Journal of Computational and Applied Mathematics 448 (2024) 115918

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A modified weak Galerkin finite element method for the Maxwell equations on polyhedral meshes

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ARTICLE INFO

MSC:
primary 65N15
65N30
76D07
secondary 35B45
35J50

Keywords:
Modified weak Galerkin
Finite element methods
Weak curl
Maxwell equations
Polyhedral meshes

ABSTRACT

We introduce a new numerical method for solving time-harmonic Maxwell's equations via the modified weak Galerkin technique. The inter-element functions of the weak Galerkin finite elements are replaced by the average of the two discontinuous polynomial functions on the two sides of the polygon, in the modified weak Galerkin (MWG) finite element method. With the dependent inter-element functions, the weak curl and the weak gradient are defined directly on totally discontinuous polynomials. Optimal-order convergence of the method is proved. Numerical examples confirm the theory and show effectiveness of the modified weak Galerkin method over the existing methods.

1. Introduction

In this paper, we introduce a new numerical method for solving the time-harmonic Maxwell equations in a heterogeneous medium $\Omega \subset \mathbb{R}^3$. The mixed formulation of this model problem seeks unknown functions $\mathbf{u}$ and p satisfying

$$\nabla \times (\mu \nabla \times \mathbf{u}) - \epsilon \nabla p = \mathbf{f}_1 \quad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot (\epsilon \mathbf{u}) = g_1 \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{3}$$

$$p = 0 \quad \text{on } \partial\Omega, \tag{4}$$

where the constant coefficients $\mu > 0$ and $\epsilon > 0$ are the magnetic permeability and the electric permittivity of the medium, respectively. The formulations (1)–(4) of the Maxwell equations have been used in [1–4], for better numerical stability.

The space $H(\text{curl}; D)$ is defined as the set of vector-valued functions on D which, together with their curl, are square integrable; i.e.,

$$H(\text{curl}; D) = \left\{ \mathbf{v} : \ \mathbf{v} \in [L^2(D)]^3, \nabla \times \mathbf{v} \in [L^2(D)]^3 \right\}.$$

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1 The research of Chunmei Wang was partially supported by National Science Foundation Grants DMS-2136380 and DMS-2206332.

https://doi.org/10.1016/j.cam.2024.115918
Received 4 August 2023; Received in revised form 15 February 2024
Available online 15 April 2024
0377-0427/

Denote the subspace of $H(\text{curl}; D)$ with vanishing trace in the tangential component by

$$H_0(\text{curl}; D) = \left\{ \mathbf{v} : \ \mathbf{v} \in [L^2(D)]^3, \nabla \times \mathbf{v} \in [L^2(D)]^3 \ : \ \mathbf{v} \times \mathbf{n}|_{\partial D} = 0 \right\}.$$

A weak formulation for (1)–(4) seeks $(\mathbf{u}, p) \in H_0(\text{curl}; \Omega) \times H_0^1(\Omega)$ such that

$$(\nu \nabla \times \mathbf{u}, \ \nabla \times \mathbf{v}) - (\mathbf{v}, \nabla p) \ = \ (\mathbf{f}, \ \mathbf{v}), \quad \forall \mathbf{v} \in H_0(\text{curl}; \Omega) \tag{5}$$

$$(\mathbf{u}, \nabla q) \ = \ -(g, q), \quad \forall q \in H_0^1(\Omega), \tag{6}$$

where $\nu = \mu/\epsilon$, $\mathbf{f} = \mathbf{f}_1/\epsilon$ and $g = g_1/\epsilon$.

The numerical methods for the Maxwell equations have been studied extensively, such as the $H(\text{curl}; \Omega)$-conforming edge elements [5–9] and the discontinuous Galerkin finite elements [2,3,10–14]. A weak Galerkin finite element method is studied for the Maxwell equations in [1].

In the weak Galerkin finite element method, the discrete finite element functions are generalized functions denoted by $u_h = \{u_0, u_b\}$ where u_0 is a polynomial on each polyhedron and u_b is an unrelated polynomial on each face-polygon. Using the usual integration by parts, we can define weak derivatives of the generalized function $\{u_0, u_b\}$ as the polynomial L^2-projections of the standard generalized derivatives. The method was first introduced in [15,16] for second order elliptic equations, and applied to other partial differential equations [17–29].

In this work, similar to the weak Galerkin finite element method in [1], the modified weak Galerkin finite element method, which was proposed in [30] and has been applied to various problems [31–34], is applied to the Maxwell equations. In the modified weak Galerkin finite element method, the weak function on the inter-element polygons is replaced by the average on the two sides, i.e., $u_h = \{u_0, u_b\} = \{u_0, \{u_0\}\}$. In particular, it is proved in [35] that any convex or nonconvex combination of two-side functions would work in the modified weak Galerkin finite element method, i.e., $u_b = \theta u_1 + (1-\theta)u_2$ for any real number θ, instead of $\theta = 1/2$ only previously.

In this paper, we prove that the modified weak Galerkin finite elements are inf-sup stable in solving the saddle-point problem (5)–(6). Consequently the optimal order of convergence for $\mathbf{u}$ and p is established. Numerical examples are provided, showing the correctness of the theory and the efficiency over the existing weak Galerkin method.

This section is ended with some basic notations. Let D be any open bounded domain with Lipschitz continuous boundary in $\mathbb{R}^3$. We use the standard definition for the Sobolev space $H^s(D)$ and their associated inner products $(\cdot, \cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$, and seminorms $|\cdot|_{s,D}$ for any $s \geq 0$. For example, for any integer $s \geq 0$, the seminorm $|\cdot|_{s,D}$ is given by

$$|v|_{s,D} = \left(\sum_{|\alpha|=s} \int_D |\partial^\alpha v|^2 dD \right)^{\frac{1}{2}}$$

with the usual notation

$$\alpha = (\alpha_1, \dots, \alpha_d), \quad |\alpha| = \alpha_1 + \cdots + \alpha_d, \quad \partial^\alpha = \prod_{j=1}^{3} \partial_{x_j}^{\alpha_j}.$$

The Sobolev norm $\|\cdot\|_{s,D}$ is given by

$$\|v\|_{s,D} = \left(\sum_{j=0}^{s} |v|_{j,D}^2 \right)^{\frac{1}{2}}.$$

The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\|\cdot\|_D$ and $(\cdot, \cdot)_D$, respectively. When $D = \Omega$, we shall drop the subscript D in the norm and the inner product notation.

2. MWG finite element scheme

Let $\mathcal{T}_h$ be a partition of the domain Ω with mesh size h that consists of polyhedra of arbitrary shape. Assume that the partition $\mathcal{T}_h$ is shape regular. Denote by $\mathcal{E}_h$ the set of all faces in $\mathcal{T}_h$ and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior faces.

For simplicity, we adopt the following notations; i.e.,

$$\begin{aligned}
(v, w)_{\mathcal{T}_h} &= \sum_{T \in \mathcal{T}_h} (v, w)_T = \sum_{T \in \mathcal{T}_h} \int_T v w d\mathbf{x}, \\
\langle v, w \rangle_{\partial \mathcal{T}_h} &= \sum_{T \in \mathcal{T}_h} \langle v, w \rangle_{\partial T} = \sum_{T \in \mathcal{T}_h} \int_{\partial T} v w ds.
\end{aligned}$$

Let $P_k(K)$ consist all the polynomials of degree less or equal to k defined on K.

Let $k \geq 1$. We define two finite element spaces V_h and W_h with boundary conditions as follows

$$\begin{aligned}
V_h &= \left\{ \mathbf{v} \in [L^2(\Omega)]^3 : \ \mathbf{v}|_T \in [P_k(T)]^3, \ T \in \mathcal{T}_h; \ \mathbf{v} \times \mathbf{n}|_{\partial\Omega} = 0 \right\}, \\
W_h &= \left\{ q \in L^2(\Omega) : \ q|_T \in P_{k-1}(T), \ T \in \mathcal{T}_h; \ q|_{\partial\Omega} = 0 \right\}.
\end{aligned}$$

For $T \in \mathcal{T}_h$, we define the jump of τ as $[\tau]_{\partial T} = \frac{1}{2}(\tau|_T - \tau|_{T_n})$ and the average of τ as $\{\tau\}_{\partial T} = \frac{1}{2}(\tau|_T + \tau|_{T_n})$, where T_n denotes the elements neighboring to T. For $e \in \partial T \cap \partial\Omega$, we define $[\tau]_e = \tau$ and $\{\tau\}_e = 0$. Then we have the following identities

$$\langle \mathbf{q}\times\mathbf{n},\ \mathbf{v}\rangle_{\partial\mathcal{T}_h} = \langle [\mathbf{q}]\times\mathbf{n},\ \mathbf{v}\rangle_{\partial\mathcal{T}_h} + \langle \{\mathbf{q}\}\times\mathbf{n},\ \mathbf{v}\rangle_{\partial\mathcal{T}_h}, \tag{7}$$

$$\langle \mathbf{v}\cdot\mathbf{n},\ q\rangle_{\partial\mathcal{T}_h} = \langle \mathbf{v}\cdot\mathbf{n},\ [q]\rangle_{\partial\mathcal{T}_h} + \langle \mathbf{v}\cdot\mathbf{n},\ \{q\}\rangle_{\partial\mathcal{T}_h}. \tag{8}$$

For a function $\mathbf{v} \in V_h$, its weak curl $\nabla_w \times \mathbf{v}$ is defined as a piecewise vector-valued polynomial such that on each $T \in \mathcal{T}_h$, $\nabla_w \times \mathbf{v} \in [P_{k-1}(T)]^3$ satisfies

$$(\nabla_w \times \mathbf{v}, \boldsymbol{\varphi})_T = (\mathbf{v}, \nabla\times\boldsymbol{\varphi})_T - \langle \{\mathbf{v}\}\times\mathbf{n}, \boldsymbol{\varphi}\rangle_{\partial T}, \qquad \forall \boldsymbol{\varphi} \in [P_{k-1}(T)]^3. \tag{9}$$

For a function $q \in W_h$, its weak gradient $\nabla_w q$ is defined as a piecewise vector-valued polynomial such that on each $T \in \mathcal{T}_h$, $\nabla_w q \in [P_k(T)]^3$ satisfies

$$(\nabla_w q, \boldsymbol{\varphi})_T = -(q, \nabla\cdot\boldsymbol{\varphi})_T + \langle \{q\}, \boldsymbol{\varphi}\cdot\mathbf{n}\rangle_{\partial T}, \qquad \forall \boldsymbol{\varphi} \in [P_k(T)]^3. \tag{10}$$

We define a bilinear form with an appropriate stabilization term as follows:

$$a(\mathbf{v},\ \mathbf{w}) = (\nu\nabla_w \times \mathbf{v},\ \nabla_w \times \mathbf{w})_{\mathcal{T}_h} + s_1(\mathbf{v},\mathbf{w}), \tag{11}$$

where

$$\begin{aligned} s_1(\mathbf{v},\ \mathbf{w}) = \sum_{T\in\mathcal{T}_h} & h_T^{-1}(\langle [\mathbf{v}]\times\mathbf{n},\ \ [\mathbf{w}]\times\mathbf{n}\rangle_{\partial T} \\ & +\langle [\mathbf{v}]\cdot\mathbf{n},\ \ [\mathbf{w}]\cdot\mathbf{n}\rangle_{\partial T\setminus\partial\Omega}). \end{aligned} \tag{12}$$

We introduce the following bilinear form

$$b(\mathbf{v},\ q) = (\mathbf{v}, \nabla_w q)_{\mathcal{T}_h}, \tag{13}$$

and a second stabilization term

$$s_2(p,\ q) = \sum_{T\in\mathcal{T}_h} h_T\langle [p],\ \ [q]\rangle_{\partial T}. \tag{14}$$

We are ready to present the MWG method for the weak formulation (5) of the time-harmonic Maxwell model Eqs. (1).

Modified Weak Galerkin Algorithm 1. Find $\mathbf{u}_h \in V_h$ and $p_h \in W_h$ satisfying

$$a(\mathbf{u}_h,\ \mathbf{v}) - b(\mathbf{v},\ p_h) = (\mathbf{f},\ \mathbf{v}) \quad \forall\ \mathbf{v} \in V_h, \tag{15}$$

$$b(\mathbf{u}_h,\ q) + s_2(p_h, q) = -(g, q) \quad \forall\ q \in W_h. \tag{16}$$

Lemma 2.1. *The modified weak Galerkin finite element algorithm (15)–(16) has a unique solution.*

Proof. It suffices to show that zero is the only solution of (15)–(16) if $\mathbf{f} = 0$ and $g = 0$. Letting $\mathbf{v} = \mathbf{u}_h$ and $q = p_h$ in (15)–(16) gives

$$\begin{aligned} &(\nu\nabla_w\times\mathbf{u}_h,\ \nabla_w\times\mathbf{u}_h)_{\mathcal{T}_h} + \sum_{T\in\mathcal{T}_h} h_T^{-1}(\langle [\mathbf{u}_h]\times\mathbf{n},\ \ [\mathbf{u}_h]\times\mathbf{n}\rangle_{\partial T} \\ &\quad +\langle [\mathbf{u}_h]\cdot\mathbf{n},\ \ [\mathbf{u}_h]\cdot\mathbf{n}\rangle_{\partial T\setminus\partial\Omega}) + \sum_{T\in\mathcal{T}_h} h_T\langle [p_h],\ [p_h]\rangle_{\partial T} = 0, \end{aligned}$$

which implies $\nabla_w \times \mathbf{u}_h = 0$ on each T, $\mathbf{u}_h$ is continuous in its tangential and normal directions at element boundary and $[p_h] = 0$ on ∂T. Then, it follows from (9) and the usual integration by parts that for any $\mathbf{v} \in V_h$,

$$\begin{aligned} 0 &= (\nabla_w\times\mathbf{u}_h, \mathbf{v}) \\ &= (\mathbf{u}_h,\ \nabla\times\mathbf{v})_{\mathcal{T}_h} - \langle \{\mathbf{u}_h\}\times\mathbf{n},\ \mathbf{v}\rangle_{\partial\mathcal{T}_h} \\ &= (\nabla\times\mathbf{u}_h,\ \mathbf{v})_{\mathcal{T}_h} + \langle \mathbf{u}_h\times\mathbf{n},\ \mathbf{v}\rangle_{\partial\mathcal{T}_h} - \langle \{\mathbf{u}_h\}\times\mathbf{n},\ \mathbf{v}\rangle_{\partial\mathcal{T}_h} \\ &= (\nabla\times\mathbf{u}_h,\ \mathbf{v})_{\mathcal{T}_h} + \langle [\mathbf{u}_h]\times\mathbf{n},\ \mathbf{v}\rangle_{\partial\mathcal{T}_h} \\ &= (\nabla\times\mathbf{u}_h,\ \mathbf{v})_{\mathcal{T}_h} + \langle \mathbf{u}_h\times\mathbf{n},\ \mathbf{v}\rangle_{\partial\Omega} \\ &= (\nabla\times\mathbf{u}_h,\ \mathbf{v})_{\mathcal{T}_h}, \end{aligned}$$

which gives $\nabla\times\mathbf{u}_h = 0$ on each $T \in \mathcal{T}_h$ due to $\mathbf{u}_h \in V_h$, which, together with $\mathbf{u}_h$ is continuous in its tangential and normal directions at element boundary, yields $\nabla\times\mathbf{u}_h = 0$ in the domain Ω. Using Recall that $[p_h] = 0$ on ∂T. It follows from (16), (10) and the facts $[\mathbf{u}_h]\cdot\mathbf{n} = 0$ on $e \in \mathcal{E}_h^0$ and $\{q\} = 0$ on $\partial\Omega$,

$$\begin{aligned} 0 &= (\mathbf{u}_h, \nabla_w q)_{\mathcal{T}_h} = -(\nabla\cdot\mathbf{u}_h, q)_{\mathcal{T}_h} + \langle \mathbf{u}_h\cdot\mathbf{n},\ \{q\}\rangle_{\partial\mathcal{T}_h} \\ &= -(\nabla\cdot\mathbf{u}_h, q)_{\mathcal{T}_h}. \end{aligned}$$

Letting $q = \nabla \cdot \mathbf{u}_h$ in the above equation yields $\nabla \cdot \mathbf{u}_h = 0$ on each $T \in \mathcal{T}_h$.

Note that $\nabla \times \mathbf{u}_h = 0$ in the domain Ω. Thus, there exists a potential function ϕ such that $\mathbf{u}_h = \nabla\phi$ in Ω. It follows from $\nabla \cdot \mathbf{u}_h = 0$ and the fact that $\mathbf{u}_h \cdot \mathbf{n}$ is continuous that $\Delta\phi = 0$ is strongly satisfied in Ω. The boundary condition of (3) implies that $\mathbf{u}_h \times \mathbf{n} = \nabla\phi \times \mathbf{n} = 0$ on $\partial\Omega$. Therefore, ϕ must be a constant on $\partial\Omega$. The uniqueness of the solution of the Laplace equation implies that $\phi = const$ is the only solution of $\Delta\phi = 0$ if Ω is simply connected. Then we must have $\mathbf{u}_h = \nabla\phi = 0$.

Since $\mathbf{u}_h = 0$, we then have $b(\mathbf{v}, p_h) = 0$ for any $\mathbf{v} \in V_h$ by (15). It follows from the definition of $b(\cdot,\cdot)$ and (10) that

$$\begin{aligned} 0 \ &= \ b(\mathbf{v},\ p_h) = (\mathbf{v}, \nabla_w p_h)_{\mathcal{T}_h} \\ &= -(\nabla \cdot \mathbf{v},\ p_h)_{\mathcal{T}_h} + \langle \mathbf{v} \cdot \mathbf{n},\ \{p_h\}\rangle_{\partial\mathcal{T}_h} \\ &= (\mathbf{v}, \nabla p_h)_{\mathcal{T}_h} - \langle \mathbf{v} \cdot \mathbf{n},\ [p_h]\rangle_{\partial\mathcal{T}_h}, \\ &= (\mathbf{v}, \nabla p_h)_{\mathcal{T}_h}, \end{aligned} \tag{17}$$

where we have used the fact that $[p_h] = 0$ on ∂T. Letting $\mathbf{v} = \nabla p_h$ in (17) gives $\nabla p_h = 0$ on each $T \in \mathcal{T}_h$, i.e. p_h is a constant on $T \in \mathcal{T}_h$. Using the fact that $p_h = [p_h] = 0$ on $\partial\Omega$, we obtain $p_h = 0$ in Ω.

This completes the proof of the Lemma. □

3. Error equations

Denote by $\mathbf{Q}_k$ and Q_{k-1} the element-wise defined L^2 projections onto $[P_k(T)]^3$ and $P_{k-1}(T)$ for each element $T \in \mathcal{T}_h$, respectively. We define two error functions

$$\mathbf{e}_h = \mathbf{Q}_k\mathbf{u} - \mathbf{u}_h, \qquad \epsilon_h = Q_{k-1}p - p_h.$$

Next we will derive the equations that $\mathbf{e}_h$ and ϵ_h satisfy. For simplicity of analysis, we assume that the coefficient ν in (5) is a piecewise constant function with respect to the finite element partition $\mathcal{T}_h$.

Lemma 3.1. *For $\mathbf{v}, \mathbf{w} \in V_h$, we have*

$$(\nabla_w \times \mathbf{v}, \mathbf{w})_{\mathcal{T}_h} = (\nabla \times \mathbf{v}, \mathbf{w})_{\mathcal{T}_h} + \langle [\mathbf{v}] \times \mathbf{n}, \mathbf{w}\rangle_{\partial\mathcal{T}_h}. \tag{18}$$

Proof. Using (9), the usual integration by parts, and (7), we have

$$\begin{aligned} (\nabla_w \times \mathbf{v},\ \mathbf{w})_{\mathcal{T}_h} \ &= \ (\mathbf{v},\ \nabla \times \mathbf{w})_{\mathcal{T}_h} - \langle \{\mathbf{v}\} \times \mathbf{n},\ \mathbf{w}\rangle_{\partial\mathcal{T}_h} \\ &= \ (\nabla \times \mathbf{v},\ \mathbf{w})_{\mathcal{T}_h} + \langle (\mathbf{v} - \{\mathbf{v}\}) \times \mathbf{n},\ \mathbf{w}\rangle_{\partial\mathcal{T}_h} \\ &= \ (\nabla \times \mathbf{v},\ \mathbf{w})_{\mathcal{T}_h} + \langle [\mathbf{v}] \times \mathbf{n}, \mathbf{w}\rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

This completes the lemma. □

Lemma 3.2. *Let $(\mathbf{u}_h, p_h) \in V_h \times W_h$ be the MWG finite element solution arising from (15) and (16). For any $\mathbf{v} \in V_h$ and $q \in W_h$, the following error equations hold true:*

$$\begin{aligned} a(\mathbf{e}_h,\ \mathbf{v}) - b(\mathbf{v},\ \epsilon_h) = &-E_1(\mathbf{u}, \mathbf{v}) - E_2(\mathbf{u}, \mathbf{v}) + E_3(p, \mathbf{v}) \\ &+ s_1(\mathbf{Q}_k\mathbf{u}, \mathbf{v}), \end{aligned} \tag{19}$$

$$b(\mathbf{e}_h,\ q) + s_2(\epsilon_h,\ q) = E_4(\mathbf{u}, q) + s_2(Q_{k-1}p, q), \tag{20}$$

where

$$\begin{aligned} E_1(\mathbf{u}, \mathbf{v}) \ &= \ \langle \nu\nabla \times \mathbf{u} - \mathbf{Q}_k(\nu\nabla \times \mathbf{u}),\ [\mathbf{v}] \times \mathbf{n}\rangle_{\partial\mathcal{T}_h}, \\ E_2(\mathbf{u}, \mathbf{v}) \ &= \ (\mathbf{Q}_k\nu\nabla \times \mathbf{u} - \nu\nabla_w \times \mathbf{Q}_k\mathbf{u},\ \nabla_w \times \mathbf{v})_{\mathcal{T}_h}, \\ E_3(p, \mathbf{v}) \ &= \ \langle \{p - Q_{k-1}p\},\ [\mathbf{v}] \cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h}, \\ E_4(\mathbf{u}, q) \ &= \ \langle (\mathbf{u} - \mathbf{Q}_k\mathbf{u}) \cdot \mathbf{n},\ [q]\rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

Proof. Testing (1) by $\mathbf{v} \in V_h$ gives

$$(\nabla \times (\nu\nabla \times \mathbf{u}),\ \mathbf{v}) - (\nabla p,\ \mathbf{v}) = (\mathbf{f},\ \mathbf{v}). \tag{21}$$

Using the usual integration by parts, (7) and (9), we have

$$\begin{aligned} &(\nabla \times (\nu\nabla \times \mathbf{u}),\ \mathbf{v})_{\mathcal{T}_h} \\ = \ &(\nu\nabla \times \mathbf{u},\ \nabla \times \mathbf{v})_{\mathcal{T}_h} + \langle \nu\nabla \times \mathbf{u},\ \mathbf{v} \times \mathbf{n}\rangle_{\partial\mathcal{T}_h} \\ = \ &(\mathbf{Q}_k(\nu\nabla \times \mathbf{u}),\ \nabla \times \mathbf{v})_{\mathcal{T}_h} + \langle \nu\nabla \times \mathbf{u},\ [\mathbf{v}] \times \mathbf{n}\rangle_{\partial\mathcal{T}_h} \\ = \ &(\nabla \times \mathbf{Q}_k(\nu\nabla \times \mathbf{u}),\ \mathbf{v})_{\mathcal{T}_h} - \langle \mathbf{Q}_k(\nu\nabla \times \mathbf{u}),\ \mathbf{v} \times \mathbf{n}\rangle_{\partial\mathcal{T}_h} \end{aligned}$$

$$
\begin{aligned}
&\quad +\langle \nu\nabla\times\mathbf{u},\ [\mathbf{v}]\times\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\
&= (\nu\nabla\times\mathbf{u},\ \nabla_w\times\mathbf{v})_{\mathcal{T}_h} - \langle \mathbf{Q}_k(\nu\nabla\times\mathbf{u}),\ [\mathbf{v}]\times\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\
&\quad +\langle \nu\nabla\times\mathbf{u},\ [\mathbf{v}]\times\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\
&= (\nu\nabla\times\mathbf{u},\ \nabla_w\times\mathbf{v})_{\mathcal{T}_h} + \langle \nu\nabla\times\mathbf{u} - \mathbf{Q}_k(\nu\nabla\times\mathbf{u}),\ [\mathbf{v}]\times\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\
&= (\mathbf{Q}_k\nu\nabla\times\mathbf{u},\ \nabla_w\times\mathbf{v})_{\mathcal{T}_h} + E_1(\mathbf{u},\mathbf{v})\\
&= (\mathbf{Q}_k\nu\nabla\times\mathbf{u},\ \nabla_w\times\mathbf{v})_{\mathcal{T}_h} + (\nu\nabla_w\times\mathbf{Q}_k\mathbf{u},\ \nabla_w\times\mathbf{v})_{\mathcal{T}_h}\\
&\quad -(\nu\nabla_w\times\mathbf{Q}_k\mathbf{u},\ \nabla_w\times\mathbf{v})_{\mathcal{T}_h} + E_1(\mathbf{u},\mathbf{v})\\
&= (\nu\nabla_w\times\mathbf{Q}_k\mathbf{u},\ \nabla_w\times\mathbf{v})_{\mathcal{T}_h} + E_2(\mathbf{u},\mathbf{v}) + E_1(\mathbf{u},\mathbf{v}).
\end{aligned}
\tag{22}
$$

Using the usual integration by parts, (8) and (10) gives

$$
\begin{aligned}
(\nabla p,\ \mathbf{v}) &= -(p,\ \nabla\cdot\mathbf{v})_{\mathcal{T}_h} + \langle p,\ \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\
&= -(Q_{k-1}p,\ \nabla\cdot\mathbf{v})_{\mathcal{T}_h} + \langle p,\ \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\
&= (\nabla_w Q_{k-1}p,\ \mathbf{v})_{\mathcal{T}_h} + \langle \{p - Q_{k-1}p\},\ \mathbf{v}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\
&= (\nabla_w Q_{k-1}p,\ \mathbf{v})_{\mathcal{T}_h} + \langle \{p - Q_{k-1}p\},\ [\mathbf{v}]\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\
&= (\nabla_w Q_{k-1}p,\ \mathbf{v})_{\mathcal{T}_h} + E_3(p,\mathbf{v}).
\end{aligned}
\tag{23}
$$

Combining the two Eqs. (22) and (23) with (21) gives

$$
\begin{aligned}
&(\nu\nabla_w\times\mathbf{Q}_k\mathbf{u},\ \nabla_w\times\mathbf{v}) - (\nabla_w Q_{k-1}p,\ \mathbf{v})_T\\
&= (\mathbf{f},\ \mathbf{v}) - E_1(\mathbf{u},\mathbf{v}) - E_2(\mathbf{u},\mathbf{v}) + E_3(p,\mathbf{v}).
\end{aligned}
$$

Further we add $s_1(\mathbf{Q}_k\mathbf{u},\mathbf{v})$ to the both sides of the above equation to obtain

$$
\begin{aligned}
a(\mathbf{Q}_k\mathbf{u},\mathbf{v}) - b(\mathbf{v},Q_{k-1}p) = (\mathbf{f},\ \mathbf{v}) &- E_1(\mathbf{u},\mathbf{v})\\
&- E_2(\mathbf{u},\mathbf{v}) + E_3(p,\mathbf{v}) + s_1(\mathbf{Q}_k\mathbf{u},\mathbf{v}).
\end{aligned}
\tag{24}
$$

Subtracting (15) from (24) yields (19).

Testing (2) by $q\in W_h$ gives

$$
(\nabla\cdot\mathbf{u},\ q) = (g,\ q). \tag{25}
$$

It follows from the usual integration by parts, (8) and (10)

$$
\begin{aligned}
(\nabla\cdot\mathbf{u},\ q)_{\mathcal{T}_h} &= -(\mathbf{u},\ \nabla q)_{\mathcal{T}_h} + \langle \mathbf{u}\cdot\mathbf{n},\ q\rangle_{\partial\mathcal{T}_h}\\
&= -(\mathbf{Q}_k\mathbf{u},\ \nabla q)_{\mathcal{T}_h} + \langle \mathbf{u}\cdot\mathbf{n},\ q\rangle_{\partial\mathcal{T}_h}\\
&= (\nabla\cdot\mathbf{Q}_k\mathbf{u},\ q)_{\mathcal{T}_h} + \langle (\mathbf{u}-\mathbf{Q}_k\mathbf{u})\cdot\mathbf{n},\ q\rangle_{\partial\mathcal{T}_h}\\
&= (\nabla\cdot\mathbf{Q}_k\mathbf{u},\ q)_{\mathcal{T}_h} - \langle \mathbf{Q}_k\mathbf{u}\cdot\mathbf{n},\ \{q\}\rangle_{\partial\mathcal{T}_h}\\
&\quad +\langle \mathbf{Q}_k\mathbf{u}\cdot\mathbf{n},\ \{q\}\rangle_{\partial\mathcal{T}_h} + \langle (\mathbf{u}-\mathbf{Q}_k\mathbf{u})\cdot\mathbf{n},\ q\rangle_{\partial\mathcal{T}_h}\\
&= -(\mathbf{Q}_k\mathbf{u},\ \nabla_w q)_{\mathcal{T}_h} - \langle (\mathbf{u}-\mathbf{Q}_k\mathbf{u})\cdot\mathbf{n},\ \{q\}\rangle_{\partial\mathcal{T}_h}\\
&\quad +\langle (\mathbf{u}-\mathbf{Q}_k\mathbf{u})\cdot\mathbf{n},\ q\rangle_{\partial\mathcal{T}_h}\\
&= -(\mathbf{Q}_k\mathbf{u},\ \nabla_w q)_{\mathcal{T}_h} + \langle (\mathbf{u}-\mathbf{Q}_k\mathbf{u})\cdot\mathbf{n},\ [q]\rangle_{\partial\mathcal{T}_h}\\
&= -(\mathbf{Q}_k\mathbf{u},\ \nabla_w q)_{\mathcal{T}_h} + E_4(\mathbf{u},q).
\end{aligned}
$$

Using the equation above, (25) becomes

$$
b(\mathbf{Q}_k\mathbf{u},q) = -(g,\ q) + E_4(\mathbf{u},q).
$$

Adding $s_2(Q_{k-1}p,q)$ to the both sides of the equation above gives

$$
b(\mathbf{Q}_k\mathbf{u},q) + s_2(Q_{k-1}p,q) = -(g,\ q) + E_4(\mathbf{u},q) + s_2(Q_{k-1}p,q). \tag{26}
$$

Subtracting (16) from (26) gives (20).

This completes the proof of the lemma. □

4. Preparation for error estimates

For $\mathbf{v}\in V_h$, we define a semi-norm

$$
|||\mathbf{v}|||^2 = a(\mathbf{v},\ \mathbf{v}) = \sum_{T\in\mathcal{T}_h} \nu\|\nabla_w\times\mathbf{v}\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^{-1}\|[\mathbf{v}]\|_{\partial T}^2. \tag{27}
$$

For $q \in W_h$, we define another semi-norm

$$|q|_{0,h}^2 = \sum_{T\in\mathcal{T}_h} h_T \|[q]\|_{\partial T}^2. \tag{28}$$

Let T be an element with e as a face. For any function $\phi \in H^1(T)$, the following trace inequality has been proved for arbitrary polyhedra T in [16]; i.e.,

$$\|\phi\|_e^2 \le C\left(h_T^{-1}\|\phi\|_T^2 + h_T\|\nabla\phi\|_T^2\right). \tag{29}$$

If ϕ is a polynomial on the element $T \in \mathcal{T}_h$, we have from the inverse inequality that

$$\|\phi\|_e^2 \le C h_T^{-1}\|\phi\|_T^2. \tag{30}$$

Lemma 4.1. *Let $(\mathbf{w}, p) \in [H^{t+1}(\Omega)]^3 \times H^t(\Omega)$ with $\mathbf{w}\times\mathbf{n} = 0$ and $p = 0$ on $\partial\Omega$ and $(\mathbf{v}, q) \in V_h \times W_h$ with $\frac{1}{2} < t \le k$. Then*

$$|s_1(\mathbf{Q}_k\mathbf{w},\ \mathbf{v})| \le Ch^t\|\mathbf{w}\|_{t+1}|||\mathbf{v}|||, \tag{31}$$
$$|s_2(Q_{k-1}p,\ q)| \le Ch^t\|p\|_t\ |q|_{0,h}, \tag{32}$$
$$|E_1(\mathbf{w},\mathbf{v})| \le Ch^t\|\mathbf{w}\|_{t+1}|||\mathbf{v}|||, \tag{33}$$
$$|E_2(\mathbf{w},\mathbf{v})| \le Ch^t\|\mathbf{w}\|_{t+1}|||\mathbf{v}|||, \tag{34}$$
$$|E_3(p,\mathbf{v})| \le Ch^t\|p\|_t|||\mathbf{v}|||, \tag{35}$$
$$|E_4(\mathbf{w},q)| \le Ch^t\|\mathbf{w}\|_{t+1}|q|_{0,h}. \tag{36}$$

Proof. Using the trace inequality (29), the Cauchy–Schwarz inequality, and the properties of the projection operators $\mathbf{Q}_k$ and Q_{k-1}, we have

$$\begin{aligned}
&|s_1(\mathbf{Q}_k\mathbf{w},\ \mathbf{v})|\\
&= |\sum_{T\in\mathcal{T}_h} h_T^{-1}(\langle[\mathbf{Q}_k\mathbf{w}]\times\mathbf{n},\ [\mathbf{v}]\times\mathbf{n}\rangle_{\partial T} + \langle[\mathbf{Q}_k\mathbf{w}]\cdot\mathbf{n},\ [\mathbf{v}]\cdot\mathbf{n}\rangle_{\partial T\setminus\partial\Omega})|\\
&= |\sum_{T\in\mathcal{T}_h} h_T^{-1}(\langle[\mathbf{Q}_k\mathbf{w}-\mathbf{w}]\times\mathbf{n},\ [\mathbf{v}]\times\mathbf{n}\rangle_{\partial T}\\
&\qquad +\langle[\mathbf{Q}_k\mathbf{w}-\mathbf{w}]\cdot\mathbf{n},\ [\mathbf{v}]\cdot\mathbf{n}\rangle_{\partial T\setminus\partial\Omega})|\\
&\le C\left(\sum_{T\in\mathcal{T}_h}(h_T^{-2}\|\mathbf{Q}_k\mathbf{w}-\mathbf{w}\|_T^2 + \|\nabla(\mathbf{Q}_k\mathbf{w}-\mathbf{w})\|_T^2)\right)^{1/2} |||\mathbf{v}|||\\
&\le Ch^t\|\mathbf{w}\|_{t+1}|||\mathbf{v}|||,
\end{aligned}$$

and

$$\begin{aligned}
|s_2(Q_{k-1}p,\ q)| &= |\sum_{T\in\mathcal{T}_h} h_T\langle[Q_{k-1}p],\ [q]\rangle_{\partial T}|\\
&\le |\sum_{T\in\mathcal{T}_h} h_T|\langle[Q_{k-1}p-p],\ [q]\rangle_{\partial T}|\\
&\le Ch^t\|p\|_t\ |q|_{0,h}.
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
|E_1(\mathbf{w},\ \mathbf{v})| &= |\langle\nu\nabla\times\mathbf{w} - \mathbf{Q}_k(\nu\nabla\times\mathbf{w}),\ [\mathbf{v}]\times\mathbf{n}\rangle_{\partial\mathcal{T}_h}|\\
&\le (\sum_{T\in\mathcal{T}_h} h_T\|(I-\mathbf{Q}_k)\nabla\times\mathbf{w}\|_{\partial T}^2)^{1/2}|||\mathbf{v}|||\\
&\le Ch^t\|\mathbf{w}\|_{t+1}|||\mathbf{v}|||,
\end{aligned}$$

and

$$\begin{aligned}
|E_3(p,\mathbf{v})| &= |\langle\{p - Q_{k-1}p\},\ [\mathbf{v}]\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}|\\
&\le (\sum_{T\in\mathcal{T}_h} h_T\|p-Q_{k-1}p\|_{\partial T}^2)^{1/2}|||\mathbf{v}|||\\
&\le Ch^t\|p\|_t|||\mathbf{v}|||,
\end{aligned}$$

and

$$\begin{aligned}
|E_4(\mathbf{w},q)| &= |\langle(\mathbf{w}-\mathbf{Q}_k\mathbf{w})\cdot\mathbf{n},\ [q]\rangle_{\partial\mathcal{T}_h}|\\
&\le (\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{w}-\mathbf{Q}_k\mathbf{w}\|_{\partial T}^2)^{1/2}|q|_{0,h}\\
&\le Ch^t\|\mathbf{w}\|_{t+1}|q|_{0,h}.
\end{aligned}$$

In order to bound $E_2(\mathbf{w},\mathbf{v})$, for $\mathbf{v}\in V_h$, it follows from (7) and (9) that

$$\begin{aligned}
&(\mathbf{Q}_k\nabla\times\mathbf{w}-\nabla_w\times\mathbf{Q}_k\mathbf{w},\ \mathbf{v})_{\mathcal{T}_h}\\
&=(\mathbf{Q}_k\nabla\times\mathbf{w},\ \mathbf{v})_{\mathcal{T}_h}-(\nabla_w\times\mathbf{Q}_k\mathbf{w},\ \mathbf{v})_{\mathcal{T}_h}\\
&=(\mathbf{w},\ \nabla\times\mathbf{v})_{\mathcal{T}_h}-\langle\mathbf{w}\times\mathbf{n},\mathbf{v}\rangle_{\partial\mathcal{T}_h}\\
&\quad-(\mathbf{Q}_k\mathbf{w},\ \nabla\times\mathbf{v})_{\mathcal{T}_h}-\langle\{\mathbf{Q}_k\mathbf{w}\}\times\mathbf{n},\mathbf{v}\rangle_{\partial\mathcal{T}_h}\\
&=(\mathbf{w},\ \nabla\times\mathbf{v})_{\mathcal{T}_h}-\langle\mathbf{w}\times\mathbf{n},[\mathbf{v}]\rangle_{\partial\mathcal{T}_h}-(\mathbf{Q}_k\mathbf{w},\ \nabla\times\mathbf{v})_{\mathcal{T}_h}\\
&\quad-\langle\{\mathbf{Q}_k\mathbf{w}\}\times\mathbf{n},[\mathbf{v}]\rangle_{\partial\mathcal{T}_h}\\
&=(\mathbf{w}-\mathbf{Q}_k\mathbf{w},\ \nabla\times\mathbf{v})_{\mathcal{T}_h}+\langle\{\mathbf{w}-\mathbf{Q}_k\mathbf{w}\},[\mathbf{v}]\times\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\
&=\langle\{\mathbf{w}-\mathbf{Q}_k\mathbf{w}\},[\mathbf{v}]\times\mathbf{n}\rangle_{\partial\mathcal{T}_h}.
\end{aligned}\tag{37}$$

Letting $\mathbf{v}=\mathbf{Q}_k\nabla\times\mathbf{w}-\nabla_w\times\mathbf{Q}_k\mathbf{w}$ in (37) and using the trace inequalities (29)–(30) gives

$$\|\mathbf{Q}_k\nabla\times\mathbf{w}-\nabla_w\times\mathbf{Q}_k\mathbf{w}\|\le Ch^t\|\mathbf{w}\|_{t+1}.\tag{38}$$

Using Cauchy–Schwarz inequality and the estimate (38) implies

$$\begin{aligned}
|E_2(\mathbf{w},\mathbf{v})| &= |(\mathbf{Q}_k(\nu(\nabla\times\mathbf{w})-\nu\nabla_w\times\mathbf{Q}_k\mathbf{w}),\ \nabla_w\times\mathbf{v})_{\mathcal{T}_h}|\\
&\le Ch^t\|\mathbf{w}\|_{t+1}|||\mathbf{v}|||.
\end{aligned}$$

This completes the proof of the lemma. □

5. Error estimates

The objective of this section is to establish the optimal order error estimates for $\mathbf{u}_h$ and p_h in certain discrete norms.

Lemma 5.1. *For any $q\in W_h$, there exist a $\mathbf{v}_q\in V_h$ with $\mathbf{v}|_T=h_T^2\nabla q$ such that*

$$b(\mathbf{v}_q,q)\ge\sum_{T\in\mathcal{T}_h}h_T^2\|\nabla q\|_T^2-C\sum_{T\in\mathcal{T}_h}h_T\|[q]\|_{\partial T}^2\tag{39}$$

and

$$|||\mathbf{v}_q|||^2\le C\sum_{T\in\mathcal{T}_h}h_T^2\|\nabla q\|_T^2,\tag{40}$$

where C is a constant independent of h.

Proof. For a given $q\in W_h$ and $\mathbf{v}\in V_h$, it follows from the usual integration by parts, (8) and (10) that

$$\begin{aligned}
b(\mathbf{v},q)&=(\mathbf{v},\nabla_w q)_{\mathcal{T}_h}\\
&=\langle\mathbf{v}\cdot\mathbf{n},\{q\}\rangle_{\partial\mathcal{T}_h}-(\nabla\cdot\mathbf{v},q)_{\mathcal{T}_h}\\
&=(\mathbf{v},\nabla q)_{\mathcal{T}_h}-\langle\mathbf{v}\cdot\mathbf{n},[q]\rangle_{\partial\mathcal{T}_h}.
\end{aligned}$$

By choosing $\mathbf{v}|_T=2h_T^2\nabla q$, we arrive at

$$b(\mathbf{v},q)=2\sum_{T\in\mathcal{T}_h}h_T^2(\nabla q,\nabla q)_T-\sum_{T\in\mathcal{T}_h}2h_T^2\langle\nabla q\cdot\mathbf{n},[q]\rangle_{\partial T}.$$

Now by the Cauchy–Schwarz inequality, the trace inequality (30) and the inverse inequality we obtain

$$\begin{aligned}
b(\mathbf{v},q)&\ge2\sum_{T\in\mathcal{T}_h}h_T^2(\nabla q,\nabla q)_T-2\sum_{T\in\mathcal{T}_h}h_T^2\|\nabla q\cdot\mathbf{n}\|_{\partial T}\|[q]\|_{\partial T}\\
&\ge\sum_{T\in\mathcal{T}_h}2h_T^2(\nabla q,\nabla q)_T-\sum_{T\in\mathcal{T}_h}Ch_T^{1.5}\|\nabla q\|_T\|[q]\|_{\partial T}\\
&\ge\sum_{T\in\mathcal{T}_h}h_T^2\|\nabla q\|_T^2-C\sum_{T\in\mathcal{T}_h}h_T\|[q]\|_{\partial T}^2,
\end{aligned}$$

which gives rise to the inequality (39). The boundedness estimate (40) can be obtained by computing the triple bar norm of $\mathbf{v}_q$ directly with the trace and the inverse inequalities. This completes the proof of the lemma. □

Theorem 5.2. *Let $(\mathbf{u},p)\in[H^{t+1}(\Omega)]^3\times[H^t(\Omega)]$ with $\frac{1}{2}<t\le k$ and $(\mathbf{u}_h,p_h)\in V_h\times W_h$ be the solution of (1)–(4) and (15)–(16) respectively. There holds*

$$|||\mathbf{e}_h|||+|\epsilon_h|_{0,h}\le Ch^t(\|\mathbf{u}\|_{t+1}+\|p\|_t),\tag{41}$$

$$\Big(\sum_{T\in\mathcal{T}_h}h_T^2\|\nabla\epsilon_h\|_T^2\Big)^{\frac{1}{2}}\le Ch^t(\|\mathbf{u}\|_{t+1}+\|p\|_t).\tag{42}$$

Proof. By letting $\mathbf{v} = \mathbf{e}_h$ in (19) and $q = \epsilon_h$ in (20) and adding the two resulting equations, we have

$$\begin{aligned}|||\mathbf{e}_h|||^2 + |\epsilon_h|_{0,h}^2 &= -E_1(\mathbf{u}, \mathbf{e}_h) - E_2(\mathbf{u}, \mathbf{e}_h) + E_3(p, \mathbf{e}_h)\\ &\quad + s_1(\mathbf{Q}_k\mathbf{u}, \mathbf{e}_h) + E_4(\mathbf{u}, \epsilon_h) + s_2(Q_{k-1}p, \epsilon_h).\end{aligned} \tag{43}$$

Substituting the estimates (31)–(36) into (43) yields

$$|||\mathbf{e}_h|||^2 + |\epsilon_h|_{0,h}^2 \le Ch^t(\|\mathbf{u}\|_{t+1} + \|p\|_t)(|||\mathbf{e}_h||| + |\epsilon_h|_{0,h}) \tag{44}$$

which implies the error estimate (41).

It follows from (19) with $\mathbf{v}|_T = \mathbf{v}_{\epsilon_h}|_T = h_T^2\nabla\epsilon_h$ that

$$\begin{aligned}b(\mathbf{v}_{\epsilon_h},\ \epsilon_h) &= a(\mathbf{e}_h,\ \mathbf{v}_{\epsilon_h}) + E_1(\mathbf{u}, \mathbf{v}_{\epsilon_h}) + E_2(\mathbf{u}, \mathbf{v}_{\epsilon_h})\\ &\quad -\ E_3(p, \mathbf{v}_{\epsilon_h}) - s_1(\mathbf{Q}_k\mathbf{u}, \mathbf{v}_{\epsilon_h}),\end{aligned}$$

which, together with Cauchy–Schwarz inequality, (39), and (31)–(36), yields

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h} h_T^2\|\nabla\epsilon_h\|_T^2\\ &\le |b(\mathbf{v}_{\epsilon_h}, \epsilon_h)| + C|\epsilon_h|_{0,h}^2\\ &\le |a(\mathbf{e}_h,\ \mathbf{v}_{\epsilon_h})| + |E_1(\mathbf{u}, \mathbf{v}_{\epsilon_h})| + |E_2(\mathbf{u}, \mathbf{v}_{\epsilon_h})|\\ &+ |E_3(p, \mathbf{v}_{\epsilon_h})| + |s_1(\mathbf{Q}_k\mathbf{u}, \mathbf{v}_{\epsilon_h})| + C|\epsilon_h|_{0,h}^2\\ &\le |||\mathbf{e}_h|||\ |||\mathbf{v}_{\epsilon_h}||| + h^t(\|\mathbf{u}\|_{t+1} + \|p\|_t)|||\mathbf{v}_{\epsilon_h}||| + C|\epsilon_h|_{0,h}^2\\ &\le C\Big(|||\mathbf{e}_h||| + h^t(\|\mathbf{u}\|_{t+1} + \|p\|_t)\Big)\big(\sum_{T\in\mathcal{T}_h} h_T^2\|\nabla\epsilon_h\|_T^2\big)^{\frac12} + C|\epsilon_h|_{0,h}^2,\end{aligned} \tag{45}$$

where we have used the estimate (40) in the last inequality. The estimate (42) can be obtained from (41) and (45).

This completes the proof of the theorem. □

6. An error estimate in L^2 norm

We consider an auxiliary problem that seeks $(\boldsymbol{\psi}, \xi)$ satisfying

$$\begin{aligned}\nabla\times(\nu\nabla\times\boldsymbol{\psi}) - \nabla\xi &= \mathbf{e}_h \quad \text{in } \Omega,\\ \nabla\cdot\boldsymbol{\psi} &= 0 \quad \text{in } \Omega,\\ \boldsymbol{\psi}\times\mathbf{n} &= 0 \quad \text{on } \partial\Omega,\\ \xi &= 0 \quad \text{on } \partial\Omega.\end{aligned} \tag{46}$$

Assume that the problem (46) has the $[H^{1+s}(\Omega)]^3 \times H^s(\Omega)$-regularity property in the sense that the solution $(\boldsymbol{\psi}, \xi) \in [H^{1+s}(\Omega)]^3 \times H^s(\Omega)$ and the following a priori estimate holds true:

$$\|\boldsymbol{\psi}\|_{1+s} + \|\xi\|_s \le C\|\mathbf{e}_h\|, \tag{47}$$

where $0 < s \le 1$.

Theorem 6.1. *Let $(\mathbf{u}, p) \in [H^{t+1}(\Omega)]^3 \times [H^t(\Omega)]$ and $(\mathbf{u}_h, p_h) \in V_h \times W_h$ be the solutions of (1)–(4) and (15)–(16), respectively. Let $\frac12 < t \le k$ and $0 < s \le 1$. Then,*

$$\|\mathbf{Q}_k\mathbf{u} - \mathbf{u}_h\| \le Ch^{t+s}(\|\mathbf{u}\|_{t+1} + \|p\|_t). \tag{48}$$

Proof. Testing the first equation of (46) by $\mathbf{e}_h$ gives

$$\|\mathbf{Q}_k\mathbf{u} - \mathbf{u}_h\|^2 = (\nabla\times(\nu\nabla\times\boldsymbol{\psi}),\ \mathbf{e}_h) - (\nabla\xi,\ \mathbf{e}_h).$$

Using (22) and (23) (with $\boldsymbol{\psi}, \xi, \mathbf{e}_h$ in the place of $\mathbf{u}, p, \mathbf{v}$, respectively), the above equation becomes

$$\begin{aligned}\|\mathbf{Q}_k\mathbf{u} - \mathbf{u}_h\|^2 &= (\nu\nabla_w\times\mathbf{Q}_k\boldsymbol{\psi},\ \nabla_w\times\mathbf{e}_h) - (\mathbf{e}_h,\ \nabla_w(Q_h\xi))\\ &\quad + E_2(\boldsymbol{\psi}, \mathbf{e}_h) + E_1(\boldsymbol{\psi}, \mathbf{e}_h) - E_3(\xi, \mathbf{e}_h)\\ &= a(\mathbf{Q}_k\boldsymbol{\psi},\ \mathbf{e}_h) - b(\mathbf{e}_h, \mathbb{Q}_h\xi) - s_1(\mathbf{Q}_k\boldsymbol{\psi},\ \mathbf{e}_h)\\ &\quad + E_2(\boldsymbol{\psi}, \mathbf{e}_h) + E_1(\boldsymbol{\psi}, \mathbf{e}_h) - E_3(\xi, \mathbf{e}_h).\end{aligned}$$

The error Eq. (20) implies

$$b(\mathbf{e}_h,\ Q_{k-1}\xi) = -s_2(\epsilon_h,\ Q_{k-1}\xi) + E_4(\mathbf{u}, Q_{k-1}\xi) + s_2(Q_{k-1}p, Q_{k-1}\xi).$$

Combining the two equations above yields

$$\begin{aligned}\|\mathbf{Q}_k\mathbf{u}-\mathbf{u}_h\|^2 = {} & a(\mathbf{Q}_k\boldsymbol{\psi},\ \mathbf{e}_h) - s_2(\epsilon_h,\ Q_{k-1}\xi) + E_4(\mathbf{u}, Q_{k-1}\xi)\\ & + s_2(Q_{k-1}p, Q_k\xi) - s_1(\mathbf{Q}_k\boldsymbol{\psi},\ \mathbf{e}_h)\\ & + E_2(\boldsymbol{\psi},\mathbf{e}_h) + E_1(\boldsymbol{\psi},\mathbf{e}_h) + E_3(\xi,\mathbf{e}_h).\end{aligned} \tag{49}$$

It now follows from the definition of $\mathbf{Q}_k$, (10), the usual integration by parts, and the second equation of (46) that

$$\begin{aligned} & b(\mathbf{Q}_k\boldsymbol{\psi},\ \epsilon_h)\\ = {} & (\mathbf{Q}_k\boldsymbol{\psi}, \nabla_w\epsilon_h)_{\mathcal{T}_h} = \langle\{\epsilon_h\},\ \mathbf{Q}_k\boldsymbol{\psi}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} - (\epsilon_h,\ \nabla\cdot(\mathbf{Q}_k\boldsymbol{\psi}))_{\mathcal{T}_h}\\ = {} & \langle\{\epsilon_h\}-\epsilon_h,\ \mathbf{Q}_k\boldsymbol{\psi}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} + (\nabla\epsilon_h,\ \boldsymbol{\psi})_{\mathcal{T}_h}\\ = {} & -\langle[\epsilon_h],\ \mathbf{Q}_k\boldsymbol{\psi}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} + \langle[\epsilon_h],\ \boldsymbol{\psi}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\ = {} & E_4(\boldsymbol{\psi},\epsilon_h),\end{aligned}$$

where we have used the fact that $\sum_{T\in\mathcal{T}_h}\langle\{\epsilon_h\},\ \boldsymbol{\psi}\cdot\mathbf{n}\rangle_{\partial T}=0$. Adding and subtracting $b(\mathbf{Q}_k\boldsymbol{\psi},\ \epsilon_h)$ to Eq. (49) and using the equation above and (19), we have

$$\begin{aligned} & \|\mathbf{Q}_k\mathbf{u}-\mathbf{u}_h\|^2\\ = {} & a(\mathbf{Q}_k\boldsymbol{\psi},\ \mathbf{e}_h) - b(\mathbf{Q}_k\boldsymbol{\psi},\ \epsilon_h) + E_4(\boldsymbol{\psi},\epsilon_h)\\ & - s_2(\epsilon_h,\ Q_{k-1}\xi) + E_4(\mathbf{u},Q_{k-1}\xi) + s_2(Q_{k-1}p, Q_{k-1}\xi)\\ & - s_1(\mathbf{Q}_k\boldsymbol{\psi},\ \mathbf{e}_h) + E_2(\boldsymbol{\psi},\mathbf{e}_h) + E_1(\boldsymbol{\psi},\mathbf{e}_h) + E_3(\xi,\mathbf{e}_h)\\ = {} & -E_1(\mathbf{u},\mathbf{Q}_k\boldsymbol{\psi}) - E_2(\mathbf{u},\mathbf{Q}_k\boldsymbol{\psi}) + E_3(p,\mathbf{Q}_k\boldsymbol{\psi})\\ & + s_1(\mathbf{Q}_k\mathbf{u},\mathbf{Q}_k\boldsymbol{\psi}) + E_4(\boldsymbol{\psi},\epsilon_h) - s_2(\epsilon_h,\ Q_{k-1}\xi)\\ & + E_4(\mathbf{u},Q_{k-1}\xi) + s_2(Q_{k-1}p,Q_{k-1}\xi) - s_1(\mathbf{Q}_k\boldsymbol{\psi},\ \mathbf{e}_h)\\ & + E_2(\boldsymbol{\psi},\mathbf{e}_h) + E_1(\boldsymbol{\psi},\mathbf{e}_h) + E_3(\xi,\mathbf{e}_h).\end{aligned} \tag{50}$$

We will provide the estimates for the twelve terms on the last line of (50). Among them, we can bound six of them by using Lemma 4.1 with $t = s$ and (41),

$$\begin{aligned}|s_1(\mathbf{Q}_k\boldsymbol{\psi},\ \mathbf{e}_h)| &\le Ch^s\|\boldsymbol{\psi}\|_{1+s}|\!|\!|\mathbf{e}_h|\!|\!| \le Ch^{t+s}(\|\mathbf{u}\|_{t+1}+\|p\|_t)\|\boldsymbol{\psi}\|_{1+s},\\ |s_2(\epsilon_h,\ Q_h\xi)| &\le Ch^s\|\xi\|_s\ |\epsilon_h|_{0,h} \le Ch^{t+s}(\|\mathbf{u}\|_{t+1}+\|p\|_t)\|\xi\|_s,\\ |E_1(\boldsymbol{\psi},\mathbf{v})| &\le Ch^s\|\boldsymbol{\psi}\|_{1+s}|\!|\!|\mathbf{e}_h|\!|\!| \le Ch^{t+s}(\|\mathbf{u}\|_{t+1}+\|p\|_t)\|\boldsymbol{\psi}\|_{1+s},\\ |E_2(\boldsymbol{\psi},\mathbf{e}_h)| &\le Ch^s\|\boldsymbol{\psi}\|_{1+s}|\!|\!|\mathbf{e}_h|\!|\!| \le Ch^{t+s}(\|\mathbf{u}\|_{t+1}+\|p\|_t)\|\boldsymbol{\psi}\|_{1+s},\\ |E_3(\xi,\mathbf{e}_h)| &\le Ch^s\|\xi\|_s|\!|\!|\mathbf{e}_h|\!|\!| \le Ch^{t+s}(\|\mathbf{u}\|_{t+1}+\|p\|_t)\|\xi\|_s,\\ |E_4(\boldsymbol{\psi},\epsilon_h)| &\le Ch^s\|\boldsymbol{\psi}\|_{1+s}|\epsilon_h|_{0,h} \le Ch^{t+s}(\|\mathbf{u}\|_{t+1}+\|p\|_t)\|\boldsymbol{\psi}\|_{1+s}.\end{aligned}$$

Using the trace inequality (29), Cauchy–Schwarz inequality, and the properties of the projection operators $\mathbf{Q}_k$ and Q_{k-1}, we have

$$\begin{aligned}|s_1(\mathbf{Q}_k\mathbf{u},\ \mathbf{Q}_k\boldsymbol{\psi})| &= |\sum_{T\in\mathcal{T}_h} h_T^{-1}(\langle[\mathbf{Q}_k\mathbf{u}]\times\mathbf{n},\ [\mathbf{Q}_k\boldsymbol{\psi}]\times\mathbf{n}\rangle_{\partial T}\\ &\qquad + \langle[\mathbf{Q}_k\mathbf{u}]\cdot\mathbf{n},\ [\mathbf{Q}_k\boldsymbol{\psi}]\cdot\mathbf{n}\rangle_{\partial T\setminus\partial\Omega})|\\ &= |\sum_{T\in\mathcal{T}_h} h_T^{-1}(\langle[\mathbf{Q}_k\mathbf{u}-\mathbf{u}]\times\mathbf{n},\ [\mathbf{Q}_k\boldsymbol{\psi}-\boldsymbol{\psi}]\times\mathbf{n}\rangle_{\partial T}\\ &\qquad + \langle[\mathbf{Q}_k\mathbf{u}-\mathbf{u}]\cdot\mathbf{n},\ [\mathbf{Q}_k\boldsymbol{\psi}-\boldsymbol{\psi}]\cdot\mathbf{n}\rangle_{\partial T\setminus\partial\Omega})|\\ &\le Ch^{t+s}\|\mathbf{u}\|_{t+1}\|\boldsymbol{\psi}\|_{1+s},\end{aligned}$$

and

$$\begin{aligned}|s_2(Q_{k-1}p,\ Q_{k-1}\xi)| &= |\sum_{T\in\mathcal{T}_h} h_T\langle[Q_{k-1}p],\ [Q_{k-1}\xi]\rangle_{\partial T}|\\ &\le |\sum_{T\in\mathcal{T}_h} h_T|\langle[Q_{k-1}p-p],\ [Q_{k-1}\xi-\xi]\rangle_{\partial T}|\\ &\le Ch^{t+s}\|p\|_t\|\xi\|_s.\end{aligned}$$

Similarly, we have that

$$\begin{aligned}|E_1(\mathbf{u},\ \mathbf{Q}_k\boldsymbol{\psi})| &= |\langle\nu\nabla\times\mathbf{u}-\mathbf{Q}_k(\nu\nabla\times\mathbf{u}),\ [\mathbf{Q}_k\boldsymbol{\psi}]\times\mathbf{n}\rangle_{\partial\mathcal{T}_h}|\\ &= |\langle\nu\nabla\times\mathbf{u}-\mathbf{Q}_k(\nu\nabla\times\mathbf{u}),\ [\mathbf{Q}_k\boldsymbol{\psi}-\boldsymbol{\psi}]\times\mathbf{n}\rangle_{\partial\mathcal{T}_h}|\\ &\le Ch^{t+s}\|\mathbf{u}\|_{t+1}\|\boldsymbol{\psi}\|_{1+s}.\end{aligned}$$

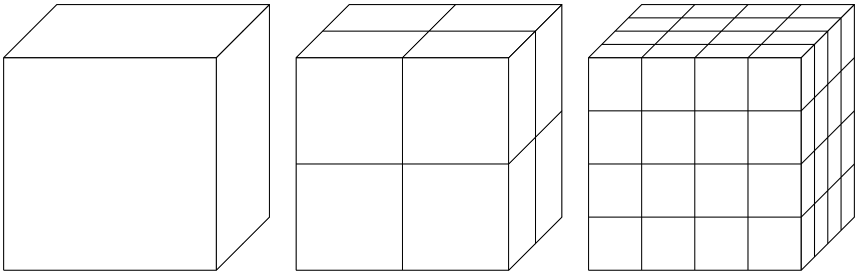

Fig. 1. The first three grids for the computation in Tables 1–4.

Table 1
Error profile on the meshes shown as in Fig. 1 for the solution (51), by the P_1 elements.

Grid	$\|Q_k\boldsymbol{u}-\boldsymbol{u}_h\|$	$O(h^r)$	$⦀Q_k\boldsymbol{u}-\boldsymbol{u}_h⦀$	$O(h^r)$	$\|p-p_h\|$	$O(h^r)$	dim
	By the P_1 weak Galerkin finite element method [1].						
1	0.124E+01	0.0	0.267E+01	0.0	0.114E+00	0.0	31
2	0.372E+00	1.7	0.156E+01	0.8	0.122E+00	0.0	296
3	0.107E+00	1.8	0.889E+00	0.8	0.817E−01	0.6	2560
4	0.315E−01	1.8	0.487E+00	0.9	0.423E−01	0.9	21 248
5	0.879E−02	1.9	0.253E+00	0.9	0.203E−01	1.1	173 056
	By the P_1 modified weak Galerkin finite element method.						
1	0.124E+01	0.0	0.267E+01	0.0	0.114E+00	0.0	31
2	0.294E+00	2.1	0.134E+01	1.0	0.120E+00	0.0	176
3	0.757E−01	2.0	0.598E+00	1.2	0.817E−01	0.6	1120
4	0.166E−01	2.2	0.236E+00	1.3	0.452E−01	0.9	7808
5	0.387E−02	2.1	0.933E−01	1.3	0.219E−01	1.0	57 856

and

$$\begin{aligned}|E_3(p,\mathbf{Q}_k\boldsymbol{\psi})| &= |\langle\{p-Q_{k-1}p\},\ [\mathbf{Q}_k\boldsymbol{\psi}]\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}|\\ &= |\langle\{p-Q_{k-1}p\},\ [\mathbf{Q}_k\boldsymbol{\psi}-\boldsymbol{\psi}]\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}|\\ &\le Ch^{t+s}\|p\|_t\|\boldsymbol{\psi}\|_{1+s}.\end{aligned}$$

and

$$\begin{aligned}|E_4(\mathbf{u},Q_{k-1}\xi)| &= |\langle(\mathbf{u}-\mathbf{Q}_k\mathbf{u})\cdot\mathbf{n},\ [Q_{k-1}\xi]\rangle_{\partial\mathcal{T}_h}|\\ &= |\langle(\mathbf{u}-\mathbf{Q}_k\mathbf{u})\cdot\mathbf{n},\ [Q_{k-1}\xi-\xi]\rangle_{\partial\mathcal{T}_h}|\\ &\le Ch^{t+s}\|\mathbf{u}\|_{t+1}\|\xi\|_s.\end{aligned}$$

Using (37), (38), and the trace inequality (29), we have

$$\begin{aligned}|E_2(\mathbf{u},\ \mathbf{Q}_k\boldsymbol{\psi})| &= |(\mathbf{Q}_k\nabla\times\mathbf{u}-\nabla_w\times\mathbf{Q}_k\mathbf{u},\ \nabla_w\times\mathbf{Q}_k\boldsymbol{\psi})|\\ &= |\langle\{\mathbf{u}-\mathbf{Q}_k\mathbf{u}\},[\nabla_w\times\mathbf{Q}_k\boldsymbol{\psi}]\times\mathbf{n}\rangle_{\partial\mathcal{T}_h}|\\ &= |\langle\{\mathbf{u}-\mathbf{Q}_k\mathbf{u}\},[\nabla_w\times\mathbf{Q}_k\boldsymbol{\psi}-\nabla\times\boldsymbol{\psi}]\times\mathbf{n}\rangle_{\partial\mathcal{T}_h}|\\ &\le Ch^{t+s}\|\mathbf{u}\|_{t+1}\|\boldsymbol{\psi}\|_{1+s}.\end{aligned}$$

Substituting all the estimates above into (50) and using (47) imply the error estimate (48). This completes the proof of the theorem. □

7. Numerical results

We solve the model problem (1)–(4) on the unit cube domain $\Omega=(0,1)^3$, with $\mu=\epsilon=1$. The exact solution is chosen as

$$\mathbf{u}=\begin{pmatrix}z^2\\x^3\\y^4\end{pmatrix},\quad p=x^4, \tag{51}$$

which defines the functions $\mathbf{f}_1$, g_1, and non-homogeneous boundary conditions in (1)–(4). The computation is done on uniform cubic meshes shown in Fig. 1. In Tables 1–2, we list the errors and the computed order of convergence in various norms, by the $P_k(k=1,2,3,4)$ weak Galerkin finite elements [1] and by the new $P_k(k=1,2,3,4)$ modified weak Galerkin finite elements. The

Table 2
Error profile on the meshes shown as in Fig. 1 for the solution (51), by the P_2 elements.

Grid	$\|Q_k\boldsymbol{u} - \boldsymbol{u}_h\|$	$O(h^r)$	$⦀Q_k\boldsymbol{u} - \boldsymbol{u}_h⦀$	$O(h^r)$	$\|p - p_h\|$	$O(h^r)$	dim
	By the P_2 weak Galerkin finite element method [1].						
1	0.475E+00	0.0	0.131E+01	0.0	0.160E+00	0.0	70
2	0.738E−01	2.7	0.372E+00	1.8	0.783E−01	1.0	668
3	0.105E−01	2.8	0.980E−01	1.9	0.241E−01	1.7	5776
	By the P_2 modified weak Galerkin finite element method.						
1	0.475E+00	0.0	0.131E+01	0.0	0.160E+00	0.0	70
2	0.695E−01	2.8	0.359E+00	1.9	0.798E−01	1.0	416
3	0.127E−01	2.4	0.106E+00	1.8	0.243E−01	1.7	2752

Table 3
Error profile on the meshes shown as in Fig. 1 for the solution (51), by the P_3 elements.

Grid	$\|Q_k\boldsymbol{u} - \boldsymbol{u}_h\|$	$O(h^r)$	$⦀Q_k\boldsymbol{u} - \boldsymbol{u}_h⦀$	$O(h^r)$	$\|p - p_h\|$	$O(h^r)$	dim
	By the P_3 weak Galerkin finite element method [1].						
1	0.138E+00	0.0	0.344E+00	0.0	0.114E+00	0.0	130
2	0.101E−01	3.8	0.432E−01	3.0	0.193E−01	2.6	1232
3	0.674E−03	3.9	0.538E−02	3.0	0.270E−02	2.8	10 624
	By the P_3 modified weak Galerkin finite element method.						
1	0.138E+00	0.0	0.344E+00	0.0	0.114E+00	0.0	130
2	0.930E−02	3.9	0.340E−01	3.3	0.195E−01	2.5	800
3	0.554E−03	4.1	0.301E−02	3.5	0.273E−02	2.8	5440

Table 4
Error profile on the meshes shown as in Fig. 1 for the solution (51), by the P_4 elements.

Grid	$\|Q_k\boldsymbol{u} - \boldsymbol{u}_h\|$	$O(h^r)$	$⦀Q_k\boldsymbol{u} - \boldsymbol{u}_h⦀$	$O(h^r)$	$\|p - p_h\|$	$O(h^r)$	dim
	By the P_4 weak Galerkin finite element method [1].						
1	0.317E−01	0.0	0.652E−01	0.0	0.319E−01	0.0	215
2	0.998E−03	5.0	0.367E−02	4.1	0.222E−02	3.8	2020
3	0.330E−04	4.9	0.209E−03	4.1	0.144E−03	3.9	17 360
	By the P_4 modified weak Galerkin finite element method.						
1	0.317E−01	0.0	0.652E−01	0.0	0.319E−01	0.0	215
2	0.892E−03	5.2	0.286E−02	4.5	0.227E−02	3.8	1360
3	0.295E−04	4.9	0.165E−03	4.1	0.147E−03	3.9	9440

optimal-order convergence is achieved in all cases. For the P_1 finite element methods, the number of unknowns in the modified weak Galerkin finite element discrete equations is about one-third of that of the weak Galerkin equations. But both methods produce solutions of about equal accuracy, in all the tests.

Data availability

No data was used for the research described in the article.

References

[1] L. Mu, J. Wang, X. Ye, S. Zhang, A weak Galerkin finite element method for the Maxwell equations, J. Sci. Comput. 65 (1) (2015) 363–386.
[2] I. Perugia, D. Schotzau, The hp-local discontinuous Galerkin method for low-frequency time-harmonic Maxwell equations, Math. Comp. 72 (2003) 1179–1214.
[3] I. Perugia, D. Schotzau, P. Monk, Stabilized interior penalty methods for the time- harmonic Maxwell equations, Comput. Methods Appl. Mech. Engrg. 191 (2002) 4675–4697.
[4] L. Vardapetyan, L. Demkowicz, hp-adaptive finite elements in electromagnetics, Comput. Methods Appl. Mech. Engrg. 169 (1999) 331–344.
[5] A. Bossavit, Bossavit Computational Electromagnetism, Academic Press, San Diego, 1998.
[6] J. Jin, The Finite Element Method in Electromagnetics, second ed., John Wiley & Sons, Inc., New York, 2002.
[7] P. Monk, Finite Element Methods for Maxwell's Equations, Oxford University Press, New York, 2003.
[8] J. Nedelec, Mixed finite elements in R3, Numer. Math. 35 (1980) 315–341.
[9] J. Nedelec, A new family of mixed finite elements in R3, Numer. Math. 50 (1986) 57–81.
[10] S. Brenner, F. Li, L. Sung, A locally divergence-free interior penalty method for two-dimensional curl-curl problems, SIAM J. Numer. Anal. 42 (2008) 1190–1211.
[11] S. Brenner, J. Cui, Z. Nan, L. Sung, Hodge decomposition for divergence-free vector fields and two-dimensional Maxwell's equations, Math. Comp. 81 (2012) 643–659.

[12] P. Houston, I. Perugia, D. Schotzau, Mixed Discontinuous Galerkin Approximation of the Maxwell Operator, Tech. Report 02-16, University of Basel, Department of Mathematics, Basel, Switzerland, 2002.
[13] P. Houston, I. Perugia, D. Schotzau, hp-DGFEM for Maxwell's equations, in: F. Brezzi, A. Buffa, S. Corsaro, A. Murli (Eds.), Numerical Mathematics and Advanced Applications: ENUMATH 2001, Springer-Verlag, Berlin, 2003, pp. 785–794.
[14] P. Houston, I. Perugia, D. Schotzau, Mixed discontinuous Galerkin approximation of the Maxwell operator, SIAM J. Numer. Anal. 42 (2004) 434–459.
[15] J. Wang, X. Ye, A weak Galerkin finite element method for second-order elliptic problems, J. Comput. Appl. Math. 241 (2013) 103–115.
[16] J. Wang, X. Ye, A weak Galerkin mixed finite element method for second-order elliptic problems, Math. Comp. 83 (2014) 2101–2126.
[17] C. Wang, J. Wang, An efficient numerical scheme for the Biharmonic equation by weak Galerkin finite element methods on polygonal or polyhedral meshes, Comput. Math. Appl. 68 (12 Part B) (2014) 2314–2330, arXiv:1309.5560v1.
[18] X. Ye, S. Zhang, A stabilizer-free weak Galerkin finite element method on polytopal meshes, J. Comput. Appl. Math. 371 (2020) 112699, 9pp.
[19] X. Ye, S. Zhang, A conforming discontinuous Galerkin finite element method, Int. J. Numer. Anal. Model. 17 (1) (2020) 110–117.
[20] X. Ye, S. Zhang, A conforming discontinuous Galerkin finite element method: Part II, Int. J. Numer. Anal. Model. 17 (2) (2020) 281–296.
[21] X. Ye, S. Zhang, A conforming discontinuous Galerkin finite element method: Part III, Int. J. Numer. Anal. Model. 17 (6) (2020) 794–805.
[22] X. Ye, S. Zhang, A stabilizer free weak Galerkin method for the biharmonic equation on polytopal meshes, SIAM J. Numer. Anal. 58 (5) (2020) 2572–2588.
[23] X. Ye, S. Zhang, A stabilizer free WG finite element method on polytopal mesh: Part III, J. Comput. Appl. Math. 394 (2021) Artical No. 113538, 9 pp.
[24] X. Ye, S. Zhang, A stabilizer-free pressure-robust finite element method for the Stokes equations, Adv. Comput. Math. 47 (2) (2021) Artical No. 28, 17 pp.
[25] X. Ye, S. Zhang, A stabilizer free WG method for the Stokes equations with order two superconvergence on polytopal mesh, Electron. Res. Arch. 29 (6) (2021) 3609–3627.
[26] X. Ye, S. Zhang, A numerical scheme with divergence free H-div triangular finite element for the Stokes equations, Appl. Numer. Math. 167 (2021) 211–217.
[27] X. Ye, S. Zhang, A C^0-conforming DG finite element method for biharmonic equations on triangle/tetrahedron, J. Numer. Math. 30 (3) (2022) 163–172.
[28] X. Ye, S. Zhang, A weak divergence CDG method for the biharmonic equation on triangular and tetrahedral meshes, Appl. Numer. Math. 178 (2022) 155–165.
[29] X. Ye, S. Zhang, Four-order superconvergent weak Galerkin methods for the biharmonic equation on triangular meshes, Commun. Appl. Math. Comput. (2022) http://dx.doi.org/10.1007/s42967-022-00201-5.
[30] X. Wang, N. Malluwawadu, F. Gao, T. McMillan, A modified weak Galerkin finite element method, J. Comput. Appl. Math. 217 (2014) 319–327.
[31] M. Cui, X. Ye, S. Zhang, A modified weak Galerkin finite element method for the biharmonic equation on polytopal meshes, Commun. Appl. Math. Comput. 3 (1) (2021) 91–105.
[32] F. Gao, S. Zhang, P. Zhu, Modified weak Galerkin method with weakly imposed boundary condition for convection-dominated diffusion equations, Appl. Numer. Math. 157 (2020) 490–504.
[33] L. Mu, X. Wang, X. Ye, A modified weak Galerkin finite element method for the Stokes equations, J. Comput. Appl. Math. 275 (2015) 79–90.
[34] X. Wang, X. Meng, S. Zhang, H. Shangyou, Zhou, A modified weak galerkin finite element method for the linear elasticity problem in mixed form, J. Comput. Appl. Math. 420 (2023) Paper No. 114743, 19 pp.
[35] F. Gao, X. Ye, S. Zhang, A discontinuous Galerkin finite element method without interior penalty terms, Adv. Appl. Math. Mech. 14 (2) (2022) 299–314.