PETERZIL-STEINHORN SUBGROUPS AND μ-STABILIZERS IN ACF

MOSHE KAMENSKY[1], SERGEI STARCHENKO[2], AND JINHE YE[3]

Abstract. We consider a linear algebraic group G defined over an algebraically closed field $\Bbbk$. By considering $\Bbbk$ as an embedded residue field of an algebraically closed valued field K, we can associate to it a compact G-space $S^{\mu}_{G}(\Bbbk)$, consisting of μ-types on G. We showed that for each $p \in S_G(\Bbbk)$, $\mathrm{Stab}^{\mu}(p)$ is a solvable infinite algebraic group when p_{μ} is centered at infinity and residually algebraic. Moreover we give a description of the dimension $\mathrm{Stab}^{\mu}(p)$ in terms of the dimension of p.

1. Introduction

Let G be a group definable in an o-minimal structure, and let $\gamma : (a, b) \to G$ be a definable curve which is unbounded, in the sense that the limit at b does not exist. In [9], it was shown that one can associate to this datum a definable one-dimensional torsion-free group $H_\gamma \subseteq G$, that can be viewed as the "stabilizer of γ at ∞". The group H_γ is called the *Peterzil–Steinhorn subgroup* associated to γ. For example, when G is a Cartesian power of the additive group, H_γ is the linear subspace whose translate is the asymptote of γ at ∞.

Assume now that G is an affine algebraic group over the complex numbers, and X is an algebraic curve embedded in G. If we view $\mathbb{C}$ as the algebraic closure of a real closed field $\mathcal{R}$, the set of complex points of X can be viewed as the set of $\mathcal{R}$-points of an $\mathcal{R}$-definable set X^{an} in the o-minimal structure $\mathcal{R}$. This set is unbounded, and we may therefore choose an unbounded curve γ inside X^{an}, and consider the corresponding PS-group H_γ. Taking its Zariski closure, we obtain an algebraic subgroup G_γ of G, of (algebraic) dimension 1.

It is natural to ask, to which extent does the subgroup G_γ depend on the non-algebraic data involved, namely, the dependence on the real closed field $\mathcal{R}$ of choice and the curve γ? And if it does not depend on the above, can the construction be described in a purely algebraic manner? We first note that choice of $\gamma : (a, b) \to X^{an}$ determines an additional *algebraic* datum: the curve X (which we may assume to be smooth) has a canonical compactification $\tilde{X}$, its projective model, which is obtained from X by adding finitely many points. Viewing γ as taking values in $\tilde{X}^{an}$ rather than X^{an}, the limit of γ at b will be precisely one of these points, and curves γ corresponding to different such points definitely might give rise to different subgroups G_γ. Hence, any hope of providing an algebraic construction of G_γ should take into account the choice of such a point at infinity.

[1]Department of Mathematics, Ben-Gurion University
[2,3]Department of Mathematics, University of Notre Dame
E-mail addresses: [1]kamenskm@math.bgu.ac.il, [2]starchenko.1@nd.edu, [3]jye@nd.edu.

The main result of this paper (stated below) provides an algebraic construction as expected above, once the additional datum of a limit point is chosen:

Theorem 1.1. Let G be a linear algebraic group over $\Bbbk$, and let $X \subseteq G$ be an irreducible curve. Then there are finitely many 1-dimensional linear subgroups of G, associated to points at infinity of a projective model of the curve X.

A more precise version is given in Theorem 3.11. The actual main result in the paper includes a generalization to higher dimensions, and some analysis of the structure of the resulting group. These results are obtained by viewing G as a definable group in ACVF, the theory of algebraically closed valued fields, and applying some results from [4].

To state the main result, we need to introduce some additional terminology. The subgroups we are interested in were introduced in an abstract setup in [7]. There, one considers (suitably defined) definable topological groups. To such a group G, one associates an "infinitesimal subgroup" μ, the intersection of all definable neighbourhoods of the identity. If P is a (partial) type on G, the set μP can be viewed geometrically as a tube around P, and the μ-stabilizer $\mathrm{Stab}^{\mu}(P)$ of P is defined to be the stabilizer of this set.

In the o-minimal context, the datum of a curve γ as above determines a "type at infinity" p_{γ}, and it is easy to see that the PS-group H_{γ} depends only on this type. It is shown in [7] that H_{γ} is precisely the μ-stabilizer of p_{γ}. Similarly, every closed point of the projective model of a smooth curve X determines an ACVF type on X, and the associated group is defined as the μ-stabilizer of this type. To see that the definition is reasonable, it is shown that the resulting group is 1-dimensional. Furthermore, it is contained in the (algebraic) stabilizer of the corresponding point in every equivariant compactification of G (Remark 3.4).

The definition of μ-stabilizer makes sense for types of higher Zariski dimensions as well. However, two types of different Zariski dimension might have the same tube (Example 2.15), so the dimension comparison is not straightforward. We say that a type is μ-reduced if it is of minimal dimension among all types with a given tube. With this terminology, we have the following generalization of Theorem 1.1:

Theorem 1.2 (Main theorem)**.** Let G be a linear algebraic group defined over $\Bbbk$ and p be a residually algebraic type. If p is centered at infinity, then $\mathrm{Stab}^{\mu}(p)$ is infinite.

Furthermore, if p is μ-reduced, then $\dim(\mathrm{Stab}^{\mu}(p)) = \dim p$. Moreover, for each type p, $\mathrm{Stab}^{\mu}(p)$ is a solvable linear algebraic group.

Here the term "centered at infinity" should be understood as "unbounded" in the o-minimal counterpart. Do note that one cannot hope for the group to be torsion-free, as in the result on PS-groups, since the underlying field may have positive characteristic.

The structure of the paper is as follows: In §2 we review some definitions and results related to group actions, and provide an alternative approach to μ-stabilizers. In §3 we consider the one-dimensional case of Theorem 1.1. Though formally included in the general case, the situation is considerably simpler in this case, and sheds light on the more complicated general case. Then in §4 we deal with the general case.

Acknowledgements. We would like to thank Antoine Ducros, François Loeser and Kobi Peterzil for their helpful conversations. M.K. was partially supported by the Israel Science foundation (Grant no. 1382/15). S.S and J.Y. was partially supported by NSF research grant DMS1500671, and J.Y was also supported by François Loeser's Institut Universitaire de France grant.

2. μ-stabilizers over ACF

Let $\Bbbk$ be an arbitrary algebraically closed field, and let G be a linear algebraic group defined over $\Bbbk$. In this section, we develop the theory of μ-types and their stabilizers in this context following [7]. Before going into μ-types, lets begin with some generality on definable group actions.

2.1. **Definable group actions.** Let us start by recalling some general facts about stabilizers of definable types following [7, Section 2], the material works in an arbitrary complete theory T. We fix a monster model $\mathbb{U}$ of T and all the models of T we consider will be elementary submodels of $\mathbb{U}$.

Let $\mathbf{X}$ be a definable set. Let A be a small set of parameters such that $\mathbf{X}$ can be defined over A. We use $\mathcal{L}_{\mathbf{X}}(A)$ to denote the set of formulas ψ over A such that $\psi(x) \Rightarrow x \in \mathbf{X}$. Such formulas will occasionally be called $\mathbf{X}$-*formulas*. And by a *(partial)* $\mathbf{X}$-*type over* A, we mean a consistent collection of formulas in $\mathcal{L}_{\mathbf{X}}(A)$.

We fix $\mathbf{H}$ be a definable group with a definable action on $\mathbf{X}$. For $\mathbf{H}$-formula $\phi(x)$ and $\mathbf{X}$-formula $\psi(y)$, let $(\phi \cdot \psi)(z)$ be the following $\mathbf{X}$-formula

$$\exists x \, \exists y \, \phi(x) \wedge \psi(y) \wedge z = x \cdot y.$$

And for a partial $\mathbf{X}$-type p, $\phi \cdot p = \{\phi \cdot \psi : \psi \in p\}$.

For a set of parameters A by a *definable* $\mathbf{X}$-*type over* A, we mean a $\mathbf{X}$-type p over A such that for any formula $\phi(x, y)$, $\{a \in A : \phi(x, a) \in p\} = \{a \in A : d_p\phi(a)\}$ for some formula $d_p\phi$ over A. Note in the above definition, a can be tuples in A.

Let $\mathbb{M}$ a model of T such that $A = \mathbf{Y}(\mathbb{M})$ for some $\mathbb{M}$-definable set $\mathbf{Y}$. Then any such definition $d_p\phi$ will be equivalent. Moreover, p can be extended to a unique type over $\mathbf{Y}(\mathbb{L})$ determined by $\{\phi(x, c) : \mathbb{U} \models d_p\phi(c)$ where c is a tuple in $\mathbf{Y}(\mathbb{L})\}$ for any $\mathbb{L}$ such that $\mathbb{M} \preceq \mathbb{L}$, and we denote it by $p|\mathbb{L}$.

Convention 2.1. For the remainder of the paper, we will assume that we are working over a set of parameters A such that $A = \mathbf{Y}(\mathbb{M})$ for some model $\mathbb{M}$, and we assume further that $\mathbf{H}$ and its action on $\mathbf{X}$ is defined over A. We assume further that $\mathbf{H} \subseteq \mathbf{Y}^n$ for some Cartesian product of $\mathbf{Y}$.

Definition 2.2. Let p be a definable partial $\mathbf{X}$-type over A as in Convention 2.1. We define

$$\mathrm{Stab}(p)(\mathbb{M}) = \{h \in \mathbf{H}(\mathbb{M}) : \text{ For any } \phi \in \mathcal{L}_{\mathbf{X}}(A) \; p \models h \cdot \phi \Leftrightarrow p \models \phi\}$$

where $h \cdot \phi$ stands for $(x = h) \cdot \phi$. We will occasionally denote $\mathrm{Stab}(p)(\mathbb{M})$ by $\mathrm{Stab}(p)(A)$.

The following is [7, Proposition 2.13].

Fact 2.3. Let $\mathbf{H}$ be a definable group with a definable action on $\mathbf{X}$ and assume we are in the setting of Convention 2.1. Let p be a partial definable $\mathbf{X}$-type over A. Then $\mathrm{Stab}(p)$ is a A-type-definable subgroup of $\mathbf{H}$ in the following sense: There is a small system $\mathbf{H}_\alpha$ of A-definable subgroups of $\mathbf{H}$ such that for every elementary extension $\mathbb{M} \preceq \mathbb{L}$, for $a \in \mathbf{H}(\mathbb{L})$, we have that $a \in \mathrm{Stab}(p|\mathbb{L})(\mathbb{L})$ if and only if $a \in \mathbf{H}_\alpha(\mathbb{L})$ for all α.

With the language set up, we will now look at the setting to talk about μ-types as in [7] over algebraically closed fields.

2.2. **μ-stabilizers over ACF.** The theory of algebraically closed fields is not rich enough to have a good notion of infinitesimal subgroups as in [7]. Hence, it is natural to work with the theory T_{loc} as introduced in [3, Section 6]. The language for T_{loc} has 3 sorts, a sort $\mathbf{VF}$ for the valued field, a sort $\mathbf{RES}$ for the residue field and a sort Γ for the value group. It is equipped with a function symbol $\mathrm{res}(x, y) : \mathbf{VF}^2 \to \mathbf{RES}$ and $i : \mathbf{RES} \to \mathbf{VF}$ and a map $\mathrm{val} : \mathbf{VF} \to \Gamma$. The theory T_{loc} asserts that the $\mathbf{VF}$ sort is an algebraically closed valued field, val is a valuation map, $\mathrm{res}(x, y) = \mathrm{res}(x/y)$, the residue of x/y if $\mathrm{val}(x) \geq \mathrm{val}(y)$ and 0 otherwise, and the map i is a field theoretic embedding of $\mathbf{RES}$ into $\mathbf{VF}$ with $\mathrm{res}(i(c), 1) = c$.

With this map, we may identify the residue field as a subfield of the valued field, where the residue map restricts to the identity on it. For notational simplicity, we will use $\mathrm{res}(x)$ to denote $\mathrm{res}(x, 1)$ for $x \in \mathcal{O}$, the valuation ring. For the remainder of the paper, we will identify freely $\mathbf{RES}$ with its image in $\mathbf{VF}$.

Fact 2.4 ([3, Lemma 6.3])**.** T_{loc} admits quantifier elimination in this language. The sorts Γ and $\mathbf{RES}$ are stably embedded and orthogonal to each other. The induced structure on Γ is divisible ordered abelian groups and on $\mathbf{RES}$ algebraically closed fields respectively.

Remark 2.5. In the paper [3], a constant symbol 1 in the Γ-sort for some positive element was included. But the proof of quantifier elimination does not rely on the constant.

Let us assume further that we have a set of constants for $\Bbbk$, in the $\mathbf{RES}$-sort. In some cases, we will work in the reduct of T_{loc} in the 3-sorted language $\mathcal{L}_{\mathrm{val}}$. The language $\mathcal{L}_{\mathrm{val}}$ consists of the valued field sort $\mathbf{VF}$, the value group sort Γ and the residue field sort $\mathbf{RES}$ and maps $\mathrm{val} : \mathbf{VF} \to \Gamma$, $\mathrm{res} : \mathbf{VF} \to \mathbf{RES}$. In this reduct, we have constants for $\Bbbk$ in both $\mathbf{VF}$-sort and $\mathbf{RES}$-sort. The theory ACVF admits quantifier elimination in $\mathcal{L}_{\mathrm{val}}$.

For general facts concerning ACVF, we refer the readers to [12]. For the purpose of the paper, T_{loc} was only introduced to discuss the definable group acting canonically on the space of types, and its definable extensions. The readers should feel free to identify our setup with working in a model of ACVF with constants for $\Bbbk$ in both the valued field and residue field sort, such that $\mathrm{res}(c_{\mathbf{VF}}) = c_{\mathbf{RES}}$ for $c \in \Bbbk$.

Recall that we have a linear algebraic group G defined over $\Bbbk$, an algebraically closed field. We denote by $\mathbf{G}$ the linear algebraic group G viewed as a definable set in $\mathbf{VF}$, and by $\mathbf{G}(\mathcal{O})$ and $\overline{\mathbf{G}}$ the definable subgroups of $\mathcal{O}$ and $\mathbf{RES}$ points, respectively. Here, remember we identify $\mathbf{RES}$ with its image in $\mathbf{VF}$-sort given by the section (this makes sense since G is defined over $\Bbbk$). By applying pointwise the residue map res, we have a map res : $\mathbf{G}(\mathcal{O}) \to \overline{\mathbf{G}}$, and since $\mathbf{G}$ is defined over $\Bbbk \subseteq \mathbf{RES}$, the map is actually a group homomorphism, whose kernel we denote by μ. Note μ is definable over $\Bbbk$ as well. Geometrically, μ can be viewed as an (infinitesimal) neighbourhood of the identity in G.

We apply the results of 2.1 to T_{loc} in the case $\mathbf{H} = \overline{\mathbf{G}}$ and $\mathbf{X} = \mathbf{P}$, a variety over $\Bbbk$ with an (algebraic) action of G, viewed as a definable set in T_{loc}. Our main example of interest will be $\mathbf{P} = \mathbf{G}$, with G acting on itself by left multiplication, but occasionally we will need the more general setup. Note that the choice of $\Bbbk$ as parameters satisfies Convention 2.1 as $\Bbbk = \mathbf{RES}(\mathbb{M})$, by taking $\mathbb{M}$ to be certain field of series with coefficients in $\Bbbk$, say for example, Hahn series.

Definition 2.6. We denote by $S_{\mathbf{P}}(\Bbbk)$ the space of complete $\mathcal{L}_{\text{val}}$-$\mathbf{P}$-types over $\Bbbk$. By quantifier elimination, it is easy to see that such types correspond to tuples of the form $(\mathbf{X}, v)$, where $\mathbf{X}$ is an irreducible subvariety of $\mathbf{P}$ and v is a valuation on the function field of $\mathbf{P}$ which is trivial on $\Bbbk$. For $p \in S_{\mathbf{P}}(\Bbbk)$, the *$\mu$-stabilizer* $\mathrm{Stab}^{\mu}(p)$ of p is $\mathrm{Stab}(\mu \cdot p)$.

By Fact 2.4, the **RES**-sort is stably embedded as an algebraically closed field. In particular, $p \in S_{\mathbf{P}}(\Bbbk)$ is definable over $\Bbbk$ in $\mathcal{L}_{\text{val}}$.

Proposition 2.7. Let $p \in S_{\mathbf{P}}(\Bbbk)$. Then $\mu \cdot p$ is a definable partial type over $\Bbbk$.

Proof. Let $\mathbf{X}$ be a $\mathcal{L}_{\text{val}}$-definable set over $\Bbbk$. Then $\mu \cdot p \models \mathbf{X}$ iff $p(x) \models \forall \epsilon \in \mu(\epsilon \cdot x \in \mathbf{X})$. The latter condition is $\mathcal{L}_{\text{val}}$-definable over $\Bbbk$, hence the result follows from the definability of p. □

By Fact 2.3 and the above discussion, $\mathrm{Stab}^{\mu}(p)$ is given by an intersection of $\mathcal{L}_{\text{val}}$-definable subgroups of $\overline{\mathbf{G}}$. However, $\overline{\mathbf{G}}$ has the descending chain condition on subgroups by Fact 2.4, hence we have:

Corollary 2.8. Let p be an $\mathcal{L}_{val}$-complete G-type over $\Bbbk$. Then the μ-stabilizer $\mathrm{Stab}^{\mu}(p)$ of p is a $\Bbbk$-definable subgroup of $\overline{\mathbf{G}}$, in the sense that there is a $\Bbbk$-definable subgroup $\mathbf{H}$ of $\overline{\mathbf{G}}$, such that $\mathrm{Stab}^{\mu}(p|\mathbb{L})(\mathbb{L}) = \mathbf{H}(\mathbb{L})$ for any model $\mathbb{L} \succeq \mathbb{M}$.

For p and $q \in S_{\mathbf{P}}(\Bbbk)$, define $p \sim q$ if $\mu \cdot p = \mu \cdot q$. It is easy to check that $\mu \cdot p = \mu \cdot q$ iff in a monster model $\mathbb{U}$, there are $a \models p$, $b \models q$ and $\epsilon \in \mu$ such that $\epsilon \cdot a = b$.

We denote by $S^{\mu}_{\mathbf{P}}(\Bbbk)$ the quotient by this equivalence relation and for each $p \in S_{\mathbf{P}}(\Bbbk)$, we denote by p_{μ} its equivalence class. Since μ is normal in $\mathbf{G}(\mathcal{O})$, the $\overline{\mathbf{G}}(\Bbbk)$ action on $S_{\mathbf{P}}(\Bbbk)$ given in Subsection 2.1 respects the equivalence relation. Hence $\overline{\mathbf{G}}(\Bbbk)$ acts on $S^{\mu}_{\mathbf{P}}(\Bbbk)$, and $\mathrm{Stab}^{\mu}(p) = \mathrm{Stab}(p_{\mu})$, where the right hand side is by considering $\overline{\mathbf{G}}(\Bbbk)$ acting on $S^{\mu}_{\mathbf{P}}(\Bbbk)$.

Lastly, we finish the section with an easy fact and some discussion.

Lemma 2.9. Let $g \in \overline{\mathbf{G}}(\Bbbk)$ be such that $g \cdot p_{\mu} = q_{\mu}$. Then $\mathrm{Stab}^{\mu}(q) = g\mathrm{Stab}^{\mu}(p)g^{-1}$.

2.3. **A different view on μ-stabilizers.** Instead of viewing the μ-stabilizers syntactically as in the previous section, we have some concrete constructions to realize them in the monster model as well. In this section, we describe the construction, following the same idea of Section 2.4 in [7]. We work in a fixed monster model of T_{loc}, $\mathbb{U}$, and identify definable sets and (partial) types with their realisations in $\mathbb{U}$. From now on, we restrict our attention to the case $\mathbf{P} = \mathbf{G}$, unless mentioned otherwise. And as before, we identify $\overline{\mathbf{G}}$ with its image under the section.

Definition 2.10. For $p \in S_G(\mathbb{k})$ we use $\overline{\mathbf{G}}_p$ to denote the following set $((\mu \cdot p) \cdot (\mu \cdot p)^{-1}) \cap \overline{\mathbf{G}}$.

Proposition 2.11. Let $a \in \mu \cdot p$. The following are equivalent for an element $b \in \overline{\mathbf{G}}(\mathbb{k})$:

(1) $b \in \mu \cdot p \cdot a^{-1} \cap \overline{\mathbf{G}}$
(2) $b = \mathrm{res}(a_1 a^{-1})$ for some $a_1 \models p$ for which $a_1 a^{-1} \in \mathbf{G}(\mathcal{O})$
(3) $b \in \overline{\mathbf{G}}_p$
(4) $b = \mathrm{res}(a_1 {a_2}^{-1})$ for some $a_1, a_2 \models p$ for which $a_1 {a_2}^{-1} \in \mathbf{G}(\mathcal{O})$

Hence, $\overline{\mathbf{G}}_p(\mathbb{k}) = \mu \cdot p \cdot a^{-1} \cap \overline{\mathbf{G}}(\mathbb{k})$

Proof. The equivalence of (1) and (2) follows directly from the definitions, and likewise for (3) and (4). Hence we need to show (4) implies (2). Assume $b = \mathrm{res}(a_1 a_2^{-1})$. Since a and a_2 satisfy the same type over $\mathbb{k}$, and **RES** is stably embedded and stable, there is an automorphism τ over $\mathbb{k}$ such that $\tau(a_2) = a$. Then $b = \tau(b) = \mathrm{res}(\tau(a_1)a^{-1})$, with $\tau(a_1)$ also satisfying p, showing (2) for b. □

We now have the following description of $\overline{\mathbf{G}}_p$.

Corollary 2.12. $\overline{\mathbf{G}}_p(\mathbb{k}) = \mathrm{Stab}^{\mu}(p)(\mathbb{k})$

Proof. Assume $g \in \overline{\mathbf{G}}$ stabilizes $\mu \cdot p$. Then for any $a \in \mu \cdot p$, $g \cdot a \in \mu \cdot p$, hence $g \in \mu \cdot pa^{-1}$, so is in $\overline{\mathbf{G}}_p$. Conversely, if $g \in \overline{\mathbf{G}}_p$ and $a \in \mu \cdot p$, writing $g = a_1 a^{-1}$ as above we obtain $g \cdot a \in \mu \cdot p$. □

Remark 2.13. We would like to have Cor. 2.12 to hold for **RES**-points, instead of just $\mathbb{k}$-points. This is not automatic, since $\overline{\mathbf{G}}_p$ is not, a-priori, a definable set. However, in the special case as in Theorem 1.1, it is indeed the case.

2.4. **μ-reduced types.** For $p \in S_G(\mathbb{k})$ we denote by $\dim(p)$ the dimension of its Zariski closure in $\mathbf{G}$ over $\mathbb{k}$. Most of the rest of this paper is devoted to comparing this dimension to the dimension of $\mathrm{Stab}^{\mu}(p)$. We first note that if $\mathbf{X}$ is a variety over a valued field L and let $\mathcal{O}$ denote its valuation ring, then the Zariski dimension of $\mathrm{res}(\mathbf{X} \cap \mathcal{O})$ is at most the dimension of $\mathbf{X}$ (this follows for example from [11, Lemma 00QK], by choosing a model of $\mathbf{X}$ over $\mathcal{O}$). Applying this observation to $\mathbf{X} = \mathbf{Y}a^{-1}$, where $\mathbf{Y}$ is a variety containing p, we obtain:

Proposition 2.14. For any $p \in S_G(\mathbb{k})$, $\dim(\mathrm{Stab}^{\mu}(p)) = \dim \overline{\mathbf{G}}_p(\mathbb{k}) \leq \dim(p)$, where dim means the Krull dimension in $\mathrm{Stab}^{\mu}(p)$ and $\dim \overline{\mathbf{G}}_p(\mathbb{k})$. $\dim(p)$ is the minimal **VF**-dimension of the formulas $\varphi \in p$

In general, the above bound will not be sharp, since types of different dimensions may have the same μ-type:

Example 2.15. Let $G = \mathbb{A}^2$ as an additive group. Let K be a large enough Hahn series in variable t over $\mathbb{k}$. Let $p = tp((t^{-1}, t^{-1} + t^r)/\mathbb{k})$ where $r > 0, r \notin \mathbb{Q}$. Then

$\dim(p) = 2$, since $t^{-1}+t^r$ is transcendental over t^{-1}. But $\mu\cdot p = \mu\cdot q$ since $(t^{-1}, t^{-1}+t^r)$ and (t^{-1}, t^{-1}) differes by $(0, -t^r) \in \mu$, where $q = tp((t^{-1}, t^{-1})/\Bbbk)$, and tp denotes the $\mathcal{L}_{\mathrm{val}}$-type. So $\dim(\overline{\mathbf{G}}_p) \leq 1$ (in fact equal). Furthermore, when $\mathrm{Char}(k) = p > 0$, we can see that $\mathrm{Stab}^\mu(p)$ is not torsion-free.

This observation motivates the following definition.

Definition 2.16. For $p \in S_G(\Bbbk)$, we say that p is *μ-reduced* if p is a type of minimal dimension in p_μ. An element $a \in \mathbf{G}$ is μ-reduced over $\Bbbk$ if $a \models p$ for some μ-reduced p.

2.5. **Bounded types.** In this sub-section, we revert to working with a general G-variety $\mathbf{P}$. We recall the following definition (e.g., from [4, §4.2]):

Definition 2.17. Let $\mathbf{V}$ be an affine variety, viewed as a definable set in ACVF, and let $\mathbf{X} \subseteq \mathbf{V}$ be a $\mathcal{L}_{\mathrm{val}}$-definable subset. We say that $\mathbf{X}$ is *bounded* if for every regular function f on $\mathbf{V}$ there is $\gamma \in \Gamma$ such that $\mathrm{val}(f(\mathbf{X})) \geq \gamma$.

For a general variety $\mathbf{V}$, a subset $\mathbf{X} \subseteq \mathbf{V}$ is bounded if it is covered by bounded subsets of an affine cover.

A partial type p in $\mathbf{V}$ is *bounded* if $p \models \mathbf{X}$ for some bounded $\mathbf{X} \subseteq \mathbf{V}$. A type in $\mathbf{V}$ is said to be *centered at infinity* if it is not bounded.

Note that the property of a definable set to be bounded depends on the ambient variety (for example, $\mathbb{A}^1$ is bounded as a subset of $\mathbb{P}^1$, but not as a subset of $\mathbb{A}^1$). However, if $\mathbf{V}$ is a closed subvariety of $\mathbf{W}$, then $\mathbf{X} \subseteq \mathbf{V}$ is bounded in $\mathbf{V}$ if and only if it is bounded in $\mathbf{W}$. Also, it suffices to check the conditions for generators of the regular functions. In particular, a subset $\mathbf{X}$ of a closed subvariety of $\mathbb{A}^n$ is bounded if and only if $\mathrm{val}(\mathbf{X}) \geq \gamma$ for some γ.

Over $\Bbbk$, we have in our situation the following:

Proposition 2.18. A $\Bbbk$-definable set $\mathbf{X} \subseteq \mathbf{V}$ is bounded if and only if it is contained in $\mathbf{V}(\mathcal{O})$

Proof. By definition, it suffices to prove the statement for $\mathbf{V}$ affine, and by the remarks above, for $\mathbf{V} = \mathbb{A}^n$.

If $\mathbf{X} \subseteq \mathcal{O}^n$ we may take $\gamma = 0$ in the definition. Conversely. We may assume $n = 1$ by projecting. If $a \in X \setminus \mathcal{O}$ then $\gamma = \mathrm{val}(a) < 0$ has the same type as any other negative value γ', so there is an automorphism of Γ taking γ to γ', and since Γ is stably embedded and Γ and $\mathbf{RES}$ are orthogonal, it extends to an automorphism over $\Bbbk$ that takes a to $a' \in \mathbf{X}$, with $\mathrm{val}(a') = \gamma'$. Thus, $\mathbf{X}$ is unbounded. □

Let p be a bounded type on $\mathbf{P}$, a variety endowed with an action of $\mathbf{G}$. A realization a of p is then an $\mathcal{O}$-point of $\mathbf{P}$, and so determines a point $\overline{a}$ of $\mathbf{P}$ in the residue field. The type of $\overline{a}$ depends only on p (since it is encoded there), and we denote it by $\overline{p}$. The group $\overline{\mathbf{G}}$ acts on the set of all types in $\overline{\mathbf{P}}$, the variety $\mathbf{P}$ viewed as a definable set in $\mathbf{RES}$. In particular, we may consider the stabilizer of $\overline{p}$.

Proposition 2.19. For any bounded type p on $\mathbf{P}$ we have $\mathrm{Stab}^\mu(p) \leq \mathrm{Stab}(\overline{p})$.

Proof. Let $\overline{a}$ be a realization of $\overline{p}$, and let a be a realization of p whose residue is $\overline{a}$. Assume that for some $g \in \overline{\mathbf{G}}$ we have $g \cdot a = \epsilon \cdot b$ for some $\epsilon \in \mu$ and b realizing

p (so that $g \in \mathrm{Stab}^{\mu}(p)$). Since all elements involved are in $\mathcal{O}$, we may apply the residue map, and obtain $g \cdot \overline{a} = \overline{b}$. Since b realizes p, $\overline{b}$ realizes $\overline{p}$. Thus $g \cdot \overline{p} = \overline{p}$, i.e., $g \in \mathrm{Stab}(\overline{p})$. □

Returning to the case $\mathbf{P} = \mathbf{G}$, we obtain:

Corollary 2.20. If p is a bounded type on $\mathbf{G}$ such that $\overline{p}$ is realized in $\Bbbk$ then its μ-stabilizer is trivial.

Proof. In this case $\overline{p}$ corresponds to a (closed) point of G, hence the stabilizer is trivial. □

Because of the last corollary, we shall concentrate on types centered at infinity.

3. Analyzing the one-dimensional case

In this section, we prove the main theorem in dimension 1 (Theorem 1.1). We think this section worth including even though it follows from the general case, since it is relatively simple, and it sheds light on the important idea in proving the general case. The result in this section is first prove by Moshe Kamensky and Sergei Starchenko in their unpublished notes via the language of places.

3.1. **Points on curves.** Each smooth curve $\mathbf{X}$ over $\Bbbk$ embeds in a unique smooth projective one over $\Bbbk$, its projective model $\tilde{\mathbf{X}}$. Every closed point c on $\tilde{\mathbf{X}}$ corresponds to a valuation val_c on the function field $\Bbbk(\mathbf{X})$, given by the order of vanishing at c. In particular, val_c is trivial on $\Bbbk$. The projective model contains a finite number of closed points outside of $\mathbf{X}$, which we call the points at infinity.

In our case, $\mathbf{X}$ is an affine curve, embedded as a closed subvariety in a fixed affine space $\mathbb{A}^n$. To any point $c \in \tilde{\mathbf{X}}$ we associate the complete type on $\mathbf{X}$ determined by

$$p_c(a) = \{\mathrm{val}(f(a)) > 0 : \mathrm{val}_c(\overline{f}) > 0\}$$

where f runs over all elements of the local ring corresponding to $\mathbf{X}$, and $\overline{f}$ is the corresponding element in $\Bbbk(\mathbf{X})$.

We would like to describe the types that occur in this way intrinsically, in a way that will be helpful later. The condition that c is a closed point corresponds to the following.

Definition 3.1. An extension of (possibly trivially) valued fields is *residually algebraic* if the corresponding residue field extension is algebraic. For L a (possibly trivially) valued field, an $\mathcal{L}_{\mathrm{val}}$-type p over L is *residually algebraic* if a/every realization a satisfies $L(a)$ is residually algebraic over L.

Proposition 3.2. Let $\mathbf{X}$ be a smooth curve embedded in $\mathbb{A}^n$ (viewed as a definable set in $\mathbf{VF}^n$). A $\mathcal{L}_{\mathrm{val}}$-type p over $\Bbbk$ on $\mathbf{X}$ is residually algebraic if and only if it is of the form p_c for a closed point c of $\tilde{\mathbf{X}}$, the projective model of $\mathbf{X}$. Furthermore, $c \in \tilde{\mathbf{X}} \setminus \mathbf{X}$ if and only if p_c is unbounded.

Proof. Let p be a residually algebraic type on $\mathbf{X}$, and let a be a realization that witnesses this. If $a \in \Bbbk$, p corresponds to the $\Bbbk$-point a of $\mathbf{X}$, and we are done. Otherwise, $\Bbbk(a)$ is isomorphic to $\Bbbk(\mathbf{X})$ as a field, and since p is residually algebraic, the valuation on $\Bbbk(a)$ is non-trivial. Thus, we obtain a $\Bbbk$-point c of $\tilde{\mathbf{X}}$ by the discussion earlier in this subsection, and it is clear that the two procedures are inverse to each other. The last statement also follows from the above. □

We have been working with smooth curves, but since we are interested in points at infinity, hence the assumption is immaterial, since the singularity of varieties are at least codimension 1, hence varieties are smooth at generic points.

Corollary 3.3. For $\mathbf{X}$ a curve, there are finitely many residually algebraic types centered at infinity. Moreover, they are isolated by $\mathcal{L}_{\mathrm{val}}$-formulas over $\Bbbk$.

Proof. It remains to prove the moreover part of the statement. Since we know that there are only finitely many types on $\mathbf{X}$ centered at infinity, call them $p_1, ..., p_m$. Without loss of generality, for each i, j, there will be regular functions f_{ij}, g_{ij} such that $p_i \models \mathrm{val}(f_{ij}) < \mathrm{val}(g_{ij})$ but $p_j \models \mathrm{val}(f_{ij}) \geq \mathrm{val}(g_{ij})$. Hence some Boolean combinations of the formulas above together with the formula $x \notin \mathbf{X}(\mathcal{O})$ will isolate the above types. □

Remark 3.4. Let $\mathbf{X}$ be an affine curve embedded in G, and assume that we are given a G-equivariant embedding of G in a G-variety P. Assume that the closure $\mathbf{X}'$ of $\mathbf{X}$ in P includes the point $c \in \tilde{\mathbf{X}}$. The type p_c is then bounded in P, and by Prop. 2.19 the μ-stabilizer of p_c is contained in the stabilizer of the residue type of p_c, which is simply c. Hence, the μ stabilizer of c is contained in the stabilizer of this point in every equivariant "compactification" where the point is realized.

This fact, along with the dimension equalities for the μ-stabilizers justifies viewing the μ-stabilizer as a "canonical stabilizer" for the corresponding point.

Let $p \in S_G(\Bbbk)$ be a residually algebraic type of Zariski dimension 1 inside $\mathbf{G}$. There is then a curve $\mathbf{X}$ in $\mathbf{G}$ containing p. We had explained in Proposition 2.11 and Remark 2.13 that $\mathrm{Stab}^{\mu}(p)(\Bbbk) = \mu \cdot p \cdot a^{-1} \cap \overline{\mathbf{G}}(\Bbbk)$ for any realization a of p (this will be shown again for residually algebraic types in Cor. 4.8). However, since p is isolated by Corollary 3.3, we see that $\mathrm{Stab}^{\mu}(p|\mathbb{L})(\mathbb{L}) = \mu \cdot p \cdot a^{-1}(\mathbb{L}) \cap \overline{\mathbf{G}}(\mathbb{L})$ for any $\mathbb{L}$ extending $\mathbb{M}$ and a. In particular, one can work with a model of T_{loc}, $\mathbb{L}$ with $\mathbf{RES}(\mathbb{L}) = \Bbbk$. Working in this model and let p_i, p_j be as above, if $g \in G(\Bbbk)$ satisfies $g \cdot p_i \in p_{j_\mu}$, then $\mu \cdot p_j \cdot a^{-1} = g \cdot \mathrm{Stab}^{\mu}(p_i)$, for any $a \models p_i$.

To complete the proof, we would like to show that this set is infinite for some realization a of p. This amounts to showing that $\mu \cdot p \cdot a^{-1}$ cannot be covered by a finite number of open balls. To do that, we will use topological methods from [4], which we review below.

3.2. **Tame topology on definable sets.** We make a slight digression into the tame topology of definable sets in ACVF, as developed in [4]. This is an important ingredient in the proof of the main result.

The results in this section can be found in [4]. In this section, the underlying theory is ACVF, and the main motivation is to study the topological structure of $\mathcal{L}_{\mathrm{val}}$-definable sets in the $\mathbf{VF}$-sort.

Definition 3.5. Let $\mathbf{V}$ be an algebraic variety over a valued field F, a subset $\mathbf{X} \subseteq \mathbf{V}$ is *v-open* if it is open for the valuative topology.

A subset $\mathbf{X} \subseteq \mathbf{V}$ is *g-open* if it is a positive Boolean combination of Zariski open, closed sets and sets of the form

$$\{x : v \circ f(x) > v \circ g(x)\}$$

where f and g are regular functions defined on $\mathbf{U}$, a Zariski open subset of $\mathbf{V}$.

If $\mathbf{Z} \subseteq \mathbf{V}$ is a definable subset of $\mathbf{V}$, a subset $\mathbf{W}$ of $\mathbf{Z}$ is said to be v(respectively g)-open if $\mathbf{W}$ is of the form $\mathbf{Z} \cap \mathbf{Y}$, where $\mathbf{Y}$ is v (respectively g)-open in $\mathbf{V}$.

The complement of a v(respectively g)-open is called v(respectively g)-closed. We say $\mathbf{X}$ is v+g-open (respectively v+g-closed) if it is both v-open and g-open (respectively both v-closed and g-closed).

Note that the v+g-opens does not form a topology, as it is not even closed under arbitrary union. However, it is still makes sense to talk about connectedness in this setting:

Definition 3.6. Let $\mathbf{X}$ be a definable subset of $\mathbf{V}$, an algebraic variety. We say that $\mathbf{X}$ is *definably connected* if $\mathbf{X}$ cannot be written as a disjoint union of two non-empty v+g-open subsets of $\mathbf{X}$.

We say that $\mathbf{X}$ has *finitely many definably connected components* if $\mathbf{X}$ can be written as a finite disjoint union of v+g-clopen definably connected subsets.

Definition 3.7. Let $f : \mathbf{V} \to \mathbf{W}$ be a definable function from $\mathbf{V}$ to $\mathbf{W}$, we say f is *v-continuous* if $f^{-1}(\mathbf{X})$ is v-open for $\mathbf{X}$ a v-open subset of $\mathbf{W}$, and we define *g-continuous* functions similarly. We say f is v+g-continuous if f is both v-continuous and g-continuous.

Proposition 3.8 (Hrushovski, Loeser)**.** If f is v+g-continuous and $\mathbf{X}$ is a definably connected and f is defined on $\mathbf{X}$, then $f(\mathbf{X})$ is definably connected.

If $\mathbf{V}$ is an geometrically/absolutely irreducible variety, then $\mathbf{V}$ is definably connected.

The following is an easy corollary of [4, Theorem 11.1.1].

Theorem 3.9 (Hrushovski, Loeser)**.** Given a definable subset $\mathbf{X} \subseteq \mathbf{V}$, where $\mathbf{V}$ is some quasi-projective variety, $\mathbf{X}$ has finitely many definably connected components.

We also have the following.

Theorem 3.10. Let $\mathbf{V} \subseteq \mathbb{A}^n$ be a closed subvariety. $\mathbf{V}$ is bounded iff $\mathbf{V}$ is zero dimensional.

Proof. If $\mathbf{V}$ is bounded, then $\mathbf{V}$ will be definably compact as in Section 4 of [4]. This implies $\mathbf{V}$ is proper by [4, Proposition 4.2.30], hence $\mathbf{V}$ is zero dimensional. The converse is clear. □

We may now prove the following more precise version of Theorem 1.1 (the case of curves).

Theorem 3.11. Let $p \in S_G(\Bbbk)$ be a residually algebraic type, centered at infinity with $\dim(p) = 1$. Then $\dim(\mathrm{Stab}^{\mu}(p)) = 1$.

Proof. By Prop. 2.14 and the discussion preceding Section 3.2, it suffices to show that the μ-stabilizer is infinite. Let $\mathbf{X}$ be the Zariski closure of p in $\mathbf{G}$. Assume to the contrary, that $\mathrm{Stab}^{\mu}(p)$ is finite. Then $\mathrm{res}(\mu \cdot p \cdot a^{-1} \cap \mathbf{G}(\mathcal{O}))$ is finite by Prop. 2.11. By Cor. 3.3, there are only finitely many types centered at infinity on $\mathbf{X}$, so the set $\mathbf{X}\cdot a^{-1} \cap \mathbf{G}(\mathcal{O})$ is the intersection of $\mathbf{X}\cdot a^{-1}$ with a (disjoint) union of finitely many balls $\mu\cdot g$, for $g \in \overline{\mathbf{G}}(\Bbbk)$.

Therefore, $\mathbf{X}\cdot a^{-1} \cap \mathbf{G}(\mathcal{O})$ is a non-empty v+g-open subset of $\mathbf{X}\cdot a^{-1}$. However, it is also a v+g-closed subset of $\mathbf{X}\cdot a^{-1}$ since $\mathbf{G}(\mathcal{O})$ is v+g-closed. By Prop. 3.8, $\mathbf{X}\cdot a^{-1}$ is definably connected, so $\mathbf{X}\cdot a^{-1} \subseteq \mathbf{G}(\mathcal{O}) \subseteq \mathbb{A}^n$. However, this is impossible since this implies that $\mathbf{X}\cdot a^{-1}$ as an affine curve is bounded in $\mathbb{A}^n$, contradicting Theorem 3.10. □

4. Proof of the main theorem

4.1. **Residually algebraic saturation.** We would like to work with saturation in a residually algebraic context, i.e., without extending the residue field. Thus we make the following definition.

Definition 4.1. A model K of T_{loc} is (sufficiently) Γ-*saturated* if every $\mathcal{L}_{\text{val}}$-residually algebraic type over a (sufficiently) small subset of K is realised in K.

Theorem 4.2. Let L be a (possibly trivially) valued field, then there is a Γ-saturated extension of L.

Proof. Let Γ be a sufficiently saturated ordered abelian group and k the algebraically closed closure of $\mathbf{RES}(L)$. Consider the Hahn series field

$$k((t^{\Gamma})) = \{\sum_{\gamma} c_{\gamma}t^{\gamma} : c_{\gamma} \in k, \{\gamma : c_{\gamma} \neq 0\} \text{ is well ordered } \}$$

Clearly L embeds into K (see [6] for example). Then, by a result of Poonen ([10, Theorem 2]), is a Γ-saturated model (with residue field k). □

From now on, K will be a fixed sufficiently Γ-saturated model K with residue field $\Bbbk$ and we will identify definable sets and $p \in S_G(\Bbbk)$ with their realizations in K.

As a first application, we note:

Lemma 4.3. Let $p \in S_G(\Bbbk)$ be residually algebraic. Then there is $q \in p_{\mu}$ which is μ-reduced and residually algebraic.

Proof. Let a be a realization of p in K. There is a variety $\mathbf{V}$ over $\Bbbk$ of minimal dimension that intersects $\mu \cdot a$. The above can be expressed as $\mathcal{L}_{\text{val}}$-formula, so it is witnessed by some element of K. Take q to be the $\mathcal{L}_{\text{val}}$-type of this element over $\Bbbk$. □

We would like to give a syntactic (or geometric) description of types realised in K. To this end, we use another result from [4], which requires the following definition.

The following is a part of Lemma 9.1.1 in [4], which will be needed in the proof.

Lemma 4.4. (Hrushovski, Loeser) Let $\mathbf{V}$ be an F-variety and $\mathbf{X} \subseteq \mathbf{V}$ be a F-definable g-open set, then $\mathbf{X}(M_2) \subseteq \mathbf{X}(M_1)$ whenever M_1 and M_2 are algebraically closed valued field extensions of F with the same underlying field, and $\mathcal{O}_{M_1} \subseteq \mathcal{O}_{M_2}$.

We now have the following description:

Proposition 4.5. Let $\Phi(x)$ be a small finitely consistent collection of g-open sets, with parameters in $L \subseteq K$. Then Φ is realised in K. In addition, if p is a $\mathcal{L}_{\text{val}}$-residually algebraic type, then it is the intersection of the g-open formulas that it implies.

In other words, every partial type Σ of g-open sets admits an extension to a $\mathcal{L}_{\text{val}}$-residually algebraic complete type p over the same set of parameters.

Remark 4.6. It is worth pointing out that Proposition 4.5 has an easy proof in the case when $L = \Bbbk = \mathbb{C}$. See [8, Section 3.1].

Proof. Let b be any realisation of Φ in $\mathbb{U}$, and let k be the residue field of $L(b)$. Then k is the function field of some variety $\mathbf{X}$ over $\Bbbk$, fix a valuation val' of k over $\Bbbk$, with residue field $\Bbbk$.

Let M_2 be the algebraic closure of $L(b)$ with the induced valuation from $\mathbb{U}$. Consider a valuation of $\mathbf{RES}(M_1)$ extending val'. Abusing notaion, we call the valuation val' as well. Let $\bar{\mathcal{O}}$ be the valuation ring of $\mathbf{RES}(M_2)$. Consider $\text{res}^{-1}(\bar{\mathcal{O}}) \subseteq M_2$, this is again a valuation ring of the underlying field of M_2 over $\Bbbk$. We use M_1 to denote the same field as M_2 with the valuation determined by $\text{res}^{-1}(\bar{\mathcal{O}})$. Note that M_1 has residue field $\Bbbk$.

Then by Lemma 4.4, $\phi(M_2) \subseteq \phi(M_1)$ for each $\phi \in \Phi$. In particular, b is a realisation of Φ in M_1. But the residue field of M_1 is $\Bbbk$, so $tp_{M_1}(b/L)$ is residually algebraic and hence realizable in K.

For the converse, let p be a complete $\mathcal{L}_{\text{val}}$-residually algebraic type. By quantifier elimination in ACVF, it is given within its Zariski closure by formulas of the form $f(x) \neq 0$, $\text{val}(f(x)) > \text{val}(g(x))$ and $\text{val}(f(x)) = \text{val}(g(x)) \neq \infty$. Each formula of the last form is equivalent to $\text{val}(f(x)/g(x)) = 0$, so that $f(x)/g(x)$ has non-zero residue. Since p is residually algebraic, the residue is actually a well determined element b of $\Bbbk$, so the original formula is equivalent to $\text{val}(f(x) - bg(x)) > \text{val}(bg(x))$. □

We now apply this result in our context:

Corollary 4.7. Let $p \in S_G(\Bbbk)$. Then $\mu(K)\cdot p(K) = (\mu \cdot p)(K)$.

Proof. Since K is contained in the monster model, $\mu(K)\cdot p(K) \subseteq (\mu \cdot p)(K)$. For the reverse containment, for p residually algebraic, fix $a \in (\mu \cdot p)(K)$. Recall that it means that for any $\phi \in p$, there is $\varepsilon_\phi \in \mu$ such that $\models \phi(\varepsilon_\phi \cdot a)$. Since p is residually algebraic, we may, by Prop. 4.5, assume that each such ϕ is g-open.

Consider the following partial type: $\Sigma(y) = \{\phi(y\cdot a) \wedge \mu(y) : \phi \in p\ g\text{-open}\}$. Each ϕ there is g-open, hence also $\phi(y\cdot a)$ (since the group is algebraic) and μ is given by strict inequalities, so this is a small collection of g-open sets, consistent by assumption. By the other direction of Prop. 4.5, we can find $\varepsilon \in \mu(K)$ such that $\varepsilon\cdot a$ satisfies p. □

Corollary 4.8. Let p be a residually algebraic G-type over $\Bbbk$, and let a be a realization in K. Then $\text{Stab}^\mu(p)(\Bbbk) = \text{res}(\mu(K)\cdot p(K)\cdot a^{-1} \cap \mathbf{G}(\mathcal{O}))$.

Proof. Since $\mathrm{res}(\mu(K)\cdot p(K)\cdot a^{-1} \cap \mathbf{G}(\mathcal{O})) \subseteq \mathrm{res}(\mu(\mathbb{U})\cdot p(\mathbb{U})\cdot a^{-1} \cap \mathbf{G}(\mathcal{O}))$, we have that $\mathrm{Stab}^{\mu}(p)(\Bbbk) \supseteq \mu(K)\cdot p(K)\cdot a^{-1} \cap \mathbf{G}(\mathcal{O})$ by Cor. 2.12. The reverse containment follows from Cor. 4.7. □

4.2. **μ-reduced types and their stabilizers.** In this section we prove Cor. 4.11, an analogue of Corollary 3.3 for types of higher dimension.

Recall that we are working within K, a Γ-saturated model, and all the elements in the statement are from K and definable sets are identified with their realization in K.

In particular, we have the following.

Lemma 4.9. If a is μ-reduced and $g \in \mathbf{G}(\mathcal{O})$, then $g \cdot a$ is also μ-reduced, of the same dimension.

Proof. Assume $\varepsilon \cdot g \cdot a \in \mathbf{W}$ with $\varepsilon \in \mu$ and $\mathbf{W}$ a variety over $\Bbbk$. Since $\varepsilon \cdot g \cdot a = \bar{g} \cdot \varepsilon' \cdot a$ for some $\bar{g} \in \overline{\mathbf{G}}(\Bbbk)$ and $\varepsilon' \in \mu$, we have $\varepsilon' \cdot a \in \bar{g}^{-1} \cdot \mathbf{W}$, a variety over $\Bbbk$ of the same dimension as $\mathbf{W}$. □

The following is an important observation about μ-reduced types.

Proposition 4.10. Let $p \in S_G(\Bbbk)$ be a μ-reduced residually algebraic type centered at infinity, and let $a \models p$. Let $\mathbf{V}$ be the unique irreducible $\Bbbk$-variety such that $a \in \mathbf{V}$ and $\dim(\mathbf{V}) = \dim(p)$. Assume that $\mathbf{X} \subseteq \mathbf{G}(\mathcal{O})\cdot a \cap \mathbf{V}$ be definably connected and $a \in X$. Then for every $b \in \mathbf{X}$ we have $tp(a/\Bbbk) = tp(b/\Bbbk)$, where $tp(\cdot/\Bbbk)$ denotes the $\mathcal{L}_{\mathrm{val}}$-type over $\Bbbk$.

Proof of 4.10. By Lemma 4.9, b is not contained in any proper subvariety of $\mathbf{V}$, so is nonzero when evaluated by any regular function on $\mathbf{V}$. Hence, every element of the function field $\Bbbk(\mathbf{V})$ is well defined as a $\Bbbk$-definable function on $\mathbf{X}$.

Assume that the types of a and b are different. By quantifier elimination in ACVF, without loss of generality, there is $f \in \Bbbk(\mathbf{V})$ such that $\mathrm{val}(f(a)) < 0 \leq \mathrm{val}(f(b))$. We may further assume that the last inequality is strict, by subtracting the residue.

By [4], it can be easily checked that rational functions are v+g-continuous on their domain, so the image $f(\mathbf{X})$ is again definably connected. As a definable subset of K, it is a union of "Swiss cheeses", and by definable connectedness, the Swiss cheese decomposition of the image will be of the form $\mathbf{B} \setminus \cup_{i \leq m} \mathbf{C}_i$, where $\mathbf{B}$ is a ball and $\mathbf{C}_i$'s are disjoint sub-balls of $\mathbf{B}$.

Claim. $f(\mathbf{X})$ contains a $\Bbbk$-point.

Proof of claim. Since $\mathbf{B}$ contains both a point with positive valuation and point with valuation ≤ 0, then it must contain $\mathcal{O}$. If $f(\mathbf{X})$ contains no $\Bbbk$-point, $\Bbbk$ must be covered by $\bigcup_{i \leq m} \mathbf{C}_i$. This implies one of the $\mathbf{C}_i$ contains at least two points in $\Bbbk$ and hence contains $\mathcal{O}$. But this is a contradiction since that means that there is no point in $f(\mathbf{X})$ with positive valuation. □

Hence, we know that there must be some $c \in \Bbbk$ such that $c \in f(\mathbf{X})$. Note however that each element in $\mathbf{X}$ is a generic point of $\mathbf{V}$ by Lemma 4.9 and we know that this would imply that the rational function f is constant, a contradiction to the assumption. Hence we know that $tp(a/\Bbbk) = tp(b/\Bbbk)$. □

Using a similar argument, we have the following, which is the key fact that will replace Cor. 3.3 for our proof of the main theorem.

Corollary 4.11. *Let $a \in \mathbf{V}$ be μ-reduced, with $\mathbf{V}$ the Zariski closure of a over $\Bbbk$. Then there are finitely many types $p_1, ..., p_m \in S_G(\Bbbk)$ for some m such that if $g \in \mathbf{G}(\mathcal{O})$ and $g \cdot a \in \mathbf{V}$, then $tp(g \cdot a/\Bbbk) = p_i$ for some i.*

Proof. From Theorem 3.9, we know that there are only finitely definably connected components of the set $\mathbf{G}(\mathcal{O}) \cdot a \cap \mathbf{V}$, call them $\mathbf{X}_i$ for $i = 1, ..., n$ for some n. By Proposition 4.10, we have that for each $b, b' \in \mathbf{X}_i$, we have that $tp(b/\Bbbk) = tp(b'/\Bbbk)$. Hence there are only finitely many types p_i's with the property stated in the statement of the corollary. □

Here, we stated a variant that is similar to the [7]'s Claim 3.13.

Corollary 4.12. *In the same setting as above, there is a $\mathcal{L}_{\text{val}}$-definable set $\mathbf{X}$ over $\Bbbk$ containing a, such that for each $b \in \mathbf{G}(\mathcal{O}) \cdot a \cap \mathbf{X}$, $tp(b/\Bbbk) = tp(a/\Bbbk)$. Furthermore, $\mathbf{X}$ is v+g-open and $\mathbf{X} \cdot a^{-1}$ is v+g-clopen in $\mathbf{G}(\mathcal{O})$.*

Proof. By the above corollary, we see that there are finitely many regular functions f_{ij}, g_{ij} such that for the formula $\text{val}(f_{ij}) < \text{val}(g_{ij})$ p_i and p_j disagrees. The set defined by some boolean combinations of the above formulas containing $tp(a/\Bbbk)$ will define the set $\mathbf{X}$. □

In particular, we have the following.

Corollary 4.13. *$\text{Stab}^{\mu}(p)(\Bbbk)$ is infinite for each p centered at infinity.*

Proof. Without loss of generality, we can assume that p is μ-reduced and let $a \models p$ be any realization and $\mathbf{V}$ denote its Zariski closure. We have $\dim \mathbf{V} > 0$ since p is centered at infinity. Also, $\mathbf{V}$ is an irreducible $\Bbbk$-variety hence v+g-connected and so is $\mathbf{V} \cdot a^{-1}$. By Proposition 2.11, if $\text{Stab}^{\mu}(p)(\Bbbk) = \overline{\mathbf{G}}_p(\Bbbk)$ is finite, $\mathbf{V} \cdot a^{-1} \cap \mathbf{G}(\mathcal{O})$ can be covered by finitely many v+g-open sets and hence is v+g-open. But $\mathbf{V} \cdot a^{-1} \setminus \mathbf{G}(\mathcal{O})$ is also v+g-open by definition, a contradiction to the fact that non-zero dimensional affine varieties are not bounded in the affine space. □

It is worth noting that the above proof uses the same idea in the 1-dimensional case where the key ingredient is the connectedness of irreducible varieties.

We are now ready to begin the proof of the main theorem on the dimension of μ-stabilizers. Let $a \in \mathbf{V}$ be μ-reduced, and let $\mathbf{V}$ be its Zariski closure. Note that if $g \in \mathbf{G}(\mathcal{O})$ and $g \cdot a \in \cdot V$, then $tp(g \cdot a/\Bbbk)$ will be the same as one of the above p_i's. Hence $\mu \cdot \mathbf{V} \cdot a^{-1} \cap \mathbf{G}(\mathcal{O})$ will be a finite union of cosets of $\mu \cdot \mathbf{G}_p$. Thus it suffices to show that $\text{res}(\mathbf{V} \cdot a^{-1} \cap \mathbf{G}(\mathcal{O}))$ and $\mathbf{V}$ have the same dimension. We will establish this fact in the next section.

4.3. **Dimension of the μ-stabilizers.** Before proving the main theorem, we need the machinery about varieties over $\mathcal{O}$, those facts can be found in [5] and [2].

Definition 4.14. Let $\mathcal{O}$ be a valuation ring, let $L = \text{Frac}(\mathcal{O})$ and $k = \text{res}(\mathcal{O})$. By an variety over $\mathcal{O}$, we mean a flat reduced scheme of finite type over $\mathcal{O}$. In particular, it will have a generic fiber, which is a variety over L, obtained by base changing with respect to the morphism $\mathcal{O} \to L$. And it has a special fiber, which is a variety over k, obtained by base changing with respect to $\mathcal{O} \to k$.

Remark 4.15. It is worth noting that since $\mathcal{O}$ is a valuation ring, $A = \mathcal{O}[x_1, ..., x_n]/I$ is flat over $\mathcal{O}$ iff no nonzero element in $\mathcal{O}$ is a zero divisor in A.

In particular, if S is any subset of $\mathcal{O}^n$, then $I = \{f \in \mathcal{O}[x_1, ..., x_n] : f(s) = 0 \, \forall s \in S\}$ is an ideal and $A = \mathcal{O}[x_1, ..., x_n]/I$ is flat over $\mathcal{O}$. We use I_L and I_k to denote the ideal generated by I in $A \otimes L$ and $A \otimes k$ respectively. Then the generic(respectively special) fiber of $\mathrm{Spec}(A)$ is $\mathrm{Spec}(A \otimes L/I_L)$(respectively $\mathrm{Spec}(A \otimes k/I_k)$).

If $\mathcal{V}$ is a variety over $\mathcal{O}$, we use $\mathbf{V}_k$ to denote its special fiber and $\mathbf{V}_K$ to denote its generic fiber. The following is [2, Theorem 3.2.4].

Theorem 4.16 (Halevi)**.** Let K be a model of ACVF and $\mathcal{V}$ be an irreducible variety over $\mathcal{O}_K$. If $\mathbf{V}_K$ has an $\mathcal{O}_K$-point then the $\mathcal{O}_K$-points are Zariski dense, and the canonical map $\mathrm{res} : \mathbf{V}_K(\mathcal{O}_K) \to \mathbf{V}_k(k)$ is surjective, where res is given by taking residue pointwise.

Now let us get back to the proof of the main theorem.

Proof of Theorem 1.2. By assumption, $\mathbf{V}$ is an affine variety over $\Bbbk$, and $a \in \mathbf{V}$ is a μ-reduced element such that the Zariski closure of a is $\mathbf{V}$. In particular, $\mathbf{V}$ is irreducible. Then $\mathbf{V}\cdot a^{-1}$ is an affine variety over K, and by multiplying the defining polynomials by elements in K, it can be viewed as the generic fiber of a variety $\mathcal{V}$ over $\mathcal{O}_K$. Furthermore, $\mathcal{V}$ has an $\mathcal{O}_K$-point, namely e, the identity of the group G.

It follows from Theorem 4.16 that the map $\mathrm{res} : \mathbf{V}\cdot a^{-1} \cap \mathbf{G}(\mathcal{O}) \to \overline{\mathbf{G}}$ maps onto the special fiber of $\mathbf{V}\cdot a^{-1}$. Also, by flatness, the special fiber has the same dimension as the generic fiber, which is the dimension of p. □

The proof also shows that the special fiber, being the image of $\mathbf{V}\cdot a^{-1} \cap \mathbf{G}(\mathcal{O})$, is a finite union of cosets of $\mathrm{Stab}^{\mu}(p)$. Therefore, we have established the following.

Corollary 4.17. Let $\mathbf{V} \subseteq \mathbf{G}$ be a variety over $\Bbbk$, let $a \models p$, where $p \in S_G(\Bbbk)$ is a μ-reduced residually algebraic type centered at infinity. Assume further that the Zariski closure of p over $\Bbbk$ is $\mathbf{V}$.

Then the special fiber of $\mathbf{V}\cdot a^{-1}$ is equi-dimensional i.e. each irreducible component of it has the same dimension. Moreover, each irreducible component of the special fiber of $\mathbf{V}\cdot a^{-1}$ is a coset of an algebraic subgroup of $\overline{\mathbf{G}}$.

4.4. **Structure of** $\mathrm{Stab}^{\mu}(p)$**.** In this section, we analyze the structure of $\mathrm{Stab}^{\mu}(p)$. Note that due to trivial constraints on characteristic, it is not possible to show in general such a group is torsion-free. However, in characteristic 0, we can indeed show it is torsion free.

Lemma 4.18. Let $p \in S_G(\Bbbk)$ be residually algebraic and let H be a $\Bbbk$-definable linear subgroup of G with $p \in H$. Then $\mathrm{Stab}^{\mu}(p)$ computed in $\mathbf{G}$ and in $\mathbf{H}$ coincide, where $\mathbf{H}$ denotes the group H viewed as a subset in $\mathbf{VF}$.

Proof. Since the Zariski closure $\mathbf{V}$ of p is contained in $\mathbf{H}$ in this case since $\mathbf{H}$ is a Zariski-closed subgroup and $\mu_G \cap \mathbf{H} = \mu_H$. Hence, the arguments of computing the

μ-stabilizers of p can be carried out in both $\mathbf{H}$ and $\mathbf{G}$ and the results will be the same. $\square$

The following is the Iwasawa Decomposition over non-archimedean fields, it can be found in [1, Proposition 4.5.2].

Theorem 4.19. Let $\mathbf{G}$ be a reductive linear algebraic group over $\Bbbk$, there is a solvable subgroup H over $\Bbbk$ such that $\mathbf{G}(K) = \mathbf{G}(\mathcal{O})\cdot \mathbf{H}(K)$.

For GL_n we may take H to be the standard Borel subgroup (upper triangular matrices).

Theorem 4.20. Let $p \in S_G(\Bbbk)$ be centered at infinity and residually algebraic. Then $\mathrm{Stab}^\mu(p)$ is solvable.

Proof. We can embed $\mathbf{G} \subseteq \mathrm{GL}_n$ over $\Bbbk$ for some n, and use the lemma 4.18 to reduce to the case $\mathbf{G} = \mathrm{GL}_n$. Let $\mathbf{H}$ be the Borel. By the Iwasawa decomposition, we have some $g \in \mathbf{G}(\mathcal{O})$ such that $g^{-1}\cdot a = \beta \in \mathbf{H}(K)$. Let $g_1 \in \overline{\mathbf{G}}(\Bbbk)$ be such that $g_1\cdot g^{-1} \in \mu$. Hence $g_1^{-1}\cdot a \in \mu(K)\cdot \beta$, so $\mathrm{Stab}^\mu(g_1^{-1}\cdot p) = \mathrm{Stab}^\mu(q) \subseteq \overline{\mathbf{H}}$. By Lemma 2.9, this group is conjugate to $\mathrm{Stab}^\mu(p)$, hence $\mathrm{Stab}^\mu(p)$ is solvable. $\square$

Corollary 4.21. If G is not solvable and G is irreducible, then there is no μ-reduced residually algebraic G-type of full dimension.

Remark. We briefly introduce the Zariski-Riemann space of a variety over k, and explain its connection with our setting.

Definition. Let V be an variety over k, the Zariski-Riemann space of V over k, is the set of valuation rings of $k(V)$ over k, denoted by $\mathbf{RZ}_k(V)$.

Note that by quantifier elimination in ACVF, for a linear algebraic group $\mathbf{G}$ over $\Bbbk$, it is not hard to see that the above set $\mathbf{RZ}_\Bbbk(\mathbf{G})$ is in canonically embeddable into the set $S_G(\Bbbk)$. Hence we can identify $\mathbf{RZ}_\Bbbk(\mathbf{G})$ with its image in $S_G(\Bbbk)$. Note further, since μ is Zariski dense in $\mathbf{G}$, we see that for each $p \in S_G(\Bbbk)$, there is some $q \in \mathbf{RZ}_\Bbbk(\mathbf{G})$ such that $p \sim_\mu q$.

The above argument implies that we can consider the quotient of $\mathbf{RZ}_\Bbbk(\mathbf{G})$ under μ, which exactly the space $S^\mu_G(\Bbbk)$. Note further that the equivalence relation induced by μ on $\mathbf{RZ}_\Bbbk(\mathbf{G})$ is independent of the $\Bbbk$-closed-immersion of $\mathbf{G}$ into $\mathbb{A}^n$, since every embedding over $\Bbbk$ will respect μ. We will explore more on the structure of $\mathbf{RZ}_\Bbbk(\mathbf{G})$ in a subsequent paper.

References

[1] Daniel Bump. *Automorphic forms and representations*, volume 55. Cambridge University Press, 1998.

[2] Yatir Halevi. On stably pointed varieties and generically stable groups in ACVF. *Ann. Pure Appl. Logic*, 170(2):180–217, 2019.

[3] Ehud Hrushovski and David Kazhdan. Motivic Poisson summation. *Mosc. Math. J.*, 9(3):569–623, back matter, 2009.

[4] Ehud Hrushovski and François Loeser. *Non-Archimedean Tame Topology and Stably Dominated Types (AM-192)*. Princeton University Press, 2016.

[5] Ehud Hrushovski and Silvain Rideau-Kikuchi. Valued fields, metastable groups. *Selecta Math. (N.S.)*, 25(3):Art. 47, 58, 2019.

[6] Irving Kaplansky. Maximal fields with valuations. *Duke Math. J.*, 9(2):303–321, 06 1942.

[7] Ya'acov Peterzil and Sergei Starchenko. Topological groups, μ-types and their stabilizers. *Journal of the European Mathematical Society*, 19(10):2965–2995, 2017.

[8] Ya'acov Peterzil and Sergei Starchenko. Algebraic and o-minimal flows on complex and real tori. *Adv. Math.*, 333:539–569, 2018.

[9] Ya'acov Peterzil and Charles Steinhorn. Definable compactness and definable subgroups of o-minimal groups. *Journal of the London Mathematical Society*, 59(3):769–786, 1999.

[10] Bjorn Poonen. Maximally complete fields. *Enseign. Math*, 39(1-2):87–106, 1993.

[11] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`, 2018.

[12] Lou van den Dries. Lectures on the model theory of valued fields. In *Model Theory in Algebra, Analysis and Arithmetic*, pages 55–157. Springer, 2014.