Pairwise-Independent Contention Resolution

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Abstract. We study online contention resolution schemes (OCRSs) and prophet inequalities for non-product distributions. Specifically, when the active set is sampled according to a *pairwise-independent* (PI) distribution, we show a $(1-o_k(1))$-selectable OCRS for uniform matroids of rank k, and $\Omega(1)$-selectable OCRSs for laminar, graphic, cographic, transversal, and regular matroids. These imply prophet inequalities with the same ratios when the set of values is drawn according to a PI distribution. Our results complement recent work of Dughmi et al. [14] showing that no $\omega(1/k)$-selectable OCRS exists in the PI setting for general matroids of rank k.

Keywords: Online Algorithms · Prophet Inequalities · Contention Resolution

1 Introduction

Consider the *prophet inequality* problem: a sequence of independent positive real-valued random variables $\mathbf{X} = \langle X_1, X_2, \ldots, X_n \rangle$ are revealed one by one. Upon seeing X_i the algorithm must decide whether to select or discard the index i; these decisions are irrevocable. The goal is to choose some subset S of the indices $\{1, 2, \ldots, n\}$ to maximize $\mathbb{E}[\sum_{i \in S} X_i]$, subject to the set S belonging to a well-behaved family $\mathcal{I} \subseteq 2^{[n]}$. The goal is to get a value close to $\mathbb{E}[\max_{S \in \mathcal{I}} \sum_{i \in S} X_i]$, the value that a clairvoyant "prophet" could obtain in expectation. This problem originally arose in optimal stopping theory, where the case of $\mathcal{I}$ being the set of all singletons was considered [23]: more recently, the search for good prophet inequalities has become a cornerstone of stochastic optimization and online decision making, with the focus being on generalizing to broad classes of downward-closed sets $\mathcal{I}$ [16,22,28], considering additional assumptions on the order in which

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these random variables are revealed [1,2,15,17], obtaining optimal approximation guarantees [11,24], and competing with nonlinear objectives [18,29].

One important and interesting direction is to reduce the requirement of independence between the random variables: *what if the r.v.s are correlated*? The case of negative correlations is benign [26,27], but general correlations present significant hurdles—even for the single-item case, it is impossible to get value much better than the trivial $\mathbb{E}[\max_i X_i]/n$ value obtained by random guessing [20]. As another example, the model with *linear correlations*, where $\mathbf{X} = A\mathbf{Y}$ for some independent random variables $\mathbf{Y} \in \mathbb{R}^d_+$ and known non-negative matrix $A \in \mathbb{R}^{d\times n}_+$, also poses difficulties in the single-item case [21].

Given these impossibilities, Caragiannis et al. [7] gave single-item prophet inequalities in the setting of *weak correlations*: specifically, for the setting of *pairwise-independent distributions*. As the name suggests, these are distributions that look like product distributions when restricted to any two random variables. While pairwise-independent distributions have long been studied in other contexts [25], they have received less attention in the context of stochastic optimization. Caragiannis et al. [7] give both algorithms and some limitations for $\mathbf{X}$ exhibiting pairwise independence. They also considered related pricing and bipartite matching problems.

We ask the question: can we extend the prophet inequalities known for richer classes of constraint families $\mathcal{I}$ to the pairwise-independent case? In particular,

Which matroids admit good pairwise-independent prophet inequalities?

Specifically, we investigate the analogous questions for *(online) contention resolution schemes (OCRSs)* [16], another central concept in online decision making, and a close relative of prophet inequalities. In an OCRS, a random subset of a ground set is marked active. Elements are sequentially revealed to be active or inactive, and the OCRS must decide irrevocably on arrival whether to select each active element, subject to the constraint that the selected element set belongs to a constraint family $\mathcal{I}$. The goal is to ensure that each element, conditioned on being active, is picked with high probability. It is intuitive from the definitions (and formalized by Feldman et al. [16]) that good OCRSs imply good prophet inequalities (see also [24]).

1.1 Our Results

Our first result is for the k-uniform matroid, where the algorithm can pick up to k items: we achieve a $(1 - o_k(1))$-factor of the expected optimal value.

Theorem 1 (Uniform Matroid PI Prophets). *There is an algorithm in the prophet model for k-uniform matroids that achieves expected value at least $(1 - O(k^{-1/5}))$ of the expected optimal value.*

We prove this by giving a $(1 - O(k^{-1/5}))$-*selectable* online contention resolution scheme for k-uniform matroids, even when the underlying generative process is only pairwise-independent. Feldman et al. [16] showed that selectable OCRSs immediately lead to prophet inequalities (against an almighty adversary) in the fully independent case, and we observe that their proofs translate to pairwise-independent distributions as well. Along the way we also show a $(1 - O(k^{-1/3}))$ (offline) CRS for the pairwise-independent k-uniform matroid case.

We then show $\Omega(1)$-selectable OCRSs for sevaral classes of matroids, again via pairwise-independent OCRSs.

Theorem 2 (Other Matroids PI Prophets). *There exist $\Omega(1)$-selectable OCRSs for laminar (Sect. 3), graphic (Sect. 4), cographic (full version), transversal (full version), and regular (Sect. 5) matroids. These immediate imply $\Omega(1)$ prophet inequalities for these matroids against almighty adversaries.*

Finally, we consider the single-item case in greater detail in the full version. For this single-item case the current best result from Caragiannis et al. [7] uses a multiple-threshold algorithm to achieve a $(\sqrt{2} - 1)$-prophet inequality; however, this bound is worse than the $1/2$-prophet inequality known for fully independent distributions. We show that no (non-adaptive) multiple-threshold algorithm (i.e., one that prescribes a sequence of thresholds τ_i up-front, and picks the first index i such that $X_i \geq \tau_i$) can beat $2(\sqrt{5} - 2) \approx 0.472$, suggesting that if $1/2$ is at all possible it will require adaptive algorithms.

Theorem 3 (Upper Bound for Multiple Thresholds). *Any multiple-threshold algorithm for the single-item PI prophet inequality is at most 0.472-competitive.*

In the full version, we also give a *single-sample* single-item PI prophet inequality.

Theorem 4 (Single-Sample Prophet Inequality). *There is an algorithm that draws a single sample from the underlying pairwise-independent distribution $\langle \widetilde{X}_1, \ldots, \widetilde{X}_n \rangle \sim \mathcal{D}$ on $\mathbb{R}_+^n$, and then faced with a second sample $\langle X_1, \ldots, X_n \rangle \sim \mathcal{D}$ (independent from $\langle \widetilde{X}_1, \ldots, \widetilde{X}_n \rangle$), picks a single item i from $X_1, \ldots, X_n$ with expected value at least $\Omega(1) \cdot \mathbb{E}_{\mathbf{X} \sim \mathcal{D}}[\max_i X_i]$.*

1.2 Related Work

In independent and concurrent work, Dughmi et al. [14] also study the pairwise-independent versions of prophet inequalities and (online) contention resolution schemes. This work can be considered complementary to ours: they show that for arbitrary linear matroids, nothing better than $O(1/r)$ factors can be achieved for pairwise-independent versions of OCRSs, and nothing better than $O(1/(\log r))$ factors can be achieved for pairwise-independent versions of matroid prophet inequalities (where r is the rank of the matroid). They also obtain $\Omega(1)$-selectable OCRSs for uniform, graphical, and bounded degree transversal matroids by observing that these have the α-partition property (see [5]), reducing to the single-item setting. Another motivation for our work is the famous matroid secretary problem, since the latter is known to be equivalent to the existence of good OCRSs for *arbitrary* distributions that admit $\Omega(1)$-balanced CRSs against a *random-order* adversary [13].

The original single-item prophet inequality for product distributions was proven by Krengel and Sucheston [23]. There is a vast literature on variants and extensions of prophet inequalities, which we cannot survey here for lack of space. Contention resolution schemes were introduced by Chekuri et al. [10] in the context of constrained submodular function maximization, and these were generalized by Feldman et al. [16] to the online setting in order to give prophet inequalities for richer constraint families.

Limited-independence versions of prophet inequalities were studied from the early days e.g. by Hill and Kertz and Rinott and Samuel-Cahn [20,27]. Many stochastic optimization problems have been studied recently in correlation-robust settings, e.g., by Bateni et al., Chawla et al., Immorlica et al. [6,9,21]; pairwise-independent prophet inequalities were introduced by Caragiannis et al. [7].

There is a line of work on single-sample prophet inequalities in the i.i.d. setting [3,4,8,19,30]. This is the first such study for pairwise-independent distributions.

1.3 Preliminaries

We provide several essential definitions here, and a more complete preliminaries section in the full version. We assume the reader is familiar with the basics of matroid theory, and refer to Schrijver [31] for definitions. For a matroid $M = (E, \mathcal{I})$ the matroid polytope is defined to be $\mathcal{P}_M := \{\mathbf{x} \in \mathbb{R}^E : \mathbf{x} \geq \mathbf{0},\ \mathbf{x}(S) \leq \text{rank}(S)\ \forall S \subseteq E\}$. For polytope $\mathcal{P}$ and scalar $b \in \mathbb{R}$, define $b\mathcal{P} := \{b\mathbf{x} : \mathbf{x} \in \mathcal{P}\}$.

We focus on pairwise independent versions of *contention resolution schemes* (CRSs), in both offline and online settings. Our setting entails a set $R \subseteq E$ drawn from a distribution $\mathcal{D}$ with marginal probabilities given by some $\mathbf{x} \in b\mathcal{P}_M$, and the goal is to select items $I \subseteq R$, $I \in \mathcal{I}$, such that $\Pr[i \in I \mid i \in R] \geq c$ for all i. An algorithm which does this is a (b, c)*-balanced* CRS.

For *online* contention resolution schemes (OCRSs) the items arrive one-at-a-time; a scheme must decide whether to include each arriving element into its independent set I or irrevocably reject it [16]. Generally the events $[i \in R]$ are taken to be independent, so that $\mathbf{x}$ determines $\mathcal{D}$. For a *pairwise-independent* (PI) OCRS these events are only pairwise independent under $\mathcal{D}$.

The Almighty Adversary. The almighty adversary knows everything. It first sees the realization of $R \sim \mathcal{D}$, as well as all randomness the algorithm will use. It then adversarially orders R. To describe PI-OCRS's with guarantees against the almighty adversary, we adopt ideas from Feldman et al. [16], and restrict our attention to a schemes which coincide with their *greedy* OCRSs:

Definition 1 ((b, c)-selectable PI-OCRS). *Let $\mathcal{P} \subseteq [0, 1]^n$ be some convex polytope. We call a (randomized) algorithm $\pi : 2^{[n]} \to 2^{[n]}$ a* (b, c)-selectable PI-OCRS *if it satisfies the following:*

1. *Algorithm π precommits to some feasible set family $\mathcal{F} \subseteq \mathcal{I}$, and then adds each arriving i to I only if the resulting set is in $\mathcal{F}$.*
2. *For any $\mathbf{x} \in b\mathcal{P}$, any distribution $\mathcal{D}$ PI-consistent with $\mathbf{x}$ and any $i \in [n]$, let $\mathcal{F}$ be the feasible set family defined by π. Let R be sampled according to $\mathcal{D}$, then*
$$\Pr_{R \sim \mathcal{D}}[I \cup \{i\} \in \mathcal{F} \quad \forall I \subseteq R, I \in \mathcal{F} \mid i \in R] \geq c. \tag{1.1}$$
Here the probability is over R and internal randomness of π in defining $\mathcal{F}$.

Notice the definition here is slightly different from [16], as we need to condition on the event $i \in R$. This is due to our limited independence over events $i \in R$. For the mutually independent case, one can prove that $\Pr_{R \sim \mathcal{D}}[I \cup \{i\} \in \mathcal{F} \quad \forall I \subseteq R, I \in \mathcal{F} \mid i \in R] = \Pr_{R \sim \mathcal{D}}[I \cup \{i\} \in \mathcal{F} \quad \forall I \subseteq R, I \in \mathcal{F}]$, but this may not hold in the pairwise-independent case.

A (b, c)-selectable PI-OCRS implies a $(1, bc)$-selectable PI-OCRS (or for short bc-selectable PI-OCRS), and gives guarantees against an almighty adversary. For details see the full version.

The Offline Adversary and Prophet Inequalities. The offline adversary does not know the randomness of π and must choose an arrival order for $[n]$ before $R \sim \mathcal{D}$ is sampled. PI-OCRSs that are (b, c)*-balanced* are effective against the offline adversary, and since the offline adversary is weaker than the almighty adversary, a (b, c)-selectable PI-OCRS is always (b, c)-balanced. Once again, a (b, c)-balanced PI-OCRS may be converted to a $(1, bc)$-balanced PI-OCRS (or a bc-balanced PI-OCRS for short) via independent subsampling of R.

Feldman et al. [16] showed connections between OCRSs and prophet inequalities. In the full version, we formally establish this connection in the pairwise-independent setting through the formulation of a *PI matroid prophet game*, and we demonstrate that balanced PI-OCRSs are enough to give prophet inequalities.

As an upshot, we show that our results imply matroid prophet inequalities for the pairwise-independent setting; for each class of matroids, any c-balanced PI-OCRS yields a c-competitive prophet inequality for values drawn from a pairwise-independent distribution. This generalizes the single-item PI prophet inequality of Caragiannis et al. [7] in the setting where the gambler knows the joint distribution as well as the marginals.

2 Uniform Matroids

Recall that the independent sets of a *uniform matroid* $M = (E, \mathcal{I})$ of rank k are all subsets of E of size at most k; hence our goal is to pick some set of size at most k. Identifying E with $[n]$, the corresponding matroid polytope is $\mathcal{P}_M := \{\mathbf{x} \in [0,1]^n : \sum_{i=1}^n x_i \leq k\}$. Our main results for uniform matroids are the following, which imply Theorem 1.

Theorem 5 (Uniform Matroids). *For uniform matroids of rank k, there is*

(i) a $(1 - O(k^{-1/3}))$-balanced PI-CRS, and
(ii) a $(1 - O(k^{-1/5}))$-selectable PI-OCRS.

A simple greedy PI-OCRS follows by choosing the feasible set family $\mathcal{F} = \mathcal{I}$, i.e. selecting R as the resulting set if $|R| \leq k$. However, conditioning on $i \in R$, pairwise independence only guarantees the marginals of the events $j \in R$ (and they might have arbitrary correlation), so we can only use Markov's inequality to bound $\Pr[|R| \leq k \mid i \in R]$. This analysis only gives a $(b, 1-b)$-selectable PI-OCRS for k-uniform matroid. (For details see the full version.)

Hence instead of conditioning on some $i \in R$ and using Markov's inequality, we consider all items together and use Chebyshev's inequality to bound $\Pr[|R| \geq k, i \in R]$. The following lemma is key for both our PI-CRS and PI-OCRS.

Lemma 1. *Let $M = (E, \mathcal{I})$ be a k-uniform matroid, where E is identified as $[n]$. Given $\mathbf{x} \in (1-\delta)\mathcal{P}_M$ and a distribution $\mathcal{D}$ of subsets of E that is PI-consistent with $\mathbf{x}$, let $R \subseteq E$ be the random set sampled according to $\mathcal{D}$. Then*

$$\sum_{i=1}^{n} \Pr[|R| \geq k, i \in R] \leq \frac{1-\delta^2}{\delta^2}.$$

Proof. The left-hand side can be written as

$$\begin{aligned}\sum_{i=1}^{n} \Pr[|R| \geq k, i \in R] &= \sum_{i=1}^{n} \sum_{t=k}^{n} \sum_{\substack{S: \\ |S|=t}} \mathbb{1}[i \in S] \Pr[R = S] = \sum_{t=k}^{n} \sum_{\substack{S: \\ |S|=t}} \Pr[R = S]|S| \\ &= \sum_{t=k}^{n} t \cdot \Pr[|R| = t] = k \Pr[|R| \geq k] + \sum_{t=k+1}^{n} \Pr[|R| \geq t].\end{aligned}$$

We now bound the two parts separately using Chebyshev's inequality. Let $X_i := \mathbb{1}[i \in R]$ be the indicator for i being active, and let $X = \sum_{i \in E} X_i$. Since X_i are pairwise independent, $\mathrm{Var}[X] = \sum_i \mathrm{Var}[X_i] \leq \sum_i \mathbb{E}[X_i^2] = \sum_i \mathbb{E}[X_i] = \mathbb{E}[X]$. For the first part, we have

$$\begin{aligned}k \cdot \Pr[|R| \geq k] = k \cdot \Pr[X \geq k] &\leq k \cdot \frac{\mathrm{Var}[X]}{(k - \mathbb{E}[X])^2} && \text{(Chebyshev's ineq.)} \\ &\leq k \cdot \frac{1-\delta}{\delta^2 k} = \frac{1-\delta}{\delta^2}. && (2.1)\end{aligned}$$

For the second part,

$$\begin{aligned}\sum_{t=k+1}^{n}\Pr[|R|\geq t] &= \sum_{t=k+1}^{n}\Pr[X\geq t]\leq\sum_{t=k+1}^{n}\frac{\mathrm{Var}[X]}{(t-\mathbb{E}[X])^2} \qquad \text{(Chebyshev's ineq.)}\\ &\leq\sum_{t=k+1}^{n}\frac{(1-\delta)k}{(t-(1-\delta)k)^2}\leq(1-\delta)k\cdot\sum_{t\geq 1}\frac{1}{(\delta k+t)^2}\\ &\leq(1-\delta)k\cdot\frac{1}{\delta k}=\frac{1-\delta}{\delta},\end{aligned} \tag{2.2}$$

where we used the inequality

$$\sum_{j\geq 1}\frac{1}{(x+j)^2}\leq\sum_{j\geq 1}\frac{1}{(x+j-1)(x+j)}=\sum_{j\geq 1}\left(\frac{1}{x+j-1}-\frac{1}{x+j}\right)=\frac{1}{x}.$$

Summing up the (2.1) and (2.2) finishes the proof. □

Using this lemma, we can bound $\min_i \Pr[|R|\geq k \mid i\in R]$ and obtain a $(1-O(k^{-1/3}))$-balanced PI-CRS (in the same way that [10, Lemma 4.13] implies a $(b,1-b)$-CRS in the i.i.d. setting). The details are deferred to the full version.

2.1 A $(1-O(k^{-1/5}))$-Selectable PI-OCRS for Uniform Matroids

Our PI-CRS has to consider the elements in a specific order, and therefore it does not work in the online setting where the items come in adversarial order. The key idea for our PI-OCRS is to separate "good" items and "bad" items, and control each part separately. Let us assume R is sampled according to some distribution $\mathcal{D}$ PI-consistent with $\mathbf{x}$, and that $\mathbf{x}$ is on a face of $(1-\varepsilon)\mathcal{P}_M$, i.e.

$$\sum_{i=1}^{n}\Pr[i\in R]=(1-\varepsilon)k. \tag{2.3}$$

We will choose the value of ε later. For some other constants $r,b\in(0,1)$ define an item i to be *good* if $\Pr[|R|>\lfloor(1-r\varepsilon)k\rfloor \mid i\in R]\leq b$. Let E_g denote the set of good items, and $E_b := E\backslash E_g$ the remaining *bad* items. Our algorithm keeps two buckets, one for the good items and one for the bad, such that

(i) the good bucket has a capacity of $\lfloor(1-r\varepsilon)k\rfloor$, and
(ii) the bad bucket has a capacity of $\lceil r\varepsilon k\rceil$.

When an item arrives, we put it into the corresponding bucket as long as that bucket is not yet full. Finally, we take the union of the items in the two buckets as the output of our OCRS. This algorithm is indeed a greedy PI-OCRS with the feasible set family $\mathcal{F}=\{I\in\mathcal{I}: |I\cap E_g|\leq\lfloor(1-r\varepsilon)k\rfloor, |I\cap E_b|\leq\lceil r\varepsilon k\rceil\}$.

We show that for any item i, $\Pr[I\cup\{i\}\in\mathcal{F}\quad\forall I\in\mathcal{F}, I\subseteq R \mid i\in R]\geq 1-o(1)$. First, for a good item i, by definition

$\Pr[I \cup \{i\} \in \mathcal{F} \ \ \forall I \in \mathcal{F},\ I \subseteq R \mid i \in R] = 1 - \Pr[|R \cap E_g| > \lfloor (1 - r\varepsilon)k \rfloor \mid i \in R]$
$\geq 1 - \Pr[|R| > \lfloor (1 - r\varepsilon)k \rfloor \mid i \in R] \geq 1 - b.$

Next, for a bad item i, we can use Markov's inequality conditioning on $i \in R$:

$$\begin{aligned} &\Pr[I \cup \{i\} \in \mathcal{F} \ \ \forall I \in \mathcal{F}, I \subseteq R \mid i \in R] = 1 - \Pr[|R \cap E_b| > \lceil r\varepsilon k \rceil \mid i \in R] \\ &\geq 1 - \frac{\sum_{j \in E_b} \Pr[j \in R \mid i \in R]}{r\varepsilon k} = 1 - \frac{\sum_{j \in E_b} \Pr[j \in R]}{r\varepsilon k}, \end{aligned} \tag{2.4}$$

where we use Markov's inequality, and the last step uses pairwise independence of events $i \in R$. We now need to bound $\sum_{j \in E_b} \Pr[j \in R]$. If we define ε' as $1 - \varepsilon' = \frac{1-\varepsilon}{1-r\varepsilon}$, then we have

$$\begin{aligned} \sum_{j \in E_b} \Pr[j \in R] &= \sum_{j \in E_b} \frac{\Pr[|R| \geq \lfloor (1 - r\varepsilon)k \rfloor, j \in R]}{\Pr[|R| \geq \lfloor (1 - r\varepsilon)k \rfloor \mid j \in R]} \\ &\leq \sum_{j \in E_b} \frac{\Pr[|R| \geq \lfloor (1 - r\varepsilon)k \rfloor, j \in R]}{b} && \text{(since } j \text{ is bad)} \\ &\overset{(\star)}{\leq} \frac{(1 - (\varepsilon')^2)/(\varepsilon')^2}{b} \leq \frac{1}{(1-r)^2 \varepsilon^2 b}, \end{aligned}$$

where $(\star)$ uses Lemma 1. Substituting back into (2.4), $\Pr[I \cup \{i\} \in \mathcal{F} \quad \forall I \in \mathcal{F}, I \subseteq R \mid i \in R] \geq 1 - ((1-r)^2 r\varepsilon^3 bk)^{-1}$.

To balance the good and bad items, we set $b = ((1-r)^2 r\varepsilon^3 bk)^{-1} = ((1-r)^2 r\varepsilon^3 k)^{-1/2}$. If we set $r = 1/3$, then we have an $(1-\varepsilon, 1-(\frac{4}{27}\varepsilon^3 k)^{-1/2})$-selectable PI-OCRS. Finally, if we set $\varepsilon = k^{-1/5}$, since a (b, c)-selectable PI-OCRS implies a (bc)-selectable PI-OCRS, we have a $(1 - O(k^{-1/5}))$-selectable PI-OCRS.

3 Laminar Matroids

In this section we give an $\Omega(1)$-selectable PI-OCRS for laminar matroids. A laminar matroid is defined by a laminar family $\mathcal{A}$ of subsets of E, and a capacity function $c : \mathcal{A} \to \mathbb{Z}$; a set $S \subseteq E$ is independent if $|S \cap A| \leq c(A)$ for all $A \in \mathcal{A}$.

The outline of the algorithm is as follows: we construct a new capacity function c' by rounding down $c(A)$ to powers of two; satisfying these more stringent constraints loses only a factor of two. Then we run greedy PI-OCRSs for uniform matroids from Sect. 2.1 *independently* for each capacity constraint $c'(A)$, $A \in \mathcal{A}$. Finally, we output the intersection of these feasible sets. For our analysis, we apply a union bound on probability of an item being discarded by some greedy PI-OCRS; this is a geometric series by our choice of c'.

As the first step, we define $c'(A)$ to be the largest power of 2 smaller than $c(A)$, for each $A \in \mathcal{A}$. (For simplicity we assume that $E \in \mathcal{A}$.) Moreover, if sets $A, B \in \mathcal{A}$ with $A \subseteq B$ and $c'(A) \geq c'(B)$, then we can discard A from the collection. In conclusion, the final constraints satisfy:

1. The new laminar family is $\mathcal{A}' \subseteq \mathcal{A}$.
2. For any $A \in \mathcal{A}'$, $c'(A)$ is power of 2, and $c(A)/2 < c'(A) \leq c(A)$.
3. (Strict Monotonicity) For any $A, B \in \mathcal{A}'$ with $A \subsetneq B$, we have $c'(A) < c'(B)$.

Let M' denote the laminar matroid defined by the new set of constraints. We can check that any c-selectable PI-OCRS for M' is a $(1/2, c)$-selectable PI-OCRS for M. Hence, it suffices to give a $\Omega(1)$-selectable PI-OCRS for M'.

Now we run greedy OCRSs for uniform matroids to get a $(1/25, 1/2.661)$-selectable PI-OCRS: for a set A with capacity $c'(A)$, from Sect. 2 we have both a $(1-b, b)$-selectable PI-OCRS and a $(1-b, 1-(\frac{4}{27}b^3c'(A))^{-1/2})$-selectable PI-OCRS: the former is better for small capacities, whereas the latter is better for larger capacities. Setting a threshold of $t = 13$ and choosing $b = 24/25$, we use the former when $c'(A) < 2^t$, else we use the latter. Now a union bound over the various sets containing an element gives us the result: the crucial fact is that we get a contribution of $t(1-b)$ from the first smallest scales and a geometric sum giving $O(2^{-t/2}b^{-3/2})$ from the larger ones. The details appear in the full version.

4 Graphic Matroids

Recall that graphic matroids correspond to forests (acyclic subgraphs) of a given (multi)graph. For these matroids we show the following.

Theorem 6. *For $b \in (0, 1/2)$, there is a $(b, 1-2b)$-selectable PI-OCRS scheme for graphic matroids.*

Let $M = (E, \mathcal{I})$ be a graphic matroid defined on (multi)graph $G = (V, E)$. Let $\mathcal{D}$ be any distribution over 2^E that is PI-consistent with some $\mathbf{x} \in b\mathcal{P}_M$, and R sampled according to $\mathcal{D}$. We follow the construction of OCRS of Feldman et al. [16]. Our goal is to construct a chain of sets: $\varnothing = E_l \subsetneq E_{l-1} \subsetneq \cdots \subsetneq E_0 = E$ where for any $i \in \{0 \cdots l-1\}$ and any $e \in E_i \backslash E_{i+1}$,

$$\Pr[e \in \operatorname*{span}_{M/E_{i+1}} (((R \cap (E_i \backslash E_{i+1}))\backslash e) \mid e \in R] \leq 2b. \tag{4.1}$$

We can now define the feasible set for our greedy PI-OCRS as $\mathcal{F} = \{I \subseteq E : \forall i, I \cap (E_i \backslash E_{i+1}) \in \mathcal{I}(M/E_{i+1})\}$. By definition of contraction, $\mathcal{F} \subseteq \mathcal{I}(M)$. To check selectability, for an edge e in $E_i \backslash E_{i+1}$, we have $\Pr[I \cup \{e\} \in \mathcal{F} \quad \forall I \in \mathcal{F}, I \subseteq R \mid e \in R] = \Pr[e \notin \text{span}_{M/E_{i+1}}(((R \cap (E_i \backslash E_{i+1}))\backslash e) \mid e \in R] \geq 1 - 2b$ (using (4.1)). Therefore this is a $(b, 1-2b)$-selectable PI-OCRS. It remains to show how to construct such a chain. We use the following recursive procedure:

1. Initialize $E_0 = E, i = 0$.
2. Set $S = \varnothing$.
3. While there exists $e \in E_i \backslash S$ such that $\Pr[e \in \text{span}_{M/S}((R \cap (E_i \backslash S))\backslash e) \mid e \in R] > 2b$, add e into S.
4. $i \leftarrow i + 1$, set $E_i = S$.
5. If $E_i \neq \varnothing$, goto step 2; otherwise set $l = i$ and terminate.

Inequality (4.1) is automatically satisfied by this procedure. It remains to show that the process always terminates, i.e. that step 3 always leaves at least one element unidentified, and hence $E_{i+1} \subsetneq E_i$. We start with the following claim.

Claim 1. *If $u_0 \in V$ satisfies $\sum_{e \in \mathcal{E}(u_0)} x_e \leq 2b$, then in the above procedure generating E_1 from E, we have that for all $e \in \mathcal{E}(u_0)$, $e \notin S$.*

Proof. We prove our claim using induction. For any edge $e \in \mathcal{E}(u_0) \cap R$, $e \in \mathrm{span}(R \backslash \{e\})$ implies the existence of a circuit $C \subseteq R$ containing e. By the definition of circuits, C must contain some edge $e' \in \mathcal{E}(u_0) \backslash \{e\}$. By the pairwise independence of events $e \in R$, we have $\Pr[e \in \mathrm{span}(R \backslash \{e\}) \mid e \in R] \leq \Pr[\exists e' \in \mathcal{E}(u_0) \backslash \{e\}, e' \in R \mid e \in R] \leq \sum_{e \in \mathcal{E}(u_0)} x_e \leq 2b$.

Therefore we do not add any $e \in \mathcal{E}(u_0)$ into S in the first iteration. Suppose no $e \in \mathcal{E}(u_0)$ has been added to S during the first i iterations, then before the $(i+1)^{th}$ iteration starts, u_0 has not been merged with any other vertex in the contracted graph G/S, so $\mathcal{E}(u_0)$ in G/S is the same as the original graph G. Thus $\sum_{e \in \mathcal{E}(u_0)} x_e \leq 2b$ still holds for u_0 in G/S, and by the same argument as the first iteration, no $e \in \mathcal{E}(u_0)$ will be added to S in the $(i+1)^{th}$ iteration. □

Since $\mathbf{x} \in b\mathcal{P}_M$, we have $\sum_{e \in E} x_e \leq b(n-1)$, which implies $\sum_{u \in V} \sum_{e \in \mathcal{E}(u)} x_e \leq 2b(n-1)$. By averaging, there exists a vertex $u_0 \in V$ such that $\sum_{e \in \mathcal{E}(u_0)} x_e \leq 2b(n-1)/n \leq 2b$, and by Claim 1, $\mathcal{E}(u_0) \cap E_1 = \varnothing$. Assuming no isolated vertex in V, $E_1 \subsetneq E_0$. Similarly, for any i, since $M|_{E_i}$ is also a graphic matroid and $\mathbf{x}|_{E_i} \in b\mathcal{P}_{M|_{E_i}}$, the same argument holds for it. Therefore $E_{i+1} \subsetneq E_i$ always holds, which finishes our proof of termination for our construction.

5 Regular Matroids

We now give a $\Omega(1)$-competitive PI-OCRS for regular matroids. We use the regular matroid decomposition theorem of Seymour [32] and its modification by Dinitz and Kortsarz [12], which decomposes any regular matroid into 1-sums, 2-sums, and 3-sums of graphic matroids, cographic matroids, and a specific 10-element matroid R_{10}. (These matroids are called the *basic* matroids of the decomposition). We now define 1,2,3-sums, and argue that it suffices to run a PI-OCRS for each of the basic matroids and to output the union of their outputs.

Definition 2 (Binary Matroid Sums [12,32]). *Given two matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$, the* matroid sum *$M$ defined on the ground set $E(M_1) \Delta E(M_2)$ is as follows. The set C is a cycle in M iff it can be written as $C_1 \Delta C_2$, where C_1 and C_2 are cycles of M_1 and M_2. respectively. Furthermore,*

1. *If $E_1 \cap E_2 = \varnothing$, then M is called 1-sum of M_1 and M_2.*
2. *If $|E_1 \cap E_2| = 1$, then we call M the 2-sum of M_1 and M_2.*
3. *If $|E_1 \cap E_2| = 3$, let $Z = E_1 \cap E_2$. If Z is a circuit of both M_1 and M_2, then we call M the 3-sum of M_1 and M_2.*

(The i-sum is denoted $M_1 \oplus_i M_2$.) Our definition differs from [12,32] as we have dropped some conditions on the sizes of M_1 and M_2 that we do not need. A $\{1,2,3\}$-*decomposition* of a matroid $\widetilde{M}$ is a set of matroids $\mathcal{M}$ called the *basic matroids*, together with a rooted binary tree T in which $\widetilde{M}$ is the root and the leaves are the elements of $\mathcal{M}$. Every internal vertex in the tree is either the 1-, 2-, or 3-sum of its children. Seymour's decomposition theorem for regular matroids [32] says that every regular matroid $\widetilde{M}$ has a (poly-time computable) $\{1,2,3\}$-decomposition with all basic matroids being graphic, cographic or R_{10}.

The Dinitz-Kortsarz Modification. Dinitz and Kortsarz [12] modified Seymour's decomposition to give an $O(1)$-competitive algorithm for the regular-matroid secretary problem, as follows. Given a $\{1,2,3\}$-decomposition T for binary matroid $\widetilde{M}$ with basic matroids $\mathcal{M}$, we define Z_M, the *sum-set* of a non-leaf vertex M in T, to be the intersection of the ground sets of its children (the sum-set is thus not in the ground set of M). A sum-set Z_M for internal vertex M is either the empty set (if M is the 1-sum of its children), a single element (for 2-sums), or three elements that form a circuit in its children (for 3-sums). A $\{1,2,3\}$-decomposition is *good* if for every sum-set Z_M of size 3 associated with internal vertex $M = M_1 \oplus_3 M_2$, the set Z_M is contained in the ground set of a single basic matroid below M_1, and in the ground set of a single basic matroid below M_2. For a given $\{1,2,3\}$-decomposition of a matroid $\widetilde{M}$ with basic matroids $\mathcal{M}$, define the *conflict graph* G_T to be the graph on $\mathcal{M}$ where basic matroids M_1 and M_2 share an edge if their ground sets intersect. [12] show that if T is a good $\{1,2,3\}$-decomposition of $\widetilde{M}$, then G_T is a forest. We can root each tree in such a forest arbitrarily, and define the parent $p(M)$ of each non-root matroid $M \in \mathcal{M}$. Let A_M be the sum-set for the edge between matroid M and its parent, i.e., $A_M = E(M) \cap E(p(M))$.

Theorem 7 (Theorem 3.8 of [12]). *There is a good $\{1,2,3\}$-decomposition T for any binary matroid $\widetilde{M}$ with basic matroids $\mathcal{M}$ such that (a) each matroid $M \in \mathcal{M}$ has no circuits of size 2 consisting of an element of A_M and an element of $E(\widetilde{M})$, and (b) every basic matroid $M \in \mathcal{M}$ can be obtained from some $M' \in \widetilde{\mathcal{M}}$ by deleting elements and adding parallel elements.*

Dinitz and Kortsarz showed that a good $\{1,2,3\}$-decomposition for a matroid $\widetilde{M}$ can be used to construct independent sets for $\widetilde{M}$ as follows. Below, $\cdot|_S$ denotes restriction to the set S.

Lemma 2 (Lemma 4.4 of [12]). *Let T be a good $\{1,2,3\}$-decomposition for $\widetilde{M}$ with basic matroids $\mathcal{M}$. For each $M \in \mathcal{M}$, let I_M be an independent set of $(M/A_M)|_{(E(M)\cap E(\widetilde{M}))}$. Then $I = \bigcup_{M\in\mathcal{M}} I_M$ is independent in $\widetilde{M}$.*

Our Algorithm. Given the input matroid $\widetilde{M}$, our idea is to take a good decomposition T and run a PI-OCRS for $(M/A_M)|_{(E(M)\cap E(\widetilde{M}))}$ for each vertex M in the conflict graph G_T. Then we need to glue the pieces together using Lemma 2. One technical obstacle is that the input to an OCRS is a feasible point in the matroid polytope, so to use the framework of [12] we need to convert it into a feasible solution to the polytopes of the (modified) basic matroids. Our insight is captured by the following lemma.

Lemma 3. *Let T be a good $\{1,2,3\}$-decomposition of regular matroid $\widetilde{M}$ with basic matroids $\mathcal{M}$, and let vector $\mathbf{x} \in \frac{1}{3}\mathcal{P}_{\widetilde{M}}$. Then for every basic matroid $M \in \mathcal{M}$, if $\widehat{M} := (M/A_M)|_{(E(M)\cap E(\widetilde{M}))}$, then $\mathbf{x}|_{\widehat{M}} \in \mathcal{P}_{\widehat{M}}$.*

Proof. Fix a set $S \subseteq E(M)\cap E(\widetilde{M})$. We will show that $\operatorname{rank}_{\widehat{M}}(S) \geq \frac{1}{3}\operatorname{rank}_{\widetilde{M}}(S)$, from which the claim follows.

Case 1: $A_M = \{z\}$. For any maximal independent set $I \subset S$ according to M, there always exists $a \in I$ such that $(I \cup \{z\})\backslash\{a\}$ is independent in M, therefore $\operatorname{rank}_{\widehat{M}}(S) \geq \operatorname{rank}_{\widetilde{M}}(S) - 1$. Also since no element in S is parallel to z, for any non-empty S we have $\operatorname{rank}_{M/A_M}(S) \geq 1$, and we can conclude that $\operatorname{rank}_{M/A_M}(S) \geq \frac{1}{3}\operatorname{rank}_{\widetilde{M}}(S)$.

Case 2: A_M is some 3-cycle $\{z_1, z_2, z_3\}$. For any maximal independent set $I \subset S$ according to M where $|I| \geq 3$, there always exists $a, b \in I$ such that $(I \cup \{z_1, z_2\})\backslash\{a, b\}$ is independent in M. Therefore $\operatorname{rank}_{M_2/Z}(S) \geq \operatorname{rank}_M(S) - 2$. We claim that there does not exist e in $E(M)\backslash A_M$ such that $e \in \operatorname{span}(A_M)$.

Suppose for contradiction such an e exists. Then there is some circuit in $A_M \cup e$ containing e. Since there are no parallel elements, this circuit have size 3. Without loss of generality, assume this circuit is $C = \{z_1, z_2, e\}$. Since A_M is a circuit, by definition of binary matroids, the set $C \Delta A_M = \{z_3, e\}$ is a cycle, and thus e is parallel to z_3, a contradiction. Therefore for any non-empty S, we have that $\operatorname{rank}_{M/A_M}(S) \geq 1$, and we conclude that $\operatorname{rank}_{M/A_M}(S) \geq \frac{1}{3}\operatorname{rank}_{\widetilde{M}}(S)$. $\square$

We conclude the main theorem of the section (see the full version for a proof).

Theorem 8 (Regular Matroids). *There is a $(1/3, 1/12)$-selectable PI-OCRS for regular matroids.*

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