Asymptotic Plateau Problem in $\mathbb{H}^2 \times \mathbb{R}$: Tall Curves

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ABSTRACT. We study the asymptotic Plateau problem in $\mathbb{H}^2 \times \mathbb{R}$ for area-minimizing surfaces, and give a fairly complete solution for finite curves.

1. INTRODUCTION

The Asymptotic Plateau Problem in $\mathbb{H}^2 \times \mathbb{R}$ studies the existence of a minimal surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ for a given curve Γ in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ with $\partial_\infty \Sigma = \Gamma$. In past years, the existence, uniqueness, and regularity of solutions to the asymptotic Plateau problem in $\mathbb{H}^2 \times \mathbb{R}$ have been studied extensively by the leading researchers of the field (e.g., [CR, CMT, Da, FMMR, KM, MMR, MoR, MRR, NR, PR, RT, ST1, ST2]).

Unlike $\mathbb{H}^3$, the asymptotic Plateau problem in $\mathbb{H}^2 \times \mathbb{R}$ is quite interesting and challenging as there are several families of curves in $S^1_\infty \times \mathbb{R}$, which do not bound any minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ [ST1]. In this paper, we finish off an important case by classifying strongly fillable, finite curves in $S^1_\infty \times \mathbb{R}$ as follows.

Theorem 1.1. *Let Γ be a finite collection of disjoint Jordan curves in $S^1_\infty \times \mathbb{R}$ with $h(\Gamma) \neq \pi$. If Γ is a tall curve, there exists an embedded area-minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$. Conversely, if Γ is a short $C^{1,\alpha}$ non-exceptional curve, then there is no area-minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$.*

The organization of the paper is as follows. In the next section, we give some definitions, and introduce the basic tools which we use in our construction. In Section 3, we introduce tall curves, and study their properties. In Section 4, we prove our main result above. In Section 5, we show that the asymptotic Plateau problem for minimal surfaces and area-minimizing surfaces are quite different,

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and we construct some explicit examples. Finally, in Section 6, we give some concluding remarks, and mention some interesting open problems in the subject. We postpone some technical steps to Appendix A at the end.

2. Preliminaries

In this section, we give the basic definitions, and a brief overview of the past results which will be used in the paper.

Throughout the paper, we use the product compactification of $\mathbb{H}^2 \times \mathbb{R}$. In particular, $\overline{\mathbb{H}^2 \times \mathbb{R}} = \overline{\mathbb{H}^2} \times \overline{\mathbb{R}} = \mathbb{H}^2 \times \mathbb{R} \cup \partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ where $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ consists of three components: the infinite open cylinder $S^1_\infty \times \mathbb{R}$, and the closed caps at infinity $\overline{\mathbb{H}^2} \times \{+\infty\}$, $\overline{\mathbb{H}^2} \times \{-\infty\}$. Hence, $\overline{\mathbb{H}^2 \times \mathbb{R}}$ is a solid cylinder under this compactification.

Let Σ be an open, complete surface in $\mathbb{H}^2 \times \mathbb{R}$, and $\partial_\infty \Sigma$ represent the asymptotic boundary of Σ in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$. Then, if $\bar{\Sigma}$ is the closure of Σ in $\overline{\mathbb{H}^2 \times \mathbb{R}}$, then $\partial_\infty \Sigma = \bar{\Sigma} \cap \partial_\infty(\mathbb{H}^2 \times \mathbb{R})$.

Definition 2.1. A surface is *minimal* if the mean curvature H vanishes everywhere. A compact surface with boundary Σ is called an *area-minimizing surface* if Σ has the smallest area among the surfaces with the same boundary. A noncompact surface is called an *area-minimizing surface* if any compact subsurface is an area-minimizing surface.

Remark 2.2 (Rectifiable Currents with $\mathbb{Z}$-coefficients and Orientation of Surfaces). Throughout the paper, all the surfaces will be orientable, and we use rectifiable currents with $\mathbb{Z}$-coefficients to represent them. In particular, in the definition above, when we say "Σ has the smallest area among the surfaces with the same boundary," we mean any competitor surface S has the same boundary and the same induced orientation, i.e., $\partial S = \partial \Sigma$ and ∂S matches with the orientation of $\partial \Sigma$. A very important point here is that when we say orientable for a nonconnected surface $S = \bigcup_{i=1}^m S_i$, we mean all its components $\{S_i\}$ have consistent orientation, particularly since $\mathbb{H}^2 \times \mathbb{R}$ has trivial topology, and any proper, complete surface S is separating. Then, $S = \bigcup_{i=1}^m$ separate $\mathbb{H}^2 \times \mathbb{R}$ into two types of regions, say blue Ω^+, and black Ω^-. These blue and black regions may not be connected, but for every component S_i, its one side is a black region, and the other is a blue region. Further, that S has consistent orientation means that at each component we chose the normal direction towards, say, blue regions. This convention is crucial when we get the area bound for the sequence $\{S_i^n\}$ via $|\partial \Delta_n|$ in Lemma 2.8.

In this paper, we study the Jordan curves in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ which bound complete, embedded, minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Throughout the paper, when we say *a curve in* $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ we mean a finite collection of pairwise disjoint Jordan curves in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$.

Definition 2.3 (Fillable Curves). Let Γ be a curve in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$. We call Γ *fillable* if Γ bounds a complete, embedded, minimal surface S in $\mathbb{H}^2 \times \mathbb{R}$, that is, $\partial_\infty S = \Gamma$. We call Γ *strongly fillable* if Γ bounds a complete, embedded, area-minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$, that is, $\partial_\infty \Sigma = \Gamma$.

Notice that a strongly fillable curve is fillable since any area-minimizing surface is minimal.

Definition 2.4 (Finite and Infinite Curves). Let Γ be a curve in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$. Decompose $\Gamma = \Gamma^+ \cup \Gamma^- \cup \bar{\Gamma}$ such that $\Gamma^\pm = \Gamma \cap (\overline{\mathbb{H}^2} \times \{\pm\infty\})$ and $\bar{\Gamma} = \Gamma \cap (S^1_\infty \times \mathbb{R})$. In particular, $\Gamma^\pm$ is a collection of closed arcs and points in the closed caps at infinity, while $\bar{\Gamma}$ is a collection of open arcs and closed curves in the infinite open cylinder. With this notation, we call a curve Γ *finite* if $\Gamma^+ = \Gamma^- = \emptyset$. We call Γ *infinite* otherwise.

Problem (***Asymptotic Plateau Problem for*** $\mathbb{H}^2 \times \mathbb{R}$). *Which* (*finite or infinite*) Γ *in* $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ *is fillable or strongly fillable?*

As the question suggests, there are mainly four versions of the problem: the classifications of "fillable finite curves," "fillable infinite curves," "strongly fillable finite curves," and "strongly fillable infinite curves." Unfortunately, we are currently far from classification of the fillable (finite or infinite) curves [FMMR].

Recently, we gave a classification for strongly fillable, infinite curves in [Co2]. In this paper, we give a fairly complete solution for the classification of *strongly fillable, finite curves* in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$.

One of the most interesting properties of the asymptotic Plateau problem in $\mathbb{H}^2 \times \mathbb{R}$ is the existence of non-fillable curves. While any curve Λ in $S^2_\infty(\mathbb{H}^3)$ is strongly fillable in $\mathbb{H}^3$ [An], Sa Earp and Toubiana showed there exist some *non-fillable curves* in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ [ST1].

Definition 2.5 (Thin Tail). Let Γ be a Jordan curve in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$, and let τ be an arc in Γ. Assume there is a vertical straight line L_0 in $S^1_\infty \times \mathbb{R}$ such that the following hold:

- $\tau \cap L_0 \neq \emptyset$ and $\partial\tau \cap L_0 = \emptyset$.
- τ stays in one side of L_0.
- $\tau \subset S^1_\infty \times (c, c + \pi)$ for some $c \in \mathbb{R}$.

Then, we call τ *a thin tail* in Γ.

Lemma 2.6 (***Non-fillable Curves*** [ST1]). *Let* Γ *be a curve in* $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$. *If* Γ *contains a thin tail, then there is no properly immersed minimal surface* Σ *in* $\mathbb{H}^2 \times \mathbb{R}$ *with* $\partial_\infty \Sigma = \Gamma$.

This nonexistence result makes the asymptotic Plateau problem quite interesting. In particular, to address the fillability question, we need to understand which curves have no thin tails. In Section 3, we introduce a notion called *tall curves* to recognize them. Note also that we recently gave the first examples of non-fillable curves with no thin tails [Co3].

To construct our sequence of compact area-minimizing surfaces in our main result, we need the following classical result of geometric measure theory.

Lemma 2.7 (Existence and Regularity of Area-Minimizing Surfaces* [Fe], *Theorems 5.1.6 and 5.4.7). *Let M be a homogeneously regular, closed (or mean convex) 3-manifold. Let γ be a nullhomologous smooth curve in M. Then, γ bounds an area-minimizing surface Σ in M. Furthermore, any such area-minimizing surface is smoothly embedded.*

Now, we state the convergence theorem for area-minimizing surfaces, which will be used throughout the paper. Note that we use convergence in the sense of geometric measure theory, that is, the convergence of rectifiable currents with $\mathbb{Z}$-coefficients in the flat metric.

Lemma 2.8 (Convergence). *Let $\{\Sigma_i\}$ be a sequence of complete area-minimizing surfaces in $\mathbb{H}^2 \times \mathbb{R}$ where $\Gamma_i = \partial_\infty \Sigma_i$ is a finite collection of closed curves in $S^1_\infty \times \mathbb{R}$. If Γ_i converges to a finite collection of closed curves $\hat{\Gamma}$ in $S^1_\infty \times \mathbb{R}$, then there exists a subsequence $\{\Sigma_{n_j}\}$ such that Σ_{n_j} converges to an area-minimizing surface $\hat{\Sigma}$ (possibly empty) with $\partial_\infty \hat{\Sigma} \subset \hat{\Gamma}$. In particular, the convergence is smooth on compact subsets of $\mathbb{H}^2 \times \mathbb{R}$.*

Proof. Let $\Delta_n = \mathbf{B}_n(0) \times [-C, C]$ be convex domains in $\mathbb{H}^2 \times \mathbb{R}$ where $\mathbf{B}_n(0)$ is the closed disk of radius n in $\mathbb{H}^2$ with center 0, and $\hat{\Gamma} \subset S^1_\infty \times (-C, C)$. For n sufficiently large, consider the surfaces $S_i^n = \Sigma_i \cap \Delta_n$. We claim that the area of the surfaces $\{S_i^n \subset \Delta_n\}$ is uniformly bounded by $|\partial \Delta_n|$. Recall that Σ_i is an orientable surface, and area minimizing. Then, as Σ_i is oriented, $\partial S_i^n \subset \partial \Delta_n$ has the induced orientation coming from the orientation of Σ_i. Notice that $\mathbb{H}^2 \times \mathbb{R}$ is a topological ball, and Σ_i separates $\mathbb{H}^2 \times \mathbb{R}$. Let $\mathbb{H}^2 \times \mathbb{R} \setminus \Sigma_i = \Omega_i^+ \cup \Omega_i^-$ (see Remark 2.2). Let $\Omega_i^{n\pm} = \Delta_n \cap \Omega_i^{\pm}$. Then, $\Delta_n \setminus \Sigma_i = \Omega_i^{n+} \cup \Omega_i^{n-}$. In particular, if $\Sigma_i \cap \mathrm{int}(\Delta_n) \neq \emptyset$, Σ_i separates Δ_n.

Now, we have $\partial \overline{\Omega_i^{n\pm}} = S_i^n \cup T_i^{n\pm}$. Hence, $S_i^n \cup T_i^n$ is a closed oriented surface $\partial \overline{\Omega_i^{n\pm}}$. This means ∂T_i^n and ∂S_i^n are oppositely oriented. Hence, ∂S_i^n and $\partial(-T_i^n)$ have the same orientation as rectifiable currents. This shows that ∂S_i^n (with the induced orientation) bounds a surface $(-T_i^n) \subset \partial \Delta_n$. As S_i^n is area minimizing, $|S_i^n| \leq |T_i^n| < |\Delta_n|$. This gives a uniform bound $|\Delta_n|$ on the sequence $\{S_i^n\}$.

Similarly, ∂S_i^n can be bounded by using standard techniques. Hence, if $\{S_i^n\}$ is an infinite sequence, we get a convergent subsequence of $\{S_i^n\}$ in Δ_n with *nonempty limit* S^n. S^n is an area-minimizing surface in Δ_n by the compactness theorem for rectifiable currents (codimension-1) with the flat metric of geometric measure theory [Fe]. By the regularity theory, the limit S^n is a smoothly embedded area-minimizing surface in Δ^n.

If the sequence $\{S_i^n\}$ is an infinite sequence for infinitely many n, we get an infinite sequence of compact area-minimizing surfaces $\{S^n\}$. Then, by using the diagonal sequence argument, we can find a subsequence of $\{\Sigma_i\}$ converging to an area-minimizing surface $\hat{\Sigma}$ with $\partial_\infty \hat{\Sigma} \subset \hat{\Gamma}$ as $\Gamma_i \to \hat{\Gamma}$. Note also that for fixed n, the curvatures of $\{S_i^n\}$ are uniformly bounded by curvature estimates for area-minimizing surfaces. Hence, with the uniform area bound, we get smooth convergence on compact subsets of $\mathbb{H}^2 \times \mathbb{R}$. (See [MW, Theorem 3.3] for further details.) $\square$

Remark 2.9 (Empty Limit). In the proof above, there might be cases like $\{S_i^n\}$ is a finite sequence for any n. In particular, assume that if, for every n, there exists $K_n \gg 0$ such that for every $i > K_n$, $\Sigma_i \cap \Delta_n = \emptyset$. In such a case the limit is empty, and we say $\{\Sigma_i\}$ *escapes to infinity*. An example to this case is a sequence of rectangles R_i in $S^1_\infty \times \mathbb{R}$ with $h(R_i) \searrow \pi$ and $R_i \to \hat{R}$ where $\hat{R}$ is a rectangle of height π. Then, the sequence of area-minimizing surfaces P_i with $\partial_\infty P_i = R_i$ escapes to infinity, as there is no area-minimizing surface Σ with $\partial_\infty \Sigma = \hat{\Gamma}$. In Theorem 4.1, we will prove that if $\hat{\Gamma}$ is a tall curve, the sequence $\{\Sigma_i\}$ does not escape to infinity, and a subsequence Σ_{i_j} converges to an area-minimizing surface $\hat{\Sigma}$ with $\partial_\infty \hat{\Sigma} \subset \hat{\Gamma}$.

Remark 2.10 (Asymptotic Regularity). Unfortunately, there is no general asymptotic regularity result for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ in the literature so far. However, for the horizontal parts of the minimal surface, Kloeckner-Mazzeo gave $C^{k,\alpha}$ regularity [KM, Proposition 3.1]. In particular, they show that if γ is a $C^{k,\alpha}$ horizontal arc in $S^1_\infty \times \mathbb{R}$ (vertical graph over a segment in $S^1_\infty \times \{0\}$), then the minimal surface bounding γ is also $C^{k,\alpha}$ regular up to the boundary.

3. Tall Curves in $S^1_\infty \times \mathbb{R}$

After the Sa Earp-Toubiana nonexistence result (Lemma 2.6), one needs to understand the curves with no thin tails in order to solve the asymptotic Plateau problem. In this section, we introduce a notion called *tall curves* to easily identify such curves. First, we study the tall rectangles. Then, by using these, we define the tall curves.

3.1. Tall rectangles.

Definition 3.1 (Tall Rectangles). Consider the asymptotic cylinder $S^1_\infty \times \mathbb{R}$ with the coordinates (θ, t) where $\theta \in [0, 2\pi)$ and $t \in \mathbb{R}$. We call a rectangle $R = [\theta_1, \theta_2] \times [t_1, t_2] \subset S^1_\infty \times \mathbb{R}$ *tall rectangle* if $t_2 - t_1 > \pi$.

In [ST1], for the boundaries of tall rectangles, Sa Earp and Toubiana further proved the following result.

Lemma 3.2 ([ST1]). *If R is a tall rectangle in $S^1_\infty \times \mathbb{R}$, then there exists a minimal surface P in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty P = \partial R$. In particular, P is a graph over R.*

Furthermore, the authors in [ST1] gave a very explicit description of P as follows. Without loss of generality, let $R = [-\theta_1, \theta_1] \times [-c, c]$ in $S^1_\infty \times \mathbb{R}$ where $c > \pi/2$ and $\theta_1 \in (0, \pi)$. Let φ_t be the hyperbolic isometry of $\mathbb{H}^2$ fixing the geodesic γ with $\partial_\infty \gamma = \{-\theta_1, \theta_1\}$ with translation length t. Let $\hat{\varphi}$ be the isometry of $\mathbb{H}^2 \times \mathbb{R}$ with $\hat{\varphi}_t(q, z) = (\varphi_t(q), z)$. The authors here proved that P is invariant under $\hat{\varphi}_t$ for any t. Let τ be geodesic in $\mathbb{H}^2$ with $\partial_\infty \tau = \{0, \pi\} \subset \partial_\infty \mathbb{H}^2$. Let $\alpha =$

$P \cap (\tau \times \mathbb{R})$. Then, α is the generating curve for P where $\partial_\infty \alpha = \{(0,c),(0,-c)\}$, that is, $P = \bigcup_t \hat{\varphi}_t(\alpha)$.

On the other hand, let P_h be the minimal plane with $\partial_\infty P_h = \partial R_h$ where the height of the rectangle R_h is h, i.e., $2c = h$. The invariance of P_h under the isometry $\hat{\varphi}$ shows that $\gamma_h = P_h \cap \mathbb{H}^2 \times \{0\}$ is an equidistant curve from the geodesic $\hat{\gamma} = \gamma \times \{0\}$ in $\mathbb{H}^2 \times \{0\}$. Let $d_h = \text{dist}(\gamma_h, \hat{\gamma})$. Then, the authors also show that if $h \nearrow \infty$ then $d_h \searrow 0$ and if $h \searrow \pi$ then $d_h \nearrow \infty$. In other words, when $h \to \infty$, P_h gets closer to the vertical geodesic plane $\gamma \times \mathbb{R}$. When $h \searrow \pi$, P_h escapes to infinity. Moreover, the upper half of P_h, $P_h \cap \mathbb{H}^2 \times [0,c]$, is a vertical graph over the component of $\mathbb{H}^2 \times \{0\} \setminus \gamma_h$ in the $R \subset S^1_\infty \times \mathbb{R}$ side.

Now, we show that tall rectangles are indeed quite special. They bound a unique area-minimizing surface which is area minimizing.

Lemma 3.3 (Tall Rectangles are Strongly Fillable). *If R is a tall rectangle in $S^1_\infty \times \mathbb{R}$, then there exists a unique minimal surface P in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty P = \partial R$. Furthermore, P is also an area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$.*

Proof. The outline of the proof is as follows. By using rectangles $\hat{R}_h \subset S^1_\infty \times \mathbb{R}$, we foliate a convex region Δ in $\mathbb{H}^2 \times \mathbb{R}$ by minimal planes $\hat{P}_h$ with $\partial_\infty \hat{P}_h = \partial \hat{R}_h$. As our minimal plane $P = \hat{P}_{h_0}$ is a leaf in this foliation, it is the unique minimal surface bounding $\Gamma_{h_0} = \partial \hat{R}_{h_0}$, and hence is area minimizing.

Step 1: *Defining the convex region* Δ. The convex region Δ will be a component of the complement of a vertical geodesic plane in $\mathbb{H}^2 \times \mathbb{R}$, that is, $\mathbb{H}^2 \times \mathbb{R} \setminus (\eta \times \mathbb{R})$. The setup is as follows. Let $R_h = [-\theta_1, \theta_1] \times [-h, h]$ be a tall rectangle in $S^1_\infty \times \mathbb{R}$, that is, $h > \pi/2$ and $0 < \theta_1 < \pi$. By Lemma 3.2, for any $h > \pi/2$, there exists a minimal surface P_h with $\partial_\infty P_h = \Gamma_h = \partial R_h$. Moreover, by the construction [ST1], $\{P_h\}$ is a continuous family of complete minimal planes with $P_h \cap P_{h'} = \emptyset$ for $h \neq h'$. Now, fix $h_0 > \pi/2$, and let $R_{h_0} = [-\theta_1, \theta_1] \times [-h_0, h_0]$.

Let τ be geodesic in $\mathbb{H}^2$ with $\partial_\infty \tau = \{0, \pi\} \subset \partial_\infty \mathbb{H}^2$. Let ψ_t be the hyperbolic isometry of $\mathbb{H}^2$ that fixes τ, where t is the translation parameter along τ. In particular, in the upper half plane model $\mathbb{H}^2 = \{(x,y) \mid y > 0\}$, $\tau = \{(0,y) \mid y > 0\}$ and $\psi_t(\mathbf{x}) = t\mathbf{x}$. Then, let $\theta_t = \psi_t(\theta_1)$. Then, for $0 < t < \infty$, $0 < \theta_t < \pi$. Hence, $\theta_t < \theta_1$ when $0 < t < 1$, and $\theta_t > \theta_1$ when $1 < t < \infty$. In particular, this implies $[-\theta_1, \theta_1] \subset [-\theta_t, \theta_t]$ for $t > 1$, and $[-\theta_1, \theta_1] \supset [-\theta_t, \theta_t]$ for $t < 1$. For notation, let $\theta_0 = 0$ and let $\theta_\infty = \pi$.

Now, define a continuous family of rectangles $\hat{R}_h$ which foliates an infinite vertical strip in $S^1_\infty \times \mathbb{R}$ as follows. Let $s : (\pi/2, \infty) \to (0,2)$ be a smooth monotone increasing function such that $s(h) \nearrow 2$ when $h \nearrow \infty$, and $s(h) \searrow 0$ when $h \searrow \pi/2$. Furthermore, let $s(h_0) = 1$.

Now, define $\hat{R}_h$ as the rectangle in $S^1_\infty \times \mathbb{R}$ with $\hat{R}_h = [-\theta_{s(h)}, \theta_{s(h)}] \times [-h, h]$. Hence, $\hat{R}_{h_0} = R_{h_0}$, and for any $h \in (\pi/2, \infty)$, $\hat{R}_h$ is a tall rectangle with height $2h > \pi$. Let $\hat{\Gamma}_h = \partial \hat{R}_h$. Then, the family of simple closed curves $\{\hat{\Gamma}_h\}$ foliates the vertical infinite strip $\Omega = ((-\theta_2, \theta_2) \times \mathbb{R}) \setminus (\{0\} \times [-\pi/2, \pi/2])$ in $S^1_\infty \times \mathbb{R}$.

Recall that $R_h = [-\theta_1, \theta_1] \times [-h, h]$ for any $h > \pi/2$, and the planes P_h are minimal surfaces with $\partial_\infty P_h = \Gamma_h$. Let $\hat{\psi}_t$ be the isometry of $\mathbb{H}^2 \times \mathbb{R}$ with $\hat{\psi}_t(p, s) = (\psi_t(p), s)$ where $p \in \mathbb{H}^2$ and $s \in \mathbb{R}$. Then, clearly $\hat{R}_h = \hat{\psi}_{s(h)}(R_h)$. In other words, $\hat{R}_h$ and R_h have the same height, but $\hat{R}_h$ is "widened R_h" in the horizontal direction via isometry $\hat{\psi}$. Similarly, define $\hat{P}_h = \hat{\psi}_{s(h)}(P_h)$. Hence, $\hat{P}_h$ is a complete minimal plane with $\partial_\infty \hat{P}_h = \hat{\Gamma}_h = \partial \hat{R}_h$.

Notice that $\hat{P}_\infty$ is the geodesic plane $\eta \times \mathbb{R}$ in $\mathbb{H}^2 \times \mathbb{R}$ where η is a geodesic in $\mathbb{H}^2$ with $\partial_\infty \eta = \{-\theta_2, \theta_2\}$. Let Δ be the component of $\mathbb{H}^2 \times \mathbb{R} \setminus \hat{P}_\infty$ containing P_{h_0}, that is, $\partial \bar{\Delta} = \hat{P}_\infty$ and $\partial_\infty \bar{\Delta} = \bar{\Omega}$. We claim that the family of complete minimal planes $\{\hat{P}_h \mid h \in (\pi/2, \infty)\}$ foliates Δ.

Step 2: *Foliating* Δ *by minimal planes* $\{\hat{P}_h\}$. Notice that as $\{P_h\}$ is a continuous family of minimal planes, and $\{\hat{\psi}_t\}$ is a continuous family of isometries, then by construction $\hat{P}_h = \hat{\psi}_{s(h)}(P_h)$ is a continuous family of minimal planes, and $\Delta = \bigcup_{h \in (\pi/2, \infty)} \hat{P}_h$. Hence, all we need to show is that $\hat{P}_h \cap \hat{P}_{h'} = \emptyset$ for $h < h'$. First, notice that $P_h \cap P_{h'} = \emptyset$ by [ST1]. Hence, $\hat{\psi}_{s(h)}(P_h) \cap \hat{\psi}_{s(h)}(P_{h'}) = \emptyset$. Let $s' = s(h')/s(h) > 1$.

Notice that both planes $\hat{\psi}_{s(h)}(P_h)$ and $\hat{\psi}_{s(h)}(P_{h'})$ are graphs over rectangles $[-\theta_{s(h)}, \theta_{s(h)}] \times [-h, h]$ and $[-\theta_{s(h)}, \theta_{s(h)}] \times [-h', h']$, respectively. For any $c \in (-h, h)$, the line $\ell_c^{h'} = \hat{\psi}_{s(h)}(P_{h'}) \cap (\mathbb{H}^2 \times \{c\})$ is on the far side ($\pi \in S^1_\infty$ side) of the line $\ell_c^h = \hat{\psi}_{s(h)}(P_h) \cap \mathbb{H}^2 \times \{c\}$ in $\mathbb{H}^2 \times \{c\}$. Hence, for any c, $\psi_{s'}(\ell_c^{h'}) \cap \ell_c^h = \emptyset$ since $\psi_{s'}$ pushes $\mathbb{H}^2$ toward $\pi \in \partial_\infty \mathbb{H}^2$ as $s' > 1$. As $\hat{\psi}_{s'} \circ \hat{\psi}_s(h) = \hat{\psi}_{s'.s(h)} = \hat{\psi}_{s(h')}$, then $\hat{\psi}_{s(h)}(P_h) \cap \hat{\psi}_{s(h')}(P_{h'}) = \emptyset$. In other words, $\hat{P}_h \cap \hat{P}_{h'} = \emptyset$ for $h < h'$. In particular, $\{\hat{P}_h\}$ is a pairwise disjoint family of planes, with $\Delta = \bigcup_{\pi/2}^\infty \hat{P}_h$. This shows that the family of minimal planes $\{\hat{P}_h \mid h \in (\pi/2, \infty)\}$ foliates Δ.

Step 3: P_{h_0} *is the unique minimal surface with asymptotic boundary* $\Gamma_{h_0} = \partial R_{h_0}$ *in* $S^1_\infty \times \mathbb{R}$*, that is,* $\partial_\infty P_{h_0} = \Gamma_{h_0}$. Assume the contrary. If there were another minimal surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \partial R_{h_0}$, then Σ would necessarily belong to the convex region Δ by the convex hull principle. In particular, one can easily see this fact by foliating $\mathbb{H}^2 \times \mathbb{R} \setminus \Delta$ by the geodesic planes $\{\hat{\psi}_t(\hat{P}_\infty) \mid t > 1\}$. Hence, if $\Sigma \not\subseteq \Delta$, then for $t_0 = \sup_t \{\Sigma \cap \hat{\psi}_t(\hat{P}_\infty) \neq \emptyset\}$, Σ would intersect the geodesic plane $\hat{\psi}_{t_0}(\hat{P}_\infty)$ tangentially with lying in one side. This contradicts the maximum principle as both are minimal surfaces.

Now, as $\Sigma \subset \Delta$ and Δ is foliated by $\hat{P}_h$, if $\Sigma \neq P_{h_o}$, then $\Sigma \cap P_h \neq \emptyset$ for some $h \neq h_o$. Then, we have either $h_1 = \sup\{h > h_o \mid \Sigma \cap \hat{P}_h \neq \emptyset\}$ or $h_1' = \inf\{h < h_o \mid \Sigma \cap \hat{P}_h \neq \emptyset\}$ exists. In either case, Σ would intersect $\hat{P}_{h_1}$ or $\hat{P}_{h_1'}$ tangentially by lying in one side. Again, this contradicts the maximum principle as both are minimal surfaces. Hence, such a Σ cannot exist, and the uniqueness follows.

Step 4: P_{h_0} *is indeed an area-minimizing surface in* $\mathbb{H}^2 \times \mathbb{R}$. Now, we finish the proof by showing that P_{h_0} is indeed an area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$. Let B_n be the n-disk in $\mathbb{H}^2$ with the center origin O in the Poincaré disk model, that is, $B_n = \{x \in \mathbb{H}^2 \mid d(x, O) < n\}$. Let $\hat{B}_n = B_n \times [-h_0, h_0]$ in $\mathbb{H}^2 \times \mathbb{R}$. We claim that $P^n_{h_0} = P_{h_0} \cap \hat{B}_n$ is an area-minimizing surface, that is, that $P^n_{h_0}$ has the smallest area among the surfaces S in $\mathbb{H}^2 \times \mathbb{R}$ with the same boundary (i.e., $\partial P^n_{h_0} = \partial S \Rightarrow |P^n_{h_0}| \le |S|$ where $|\cdot|$ represents the area).

Let $\Omega_n = \hat{B}_n \cap \bar{\Delta}$ be a compact, convex subset of $\mathbb{H}^2 \times \mathbb{R}$. Let $\beta_n = \partial P^n_{h_0}$ be a simple closed curve in $\partial \Omega_n$. Notice that by the existence theorem of area-minimizing surfaces (Lemma 2.7), there exists an area-minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial \Sigma = \beta_n$. Furthermore, as Ω_n is convex, $\Sigma \subset \Omega_n$. However, as $\{\hat{P}_h \mid h \in (\pi/2, \infty)\}$ foliates Δ, $\{\hat{P}_h \cap \Omega_n\}$ foliates Ω_n. Much as in the above argument, if Σ is not a leaf of this foliation, there must be a last point of contact with the leaves, which gives a contradiction with the maximum principle. Hence, $\Sigma = P^n_{h_0}$, and $P^n_{h_0}$ is an area-minimizing surface. This shows that any compact subsurface of P_{h_0} is an area-minimizing surface, as it must belong to $P^n_{h_0}$ for sufficiently large $n > 0$. This proves P_{h_0} is an area-minimizing surface with $\partial_\infty P_{h_0} = \Gamma_{h_0}$, and it is the unique minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with asymptotic boundary ∂R_{h_0} in $S^1_\infty \times \mathbb{R}$. As any tall rectangle in $S^1_\infty \times \mathbb{R}$ is the isometric image of R_h for some $\pi/2 < h < \infty$, the proof follows. ◻

3.2. Tall curves. After defining, and studying tall rectangles in $S^1_\infty \times \mathbb{R}$ (Section 3.1), we are now ready to define tall curves in $S^1_\infty \times \mathbb{R}$.

Definition 3.4 (Tall Curves). We call a finite collection of disjoint simple closed curves Γ in $S^1_\infty \times \mathbb{R}$ a *tall curve* if the region $\Gamma^c = S^1_\infty \times \mathbb{R} \setminus \Gamma$ can be written as a union of open tall rectangles $R_i = (\theta^i_1, \theta^i_2) \times (t^i_1, t^i_2)$, that is, $\Gamma^c = \bigcup_i R_i$ (see Figure 3.1). If Γ has more than one component, then we assume Γ has consistent orientation (see Remark 2.2).

We call a region Ω in $S^1_\infty \times \mathbb{R}$ a *tall region*, if Ω can be written as a union of tall rectangles, that is, $\Omega = \bigcup_i R_i$ where R_i is a tall rectangle.

On the other hand, by using the idea above, we can define a notion called *the height of a curve* as follows.

Definition 3.5 (Height of a Curve). Let Γ be a collection of simple closed curves in $S^1_\infty \times \mathbb{R}$, and let $\Omega = S^1_\infty \times \mathbb{R} \setminus \Gamma$. For any $\theta \in [0, 2\pi)$, let $L_\theta = \{\theta\} \times \mathbb{R}$ be the vertical line in $S^1_\infty \times \mathbb{R}$. Let $L_\theta \cap \Omega = \ell^1_\theta \cup \cdots \cup \ell^{i_\theta}_\theta$ where ℓ^i_θ is a component of $L_\theta \cap \Omega$. Define the height $h(\Gamma) = \inf_\theta\{|\ell^i_\theta|\}$.

Notice that Γ is a tall curve if and only if $h(\Gamma) > \pi$. Now, we say Γ is a *short curve* if $h(\Gamma) < \pi$.

Remark 3.6. Note that if Γ is a finite collection of disjoint simple closed curves in $S^1_\infty \times \mathbb{R}$, then we can always write $\Gamma^c = \Omega^+ \cup \Omega^-$ where $\Omega^\pm$ are (possibly disconnected) tall regions with $\partial \overline{\Omega^+} = \partial \overline{\Omega^-} = \Gamma$. Notice that if Γ has more than

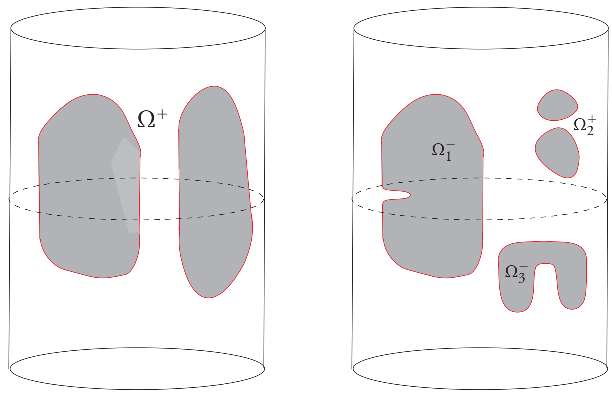


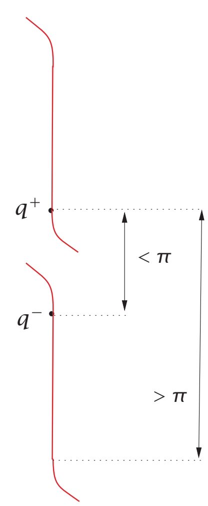
FIGURE 3.1. In the left, Γ is a tall curve with two components. In the right, there are three nonexamples of tall curves. Shaded regions describe the Ω_i^- where $S^1_\infty \times \mathbb{R} \setminus \Gamma_i = \Omega_i^+ \cup \Omega_i^-$.

one component, than Ω^+ or Ω^- may not be connected. Throughout the paper, we assume Γ has the orientation induced by $\Omega^\pm$.

Note also that any curve containing a thin tail is a short curve by definition. However, there are some short curves with no thin tails, like Γ_3 in Figure 3.1-right and Figure 5.1-right.

Notice also that for each nullhomotopic component γ_i of a tall curve Γ, if θ_i^+ (θ_i^-) is a local maximum (minimum) of horizontal coordinates of γ_i, then by Lemma 2.6, $L_{\theta_i^\pm} \cap \gamma_i$ must be a pair of vertical line segments of length greater than π (See Figure 3.1 left). Also, in Figure 3.1 right, three non-tall curves Γ_1, Γ_2 and Γ_3 are pictured as examples. If we name the shaded regions as Ω_i^-, Γ_1 is not tall as Ω_1^+ is not tall because of the small cove. Also, Γ_2 has two components, and it is not tall because Ω_2^+ is not tall (the two components are very close to each other). Finally, Γ_3 is not tall, since Ω_3^- is not a tall region because of the short neck.

Note also that recently in [KMR], Klaser, Menezes, and Ramos generalized the *tall curve* and *height of a curve* notions to the other $\mathbb{E}(-1,\tau)$ homogeneous spaces, and so obtained several existence and nonexistence results for the asymptotic Plateau problem in these spaces.



Remark 3.7 (Exceptional Curves). We call a short curve Γ *exceptional* in the following case. Assume Γ is short. Then, there is a pair of points $\{p^+, p^-\} \subset \Gamma$ where $p^+ = (\theta, c^+)$ and $p^- = (\theta, c^-)$ where $0 < c^+ - c^- < \pi$. Call such a pair of points a *short pair*. If there are points $q^+ = (\theta, d^+)$ and $q^- = (\theta, d^-)$ with $d^- < c^- < c^+ < d^+ < d^- + \pi$ such that $\{q^+, q^-\} \subset S^1_\infty \times \mathbb{R} \setminus \Gamma$, then we call $\{p^-, p^+\}$ a *regular*

short pair. Otherwise, we call it an *exceptional short pair*. If all short pairs are exceptional in a curve, we call it an *exceptional curve*. In other words, if a short curve contains at least one regular pair, it is not exceptional. (See the figure in the right.)
This condition is only used in the nonexistence part of Theorem 1.1 (Step 2). Thus, throughout the paper, we will assume that closed curves in $S^1_\infty \times \mathbb{R}$ are not exceptional unless otherwise stated.[1]

4. ASYMPTOTIC PLATEAU PROBLEM IN $\mathbb{H}^2 \times \mathbb{R}$

In this section, we prove our main result. Note that the converse side of the theorem is true for more general curves (See Remark 4.2).

Theorem 4.1. *Let Γ be a finite collection of disjoint Jordan curves in $S^1_\infty \times \mathbb{R}$ with $h(\Gamma) \neq \pi$. If Γ is a tall curve, there exists an embedded area-minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$. Conversely, if Γ is a short $C^{1,\alpha}$ non-exceptional curve, then there is no area-minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$.*

Outline of the proof: We use the standard techniques for the asymptotic Plateau problem [An]. In particular, we construct a sequence of compact area-minimizing surfaces $\{\Sigma_n\}$ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial \Sigma_n \to \Gamma$, and in the limit, we aim to obtain an area-minimizing surface Σ with $\partial_\infty \Sigma = \Gamma$. Notice that the main issue here is not to show that Σ is an area-minimizing surface, but to show that Σ is not escaping to infinity, that is, $\Sigma \neq \emptyset$ and $\partial_\infty \Sigma = \Gamma$ (see Remark 2.9). Recall that by Lemma 2.6 if a simple closed curve γ in $S^1_\infty \times \mathbb{R}$ has a thin tail, then there is no minimal surface S in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty S = \gamma$. This means that if one similarly constructs area-minimizing surfaces S_n with $\partial S_n \to \gamma$, then either $S = \lim S_n = \emptyset$ or $\partial_\infty S \neq \gamma$; that is, the sequence S_n escapes to infinity completely ($S = \emptyset$), or some parts of the sequence S_n escape to infinity ($\partial_\infty S \neq \gamma$).

In particular, in the following, we aim to show that for a tall curve Γ, the limit surface Σ does not escape to infinity, and $\partial_\infty \Sigma = \Gamma$. We achieve this by constructing barriers near infinity preventing escaping to infinity.

Proof. We split the proof into two parts. In the first part, we show the "if" part. In the second part, we prove the converse.

Step 1: [Existence] If Γ is tall ($h(\Gamma) > \pi$), then there exists an area-minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$.

Step 1A: Construction of the barrier $\mathcal{X}$ near $S^1_\infty \times \mathbb{R}$.

Proof of Step 1A. In this part, we construct a barrier $\mathcal{X}$ which prevents the limit escape to infinity.

Since Γ is a tall curve, by definition, $\Gamma^c = S^1_\infty \times \mathbb{R} \setminus \Gamma = \Omega^+ \cup \Omega^-$ where $\Omega^\pm$ is a tall open region with $\partial \overline{\Omega^\pm} = \Gamma$. Notice that if Γ has more than one component,

[1] We thank the referee for pointing out these exceptional curves.

Ω^+ or Ω^- may not be connected. Note also that for any component of Γ, one side belongs to Ω^+, and the other side belongs to Ω^- by assumption.

Cover $\Omega^\pm$ by tall rectangles $\{R_\alpha\}$ in $S^1_\infty \times \mathbb{R}$ such that $\Omega^\pm = \bigcup_{\alpha \in \mathcal{A}^\pm} R_\alpha$. Further assume that for any $R_\alpha = [\theta_1^\alpha, \theta_2^\alpha] \times I$, the width $|\theta_2^\alpha - \theta_1^\alpha| < \pi$. For each tall rectangle R_α, by Lemma 3.3, there exists a unique area-minimizing surface P_α with $\partial_\infty P_\alpha = \partial R_\alpha$. Let Δ_α be the component of $\mathbb{H}^2 \times \mathbb{R} \setminus P_\alpha$ with $\partial_\infty \Delta_\alpha = R_\alpha$. Define $\mathcal{X}^\pm = \bigcup_{\alpha \in \mathcal{A}^\pm} \Delta_\alpha$. Then, by construction $\partial_\infty \mathcal{X}^\pm = \Omega^\pm$. Let $\mathcal{X} = \mathcal{X}^+ \cup \mathcal{X}^-$. We call $\mathcal{X}$ a barrier near infinity. ◻

Step 1B: Construction of the sequence $\{\Sigma_n\}$.

Proof of Step 1B. Let $C > 0$ be sufficiently large that $\Gamma \subset \partial_\infty \mathbb{H}^2 \times (-C, C)$. Let B_n be the n-disk in $\mathbb{H}^2$ with the center origin, and $\hat{B}_n = B_n \times [-C, C]$ be an compact solid cylinder in $\mathbb{H}^2 \times \mathbb{R}$. Let Γ_n be the radial projection of Γ into the cylinder $\partial B_n \times [-C, C]$. We define radial projection $\psi_n : S^1_\infty \times [-C, C] \to \partial \hat{B}_n$ for $(\theta, t) \in S^1_\infty \times [-2m, 2m]$ as follows. For $\mathbb{H}^2 \times [-C, C]$, use polar coordinates where $q = (r, \theta, t)$ represents q is a point the plane $\mathbb{H}^2 \times \{t\}$ with $d(q, O_t) = r$ where O_t is the origin in the plane $\mathbb{H}^2 \times \{t\}$. Then, we define $\psi_n(\theta, t) = (n, \theta, t) \in \hat{B}_n$, and then $\Gamma_n = \psi_n(\Gamma)$. For a given Γ, we can choose N_0 sufficiently large so that for any α and $n > N_0$, $\Delta_\alpha \cap \partial \hat{B}_n \subset \psi_n(R_\alpha)$ as we assume the width of any R_α is $< \pi$.

Now, Γ_n is a finite union of disjoint Jordan curves in $\partial \hat{B}_n$. Notice that for any $\alpha \in \mathcal{A}^\pm$, P_α is a graph over R_α by Section 3. Hence, as $R_\alpha \cap \Gamma = \emptyset$ in $S^1_\infty \times \mathbb{R}$, their radial projections are also disjoint, that is, $\psi_n(R_\alpha) \cap \psi_n(\Gamma) = \emptyset$ in $\partial \hat{B}_n$. Again, by construction, $\Delta_\alpha \cap \partial \hat{B}_n \subset \psi_n(R_\alpha)$. This implies $\Gamma_n \cap \mathcal{X} = \emptyset$ for any n.

Let Σ_n be the area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial \Sigma_n = \Gamma_n$ by Lemma 2.7. Then, as $\hat{B}_n$ is convex, $\Sigma_n \subset \hat{B}_n$. ◻

Step 1C: For any n, $\Sigma_n \cap \mathcal{X} = \emptyset$.

Proof of Step 1C. Recall that $\mathcal{X}^\pm = \bigcup_{\alpha \in \mathcal{A}^\pm} \Delta_\alpha$. Hence, if we can show that for any $\alpha \in \mathcal{A}^\pm$, $\Sigma_n \cap \Delta_\alpha = \emptyset$, we are done. Notice that for any α, Δ_α is foliated by tall rectangles by Lemma 3.3 (Step 2). This means if $\Sigma_n \cap \Delta_\alpha \neq \emptyset$, Σ_n will have a last point of touch in the minimal foliation, and tangential intersection with a minimal leaf. This contradicts the maximum principle. Step 1C follows. ◻

Step 1D: The limit of area-minimizing surface Σ with $\partial_\infty \Sigma = \Gamma$.

Proof of Step 1D. Let Σ be the limit of $\{\Sigma_n\}$ in $\mathbb{H}^2 \times \mathbb{R}$ given by Lemma 2.8. Notice that the limit might be empty. When showing that $\partial_\infty \Sigma = \Gamma$, we also prove that Σ is not an empty limit.

First, we show that $\partial_\infty \Sigma \subset \Gamma$. By Step 1C, $\Sigma_n \cap \mathcal{X} = \emptyset$. We claim that $\partial_\infty \Sigma \subset \Gamma$. Let p be a point in $S^1_\infty \times \mathbb{R} \setminus \Gamma$. Then, by construction, p_0 belongs to the interior of $R_{\alpha_0} \subset S^1_\infty \times \mathbb{R}$. Since $\Delta_{\alpha_0} \cap \Sigma_n = \emptyset$ for any n. Hence, p cannot be in $\partial_\infty \Sigma$. This proves $\partial_\infty \Sigma \subset \Gamma$. ◻

We finish the proof by showing that $\partial_\infty \Sigma \supset \Gamma$. Let $p \in \Gamma$. We will show that $p \in \bar{\Sigma}$. Let p be in the component γ in Γ. As $\Gamma^c = \Omega^+ \cup \Omega^-$, let $\{p_i^\pm\} \subset \Omega^\pm$ be two sequences in opposite sides of γ with $\lim p_i^\pm = p$. Let α_i be a small circular arc in $\overline{\mathbb{H}^2 \times \mathbb{R}}$ with $\partial \alpha_i = \{p_i^+, p_i^-\}$ and $\alpha_i \perp S^1_\infty \times \mathbb{R}$. Then, for any i, there exists N_i such that for any $n > N_i$, Γ_n links α_i, that is, Γ_n is not nullhomologous in $\mathbb{H}^2 \times \mathbb{R} \setminus \alpha_i$. Hence, for any $n > N_i$, $\alpha_i \cap \Sigma_n \neq \emptyset$. This implies $\Sigma \cap \alpha_i \neq \emptyset$ for any i by construction. As above, let $R_i^\pm \subset \Omega^\pm$ be the tall rectangle with $p_i^\pm \subset R_i^\pm$. Similarly, let $P_i^\pm$ be the unique area-minimizing surface with $\partial_\infty P_i^\pm = \partial R_i^\pm$. Let $\alpha_i' \subset \alpha_i$ be a subarc with $\partial \alpha_i' \subset P_i^+ \cup P_i^-$. Hence, α_i' is a compact arc in $\mathbb{H}^2 \times \mathbb{R}$. Moreover, as $\Sigma \cap P_i^\pm = \emptyset$, then there exists a point x_i in $\Sigma \cap \alpha_i'$ for any i. Then, $\lim x_i = p$. Hence, $p \in \bar{\Sigma}$, and $\partial_\infty \Sigma = \Gamma$. Notice that this shows the limit does not escape to infinity, and the area-minimizing surface Σ is not empty, either. Step 1 follows. ◻

Step 2: [Nonexistence] If Γ is a short $C^{1,\alpha}$ curve, then there is no area-minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$.

Proof. Assume there is an area-minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$. Recall that, here, Γ being short means $h(\Gamma) < \pi$. Since Γ is a non-exceptional short curve in $S^1_\infty \times \mathbb{R}$, there is a regular short pair $q^\pm = (\theta_0, d^\pm) \in \Gamma$ where $0 < d^+ - d^- < \pi - 2\varepsilon$ for some $\varepsilon > 0$. Since $q^\pm$ is a regular short pair, there exists $p^\pm \notin \Gamma$ with $p^\pm = (\theta_0, c^\pm)$ and $c^- < d^- < d^+ < c^+ < c^- + \pi$ along the line $\{\theta_0\} \times \mathbb{R} \in S^1_\infty \times \mathbb{R}$ (see Remark 3.7).

Let $O^\pm$ be an open neighborhood of $p^\pm$ in $\overline{\mathbb{H}^2 \times \mathbb{R}}$ such that $O^\pm \cap \bar{\Sigma} = \emptyset$. Let $D^\pm = (\mathbb{H}^2 \times \{c^\pm\}) \cap O^\pm$. By construction, $D^\pm$ contains a half plane in the hyperbolic plane $\mathbb{H}^2 \times \{c^\pm\}$.

By Lemma A.1 and Lemma A.3 in the Appendix, for any $h < \pi$, there exists an *area-minimizing* compact catenoid S of height h. For $h = c^+ - c^- < \pi$, let S be the area-minimizing compact catenoid with $\partial S \subset \mathbb{H}^2 \times \{c^-, c^+\}$. In other words, ∂S consists of two curves γ^+ and γ^- where $\gamma^\pm$ is a round circle of radius $\hat{\rho}(d)$ in $\mathbb{H}^2 \times \{c^\pm\}$ centered at the origin. Let θ_1 be the antipodal point of θ_0 in S^1_∞. Let ψ_t be the hyperbolic isometry fixing the geodesic between θ_0 and θ_1. In particular, ψ_t corresponds to $\psi_t(x, y) = (tx, ty)$ in the upper half space model where θ_1 corresponds to the origin, and θ_0 corresponds to the point at infinity. Let $\hat{\psi}_t : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$ be the isometry of $\mathbb{H}^2 \times \mathbb{R}$ where $\hat{\psi}_t(p, z) = (\psi_t(p), z)$.

Let $S_t = \hat{\psi}_t(S)$ be the isometric image of the area-minimizing catenoid S in $\mathbb{H}^2 \times \mathbb{R}$. Let $\partial S_t = \gamma_t^+ \cup \gamma_t^-$ where $\gamma_t^\pm = \psi_t(\gamma^\pm)$. Notice that $\gamma_t^\pm \subset \mathbb{H}^2 \times \{c^\pm\}$. Let $N_o > 0$ be sufficiently large that $\gamma_t^+ \subset D^+$ and $\gamma_t^- \subset D^-$ for any $t > N_o$. Then, for any $t > N_o$, $\partial S_t \subset D^+ \cup D^-$, and $\partial S_t \cap \Sigma = \emptyset$.

Recall $\partial_\infty(\mathbb{H}^2 \times \mathbb{R}) \setminus \Gamma = \Omega^+ \cup \Omega^-$ where $\partial \Omega^+ = \partial \Omega^- = \Gamma$. Let $\overline{\mathbb{H}^2 \times \mathbb{R}} \setminus \bar{\Sigma} = \Delta^+ \cup \Delta^-$ where $\partial_\infty \Delta^\pm = \Omega^\pm$. Let $\beta = \{\theta_0\} \times (d^+, d^-)$ be the vertical line segment in $S^1_\infty\times\mathbb{R}$, and let $\beta\subset\Omega^+$. Since Δ^+ is an open subset in $\overline{\mathbb{H}^2\times\mathbb{R}}$ and $\beta\subset\Delta^+$, then an open neighborhood O_β of β in $\mathbb{H}^2\times\mathbb{R}$ must belong to Δ^+. Then, by construction, we can choose $t_o > N_o$ sufficiently large that $S_{t_o}\cap O_\beta\neq\emptyset$ and $S_{t_o}\cap O_\beta$ is connected. As Γ is $C^{1,\alpha}$, Σ is $C^{1,\alpha}$ regular up to the boundary by [KM, Proposition 3.1]. As Γ is not exceptional, we can choose S_{t_0} such that $\partial S_{t_0}\cap\Sigma=\emptyset$. Therefore, $S_{t_o}\cap\Sigma\neq\emptyset$.

Let $D^\pm$ be the caps of the compact catenoid S_{t_0}. Let U be the compact region enclosed by $S_{t_0}\cup D^+\cup D^-$. Let $S_{t_o}\cap\Sigma=\alpha$. Notice that as both Σ and S_{t_o} are area-minimizing surfaces and $\partial S_t\cap\Sigma=\emptyset$, α is a collection of closed curves, and contains no isolated points because of the maximum principle.

Let $E=U\cap\Sigma$. Then, E is a compact subsurface of Σ with $\partial E=\alpha$. In other words, S_{t_o} separates E from Σ. Similarly, let T be the subsurface of S_{t_o} with $\partial T=\alpha$. In particular, $T=S_{t_o}\cap\overline{\Delta^+}$. Since S_{t_o} and Σ are area-minimizing surfaces, so are T and E. As $\partial T=\partial E=\alpha$, and both are area-minimizing surfaces, both have the same area, that is, $|E|=|T|$.

Let $S'=(S_{t_o}\setminus T)\cup E$. Then, clearly $\partial S_{t_o}=\partial S'$ and $|S_{t_o}|=|S'|$. Hence, as S_{t_o} is an area-minimizing surface, so is S'. However, S' has singularity along α. This contradicts the regularity of area-minimizing surfaces (Lemma 2.7). Step 2 follows. ◻

Remark 4.2 (Generalization to C^0 Curves). Note that in the proof of the converse, we use the $C^{1,\alpha}$ condition only in the neighborhood of the short segment, that is, near $\{q^-,q^+\}\in\Gamma$. Thus, one can generalize the statement to C^0 curves containing a pair of $C^{1,\alpha}$ segments $\gamma^\pm$ which are vertically close ($<\pi$).

Remark 4.3 (The $h(\Gamma)=\pi$ Case). Notice that the theorem finishes off the asymptotic Plateau problem for $\mathbb{H}^2\times\mathbb{R}$ except the case $h(\Gamma)=\pi$. Note that this case is delicate as there are strongly fillable and strongly non-fillable curves of height π. For example, if Γ_1 is a rectangle in $S^1_\infty\times\mathbb{R}$ with height π, then the discussion in Remark 2.9 shows that Γ_1 bounds no minimal surface; hence, such a Γ is nonfillable. On the other hand, in Theorem 5.1, if we take $h_0=\pi$ and use the parabolic catenoid ([Da]), it is not hard to show that the constructed surface is also area minimizing in $\mathbb{H}^2\times\mathbb{R}$ since the parabolic catenoid is also area minimizing. (See Figure 5.1-right.) These two examples show the case $h(\Gamma)=\pi$ is very delicate. Note also that Sa Earp and Toubiana studied a relevant problem in [ST1, Corollary 2.1].

Remark 4.4 (Minimal vs. Area Minimizing). Notice that the theorem above does not say that *If γ is a short curve, then there is no* minimal *surface S in $\mathbb{H}^2\times\mathbb{R}$ with $\partial_\infty S=\gamma$.* There are many examples of complete embedded minimal surfaces S in $\mathbb{H}^2\times\mathbb{R}$ where the asymptotic boundary γ is a short curve (e.g., butterfly curves). We postpone this question to Section 5 to discuss in detail.

4.1. Convex hull property for tall curves. In this part, we give a natural generalization of the convex hull property for the asymptotic Plateau problem in $\mathbb{H}^2\times\mathbb{R}$.

Definition 4.5 (Mean Convex Hull). Let Γ be a tall curve in $S^1_\infty \times \mathbb{R}$. Consider the barrier $\mathcal{X}$ constructed in Step 1A in the proof of Theorem 4.1. Define the mean convex hull of Γ as $\mathrm{MCH}(\Gamma) = \mathbb{H}^2 \times \mathbb{R} \setminus \mathcal{X}$. Notice that $\mathrm{MCH}(\Gamma)$ is a mean convex region in $\mathbb{H}^2 \times \mathbb{R}$ by construction. Furthermore, $\partial_\infty \mathrm{MCH}(\Gamma) = \Gamma$.

Analogous to the convex hull property in $\mathbb{H}^3$, we have the following property in $\mathbb{H}^2 \times \mathbb{R}$.

Corollary 4.6 (Convex Hull Property). *Let Γ be a tall curve in $S^1_\infty \times \mathbb{R}$. Let S be a complete, embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty S = \Gamma$. Then, we have $S \subset \mathrm{MCH}(\Gamma)$.*

Proof. The proof is similar to the convex hull property in other homogeneous ambient spaces. We use the same notation as in the proof of Theorem 4.1. In that proof, we proved that for our special sequence $\{\Sigma_n\}$ and the limit Σ, $\Sigma \cap \mathcal{X} = \emptyset$. However, the same proof works for any area-minimizing surface S with $\partial_\infty S = \Gamma$.

Recall that $\Gamma^c = \bigcup R_\alpha$ where R_α are tall rectangles. Let P_α be the unique area-minimizing surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty P_\alpha = \partial R_\alpha$. Let Δ_α be the components of $\mathbb{H}^2 \times \mathbb{R} \setminus P_\alpha$ with $\partial_\infty \Delta_\alpha = \mathrm{int}(R_\alpha)$. Then, $\mathcal{X} = \bigcup_\alpha \Delta_\alpha$.

Assume $S \not\subset \mathrm{MCH}(\Gamma) = \mathcal{X}^c$. Then, $S \cap \Delta_\alpha \neq \emptyset$ for some α. However, by the proof of Lemma 3.3, we know that Δ_α is foliated by minimal surfaces $\{P_t \mid t \in [0, \infty)\}$. Let $t_0 = \sup_t \{P_t \cap \Sigma \neq \emptyset\}$. Again, by maximum principle, this is a contradiction as both Σ and P_{t_0} are minimal surfaces. The proof follows. ∎

One can visualize $\mathrm{MCH}(\Gamma)$ as follows. Assume $\Gamma \subset S^1_\infty \times [c_1, c_2]$ for the smallest $[c_1, c_2]$ possible. Then, $\mathrm{MCH}(\Gamma)$ is the region in $\mathbb{H}^2 \times [c_1, c_2]$ where we carve out all Δ_α defined by rectangles $R_\alpha \subset \Gamma^c$.

5. Asymptotic Plateau Problem for Minimal Surfaces

So far, we only dealt with the strong fillability question, that is, detecting curves in $S^1_\infty \times \mathbb{R}$ bounding *area-minimizing surfaces* in $\mathbb{H}^2 \times \mathbb{R}$. If we relax the question from "strong fillability" to only "fillability," the picture completely changes. In other words, we will see that detecting curves in $S^1_\infty \times \mathbb{R}$ bounding embedded *minimal surfaces* is much more complicated than detecting the curves bounding embedded *area-minimizing surfaces*. In Theorem 4.1, we gave a fairly complete answer to the asymptotic Plateau problem in the strong fillability case. In this section, we will see that the classification of fillable curves is highly different.

A simple example to show the drastic change in the problem is the following. Let $\Gamma = \gamma_1 \cup \gamma_2$ where $\gamma_i = S^1_\infty \times \{c_i\}$ and $|c_1 - c_2| < \pi$. Then clearly, Γ is a short curve and it bounds a complete minimal catenoid $\mathcal{C}_d$ by [NSST] (See also Appendix A for further discussion on catenoids.) On the other hand, the pair of geodesic planes $\mathbb{H}^2 \times \{c_1\} \cup \mathbb{H}^2 \times \{c_2\}$ also bounds $\Gamma = \gamma_1 \cup \gamma_2$. However, there is no area-minimizing surface Σ with $\partial_\infty \Sigma = \gamma_1 \cup \gamma_2$ by Theorem 4.1. This means neither the catenoid, nor pair of geodesic planes are area minimizing, but just minimal surfaces. Hence, the following version of the problem becomes very interesting.

Problem (***Asymptotic Plateau Problem for Minimal Surfaces in*** $\mathbb{H}^2 \times \mathbb{R}$). *For which curves* Γ *in* $S^1_\infty \times \mathbb{R}$*, there exists an embedded minimal surface* S *in* $\mathbb{H}^2 \times \mathbb{R}$ *with* $\partial_\infty S = \Gamma$.

In other words, *which curves in* $S^1_\infty \times \mathbb{R}$ *are fillable?* (Note that here we only discuss the finite curve case ($\Gamma \subset S^1_\infty \times \mathbb{R}$). For infinite curves case for the same question, see [Co2].)

Recall that by Lemma 2.6, for any short curve γ in $S^1_\infty \times \mathbb{R}$ containing a thin tail, there is no complete minimal surface S in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty S = \gamma$. Thus, this result suggests that the minimal surface case is similar to the area-minimizing surface case.

On the other hand, unlike the area-minimizing surface case, it is quite easy to construct short curves with more than one component, bounding minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Let $\Gamma = \gamma_1 \cup \cdots \cup \gamma_n$ be a finite collection of disjoint tall curves γ_i. Even though every component γ_i is tall, because of the vertical distances between the components γ_i and γ_j, the height $h(\Gamma)$ can be very small. Thus, Γ itself might be a short curve, even though every component is a tall curve. For each component γ_i, our existence theorem (Theorem 4.1) already gives an area-minimizing surface Σ_i with $\partial_\infty \Sigma_i = \gamma_i$. Hence, the surface $\hat{S} = \Sigma_1 \cup \cdots \cup \Sigma_n$ is automatically a minimal surface with $\partial_\infty \hat{S} = \Gamma$. By using this idea, for any height $h_0 > 0$, we can trivially produce short curves Γ with height $h(\Gamma) = h_0$ by choosing the components sufficiently close, for example, the pair of horizontal geodesic planes $\mathbb{H}^2 \times \{c_1\} \cup \mathbb{H}^2 \times \{c_2\}$ with $|c_1 - c_2| = h_0$.

Naturally, the next question would be what if Γ has only one component. Does Γ need to be a tall curve to bound a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$? The answer is again no. Now, we also construct *simple closed short curves* which bound complete minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. The following result with the observation above shows that the asymptotic Plateau problem for minimal surfaces is very different from the asymptotic Plateau problem for area-minimizing surfaces.

Theorem 5.1. *For any* $h_0 > 0$*, there exists a nullhomotopic simple closed curve* Γ *with height* $h(\Gamma) = h_0$ *such that there exists a minimal surface* S *in* $\mathbb{H}^2 \times \mathbb{R}$ *with* $\partial_\infty S = \Gamma$.

Proof. For $h_0 > \pi$, we have tall rectangles with height h_0. Thus, we assume $0 < h_0 \leq \pi$. Consider the rectangles $R^+ = [s, \pi/2] \times [-m, m]$ and $R^- = [-\pi/2, -s] \times [-m, m]$ with $s > 0$ sufficiently small, and $m \gg 0$ sufficiently large, which will be fixed later. Consider another rectangle $Q = [-s, s] \times [0, h_0]$. Consider the area-minimizing surfaces P^+ and P^- with $\partial_\infty P^\pm = \partial R^\pm$. Let $\Gamma = (\partial R^+ \cup \partial R^-) \triangle \partial Q$ where $\triangle$ represents the symmetric difference (see Figure 5.1). Notice that $h(\Gamma) = h_0$. We claim there exists a complete embedded minimal surface S in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty S = \Gamma$.

Consider now the minimal catenoid C_{h_0} that has an asymptotic boundary $S^1_\infty \times \{0\} \cup S^1_\infty \times \{h_0\}$. If $h_0 = \pi$, take C_{h_0} to be the Daniel's parabolic catenoid where $\partial_\infty C_\pi = (S^1_\infty \times \{-\pi/2, +\pi/2\}) \cup (\{\pi\} \times [-\pi/2, +\pi/2])$. Let φ_t be the isometry of $\mathbb{H}^2 \times \mathbb{R}$ which keeps $\mathbb{R}$ coordinates the same, and fixes the geodesic ℓ in $\mathbb{H}^2$ with $\partial_\infty \ell = \{0, \pi\}$ with translation length $\log t$. In particular, we have $\varphi_t|_{\mathbb{H}^2 \times \{c\}} : \mathbb{H}^2 \times \{c\} \to \mathbb{H}^2 \times \{c\}$. Furthermore, for any $p \in \mathbb{H}^2 \times \{c\}$, we have $\varphi_t(p) \to (0, c) \in S^1_\infty \times \mathbb{R}$ as $t \searrow 0$ and $\varphi_t(p) \to (\pi, c) \in S^1_\infty \times \mathbb{R}$ as $t \searrow \infty$. Now, we can choose $t > 0$ and $s > 0$ sufficiently small, and $m > 0$ sufficiently large, so that $P^+ \cup P^-$ separates $\varphi_t(C_{h_0}) = C^t_{h_0}$ into four disks (see Figure 5.1). In other words, there is a component Δ in $\mathbb{H}^2 \times \mathbb{R} \setminus (P^+ \cup P^- \cup C^t_{h_0})$ such that $\partial_\infty \Delta = Q$.

Now, let Ω^+ be the component of $\mathbb{H}^2 \times \mathbb{R} \setminus P^+$ such that $\partial_\infty \Omega^+ = R^+$. Similarly, let Ω^- be the component of $\mathbb{H}^2 \times \mathbb{R} \setminus P^-$ such that $\partial_\infty \Omega^- = R^-$. Let $\mathcal{X} = \mathbb{H}^2 \times \mathbb{R} \setminus (\Omega^+ \cup \Omega^- \cup \Delta)$. Hence, $\mathcal{X}$ is a mean convex domain in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \mathcal{X} = \partial_\infty(\mathbb{H}^2 \times \mathbb{R}) \setminus \text{int}(R^+ \cup R^- \cup Q)$. Hence, $\Gamma \subset \partial_\infty \mathcal{X}$.

Now, let $\mathbf{B}_n(0)$ be the ball of radius n in $\mathbb{H}^2$ with center 0. Let also $\mathbf{D}_n = \mathbf{B}_n(0) \times [-2m, 2m]$. Let $\hat{\mathbf{D}}_n = \mathbf{D}_n \cap \mathcal{X}$. Let Γ_n be the radial projection of Γ to $\partial \hat{\mathbf{D}}_n$. In particular, we define radial projection $\psi_n : S^1_\infty \times [-2m, 2m] \to \partial \hat{\mathbf{D}}_n$ for $(\theta, t) \in S^1_\infty \times [-2m, 2m]$ as follows. For $\mathbb{H}^2 \times [-2m, 2m]$, use polar coordinates where $q = (r, \theta, t)$ represents q as a point in the plane $\mathbb{H}^2 \times \{t\}$ with $d(q, O_t) = r$ where O_t is the origin in the plane $\mathbb{H}^2 \times \{t\}$. Define

$$R_n(\theta_0, t_0) = \sup\{r \in \mathbb{R}^+ \mid (r, \theta_0, t_0) \in \hat{\mathbf{D}}_n\} \leq n.$$

Then, define $\psi_n(\theta, t) = (R_n(\theta), \theta, t)$. Then, $\Gamma_n = \psi_n(\Gamma)$.

Let S_n be the area-minimizing surface in $\hat{\mathbf{D}}_n$ with $\partial S_n = \Gamma_n$. Since $\hat{\mathbf{D}}_n$ is mean convex, S_n is a smooth embedded surface in $\hat{\mathbf{D}}_n$. Again, by using Lemma 2.8, we get an area-minimizing surface S in $\mathcal{X}$. By using similar ideas in Theorem 4.1

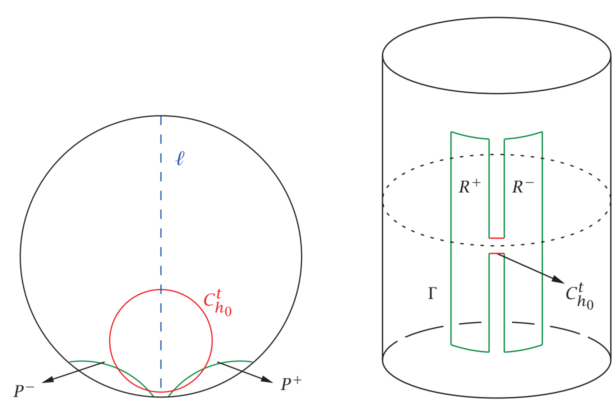


FIGURE 5.1. In the left, the horizontal slice $\mathbb{H}^2 \times \{h_0/2\}$ is given. In the right, $\Gamma \subset S^1_\infty \times \mathbb{R}$ is pictured.

(Step 1), it can be shown that $\partial_\infty S = \Gamma$. While S is an area-minimizing surface in $\mathcal{X}$, it is only a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. The proof follows. ◻

Remark 5.2. Notice that for smaller choice of $h_0 > 0$ in the theorem above, one needs to choose the height $2m$ of the rectangles large, and the distance s of the rectangles small by the construction (see Figure 5.1).

Recently, Kloeckner and Mazzeo have also studied these curves more extensively in [KM], where they call these curves *butterfly curves*. In [KM], they also constructed different families of complete minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Furthermore, they studied the asymptotic behavior of the minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Remark 5.3. Recently, we showed that when we choose $h_0 < \pi$ and $2m > \pi$ sufficiently close, the butterfly curve $\Gamma^m_{h_0}$ constructed above does not bound any minimal surface, either. This example is the first non-fillable example in $S^1_\infty \times \mathbb{R}$ with no thin tail [Co3].

6. Final Remarks

6.1. Infinite curves. In this paper, we only dealt with the finite curves, that is, $\Gamma \subset S^1_\infty \times \mathbb{R}$. On the other hand, the infinite curve case is also very interesting ($\Gamma \cap (\overline{\mathbb{H}^2} \times \{\pm\infty\}) \neq \emptyset$). In [Co2], we studied this problem, and gave a fairly complete solution in the strongly fillable case. Kloeckner and Mazzeo studied this problem in [KM], and constructed a rich and interesting family of fillable infinite curves.

On the other hand, strong fillability questions and fillability questions are quite different in both finite and infinite curve case. While we gave a classification result for strongly fillable infinite curves in [Co2], the examples in Section 4 of [Co2] show that there are many fillable and non-fillable infinite curve families, and we are far from classification of these infinite curves in the fillable case.

6.2. Fillable curves. In Section 5, when we relax the question from "existence of area-minimizing surfaces" to "existence of minimal surfaces," we see that the picture completely changes. While Theorem 4.1 shows that if $h(\Gamma) < \pi$, there is no area-minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$, we constructed many examples of short fillable curves Γ in $S^1_\infty \times \mathbb{R}$ for any height in Section 5.

Again, by the Sa Earp and Toubiana nonexistence theorem (Lemma 2.6), if Γ contains a thin tail, then there is no minimal surface S in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty S = \Gamma$. Hence, the following classification problem is quite interesting and wide open.

Problem (***Classification of Fillable Curves in*** $\mathbb{H}^2 \times \mathbb{R}$). *For which curves* Γ *in* $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ *does there exist a minimal surface* S *in* $\mathbb{H}^2 \times \mathbb{R}$ *with* $\partial_\infty S = \Gamma$*?*

Note that Kloeckner and Mazzeo have studied this problem, and constructed many families of examples. They have also studied the asymptotic behavior of these complete minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ in [KM]. Further, in [FMMR], the authors have recently studied the existence of vertical minimal annuli in $\mathbb{H}^2 \times \mathbb{R}$, and given a very interesting classification. Note also that we constructed the first examples of non-fillable finite curves with no thin tail in [Co3].

APPENDIX A.

In this part, we give some technical lemmas used in the proof of the main theorem.

1.1. Area minimizing catenoids in $\mathbb{H}^2 \times \mathbb{R}$**.** In this section, we study the family of minimal catenoids C_d described in [NSST], and show that for sufficiently large $d > 0$, a compact subsurface $S_d \subset C_d$ near girth of the catenoid C_d is an area-minimizing surface.

First, we recall some results on the rotationally symmetric minimal catenoids C_d [NSST, Proposition 5.1]. Let (ρ, θ, z) represent the coordinates on $\mathbb{H}^2 \times \mathbb{R}$ with the metric $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \sinh^2 \rho \, \mathrm{d}\theta^2 + \mathrm{d}z^2$. Then,

$$C_d = \{(\rho, \theta, \pm\lambda_d(\rho)) \mid \rho \geq \sinh^{-1} d\}$$
$$\text{with } \lambda_d(\rho) = \int_{\sinh^{-1} d}^{\rho} \frac{d}{\sqrt{\sinh^2 x - d^2}} \, \mathrm{d}x.$$

The catenoid C_d is obtained by rotating the generating curve γ_d about z-axis where $\gamma_d = \{(\rho, 0, \pm\lambda_d(\rho)) \mid \rho \geq \sinh^{-1} d\}$. Here, $\sinh^{-1} d$ is the distance of the rotation axis to the catenoid C_d, that is, the necksize of C_d.

On the other hand, the asymptotic boundary of the catenoid C_d is the pair of circles of height $\pm h(d)$, that is, $\partial_\infty C_d = S^1_\infty \times \{-h(d), +h(d)\} \subset S^1_\infty \times \mathbb{R}$. Here, $h(d) = \lim_{\rho \to \infty} \lambda_d(\rho)$. By [NSST], $h(d)$ is a monotone-increasing function with $h(d) \searrow 0$ when $d \searrow 0$, and $h(d) \nearrow \pi/2$ when $d \nearrow \infty$. Hence, for any $d > 0$, the catenoid C_d has height $2h(d) < \pi$ (See Figure A.1).

By Theorem 4.1, we know that the minimal catenoid C_d is not area minimizing as $\partial_\infty C_d$ is a short curve. However, we claim that for sufficiently large $d > 0$, the compact subsurfaces near the girth of C_d are indeed area minimizing. In particular, we prove the following result.

Lemma A.1. *Let* $S^\rho_d = C_d \cap \mathbb{H}^2 \times [-\lambda_d(\rho), +\lambda_d(\rho)]$ *be a compact subsurface of* C_d*. Then, for sufficiently large* $d > 0$*, there is a* $\hat{\rho}(d) \geq \frac{3}{2} \log d > \sinh^{-1} d$ *such that* $S^{\hat{\rho}(d)}_d$ *is an area-minimizing surface.*

Proof. Consider now the upper half of the minimal catenoid C_d with the parametrization $\varphi_d(\rho, \theta) = (\rho, \theta, \lambda_d(\rho))$ where $\rho \geq \sinh^{-1} d$. Hence, the area of S^ρ_d can be written as

$$|S^{\rho_0}_d| = 2 \int_0^{2\pi} \int_{\sinh^{-1} d}^{\rho_0} \sinh x \sqrt{1 + \frac{d^2}{\sinh^2 x - d^2}} \, \mathrm{d}x \, \mathrm{d}\theta.$$

Notice that $\partial S^{\rho_o}_d = \gamma^+_{d,\rho_o} \cup \gamma^-_{d,\rho_o}$ is a pair of round circles of radius ρ_o in C_d where $\gamma^\pm_{d,\rho_o} = \{(\rho_o, \theta, \pm\lambda_d(\rho_o)) \mid 0 \leq \theta \leq 2\pi\}$. By [NSST], only minimal surfaces bounding $\gamma^+_{d,\rho_o} \cup \gamma^-_{d,\rho_o}$ in $\mathbb{H}^2 \times \mathbb{R}$ are subsurfaces of minimal catenoids C_d and a pair of closed horizontal disks $D^+_{d,\rho_o} \cup D^-_{d,\rho_o}$ where

$$D^{\pm}_{d,\rho_o} = \{(\rho, \theta, \pm\lambda_d(\rho_o)) \mid 0 \le \rho \le \rho_o,\ 0 \le \theta \le 2\pi\}.$$

In other words, here $D^{\pm}_{d,\rho_o}$ is a hyperbolic disk of radius ρ_0 with $z = \pm\lambda_d(\rho_o)$ in $\mathbb{H}^2 \times \mathbb{R}$. Recall the area of a hyperbolic disk of radius ρ is equal to $2\pi(\cosh\rho - 1)$.

Hence, if we can show that

$$|S^\rho_d| < 2|D_\rho| = 4\pi(\cosh\rho - 1)$$

for some $\rho > \sinh^{-1} d$, this implies $S^\rho_d \subset C_d$ is an area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$, and we are done. Hence, we claim there is a $\hat{\rho}(d) > \sinh^{-1} d$ such that $|S^\rho_d| < 2|D_\rho| = 4\pi(\cosh\rho - 1)$ where $\sinh^{-1} d < \rho < \hat{\rho}(d)$. In other words, we claim the following inequality:

$$I = \int_{\sinh^{-1} d}^{\rho} \sinh x \sqrt{1 + \frac{d^2}{\sinh^2 x - d^2}}\, \mathrm{d}x < \cosh\rho - 1.$$

Now, we separate the integral into two parts:

$$\int_{\sinh^{-1} d}^{\rho} = \int_{\sinh^{-1} d}^{\sinh^{-1}(d+1)} + \int_{\sinh^{-1}(d+1)}^{\rho},$$

that is, $I = I_1 + I_2$.

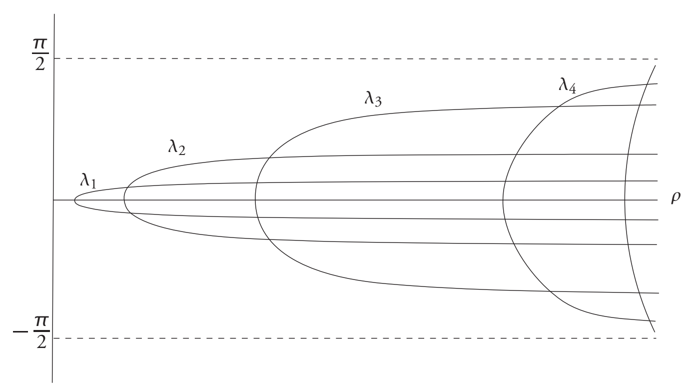


FIGURE A.1. For $d_i < d_{i+1}$, λ_i represents the graphs of functions $\lambda_{d_i}(\rho)$ which are generating curves for the minimal catenoids C_d. If $h(d) = \lim_{\rho\to\infty} \lambda_d(\rho)$, then $h(d)$ is monotone increasing with $h(d) \nearrow \pi/2$ as $d \to \infty$.

For the first part, clearly

$$\begin{aligned} I_1 &= \int_{\sinh^{-1} d}^{\sinh^{-1}(d+1)} \sinh x \sqrt{1 + \frac{d^2}{\sinh^2 x - d^2}}\, \mathrm{d}x \\ &< d + 1 \int_{\sinh^{-1} d}^{\sinh^{-1}(d+1)} \frac{\sinh x \,\mathrm{d}x}{\sqrt{\sinh^2 x - d^2}}. \end{aligned}$$

Recall that $\cosh^2 x - \sinh^2 x = 1$. By substituting $u = \cosh x$, we get

$$\begin{aligned} I_1 &< (d+1) \int_{\sqrt{1+d^2}}^{\sqrt{1+(d+1)^2}} \frac{\mathrm{d}u}{\sqrt{u^2 - (1+d^2)}} \\ &= (d+1) \log\left[u + \sqrt{u^2 - (1+d^2)}\right] \Big|_{\sqrt{1+d^2}}^{\sqrt{1+(d+1)^2}}. \end{aligned}$$

This implies

$$I_1 < F(d) = (d+1) \log \frac{\sqrt{1+(d+1)^2} + \sqrt{2d+1}}{\sqrt{1+d^2}}.$$

Now, let $F(d) = \sqrt{d+1} \cdot G(d)$. Consider

$$\begin{aligned} \lim_{d\to\infty} G(d) &= \lim_{d\to\infty} \sqrt{d+1} \log \frac{\sqrt{1+(d+1)^2} + \sqrt{2d+1}}{\sqrt{1+d^2}} \\ &= \lim_{d\to\infty} \sqrt{d} \log \frac{d + \sqrt{2d}}{\sqrt{d}} = \lim_{d\to\infty} \sqrt{d} \log \frac{(\sqrt{d} + 1/\sqrt{2})^2}{d} \\ &= \lim_{d\to\infty} \log \left(1 + \frac{1}{\sqrt{2d}}\right)^{2\sqrt{d}} = \sqrt{2}. \end{aligned}$$

Hence, $G(d) < 2$ for $d > C_0$ where C_0 is sufficiently large. Then, we have $I_1 < F(d) \le 2\sqrt{d}$ for $d > C_0$.

For the second integral, we have

$$I_2 = \int_{\sinh^{-1}(d+1)}^{\rho} \sinh x \sqrt{1 + \frac{d^2}{\sinh^2 x - d^2}}\, \mathrm{d}x.$$

Notice that the integrand

$$\sinh x \sqrt{1 + \frac{d^2}{\sinh^2 x - d^2}} = \frac{\sinh^2 x}{\sqrt{\sinh^2 x - d^2}}.$$

Hence, as $\sinh x < e^x/2$ and $(\sinh^2 x > e^{2x} - 2)/4$, we obtain

$$\int \frac{\sinh^2 x}{\sqrt{\sinh^2 x - d^2}}\, \mathrm{d}x < \int \frac{e^{2x}}{2\sqrt{e^{2x} - (2+4d^2)}}\, \mathrm{d}x = \frac{\sqrt{e^{2x} - (2+4d^2)}}{2}.$$

As $\sinh^{-1} y = \log(y + \sqrt{1 + y^2})$, after cancellations, we obtain

$$I_2 < \frac{\sqrt{e^{2\rho} - (2 + 4d^2)} - \sqrt{8d + 2}}{2} < \frac{\sqrt{e^{2\rho} - (2 + 4d^2)}}{2} - \sqrt{2d}.$$

This implies, for $d > C_0$,

$$\begin{aligned} I = I_1 + I_2 &< 2\sqrt{d} + \left(\frac{\sqrt{e^{2\rho} - (2 + 4d^2)}}{2} - \sqrt{2d} \right) \\ &= \frac{\sqrt{e^{2\rho} - (2 + 4d^2)}}{2} + (2 - \sqrt{2})\sqrt{d}. \end{aligned}$$

Now, by taking $\rho = \frac{3}{2} \log d$ for $d > C_0$, we obtain

$$\begin{aligned} I &< \frac{\sqrt{d^3 - (2 + 4d^2)}}{2} + (2 - \sqrt{2})\sqrt{d} \\ &< \frac{d^{3/2} - 2\sqrt{d}}{2} + (2 - \sqrt{2})\sqrt{d} \\ &= \frac{d^{3/2}}{2} - (\sqrt{2} - 1)\sqrt{d}. \end{aligned}$$

On the other hand,

$$\cosh \rho - 1 = \cosh\left(\frac{3}{2} \log d\right) - 1 = \frac{d^{3/2} + d^{-3/2}}{2} - 1.$$

This shows that for $\hat{\rho}(d) = \frac{3}{2} \log d$, we have $I < \cosh \hat{\rho}$. Therefore, the total area of the compact catenoid $|S_d^{\hat{\rho}(d)}| < 2|D_{\hat{\rho}(d)}|$. Hence, the compact catenoid $S_d^{\hat{\rho}(d)}$ is an area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$. ◻

Remark A.2. Notice that in the lemma above, for $\hat{\rho}(d)$ about $\frac{3}{2}$ times the neck radius of the catenoid C_d, we showed that the compact slice $S_d^{\hat{\rho}(d)}$ in C_d is an area-minimizing surface. However, the comparison between $\sqrt{e^{2\rho} - (2 + 4d^2)}/2$ and $\cosh \rho$ indicates that if ρ_0 is greater than twice the neck radius of the catenoid C_d (i.e., $\rho_0 > 2\log(d)$), the estimates above become very delicate, and $S_d^{\rho_0}$ is no longer area minimizing. (See Remark A.5 for further discussion.) Note also that any subsurface of an area-minimizing surface is automatically area minimizing. Thus, the for any $\sinh^{-1}(d) < \rho < \hat{\rho}(d)$, S_d^{ρ} is also an area-minimizing surface.

Now, we show that as $d \to \infty$, the height $2\hat{h}(d)$ of the compact area minimizing catenoids $S_d^{\hat{\rho}(d)}$ goes to π, that is, $\hat{h}(d) \to \pi/2$.

Lemma A.3. *Let* $\hat{h}(d) = \lambda_d(\hat{\rho}(d))$. *Then,* $\lim_{d\to\infty} \hat{h}(d) = \pi/2$.

Proof. By [NSST, Proposition 5.1],

$$\lim_{d\to\infty} \hat{h}(d) = \int_0^{s(\hat{\rho}(d))} \frac{\mathrm{d}t}{\cosh t}.$$

By the same proposition, $s(\rho) = \cosh^{-1}(\cosh\rho/\sqrt{1+d^2})$. As $\hat{\rho}(d) = \frac{3}{2}\log d$, then $s(\hat{\rho}(d)) \sim \sqrt{d}$. This implies

$$\lim \hat{h}(d) = \int_0^{\infty} \frac{\mathrm{d}t}{\cosh t} = \int_0^{\infty} \frac{\mathrm{d}u}{u^2+1} = \frac{\pi}{2}. \qquad \square$$

Remark A.4. Notice this lemma implies that, for any height $h_o \in (0,\pi)$, there is an area-minimizing compact catenoid S_d^{ρ} of height h_o. In other words, for any $h_o \in (0,\pi)$, there exists $d > 0$ with $h(d) > h_o$ such that

$$C_d \cap \mathbb{H}^2 \times [-h_o/2, h_o/2]$$

is an area-minimizing compact catenoid in $\mathbb{H}^2 \times \mathbb{R}$. Recall also that any subsurface of a area-minimizing surface is also area minimizing.

Remark A.5 (Pairwise Intersections of Minimal Catenoids $\{C_d\}$***).*** With these results on the area-minimizing subsurfaces S_d^{ρ} in the minimal catenoids C_d in the previous part, a very interesting point deserves a brief discussion. Notice that by definition [NSST], for $d_1 < d_2$, the graphs of the monotone increasing functions $\lambda_{d_1} : [\sinh^{-1} d_1, \infty) \to [0, h(d_1))$ and $\lambda_{d_2} : [\sinh^{-1} d_2, \infty) \to [0, h(d_2))$ intersect at a unique point $\rho_o \in (\sinh^{-1} d_2, \infty)$, that is, $\lambda_{d_1}(\rho_o) = \lambda_{d_2}(\rho_o)$ (see Figure A.1).

This implies the minimal catenoids C_{d_1} and C_{d_2} intersect at two round circles $\alpha^{\pm}$ of radius ρ_o, where $\alpha^{\pm} = (\rho_o, \theta, \pm\lambda_{d_1}(\rho_o))$, that is, $C_{d_1} \cap C_{d_2} = \alpha^+ \cup \alpha^-$.

Recall the well-known fact that two area-minimizing surfaces with disjoint boundaries cannot "separate" a compact subsurface from interiors of each other. In other words, let Σ_1 and Σ_2 be two area-minimizing surfaces with disjoint boundaries. If $\Sigma_1 \setminus \Sigma_2$ has a compact subsurface S_1 with $\partial S_1 \cap \partial\Sigma_1 = \emptyset$ and similarly $\Sigma_2 \setminus \Sigma_1$ has a compact subsurface S_2 with $\partial S_2 \cap \partial\Sigma_2 = \emptyset$, then $\Sigma_1' = (\Sigma_1 \setminus S_1) \cup S_2$ is an area-minimizing surface with a singularity along ∂S_1, which contradicts the regularity of area-minimizing surfaces (Lemma 2.7).

This argument shows that if both C_{d_1} and C_{d_2} were area-minimizing surfaces, then they would have to be disjoint. Hence, both C_{d_1} and C_{d_2} cannot be area-minimizing surfaces at the same time. In particular, the compact area-minimizing surfaces $S_{d_1}^{\rho_1} \subset C_{d_1}$ and $S_{d_2}^{\rho_2} \subset C_{d_2}$ must be disjoint, too.

This observation suggest an upper bound for $\hat{\rho}(d)$ we obtained in the previous part. Let $\iota(d)$ be the *intersection number* for C_d defined as follows:

$$\iota(d) = \inf_{t>d}\{\rho_t \mid \lambda_d(\rho_t) = \lambda_t(\rho_t)\} = \sup_{t<d}\{\rho_t \mid \lambda_d(\rho_t) = \lambda_t(\rho_t)\}.$$

The discussion above implies that $\hat{\rho}(d) < \iota(d)$, as the area-minimizing surfaces $S_{d_1}^{\rho_1} \subset C_{d_1}$ and $S_{d_2}^{\rho_2} \subset C_{d_2}$ must be disjoint.

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