NON-ISOMORPHISM OF $A^{*n}, 2 \leq n \leq \infty$, FOR A NON-SEPARABLE ABELIAN VON NEUMANN ALGEBRA A

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Abstract. We prove that if A is a non-separable abelian tracial von Neuman algebra then its free powers $A^{*n}, 2 \leq n \leq \infty$, are mutually non-isomorphic and with trivial fundamental group, $\mathcal{F}(A^{*n}) = 1$, whenever $2 \leq n < \infty$. This settles the non-separable version of the free group factor problem.

1. Introduction

The free group factor problem, asking whether the II_1 factors $L\mathbb{F}_n$ arising from the free groups with n generators $\mathbb{F}_n$, $2 \leq n \leq \infty$, are isomorphic or not, is perhaps the most famous in operator algebras, being in a way emblematic for this area, broadly known even outside of it.

It is generally believed that the free group factors are not isomorphic. Since $L\mathbb{F}_n = L\mathbb{Z} * \cdots * L\mathbb{Z}$, this amounts to $A^{*n}, 2 \leq n \leq \infty$, being non-isomorphic, where $A = L\mathbb{Z}$ is the unique (up to isomorphism) separable diffuse abelian von Neumann algebra. Due to work in [Rad94, Dyk94], based on Voiculescu's free probability methods, this is also equivalent to the fundamental group of A^{*n} being trivial for some (equivalently, all) $2 \leq n < \infty$, $\mathcal{F}(A^{*n}) = 1$.

We study here the non-separable version of the free group factor problem, asking whether the II_1 factors $A^{*n}, 2 \leq n \leq \infty$, are non-isomorphic when A is an abelian but non-separable von Neumann algebra (always assumed tracial, i.e., endowed with a given normal faithful trace). Examples of such algebras A include the ultrapower von Neumann algebra $(L\mathbb{Z})^\omega$ and the group von Neumann algebra LH, where ω is a free ultrafilter on $\mathbb{N}$ and H is an uncountable discrete abelian group, such as $\mathbb{R}$ or $\mathbb{Z}^\omega$. We obtain the following affirmative answer to the problem:

Theorem 1.1. *Let A be a diffuse non-separable abelian tracial von Neumann algebra.*

Then the II_1 factors $A^{*n}, 2 \leq n \leq \infty$, are mutually non-isomorphic, and have trivial fundamental group, $\mathcal{F}(A^{*n}) = 1$, whenever $2 \leq n < \infty$.

In other words, if the abelian components of a free product A^{*n} are being "magnified" from separable to non-separable, then the corresponding II_1 factors do indeed remember the number of terms involved. One should note that if $2 \leq n \leq \infty$, then any II_1 factor A^{*n}, with A diffuse abelian, is an inductive limit of subfactors isomorphic to $L\mathbb{F}_n$.

To prove Theorem 1.1, we show that the II_1 factors of the form $M = A_1 * \cdots * A_n$, with $A_1, A_2, \cdots, A_n$ non-separable abelian, have a remarkably rigid structure. Specifically, we prove that given any unital abelian von Neumann subalgebra $B \subset pMp$ that is purely non-separable (i.e., has no separable direct summand) and singular (i.e., has trivial normalizer), there is a partition of p into projections $p_i \in B$ such that Bp_i is unitarily conjugate to a direct summand of A_i, for every $1 \leq i \leq n$ (see Corollary 3.7). This implies that the family $\{A_i p_i\}_i$, consisting of the maximal purely non-separable direct summands of A_i, $1 \leq i \leq n$, coincides with the *sans-core* of M, a term we use to denote the maximal family $\mathcal{A}_M^{ns} = \{B_j\}_j$ of pairwise disjoint,

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RB is supported by ANR grant AODynG 19-CE40-0008.

DD holds the postdoctoral fellowship fundamental research 12T5221N of the Research Foundation Flanders.

AI was supported by NSF DMS grants 1854074 and 2153805, and a Simons Fellowship.

SP Supported in part by NSF Grant DMS-1955812 and the Takesaki Endowed Chair at UCLA.

singular, purely non-separable abelian subalgebras B_j of M. The uniqueness (up to unitary conjugacy, cutting and gluing) of this family ensures that the *sans-rank* of M, defined by

$$\mathrm{r}_{ns}(M) := \sum_j \tau(1_{B_j}) \in [0, +\infty],$$

is an isomorphism invariant for M. This shows in particular that if A is a diffuse non-separable abelian von Neumann algebra and Ap is its maximal purely non-separable direct summand, then $\mathrm{r}_{\mathrm{ns}}(A^{*n}) = n\tau(p)$, for every $2 \le n \le \infty$, implying the non-isomorphism in the first part of Theorem 1.1. Since the sans-rank is easily seen to satisfy the amplification formula $\mathrm{r}_{ns}(M^t) = \mathrm{r}_{ns}(M)/t$, for every $t > 0$, the last part of the theorem follows as well.

We define the sans-core and sans-rank of a II_1 factor in Section 2, where we also discuss some basic properties, including the amplification formula for the sans-rank. In Section 3 we prove that $\mathrm{r}_{ns}(*_{i\in I} M_i) = \sum_{i\in I} \mathrm{r}_{ns}(M_i)$, for any family $M_i, i \in I$, of tracial von Neumann algebras (see Theorem 3.8) and use this formula to deduce Theorem 1.1. The proof of Theorem 3.8 uses intertwining by bimodules techniques and control of relative commutants in amalgamated free product II_1 factors from [IPP08]. Notably, we use results from [IPP08] to show that any von Neumann subalgebra P of a tracial free product $M = M_1 * M_2$ which has a non-separable relative commutant, $P' \cap M$, must have a corner which embeds into M_1 or M_2 (see Theorem 3.4). The last section of the paper, Section 4, records some further remarks and open problems.

2. The singular abelian core of a II_1 factor

The aim of this section is to define the singular abelian core a II_1 factor and its non-separable analogue. We start by recalling some terminology involving von Neumann algebras. We will always work with tracial von Neumann algebras, i.e., von Neumann algebras M endowed with a fixed faithful normal trace τ. We endow M with the 2-norm given by $\|x\|_2 = \tau(x^*x)^{1/2}$ and denote by $\mathcal{U}(M)$ its group of unitaries and by $(M)_1 = \{x \in M \mid \|x\| \le 1\}$ its (uniform) unit ball. We assume that all von Neumann subalgebras are unital. For a von Neumann subalgebra $A \subset M$, we denote by $\mathrm{E}_A : M \to M$ the conditional expectation onto A and by $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ the normalizer of A in M. We say that a von Neumann algebra M is *purely non-separable* if pMp is non-separable, for every nonzero projection $p \in M$.

2.1. **Interwining by bimodules.** We recall the intertwining by bimodules theory from [Pop06b, Theorem 2.1 and Corollary 2.3].

Theorem 2.1 ([Pop06b]). *Let (M, τ) be a tracial von Neumann algebra and $A \subset pMp, B \subset qMq$ be von Neumann subalgebras. Then the following conditions are equivalent.*

(1) There exist nonzero projections $p_0 \in A, q_0 \in B$, a $*$-homomorphism $\theta\colon p_0Ap_0 \to q_0Qq_0$ and a nonzero partial isometry $v \in q_0Mp_0$ such that $\theta(x)v = vx$, for all $x \in p_0Ap_0$.
(2) There is no net $u_n \in \mathcal{U}(A)$ satisfying $\|\mathrm{E}_B(x^*u_ny)\|_2 \to 0$, for all $x, y \in pM$.

If (1) or (2) hold true, we write $A \prec_M B$ and say that *a corner of A embeds into B inside M*. If $Ap' \prec_M B$, for any nonzero projection $p' \in A \cap pMp$, we write $A \prec^{\mathrm{f}}_M B$.

2.2. **Singular MASAs.** Let (M, τ) be a tracial von Neumann algebra. An abelian von Neumann subalgebra $A \subset M$ is called a *MASA* if it is maximal abelian and *singular* if it satisfies $\mathcal{N}_M(A) = \mathcal{U}(A)$ [Dix54]. Note that a singular abelian von Neumann subalgebra $A \subset M$ is automatically a MASA.

Two MASAs $A \subset pMp, B \subset qMq$ are called *disjoint* if $A \nprec_M B$. The following result from [Pop06a, Theorem A.1] shows that disjointness for MASAs is the same as having no unitarily conjugated corners. In particular, disjointness of MASAs is a symmetric relation.

Theorem 2.2 ([Pop06a]). *Let (M, τ) be a tracial von Neumann algebra and $A \subset pMp, B \subset qMq$ be MASAs. Then $A \prec_M B$ if and only if $B \prec_M A$ and if and only if there exist nonzero projections $p_0 \in A, q_0 \in B$ such that $u(Ap_0)u^* = Bq_0$, for some $u \in \mathcal{U}(M)$.*

2.3. **The singular abelian core.** We are now ready to give the following:

Definition 2.3. Let (M,τ) be a tracial von Neumann algebra. We denote by $\mathcal{S}(M)$ the set of all families $\mathcal{A}=\{A_i\}_{i\in I}$, where $p_i\in M$ is a projection, $A_i\subset p_iMp_i$ is a singular MASA, for every $i\in I$, and $A_i, A_{i'}$ are disjoint, for every $i,i'\in I$ with $i\neq i'$. We denote $\mathrm{d}(\mathcal{A})=\sum_{i\in I}\tau(p_i)$, the *size* of the family $\mathcal{A}$. Given $\mathcal{A}=\{A_i\}_{i\in I},\mathcal{B}=\{B_j\}_{j\in J}\in\mathcal{S}(M)$ we write $\mathcal{A}\leq\mathcal{B}$ if for every $i\in I$ and nonzero projection $p\in A_i$, there exists $j\in J$ such that $A_ip\prec_M B_j$. We say that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* and write $\mathcal{A}\sim\mathcal{B}$ if $\mathcal{A}\leq\mathcal{B}$ and $\mathcal{B}\leq\mathcal{A}$.

Lemma 2.4. *Let (M,τ) be a tracial von Neumann algebra. Then $\mathcal{S}(M)$ admits a maximal element with respect $\leq$. Moreover, any two maximal elements of $\mathcal{S}(M)$ with respect to $\leq$ are equivalent.*

Proof. Let $\mathcal{A}=\{A_i\}_{i\in I}\in\mathcal{S}(M)$ be a maximal family with respect to inclusion. Then $\mathcal{A}$ is maximal with respect to $\leq$. To see this, let $\mathcal{B}=\{B_j\}_{j\in J}\in\mathcal{S}(M)$. If $\mathcal{B}\not\leq\mathcal{A}$, then there are $j\in J$ and a nonzero projection $q\in B_j$ with $B_jq\nprec_M A_i$, for every $i\in I$. As $B_jq\subset qMq$ is a singular MASA, we get that $\mathcal{A}\cup\{B_jq\}\in\mathcal{S}(M)$, contradicting the maximality of $\mathcal{A}$ with respect to inclusion. The moreover assertion follows. □

Definition 2.5. Let (M,τ) be a tracial von Neumann algebra. We denote by $\mathcal{A}_M$ the equivalence class consisting of all maximal elements of $\mathcal{S}(M)$ with respect to $\leq$, and call it the *singular abelian core* of M. We define the *rank* $\mathrm{r}(M)$ of M as the size, $\mathrm{d}(\mathcal{A})$, of any $\mathcal{A}\in\mathcal{A}_M$. Note that $\mathrm{r}(M)$ is a well-defined isomorphism invariant of M since the map $\mathcal{A}\mapsto\mathrm{d}(\mathcal{A})$ is constant on equivalence classes.

Remark 2.6. Definition 2.3 presents the *folded form* of $\mathcal{S}(M)$, for a tracial von Neumann algebra (M,τ). Let K be a large enough set, which contains the index set I of any element $\mathcal{A}=\{A_i\}_{i\in I}$ of $\mathcal{S}(M)$. For instance, take K to be the collection of all singular MASAs $A\subset pMp$, for all projections $p\in M$. We identify every $\mathcal{A}=\{A_i\}_{i\in I}$ of $\mathcal{S}(M)$ with the singular abelian von Neumann subalgebra $\mathcal{A}=\oplus_{i\in I}A_i$ of $p\mathcal{M}p$, where $\mathcal{M}=M\overline{\otimes}\mathbb{B}(\ell^2K)$ and $p=\oplus_{i\in I}p_i\in\mathcal{M}$. This is the *unfolded form* of $\mathcal{S}(M)$. In this unfolded form, given $\mathcal{A},\mathcal{B}\in\mathcal{S}(M)$, we have that $\mathcal{A}\leq\mathcal{B}$ (respectively, $\mathcal{A}\sim\mathcal{B}$) if and only if $\mathcal{A}\subset u\mathcal{B}qu^*$ (respectively, $\mathcal{A}=u\mathcal{B}u^*$), for a projection $q\in\mathcal{B}$ and unitary $u\in\mathcal{M}$.

The unfolded form of the singular abelian core $\mathcal{A}_M$ of M is then the unique (up to unitary conjugacy) singular abelian von Neumann subalgebra $\mathcal{A}\subset p\mathcal{M}p$ generated by finite projections such that for any singular abelian von Neumann subalgebra $\mathcal{B}\subset q\mathcal{M}q$, for a finite projection q, we have that $\mathcal{B}\prec_{\mathcal{M}}\mathcal{A}$. The rank $\mathrm{r}(M)$ is then equal to the semifinite trace, $(\tau\otimes\mathrm{Tr})(p)$, of the unit p of $\mathcal{A}_M$. Notice that if the semifinite trace $(\tau\otimes\mathrm{Tr})(p)$ of the support of $\mathcal{A}$ is infinite, then it can be viewed as a cardinality $\leq|K|$. We will in fact view $\mathrm{r}(M)$ this way, when infinite.

Remark 2.7. Let M be an arbitrary separable II_1 factor. By a result in [Pop83c], M admits a singular MASA. This result was strengthened in [Pop19, Theorem 1.1] where it was shown that M contains an uncountable family of pairwise disjoint singular MASAs. Consequently, $\mathrm{r}(M)>\aleph_0$. More recently, it was shown in [Pop21, Theorem 1.1] that M contains a copy of the hyperfinite II_1 factor $R\subset M$ which is coarse, i.e., such the R-bimodule $\mathrm{L}^2(M)\ominus\mathrm{L}^2(R)$ is a multiple of the coarse R-bimodule $\mathrm{L}^2(R)\overline{\otimes}\mathrm{L}^2(R)$. In combination with [Pop21, Proposition 2.6.3] and [Pop14, Theorem 5.1.1], this implies that M has a continuous family of disjoint singular MASAs. Since the set of distinct self-adjoint elements in a separable II_1 factor has continuous cardinality $c=2^{\aleph_0}$ and each singular MASA is generated by a self-adjoint element, it follows that $\mathrm{r}(M)=c$, for every separable II_1 factor M.

2.4. **The singular abelian non-separable core.** Remark 2.7 shows that the rank $\mathrm{r}(M)$ is equal to the continuous cardinality c for any separable II_1 factor M, and thus cannot be used to distinguish such factors up to isomorphism. In contrast, we define in this section a non-separable analogue of $\mathrm{r}(M)$, which will later enable us to prove the non-isomorphisms asserted by Theorem 1.1.

Definition 2.8. Let (M,τ) be a tracial von Neumann algebra. We say that a von Neumann subalgebra $A \subset pMp$ is a *sans*-subalgebra of M if it is singular abelian in pMp and purely non-separable. We denote by $\mathcal{S}_{\rm ns}(M) \subset \mathcal{S}(M)$ the set of $\mathcal{A} = \{A_i\}_{i\in I} \in \mathcal{S}(M)$ such that A_i is a sans-subalgebra, for every $i \in I$. We call any $\mathcal{A} \in \mathcal{S}_{\rm ns}(M)$ a *sans family* in M.

Since Lemma 2.4 trivially holds true if we replace $\mathcal{S}(M)$ by $\mathcal{S}_{\rm ns}(M)$, we can further define:

Definition 2.9. Let (M,τ) be a tracial von Neumann algebra. We denote by $\mathcal{A}_M^{\rm ns}$ the equivalence class consisting of all maximal elements of $\mathcal{S}_{\rm ns}(M)$ with respect to $\leq$, and call it the *singular abelian non-separable core* (abbreviated, the *sans-core*) of M. We define the *sans-rank* $\mathrm{r}_{\rm ns}(M)$ of M as the size, $\mathrm{d}(\mathcal{A})$, of any $\mathcal{A} \in \mathcal{A}_M^{\rm ns}$.

Remark 2.10. Like in Remark 2.6, consider $\mathcal{M} = M\overline{\otimes}\mathbb{B}(\ell^2 K)$, for a large enough set K. In the unfolded form of $\mathcal{S}_{\rm ns}(M)$, the sans-core $\mathcal{A}_M^{\rm ns}$ of M is the unique (up to unitary conjugacy) sans-subalgebra $\mathcal{A} \subset p\mathcal{M}p$ generated by finite projections such that for any sans-subalgebra $\mathcal{B} \subset q\mathcal{M}q$, for a finite projection q, we have that $\mathcal{B} \prec_{\mathcal{M}} \mathcal{A}$. The sans-rank $\mathrm{r}_{\rm ns}(M)$ is then the semifinite trace, $(\tau \otimes \mathrm{Tr})(p)$, of the unit p of $\mathcal{A}_M^{\rm ns}$. Like in Remark 2.6, when the semifinite trace of the support of the sans-core in this unfolded form is infinite, then we will view $\mathrm{r}_{\rm ns}(M)$ as a cardinality $\leq |K|$.

Remark 2.11. If M is a separable II$_1$ factor, then we clearly have $\mathrm{r}_{\rm ns}(M) = 0$. If $A \subset M$ is a singular MASA and ω is a free ultrafilter on $\mathbb{N}$, then $A^\omega \subset M^\omega$ is a purely non-separable singular MASA, see [Pop83c, 5.3]. Moreover, disjoint MASAs in M give rise to disjoint ultrapower MASAs in M^ω. By using these facts and results from [Pop14, Pop21] as in Remark 2.7 we get that $\mathrm{r}_{\rm ns}(M^\omega) \geq c$, for every separable II$_1$ factor M. But getting $\mathrm{r}_{\rm ns}(M^\omega) \leq c$ is problematic, as besides the family of disjoint ultraproduct singular MASAs in M^ω, which has cardinality c, one may have singular MASAs that are not of this form.

The expression of $\mathrm{r}_{\rm ns}(M)$ as the semifinite trace of the support of the sans-core in unfolded form, as in Remark 2.10, implies the following scaling formula for $\mathrm{r}_{\rm ns}(M)$. We include below an alternative short proof using the folded form of $\mathcal{S}_{\rm ns}(M)$.

Proposition 2.12. *Let M be any II$_1$ factor and $t \in \mathbb{R}_+^*$. Then we have*

$$\mathrm{r}_{ns}(M^t) = \mathrm{r}_{ns}(M)/t.$$

In particular, if $0 < \mathrm{r}_{ns}(M) < \infty$, then M has trivial fundamental group, $\mathcal{F}(M) = \{1\}$.

Proof. It is enough to argue that $\mathrm{r}_{\rm ns}(qMq) = \mathrm{r}_{\rm ns}(M)/\tau(q)$, for every nonzero projection $q \in M$. This follows immediately by using the fact that any $\mathcal{A} = \{A_i\}_{i\in I} \in \mathcal{S}(M)$ is equivalent to some $\mathcal{B} = \{B_j\}_{j\in J} \in \mathcal{S}(M)$, such that $B_j \subset q_j M q_j$, for some $q_j \leq q$, for every $j \in J$. □

3. Main results

3.1. **Main technical result.** This subsection is devoted to proving our main technical result. Throughout the subsection we use the following notation. Let (M_1,τ_1) and (M_2,τ_2) tracial von Neumann algebras and denote by $M = M_1 * M_2$ their free product with its canonical trace τ.

Theorem 3.1. *Let $P \subset pMp$ be a von Neumann subalgebra such that $P'\cap pMp$ is non-separable. Then $P \prec_M M_1$ or $P \prec_M M_2$.*

The proof of Theorem 3.1 is based on the main technical result of [IPP08]. By [PV10, Section 5.1], given $\rho \in (0,1)$, we have a unital tracial completely positive map $\mathrm{m}_\rho : M \to M$ such that $\mathrm{m}_\rho(x_1 x_2 \cdots x_n) = \rho^n x_1 x_2 \cdots x_n$, for every $n \in \mathbb{N}$ and $x_i \in M_{i_j} \ominus \mathbb{C}1$, where $i_j \in \{1,2\}$, for every $1 \leq j \leq n$, and $i_j \neq i_{j+1}$, for every $1 \leq j \leq n-1$. Note that $\lim\limits_{\rho\to 1} \|\mathrm{m}_\rho(x) - x\|_2 = 0$ and the map $(0,1) \ni \rho \mapsto \|\mathrm{m}_\rho(x)\|_2$ is increasing, for every $x \in M$. The implication (1) $\Rightarrow$ (2) follows from [IPP08, Theorem 4.3], formulated here as in [PV10, Theorem 5.4], see also [Hou09, Section 5].

Theorem 3.2 ([IPP08]). *Let $P \subset pMp$ be a von Neumann subalgebra. Then the following two conditions are equivalent:*

(1) There exists $\rho \in (0,1)$ such that $\inf_{u\in\mathcal{U}(P)} \|\mathrm{m}_\rho(u)\|_2 > 0$.
(2) $P \prec_M M_1$ or $P \prec_M M_2$.

Proof. Assume that (1) holds. Since $\tau(x^*\mathrm{m}_{\rho^2}(x)) = \|\mathrm{m}_\rho(x)\|_2^2$, for every $x \in M$, we get that $\inf_{u\in\mathcal{U}(P)} \tau(u^*\mathrm{m}_{\rho^2}(u)) > 0$ and [PV10, Theorem 5.4] implies (2).

To see that (2) ⇒ (1), assume that $P \prec_M M_i$, for some $i \in \{1,2\}$. By Theorem 2.1 we find a nonzero partial isometry $v \in M$ such that $v^*v = p_0p'$, for some projections $p_0 \in P, p' \in P'\cap pMp$, and $(p_0Pp_0)_1p' \subset v^*(M_i)_1v$. Since $\|\mathrm{m}_\rho(x) - x\|_2 \leq |\rho - 1|$, for every $x \in (M_i)_1$, we get that $\lim_{\rho\to1}(\sup_{x\in(p_0Pp_0)_1p'} \|\mathrm{m}_\rho(x) - x\|_2) = 0$. Let p_1 be the central support of p_0 in P and denote $p'' = p_1p' \in P' \cap pMp$. It follows that $\lim_{\rho\to1}(\sup_{x\in(Pp'')_1} \|\mathrm{m}_\rho(x) - x\|_2) = 0$. From this it is easy to deduce that $\liminf_{\rho\to1}(\inf_{u\in\mathcal{U}(P)} \|\mathrm{m}_\rho(u)\|_2) \geq \|p''\|_2 > 0$, which clearly implies (1). □

Corollary 3.3. *Let $P \subset pMp$ be a von Neumann subalgebra such that $P \nprec_M M_1$ and $P \nprec_M M_2$. Then there exists a separable von Neumann subalgebra $Q \subset P$ such that $Q \nprec_M M_1$ and $Q \nprec_M M_2$.*

Proof. Since $P \nprec_M M_1$ and $P \nprec_M M_2$, by Theorem 3.2 we find a sequence $u_n \in \mathcal{U}(P)$ such that $\|m_{1-1/n}(u_n)\|_2 \leq 1/n$. Let $Q \subset P$ be the separable von Neumann subalgebra generated by $\{u_n\}_{n\geq1}$. Let $\rho \in (0,1)$. Then for every $n \geq 1$ such that $\rho \leq 1 - 1/n$ we have that $\|\mathrm{m}_\rho(u_n)\|_2 \leq \|\mathrm{m}_{1-1/n}(u_n)\|_2 \leq 1/n$. This implies $\inf_{u\in\mathcal{U}(Q)} \|\mathrm{m}_\rho(u)\|_2 = 0$. Since this holds for every $\rho \in (0,1)$, Theorem 3.2 implies that $Q \nprec_M M_1$ and $Q \nprec_M M_2$. □

Lemma 3.4. *Let $Q \subset M$ be a separable von Neumann subalgebra. Then we can find separable von Neumann subalgebras $N_1 \subset M_1$ and $N_2 \subset M_2$ such that $Q \subset N_1 * N_2$.*

Proof. For $i \in \{1,2\}$ let $\mathcal{B}_i$ be an orthonormal basis of $\mathrm{L}^2(M_i) \ominus \mathbb{C}1$ such that $\mathcal{B}_i \subset M_i \ominus \mathbb{C}1$. Let $\mathcal{B}_0$ be the set of $\xi_1\xi_2\cdots\xi_n$, where $n \in \mathbb{N}$, $\xi_i \in \mathcal{B}_{i_j}$, for some $i_j \in \{1,2\}$, for every $1 \leq j \leq n$, and $i_j \neq i_{j+1}$, for every $1 \leq j \leq n-1$. Then $\mathcal{B} = \mathcal{B}_0 \cup \{1\}$ is an orthonormal basis of $\mathrm{L}^2(M)$.

Let $\{x_k\}_{k\geq1}$ be a sequence which generates Q. Then $\mathcal{C} = \cup_{k\geq1}\{\xi \in \mathcal{B} \mid \langle x_k, \xi\rangle \neq 0\}$ is countable. For $i \in \{1,2\}$, let $\mathcal{C}_i$ be the countable set of all $\xi \in \mathcal{B}_i$ which appear in the decomposition of some element of $\mathcal{C}$. The von Neumann subalgebra N_i of M_i generated by $\mathcal{C}_i$ is separable, for every $i \in \{1,2\}$. Since by construction we have that $Q \subset N_1 * N_2$, this finishes the proof. □

Proof of Theorem 3.1. Assume by contradiction that $P \nprec_M M_1$ and $P \nprec_M M_2$. By applying Corollary 3.3, we can find a separable von Neumann subalgebra $Q \subset P$ such that $Q \nprec_M M_1$ and $Q \nprec_M M_2$. By Lemma 3.4, we can further find separable von Neumann subalgebras $N_1 \subset M_1$ and $N_2 \subset M_2$, such that $Q \subset N := N_1 * N_2$. Denote $R = M_1 * N_2$.

Since $Q \nprec_M M_1$, $Q \subset R \subset M$ and $N_1 \subset M_1$, we get that $Q \nprec_R N_1$. Since $Q \subset N$ and $R = M_1 *_{N_1} N$, [IPP08, Theorem 1.1] implies that $Q' \cap R = Q' \cap N$. Next, since $Q \nprec_M M_2$ and $N_2 \subset M_2$, we get that $Q \nprec_M N_2$. Since $Q \subset R$ and $M = R *_{N_2} M_2$, applying [IPP08, Theorem 1.1] again gives that $Q' \cap M = Q' \cap R$. Altogether, we get that $Q' \cap M = Q' \cap N$. Since N and thus $Q' \cap N$ is separable, using that $P' \cap M \subset Q' \cap M$, we conclude that $P' \cap M$ is separable. □

3.2. **Non-separable MASAs in free product algebras.** In this subsection, we derive some consequences of Theorem 3.1 to the structure of non-separable MASAs in free product algebras.

Corollary 3.5. *Let (M_1, τ_1) and (M_2, τ_2) be tracial von Neumann algebras, and denote by $M = M_1 * M_2$ their free product. Let $A \subset pMp$ be a purely non-separable MASA. Then there exist projections $(p_k)_{k\in K} \subset A$ and unitaries $(u_k)_{k\in K} \subset M$ such that $\sum_{k\in K} p_k = p$ and for every $k \in K$, $u_kAp_ku_k^* \subset M_i$, for some $i \in \{1,2\}$.*

Proof. By a maximality argument, it suffices to prove that if $q \in A$ is a nonzero projection, then there are a nonzero projection $r \in Aq$, a unitary $u \in M$ and $i \in \{1,2\}$ such that $uAru^* \subset M_i$.

To this end, let $q \in A$ be a nonzero projection. Since $(Aq)' \cap qMq = Aq$ is non-separable, Theorem 3.1 implies that there is $i \in \{1,2\}$ such that $Aq \prec_M M_i$. By Theorem 2.1, we can find nonzero projections $e \in Aq$, $f \in M_i$, a nonzero partial isometry $v \in fMe$ and a $*$-homomorphism $\theta : Ae \to fM_if$ such that $\theta(x)v = vx$, for every $x \in Ae$. Then $r := v^*v \in (Ae)' \cap eMe = Ae$ and $vv^* \in \theta(Ae)' \cap fMf$. Since $\theta(Ae) \subset fM_if$ is diffuse, by applying [IPP08, Theorem 1.1] (see also [Pop83b, Remarks 6.3.2)]) we get that $vv^* \in fM_if$. Finally, let $u \in M$ be any unitary such that $ur = v$. Then $uAru^* = vArv^* = vAev^* = \theta(Ae)vv^* \subset M_i$, which finishes the proof. □

We continue by generalizing Corollary 3.5 to arbitrary tracial free products.

Corollary 3.6. *Let (M_i, τ_i), $i \in I$, be a collection of tracial von Neumann algebras, and denote by $M = *_{i\in I} M_i$ their free product. Let $A \subset pMp$ be a purely non-separable MASA. Then there exist projections $(p_k)_{k\in K} \subset A$ and unitaries $(u_k)_{k\in K} \subset M$ such that $\sum_{k\in K} p_k = p$ and for every $k \in K$, $u_kAp_ku_k^* \subset M_i$, for some $i \in I$.*

Proof. Let $A_0 \subset A$ be a separable diffuse von Neumann subalgebra. Reasoning similarly to the proof of Lemma 3.4 yields a countable set $J \subset I$ such that $A_0 \subset *_{j\in J} M_j$. Since A_0 is diffuse, [IPP08, Theorem 1.1] gives that $A \subset A_0' \cap pMp \subset *_{j\in J} M_j$. Thus, in order to prove the conclusion, after replacing I with J, we may take I countable. Enumerate $I = \{i_m\}_{m\geq 1}$.

Let $\{p_k\}_{k\in K} \subset A$ be a maximal family, with respect to inclusion, of pairwise orthogonal projections such that for every $k \in K$, there are a unitary $u_k \in M$ and $i \in I$ such that $u_kAp_ku_k^* \subset M_i$. In order to prove the conclusion it suffices to argue that $\sum_{k\in K} p_k = p$. Put $r := p - (\sum_{k\in K} p_k)$. Assume by contradiction that $r \neq 0$. We claim that

$$Ar \nprec_M *_{m\leq n} M_{i_m}, \text{ for every } n \geq 1. \tag{3.1}$$

Otherwise, if (3.1) fails for some $n \geq 1$, then the proof of Corollary 3.5 gives a nonzero projection $s \in Ar$ and a unitary $u \in M$ such that $uAsu^* \subset *_{m\leq n} M_{i_m}$. Applying Corollary 3.5 repeatedly gives a nonzero projection $t \in As$ and a unitary $v \in *_{m\leq n} M_{i_m}$ such that $vuAtu^*v^* \subset M_{i_m}$, for some $1 \leq m \leq n$. This contradicts the maximality of the family $\{p_k\}_{k\in K}$, and proves (3.1).

If $e \in (Ar)' \cap rMr = Ar$ is a nonzero projection, then $(Ae)' \cap eMe = Ae$ is nonseparable. Since $Ae \nprec_M *_{m\leq n} M_{i_m}$ by (3.1), Theorem 3.4 implies that $Ae \prec_M *_{m>n} M_{i_m}$ and thus

$$Ar \prec^{\mathrm{f}}_M *_{m>n} M_{i_m}, \text{ for every } n \geq 1. \tag{3.2}$$

To get a contradiction, we follow the proof of [HU16, Proposition 4.2]. Let $\widetilde{M} = M * M$, identify M with $M * 1 \subset \widetilde{M}$, and denote by θ the free flip automorphism of $\widetilde{M}$. Endow $\mathcal{H} = \mathrm{L}^2(\widetilde{M})$ with the M-bimodule structure given by $x \cdot \xi \cdot y = \theta(x)\xi y$, for every $x, y \in M$ and $\xi \in \mathcal{H}$. Using (3.2), the proof of [HU16, Proposition 4.2] yields a sequence of vectors $\eta_n \in r \cdot \mathcal{H} \cdot r$ such that $\|\eta_n\|_2 \to \|r\|_2$, $\|x \cdot \eta_n\|_2 \leq \|x\|_2$ and $\|a \cdot \eta_n - \eta_n \cdot a\|_2 \to 0$, for every $x \in rMr$ and $a \in Ar$.

Next, we note that the Ar-bimodule $r\cdot\mathcal{H}\cdot r$ is isomorphic to a multiple of the coarse Ar-bimodule, $\oplus_S(\mathrm{L}^2(Ar) \otimes \mathrm{L}^2(Ar))$, for some (possibly uncountable) set S. If $\zeta \in \oplus_S(\mathrm{L}^2(Ar) \otimes \mathrm{L}^2(Ar))$, then we can find a countable subset $T \subset S$ such that $\zeta \oplus_T (\mathrm{L}^2(Ar) \otimes \mathrm{L}^2(Ar))$. By combining these two facts with the previous paragraph, we obtain a sequence of vectors $\zeta_n \in \oplus_{\mathbb{N}}(\mathrm{L}^2(Ar) \otimes \mathrm{L}^2(Ar))$ such that $\|\zeta_n\|_2 \to \|r\|_2$, $\|a \cdot \zeta_n\|_2 \leq \|a\|_2$ and $\|a \cdot \zeta_n - \zeta_n \cdot a\|_2 \to 0$, for every $a \in Ar$. By reasoning similarly to the proof of Lemma 3.4, we find a separable von Neumann subalgebra $A_0 \subset Ar$ such that $\zeta_n \in \oplus_{\mathbb{N}}(\mathrm{L}^2(A_0) \otimes \mathrm{L}^2(A_0))$.

As A_0 is separable and Ar is purely non-separable, we derive that $Ar \nprec_{Ar} A_0$. Theorem 2.1 gives a unitary $u \in Ar$ with $\|\mathrm{E}_{A_0}(u)\|_2 \leq \|r\|_2/2$. Put $a = u - \mathrm{E}_{A_0}(u) \in A$. Since $a \cdot \zeta_n \in \oplus_{\mathbb{N}}((\mathrm{L}^2(Ar) \ominus \mathrm{L}^2(A_0)) \otimes \mathrm{L}^2(A_0))$ and $\zeta_n \cdot a \in \oplus_{\mathbb{N}}(\mathrm{L}^2(A_0) \otimes (\mathrm{L}^2(Ar) \ominus \mathrm{L}^2(A_0)))$, we have that $\langle a \cdot \zeta_n, \zeta_n \cdot a\rangle = 0$, for every n. Using that $\|a \cdot \zeta_n - \zeta_n \cdot a\|_2 \to 0$, we get that $\|a \cdot \zeta_n\|_2 \to 0$. On the other hand, $\|a \cdot \zeta_n\|_2 \geq \|u \cdot \zeta_n\|_2 - \|\mathrm{E}_{A_0}(u) \cdot \zeta_n\|_2 \geq \|\zeta_n\|_2 - \|\mathrm{E}_{A_0}(u)\|_2 \geq \|\zeta_n\|_2 - \|r\|_2/2$. Since $\|\zeta_n\|_2 \to \|r\|_2 > 0$, we altogether get a contradiction, which finishes the proof. □

We end this subsection by noticing that in the case $A \subset pMp$ is a singular MASA and M_i is abelian, for every $i \in I$, the conclusion of Corollay 3.6 can be strengthened as follows:

Corollary 3.7. *In the context of Corollary 3.6, assume additionally that $A \subset pMp$ is singular and M_i is abelian, for every $i \in I$. Then there exist projections $(q_i)_{i\in I} \subset A$ and unitaries $(v_i)_{i\in I} \subset M$ such that $\sum_{i\in I} q_i = p$, $e_i = v_i q_i v_i^* \in M_i$ and $v_i A q_i v_i^* = M_i e_i$, for every $i \in I$.*

Proof. By applying Corollary 3.6 we find projections $(p_k)_{k\in K} \subset A$ and unitaries $(u_k)_{k\in K} \subset M$ such that $\sum_{k\in K} p_k = p$ and for every $k \in K$, $u_k A p_k u_k^* \subset M_{i_k}$, for some $i_k \in I$. Let $k \in K$ and put $r_k := u_k p_k u_k^* \in M_{i_k}$. Since $u_k A p_k u_k^* \subset r_k M r_k$ is a MASA and M_{i_k} is abelian we deduce that $u_k A p_k u_k^* = M_{i_k} r_k$, for every $k \in K$. Let $k, k' \in K$ such that $k \neq k'$ and $i_k = i_{k'}$. Since $A \subset pMp$ is singular and $p_k p_{k'} = 0$, there are no nonzero projections $s \in Ap_k, s' \in Ap_{k'}$ such that As and As' are unitarily conjugated in M. This implies that $r_k r_{k'} = 0$. Using this fact, it follows that if we denote $q_i = \sum_{k\in K, i_k = i} p_k$, then $v_i A q_i v_i^* \subset M_i$, for every $i \in I$. For $i \in I$, let $e_i = v_i q_i v_i^* \in M_i$. Then $v_i A q_i v_i^* \subset M_i e_i$ and since $v_i A q_i v_i^* \subset M_i e_i$ is a MASA, while $M_i e_i$ is abelian, it follows that $v_i A q_i v_i^* = M_i e_i$, as claimed. □

3.3. **The non-separable rank of free product von Neumann algebras.** In this section, we show that the sans core of a free product of tracial von Neumann algebras $M = *_{i\in I} M_i$ is the union of the sans cores of $M_i, i \in I$. This allows us to deduce that the sans rank of M is the sum of the sans ranks of $M_i, i \in I$.

Theorem 3.8. *Let (M_i, τ_i), $i \in I$, be a colection of tracial von Neumann algebras, and denote by $M = *_{i\in I} M_i$ their free product. Then $\mathrm{r_{ns}}(M) = \sum_{i\in I} \mathrm{r_{ns}}(M_i)$. Moreover, if $\mathcal{A}_i \in \mathcal{A}^{\mathrm{ns}}_{M_i}$, for every $i \in I$, then $\cup_{i\in I} \mathcal{A}_i \in \mathcal{A}^{\mathrm{ns}}_M$.*

The moreover assertion uses implicitly the fact, explained in the proof, that every sans family in M_i is naturally a sans family in M, for every $i \in I$.

Proof. We have two inequalities to prove.

Inequality 1. $\mathrm{r_{ns}}(M) \geq \sum_{i\in I} \mathrm{r_{ns}}(M_i)$.

This inequality relies on several facts on free products, all of which follow from [IPP08, Theorem 1.1]. Let $i, j \in I$ with $i \neq j$.

(1) If $A \subset pM_i p$ is a MASA, then $A \subset pMp$ is a MASA.
(2) If $A \subset pM_i p$ is a singular diffuse von Neumann subalgebra, then $A \subset pMp$ is singular.
(3) If $A \subset pM_i p$, $B \subset qM_i q$ are von Neumann subalgebras with $A \prec_M B$, then $A \prec_{M_i} B$.
(4) If $A \subset pM_i p$ and $B \subset qM_j q$ are diffuse von Neumann subalgebras, then $A \not\prec_M B$.

For $i \in I$, let $\mathcal{A}_i \in \mathcal{A}^{\mathrm{ns}}_{M_i}$ be a maximal sans family in M_i. We view every (not necessarily unital) subalgebra of M_i as a subalgebra of M. Then facts (1)-(3) imply that $\mathcal{A}_i$ is a sans family in M. Moreover, fact (4) implies that $\mathcal{A} := \cup_{i\in I} \mathcal{A}_i$ is a sans family in M. Thus,

$$\mathrm{r_{ns}}(M) \geq \mathrm{d}(\mathcal{A}) = \sum_{i\in I} \mathrm{d}(\mathcal{A}_i) = \sum_{i\in I} \mathrm{r_{ns}}(M_i).$$

Inequality 2. $\mathrm{r}_{ns}(M) \leq \sum_{i\in I} \mathrm{r}_{ns}(M_i)$.

Let $\mathcal{A} = \{A_l\}_{l\in L} \in \mathcal{A}^{\mathrm{ns}}_M$ be a maximal sans family in M. Let $l \in L$. Applying Corollary 3.6 to A_l gives projections $(p_{k,l})_{k\in K_l}$ and unitaries $(u_{k,l})_{k\in K_l}$ such that for every $k \in K_l$ we have $u_{k,l} A_l p_{k,l} u_{k,l}^* \subset M_i$, for some $i \in I$. For $i \in I$, let $\mathcal{A}_i \in \mathcal{S}_{\mathrm{ns}}(M_i)$ be the collection of sans-subalgebras of M_i of the form $u_{k,l} A_l p_{k,l} u_{k,l}^*$, for all $l \in L, k \in K_l$ such that $u_{k,l} A_l p_{k,l} u_{k,l}^* \subset M_i$. Then $\mathcal{A}$ is equivalent to $\cup_{i\in I} \mathcal{A}_i$, which allows us to conclude that

$$\mathrm{r_{ns}}(M) = \mathrm{d}(\mathcal{A}) = \sum_{i\in I} \mathrm{d}(\mathcal{A}_i) \leq \sum_{i\in I} \mathrm{r_{ns}}(M_i).$$

This finishes the proof of the main assertion. The moreover assertion now follows by combining the proofs of inequalities 1 and 2. □

3.4. **Proof of Theorem 1.1.** In preparation for the proof of Theorem 1.1, we first record the following direct consequence of Theorem 3.8:

Corollary 3.9. *Let (A_i, τ_i), $i \in I$, be a collection of diffuse tracial abelian von Neumann algebras, and denote by $M = *_{i\in I} A_i$ their free product. For $i \in I$, let $p_i \in A_i$ be the maximal (possibly zero) projection such that $A_i p_i$ is purely non-separable. Then $\mathrm{r}_{\mathrm{ns}}(M) = \sum_{i\in I} \tau_i(p_i)$. Moreover, if $|I| \geq 2$ and $\sum_{i\in I} \tau_i(p_i) \in (0, +\infty)$, then M is a II_1 factor with $\mathcal{F}(M) = \{1\}$. Also, the sans-core of M is given by $\mathcal{A}_M^{ns} = \{A_i p_i\}_{i\in I}$.*

Proof. Let $i \in I$. Since $\{A_i p_i\} \in \mathcal{S}_{\mathrm{ns}}(A_i)$ is a maximal element, we get that $\mathrm{r}_{\mathrm{ns}}(A_i) = \tau_i(p_i)$. The assertions now follow by using Theorem 3.8, Proposition 2.12, and the fact that any free product of diffuse tracial von Neumann algebras is a II_1 factor. □

Proof of Theorem 1.1. Let (A, τ) be a diffuse non-separable tracial abelian von Neumann algebra. Let $p \in A$ be the maximal, necessarily non-zero, projection such that Ap is purely non-separable. By Corollary 3.9, $\mathrm{r}_{\mathrm{ns}}(A^{*n}) = n\tau(p)$, for every $2 \leq n \leq \infty$. Since $p \neq 0$, we get that A^{*n}, $2 \leq n \leq \infty$, are mutually non-isomorphic, and $\mathcal{F}(A^{*n}) = \{1\}$, for $2 \leq n < \infty$. □

4. Further remarks and open problems

4.1. **Freely complemented maximal amenable MASAs in A^{*n}.** The question of whether the II_1 factors A^{*n}, $2 \leq n \leq \infty$, are non-isomorphic for a non-separable diffuse tracial abelian von Neumann algebra A was asked in [BP]. This was motivated by the consideration of certain "radial-like" von Neumann subalgebras of $M = A^{*n}$, for $2 \leq n \leq \infty$. Specifically, for every $1 \leq k \leq n$, let s_k be a semicircular self-adjoint element belonging to A_k, the k^{th} copy of A in M. For an ℓ^2-summable family of real numbers $t = (t_k)$ with at least two non-zero entries, denote by $A(t)$ the abelian von Neumann subalgebra of M generated by $\sum_k t_k s_k$. It was shown in [BP] that $A(t) \subset M$ is maximal amenable and $A(t), A(t')$ are disjoint if t and t' are not proportional. A key point in proving this result was to show that $A(t) \not\prec_M A_k$, for every k. Since the MASAs $A(t)$ are separable, despite A being non-separable, this suggested that the only way to obtain a purely non-separable MASA in M is to "re-pack" pieces of A_k, $1 \leq k \leq n$. This further suggested the possibility of recovering n from the isomorphism class of M.

The construction of the family of radial-like maximal amenable MASAs $A(t) \subset M$ in [BP] was triggered by an effort to obtain examples of non freely complemented maximal amenable MASAs in the free group factors $L\mathbb{F}_n$. However, this remained open (see though [BP, Remark 1.4] for further comments concerning the inclusions $A(t) \subset A^{*n}$). Thus, there are no known examples of non freely complemented maximal amenable von Neumann subalgebras of $L\mathbb{F}_n$. It may be that in fact any maximal amenable $B \subset L\mathbb{F}_n$ is freely complemented (a property/question which we abbreviate as *FC*), see [Pop21, Question 5.5] and the introduction of [BP].

A test case for the FC question is proposed in the last paragraph of [Pop21]. There it is pointed out that if $\{B_i\}_i$ are diffuse amenable von Neumann subalgebras of $L\mathbb{F}_n$ with B_i freely complemented and $B_i \not\prec_{L\mathbb{F}_n} B_j$, for every $i \neq j$, then $B = \oplus_i u_i p_i B_i p_i u_i^*$ is maximal amenable in M by [Pop83a], for any projections $p_i \in B_i$ and unitaries $u_i \in M$ satisfying $\sum_i u_i p_i u_i^* = 1$. Thus, if FC is to hold then B should be freely complemented as well.

The FC question is equally interesting for the factors $M = A^{*n}$ with A purely non-separable abelian. If A_k denotes the k^{th} copy of A in M, for every $1 \leq k \leq n$, then by Theorem 3.8, any purely non-separable singular abelian $B \subset M$ is of the form $B = \sum_k u_k A_k p_k u_k^*$ for some projections $p_k \in A_k$ and unitaries $u_k \in M$ with $\sum_k u_k p_k u_k^* = 1$. Thus, B is maximal amenable by [Pop83a]. Hence, if FC is to hold, then Theorem 3.8 suggests that the free complement of B could be obtained by a "free reassembling" of unitary conjugates of pieces of $\{A_k(1 - p_k)\}_{k=1}^n$.

4.2. **On the calculation of symmetries of A^{*n}.** Let $M = A^{*n}$ with A purely non-separable abelian. Theorem 3.8 shows that if $\theta \in \mathrm{Aut}(M)$ then $\theta(\mathcal{A}_M^{\mathrm{ns}}) = \mathcal{A}_M^{\mathrm{ns}}$, modulo the equivalence in $\mathcal{S}_{\mathrm{ns}}(M)$ defined in Subsection 2.4. This suggests that one could perhaps explicitly calculate $\mathrm{Out}(M)$, for instance by identifying it with the Tr-preserving automorphisms α of the sans-core $\mathcal{A}_M^{\mathrm{ns}}$, viewed in its unfolded form. In order to obtain from an arbitrary such α an automorphism θ_α of M it would be sufficient to solve the FC question in its "free repacking" form explained in Remark 4.1 above. To prove that such a map $\alpha \mapsto \theta_\alpha$ is surjective one would need to show that if $\theta \in \mathrm{Aut}(M)$ implements the identity on the sans-core $\mathcal{A}_M^{\mathrm{ns}}$, then θ is inner on M.

This heuristic is supported by the case of automorphisms θ of the free group $\mathbb{F}_2$: if $\theta(a) = a$ and $\theta(b) = gbg^{-1}$, for some $g \in \mathbb{F}_2$, where a, b denote the free generators of $\mathbb{F}_2$, then g must be of the form $g = a^k$, and so $\theta = \mathrm{Ad}(g)$ is inner.

However, this phenomenon fails for the free groups $\mathbb{F}_n$ on $n \geq 3$ generators. Specifically, any $e \neq g \in \mathbb{F}_{n-1} = \langle a_1, \cdots, a_{n-1} \rangle$ gives rise to an outer automorphism θ_g on $\mathbb{F}_n$ defined by $\theta_g(a_i) = a_i$, if $1 \leq i \leq n-1$, and $\theta_g(a_n) = g a_n g^{-1}$, where $a_1, \cdots, a_n$ are the free generators of $\mathbb{F}_n$. Similarly, if $M = A_1 * ... * A_n$, with A_i abelian diffuse, and $n \geq 3$, then any non-scalar unitary $u \in A_1 * ... * A_{n-1} * 1$ gives rise to an outer automorphism θ_u of M defined by $\theta_u(x) = x$, if $x \in A_1 * \cdots * A_{n-1} * 1$, and $\theta_u(x) = uxu^*$, if $x \in 1 * A_n$.

A related problem is to investigate the structure of irreducible subfactors of finite Jones index $N \subset M = A^{*n}$, for A purely non-separable abelian, with an identification of the sans-core, the sans-rank of N and of the set of possible indices $[M : N]$, in the spirit of [Pop06a, Section 7].

4.3. **Amplifications of A^{*n}.** While Theorem 1.1 shows that $\mathcal{F}(A^{*n}) = 1$ if A is non-separable abelian and $n \geq 2$ is finite, it is still of interest to identify the amplifications $(A^{*n})^t$, for $t > 0$. For arbitrary t this remains open, but for $t = 1/k$, $k \in \mathbb{N}$, we have the following result. We are very grateful to Dima Shlyakhtenko for pointing out to us that the 1/2-amplification of A^{*n} can be explicitly calculated for arbitrary diffuse A by using existing models in free probability, a fact that stimulated us to investigate the general $1/k$ case.

Proposition 4.1. *Let (A_i, τ_i), $i \in I$, be a countable collection of diffuse tracial abelian von Neumann algebras. Put $M = *_{i \in I} A_i$ and assume that $|I| \geq 2$. Let $k \geq 2$ and for every $i \in I$, let $p_{i,1}, \cdots, p_{i,k} \in A_i$ be projections such that $\tau(p_{i,j}) = 1/k$, for every $1 \leq j \leq k$, and $\sum_{j=1}^k p_{i,j} = 1$.*

Then M is a II$_1$ factor and $M^{1/k} \cong (*_{i \in I, 1 \leq j \leq k} A_i p_{i,j}) * D$, where

(1) $D = L\mathbb{F}_{1+|I|k(k-1)-k^2}$, if I is finite, and
(2) $D = \mathbb{C}1$, if I is infinite.

Recall that the *interpolated free group factors*, $L\mathbb{F}_r$, $1 < r \leq \infty$, introduced in [Rad94, Dyk94], satisfy the formulas

$$L\mathbb{F}_r * L\mathbb{F}_{r'} \cong L\mathbb{F}_{r+r'} \quad \text{and} \quad (L\mathbb{F}_r)^t \cong L\mathbb{F}_{1+\frac{(r-1)}{t^2}}, \quad \text{for every } 1 \leq r, r' \leq \infty \text{ and } t > 0. \tag{4.1}$$

Proof. We will use the following consequence of [Dyk93, Theorem 1.2]:

Fact 4.2 ([Dyk93])**.** Let P, Q be two tracial von Neumann algebras, and $e \in P$ be a central projection (hence, $P = Pe \oplus P(1-e)$). Denote $R = P * Q$ and $S = (\mathbb{C}e \oplus P(1-e)) * Q \subset R$. Then Pe and eSe are free and together generate eRe, hence $eRe \cong Pe * eSe$.

Specifically, we will use the following consequence of Fact 4.2:

Claim 4.3. Let P, Q be tracial von Neumann algebras and $k \geq 2$. Assume that P and Q admit projections $e_1, \cdots, e_k \in P$ and $f_1, \cdots, f_k \in Q$ such that e_i is central in P, $\tau(e_i) = \tau(f_i) = 1/k$, for every $1 \leq i \leq k$, $\sum_{j=1}^k e_j = 1$ and $\sum_{j=1}^k f_j = 1$. Then $e_1(P * Q)e_1 \cong Pe_1 * \cdots * Pe_k * e_1((\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k) * Q)e_1$.

Proof of Claim 4.3. Note that e_1 is equivalent to e_j in $(\mathbb{C}e_1\oplus\cdots\oplus\mathbb{C}e_k)*(\mathbb{C}f_1\oplus\cdots\oplus\mathbb{C}f_k)$ and so in $(\mathbb{C}e_1\oplus\cdots\oplus\mathbb{C}e_k)*Q$, for every $2\leq j\leq k$. This follows from [Dyk94, Remark 3.3] if $k=2$ and because $(\mathbb{C}e_1\oplus\cdots\oplus\mathbb{C}e_k)*(\mathbb{C}f_1\oplus\cdots\oplus\mathbb{C}f_k)\cong \mathrm{L}(\mathbb{Z}/k\mathbb{Z}*\mathbb{Z}/k\mathbb{Z})$ is a II_1 factor if $k\geq 3$.

Denote $e_j'=1-\sum_{l=1}^j e_l$ and $P_j=\mathbb{C}e_1\oplus\cdots\oplus\mathbb{C}e_j\oplus Pe_j'$, for every $1\leq j\leq k$. We claim that

$$e_1(P*Q)e_1\cong Pe_1*\cdots*Pe_j*e_1(P_j*Q)e_1, \text{ for every } 1\leq j\leq k. \tag{4.2}$$

When $j=1$, $e_1'=1-e_1$ and thus equation (4.2) follows from Fact 4.2. Assume that (4.2) holds for some $1\leq j\leq k-1$. Since $e_{j+1}\in P_j$ is a central projection, $P_je_{j+1}=Pe_{j+1}$ and $\mathbb{C}e_{j+1}\oplus P_j(1-e_{j+1})=P_{j+1}$, Fact 4.2 gives that $e_{j+1}(P_j*Q)e_{j+1}\cong Pe_{j+1}*e_{j+1}(P_{j+1}*Q)e_{j+1}$. The observation made in the beginning of the proof implies that e_1 is equivalent to e_{j+1} in P_j*Q and $P_{j+1}*Q$. Thus, $e_1(P_j*Q)e_1\cong e_{j+1}(P_j*Q)e_{j+1}$ and $e_1(P_{j+1}*Q)e_1\cong e_{j+1}(P_{j+1}*Q)e_{j+1}$. Altogether, $e_1(P_j*Q)e_1\cong Pe_{j+1}*e_1(P_{j+1}*Q)e_1$. This implies that (4.2) holds for $j+1$ and, by induction, proves (4.2). For $j=k$, (4.2) gives the claim. □

To prove the proposition, assume first that I is finite. Take $I=\{1,\cdots,n\}$, for some $n\geq 2$. For $1\leq i\leq n$, put $B_i=\mathbb{C}p_{i,1}\oplus\cdots\oplus\mathbb{C}p_{i,k}$ and $C_i=B_1*\cdots*B_i*A_{i+1}*\cdots*A_n$. We claim that

$$p_{i,1}Mp_{i,1}\cong(*_{1\leq l\leq i,1\leq j\leq k}A_lp_{l,j})*p_{i,1}C_ip_{i,1}, \text{ for every } 1\leq i\leq n. \tag{4.3}$$

The case $i=1$ follows from Claim 4.3. Assume that (4.3) holds for some $1\leq i\leq n-1$. Since the projections $p_{i,1}$ and $p_{i+1,1}$ are equivalent in C_i by the observation made in the beginning of the proof of Claim 4.3, we get that $p_{i,1}Mp_{i,1}\cong p_{i+1,1}Mp_{i+1,1}$ and $p_{i,1}C_ip_{i,1}\cong p_{i+1,1}C_ip_{i+1,1}$. By applying Claim 4.2 to $C_i=A_{i+1}*(B_1*\cdots*B_i*A_{i+2}*\cdots*A_k)$ and the projections $(p_{i+1,j})_{j=1}^k\subset A_{i+1}$, we get that $p_{i+1,1}C_ip_{i+1,1}\cong(*_{1\leq j\leq k}A_{i+1}p_{i+1,j})*p_{i+1,1}C_{i+1}p_{i+1,1}$. The last three isomorphisms together imply that (4.3) holds for $i+1$. By induction, this proves (4.3).

Next, (4.3) for $i=n$ gives that $M^{1/k}\cong(*_{1\leq i\leq n,1\leq j\leq k}A_ip_{i,j})*p_{n,1}C_np_{n,1}$. We will prove that

$$p_{n,1}C_np_{n,1}\cong L\mathbb{F}_{nk(k-1)-k^2+1} \tag{4.4}$$

and thus finish the proof of case (1) by analyzing three separate cases.

If $n=k=2$, then $C_2\cong L\mathbb{Z}\otimes\mathbb{M}_2(\mathbb{C})$ and [Dyk94, Proposition 3.2] impies that $p_{2,1}C_2p_{2,1}\cong L\mathbb{Z}$. If $n>2$ or $k>2$, then $C_n\cong L(*_{i=1}^n\mathbb{Z}/k\mathbb{Z})$ is a II_1 factor. Since $\tau(p_{n,1})=1/k$, we get that $p_{n,1}C_np_{n,1}\cong L(*_{i=1}^n\mathbb{Z}/k\mathbb{Z})^{1/k}$. Assume first that $k=2$ and $n>2$. Recall that $L(*_{j=1}^2\mathbb{Z}/2\mathbb{Z})\cong L\mathbb{Z}\otimes\mathbb{M}_2(\mathbb{C})$ and $(A\otimes\mathbb{M}_2(\mathbb{C}))*L(\mathbb{Z}/2\mathbb{Z})\cong(A*L\mathbb{F}_2)\otimes\mathbb{M}_2(\mathbb{C})$, for every tracial von Neumann algebra A, by [Dyk94, Theorem 3.5 (ii)]. Combining these facts with (4.1) and using induction gives that $L(*_{i=1}^n\mathbb{Z}/2\mathbb{Z})\cong L\mathbb{F}_{n/2}$, thus $L(*_{i=1}^n\mathbb{Z}/2\mathbb{Z})^{1/2}\cong L\mathbb{F}_{2n-3}$. Finally, assume that $k>2$. Then [Dyk93, Corollary 5.3] gives that $L(\mathbb{Z}/k\mathbb{Z}*\mathbb{Z}/k\mathbb{Z})\cong L\mathbb{F}_{2(1-1/k)}$, while [Dyk93, Proposition 2.4] gives that $L\mathbb{F}_r*L(\mathbb{Z}/k\mathbb{Z})\cong L\mathbb{F}_{r+1-1/k}$, for every $r>1$. By combining these facts, we get that $L(*_{i=1}^n\mathbb{Z}/k\mathbb{Z})\cong L\mathbb{F}_{n(1-1/k)}$. Hence, using (4.1) we derive that $L(*_{i=1}^n\mathbb{Z}/k\mathbb{Z})^{1/k}\cong L\mathbb{F}_{1+k^2[n(1-1/k)-1]}=L\mathbb{F}_{1+nk(k-1)-k^2}$. This altogether proves (4.4).

To treat case (2), assume that I is infinite. Take $I=\mathbb{N}$. For $i\geq 0$, let $M_i=A_{2i+1}*A_{2i+2}$. By applying case (1), we get that M_i is a II_1 factor and $M_i^{1/k}\cong(*_{i\leq l\leq i+1,1\leq j\leq k}A_lp_{l,j})*L\mathbb{F}_{(k-1)^2}$, for every $i\geq 0$. Since $M=*_{i\geq 0}M_i$, [DR00, Theorem 1.5] implies that $M^{1/k}\cong *_{k\geq 0}M_i^{1/k}$. Thus, $M^{1/k}\cong(*_{1\leq i,1\leq j\leq k}A_ip_{i,j})*L\mathbb{F}_\infty$. Since $*_{1\leq i,1\leq j\leq k}A_ip_{i,j}$ is a free product of infinitely many II_1 factors, it freely absorbs $L\mathbb{F}_\infty$ by [DR00, Theorem 1.5]. This finishes the proof of case (2). □

We say that an abelian tracial von Neumann algebra (A,τ) is *homogeneous* if for every $k\in\mathbb{N}$, there exists a partition of unity into k projections $p_1,\cdots,p_k\in A$ such that for every $1\leq i\leq k$ we have that $\tau(p_i)=1/k$ and $(Ap_i,k\,\tau_{|Ap_i})$ is isomorphic to (A,τ). A homogeneous abelian von Neumann algebra is necessarily diffuse. Also, note that $L\mathbb{Z}$ and $(L\mathbb{Z})^\omega$ are homogenous, and that the direct sum of two homogeneous abelian von Neumann algebras is homogeneous.

Corollary 4.4. *Let A be a homogeneous abelian tracial von Neumann algebra. Then we have:*

(1) *If $2\leq n<\infty$ and $k\geq 1$, then $(A^{*n})^{1/k}\simeq A^{*nk}*L\mathbb{F}_{1+nk(k-1)-k^2}$.*

(2) $\mathbb{Q} \subset \mathcal{F}(A^{*\infty})$.

Proof. Part (1) follows from Proposition 4.1. Proposition 4.1 also implies that $1/k \in \mathcal{F}(A^{*\infty})$, for every $k \in \mathbb{N}$, and thus part (2) also follows. □

When A is separable (and thus $A \cong L\mathbb{Z}$), Corollary 4.4 recovers two results of Voiculescu [Voi]: the amplification formula $L\mathbb{F}_n^{1/k} \cong L\mathbb{F}_{nk^2-k+1}$ and the fact that $\mathbb{Q} \subset \mathcal{F}(L\mathbb{F}_\infty)$. Corollary 4.4 extends these results to non-separable homogenenous abelian von Neumann algebras A. Recall that Radulescu [Rad92] showed that in fact $\mathcal{F}(L\mathbb{F}_\infty) = \mathbb{R}_+^*$. By analogy with this result, we expect that $\mathcal{F}(A^{*\infty}) = \mathbb{R}_+^*$, for any homogenenous abelian von Neumann algebras A.

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