\leq\|\langle\xi,\xi\rangle_M\|^{1/2}\|\eta_i\|\to 0$. Similarly, if $(\xi_i)\subset\mathcal{H}$ is a net with $\|\xi_i\|\to 0$ and $\eta\in{}^0\mathcal{K}$, then $\|\xi_i\otimes\eta\|\to 0$. These observations imply that if $\mathcal{H}^1\subset\mathcal{H}^0$ and $\mathcal{K}^1\subset{}^0\mathcal{K}$ are dense subspaces, then the span of $\{\xi\otimes\eta\mid\xi\in\mathcal{H}^1,\eta\in\mathcal{K}^1\}$ is dense in $\mathcal{H}\otimes_M\mathcal{K}$.

The following lemma is well-known, and its proof follows a standard recipe (see for instance the proof of [AP18, Lemma 13.3.11]), but we include details here for completeness.

Lemma 2.5. *Let (M,τ) be a tracial von Neumann algebra, $\mathcal{H},\mathcal{K}$ be Hilbert bimodules, $\xi\in\mathcal{H}$ a subtracial vector and $(\xi_n)\subset\mathcal{K}$ a net with $\lim_n\langle x\xi_n y,\xi_n\rangle=\langle x\xi y,\xi\rangle$, for every $x,y\in M$. Then there is a net of subtracial vectors $(\eta_n)\in\mathcal{K}^{\oplus\infty}$ such that $\lim_n\langle x\eta_n y,\eta_n\rangle=\langle x\xi y,\xi\rangle$, for every $x,y\in M$*

Proof. We follow the proof of [IV14, Theorem]. Define normal positive linear functionals $\omega,\omega',\omega_n,\omega_n'$ on M by letting $\omega(x)=\langle x\xi,\xi\rangle$, $\omega'(x)=\langle\xi x,\xi\rangle$, $\omega_n(x)=\langle x\xi_n,\xi_n\rangle$ and $\omega_n'(x)=\langle\xi_n x,\xi_n\rangle$, for every $x\in M$ and every n. Then $\omega_n\to\omega$ and $\omega_n'\to\omega'$, in the weak topology on M_*. Since the weak and norm closures of convex subsets of M_* coincide, after replacing (ω_n,ω_n') by a convex combination of (ω_n,ω_n') and the vectors (ξ_n) by vectors in $\mathcal{K}^{\oplus\infty}$, we can ensure that in addition to $\lim_n\langle x\xi_n y,\xi_n\rangle=\langle x\xi y,\xi\rangle$, for every $x,y\in M$, we have $\|\omega_n-\omega\|\to 0$ and $\|\omega_n'-\omega'\|\to 0$.

Write $\omega=\tau(\cdot T),\omega'=\tau(\cdot T'),\omega_n=\tau(\cdot T_n)$ and $\omega_n'=\tau(\cdot T_n')$, where $T,T',T_n,T_n'\in\mathrm{L}^1(M)$ are positive elements. Then $0\leq T,T'\leq 1$ as ξ is subtracial and $\varepsilon_n:=\|T_n-T\|_1+\|T_n'-T'\|_1\to 0$. Let p_n and p_n' be the spectral projections of T_n and T_n' corresponding to the interval $[0,1+\sqrt{\varepsilon_n}]$.

We claim that there is n_0 such that

$$\|(1-p_n)\xi_n\|\leq 2\sqrt[4]{\varepsilon_n}\text{ and }\|\xi_n(1-p_n')\|\leq 2\sqrt[4]{\varepsilon_n},\text{ for every }n\geq n_0. \tag{2.1}$$

If $q_n=1-p_n$, then $q_nT_nq_n\geq(1+\sqrt{\varepsilon_n})q_n$ and $q_nTq_n\leq q_n$, hence $q_n(T_n-T)q_n\geq\sqrt{\varepsilon_n}q_n$. This gives $\sqrt{\varepsilon_n}\|q_n\|_1\leq\|T_n-T\|_1\leq\varepsilon_n$, hence $\|q_n\|_1\leq\sqrt{\varepsilon_n}$ and $\|q_nT_n\|_1\leq\|T_n-T\|_1+\|q_n\|_1\leq\varepsilon_n+\sqrt{\varepsilon_n}$. Thus, $\|(1-p_n)T_n\|_1=\|q_nT_n\|_1\leq 2\sqrt{\varepsilon_n}$ and therefore $\|(1-p_n)\xi_n\|=\sqrt{\|(1-p_n)T_n\|_1}\leq 2\sqrt[4]{\varepsilon_n}$, for every n large enough. This and a similar inequality prove the claim.

Let $c_n=(1+\sqrt{\varepsilon_n})^{-1/2}$ and define $\eta_n=c_np_n\xi_np_n'$. Since $p_nT_np_n\leq(1+\sqrt{\varepsilon_n})$, for every $x\in M$, we have that $\|x\eta_n\|\leq c_n\|xp_n\xi_n\|=c_n\sqrt{\tau(x^*xp_nTp_n)}\leq\sqrt{\tau(x^*x)}=\|x\|_2$ and similarly that $\|\eta_nx\|\leq\|x\|_2$. Thus, η_n is subtracial. Since $\|\xi_n-\eta_n\|\leq(1-c_n)+\|\xi_n-p_n\xi_np_n'\|$ and $\lim_n c_n=1$, (2.1) implies that $\lim_n\|\xi_n-\eta_n\|=0$. Thus, $\lim_n\langle x\eta_ny,\eta_n\rangle=\lim_n\langle x\xi_ny,\xi_n\rangle=\langle x\xi y,\xi\rangle$, for every $x,y\in M$, which finishes the proof of the lemma. □

Proposition 2.6. *Let $\mathcal{A}$ be the class of separable II_1 factors M for which $\mathrm{L}^2(M)$ weakly contains every M-bimodule. Let M and M_n, $n\in\mathbb{N}$, be separable II_1 factors. Then the following hold:*

1. *If M is existentially closed, then $M\in\mathcal{A}$.*
2. *The hyperfinite II_1 factor $R\in\mathcal{A}$.*
3. *If $M\in\mathcal{A}$, then $M^t\in\mathcal{A}$ for every $t>0$.*
4. *If $M_n\subset M_{n+1}$ and $M_n\in\mathcal{A}$, for every $n\in\mathbb{N}$, then $\overline{\cup_{n\in\mathbb{N}}M_n}^{\mathrm{WOT}}\in\mathcal{A}$.*
5. *If $M\in\mathcal{A}$, then $M\overline{\otimes}R\in\mathcal{A}$.*

Proof. (1) and (2) follow from Theorem A and [Po86, Proposition 2.3.2.] (see also Corollary B), respectively.

(3) Assume that $M \in \mathcal{A}$ and denote $N = M^t$, for some $t > 0$. Let $d \geq 1/t$ be a integer, and p be a projection in $\mathbb{M}_d(\mathbb{C})\overline{\otimes}N$ with $(\mathrm{Tr}\otimes\tau)(p) = 1/t$, where Tr is the standard (non-normalized) trace on $\mathbb{M}_d(\mathbb{C})$ and τ denote the trace of N. Then $M \simeq N^{1/t} \simeq p(\mathbb{M}_d(\mathbb{C})\overline{\otimes}N)p$. Since the isomorphism class of $N^{1/t}$ only depends on $(\mathrm{Tr}\otimes\tau)(p)$, we can take $p = \mathrm{diag}(p_1, \cdots, p_d)$ where $p_1, \cdots, p_d$ are projections in N. Given an N-bimodule $\mathcal{H}$, we consider the M-bimodule $\mathcal{K} = p(\mathbb{M}_d(\mathbb{C}) \otimes \mathcal{H})p$ with the natural left and right $p(\mathbb{M}_d(\mathbb{C})\overline{\otimes}N)p$-actions. More specifically,

$$x \cdot \xi \cdot y = \big((\sum_{1\leq k,l\leq n} x_{i,k} \cdot \xi_{k,l} \cdot x_{l,j})_{i,j}\big),$$

for all $x = (x_{i,j})$, $y = (y_{i,j}) \in p(\mathbb{M}_d(\mathbb{C})\overline{\otimes}N)p$, $\xi = (\xi_{i,j}) \in \mathcal{K}$, where $x_{i,j}, y_{i,j} \in p_iNp_j$, $\xi_{i,j} \in p_i\mathcal{H}p_j$.

To show that $\mathcal{H}$ is weakly contained in $\mathrm{L}^2(N)$, it suffices to find, given $F \subset N, S \subset \mathcal{H}$ finite and $\varepsilon > 0$, a map $T : S \to \mathrm{L}^2(N)^{\oplus\infty}$, such that $|\langle x\xi x', \eta\rangle - \langle xT(\xi)x', T(\eta)\rangle| < \varepsilon$, for every $x, x' \in F$ and $\xi, \eta \in S$. If $(\mathrm{Tr}\otimes\tau)(p) \geq 1$, we can assume that $p_1 = 1$, and the conclusion follows from applying the fact that $\mathcal{K}$ is weakly contained in $\mathrm{L}^2(M)$ as an M-bimodule to $e_{1,1} \otimes x, e_{1,1} \otimes x' \in M = p(\mathbb{M}_d(\mathbb{C})\overline{\otimes}N)p$, and $e_{1,1} \otimes \xi, e_{1,1} \otimes \eta \in \mathcal{K} = p(\mathbb{M}_d(\mathbb{C}) \otimes \mathcal{H})p$, for $x, x' \in F$ and $\xi, \eta \in S$.

If $(\mathrm{Tr}\otimes\tau)(p) < 1$, then we can assume that $p \in N$ and $M = pNp$. Let $\sum q_i = 1$ be a finite partition of 1 by projections in N such that $\tau(q_i) \leq \tau(p)$. For each i, let v_i be a partial isometry in N such that $v_iv_i^* = q_i$ and $v_i^*v_i \leq p$. Let $\xi, \eta \in S$. Then we have

$$\langle x\xi x', \eta\rangle = \sum_{k,i,j,l} \langle (q_kxq_i)(q_i\xi q_j)(q_jx'q_l), q_k\eta q_l\rangle = \sum_{k,i,j,l} \langle (v_k^*xv_i)(v_i^*\xi v_j)(v_j^*x'v_l), v_k^*\eta v_l\rangle.$$

Since $p\mathcal{H}p$ is weakly contained in $\mathrm{L}^2(M) = \mathrm{L}^2(pNp)$ as an M-bimodule, for some $\varepsilon_{i,j,k,l} > 0$ with $\sum\limits_{i,j,k,l} \varepsilon_{i,j,k,l} < \varepsilon$, there exists $T_{i,j}(\xi) \in \mathrm{L}^2(pNp)^{\oplus\infty}$ for every (i, j) and $\xi \in S$, such that for every (i, j, k, l) we have

$$|\langle (v_k^*xv_i)(v_i^*\xi v_j)(v_j^*x'v_l), v_k^*\eta v_l\rangle - \langle (v_k^*xv_i)T_{i,j}(\xi)(v_j^*x'v_l), T_{k,l}(\eta)\rangle| < \varepsilon_{i,j,k,l}, \forall x, x' \in F, \xi, \eta \in S.$$

Note that $\langle (v_k^*xv_i)T_{i,j}(\xi)(v_j^*x'v_l), T_{k,l}(\eta)\rangle = \langle x(v_iT_{i,j}(\xi)v_j^*)x', v_kT_{k,l}(\eta)v_l^*\rangle$. Let $T(\xi) = \sum\limits_{i,j} v_iT_{i,j}(\xi)v_j^* \in \mathrm{L}^2(N)^{\oplus\infty}$ for $\xi \in S$. Then we have

$$\begin{aligned} &|\langle x\xi x', \eta\rangle - \langle xT(\xi)x', T(\eta)\rangle| \\ &\leq \sum_{i,j,k,l} |\langle (v_k^*xv_i)(v_i^*\xi v_j)(v_j^*x'v_l), v_k^*\eta v_l\rangle - \langle x(v_iT_{i,j}(\xi)v_j^*)x', v_kT_{k,l}(\eta))v_l^*\rangle| < \varepsilon \end{aligned}$$

for every $\xi, \eta \in S$, which finishes the proof of (3).

(4) Let $M = \overline{\cup M_n}^{\mathrm{WOT}}$ and E_n be the conditional expectation from M onto M_n, for every $n \in \mathbb{N}$. Let $\mathcal{H}$ be an M-bimodule. In order to show that $\mathcal{H}$ is weakly contained in $\mathrm{L}^2(M)$, it suffices to prove that for every $F \subset M, S \subset \mathcal{H}$ finite and $\varepsilon > 0$ we can find a map $T : S \to \mathrm{L}^2(M)^{\oplus\infty}$ such that $|\langle x\xi y, \xi\rangle - \langle xT(\xi)y, T(\xi)\rangle| < \varepsilon$, for every $x, y \in F$ and $\xi \in S$. To this end, we may moreover assume that $\|x\| \leq 1$, for every $x \in F$, and that S consists of subtracial vectors.

Let $n \in \mathbb{N}$ be such that $\|\mathrm{E}_n(x) - x\|_2 < \varepsilon/6$, for every $x \in F$. Then $|\langle x\xi y, \xi\rangle - \langle \mathrm{E}_n(x)\xi\mathrm{E}_n(y), \xi\rangle| < \varepsilon/3$, for every $x, y \in F$ and $\xi \in S$. Since $M_n \in \mathcal{A}$, $\mathcal{H}$ is weakly contained in $\mathrm{L}^2(M_n)$ as an M_n-bimodule. If $\xi \in S$, then since ξ is a subtracial vector, Lemma 2.5 gives a subtracial vector $T(\xi) \in \mathrm{L}^2(M_n)^{\oplus\infty}$ such that $|\langle \mathrm{E}_n(x)\xi\mathrm{E}_n(y), \xi\rangle - \langle \mathrm{E}_n(x)T(\xi)\mathrm{E}_n(y), T(\xi)\rangle| < \varepsilon/3$, for every $x, y \in F$. Consider the usual embedding $\mathrm{L}^2(M_n)^{\oplus\infty} \subset \mathrm{L}^2(M)^{\oplus\infty}$ and view $\mathrm{L}^2(M)^{\oplus\infty}$ as an M-bimodule.

Then $T(\xi) \in \mathrm{L}^2(M_n)^{\oplus\infty} \subset \mathrm{L}^2(M)^{\oplus\infty}$ is still a subtracial vector. This is because for every $x \in M$, we have that $\langle xT(\xi), T(\xi)\rangle = \langle \mathrm{E}_{M_n}(x)T(\xi), T(\xi)\rangle \leq \tau(\mathrm{E}_{M_n}(x)) = \tau(x)$ and similarly that $\langle T(\xi)x, T(\xi)\rangle = \langle T(\xi)\mathrm{E}_{M_n}(x), T(\xi)\rangle \leq \tau(\mathrm{E}_{M_n}(x)) = \tau(x)$.

Since $\|\mathrm{E}_n(x)-x\|_2 < \varepsilon/6$, for every $x \in F$, we get that $|\langle \mathrm{E}_n(x)T(\xi)\mathrm{E}_n(y), T(\xi)\rangle - \langle xT(\xi)y, T(\xi)\rangle| < \varepsilon/3$, for every $x, y \in F$. Altogether, we get that $|\langle x\xi y, \eta\rangle - \langle xT(\xi)y, T(\eta)\rangle| < \varepsilon$, for every $x, y \in F$ and $\xi \in S$, which finishes the proof of (4).

(5) Write $R = \overline{\cup_{n\in\mathbb{N}}\mathbb{M}_{2^n}(\mathbb{C})}^{\mathrm{WOT}}$, for the usual inclusions $\mathbb{M}_{2^n}(\mathbb{C}) \subset \mathbb{M}_{2^{n+1}}(\mathbb{C})$, for every $n \in \mathbb{N}$. Then $M\overline{\otimes}R = \overline{\cup_{n\in\mathbb{N}}(M\overline{\otimes}\mathbb{M}_{2^n}(\mathbb{C}))}^{\mathrm{WOT}}$, and the conclusion follows from (3) and (4). $\square$

2.3. **Completely positive maps.** Let (M, τ) be a tracial von Neumann algebra. We now recall the well-known correspondence between M-bimodules $\mathcal{H}$ and normal, completely positive maps on M. Let $\xi \in \mathcal{H}$ be a bounded vector. Then $\Phi_\xi(x) = T_\xi^* x T_\xi$ belongs to M, for every $x \in M$. The map $\Phi_\xi : M \to M$, called a *coefficient* of $\mathcal{H}$, is normal, completely positive and satisfies

$$\tau(\Phi_\xi(x)y) = \langle x\xi y, \xi\rangle, \text{ for every } x, y \in M.$$

Moreover, Φ_ξ extends to a bounded operator $\Phi_\xi : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$. Let $C > 0$ such that $\|x\xi\|, \|\xi x\| \leq C\|x\|_2$, for every $x \in M$. Then $|\tau(\Phi_\xi(x)y)| = |\langle x\xi, \xi y^*\rangle| \leq \|x\xi\|\,\|\xi y^*\| \leq C^2\|x\|_2\,\|y\|_2$, for every $x, y \in M$. Thus, we get that $\|\Phi_\xi(x)\|_2 \leq C^2\|x\|_2$, for every $x \in M$.

Definition 2.7. Let $\mathcal{H}$ be an M-bimodule.

(1) We call $\mathcal{H}$ a *mixing* M-bimodule if for every sequence $u_n \in \mathcal{U}(M)$ with $u_n \to 0$ weakly we have $\lim_{n\to\infty}\big(\sup_{x\in M, \|x\|\leq 1}|\langle u_n\xi x, \eta\rangle|\big) = \lim_{n\to\infty}\big(\sup_{x\in M, \|x\|\leq 1}|\langle x\xi u_n, \eta\rangle|\big) = 0$, for every $\xi, \eta \in \mathcal{H}$.
(2) Denote by $\mathcal{H}_{\mathrm{mix}}$ the set of bounded vectors $\xi \in \mathcal{H}$ such that $\Phi_\xi \in \mathbb{B}(\mathrm{L}^2(M))$ is compact. We call $\mathcal{H}$ a *strictly mixing* M-bimodule if the span of $M\mathcal{H}_{\mathrm{mix}}M$ is dense in $\mathcal{H}$.

The notions of mixing and strictly mixing bimodules were introduced respectively in [PS09, Definition 2.3] and [OOT15, Definition 4] (see also [BF07, Definition 3.1]). These two notions have recently been shown to be equivalent in [DKEP22, Theorem 5.10].

For future reference we note that if $\xi \in \mathcal{H}$ is subtracial, then Φ_ξ is subunital and subtracial.

Conversely, given a normal, completely positive map $\Phi : M \to M$, there is a Hilbert M-bimodule $\mathcal{H}_\Phi$ together with a vector $\xi_\Phi \in \mathcal{H}_\Phi$ such that $\overline{\mathrm{span}(M\xi_\Phi M)} = \mathcal{H}_\Phi$ and $\langle x\xi_\Phi y, \xi_\Phi\rangle = \tau(\Phi(x)y)$, for every $x, y \in M$. Assume that Φ is symmetric, i.e., $\tau(\Phi(x)y) = \tau(x\Phi(y))$, for every $x, y \in M$. Then $\mathcal{H}_\Phi$ is a symmetric M-bimodule as witnessed by the anti-unitary involution $J : \mathcal{H}_\Phi \to \mathcal{H}_\Phi$ given by $J(x\xi_\Phi y) = y^*\xi_\Phi x^*$. Moreover, ξ_Φ is a bounded vector, $J(\xi_\Phi) = \xi_\Phi$, and

$$\langle \xi_\Phi, x\xi_\Phi y\rangle_M = \Phi(x)y, \text{ for every } x, y \in M. \tag{2.2}$$

To justify (2.2), let $a = \langle \xi_\Phi, x\xi_\Phi y\rangle_M$. The definition of a gives that $\tau(az) = \langle x\xi_\Phi yz, \xi_\Phi\rangle$, for every $z \in M$. Since $\langle x\xi_\Phi yz, \xi_\Phi\rangle = \tau(\Phi(x)yz)$, it follows that $a = \Phi(x)y$, as claimed.

The assignment $\Phi \mapsto \mathcal{H}_\Phi$ is a bijection between normal, completely positive maps and cyclic M-bimodules $\mathcal{H}$ which are generated by a bounded vector ξ, i.e., satisfy $\mathcal{H} = \overline{\mathrm{span}(M\xi M)}$. Moreover, if $\mathcal{H}$ is an M-bimodule and $\xi \in \mathcal{H}$ is a bounded vector, then the M-bimodules $\mathcal{H}_{\Phi_\xi}$ and $\overline{\mathrm{span}(M\xi M)}$ are isomorphic.

Lemma 2.8. *Let (M, τ) be a tracial von Neumann algebra, $\Phi : M \to M$ a normal completely positive map, and $\mathcal{K}$ an M-bimodule. Then $\mathcal{H}_\Phi \subset_{\mathrm{weak}} \mathcal{K}$ if and only if there is a net (Φ_i) of coefficients of $\mathcal{K}^{\oplus\infty}$ such that $\|\Phi_i(x) - \Phi(x)\|_2 \to 0$, for every $x \in M$.*

Moreover, if Φ is subunital, subtracial and $\mathcal{H}_\Phi \subset_{\rm weak} \mathcal{K}$, then there is a net (Φ_i) of subunital, subtracial coefficients of $\mathcal{K}^{\oplus\infty}$ such that $\|\Phi_i(x)-\Phi(x)\|_2 \to 0$, for every $x \in M$. Furthermore, in this case, if Φ is symmetric, then we can assume that Φ_i is symmetric, for every i.

Finally, assume that M is separable. Then we may take (Φ_i) to be a sequence in the above assertions.

Proof. First, we prove the "if" part of the main assertion. Assume that (Φ_i) is a net of coefficients of $\mathcal{K}^{\oplus\infty}$ so that $\|\Phi_i(x)-\Phi(x)\|_2 \to 0$, for every $x \in M$. Let $\xi_i \in \mathcal{K}^{\oplus\infty}$ such that $\tau(\Phi_i(x)y) = \langle x\xi_i y, \xi_i\rangle$. Since $\lim_i \tau(\Phi_i(x)y) = \tau(\Phi(x)y)$, we conclude that $\langle x\xi_\Phi y, \xi_\Phi\rangle = \lim_i \langle x\xi_i y, \xi_i\rangle$, for every $x, y \in M$. This implies that $\mathcal{H}_\Phi \subset_{\rm weak} \mathcal{K}$.

Now, assume that Φ is subunital, subtracial and $\mathcal{H}_\Phi \subset_{\rm weak} \mathcal{K}$. Then $\xi_\Phi \in \mathcal{H}_\Phi$ is a subtracial vector. By Lemma 2.5 we find a net of subtracial vectors $(\eta_i) \subset \mathcal{K}^{\oplus\infty}$ with $\langle x\xi_\Phi y, \xi_\Phi\rangle = \lim_i \langle x\eta_i y, \eta_i\rangle$, for every $x, y \in M$. Thus, if $\Phi_i := \Phi_{\eta_i}$, then $\tau(\Phi(x)y) = \lim_i \tau(\Phi_i(x)y)$, for every $x, y \in M$. Since Φ_i is completely positive, $\Phi_i(x)^*\Phi_i(x) \leq \Phi_i(x^*x)$ and thus $\|\Phi_i(x)\| \leq \|x\|$, for every $x \in M$ and every i. This implies that $\Phi_i(x) \to \Phi(x)$, in the weak topology on $\mathrm{L}^2(M)$, for every $x \in M$. Since the set of subunital, subtracial coefficients of $\mathcal{K}^{\oplus\infty}$ is a convex set, after taking convex combinations of (Φ_i), the moreover assertion follows. Assume additionally that Φ is symmetric. Since η_i is subtracial, there is a completely positive map $\Phi_i^* : M \to M$ such that $\langle x\eta_i y, \eta_i\rangle = \tau(x\Phi_i^*(y))$, for every $x \in M$ and every i. Let $\Psi_i = \frac{1}{2}(\Phi_i + \Phi_i^*) : M \to M$. Then $\Psi_i(x) \to \Phi(x)$, in the weak topology on $\mathrm{L}^2(M)$, for every $x \in M$. Since Ψ_i is symmetric for every i and the set of symmetric, subunital, subtracial coefficients of $\mathcal{K}^{\oplus\infty}$ is a convex set, the furthermore assertion follows by taking convex combinations of (Ψ_i).

Next, we prove the "only if" part of the main assertion. Let $\Phi : M \to M$ be a normal completely positive map such that $\mathcal{H}_\Phi \subset_{\rm weak} \mathcal{K}$. Let $T \in \mathrm{L}^1(M)$ be a positive element such that $\tau \circ \Phi = \tau(\cdot T)$. For $n \in \mathbb{N}$, let $T_n \in M$ such that $\|T_n - T\|_1 \leq (8n^2)^{-1}$. Let $p_n \in M$ be a projection such that $p_nTp_n \in M$ and $\|p_n - 1\|_2 \leq (8n^2(\|T_n\|+1))^{-1/2}$. Define $\Psi_n : M \to M$ by $\Psi_n(x) = \Phi(p_nxp_n)$. Let $x \in (M)_1$. Since $\|\Phi(x)\|_2^2 = \tau(\Phi(x)^*\Phi(x)) \leq \tau(\Phi(x^*x)) = \tau(x^*xT)$, $\|p_nxp_n - x\| \leq 2$, and $\|p_nxp_n - x\|_2 \leq 2\|p_n - 1\|_2$, we get that

$$\|\Psi_n(x) - \Phi(x)\|_2^2 \leq \tau((p_nxp_n - x)^*(p_nxp_n - x)T) \leq 4\|T_n - T\|_1 + 4\|T_n\|\|p_n - 1\|_2^2 \leq (n^2)^{-1}.$$

Let $c_n := \max\{\|\Psi_n(1)\|, \|p_nTp_n\|\} > 0$. Since $\tau \circ \Psi_n = \tau(\cdot(p_nTp_n))$, we get that $c_n^{-1}\Psi_n$ is a subunital, subtracial completely positive map. By applying the moreover assertion to $c_n^{-1}\Psi_n$ and using that $\sup_{x\in(M)_1}\|\Psi_n(x) - \Phi(x)\|_2 \leq 1/n$, we find a net (Φ_i) of coefficients of $\mathcal{K}^{\oplus\infty}$ such that $\|\Phi_i(x) - \Phi(x)\|_2 \to 0$, for every $x \in M$. Specifically, let I be set of pairs $i = (F, n)$, where $F \subset M$ is a finite set and $n \in \mathbb{N}$, ordered by $i \leq i' = (F', n')$ if and only if $F \subset F'$ and $n \leq n'$. Then for every $i = (F, n) \in I$, we can find a coefficient $\widetilde{\Phi}_i$ of $\mathcal{K}^{\oplus\infty}$ such that $\|c_n^{-1}\Psi_n(x) - \widetilde{\Phi}_i(x)\|_2 \leq 1/(nc_n)$, for every $x \in F$. Then $\Phi_i = c_n\widetilde{\Phi}_i$ is a coefficient of $\mathcal{K}^{\oplus\infty}$ such that $\|\Psi_n(x) - \Phi_i(x)\|_2 \leq 1/n$, for every $x \in F$. Then $\|\Phi(x) - \Phi_i(x)\|_2 \leq 2/n$, for every $x \in F$, and our assertion follows.

Finally, assume that M is separable. Let $\Phi, (\Phi_i)$ be subunital, subtracial completely positive maps on M such that $\|\Phi_i(x) - \Phi(x)\|_2 \to 0$, for every $x \in M$. Since $\|\Phi(x)\|_2 \leq \|x\|_2$ and $\|\Phi_i(x)\|_2 \leq \|x\|_2$, we get that $\|\Phi_i(x) - \Phi(x)\|_2 \leq 2\|x\|_2$, for every $x \in M$. It follows that we can find a subsequence (Φ_{i_n}) of (Φ_i) such that $\|\Phi_{i_n}(x) - \Phi(x)\|_2 \to 0$, for every $x \in M$. This implies that (Φ_i) can be taken to be a sequence in the assertions of Lemma 2.8. ☐

2.4. **An elementary lemma on homomorphisms to ultrapowers.** We conclude this section by recording some terminology and a lemma that will be used in the proofs of Theorems E and F. Let S be a set. A $*$-*monomial* in variables $X_s, s \in S$, is an expression of the form $Y_1Y_2\cdots Y_k$, where

$k \in \mathbb{N}$ and $Y_i \in \{X_s, X_s^* \mid s \in S\}$, for every $1 \leq i \leq k$. A $*$-*polynomial* p in variables $X_s, s \in S$, is a complex linear combination of $*$-monomials. If A is a $*$-algebra, we denote by $p(x_s, s \in S)$ the *evaluation* of p at some $\{x_s \mid s \in S\} \subset A$.

Lemma 2.9. *Let (M, τ_M) and (N, τ_N) be tracial von Neumann algebras, $\{x_s \mid s \in S\} \subset (N)_1$ a generating set of N, and $\mathcal{U}$ be a cofinal ultrafilter on a directed set I. For every $i \in I$, let $\{x_{i,s} \mid s \in S\} \subset (M)_1$ such that $\lim_i \tau_M(p(x_{i,s}, s \in S)) = \tau_N(p(x_s, s \in S))$, for every $*$-monomial p in variables $X_s, s \in S$. Then there is a trace preserving $*$-homomorphism $\pi : N \to M^{\mathcal{U}}$ such that $\pi(x_s) = (x_{i,s})$, for every $s \in S$.*

Proof. Let $A = \{p(x_s, s \in S) \mid p \text{ $*$-polynomial in } X_s, s \in S\}$ and define a map $\pi : A \to M^{\mathcal{U}}$ by letting $\pi(p(x_s, s \in S)) = (p(x_{i,s}, s \in S))$, for every $*$-polynomial p in $X_s, s \in S$.

Then π is well-defined. If $p_1(x_s, s \in S) = p_2(x_s, s \in S)$, for $*$-polynomials p_1, p_2 on $X_s, s \in S$, then $p(x_s, s \in S) = 0$, with $p = p_1 - p_2$. Thus, we have that

$$\lim_{i \to \mathcal{U}} \|p(x_{i,s}, s \in S)\|_{2,\tau_M} = \lim_{i \to \mathcal{U}} \tau_M(p^* p(x_{i,s}, s \in S))^{1/2} = \tau_N(p^* p(x_s, s \in S))^{1/2} = 0.$$

Hence, $\lim_{i \to \mathcal{U}} \|p_1(x_{i,s}, s \in S) - p_2(x_{i,s}, s \in S)\|_{2,\tau_M} = 0$, so $\pi(p_1(x_s, s \in S)) = \pi(p_2(x_s, s \in S))$ in $M^{\mathcal{U}}$. Since π is a trace preserving $*$-homomorphism on A and A is SOT-dense in N, π extends to a trace preserving $*$-homomorphism $\pi : N \to M^{\mathcal{U}}$. By definition, $\pi(x_s) = (x_{i,s})$, for every $s \in S$. $\square$

3. Shlyakhtenko's M-valued semicircular systems

In [Sh97], Shlyakhtenko introduced a construction which generalizes Voiculescu's free Gaussian functor ([Vo83]) and associates to every von Neumann algebra M and symmetric M-bimodule $\mathcal{H}$, a von Neumann algebra which contains M. We recall this construction here in the case when M is tracial, following closely [KV15, Section 3].

Definition 3.1. Let (M, τ) be a tracial von Neumann algebra, $(\mathcal{H}, J)$ a symmetric M-bimodule and denote by $\mathcal{H}^0 \subset \mathcal{H}$ the set of bounded vectors. Define the Fock space associated to $\mathcal{H}$ by

$$\mathcal{F}_M(\mathcal{H}) = \mathrm{L}^2(M) \oplus \Big(\bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes_M^n} \Big).$$

For every $\xi \in \mathcal{H}^0$, we define $\ell(\xi) \in \mathbb{B}(\mathcal{F}_M(\mathcal{H}))$ by letting $\ell(\xi)(x) = \xi x$ and

$$\ell(\xi)(\xi_1 \otimes_M \cdots \otimes_M \xi_n) = \xi \otimes_M \xi_1 \otimes_M \cdots \otimes_M \xi_n, \text{ for every } x \in M \text{ and } \xi_1, \cdots, \xi_n \in \mathcal{H}^0.$$

Then $\ell(\xi)^*(x) = 0$ and

$$\ell(\xi)^*(\xi_1 \otimes_M \cdots \otimes_M \xi_n) = \langle \xi, \xi_1 \rangle_M \xi_2 \otimes_M \cdots \otimes_M \xi_n, \text{ for every } x \in M \text{ and } \xi_1, \cdots, \xi_n \in \mathcal{H}^0.$$

For $\xi \in \mathcal{H}^0$ with $J(\xi) = \xi$, we denote $s(\xi) = \ell(\xi) + \ell(\xi)^*$. Since $\ell(\xi)^* \ell(\xi) = \langle \xi, \xi \rangle_M$, we get that $\|\ell(\xi)\| = \|\langle \xi, \xi \rangle_M\|^{1/2}$ and thus $\|s(\xi)\| \leq 2\|\langle \xi, \xi \rangle_M\|^{1/2}$.

Shlyakhtenko's M-valued semicircular system associated to $\mathcal{H}$ is then defined as

$$\Gamma(M, \mathcal{H})'' := \Big(M \cup \{s(\xi) \mid \xi \in \mathcal{H}^0, J(\xi) = \xi\} \Big)'' \subset \mathbb{B}(\mathcal{F}_M(\mathcal{H})).$$

Let $\Omega \in \mathcal{F}_M(\mathcal{H})$ be the vacuum unit vector given by $\Omega = 1 \in \mathrm{L}^2(M)$. Then $\tau : \Gamma(M, \mathcal{H})'' \to \mathbb{C}$ given by $\tau(x) = \langle x\Omega, \Omega \rangle$ is a faithful normal tracial state (see [Sh97] and [KV15, Proposition 3.2]) The map $\Gamma(M, \mathcal{H})'' \ni x \mapsto x\Omega \in \mathcal{F}_M(\mathcal{H})$ extends to a unitary operator $\mathrm{L}^2(\Gamma(M, \mathcal{H})'') \to \mathcal{F}_M(\mathcal{H})$. Moreover, this map is an M-bimodule isomorphism. Clearly, the semicircular operators $s(\xi)$ are orthogonal to M, i.e., satisfy $\mathrm{E}_M(s(\xi)) = 0$.

Lemma 3.2. *In the above notation, let $\mathcal{H}_1 \subseteq \mathcal{H}^0$ be a dense subspace such that $M\mathcal{H}_1M \subset \mathcal{H}_1$ and $J(\mathcal{H}_1) = \mathcal{H}_1$. Then for every $\eta \in \mathcal{H}^0$ with $J(\eta) = \eta$, there exists a net $(\eta_n) \subset \mathcal{H}_1$ such that $J(\eta_n) = \eta_n$, for every n, $||\eta_n - \eta|| \to 0$, $\sup_n ||s(\eta_n)|| < \infty$ and $s(\eta_n) \to s(\eta)$ in the SOT.*

In the proof of Lemma 3.2 we will use the following fact: if $\xi \in \mathcal{H}$ satisfies $J(\xi) = \xi$, then for every $x \in M$ we have $\langle x\xi, \xi\rangle = \langle J(\xi), J(x\xi)\rangle = \langle \xi, \xi x^*\rangle = \langle \xi x, \xi\rangle$.

Proof. Let $\eta \in \mathcal{H}^0$ such that $J(\eta) = \eta$. If $a = \langle \eta, \eta\rangle_M \in M$, then the above fact gives that for every $x \in M$ we have $\langle x\eta, \eta\rangle = \langle \eta x, \eta\rangle = \tau(xa)$. After rescaling η, we may assume that $\|T_\eta\| \leq 1$. Thus, $\|\eta\| \leq 1$ and $0 \leq a \leq 1$.

Let $(\xi_n) \subset \mathcal{H}_1$ be a sequence such that $\|\xi_n\| \leq 1$, for every n, and $||\xi_n - \eta|| \to 0$. Since $J(\eta) = \eta$, we get that $\|J(\xi_n) - \eta\| = \|\xi_n - \eta\| \to 0$ and thus $\|(\xi_n + J(\xi_n))/2 - \eta\| \to 0$. As $(\xi_n + J(\xi_n))/2 \in \mathcal{H}_1$, after replacing ξ_n by $(\xi_n + J(\xi_n))/2 \in \mathcal{H}_1$, we may also assume that $J(\xi_n) = \xi_n$, for every n.

For every n, let $a_n = \langle \xi_n, \xi_n\rangle_M$. Since $J(\xi_n) = \xi_n$, by using the fact stated just before this proof we get that $\langle x\xi_n, \xi_n\rangle = \langle \xi_n x, \xi_n\rangle = \tau(xa_n)$, for every $x \in M$. Since $\|\eta\| \leq 1$ and $\|\xi_n\| \leq 1$, for every $x \in M$ we have that $|\tau(x(a_n - a))| = |\langle x\xi_n, \xi_n\rangle - \langle x\eta, \eta\rangle| \leq 2\|x\|\,\|\xi_n - \eta\|$. This implies that $\|a_n - a\|_1 \leq 2\|\xi_n - \eta\|$ and since $\|\xi_n - \eta\| \to 0$, we conclude that $\|a_n - a\|_1 \to 0$.

Next, for every n, define $s_n = f(a_n)$, where

$$f(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ t^{-\frac{1}{2}} & \text{if } t \geq 1 \end{cases}.$$

Let $\eta_n = s_n\xi_n s_n \in M\mathcal{H}_1M \subset \mathcal{H}_1$. Then $J(\eta_n) = \eta_n$, η_n is subtracial and thus $\|s(\eta_n)\| \leq 2$, for every n. We claim that $||\eta_n - \xi_n|| \to 0$. To this end, note that

$$||\eta_n - \xi_n|| \leq ||(1 - s_n)\xi_n|| + ||s_n\xi_n(1 - s_n)|| \leq ||(1 - s_n)\xi_n|| + ||\xi_n(1 - s_n)||.$$

Let p_n be the spectral projection of a_n corresponding to the interval $[1, \infty)$. Since we have that $1_{[1,\infty)}(t)(t - 1) \geq (1 - f(t))^2 t$, we get that $(1 - s_n)^2 a_n \leq p_n(a_n - 1)$ and therefore

$$\begin{aligned} ||(1 - s_n)\xi_n||^2 &= \tau((1 - s_n)^2 a_n) \\ &\leq \tau(p_n(a_n - 1)) \\ &\leq \tau(p_n(a_n - a)) \\ &\leq ||a_n - a||_1, \end{aligned}$$

where we used that $\tau(p_n a) \leq \tau(p_n)$ as $0 \leq a \leq 1$. Similarly, $||\xi_n(1 - s_n)||^2 \leq \|a_n - a\|_1$. Thus, $||\eta_n - \xi_n|| \leq 2\|a_n - a\|_1^{1/2}$. Since $\|a_n - a\|_1 \to 0$, this implies our claim that $\|\eta_n - \xi_n\| \to 0$. This further implies that $\|\eta_n - \eta\| \to 0$. Thus, $\|s(\eta_n) - s(\eta)\|_2 = \|\eta_n - \eta\| \to 0$. As $\sup_n \|s(\eta_n)\| < \infty$, it follows that $s(\eta_n) \to s(\eta)$ in the SOT. □

Lemma 3.3. *In the above notation, let $\mathcal{H}^1 \subset \mathcal{H}^0$ be a dense subspace such that $J(\mathcal{H}^1) = \mathcal{H}^1$. Then $\Gamma(M, \mathcal{H})'' = \Big(M \cup \{s(\xi) \mid \xi \in \mathcal{H}^1, J(\xi) = \xi\}\Big)''$.*

Moreover, let $\{\xi_i\}_{i\in I} \subset \mathcal{H}^0$ be a family of vectors such that $J(\xi_i) = \xi_i$, for every $i \in I$, and the span of $\{M\xi_i M\}_{i\in I}$ is dense in $\mathcal{H}$. Then $\Gamma(M, \mathcal{H})'' = \Big(M \cup \{s(\xi_i) \mid i \in I\}\Big)''$.

Proof. The main assertion follows by applying Lemma 3.2 to $\mathcal{H}_1 = M\mathcal{H}^1M$.

We give an alternative proof of the main assertion which does not rely on Lemma 3.2. We let $\mathcal{K} \subset \mathcal{F}_M(\mathcal{H})$ be the span of $\mathrm{L}^2(M) \cup (\bigcup_{n\geq 1}\{\xi_1 \otimes_M \cdots \otimes_M \xi_n \mid \xi_1, \cdots, \xi_n \in \mathcal{H}^1\})$ and denote

$\mathcal{M}=\Big(M\cup\{s(\xi)\mid \xi\in\mathcal{H}^1, J(\xi)=\xi\}\Big)''$. We claim that $\mathcal{K}\subset\mathcal{M}\Omega$. To this end, for $n\geq 1$, denote by $\mathcal{K}_n\subset\mathcal{F}_M(\mathcal{H})$ the span of $\mathrm{L}^2(M)\cup(\bigcup_{1\leq k\leq n}\{\xi_1\otimes_M\cdots\otimes_M\xi_k\mid \xi_1,\cdots,\xi_k\in\mathcal{H}^1\})$. Proceeding by induction, assume that $\mathcal{K}_n\subset\mathcal{M}\Omega$, for $n\geq 1$. If $\xi,\xi_1,\cdots,\xi_n\in\mathcal{H}^1$ and $J(\xi)=\xi$, then $\xi\otimes_M\xi_1\otimes_M\cdots\otimes_M\xi_n=s(\xi)(\xi_1\otimes_M\cdots\otimes_M\xi_n)-\langle\xi,\xi_1\rangle_M(\xi_2\otimes_M\cdots\otimes_M\xi_n)\in s(\xi)\mathcal{M}\Omega+M(\mathcal{M}\Omega)=\mathcal{M}\Omega$. Since every $\eta\in\mathcal{H}^1$ can be written as $\eta=\eta_1+i\eta_2$, where $\eta_1=(\eta+J(\eta))/2$ and $\eta_2=(\eta-J(\eta))/(2i)$ satisfy $J(\eta_1)=\eta_1$ and $J(\eta_2)=\eta_2$, this proves the claim. Since $\mathcal{H}^1$ is dense in $\mathcal{H}^0$, Remark 2.4 implies that $\mathcal{K}$ is dense $\mathcal{F}_M(\mathcal{H})$. Using the claim, we deduce that $\mathcal{M}\Omega$ is dense in $\mathcal{F}_M(\mathcal{H})$. Hence, $\mathcal{M}$ is dense in $\mathrm{L}^2(\Gamma(M,\mathcal{H})'')$ and therefore $\mathcal{M}=\Gamma(M,\mathcal{H})''$. This proves the main assertion.

Let $\mathcal{H}^2\subset\mathcal{H}^0$ be the span of $\{M\xi_iM\}_{i\in I}$. Then $J(\mathcal{H}^2)=\mathcal{H}^2$. Thus, the main assertion implies that $\Gamma(M,\mathcal{H})''$ is equal to $\Big(M\cup\{s(\xi)\mid\xi\in\mathcal{H}^2, J(\xi)=\xi\}\Big)''$. Denote $\mathcal{N}=\Big(M\cup\{s(\xi_i)\mid i\in I\}\Big)''$. Since $\ell(x\xi)=x\ell(\xi)$, $\ell(\xi x)=\ell(\xi)x$, and $J(x\xi x^*)=xJ(\xi)x^*$, for every $x\in M$ and $\xi\in\mathcal{H}^0$, we get that $s(y\xi_iy^*)=ys(\xi_i)y^*\in\mathcal{N}$, for every $y\in M$ and $i\in I$. Note that $\{\xi\in\mathcal{H}^2\mid J(\xi)=\xi\}$ is equal to the linear span of $\{v\xi_iw^*+w\xi_iv^*\mid v,w\in M, i\in I\}$ and further that of $\{y\xi_iy^*\mid y\in M, i\in I\}$. Since $s(\xi_1+\xi_2)=s(\xi_1)+s(\xi_2)$, for every $\xi_1,\xi_2\in\mathcal{H}^0$, we conclude that $s(\xi)\in\mathcal{N}$, for every $\xi\in\mathcal{H}^2$ with $J(\xi)=\xi$. Since $\mathcal{N}$ also contains M, we get that $\mathcal{M}=\Gamma(M,\mathcal{H})''$. ☐

An ultrafilter on a directed set $(I,\leq)$ is called *cofinal* if it contains $\{i\in I\mid i\geq i_0\}$, for every $i_0\in I$. If $(x_i)_{i\in I}\subset\mathbb{C}$ is a net such that $\lim_i x_i=x$ and $\mathcal{U}$ is a cofinal ultrafilter on I, then $\lim_{i\to\mathcal{U}}x_i=x$. If (M,τ) is a tracial von Neumann algebra, then we say that a positive map $\Phi:M\to M$ is subunital if $\Phi(1)\leq 1$. We now arrive at the main result of this section.

Lemma 3.4. *Let (M,τ) be a tracial von Neumann algebra, $\Phi:M\to M$ a subunital, symmetric completely positive map, and $\Phi_i:M\to M$, $i\in I$, a net of subunital, symmetric completely positive maps such that $\|\Phi_i(x)-\Phi(x)\|_2\to 0$, for every $x\in M$. Let $\mathcal{U}$ be a cofinal ultrafilter on I. Then there is a trace preserving $*$-homomorphism $\pi:\Gamma(M,\mathcal{H}_\Phi)''\to\prod_{\mathcal{U}}\Gamma(M,\mathcal{H}_{\Phi_i})''$ such that $\pi_{|M}=\mathrm{Id}_M$ and $\pi(s(\xi_\Phi))=(s(\xi_{\Phi_i}))$. Moreover, $\mathcal{H}_\Phi\subset\mathrm{L}^2(\prod_{\mathcal{U}}\Gamma(M,\mathcal{H}_{\Phi_i})'')\ominus\mathrm{L}^2(M^{\mathcal{U}})$, as an M-bimodule.*

In order to prove this result, we first introduce a definition and then establish a lemma.

Definition 3.5. Let $n\in\mathbb{N}$. We define sets of formulas $S_1\subset\cdots\subset S_n$ involving non-commuting variables $X_0,X_1,\cdots,X_n$ belonging to an algebra and a function Ψ having the same algebra as domain and range, as follows. Let $S_1=\{1,X_0,X_1\}$ and define inductively for every $2\leq i\leq n$:

$$S_i=\{X_i\}\cup S_{i-1}\cup\{X_i\Psi(a)b\mid a,b\in S_{i-1}\}.$$

For instance, S_2 and S_3 contain the formulas $X_2\Psi(X_0)X_1$ and $X_3\Psi(X_2\Psi(X_1)X_0)$, respectively.

Given an algebra M, a function $\Phi:M\to M$ and an $(n+1)$-tuple $\mathbf{x}=(x_0,x_1,\cdots,x_n)\in M^{n+1}$, for every formula $f\in S_n$ we denote by $f^\Phi(\mathbf{x})$ the element of M obtained by replacing $X_0,X_1,\cdots,X_n$ with $x_0,x_1,\cdots,x_n$ and Ψ with Φ.

Lemma 3.6. *There exist $\{\varepsilon(a)\mid a\in S_n\},\{\varepsilon(a_1,b_1,\cdots,a_k,b_k)\mid a_1,b_1,\cdots,a_k,b_k\in S_n\}\subset\mathbb{Z}$, for every $1\leq k\leq n$, such that the following holds. Let $\Phi:M\to M$ be a normal, symmetric completely positive map, where (M,τ) is a tracial von Neumann algebra. Consider the associated M-bimodule $(\mathcal{H}_\Phi,\xi_\Phi)$ and the von Neumann algebra $\Gamma(M,\mathcal{H}_\Phi)''\subset\mathbb{B}(\mathcal{F}_M(\mathcal{H}_\Phi))$. Then for every $\mathbf{x}=(x_0,x_1,\cdots,x_n)\in M^{n+1}$, the vector $x_ns(\xi_\Phi)x_{n-1}\cdots x_1s(\xi_\Phi)x_0\Omega\in\mathcal{F}_M(\mathcal{H}_\Phi)$ is equal to*

$$\sum_{a\in S_n}\varepsilon(a)\,a^\Phi(\mathbf{x})+\sum_{k=1}^n\sum_{\substack{a_0,a_1,b_1,\cdots,\\ a_k,b_k\in S_n}}\varepsilon(a_0,a_1,b_1,\cdots,a_k,b_k)\,a_0^\Phi(\mathbf{x})(a_1^\Phi(\mathbf{x})\xi_\Phi b_1^\Phi(\mathbf{x}))\otimes\cdots\otimes(a_k^\Phi(\mathbf{x})\xi_\Phi b_k^\Phi(\mathbf{x})).$$

In particular, $\tau(x_ns(\xi_\Phi)x_{n-1}\cdots x_1s(\xi_\Phi)x_0)=\sum_{a\in S_n}\varepsilon(a)\,\tau(a^\Phi(\mathbf{x}))$.

Proof. Let $x, a, b, c \in M$ and $\eta \in \mathcal{H}_\Phi^{\otimes_M^m}$, for $m \geq 1$. Then $xs(\xi_\Phi)(a) = x\xi_\Phi a$ and (2.2) implies that

$$xs(\xi_\Phi)(b\xi_\Phi c \otimes \eta) = x\xi_\Phi \otimes b\xi_\Phi c \otimes \eta + x\langle \xi_\Phi, b\xi_\Phi c\rangle_M \eta = x\xi_\Phi \otimes b\xi_\Phi c \otimes \eta + (x\Phi(b)c)\eta.$$

The proof is now immediate by induction. □

Proof of Lemma 3.4. Let $n \in \mathbb{N}$ and $\mathbf{x} = (x_0, x_1, \cdots, x_n) \in M^{n+1}$. Since $\|\Phi_i(y) - \Phi(y)\|_2 \to 0$, for every $y \in M$, it follows that $\|a^{\Phi_i}(\mathbf{x}) - a^{\Phi}(\mathbf{x})\|_2 \to 0$, for every $a \in S_n$. By combining this fact with Lemma 3.6, we conclude that

$$\lim_i \tau(x_n s(\xi_{\Phi_i})x_{n-1} \cdots x_1 s(\xi_{\Phi_i})x_0) = \tau(x_n s(\xi_\Phi)x_{n-1} \cdots x_1 s(\xi_\Phi)x_0),\ \forall x_0, x_1, \cdots, x_n \in M. \tag{3.1}$$

Since Φ_i is subunital, $\|s(\xi_{\Phi_i})\| \leq 2$, for every $i \in I$. This implies that

$$(x_n s(\xi_{\Phi_i})x_{n-1} \cdots x_1 s(\xi_{\Phi_i})x_0) \in \prod_{\mathcal{U}} \Gamma(M, \mathcal{H}_{\Phi_i})'', \forall x_0, x_1, \cdots, x_n \in M. \tag{3.2}$$

Since $\text{span}(M\xi_\Phi M)$ is dense in $\mathcal{H}_\Phi$, Lemma 3.3 implies that the linear span of

$$\{x_n s(\xi_\Phi)x_{n-1} \cdots x_1 s(\xi_\Phi)x_0 \mid x_0, x_1, \cdots, x_n \in M, n \in \mathbb{N}\}$$

is $\|\cdot\|_2$ -dense in $\Gamma(M, \mathcal{H}_\Phi)''$. This fact, (3.1) and (3.2) together imply the existence of a trace preserving $*$-homomorphism $\pi : \Gamma(M, \mathcal{H}_\Phi)'' \to \prod_{\mathcal{U}} \Gamma(M, \mathcal{H}_{\Phi_i})''$ such that

$$\pi(x_n s(\xi_\Phi)x_{n-1} \cdots x_1 s(\xi_\Phi)x_0) = (x_n s(\xi_{\Phi_i})x_{n-1} \cdots x_1 s(\xi_{\Phi_i})x_0)$$

and $\pi(x) = x$, for every $x_0, x_1, \cdots, x_n, x \in M$. This finishes the proof of the main assertion.

Finally, since $\mathrm{E}_M(s(\xi_{\Phi_i})) = 0$, for every $i \in I$, the definition of π implies that we have $\pi(Ms(\xi_\Phi)M) \subset \prod_{\mathcal{U}} \Gamma(M, \mathcal{H}_{\Phi_i})'' \ominus M^{\mathcal{U}}$. Since the M-bimodule $\overline{\text{span}(Ms(\xi_\Phi)M)}$ is isomorphic to $\mathcal{H}_\Phi$, the moreover assertion follows. □

4. Proofs of main results

4.1. **Proof of Theorem A.** We will prove (1) ⇒ (2) in the case of existentially closed separable II_1 factors M, and that (2) ⇒ (3) ⇒ (4) ⇔ (5), for general separable II_1 factors M.

(1) ⇒ (2) Assume that M is an existentially closed separable II_1 factor. By Remark 2.3, for the purpose of proving (2), after replacing $\mathcal{H}$ with $\mathcal{H} \oplus \overline{\mathcal{H}}$, we may assume $\mathcal{H}$ to be a symmetric M-bimodule. Put $N := \Gamma(M, \mathcal{H})''$. Since $M \subset N$, we can find an embedding $\pi : N \to M^{\mathcal{U}}$, for some ultrafilter $\mathcal{U}$ on a set I, whose restriction $\pi_{|M}$ is the diagonal embedding of M. Moreover, if $\mathcal{H}$ is separable, then so is N, and we can take $I = \mathbb{N}$. Then π extends to an embedding of Hilbert M-bimodules $\mathrm{L}^2(N) \subset \mathrm{L}^2(M^{\mathcal{U}})$. Since $\mathcal{H} \subset \mathrm{L}^2(N)$, part (1) follows.

(2) ⇒ (3) Assume that (2) holds. Let $\mathcal{H}$ be an M-bimodule. Then $\mathcal{H} \subset \mathrm{L}^2(M^{\mathcal{U}})$, for a free ultrafilter $\mathcal{U}$ on a set I. Fix $(\xi_j)_{j=1}^k \subset \mathcal{H}$ such that $\|\xi_j\|_2 \leq 1$ for every $1 \leq j \leq k$, a finite set $F \subset (M)_1$ and $\varepsilon > 0$. Then we can find $\eta_j \in M^{\mathcal{U}}$ such that $\|\eta_j\|_2 \leq 1$ and $\|\xi_j - \eta_j\|_2 < \frac{\varepsilon}{2}$, for every $1 \leq j \leq k$. Then $|\langle x\xi_j y, \xi_{j'}\rangle - \langle x\eta_j y, \eta_{j'}\rangle| < \varepsilon$, for every $1 \leq j, j' \leq k$ and $x, y \in F$. Represent $\eta_j = (\eta_{i,j})$, where $\eta_{i,j} \in M$, for every $i \in I, 1 \leq j \leq k$. Since $\langle x\eta_j y, \eta_{j'}\rangle = \lim_{i \to \mathcal{U}} \langle x\eta_{i,j} y, \eta_{i,j'}\rangle$, there is $i \in I$ such that

$$|\langle x\xi_j y, \xi_{j'}\rangle - \langle x\eta_{i,j} y, \eta_{i,j'}\rangle| < \varepsilon, \text{ for every } x, y \in F \text{ and } 1 \leq j, j' \leq k.$$

This implies that $\mathcal{H}$ belongs to the closure of $\mathrm{L}^2(M)$ in the Fell topology, thus proving (3).

(3) ⇒ (4) This implication is obvious.

(4) ⇒ (5) Assume that (4) holds. In order to prove (5), we first establish a claim. Let $\widetilde{\mathcal{P}}_M$ be the set of coefficients of $\mathrm{L}^2(M)^{\oplus\infty}$ and $\widetilde{\mathcal{S}}_M$ be the set of subunital, subtracial coefficients of $\mathrm{L}^2(M)^{\oplus\infty}$.

Claim 4.1. *Let $\Psi \in \widetilde{\mathcal{P}}_M$. Then there exists a sequence $(\Psi_n) \subset \mathcal{P}_M$ such that*

$$\lim_{n\to\infty}(\sup\{\|\Psi(x)-\Psi_n(x)\|_2 \mid x \in M, \|x\| \leq 1\}) = 0.$$

Moreover, if $\Psi \in \widetilde{\mathcal{S}}_M$, then we can take $(\Psi_n) \in \mathcal{S}_M$.

Proof of Claim 4.1. As $\Psi \in \widetilde{\mathcal{P}}_M$, we can find $(a_k)_{k=1}^{\infty} \subset M$ such that $\sum_{k=1}^{\infty} a_k a_k^* \leq C, \sum_{k=1}^{\infty} a_k^* a_k \leq C$, for some $C > 0$, and $\Psi(x) = \sum_{k=1}^{\infty} a_k x a_k^*$, for every $x \in M$. For every $n \in \mathbb{N}$ and $x \in M$, let $\Psi_n(x) = \sum_{k=1}^{n} a_k x a_k^*$ and $\Psi^n(x) = \sum_{k=n+1}^{\infty} a_k x a_k^*$. Then $(\Psi_n) \subset \mathcal{P}_M$. Moreover, if $\Psi \in \widetilde{\mathcal{S}}_M$, then we can take $C = 1$ and thus we have $(\Psi_n) \subset \mathcal{S}_M$. By [Po01, Lemma 1.1.2], for any normal completely positive map $\varphi : M \to M$ we have $\|\varphi(x)\|_2 \leq \|\varphi(1)\|_2$, for every $x \in M$ with $\|x\| \leq 1$. If $n \in \mathbb{N}$, then by using this fact and that Ψ^n is normal and completely positive we deduce that

$$\|\Psi(x) - \Psi_n(x)\|_2 = \|\Psi^n(x)\|_2 \leq \|\Psi^n(1)\|_2, \text{ for every } x \in M \text{ with } \|x\| \leq 1. \tag{4.1}$$

Since $\sum_{k=1}^{\infty} \tau(a_k a_k^*) \leq C$, we get $\lim_{n\to\infty} \tau(\Psi^n(1)) = \lim_{n\to\infty}(\sum_{k=n+1}^{\infty} \tau(a_k a_k^*)) = 0$. Since for every $n \in \mathbb{N}$, $0 \leq \Psi^n(1) \leq C$, we get that $\lim_{n\to\infty} \|\Psi^n(1)\|_2 = 0$. Together with (4.1), the claim follows. □

Let $\Phi : M \to M$ be a normal completely positive map. Then (4) implies that $\mathcal{H}_\Phi \subset_{\text{weak}} \mathrm{L}^2(M)$. By Lemma 2.8, there exists a sequence $(\Phi_n) \subset \widetilde{\mathcal{P}}_M$ such that $\lim_{n\to\infty} \|\Phi_n(x) - \Phi(x)\|_2 = 0$, for every $x \in M$. By Claim 4.1, for every $n \in \mathbb{N}$, we can find $\widetilde{\Phi}_n \in \mathcal{P}_M$ such that $\|\widetilde{\Phi}_n(x) - \Phi(x)\|_2 \leq 1/n$, for every $x \in M$ with $\|x\| \leq 1$. Then it is clear that $\lim_{n\to\infty} \|\widetilde{\Phi}_n(x) - \Phi(x)\|_2 = 0$, for every $x \in M$. Moreover, if Φ is subunital, subtracial, then we can take $(\Phi_n) \subset \widetilde{\mathcal{S}}_M$ by Lemma 2.8 and $(\widetilde{\Phi}_n) \subset \mathcal{S}_M$ by Claim 4.1. This altogether proves (5).

(5) ⇒ (4) Assume that (5) holds. If $\mathcal{H}$ is an M-bimodule, then $\mathcal{H} = \bigoplus_i \overline{\text{span}(M\xi_i M)}$, for a family of bounded vectors $(\xi_i) \subset \mathcal{H}$. Thus, in order to prove (4), it suffices to argue that any M-bimodule of the form $\mathcal{H} = \overline{\text{span}(M\xi M)}$, for some bounded vector $\xi \in \mathcal{H}$, is weakly contained in $\mathrm{L}^2(M)$. We denote by $\Phi := \Phi_\xi : M \to M$ the associated normal completely positive map. Then $\mathcal{H}$ is isomorphic to $\mathcal{H}_\Phi$. Since (5) holds, we can find a sequence (Φ_n) of coefficients of $\mathrm{L}^2(M)^{\oplus\infty}$ such that $\|\Phi_n(x) - \Phi(x)\|_2 \to 0$, for every $x \in M$. Thus, $\mathcal{H}_\Phi \subset_{\text{weak}} \mathrm{L}^2(M)$ and so $\mathcal{H} \subset_{\text{weak}} \mathrm{L}^2(M)$. □

In preparation for the proofs of Theorem C, Corollary B and Corollary D, we recall from [Sh97] the following result that allows to explicitly identify $\Gamma(M, \mathcal{H})''$ for certain symmetric M-bimodules $\mathcal{H}$:

Proposition 4.2 ([Sh97])**.** *Let (M, τ) be a tracial von Neumann algebra. Then we have*

(1) $\Gamma(M, \mathrm{L}^2(M))'' \cong M\overline{\otimes}\mathrm{L}(\mathbb{Z})$.
(2) $\Gamma(M, \mathrm{L}^2(M) \otimes \mathrm{L}^2(M))'' \cong M * \mathrm{L}(\mathbb{Z})$.
(3) If $Q \subset M$ is a von Neumann subalgebra, then $\Gamma(M, \mathrm{L}^2(M) \otimes_Q \mathrm{L}^2(M))'' \cong M *_Q (Q\overline{\otimes}L(\mathbb{Z}))$.
(4) If $(\mathcal{H}_i)_{i\in I}$ are symmetric M-bimodules and $\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$, then $\Gamma(M, \mathcal{H})'' \cong *_{M,i\in I}\Gamma(M, \mathcal{H}_i)''$.

Proof. We have $\mathrm{L}^2(M) \cong \mathcal{H}_\tau$, $\mathrm{L}^2(M)\otimes\mathrm{L}^2(M) \cong \mathcal{H}_{\mathrm{Id}}$ and $\mathrm{L}^2(M)\otimes_Q\mathrm{L}^2(M) \cong \mathcal{H}_{\mathrm{E}_Q}$, as M-bimodules, where we consider the following normal, completely positive maps: $\tau : M \to \mathbb{C} \subset M, \mathrm{Id} : M \to M$ and $\mathrm{E}_Q : M \to Q \subset M$. Then (1), (2) and (3) follow from parts (a), (b) and (c) of [Sh97, Example 3.3], respectively. In turn, (4) follows from [Sh97, Proposition 2.18]. □

Next we prove Theorem C and then use it to deduce Corollary B.

4.2. **Proof of Theorem C.** Let $\mathcal{H},\mathcal{K}$ be symmetric M-bimodules such that $\mathcal{H}\subset_{\text{weak}}\mathcal{K}$. Let $(\xi_j)_{j\in J}\subset\mathcal{H}$ be a family of non-zero subtracial vectors such that denoting $\mathcal{H}_j=\overline{\text{span}(M\xi_jM)}$, we have $\mathcal{H}=\bigoplus_{j\in J}\mathcal{H}_j$. For $j\in J$, consider the subunital, subtracial completely positive map $\Phi_j:=\Phi_{\xi_j}:M\to M$. Then $\mathcal{H}_{\Phi_j}$ is isomorphic to $\mathcal{H}_j$ and so $\mathcal{H}_{\Phi_j}\subset_{\text{weak}}\mathcal{K}$. By Lemma 2.8, we can find a net of subunital, subtracial coefficients $(\Phi_{i,j})_{i\in I}$ of $\mathcal{K}^{\oplus\infty}$ such that $\lim_i\|\Phi_{i,j}(x)-\Phi_j(x)\|_2\to 0$, for every $x\in M$. Here, we can take I to be independent of j by taking it to be the set of pairs (F,ε), with $F\subset M$ finite and $\varepsilon>0$, ordered by $(F,\varepsilon)\leq(F',\varepsilon')$ if and only if $F\subset F'$ and $\varepsilon\geq\varepsilon'$. Specifically, for every $i=(F,\varepsilon)\in I$, let $\Phi_{i,j}$ be any subunital, subtracial coefficient of $\mathcal{K}^{\oplus\infty}$ such that $\|\Phi_{i,j}(x)-\Phi_j(x)\|_2\leq\varepsilon$, for every $x\in F$. If M is separable, we can take $I=\mathbb{N}$. Let $\mathcal{U}$ be a cofinal ultrafilter on I.

Since $\mathcal{H}_{\Phi_{i,j}}\subset\mathcal{K}^{\oplus\infty}$, we have a natural trace preserving embedding $\Gamma(M,\mathcal{H}_{\Phi_{i,j}})''\subset\Gamma(M,\mathcal{K}^{\oplus\infty})''$. By Lemma 3.4, we find a trace preserving $*$-homomorphism $\pi_j:\Gamma(M,\mathcal{H}_j)''\to(\Gamma(M,\mathcal{K}^{\oplus\infty})'')^{\mathcal{U}}$ whose restriction to M is the diagonal embedding of M. Note that if $(M_j,\tau_j)_{j\in J}$ are tracial von Neumann algebras containing (M,τ) such that $\tau=\tau_{j|M}$, for every $j\in J$, then we have a natural embedding $*_{M,j\in J}M_j^{\mathcal{U}}\subset(*_{M,j\in J}M_j)^{\mathcal{U}}$. This implies the existence of a $*$-homomorphism

$$\pi=*_{M,j\in J}\pi_j:*_{M,j\in J}\Gamma(M,\mathcal{H}_j)''\to *_{M,j\in J}(\Gamma(M,\mathcal{K}^{\oplus\infty})'')^{\mathcal{U}}\subset(*_{M,j\in J}\Gamma(M,\mathcal{K}^{\oplus\infty})'')^{\mathcal{U}}$$

whose restriction to M is the diagonal embedding of M. By Proposition 4.2(4), we get that $*_{M,j\in J}\Gamma(M,\mathcal{H}_j)''=\Gamma(M,\mathcal{H})''$ and $*_{M,j\in J}\Gamma(M,\mathcal{K}^{\oplus\infty})''=\Gamma(M,\bigoplus_{j\in J}\mathcal{K}^{\oplus\infty})''=\Gamma(M,\mathcal{K}\otimes\ell^2(S))''$, where $S=\mathbb{N}\times J$. This proves the main assertion.

Finally, assume that M and $\mathcal{H}$ are separable. Since M is separable, we can take $I=\mathbb{N}$. Since $\mathcal{H}$ is separable, we can take $J=\mathbb{N}$. Altogether, this implies that we can take $I=S=\mathbb{N}$, proving the moreover assertion. $\square$

4.3. **Proof of Corollary B.** Let $\mathcal{H}$ be an R-bimodule. By Remark 2.3, after replacing $\mathcal{H}$ with $\mathcal{H}\oplus\overline{\mathcal{H}}$, we may assume that $\mathcal{H}$ is symmetric. Then $\mathcal{H}\subset\mathrm{L}^2(\Gamma(R,\mathcal{H})'')$, as R-bimodules. Let $\mathcal{K}=\mathrm{L}^2(R)\otimes\mathrm{L}^2(R)$. Then by Proposition 4.2(1), $\Gamma(R,\mathcal{K})''=R*\mathrm{L}(\mathbb{Z})$. Further, applying Proposition 4.2(4) implies that $\Gamma(R,\mathcal{K}\otimes\ell^2(S))''=R*\mathrm{L}(\mathbb{F}_S)$, for every set S. Since R is hyperfinite, we have that $\mathcal{H}\subset_{\text{weak}}\mathcal{K}$. Theorem C thus provides a trace preserving $*$-homomorphism $\pi:\Gamma(R,\mathcal{H})''\to(R*\mathrm{L}(\mathbb{F}_S))^{\mathcal{U}}$ such that $\pi_{|R}=\mathrm{Id}_R$, for a set S and a cofinal ultrafilter $\mathcal{U}$ on a set I. Thus, $\mathrm{L}^2(\Gamma(R,\mathcal{H})'')\subset\mathrm{L}^2((R*\mathrm{L}(\mathbb{F}_S))^{\mathcal{U}})$ and therefore

$$\mathcal{H}\subset\mathrm{L}^2((R*\mathrm{L}(\mathbb{F}_S))^{\mathcal{U}}),\text{ as }R\text{-bimodules.}\tag{4.2}$$

To finish the proof, we will use the following fact (see [CP09]):

Fact 4.3. Let $\mathcal{V}_j$ be an ultrafilter on a set I_j, for all $j\in\{1,2\}$. Let $\mathcal{V}_1\otimes\mathcal{V}_2$ be the ultrafilter on $I_1\times I_2$ defined as follows: $\lim_{(i_1,i_2)\to\mathcal{V}_1\otimes\mathcal{V}_2}f(i_1,i_2)=\lim_{i_1\to\mathcal{V}_1}(\lim_{i_2\to\mathcal{V}_2}f(i_1,i_2))$, for every $f\in\ell^\infty(I_1\times I_2)$. Then for every tracial von Neumann algebra (N,τ), we have a trace preserving $*$-isomorphism $N^{\mathcal{V}_1\otimes\mathcal{V}_2}\cong(N^{\mathcal{V}_2})^{\mathcal{V}_1}$ given by $x\mapsto((x_{i_1,i_2})_{i_2\in I_2})_{i_1\in I_1}$, for $x=(x_{i_1,i_2})_{(i_1,i_2)\in I_1\times I_2}\in\ell^\infty(I_1\times I_2,N)$.

Let J be the collection of finite subsets $T\subset S$ ordered by inclusion, and $\mathcal{V}$ a cofinal ultrafilter on J. Then the map $R*\mathrm{L}(\mathbb{F}_S)\ni x\mapsto(\mathrm{E}_{R*\mathrm{L}(\mathbb{F}_T)}(x))_{T\in J}\in\prod_{\mathcal{V}}(R*\mathrm{L}(\mathbb{F}_T))$ is a trace preserving $*$-homomorphism. For every finite subset $T\subset S$, view $\mathbb{F}_T$ as a subgroup of $\mathbb{F}_\infty$. By combining the last two facts, we get a trace preserving $*$-homomorphism $\delta:R*\mathrm{L}(\mathbb{F}_S)\to(R*\mathrm{L}(\mathbb{F}_\infty))^{\mathcal{V}}$ such that $\delta_{|R}=\mathrm{Id}_R$. Since $\mathrm{L}(\mathbb{F}_\infty)$ is Connes embeddable, we can find a trace preserving $*$-homomorphism $\rho:R*\mathrm{L}(\mathbb{F}_\infty)\to R^{\mathcal{W}}$, with $\mathcal{W}$ a free ultrafilter on $\mathbb{N}$. Since any embedding of R into $R^{\mathcal{W}}$ is unitarily conjugate to the diagonal embedding, we may assume that $\rho_{|R}=\mathrm{Id}_R$. Thus, $\rho^{\mathcal{V}}:(R*\mathrm{L}(\mathbb{F}_\infty))^{\mathcal{V}}\to(R^{\mathcal{W}})^{\mathcal{V}}$ given by $\rho^{\mathcal{V}}((x_j)_{j\in J})=(\rho(x_j))_{j\in J}$ is a trace preserving $*$-homomorphism.

Define $\sigma := (\rho^{\mathcal{V}} \circ \delta)^{\mathcal{U}} : (R * \mathrm{L}(\mathbb{F}_S))^{\mathcal{U}} \to ((R^{\mathcal{W}})^{\mathcal{V}})^{\mathcal{U}}$. Using Fact 4.3, we can view σ as trace preserving $*$-homomorphism $\sigma : (R * \mathrm{L}(\mathbb{F}_S))^{\mathcal{U}} \to R^{\mathcal{U}\otimes\mathcal{V}\otimes W}$. Moreover, it is immediate that $\sigma_{|R} = \mathrm{Id}$. In particular, we derive that

$$\mathrm{L}^2((R * \mathrm{L}(\mathbb{F}_S))^{\mathcal{U}}) \subset \mathrm{L}^2(R^{\mathcal{U}\otimes\mathcal{V}\otimes W}), \text{ as } R\text{-bimodules.} \tag{4.3}$$

By combining (4.2) and (4.3), we conclude that $\mathcal{H} \subset \mathrm{L}^2(R^{\mathcal{U}\otimes\mathcal{V}\otimes\mathcal{W}})$, as R-bimodules. This proves the main assertion.

Finally, assume that $\mathcal{H}$ is separable. The moreover part of Theorem C implies that we can take $S = I = \mathbb{N}$. Then J is countable. Thus, $\mathcal{U}, \mathcal{V}$, $\mathcal{W}$, and consequently $\mathcal{U} \otimes \mathcal{V} \otimes \mathcal{W}$ are ultrafilters on countable sets. This implies the moreover assertion. □

We next use Theorem C and its proof to derive Corollary D.

4.4. **Proof of Corollary D.** By Remark 2.3, in order to prove (1) and (2), we may assume that $\mathcal{H}$ is a symmetric M-bimodule.

(1) Assume that $\mathcal{H}$ is a symmetric M-bimodule with $\mathcal{H} \subset_{\text{weak}} \mathrm{L}^2(M)$. By combining parts (1) and (4) of Proposition 4.2 we get that $\Gamma(M, \mathrm{L}^2(M) \otimes \ell^2(\mathbb{N}))'' = \widehat{M}$, where $\widehat{M} = M \overline{\otimes} \mathrm{L}(\mathbb{F}_\infty)$. By Theorem C, there exists a trace preserving $*$-homomorphism $\pi : \Gamma(M, \mathcal{H})'' \to \widehat{M}^{\mathcal{U}}$, where $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$, such that $\pi_{|M} = \mathrm{Id}_M$. This implies that $\mathcal{H} \subset \mathrm{L}^2(\Gamma(M, \mathcal{H})'') \subset \mathrm{L}^2(\widehat{M}^{\mathcal{U}})$, as M-bimodules.

Conversely, if $\mathcal{H} \subset \mathrm{L}^2(\widehat{M}^{\mathcal{U}})$, then the same argument as in the proof of the implication (2) ⇒ (3) from Theorem A shows that $\mathcal{H} \subset_{\text{weak}} \mathrm{L}^2(\widehat{M}) = \mathrm{L}^2(M) \otimes \ell^2(\mathbb{N})$ and therefore $\mathcal{H} \subset_{\text{weak}} \mathrm{L}^2(M)$.

(2) Assume that $\mathcal{H}$ is a symmetric M-bimodule with $\mathcal{H} \subset_{\text{weak}} \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$. By combining parts (2) and (4) of Proposition 4.2, we get that $\Gamma(M, (\mathrm{L}^2(M) \otimes \mathrm{L}^2(M)) \otimes \ell^2(\mathbb{N}))'' = \overline{M}$, where $\overline{M} = M * \mathrm{L}(\mathbb{F}_\infty)$. By Theorem C, there exists a $*$-homomorphism $\pi : \Gamma(M, \mathcal{H})'' \to \overline{M}^{\mathcal{U}}$ such that $\pi_{|M} = \mathrm{Id}_M$. This implies that $\mathcal{H} \subset \mathrm{L}^2(\Gamma(M, \mathcal{H})'') \subset \mathrm{L}^2(\overline{M}^{\mathcal{U}})$, as M-bimodules. Moreover, a close inspection of the proof of Theorem C (see the moreover assertion of Lemma 3.4) shows that $\mathcal{H} \subset \mathrm{L}^2(\overline{M}^{\mathcal{U}}) \ominus \mathrm{L}^2(M^{\mathcal{U}})$, as desired.

Conversely, assume that $\mathcal{H} \subset \mathrm{L}^2(\overline{M}^{\mathcal{U}}) \ominus \mathrm{L}^2(M^{\mathcal{U}})$. We claim that $\mathcal{H} \subset_{\text{weak}} \mathrm{L}^2(\overline{M}) \ominus \mathrm{L}^2(M)$. To see this, note that if $\varepsilon > 0$ and $\xi \in \mathrm{L}^2(\overline{M}^{\mathcal{U}}) \ominus \mathrm{L}^2(M^{\mathcal{U}})$ satisfies $\|\xi\|_2 \leq 1$, then we can find $\eta \in \overline{M}^{\mathcal{U}} \ominus M^{\mathcal{U}}$ such that $\|\eta\|_2 \leq 1$ and $\|\xi - \eta\|_2 \leq \varepsilon$. Moreover, we can represent $\eta = (\eta_i)$, where $\eta_i \in \overline{M} \ominus M$, for every i. The claim now follows by repeating the same argument as in the proof of the implication (2) ⇒ (3) from Theorem A.

On the other hand, we have $\mathrm{L}^2(\overline{M}) \ominus \mathrm{L}^2(M) \cong (\mathrm{L}^2(M) \otimes \mathrm{L}^2(M)) \otimes \ell^2(\mathbb{N})$, and thus the claim allows us to conclude that $\mathcal{H} \subset_{\text{weak}} \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$.

To prove (3) and (4), let $\mathcal{K} = \mathrm{L}^2(M) \otimes_Q \mathrm{L}^2(M)$ and recall that we defined $\widetilde{M} = M *_Q (Q \overline{\otimes} \mathrm{L}(\mathbb{Z}))$. Then $\widetilde{M} \cong \Gamma(M, \mathcal{K})''$ by Proposition 4.2(3).

(3) If P is amenable relative to Q inside M, by [PV11, Proposition 2.4] we find a net of subtracial vectors $(\xi_n) \subset \mathcal{K}$ such that we have $\lim_n \|\langle \xi_n, \xi_n \rangle_M - 1\|_1 = 0$ and $\lim_n \|y\xi_n - \xi_n y\| = 0$, for every $y \in P$. Moreover, since M is separable, we can take (ξ_n) to be a sequence.

Define $x_n = s(\xi_n) \in \widetilde{M}$. Since $\|s(\xi_n)\| \leq 2\|\langle \xi_n, \xi_n \rangle_M\| \leq 2$ and $\mathrm{E}_M(s(\xi_n)) = 0$, for every n, we get that $x = (x_n)$ belongs to $\widetilde{M}^{\mathcal{U}}$ and $\mathrm{E}_{M^{\mathcal{U}}}(x) = 0$. If $\Omega = \widehat{1} \in \mathcal{F}_M(\mathcal{K})$ is the vacuum vector, then $(yx_n - x_n y)\Omega = y\xi_n - \xi_n y$ and thus $\lim_n \|yx_n - x_n y\|_2 = 0$, for every $y \in P$. This implies that

$x \in P' \cap \widetilde{M}^{\mathcal{U}}$. Since $\langle \xi_n, \xi_n \rangle_M \leq 1$, we get that $\|\langle \xi_n, \xi_n \rangle_M - 1\|_2 \leq 2\|\langle \xi_n, \xi_n \rangle_M - 1\|_1^{1/2}$ and thus $\|\langle \xi_n, \xi_n \rangle_M - 1\|_2 \to 0$. Note that for every $y \in M$, we have

$$\tau(x_n^* x_n y) = \tau(s(\xi_n)^* s(\xi_n) y) = \langle s(\xi_n)^* s(\xi_n) y\Omega, \Omega\rangle = \langle \xi_n y, \xi_n \rangle = \tau(\langle \xi_n, \xi_n \rangle_M y)$$

Thus we have $\mathrm{E}_M(x_n^* x_n) = \langle \xi_n, \xi_n \rangle_M$, from which we get that $\mathrm{E}_{M^{\mathcal{U}}}(x^*x) = (\mathrm{E}_M(x_n^* x_n)) = 1$. This proves the only if assertion.

Conversely, assume that there exists $x \in P' \cap \widetilde{M}^{\mathcal{U}}$ such that $\mathrm{E}_{M^{\mathcal{U}}}(x) = 0$ and $\mathrm{E}_{M^{\mathcal{U}}}(x^*x) = 1$. Then we can find a net $(x_n) \subset \widetilde{M} \ominus M$ such that $\sup \|x_n\| < \infty$, $\lim_n \|\mathrm{E}_M(x_n^* x_n) - 1\|_2 = 0$ and $\lim_n \|y x_n - x_n y\|_2 = 0$, for every $y \in P$. This gives that $\tau(x_n^* y x_n z) = \tau(yz)$, for every $y \in M$ and $z \in P$. Thus, the M-P-bimodule $\mathrm{L}^2(M)$ is weakly contained in $\mathrm{L}^2(\widetilde{M}) \ominus \mathrm{L}^2(M)$. On the other hand, [Io12, Lemma 2.10 (1)] gives that $\mathrm{L}^2(\widetilde{M}) \ominus \mathrm{L}^2(M) = \mathrm{L}^2(M) \otimes_Q \mathcal{L}$, as M-bimodules, for a Q-M-bimodule $\mathcal{L}$. By combining the last two facts we derive that M-P-bimodule $\mathrm{L}^2(M)$ is weakly contained in $\mathrm{L}^2(M) \otimes_Q \mathcal{L}$. By applying [PV11, Proposition 2.4, (4) ⇒ (1)], we get that the M-Q-bimodule $\mathrm{L}^2(M)$ is left P-amenable, in the sense of [PV11, Definition 2.3]. As explained in the paragraph following [PV11, Definition 2.3], this gives that P is amenable relative to Q inside M.

(4) Assume that there exists a $*$-homomorphism $\pi : \widetilde{M} \to \widehat{M}^{\mathcal{U}}$ whose restriction to M is the diagonal embedding of M. Then $\mathrm{L}^2(\widetilde{M}) \subset \mathrm{L}^2(\widehat{M}^{\mathcal{U}})$, as M-bimodules. If $u \in \mathrm{L}(\mathbb{Z}) \subset \widetilde{M}$ is a Haar unitary, then the M-bimodule $\overline{\text{span}(MuM)}$ is isomorphic to $\mathrm{L}^2(M) \otimes_Q \mathrm{L}^2(M)$. This follows by noticing that $\tau(xuyu^*) = \tau(\mathrm{E}_Q(x)y)$, for every $x, y \in M$, and so $\overline{\text{span}(MuM)} \cong \mathcal{H}_{\mathrm{E}_Q} \cong \mathrm{L}^2(M) \otimes_Q \mathrm{L}^2(M)$. Thus, $\mathrm{L}^2(M) \otimes_Q \mathrm{L}^2(M) \subset \mathrm{L}^2(\widehat{M}^{\mathcal{U}})$, which implies that $\mathrm{L}^2(M) \otimes_Q \mathrm{L}^2(M) \subset_{\text{weak}} \mathrm{L}^2(\widehat{M}) = \mathrm{L}^2(M) \otimes \ell^2(\mathbb{N})$. Hence $\mathrm{L}^2(M) \otimes_Q \mathrm{L}^2(M) \subset_{\text{weak}} \mathrm{L}^2(M)$, as claimed.

Conversely, assume that $\mathcal{K} \subset_{\text{weak}} \mathrm{L}^2(M)$. Since $\Gamma(M, \mathcal{K})'' = \widetilde{M}$, $\Gamma(M, \mathrm{L}^2(M) \otimes \ell^2(\mathbb{N}))'' = \widehat{M}$ and $\mathcal{K} = \mathcal{H}_{\mathrm{E}_Q}$, by combining Lemma 2.8 and Lemma 3.4 we deduce the existence of a trace preserving $*$-homomorphism $\pi : \widetilde{M} \to \widehat{M}^{\mathcal{U}}$ such that $\pi_{|M} = \mathrm{Id}_M$. □

4.5. **Proof of Theorem E.** Let M_0 be a separable II_1 factor. We will construct an existentially closed separable II_1 factor $M \supset M_0$. To this end, let $X = \{x_k \mid k \in \mathbb{N}\} \subset (M_0)_1$ be a sequence which generates M_0 and $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. The proof relies on the following claim:

Claim 4.4. *There exists a separable II_1 factor $M_1 \supset M_0$ such that the following holds: for any separable tracial von Neumann algebra $N \supset M_0$, there is an embedding of N into $M_1^{\mathcal{U}}$ whose restriction to M_0 is the diagonal embedding.*

Proof of Claim 4.4. We denote by $\mathcal{F}$ the set of pairs (N, Y), where N is a separable tracial von Neumann algebra which contains M_0 and $Y = X \cup \{y_k \mid k \in \mathbb{N}\} \subset (N)_1$ is a sequence which generates N and contains X. For $m \in \mathbb{N}$, we denote by $\mathcal{P}_m$ the set of $*$-monomials in the variables $\{X_1, \cdots, X_m, Y_1, \cdots, Y_m\}$ of degree at most m. For $(N, Y) \in \mathcal{F}$, $m \in \mathbb{N}$ and $\varepsilon > 0$, we denote by $\mathcal{U}_{(N,Y)}(m, \varepsilon)$ the set of $(\bar{N}, \bar{Y}) \in \mathcal{F}$ such that writing $\bar{Y} = X \cup \{\bar{y}_k \mid k \in \mathbb{N}\}$ we have:

$$\sum_{p \in \mathcal{P}_m} |\tau_{\bar{N}}(p(x_1, ..., x_m, \bar{y}_1, ..., \bar{y}_m)) - \tau_N(p(x_1, ..., x_m, y_1, ..., y_m))| < \varepsilon.$$

Consider the topology on $\mathcal{F}$ which has $\{\mathcal{U}_{(N,Y)}(m, \varepsilon)\}_{m \in \mathbb{N}, \varepsilon > 0}$ as a neighborhood basis of any $(N, Y) \in \mathcal{F}$. Note that the elements of $\mathcal{F}$ can be viewed as representatives of non-commutative laws on infinitely many variables; the topology we defined on $\mathcal{F}$ is precisely the weak*-topology on the space of laws. Since $\mathcal{F}$ is separable in this topology, it admits a dense sequence $\{(N_n, Y_n)\}_{n \in \mathbb{N}}$. Define $M_1 = *_{M_0, n \in \mathbb{N}} N_n$. Write $Y_n = X \cup \{y_{n,k} \mid k \in \mathbb{N}\}$, for every $n \in \mathbb{N}$.

Let N be a separable tracial von Neumann algebra which contains M_0. Choose $Y \subset (N)_1$ such that $(N, Y) \in \mathcal{F}$. Then (N, Y) is the limit of a subsequence (N_{n_l}, Y_{n_l}) of (N_n, Y_n). Thus, we have

$$\tau_{N_{n_l}}(p(x_1, ..., x_m, y_{n_l,1}, ..., y_{n_l,m})) \to \tau_N(p(x_1, ..., x_m, y_1, ..., y_m)), \text{ as } l \to \infty,$$

for every $m \in \mathbb{N}$ and $*$-monomial p in $X_1, ..., X_m, Y_1, ..., Y_m$.

Since $\cup_{n\in\mathbb{N}} Y_n \subset M_1$, Lemma 2.9 gives a trace preserving $*$-homomorphism $\pi : N \to M_1^{\mathcal{U}}$ such that $\pi(p(x_1, ..., x_m, y_1, ..., y_m)) = (p(x_1, ..., x_m, y_{n_l,1}, ..., y_{n_l,m}))$, for every $m \in \mathbb{N}$ and $*$-monomial p in $X_1, ..., X_m, Y_1, ..., Y_m$. In particular, $\pi(x_k) = x_k$, for every $k \in \mathbb{N}$, and thus the restriction of π to M_0 is the diagonal embedding of M_0 into $M_1^{\mathcal{U}}$. Moreover, since this property still holds if we replace M_1 by $M_1 * \mathrm{L}(\mathbb{Z})$, we can assume that M_1 is a II_1 factor. Here we are using the fact that $P * \mathrm{L}(\mathbb{Z})$ is a II_1 factor for any non-trivial tracial von Neumann algebra P. □

By Claim 4.4, we can inductively construct an increasing sequence of II_1 factors $M_n \supset M_0$, $n \geq 1$, such that for every $n \geq 0$ and any separable tracial von Neumann algebra $N \supset M_n$, there is an embedding of N into $M_{n+1}^{\mathcal{U}}$ whose restriction to M_n is the diagonal embedding.

Let $M = (\bigcup_{n\in\mathbb{N}} M_n)''$. Then $M \supset M_0$ is a separable II_1 factor. We claim that M is existentially closed. Let N be a II_1 factor containing M. Let $\{z_k \mid k \in \mathbb{N}\} \subset (M)_1$ and $\{t_m \mid m \in \mathbb{N}\} \subset (N)_1$ be sequences which generate M and N, respectively. We may assume that $z_k \in M_k$, for every $k \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and denote by P the von Neumann subalgebra of N generated by M and $\{t_1, \cdots, t_n\}$. Since $M_n \subset M \subset P$ and P is separable there is a trace preserving $*$-homomorphism $\pi : P \to M_{n+1}^{\mathcal{U}}$ such that $\pi(x) = x$, for every $x \in M_n$. In particular, $\pi(z_k) = z_k$, for every $1 \leq k \leq n$. Write $\pi(t_m) = (t_{l,m})$, where $t_{l,m} \in (M_{n+1})_1$, for every $l \in \mathbb{N}$ and $1 \leq m \leq n$. Since π is trace preserving, for every $*$-polynomial p in variables $Z_k, T_m, 1 \leq k, m \leq n$, we have

$$\lim_{l\to\mathcal{U}} \tau_M(p(z_k, t_{l,m}, 1 \leq k, m \leq n)) = \tau_N(p(z_k, t_m, 1 \leq k, m \leq n)).$$

This implies that we can find $l \in \mathbb{N}$ such that denoting $u_{n,m} := t_{l,m}$, for every $1 \leq m \leq n$, we have

$$|\tau_M(p(z_k, u_{n,m}, 1 \leq k, m \leq n)) - \tau_N(p(z_k, t_m, 1 \leq k, m \leq n))| < \frac{1}{n},$$

for every $*$-monomial p of degree $\leq n$ in variables $Z_k, X_m, 1 \leq k, m \leq n$.

Then $\lim_n \tau_M(p(z_k, u_{n,m}, k, m \in \mathbb{N})) = \tau_N(p(z_k, t_m, k, m \in \mathbb{N}))$, for every $*$-monomial p in variables $Z_k, X_m, k, m \in \mathbb{N}$. Since the set $\{z_k \mid k \in \mathbb{N}\} \cup \{t_m \mid m \in \mathbb{N}\}$ generates N, Lemma 2.9 gives a trace preserving $*$-homomorphism $\rho : N \to M^{\mathcal{V}}$ such that $\rho(z_k) = z_k$ and $\rho(t_m) = (u_{n,m})$, for every $k, m \in \mathbb{N}$. Since the set $\{z_k \mid k \in \mathbb{N}\}$ generates M, the restriction of ρ to M is the diagonal embedding into $M^{\mathcal{V}}$. Thus, M is existentially closed in N, which finishes the proof. □

4.6. **Proof of Theorem F.** Our next goal is to prove Theorem F. We first establish the following lemma.

Lemma 4.5. *Let $(M_1, \tau_1), (M_2, \tau_2)$ be tracial von Neumann algebras with a common von Neumann subalgebra (B, τ). Assume that $\mathrm{L}^2(M_i) \ominus \mathrm{L}^2(B) \subset_{\text{weak}} \mathrm{L}^2(B) \otimes \mathrm{L}^2(B)$, as B-bimodules, for every $i \in \{1, 2\}$. Then $\mathrm{L}^2(M_1 *_B M_2) \ominus \mathrm{L}^2(B) \subset_{\text{weak}} \mathrm{L}^2(B) \otimes \mathrm{L}^2(B)$, as B-bimodules.*

Proof. If two B-bimodules $\mathcal{H}_1$ and $\mathcal{H}_2$ are weakly contained in $\mathrm{L}^2(B) \otimes \mathrm{L}^2(B)$, then so is $\mathcal{H}_1 \otimes_B \mathcal{H}_2$. Indeed, the Connes tensor product preserves weak containment of bimodules (see [AP18, Exercise 13.15]) and the B-bimodule $(\mathrm{L}^2(B) \otimes \mathrm{L}^2(B)) \otimes_B (\mathrm{L}^2(B) \otimes \mathrm{L}^2(B))$ is a multiple of $\mathrm{L}^2(B) \otimes L^2(B)$.

Since the B-bimodule $\mathrm{L}^2(M_1 *_B M_2) \ominus \mathrm{L}^2(B)$ decomposes as

$$\bigoplus_{n\geq 1} \bigoplus_{\substack{i_1,i_2,\cdots,i_n\in\{1,2\}\\ i_1\neq i_2\neq\cdots\neq i_n}} (\mathrm{L}^2(M_{i_1}) \ominus \mathrm{L}^2(B)) \otimes_B \cdots \otimes_B (\mathrm{L}^2(M_{i_n}) \ominus \mathrm{L}^2(B))$$

(see [Vo83] and [Po91, Section 3]), the conclusion follows. □

Proof of Theorem F. Let M_0 be a separable II$_1$ factor. After replacing M_0 by $M_0 * \mathrm{L}(\mathbb{Z})$, we may assume that M_0 is non-Gamma. We will construct a non-Gamma separable II$_1$ factor M containing M_0 such that M is existentially closed in any II$_1$ factor N which contains M and satisfies that $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M) \subset_{\text{weak}} \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$, as M-bimodules. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$.

The rest of the proof follows closely the proof of Theorem E, whose notation we keep here. Let $\mathcal{G}$ be the subset of $\mathcal{F}$ consisting of $(N, Y) \in \mathcal{F}$ such that $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M_0) \subset_{\text{weak}} \mathrm{L}^2(M_0) \otimes \mathrm{L}^2(M_0)$ as M_0-bimodules. Since $\mathcal{G}$ is separable in the topology defined in the proof of Theorem E, it admits a dense sequence $\{(N_n, Y_n)\}_{n\in\mathbb{N}}$. Define $M_1 = (*_{M_0,n\in\mathbb{N}} N_n) * \mathrm{L}(\mathbb{Z})$. Then M_1 is a II$_1$ factor. Using Lemma 4.5 and induction implies that $\mathrm{L}^2(M_1) \ominus \mathrm{L}^2(M_0) \subset_{\text{weak}} \mathrm{L}^2(M_0) \otimes \mathrm{L}^2(M_0)$, as M_0-bimodules.

The proof of Claim 4.4 shows that for every separable tracial von Neumann algebra $N \supset M_0$ such that $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M_0) \subset_{\text{weak}} \mathrm{L}^2(M_0) \otimes \mathrm{L}^2(M_0)$ as M_0-bimodules, there is a trace preserving $*$-homomorphism $\pi : N \to M_1^{\mathcal{U}}$ whose restriction to M_0 is the diagonal embedding of M_0 into $M_1^{\mathcal{U}}$.

We then inductively construct an increasing sequence of II$_1$ factors M_n, $n \geq 1$, containing M_0 such that for every $n \geq 0$ the following properties hold:

1. For any separable tracial von Neumann algebra N which contains M_n and satisfies that $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M_n) \subset_{\text{weak}} \mathrm{L}^2(M_n) \otimes \mathrm{L}^2(M_n)$, as M_n-bimodules, there is a trace preserving $*$-homomorphism $\pi : N \to M_{n+1}^{\mathcal{U}}$ whose restriction to M_n is the diagonal embedding of M_n.
2. $\mathrm{L}^2(M_{n+1}) \ominus \mathrm{L}^2(M_n) \subset_{\text{weak}} \mathrm{L}^2(M_n) \otimes \mathrm{L}^2(M_n)$, as M_n bimodules.

Then $M = (\bigcup_{n\in\mathbb{N}} M_n)''$ is a separable II$_1$ factor which contains M_0. We will prove that M has the desired properties.

First, let $N \supset M$ be a II$_1$ factor with $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M) \subset_{\text{weak}} \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$, as M-bimodules. We claim that M is existentially closed in N. Let $\{t_m \mid m \in \mathbb{N}\} \subset (N)_1$ be a sequence which generates N. Let $n \in \mathbb{N}$ and denote by $P \subset N$ the von Neumann subalgebra generated by M and $\{t_1, \cdots, t_n\}$. For every $k \geq n$, since $M_n \subset M_k$, $\mathrm{L}^2(M_k)$ is both a sub-module of a multiple of the left M_n-module $\mathrm{L}^2(M_n)$ and a sub-module of a multiple of the right M_n-module $\mathrm{L}^2(M_n)$. Therefore, the M_n-bimodule $\mathrm{L}^2(M_k) \otimes \mathrm{L}^2(M_k)$ is a sub-bimodule of a multiple of the coarse M_n-bimodule. In combination with (2), we get that $\mathrm{L}^2(M_{k+1}) \ominus \mathrm{L}^2(M_k) \subset_{\text{weak}} \mathrm{L}^2(M_n) \otimes \mathrm{L}^2(M_n)$, as M_n-bimodules, for every $k \geq n$. Since we also have $\mathrm{L}^2(M) \ominus \mathrm{L}^2(M_n) = \bigoplus_{k\geq n}(\mathrm{L}^2(M_{k+1}) \ominus \mathrm{L}^2(M_k))$, we get $\mathrm{L}^2(M) \ominus \mathrm{L}^2(M_n) \subset_{\text{weak}} \mathrm{L}^2(M_n) \otimes \mathrm{L}^2(M_n)$, as M_n-bimodules. Since $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M) \subset_{\text{weak}} \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$, as M-bimodules, and $M_n \subset M$, we also get that $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M) \subset_{\text{weak}} \mathrm{L}^2(M_n) \otimes \mathrm{L}^2(M_n)$, as M_n-bimodules. Using that $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M_n) = (\mathrm{L}^2(N) \ominus \mathrm{L}^2(M)) \oplus (\mathrm{L}^2(M) \ominus \mathrm{L}^2(M_n))$ and the last two inclusions, we further derive that $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M_n) \subset_{\text{weak}} \mathrm{L}^2(M_n) \otimes \mathrm{L}^2(M_n)$, and hence $\mathrm{L}^2(P) \ominus \mathrm{L}^2(M_n) \subset_{\text{weak}} \mathrm{L}^2(M_n) \otimes \mathrm{L}^2(M_n)$, as M_n-bimodules. We can now apply (1) to get a trace preserving $*$-homomorphism $\pi : P \to M_{n+1}^{\mathcal{U}}$ whose restriction to M_n is the diagonal embedding of M_n into $M_{n+1}^{\mathcal{U}}$. Proceeding as in the proof of Theorem E gives that M is existentially closed in N.

Second, assume by contradiction that M has property Gamma. Then $M' \cap M^{\mathcal{U}}$ is diffuse, hence there is a unitary $u \in M' \cap M^{\mathcal{U}}$ of trace zero. Since M_0 is non-Gamma, $M_0' \cap M_0^{\mathcal{U}} = \mathbb{C}1$. Since $\mathrm{E}_{M_0^{\mathcal{U}}}(u) \in M_0' \cap M_0^{\mathcal{U}}$, it follows that $\mathrm{E}_{M_0^{\mathcal{U}}}(u) = \tau(u)1 = 0$. Thus, we get that $u \in M^{\mathcal{U}} \ominus M_0^{\mathcal{U}}$. Since u commutes with M_0, we deduce the existence of a tracial M_0-central vector in $\mathrm{L}^2(M^{\mathcal{U}}) \ominus \mathrm{L}^2(M_0^{\mathcal{U}})$, that is, $\mathrm{L}^2(M_0)$ is contained in the M_0-bimodule $\mathrm{L}^2(M^{\mathcal{U}})\ominus\mathrm{L}^2(M_0^{\mathcal{U}})$. As in the proof of part (2) of Corollary D it follows that $\mathrm{L}^2(M_0)\subset_{\text{weak}}\mathrm{L}^2(M)\ominus\mathrm{L}^2(M_0)$, as M_0-bimodules. On the other hand, (2) implies that $\mathrm{L}^2(M_{n+1})\ominus\mathrm{L}^2(M_n)\subset_{\text{weak}}\mathrm{L}^2(M_0)\otimes\mathrm{L}^2(M_0)$, as M_0-bimodules, for every $n\geq 0$. Since $\mathrm{L}^2(M)\ominus\mathrm{L}^2(M_0)=\bigoplus_{n\geq 0}(\mathrm{L}^2(M_{n+1})\ominus\mathrm{L}^2(M_n))$, we derive that $\mathrm{L}^2(M)\ominus\mathrm{L}^2(M_0)\subset_{\text{weak}}\mathrm{L}^2(M_0)\otimes\mathrm{L}^2(M_0)$, as M_0-bimodules. Combining these facts gives that $\mathrm{L}^2(M_0)\subset_{\text{weak}}\mathrm{L}^2(M_0)\otimes\mathrm{L}^2(M_0)$, as M_0-bimodules. In other words, M_0 is amenable (see [AP18, Theorem 13.4.1]), which contradicts the fact that M_0 is non-Gamma. □

We continue by proving the following result mentioned in Remark 1.5.

Lemma 4.6. *Let $M=\mathrm{L}(\mathbb{F}_n)$, for some $n\geq 1$, and $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Then we have that $\mathrm{L}^2(M^{\mathcal{U}})\ominus\mathrm{L}^2(M)\subset_{\text{weak}}\mathrm{L}^2(M)\otimes\mathrm{L}^2(M)$, as M-bimodules.*

Proof. The proof uses Popa's malleable deformation of M into $\widetilde{M}=M*M$. More precisely, by [Po86, Po06] there exist automorphisms $(\alpha_t)_{t\in\mathbb{R}}$ of $\widetilde{M}$ such that

(1) $\lim_{t\to 0}\|\alpha_t(x)-x\|_2=0$, for every $x\in M$, and
(2) the map $M\ni x\mapsto \mathrm{E}_M(\alpha_t(x))\in M$ extends to a compact operator on $\mathrm{L}^2(M)$, for any $t\neq 0$.

To prove the conclusion it suffices to show that $\overline{\text{span}(M\xi M)}\subset_{\text{weak}}\mathrm{L}^2(M)\otimes\mathrm{L}^2(M)$, as M-bimodules, for every $\xi\in\mathrm{L}^2(M^{\mathcal{U}})\ominus\mathrm{L}^2(M)$ with $\|\xi\|_2\leq 1$. To this end, let $F\subset(M)_1$ be a finite set and $\varepsilon>0$. Let $\eta\in M^{\mathcal{U}}\ominus M$ such that $\|\xi-\eta\|_2<\frac{\varepsilon}{4}$ and $\|\eta\|_2\leq 1$. Then

$$|\langle x\eta y,\eta\rangle-\langle x\xi y,\xi\rangle|<\frac{\varepsilon}{2},\text{ for every }x,y\in F. \tag{4.4}$$

By (1), we can find $t>0$ such that $\|\alpha_t(x)-x\|_2<\frac{\varepsilon}{4(\|\eta\|+1)}$, for every $x\in F$. Since $F\subset(M)_1$ and $\|\eta\|_2\leq 1$, we get that

$$|\langle\alpha_t(x)\eta\alpha_t(y),\eta\rangle-\langle x\eta y,\eta\rangle|\leq(\|\alpha_t(x)-x\|_2+\|\alpha_t(y)-y\|_2)\|\eta\|<\frac{\varepsilon}{2},\text{ for every }x,y\in F. \tag{4.5}$$

Write $\eta=(\eta_n)$, where $(\eta_n)\subset M$ is a sequence with $\sup\|\eta_n\|<\infty$. Since $\eta\in M^{\mathcal{U}}\ominus M$, we have that $\lim_{n\to\mathcal{U}}\eta_n=0$, weakly. Then (2) implies that $\lim_{n\to\mathcal{U}}\|\alpha_{-t}(\eta_n)\|_2=0$. Thus, denoting $\zeta_n=\alpha_{-t}(\eta_n)-\mathrm{E}_M(\alpha_{-t}(\eta_n))\in\widetilde{M}\ominus M$ we derive that for every $x,y\in M$ we have that

$$\langle\alpha_t(x)\eta\alpha_t(y),\eta\rangle=\lim_{n\to\mathcal{U}}\langle\alpha_t(x)\eta_n\alpha_t(y),\eta_n\rangle=\lim_{n\to\mathcal{U}}\langle x\alpha_{-t}(\eta_n)y,\alpha_{-t}(\eta_n)\rangle=\lim_{n\to\mathcal{U}}\langle x\zeta_n y,\zeta_n\rangle. \tag{4.6}$$

By combining (4.4), (4.5) and (4.6), we deduce that there is $n\in\mathbb{N}$ such that $\zeta=\zeta_n\in\widetilde{M}\ominus M$ satisfies $|\langle x\zeta y,\zeta\rangle-\langle x\xi y,\xi\rangle|<\varepsilon$, for every $x,y\in F$. This shows that $\overline{\text{span}(M\xi M)}\subset_{\text{weak}}\mathrm{L}^2(\widetilde{M})\ominus\mathrm{L}^2(M)$, as M-bimodules. Since the M-bimodule $\mathrm{L}^2(\widetilde{M})\ominus\mathrm{L}^2(M)$ is isomorphic to $(\mathrm{L}^2(M)\otimes\mathrm{L}^2(M))\otimes\ell^2(\mathbb{N})$, the conclusion follows. □

4.7. **Proof of Theorem G.** Let M be a separable II_1 factor.

Assume first that M has Haagerup's property. By applying [OOT15, Theorem 9] or [BF07, Theorem 3.4] we derive the existence of a strictly mixing M-bimodule $\mathcal{H}$ such that $\mathrm{L}^2(M)\subset_{\text{weak}}\mathcal{H}$. By using Lemma 2.5 and after replacing $\mathcal{H}$ with $\mathcal{H}^{\oplus\infty}$, we get a strictly mixing M-bimodule $\mathcal{H}$ admitting a sequence of subtracial vectors (η_n) such that $\lim_n\langle x\eta_n y,\eta_n\rangle=\tau(xy)$, for every $x,y\in M$. Next, note that the M-bimodule $\mathcal{K}:=\mathcal{H}\otimes_M\overline{\mathcal{H}}$ is symmetric, as witnessed by the involution $J(\xi\otimes_M\overline{\zeta})=\zeta\otimes_M\overline{\xi}$, and strictly mixing (see [OOT15, Proposition 7]). Then the vectors $\xi_n:=\eta_n\otimes_M\overline{\eta}_n\in\mathcal{K}$ are subtracial and satisfy $\lim_n\langle x\xi_n y,\xi_n\rangle=\tau(xy)$, for every $x,y\in M$.

Let $\widetilde{M}=\Gamma(M,\mathcal{K})''$. Then $\widetilde{M}$ is a factor (see [KV15], Theorem 5.1). We will prove that $\widetilde{M}$ has the desired properties. First, note that the M-bimodule $\mathrm{L}^2(\widetilde{M})\ominus\mathrm{L}^2(M)$ is isomorphic to

$\mathcal{F}_M(\mathcal{K}) \ominus \mathrm{L}^2(M)$ and thus to $\mathcal{K} \otimes_M \mathcal{F}_M(\mathcal{K})$. Since $\mathcal{K}$ is strictly mixing, so is $\mathrm{L}^2(\widetilde{M}) \ominus \mathrm{L}^2(M)$ by [OOT15, Proposition 7]. Second, if $\Omega = 1 \in \mathrm{L}^2(M) \subset \mathcal{F}_M(\mathcal{K})$ is the vacuum vector, then $(xs(\xi_n) - s(\xi_n)x)\Omega = x\xi_n - \xi_n x$, for every n, and thus

$$\lim_n \|xs(\xi_n) - s(\xi_n)x\|_2 = \lim_n \|(xs(\xi_n) - s(\xi_n)x)\Omega\| = \lim_n \|x\xi_n - \xi_n x\| = 0, \text{ for every } x \in M.$$

Since ξ_n is subtracial, $\langle \xi_n, \xi_n \rangle_M \leq 1$ and so $\|s(\xi_n)\| \leq 2$, for every n. Thus, $(s(\xi_n)) \in M' \cap \widetilde{M}^{\mathcal{U}}$. Since $\mathrm{E}_M(s(\xi_n)) = 0$ and $\lim_n ||s(\xi_n)||_2 = \lim_n ||s(\xi_n)\Omega|| = \lim_n \|\xi_n\| = 1$, we derive that $(s(\xi_n)) \notin M^{\mathcal{U}}$.

Conversely, assume that there exists a separable II_1 factor $\widetilde{M} \supset M$ such that the M-bimodule $\mathrm{L}^2(\widetilde{M}) \ominus \mathrm{L}^2(M)$ is strictly mixing and $M' \cap \widetilde{M}^{\mathcal{U}} \not\subset M^{\mathcal{U}}$. Let $y = (y_n) \in (M' \cap \widetilde{M}^{\mathcal{U}}) \setminus M^{\mathcal{U}}$. Then $z = y - \mathrm{E}_{M^{\mathcal{U}}}(y) \neq 0$ and $z = (z_n)$, where $z_n = y_n - \mathrm{E}_M(y_n) \in \mathrm{L}^2(\widetilde{M}) \ominus \mathrm{L}^2(M)$. Since $z \in M' \cap \widetilde{M}^{\mathcal{U}}$, we get that $\lim_{n\to\mathcal{U}} \|xz_n - z_n x\|_2 = 0$, for every $x \in M$. Since M is a factor, and $\mathrm{E}_M(zz^*) \in M' \cap M$, we get that $\mathrm{E}_M(zz^*) = \tau(zz^*)1$. Thus, for every $x \in M$ we have that

$$\lim_{n\to\mathcal{U}} \langle xz_n, z_n \rangle = \lim_{n\to\mathcal{U}} \tau(xz_n z_n^*) = \tau(xzz^*) = \tau(x\mathrm{E}_M(zz^*)) = \tau(x)\tau(zz^*).$$

It follows that if we let $\zeta_n = \tau(zz^*)^{-\frac{1}{2}} z_n \in \mathrm{L}^2(\widetilde{M}) \ominus \mathrm{L}^2(M)$, then $\lim_{n\to\mathcal{U}} \|x\zeta_n - \zeta_n x\|_2 = 0$ and $\lim_{n\to\mathcal{U}} \langle x\zeta_n, \zeta_n \rangle = \tau(x)$, for every $x \in M$. This implies that $\mathrm{L}^2(M) \subset_{\text{weak}} \mathrm{L}^2(\widetilde{M}) \ominus \mathrm{L}^2(M)$. Since the M-bimodule $\mathrm{L}^2(\widetilde{M}) \ominus \mathrm{L}^2(M)$ is also strictly mixing, applying [OOT15, Theorem 9] or [BF07, Theorem 3.4] gives that M has Haagerup's property. □

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Department of Mathematics, University of California San Diego, 9500 Gilman Drive, La Jolla, CA 92093, USA

Email address: `aioana@ucsd.edu`

Department of Mathematics, University of California San Diego, 9500 Gilman Drive, La Jolla, CA 92093, USA

Email address: `hutan@ucsd.edu`